# Contact structures on $\mathbb{R}^3$ from Trkalian fields and Maxwell's equations

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## 1 Introduction

The Beltrami equation $\nabla \times X = kX$ states that the rotation of $X$ is everywhere parallel to the field. Due to this condition, a characteristic feature of all Beltrami fields is a constant twisting. Such fields are found in many areas of physics. For instance, particle movement in tornadoes and waterspouts can be approximated by Beltrami fields [8]. Also, with suitable boundary conditions, the magnetic field inside a fusion reactor can be modeled as a Beltrami field [4]. Beltrami fields are also used to solve Maxwell's equations in bi-isotropic media [7, 8].

In reference [1] a correspondence between Beltrami fields and contact structures on 3-manifolds is established. Here, this result is used to derive contact structures on $\mathbb{R}^3$ from a special class of Beltrami fields, namely Trkalian vector fields. In Sections 2 and 3 of this study, we give the necessary definitions for Beltrami fields, Trkalian fields, and contact structures. The rest of the study is divided into two parts. In the Section 4, we combine the above correspondence with a result about Trkalian vector fields on $\mathbb{R}^3$ [3]. As a result, we obtain Proposition 4.2 from which contact structures on $\mathbb{R}^3$ can be generated. As a special case, it follows that every analytic function on $\mathbb{C}$ generates a contact structure on $\mathbb{C}\times\mathbb{R}$ (Corollary 4.5.) We also show that all the standard contact forms $dz - ydx$, $dz + \frac{1}{2}(xdy - ydx)$, and $\cos z dx - \sin z dy$ on $\mathbb{R}^3$ can be generated from Proposition 4.2. In the second part (Section 5), the result of Section 4 is used to derive a contact structure on $\mathbb{R}^3$ from Maxwell's equations in isotropic medium. The conclusions and suggestions for further work are given in Section 6.

Throughout this study, we assume that the underlying space is a real 3-dimensional manifold as defined in reference [15]; an *$n$-dimensional manifold*, denoted by $M^n$, is a topological Hausdorff space with countable base that is locally homeomorphic to $\mathbb{R}^n$. The space of differential $p$-forms on $M^n$ is denoted by $\Omega^p(M^n)$, and the tangent space at a point $x \in M^n$ is denoted by $T_xM^n$. All functions, $p$-forms and vector fields are assumed $C^\infty$-smooth. The Einstein summing convention is used throughout this work. By $\Re\{x\}$ and $\Im\{x\}$ we denote the real and imaginary parts of a complex number $x$. The complex unit is denoted by $i = \sqrt{-1}$.

## 2 Trkalian fields

For completeness, we next define both Beltrami fields and Trkalian fields. In this work we shall, however, only study Trkalian fields.

**Definition 2.1** *[1, 3] Let $M^3$ be a Riemann manifold, and let $X$ be a vector field (with possibly complex component functions.) If $\nabla \times X = fX$ for some function $f : M^3 \to \mathbb{R}$, then $X$ is said to be a* Beltrami vector field*. If $f$ is a non-zero real scalar, then $X$ is said to be a* Trkalian vector field.

In the above definition, the Riemann metric plays a crucial role. Unfortunately this metric dependence is not explicitly seen from the equation $\nabla \times X = fX$ (a statement for the vector field $X$.) For this reason, we shall use differential forms [9, 11, 14, 15].

With differential forms all the traditional differential operators $\nabla$, $\nabla\times$, and $\nabla\cdot$ can be expressed using only two operators: the *Hodge star operator* $*$ and the *exterior derivative* $d$. The Hodge star operator, defined below, depends only on the Riemann metric. The exterior derivative, on the other hand, is purely topological in the sense that it does not depend on the metric.

**Definition 2.2** *[13] Let $M^n$ be a $n$-dimensional manifold with a Riemann metric $g = g_{ij}dx^i \otimes dx^j$. The* Hodge star operator *is the linear operator $* : \Omega^p(M^n) \to \Omega^{n-p}(M^n)$ that maps the basis elements of $\Omega^p(M^n)$ as*

$$*(dx^{i_1} \wedge \cdots \wedge dx^{i_p}) \quad = \quad \frac{\sqrt{|g|}}{(n-p)!} g^{i_1 l_1} \cdots g^{i_p l_p} \varepsilon_{l_1 \cdots l_p \, l_{p+1} \cdots l_n} dx^{l_{p+1}} \wedge \cdots \wedge dx^{l_n},$$

