Solutions to the Hamilton-Jacobi equation as Lagrangian submanifolds

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1 Introduction

In this essay we shall study the following problem:

> Suppose Q is a smooth n-manifold, $H : T^*Q \to \mathbb{R}$ is a function, π is the canonical projection $\pi : T^*Q \to Q$, and $\lambda \in T^*Q$ is an initial condition such that $H(\lambda) = 0$. Find a function $S : Q \to \mathbb{R}$ such that
>
> $$\begin{aligned} (H \circ dS)(x) &= 0, \quad x \in Q \\ dS_{\pi(\lambda)} &= \lambda. \end{aligned} \tag{1}$$

Equation 1 is known the time-independent *Hamilton-Jacobi equation*. We shall not study any applications. However, let point out that the above problem arises in many areas of mathematics and physics such as mechanics [1, 2, 3], geometric optics [1, 4], the theory of Fourier integral operators [5], and control theory [6]. The Hamilton-Jacobi equation is also used in the development of numerical symplectic integrators [3].

We will show that under suitable conditions on H, the Hamilton-Jacobi equation has a local solution, and this solution is in a natural way represented as a Lagrangian submanifold. By local we mean the following. Once an initial condition λ is fixed, we seek a function S defined in a neighborhood of $\pi(\lambda) \in Q$ solving the Hamilton-Jacobi equation in this set. Global questions are considerably more involved, and a discussion on these can be found in [4].

In Section 2, we go trough the symplectic geometry of the cotangent bundle, which will provide the underlying mathematical space for our analysis. In Section

3 we prove a theorem known as the Hamilton-Jacobi theorem [2]. It shows that a solution to the Hamilton-Jacobi equation simplifies the solution process for the corresponding Hamilton equations; instead of seeking curves in phase-space, one only needs to search for a curve in physical space. This result can also be seen as a motivation for the Hamilton-Jacobi equation.

In Section 4, we define Lagrangian submanifolds and give some examples. In Section 5, we show that solutions to the Hamilton-Jacobi equation can (essentially) be identified with Lagrangian submanifolds. We also show that under suitable restrictions on H we can always construct such a Lagrangian submanifold. Most of the work in this approach will be in translating the Hamilton-Jacobi equation into the Lagrangian formalism. Once this translation is done, the proof that such a Lagrangian submanifold exists is, although not trivial, rather easy. The proof is also completely constructive. Both of these facts motivates the use of Lagrangian submanifolds in the study of the Hamilton-Jacobi equation.

The avoid misunderstanding, let us briefly mention that there exists also another equation known as the Hamilton-Jacobi equation where the function S depends on $2n$ parameters. In modern language, a solution to this "$2n$-Hamilton-Jacobi equation" is a generating function [1] for a symplectomorphism that maps the Hamiltonian vector field to the zero vector field [3]. Such solutions are important since the zero vector field is trivial to integrate. Thus, from a solution to the $2n$-Hamilton-Jacobi equation, one can directly solve the corresponding Hamilton equations. Historically, this equation was discovered by Hamilton, and Jacobi made the equation useful [7].

1.1 Notation and conventions

By a manifold we mean a a topological second countable Hausdorff space that locally is homeomorphic to $\mathbb{R}^n$ for some $n = 1, 2, \ldots$. In addition, we shall assume that all transition functions are C^∞-smooth. That is, we shall only consider C^∞-smooth manifolds. The space of differential p-forms, $p = 0, 1, \ldots$, on a manifold M is denoted by $\Omega^p M$, vector fields on M by $\mathfrak{X}(M)$, the tangent space of M by TM, and the cotangent space of M by T^*M. We also use the identification of T^*M and $\Omega^1 M$ and treat 1-forms as mappings $M \to T^*M$. If $f : M \to N$ is a mapping between manifolds M and N, we denote by Df the push-forward map $Df : TM \to TN$, and by f^* the pull-back map $f^* : \Omega^k(N) \to \Omega^k(M)$. All mathematical objects (manifolds, functions, p-forms, and vector fields) are assumed to be C^∞-smooth.

When we consider an object at a point x in M, we use x as a sub-index on the object. For example, $\Omega^1_x M$ is the set of 1-forms originating from x.