*where $|g| = \det g_{ij}$ and $\varepsilon$ is the* Levi-Civita permutation symbol*,*

$$\varepsilon_{k_1 \cdots k_m} = \varepsilon^{k_1 \cdots k_m} = \begin{cases} +1 & \text{when } k_1 \cdots k_m \text{ is an even permutation,} \\ -1 & \text{when } k_1 \cdots k_m \text{ is an odd permutation,} \\ 0 & \text{when } k_i = k_j, \text{ for some } i \neq j. \end{cases}$$

To transform vectors into 1-forms and vice-versa, we use the standard isomorphism induced by the Riemann metric $g = g_{ij}dx^i \otimes dx^j$ [11]: By contracting the metric with the vector field $X = X^i \frac{\partial}{\partial x^i}$, we obtain the 1-form $X^\flat = g(X, \cdot) = g_{ij}X^i dx^j$. This $\flat$-mapping transforms vector fields into 1-forms. Since $g_{ij}$ is positive definite, the mapping also has an inverse, a $\sharp$-mapping. Let $g^{ij}$ denote the elements of the $(g_{ij})^{-1}$. For the 1-form $\omega = \omega_i dx^i$, $\omega^\sharp = g^{ij}\omega_i \frac{\partial}{\partial x^j}$.

**Definition 2.3** *[11] Let $M^3$ be a Riemann manifold with a metric tensor $g$. The* curl *of the vector field $X$ is the vector field $\nabla \times X$ for which $(\nabla \times X)^\flat = *dX^\flat$. The* gradient *of a function $f : M^3 \to \mathbb{R}$ is the vector field $\nabla f$ for which $(\nabla f)^\flat = dF$.*

From the definition of curl, we see that if $X$ is a Trkalian vector field, then $*dX^\flat = kX^\flat$. Reading this as an equation for the 1-form $X^\flat$, it will be motivated (in Section 4) to call $X^\flat$ a Trkalian 1-form.

**Definition 2.4** *Let $M^3$ be a Riemann manifold. A (possibly complex valued) 1-form $\alpha$ on $M^3$ is a* Trkalian 1-form *if $*d\alpha = k\alpha$ for some non-zero $k \in \mathbb{R}$.*

For easy reference, we end this section with some useful properties of the Hodge star operator.

Generally $** = (-1)^{p(n-p)}\mathrm{Id}_{\Omega^p(M^n)}$ [13]. In three dimensions, $** = \mathrm{Id}_{\Omega^p(M^3)}$ for $p = 0, \cdots, 3$. In cartesian coordinates, the metric tensor is $g = dx \otimes dx + dy \otimes dy + dz \otimes dz$, and the Hodge star operator is

$$*dx = dy \wedge dz, \qquad *dy = dz \wedge dx, \qquad *dz = dx \wedge dy.$$

# 3 Contact Structures

Suppose that for each $p$ in $M^3$ we associate in a smooth manner a two-dimensional subspace $\xi_p$ of $T_pM$. We then say that $\xi$ is a *plane field* on $M^3$. An *integral surface* of $\xi$ is a two dimensional submanifold $S$ of $M^3$ such that for all $p \in M^3$, we have $T_pS \subset \xi_p$ [14]. An integral surface to $\xi$ is, in other words, a set of smooth surfaces everywhere tangential to $\xi$. An arbitrary plane field usually does not have an integral surface. If the planes twist, the planes can not be "stitched" together to form smooth surfaces. A plane field $\xi$ that has no integral surfaces, not even at a single point, is called a *contact structure* on $M^3$ [1, 5].

For the remainder of this study, we only consider plane fields $\xi$ that globally can be written as the kernel of a (real valued) 1-form $\alpha \in \Omega^1(M^3)$, i.e., $\xi = \ker \alpha$. The plane field $\xi$ is then said to be *transversally orientable*, and $\alpha$ is said to be a *contact form* for $\xi$ [5]. The reason for this restriction is given by Frobenius theorem below, which gives a sufficient and necessary condition for $\xi$ to be a contact structure in terms of $\alpha$.

**Theorem 3.1 (Frobenius theorem [11, 14])** *Let $\alpha$ be a 1-form on a 3-manifold. The plane field $\xi = \ker \alpha$ is a contact structure if and only if $\alpha \wedge d\alpha \not\equiv 0$.*

Clearly, rescaling $\alpha$ by a non-vanishing function does not modify the actual contact structure $\ker \alpha$. It should also be pointed out that a contact structure does not require a metric structure.

Here, we have only defined contact structures in three dimensions. However, the definition of a contact structure and the Frobenius theorem both generalize to manifolds of odd dimensions [5, 11, 14].

The important correspondence between Beltrami fields and contact structures is established in reference [1]. Here we are only interested in the result that Beltrami fields induce contact structures. The related result for Trkalian 1-forms reads:

**Proposition 3.2** *Let $M^3$ be a Riemann manifold, and let $\alpha$ be a real valued Trkalian 1-form on $M^3$. If $\alpha$ is identically non-vanishing, then $\alpha$ is a contact form on $M^3$.*

*Proof:* Since $\alpha$ is a Trkalian 1-form, $*d\alpha = k\alpha$ for some non-zero $k \in \mathbb{R}$. Then $\alpha \wedge d\alpha = k\alpha \wedge *\alpha$, which is the *pointwise norm* of $\alpha$ [11]. Since $\alpha$ is non-vanishing, $\alpha \wedge d\alpha \not\equiv 0$ and, by Frobenius theorem, $\alpha$ is a contact form. $\square$

# 4 Contact structures from Trkalian vector fields

In reference [3] it is shown that Trkalian fields on $\mathbb{R}^3$ either have a cartesian symmetry or a spherical symmetry. Moreover, general expressions are given from which every Trkalian vector field on $\mathbb{R}^3$ can be derived; either in cartesian coordinates, or in spherical coordinates. In this section, we will use Proposition 3.2 to derive a contact structure for Trkalian fields with a cartesian symmetry. We begin by presenting the result of reference [3] in more detail. The main result of this section is Proposition 4.2.

In reference [3] it is shown, by means of *Monge potentials*, that an arbitrary real vector field on $\mathbb{R}^3$ can be written as the real part of a complex vector field of the form

$$X = e^{ih}\nabla F,$$

where $h : \mathbb{R}^3 \to \mathbb{R}$ and $F : \mathbb{R}^3 \to \mathbb{C}$. The main result of reference [3] is the complete classification of $h$, $F$, and of the Riemann metric of $\mathbb{R}^3$ when $X$ is a solution to the Trkalian equation

$$X = \nabla \times X.$$

It is shown that there are only two possible cases:

*a)* There are Cartesian coordinates $x, y, z$ such that $h = z$, $F : \mathbb{R}^3 \to \mathbb{C}$ is an analytic function of $x + iy$, $F$ is independent of $z$, and

$$X = e^{iz}\nabla F(x + iy). \tag{1}$$

or

*b)* There are spherical coordinates $r, \theta, \phi$ such that $h = r$, $F : \mathbb{R}^3\backslash\{0\} \to \mathbb{C}$ is function of $i\phi - \text{arc tanh}(\cos\theta)$, and

$$X = e^{ir}\nabla F\big(i\phi - \text{arc tanh}(\cos\theta)\big). \tag{2}$$

For $X$ to be single valued in the spherical geometry, $F$ must be either a $2\pi$ periodic function of $\phi$ or a complex scalar multiple of the identity function [3].

Trkalian vector fields and 1-forms are closely related. Suppose $X$ is a complex Trkalian vector field. Then $\alpha = X^\flat$ is a complex Trkalian 1-form, $*d\alpha = k\alpha$, and $\Re\{\alpha\}$ is a real Trkalian 1-form. Furthermore, if $\Re\{\alpha\} \not\equiv 0$, then, by Proposition 3.2, $\Re\{\alpha\}$ is a contact form on $\mathbb{R}^3$. We next use this observation to derive a contact structure for Trkalian fields with cartesian symmetry.

**A. Cartesian coordinates**

We assume the vector field has the slightly more general form

$$X = e^{i\kappa z}\nabla F(x, y, z),$$

where $x, y, z$ are the cartesian coordinates of $\mathbb{R}^3$, $\kappa \in \mathbb{R}$ is non-zero, and $F : \mathbb{R}^3 \to \mathbb{C}$. Furthermore, we assume that $\mathbb{R}^3$ is equipped with the standard cartesian metric.