A general mathematical manifold will be denoted by M. However, when we wish to indicate that a manifold could be interpreted as a physical space, we shall denote the manifold by Q. A general interval containing zero, will be denoted by I. The tangent vector of a curve $c : I \to M$ at $t \in I$ will be written as $c'(t)$ whence $c'(t) = (Dc)(t, 1)$ where $(t, 1)$ is the tangent vector $\frac{\partial}{\partial \tau}|_t \in T_t I$. The Einstein summing convention is used throughout.

2 Symplectic geometry

Definition 2.1 *Suppose ω is a 2-form on a manifold M. Then ω is* non-degenerate, *if for each $x \in M$, we have the implication: If $a \in T_x(M)$, and $\omega_x(a, b) = 0$ for all $b \in T_x(M)$, then $a = 0$.*

Definition 2.2 (Symplectic manifold) *Let M be an even dimensional manifold, and let ω be a closed non-degenerate 2-form on M. Then (M, ω) is a* symplectic manifold, *and ω is a* symplectic form *for M.*

In Section 2.1, we shall see that for any manifold, it's cotangent bundle is always a symplectic manifold. This will provide the basic setting, or underlying space, where we shall analyse the Hamilton-Jacobi equation. Since the cotangent bundle with this symplectic structure can be seen as a natural mathematical structure generalizing phasespace in mechanics, this setting is well motivated. There are, however, also other symplectic manifolds than cotangent bundles. The next example shows, maybe, the most simple example.

Example 2.3 (Every orientable surface is a symplectic manifold) Let Σ be a 2-dimensional orientable manifold, and let ω be a volume form on Σ. (Such a form can always by constructed, for instance, by an auxiliary Riemannian metric.) Then ω makes Σ into a symplectic manifold; $d\omega$ vanishes as a 3-form. In addition, ω is non-degenerate; if $x \in \Sigma$ and $a \in T_x\Sigma \setminus \{0\}$, then there is a linearly independent $b \in T_x\Sigma$ whence $\omega(a, b) \neq 0$. □

Definition 2.4 (Hamiltonian vector field) *Suppose (M, ω) is a symplectic manifold, and suppose H be a function $H : M \to \mathbb{R}$. Then the* Hamiltonian vector field *induced by H is the unique (see next paragraph) vector field $X_H \in \mathfrak{X}(M)$ determined by the condition $dH = -\iota_{X_H}\omega$.*

In the above, ι is the *contraction mapping* $\iota_X : \Omega^r(M) \to \Omega^{r-1}(M)$ defined by $(\iota_X\omega)(\cdot) = \omega(X, \cdot)$. To see that the above definition is well defined, let us consider the mapping $X \mapsto \omega(X, \cdot)$; by non-degeneracy, it is injective, and by the rank-nullity theorem, it is surjective. Thus the Hamiltonian vector field X_H is uniquely determined by H. In Example 2.12, we will see that the integral curves of the Hamiltonian vector field correspond to solutions to Hamilton's equations. This motivates the minus sign in $dH = -\iota_{X_H}\omega$.

An important property of the Hamiltonian field is that it's flow is a symplectic mapping.

Definition 2.5 (Symplectic mapping) *Suppose (M, ω) and (N, η) are symplectic manifolds of the same dimension, and f is a diffeomorphism $f : M \to N$. Then f is a* symplectic mapping *if $f^*\eta = \omega$.*

Proposition 2.6 *Suppose (M, ω) is a symplectic manifold, and X_H is a Hamiltonian vector field corresponding to a function $H : M \to \mathbb{R}$. Further, let $\Phi : I \times U \to M$ be a local flow of X_H defined in some open $U \subset M$ and open interval I containing 0. Then for all $x \in U$, $t \in I$, we have*

$$(\Phi_t^*\omega)_x = \omega_x,$$

where $\Phi_t = \Phi(t, \cdot)$.