Next, we derive conditions on $F$ for $\alpha = X^\flat$ to be a Trkalian 1-form, i.e., for $*d\alpha = k\alpha$. From Definition 2.3, we have

$$\begin{aligned} \alpha &= X^\flat \\ &= e^{i\kappa z}dF. \end{aligned}$$

Here we can make an important observation. Even though the gradient and the $\flat$-operator both depend on the Riemann metric, the 1-form $(\nabla F)^\flat = dF$ does not depend on the metric. In consequence, the Trkalian 1-forms corresponding to the vector fields 1 and 2 do not depend on the metric. This motivates our choice to work with Trkalian 1-forms and not with Trkalian vector fields. In fact, the above observation shows that the gradient of a function $F$ should be defined as the 1-form $dF$ and not as the vector field $(dF)^\sharp$ [11].

Using the properties of the Hodge star operator in cartesian coordinates,

$$\begin{aligned} *d\alpha &= i\kappa e^{i\kappa z}\big(F_x *(dz\wedge dx) + F_y *(dz\wedge dy)\big) \\ &= i\kappa e^{i\kappa z}\big(F_x dy - F_y dx\big), \end{aligned}$$

where partial derivatives are written as $\frac{\partial F}{\partial x} = F_x$. From $*d\alpha = k\alpha$ we have

$$F_z = 0, \qquad iF_x = F_y, \qquad -iF_y = F_x,$$

and $k = \kappa$. From the first condition, we see that $F$ must not depend on $z$. Also, by multiplying the third condition by $i$, we see that the second and third conditions are equivalent. Then, writing $F = u + iv$ ($u$,$v$ real), the second condition reads

$$-v_x + iu_x = u_y + iv_y.$$

These are the Cauchy-Riemann equations for $u$ and $v$, i.e., $F$ should be an analytic function of $x + iy$.

**Proposition 4.1** *Let $x, y, z$ be the cartesian coordinates of $\mathbb{R}^3$, and let $*$ be the Hodge star operator derived from the cartesian metric. Furthermore, let $F : \mathbb{R}^3 \to \mathbb{C}$ be an analytic function of $x + iy$, and let $F$ be independent of $z$. If $k \in \mathbb{R}$ is non-zero, then $\alpha = e^{ikz}dF$ is a complex Trkalian 1-form, $*d\alpha = k\alpha$.*

The above proposition, together with Proposition 3.2, shows that $\Re\left\{e^{ikz}dF\right\}$ is a contact form on $\mathbb{R}^3$ provided that $\Re\left\{e^{ikz}dF\right\} \not\equiv 0$. The next proposition gives sufficient conditions on $F$ for $\Re\left\{e^{ikz}dF\right\}$ to be a contact form on $\mathbb{R}^3$. In particular, it shows that for $\Re\left\{e^{ikz}dF\right\}$ to be a contact form, $F$ can depend on $z$.

**Proposition 4.2** *Let $x, y, z$ be the cartesian coordinates of $\mathbb{R}^3$, and let $F : \mathbb{R}^3 \to \mathbb{C}$ be a function whose real and imaginary parts are $u$ and $v$. If the Jacobian of the mapping $(x, y) \mapsto (u(x, y, z), v(x, y, z))$ is identically non-vanishing, and if $k \in \mathbb{R}$ is non-zero, then $\Re\left\{e^{ikz}dF\right\}$ is a contact form on $\mathbb{R}^3$.*

The proof is based on the following lemma.

**Lemma 4.3** *Let $x, y, z$ be coordinates of $\mathbb{R}^3$, and let $F : \mathbb{R}^3 \to \mathbb{C}$ be a function whose real and imaginary parts are $u$ and $v$. If $\alpha = \Re\left\{e^{ikz}dF\right\}$, where $k \in \mathbb{R}$ is non-zero, then $\alpha \wedge d\alpha = kdu \wedge dv \wedge dz$.*

*Proof of lemma:* From

$$\alpha = \cos kz du - \sin kz dv,$$

and from

$$d\alpha = -k\sin kz dz\wedge du - k\cos kz dz\wedge dv,$$

it follows that

$$\begin{aligned}\alpha\wedge d\alpha &= -k\cos^2 kz du\wedge dz\wedge dv + k\sin^2 kz dv\wedge dz\wedge du\\ &= k du\wedge dv\wedge dz.\end{aligned}$$

□

*Proof of Proposition 4.2:* Let $F = u + iv$ ($u, v$ real). By the lemma,

$$\begin{aligned}\alpha\wedge d\alpha &= k du\wedge dv\wedge dz\\ &= k\big(u_x dx + u_y dy + u_z dz\big)\wedge\big(v_x dx + v_y dy + v_z dz\big)\wedge dz\\ &= k\big(u_x v_y - u_y v_x\big) dx\wedge dy\wedge dz.\end{aligned}$$

The last parenthesis is the Jacobian of the mapping $(x + iy)\mapsto\big(u(x,y,z), v(x,y,z)\big)$. By assumption, it is identically non-vanishing, and, by Frobenius theorem, $\alpha$ is a contact form on $\mathbb{R}^3$. □

By the identification $(x,y,z)\mapsto(x+iy,z)$, the above proposition gives a contact structure on $\mathbb{C}\times\mathbb{R}$. It is then natural to ask whether Proposition 4.2 generalizes to higher dimensions. For instance, consider the $(2N+1)-$dimensional space $\mathbb{C}^N\times\mathbb{R}$ with $N > 1$. In view of Proposition 4.2, it would be natural to define a contact form on $\mathbb{C}^N\times\mathbb{R}$ by $\alpha = \Re\left\{e^{ikz}dF\right\}$ for a suitable function $F:\mathbb{C}^N\to\mathbb{C}$. To show that this is a contact form we need Frobenius theorem for $2N+1$ dimensions.

**Theorem 4.4 (Frobenius theorem, [11, 14])** *Let $M^{2N+1}$ ($N\geq 1$) be an odd dimensional manifold, and let $\alpha$ be a 1-form on $M^{2N+1}$. The hyperplane $\xi = \ker\alpha$ is a contact structure if and only if the $(2N+1)$-form $\alpha\wedge d\alpha\wedge\cdots\wedge d\alpha$ is identically non-vanishing on $M^{2N+1}$.*

From $\alpha = \Re\left\{e^{ikz}dF\right\}$, we see that $d\alpha = \Re\left\{-ike^{ikz}dF\right\}\wedge dz$. Therefore $d\alpha\wedge d\alpha\equiv 0$, and Proposition 4.2 does not, as such, generalize to higher dimensions.

Proposition 4.2 is stated in the most general form. However, if we assume that $F$ is an (non-constant) analytic function with respect of $x + iy$, then the Jacobian in the assumption is identically non-vanishing. This gives the following corollary of Proposition 4.2.

**Corollary 4.5** *Let $x, y, z$ be the cartesian coordinates of $\mathbb{R}^3$, and let $F:\mathbb{R}^3\to\mathbb{C}$ be a non-constant analytic function with respect of $x + iy$ for all values of $z$. If $k\in\mathbb{R}$ is non-zero, then $\alpha = \Re\left\{e^{ikz}dF\right\}$ is a contact form on $\mathbb{R}^3$.*

**B. Spherical coordinates**

To model the spherical symmetry of the vector field 2, it is necessary to introduce coordinate functions $r(x,y,z)$, $\theta(x,y,z)$ and $\phi(x,y,z)$. These are one-to-one only when the point $r = 0$ is removed. One then obtains the space $\mathbb{R}^3\backslash\{0\}$, which no longer can be covered by only one chart. As noted in reference [12], the transition functions for the above coordinate function are complicated. Since the analysis seems to be quite involved, we shall not study contact structures induced by spherical Trkalian fields.

The space $\mathbb{R}^3\backslash\{0\}$ can be identified with $S^2\times\mathbb{R}_+$, where $S^2$ is the unit sphere in $\mathbb{R}^3$ and $\mathbb{R}_+$ is the radial coordinate. By Proposition 3.2, Trkalian fields with spherical symmetry are then seen to generate contact structures on $S^2\times\mathbb{R}_+$.

### 4.1 Standard structures on $\mathbb{R}^3$.

We next show that the standard contact forms $dz - ydx$, $dz + \frac{1}{2}\big(ydx - xdy\big)$, and the overtwisted contact form $\cos z\, dx - \sin z\, dy$ on $\mathbb{R}^3$ can all be generated from the above propositions.

Finding expressions for $F$ for which $\Re\left\{e^{ikx}dF\right\}$, $\Re\left\{e^{iky}dF\right\}$, or $\Re\left\{e^{ikz}dF\right\}$ equals the above contact forms involves some algebra best done with a computer. Omitting the details, we here only give expressions for $F$ and verify that these indeed generate the sought contact forms.