Proof. As $\Phi_0 = \mathrm{id}_M$, we know that the relation holds, when $t = 0$. Therefore, let us fix $x \in U$, $a, b \in T_x(M)$, and consider the function $r(t) = (\Phi_t^*\omega)_x(a, b)$ with $t \in I$. Then,

$$\begin{aligned} r'(t) &= \frac{d}{ds}\Big[r(s+t)\Big]\Big|_{s=0} \\ &= \frac{d}{ds}\Big[(\Phi_s^*\omega)_y(a', b')\Big]\Big|_{s=0} \\ &= \big(\mathscr{L}_{X_H}\omega\big)_y(a', b'), \end{aligned}$$

where $y = \Phi_t(x)$, $a' = (D\Phi_t)(a)$, $b' = (D\Phi_t)(b)$, and the last line is the definition of the Lie derivative. Using Cartan's formula, $\mathscr{L}_X = \iota_X \circ d + d \circ \iota_X$, we have

$$\begin{aligned} \mathscr{L}_{X_H}\omega &= \iota_{X_H} d\omega + d\iota_{X_H}\omega \\ &= \iota_{X_H} 0 + ddH \\ &= 0, \end{aligned}$$

so $r'(t) = 0$. Thus $r(t) = r(0) = \omega_x(a, b)$ and the claim follows. $\square$

Proposition 2.7 (Convervation of Energy) *Suppose (M, ω) is a symplectic manifold, X_H is the Hamiltonian vector field $X_H \in \mathfrak{X}(M)$ corresponding to a function $H : M \to \mathbb{R}$, and $c : I \to M$ is an integral curve of X_H. Then*

$$H \circ c = \textit{constant}.$$

Proof. Let $t \in I$. Since c is an integral curve, we have $(Dc)(t,1) = (X_H \circ c)(t)$, so for $\tau = (t,1) \in T_t(I)$, we have

$$
\begin{aligned}
d(H \circ c)_t(\tau) &= (c^* dH)_t(\tau) \\
&= (dH)_{c(t)}\big((Dc)(\tau)\big) \\
&= (dH)_{c(t)}\big((X_H \circ c)(t)\big) \\
&= 0,
\end{aligned}
$$

since ω is antisymmetric. The claim follows since $d(H \circ c)$ is linear. $\square$

2.1 Poincaré 1-form on the cotangent bundle

The main result of this section is Proposition 2.10, which shows that the cotangent bundle of any manifold is a symplectic manifold.

Proposition 2.8 *Suppose Q is an manifold, and T^*Q is it's cotangent bundle with standard coordinates (x^i, y_i). Then the 1-form $\theta \in \Omega^1\big(T^*Q\big)$,*

$$\theta = y_i dx^i,$$

is well defined.

Proof. Suppose $(\tilde{x}_i, \tilde{y}^i)$ are other standard coordinates for T^*Q overlapping the (x^i, y_i) coordinates. Then we have the transformation rules $x^i = x^i(\tilde{x})$ and $y_i = \frac{\partial \tilde{x}^r}{\partial x^i}\tilde{y}_r$. Thus $y_i dx^i = \frac{\partial \tilde{x}^r}{\partial x^i}\tilde{y}_r \frac{\partial x^i}{\partial \tilde{x}^l} d\tilde{x}^l = \delta^r_l \tilde{y}_r d\tilde{x}^l = \tilde{y}_i d\tilde{x}^i$. $\square$

Definition 2.9 (Poincaré 1-form) *The 1-form θ in Proposition 2.8 is called the* Poincaré 1-form.

Proposition 2.10 *The cotangent bundle T^*Q of an manifold Q is a symplectic manifold with a symplectic form ω given by*

$$
\begin{aligned}
\omega &= d\theta \\
&= dy_i \wedge dx^i,
\end{aligned}
$$

where θ is the Poincaré 1-form $\theta \in \Omega^1\big(T^*Q\big)$.