- To obtain $dz - ydx$, let $F(x,y,z) = -i(y+iz)e^{-ix}$. Then $F$ is an analytic function of $y + iz$ for all $x$. By Corollary 4.5 with $k = 1$,

$$\begin{aligned} \Re\left\{e^{ikx}dF\right\} &= \Re\left\{-id\big(e^{-ix}(y+iz)\big)e^{ix}\right\} \\ &= \Re\left\{-(y+iz)dx - idy + dz\right\} \\ &= dz - ydx \end{aligned}$$

  is a contact form.

- To obtain $dz + \frac{1}{2}\big(xdy - ydx\big)$, let $F(x,y,z) = e^{-iy}\big((z+ix) - \frac{xy}{2}\big)$. Now $F$ is not analytic. However, the real and imaginary parts of $F$ are

$$\begin{aligned} u &= \Re\left\{F\right\} \\ &= (z - \frac{xy}{2})\cos y + x\sin y, \\ v &= \Im\{F\} \\ &= (\frac{xy}{2} - z)\sin y + x\cos y \end{aligned}$$

  whence the Jacobian of the mapping $(z,x) \mapsto \big(u(x,y,z), v(x,y,z)\big)$ equals

$$\begin{aligned} u_z v_x - u_x v_z &= \cos y\big(\cos y + \frac{y}{2}\sin y\big) + \sin y\big(\sin y - \frac{y}{2}\cos y\big) \\ &= 1. \end{aligned}$$

  We can therefore use Proposition 4.2 with $k = 1$. It follows that

$$\begin{aligned} \Re\left\{e^{iky}dF\right\} &= \Re\left\{e^{iy}d\big(e^{-iy}(z+ix) - \frac{xy}{2}\big)\right\} \\ &= \Re\left\{(i - \frac{y}{2})dx + \Big[(-i)\big((z+ix) - \frac{xy}{2}\big) - \frac{x}{2}\Big]dy + dz\right\} \\ &= dz + \frac{1}{2}(xdy - ydx) \end{aligned}$$

  is a contact form.

- To obtain the overtwisted contact form $\cos z\, dx - \sin z\, dy$, let $F(x,y,z) = x + iy$. Then, by Corollary 4.5 with $k = 1$,

$$\Re\left\{e^{ikz}dF\right\} = \cos z\, dx - \sin z\, dy$$

  is a contact form.

# 5 Contact structures from Maxwell's equation

In this section, we show that solutions to Helmholtz's equation $\nabla \times (\nabla \times X) = k^2 X$ induce solutions to the Trkalian equation $\nabla \times X = kX$. As a consequence from the previous section, solutions to Helmholtz's equation then induce contact structures on $\mathbb{R}^3$. Since the electric and magnetic fields in Maxwell's equations both solve Helmholtz's equation, we obtain two contact structures, one related to the electric field, and one related to the magnetic field.

## 5.1 Helmholtz's equations

To apply the theory from the previous section, Helmholtz's equation must first be written using differential forms. For this reason, let $X$ be a vector field (with possibly complex component functions,) and let $k \in \mathbb{R}$ be non-zero. Helmholtz's equation for $X$ then reads $\nabla \times (\nabla \times X) = k^2 X$. From Definition 2.3, we find that $(\nabla \times (\nabla \times X))^\flat = *d(\nabla \times X)^\flat = *d * dX^\flat$. We then say that the (possibly complex valued) 1-form $\alpha$ is a solution to Helmholtz's equation if $*d * d\alpha = k^2\alpha$ for some non-zero $k \in \mathbb{R}$.

In reference [2] it is shown that Trkalian vector fields and solutions to Helmholtz equation are closely related. Translated into differential forms we have the following:

**Proposition 5.1** *[2] Let $M^3$ be a Riemann manifold. Furthermore, let $\beta$ be a possibly complex valued 1-form on $M^3$, and let $k \in \mathbb{R}$ be non-zero.*

*a) If $\beta$ is a Trkalian 1-form, $*d\beta = k\beta$, then $\beta$ is a solution to Helmholtz's equations $*d * d\beta = k^2\beta$.*

*b) If $\beta$ is a solution to Helmholtz's equation $*d * d\beta = k^2\beta$, then $\alpha = *d\beta + k\beta$ is a Trkalian 1-form, $*d\alpha = k\alpha$.*