Proof. Since $dd = 0$, $d\theta$ is closed, and we only need to check that ω is non-degenerate on T^*M. Suppose $\xi \in T^*Q$, and (x^i, y_i) are local coordinates around ξ. Then, for $X = a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y_i}$, and $Y = v^i \frac{\partial}{\partial x^i} + w^i \frac{\partial}{\partial y_i}$, we have $\omega(X,Y) = b \cdot v - a \cdot w$, so if $\omega(X,Y) = 0$ for all v, w, then with $v = w$, we obtain $a = b$. Finally, setting $w = 0$ yields $a = b = 0$. $\square$

Remark 2.11 In what follows, we shall frequently study cotangent bundles. We shall then, without always mentioning it, assume that they are equipped with the above symplectic form, which, as above, we systematically denote by ω. In the same way we shall reserve the symbol π for the canonical projection $\pi : T^*Q \to Q$. If (x^i, y_i) are local coordinates for T^*Q, then $\pi(x^1, \dots, x^n, y_1, \dots, y_n) = (x^1, \dots, x^n)$.

Example 2.12 (Hamilton's equations) Suppose Q is a manifold, H is a function $T^*Q \to \mathbb{R}$, and X_H is the corresponding Hamiltonian vector field. If (x^i, y_i) are standard coordinates for the cotangent bundle T^*Q, we obtain

$$X_H = \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial y_i}.$$

Suppose $c = (x^i, y_i) : I \to T^*Q$ is an integral curve to X_H, then

$$\begin{aligned} \dot{x}^i(t) &= \frac{\partial H}{\partial y_i} \circ c(t), \\ \dot{y}_i(t) &= -\frac{\partial H}{\partial x^i} \circ c(t), \end{aligned}$$

that is, integral curves to X_H are solutions Hamilton's equations. $\Box$

For future reference, let us prove three technical lemmas for the Poincaré 1-form.

Lemma 2.13 (Coordinate-free expression for θ) *[2] Let Q be a manifold, let θ be the the Poincaré 1-form $\theta \in \Omega^1(T^*Q)$, and let π be the canonical projection $\pi : T^*Q \to Q$. Then, for $\xi \in T^*Q$ and $v \in T_\xi\big(T^*Q\big)$, we have*

$$\theta_\xi(v) = \xi\big((D\pi)(v)\big).$$

Proof. Let (x^i, y_i) be standard coordinates for T^*Q near ξ. Then we can write $\xi = \xi_i dx^i|_{\pi(\xi)}$ and $v = \alpha^i \frac{\partial}{\partial x^i}|_\xi + \beta^i \frac{\partial}{\partial y_i}|_\xi$, Thus $(D\pi)(v) = \alpha^i \frac{\partial}{\partial x^i}|_{\pi(\xi)}$, and $\theta_\xi(v) = y_i(\xi) dx^i|_\xi(v) = \xi_i \alpha^i = \xi\big((D\pi)(\alpha)\big)$. $\Box$

Lemma 2.14 *[2] Suppose Q is a manifold, and $\theta \in \Omega^1(T^*Q)$ is the Poincaré 1-form. Then*

$$\beta^*\theta = \beta$$

for all 1-forms $\beta \in \Omega^1(Q)$.

Proof. Suppose $x \in Q$ and $v \in T_x Q$. Then, by Lemma 2.13,

$$\begin{aligned}(\beta^*\theta)_x(v) &= \theta_{\beta(x)}\big((D\beta)(v)\big)\\ &= \beta_x\big((D\pi)\circ(D\beta)(v)\big)\\ &= \beta_x\big(D(\pi\circ\beta)(v)\big)\\ &= \beta_x(v)\end{aligned}$$

since $\pi \circ \beta = \mathrm{id}_Q$. $\square$

Lemma 2.15 *[2] Suppose Q is a manifold, π is the canonical projection $\pi : T^*Q \to Q$, ω is the standard symplectic form on T^*Q, and S is a function $S : Q \to \mathbb{R}$. Then for any vectors $a, b \in T_{dS(x)}(T^*Q)$ with $x \in Q$, we have*

$$\omega(D(dS \circ \pi)(a), b) = \omega(a, b - D(dS \circ \pi)(b)).$$

Proof. Suppose we can prove the identity

$$\omega\big(a - D(dS \circ \pi)(a), b - D(dS \circ \pi)(b)\big) = 0. \tag{2}$$

Then the lemma follows using linearity and the relation $(dS)^*\omega = 0$ (which, in turn, follows from Lemma 2.14). Thus it suffices to prove identity 2. First, let us note that since $\pi \circ dS = \mathrm{id}_Q$, the vectors $a - D(dS \circ \pi)(a)$ and $b - D(dS \circ \pi)(b)$ map to zero under $D\pi$. For instance,

$$\begin{aligned}(D\pi)\big(a - D(dS\circ\pi)(a)\big) &= (D\pi)(a) - (D\pi)D(dS\circ\pi)(a)\\ &= (D\pi)(a) - D(\pi\circ dS\circ\pi)(a)\\ &= 0.\end{aligned}$$

Therefore, if (x^i, y_i) are standard coordinates near dS_x, then $a - D(dS \circ \pi)(a)$ and $b - D(dS \circ \pi)(b)$ are spanned by $\{\frac{\partial}{\partial y_i}|_{dS(x)} \mid i = 1, \ldots, n\}$. Thus equation 2 follows since the local coordinate expression for ω is $\omega = \sum_{i=1}^n dy_i \wedge dx^i$. $\square$

3 Hamilton-Jacobi theorem

In this section we prove the Hamilton-Jacobi theorem, which establishes a link between the non-linear Hamilton-Jacobi equation and first order ordinary differential equations. The underlying idea of the theorem (and it's proof) is the method of characteristics, which is a general method for solving non-linear partial differential equations.

Theorem 3.1 (Hamilton-Jacobi theorem) *[2] Suppose Q is a connected manifold, and T^*Q is the cotangent bundle equipped with the standard symplectic form $\omega \in \Omega^2(T^*Q)$ and the canonical projection $\pi : T^*Q \to Q$. Further, suppose H is function $H : T^*Q \to \mathbb{R}$ with corresponding Hamiltonian vector field $X_H \in \mathfrak{X}(T^*Q)$.*

If S is a function $S : Q \to \mathbb{R}$, then the following conditions are equivalent:

1. *S is a solution to the Hamilton-Jacobi equation $H \circ dS =$ constant.*
2. *If $c : I \to Q$ is an integral curve of the vector field $(D\pi)\big(X_H \circ dS\big) \in \mathfrak{X}(Q)$, that is,*
$$c'(t) = (D\pi)\big(X_H \circ dS\big)\big(c(t)\big), \tag{3}$$
then $dS \circ c : I \to T^*Q$ is an integral curve of X_H.

Proof. Suppose 1. holds, $c : I \to Q$ is a curve satisfying equation 3, and $r : I \to T^*Q$ is the curve $r = dS \circ c$. Our claim that if $t \in I$, then $r'(t) = X_H \circ r(t)$. Then, fixing t and expanding the left hand side yields

$$\begin{aligned} r'(t) &= (Dr)(t,1) \\ &= (DdS)(Tc)(t,1) \\ &= (DdS)(c'(t)) \\ &= (DdS)(D\pi)(X_H \circ dS)(c(t)) \\ &= D(dS \circ \pi)(X_H \circ r(t)). \end{aligned}$$

Therefore, if $b \in T_{r(t)}\big(T^*Q\big)$, we have, by Lemma 2.15,

$$\begin{aligned} \omega(r'(t), b) &= \omega(X_H \circ r(t), b) - \omega(X_H \circ r(t), D(dS \circ \pi)(b)) \\ &= \omega(X_H \circ r(t), b), \end{aligned} \tag{4}$$

since

$$\begin{aligned} \omega(X_H \circ r(t), D(dS \circ \pi)(b)) &= -(dH)_{r(t)}\big(D(dS)D(\pi)(b)\big) \\ &= -\big((dS)^*(dH)\big)_{c(t)}\big(D(\pi)(b)\big) \\ &= -\big(d(H \circ dS)\big)_{c(t)}\big(D(\pi)(b)\big) \\ &= 0. \end{aligned}$$

Since b was arbitrary, the claim follows from equation 4.