*Proof:* If $*d\beta = k\beta$, then $*d * d\beta = *dk\beta = k^2\beta$. Conversely, if $*d * d\beta = k^2\beta$, then

$$\begin{aligned} *d\alpha &= *d * d\beta + *dk\beta \\ &= k\big( * d\beta + k\beta) \\ &= k\alpha. \end{aligned}$$

□

## 5.2 Maxwell's equations

Traditionally Maxwell's equations are written using vectors fields in the inner product space $\mathbb{R}^3$. They can, however, equivalently be formulated using differential forms, the external derivative, and the Hodge star operator. The main advantage of the latter formalism is that Maxwell's equations split into two sets of equations: One, which is purely topological depending only on the exterior derivative, and another, which is purely metrical depending only on the Hodge star operator. With differential forms, the different physical fields also divide into forms of different degrees with different geometrical interpretations [9, 10].

Here, we only consider the source-less Maxwell's equations. We also assume time harmonic fields with time convention $e^{i\omega t}$, where $\omega > 0$ is the angular frequency of the solution [6]. Maxwell's equations then read

$$\begin{aligned} dE &= -i\omega B, & (3)\\ dH &= i\omega D. & (4)\end{aligned}$$

In the above, $E$ and $H$ are the electric and magnetic field 1-forms, and $D$ and $B$ are the electric and magnetic flux 2-forms. The forms $E, H, B$ and $D$ are all complex valued on $\mathbb{R}^3$ depending only on the spatial variables.

Equations 3-4 are purely topological and do not depend on the metric of the underlying space. The *constitutive equations*, on the other hand, are purely metrical and describe the electrical properties of the medium. In the most simple medium, the homogeneous linear non-dispersive isotropic medium [6], they read

$$\begin{aligned} B &= \mu * H, & (5)\\ D &= \epsilon * E. & (6)\end{aligned}$$

We also assume that $\epsilon > 0$ and $\mu > 0$.

From equations 3 and 5, and from $** = 1$, it follows that

$$\begin{aligned} *d*dE &= k^2 E,\\ *d*dH &= k^2 H,\end{aligned}$$

i.e., $E$ and $H$ satisfy Helmholtz equation. Here, $k = \omega\sqrt{\mu\epsilon} > 0$ is the *wave-number*. We also define the *intrinsic impedance of free space*, $\eta = \sqrt{\mu/\epsilon}$. By Proposition 5.1,

$$\begin{aligned} \alpha_E &= *dE + kE\\ &= k\big(E - i\eta H\big),\\ \alpha_H &= *dH + kH\\ &= k\big(H + \frac{i}{\eta}E\big)\end{aligned}$$

are both complex Trkalian 1-forms: $*d\alpha_E = k\alpha_E$, and $*d\alpha_H = k\alpha_H$. Since $k$ is a real number, $\Re\{E - i\eta H\}$ is a real Trkalian 1-form. If $\Re\{E - i\eta H\} \not\equiv 0$, then $\Re\{E - i\eta H\}$ is a contact form on $\mathbb{R}^3$. Similarly, if $\Re\left\{H + \frac{i}{\eta}E\right\} \not\equiv 0$, then $\Re\left\{H + \frac{i}{\eta}E\right\}$ is a contact form on $\mathbb{R}^3$. Since $\frac{i}{\eta}\alpha_E = \alpha_H$, both contact structures contain essentially the same "information."

## 5.3 Plane-wave in isotropic medium

We next study the contact structure $\alpha_E$ for the special case of a plane-wave solution to Maxwell's equations in cartesian coordinates $x, y, z$ with corresponding cartesian metric. We assume the electric field is of the form

$$E = (Adx + Bdy)e^{ikz},$$

where $A = A_r + iA_i$ and $B = B_r + iB_i$ are complex constants determining the *polarization* of the wave. The corresponding magnetic field is

$$\begin{aligned} H &= \frac{i}{\omega\mu} * dE \\ &= \frac{1}{\eta}\big(Bdx - Ady\big)e^{ikz}. \end{aligned}$$

The contact form is then

$$\begin{aligned} \alpha_E &= \Re\left\{E - i\eta H\right\} \\ &= \Re\Big\{\Big[\big((A_r + B_i)dx + (B_r - A_i)dy\big) + \\ &\qquad i\big((A_i - B_r)dx + (A_r + B_i)dy\big)\Big]e^{ikz}\Big\}. \end{aligned}$$