For the other direction, let us assume that 2. holds. To show that $d(H \circ dS) = 0$, let us fix $x \in Q$, and let $c : I \to Q$ be the integral curve of $(D\pi)(X_H \circ dS)$

such that $c(0) = x$. Then, by our assumption, the curve $r = dS \circ c$ satisfies $r' = X_H \circ r$. Hence, by the same calculations as above, we find that $r'(0) = D(dS \circ \pi)(X_H \circ r)(0)$ and $\omega\big(X_H \circ r(0), D(dS \circ \pi)(b)\big) = 0$ for any $b \in T_{r(0)}\big(T^*Q\big)$. Thus

$$\begin{aligned} d(H \circ dS)_x\big((D\pi)(b)\big) &= \big((dS)^* dH\big)_x\big((D\pi)(b)\big) \\ &= dH_{r(0)}\big(D(dS \circ \pi)(b)\big) \\ &= -\omega_{r(0)}\big((X_H \circ r)(0), D(dS \circ \pi)(b)\big) \\ &= 0. \end{aligned}$$

Since any $v \in T_x Q$ can be written as $(D\pi)(b)$ for some b as above, we see that S is a solution to the Hamilton-Jacobi equation. $\square$

4 Lagrangian submanifolds

This section we define Lagrangian submanifolds and prove some results that will be needed in Section 5, where these will play a crucial role in the analysis of the Hamilton-Jacobi equation.

Definition 4.1 (Lagrangian submanifold) *Suppose Λ is a submanifold of a symplectic manifold (M, ω). Then Λ is a* Lagrangian submanifold *if we have*

1. $\dim \Lambda = \frac{1}{2} \dim M$,
2. *ω vanishes for tangent vectors to Λ, that is, if $x \in L$, then $\omega_x|_{T_x\Lambda} = 0$.*

Example 4.2 Every curve on an orientable surface is a Lagrangian submanifold.

Let (Σ, ω) be as in Example 2.3, and let $\Gamma \subset \Sigma$ be a curve, that is, a 1-dimensional submanifold. If $x \in \Gamma$, and $a, b \in T_x(\Gamma)$, then a and b are proportional since $\dim \Gamma = 1$, say $a = Cb$ for some real C. Thus

$$(\iota^*\omega)_x(a, b) = C\omega_{\iota(x)}((D\iota)a, (D\iota)a) = 0,$$

where ι is the inclusion $\iota : \Gamma \hookrightarrow \Sigma$. $\square$

4.1 Conormal bundles as Lagrangian submanifolds

In this section, we show that if Σ is a submanifold of a manifold M, then the conormal bundle of Σ in T^*M is a Lagrangian submanifold. This demonstrates that every cotangent bundle contains an abundance of Lagrangian submanifolds. The material in this section is not needed in the subsequent sections.

Definition 4.3 (Conormal bundle) *Suppose Σ is a submanifold of a manifold M. Then, if $\sigma \in \Sigma$, we define the* conormal space *of Σ at σ as the vector space*

$$S^*_\sigma(\Sigma) = \{\alpha \in T^*_\sigma M \mid \alpha(v) = 0 \textit{ for all } v \in T_s(\Sigma)\}$$

and the conormal bundle *as*

$$S^*(\Sigma) = \bigcup_{\sigma \in \Sigma} S^*_\sigma(\Sigma).$$

Example 4.4 If (r, ϕ) are polar coordinates for $\mathbb{R}^2$, and $S^1 \subset \mathbb{R}^2$ is the unit circle, then

$$S^*(S^1) = \{C(dr)_{(1,\phi)} \mid C \in \mathbb{R},\ \phi \in [0, 2\pi)\}.$$

This shows that the conormal bundle consists of 1-forms that are completely "orthogonal" to the given submanifold in the vector — covector pairing. □

Suppose Σ is a k-dimensional submanifold of a n-dimensional manifold M, and suppose $s \in S^*(\Sigma)$, so $s \in T^*_\sigma(M)$ for $\sigma = \pi(s)$. We can then find submanifold coordinates $(x^1, \ldots, x^n)$ for M such that near σ, Σ is parametrized by $(x^1, \ldots, x^k)$ when $x^{k+1} = \cdots = x^n = 0$. Then we have $T_\sigma(\Sigma) = \text{span}\{\frac{\partial}{\partial x^i}\}_{i=1}^k$ whence $S^*_\sigma(\Sigma) = \text{span}\{dx^i\}_{i=k+1}^n$. Therefore, if $(x^1, \ldots, x^n, y_1, \ldots, y_n)$ are standard coordinates for T^*M, then $S^*(\Sigma)$ is parametrized by $(x^1, \ldots, x^k, y_{k+1}, \ldots, y_n)$ when $x^{k+1} = \cdots = x^n = 0$ and $y_1 = \cdots = y_k = 0$. In conclusion, $N^*(\Sigma)$ is a n-dimensional submanifold of T^*M [8].