We now see that $\alpha_E$ can be interpreted as a plane-wave since it has the same form as $E$ and $H$. We next show that independent on the polarization of $E$ and $H$, $\alpha_E$ is always *circulary polarized*. In traditional vector notation the plane-wave

$$\mathbf{X} = (\mathbf{X}_r + i\mathbf{X}_i)e^{ikz} \tag{7}$$

($\mathbf{X}_r, \mathbf{X}_i$ real vectors) is said to be circulary polarized if $|\mathbf{X}_i| = |\mathbf{X}_r|$ and $\mathbf{X}_i \cdot \mathbf{X}_r = 0$. It is readily seen that $\alpha_E^\sharp$ satisfies these conditions.

Lemma 4.3 yields

$$\alpha_E \wedge d\alpha_E = k\big((A_r + B_i)^2 + (A_i - B_r)^2\big)dx \wedge dy \wedge dz.$$

In particular we see that the contact condition $\alpha_E \wedge d\alpha_E \not\equiv 0$ does not involve the medium parameters $\epsilon$ and $\mu$.

The energy of the plane-wave 7 is defined as

$$|\mathbf{X}_r|^2 + |\mathbf{X}_i|^2.$$

If $A$ and $B$ are real, the condition $\alpha_E \wedge d\alpha_E \not\equiv 0$ states that the energy of plane-wave should not be zero. For a plane-wave, this is not very interesting. However, it does suggests that the contact condition $\alpha_E \wedge d\alpha_E \not\equiv 0$ is related to the non-vanishing of the electromagnetic energy also for more general solutions.

If we take $A = 1$, $B = 0$, we obtain the overtwisted contact structure

$$\alpha_E = \cos kzdx - \sin kzdy.$$

The contact form $\alpha_E$ can be interpreted as a point in $\{dx, dy\}$ space rotating around the $z$-axis with an angular frequency $k = \omega\sqrt{\epsilon\mu}$. If we assume that $kz \approx 0$ in $\alpha_E$, and use first order approximations for $\sin$ and $\cos$, we obtain

$$\Re\left\{\alpha\right\} \rightarrow dx - kzdy,$$

which resembles the standard contact form on $\mathbb{R}^3$ [5]. It is straightforward to show that the approximation of $\alpha_E$ is a contact form, but not a Trkalian 1-form. In other words, in the approximation, the contact property is preserved, and the Trkalian property is lost.

In reference [1] Beltrami fields are qualitatively described as being highly unstable to perturbations. Contact structures, on the other hand, are described as structurally stable. Since Beltrami fields involve both the metric and the topology of the underlying space, and since contact structures only depend on the topology of the space, this description seems natural.

# 6 Conclusions

In Section 4, we combined results from references [1] and [3]. As a result, we obtained Proposition 4.2 from which contact structures on $\mathbb{R}^3$ can be generated. We have also shown that all the usual standard structures on $\mathbb{R}^3$ can be generated from the proposition. A natural way to extend the proposition would be to prove, or disprove, that every contact structure on $\mathbb{R}^3$ can be generated by the proposition.

Proposition 4.2 was derived from Trkalian vector fields with cartesian symmetry. In reference [3], it was, however, shown that there also exists another class of Trkalian fields with spherical symmetry. In this work, these have not been studied. It would, of course, be interesting to derive a similar proposition to 4.2 for these fields.

With differential forms Maxwell's equations split into a topological part (depending on the exterior derivative) and a metrical part (depending on medium parameters and the Hodge star operator.) In Section 5, we have shown that solutions to Maxwell's equations generate contact structures on $\mathbb{R}^3$. Since contact structures do not require a metric structure, we could, at least in some sense, claim that the induced contact structures extract a topological component from the solutions. In Section 5.3 we have shown that for plane waves in isotropic medium, the contact structure does not depend on the medium parameters $\epsilon$ and $\mu$. Instead, it is related to the polarization and the energy of the wave. One would expected that for more general solutions to Maxwell's equations, the contact structure is also related to polarization and energy.

Here, we have only studied Maxwell's equations in isotropic media. We can therefore not deduce that in a more general medium, the contact structures would not depend on the medium parameters. A natural way to continue would therefore be to calculate the contact structures for more general media [6, 7, 8]. If one would find media for which the contact structure depends on the medium's parameters, it could be possible to gain further insight on these parameters.

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