Proposition 4.5 *Let M be an n-manifold, and let Σ be a k-dimensional submanifold of M. Then $S^*(\Sigma)$ is a Lagrangian submanifold of T^*Q.*

Proof. Suppose $s \in S^*(\Sigma)$ and suppose (x^i, y_i) are local coordinates for T^*M adopted to $S^*(\Sigma)$ as above. In other words, $(x^1, \ldots, x^k, y_{k+1}, \ldots, y_n)$ parametrize $S^*(\Sigma)$ near s when $x^{k+1} = \cdots = x^n = 0$ and $y_1 = \cdots = y_k = 0$. We then have $s = \sum_{i=k+1}^n s_i dx^i|_{\pi(s)}$. Furthermore, if $v \in T_s(S^*(\Sigma))$, it follows that $v \in \text{span}\big\{\{\frac{\partial}{\partial x^i}|_s\}_{i=1}^k \cup \{\frac{\partial}{\partial y_i}|_s\}_{i=k+1}^n\big\}$. Thus, for the Poincaré 1-form $\theta \in \Omega^1(T^*Q)$, and the natural inclusion $\iota : S^*(Q) \hookrightarrow T^*(Q)$, we have

$$\begin{aligned}
(\iota^*\theta)_s(v) &= \theta_{\iota(s)}\big((D\iota)(v)\big) \\
&= \iota(s)\big(D(\pi \circ \iota)(v)\big) \\
&= 0
\end{aligned}$$

since $D(\pi \circ \iota)(v) \in \text{span}\{\frac{\partial}{\partial x^i}|_{\pi(s)}\}_{i=1}^k$. □

4.2 Graph of a 1-form as a Lagrangian submanifold

Definition 4.6 (Graph of a 1-form) *Suppose M is a manifold, and h is a function $h : M \to \mathbb{R}$. Then we define*

$$\Lambda_h = \{(dh)_x \in T_x^*M \mid x \in M\} \subset T^*M.$$

Recall that for a manifold M, we denote the canonical projection $T^*M \to M$ by π. More generally, suppose Λ is a submanifold of T^*M with the inclusion mapping $\iota : \Lambda \hookrightarrow T^*M$. In this setting, we define the *projection* $p : \Lambda \to M$ by $p = \pi \circ \iota$. In the special case $\Lambda = T^*M$, we recover $\pi = p$.

Proposition 4.7 *Suppose M is a manifold, h is a function $h : M \to \mathbb{R}$, and T^*M is endowed with the canonical symplectic structure. Then Λ_g is a Lagrangian submanifold of T^*M.*

Proof. Let p be the projection $p : \Lambda_h \to M$, $p = \pi \circ \iota$ defined as above. Then $p^{-1} : M \to \Lambda_h$ is $p^{-1}(x) = (dh)_x$, whence Λ_h and M are diffeomorphic, so $\dim \Lambda_h = \frac{1}{2} \dim T^*M$. Next, suppose $\lambda \in \Lambda_h$ and $v \in T_\lambda \Lambda_h$. Then we have $\iota(\lambda) = (dh)_{p(\lambda)}$, and by Lemma 2.14,

$$\begin{aligned}
(\iota^*\theta)_\lambda(v) &= \theta_{\iota(\lambda)}\big((D\iota)(v)\big) \\
&= \iota(\lambda)\big((D(\pi \circ \iota)(v)\big) \\
&= (dh)_{p(\lambda)}\big((Dp)(v)\big) \\
&= (p^*dh)_\lambda(v).
\end{aligned}$$

Thus $\iota^*\theta = p^*dh$ and $\iota^*d\theta = (d \circ d)(f \circ p) = 0$. $\square$

Proposition 4.8 *Suppose M is a connected manifold.*

1. *If g, h are functions $g, h : M \to \mathbb{R}$ such that $\Lambda_f = \Lambda_g$. Then $g - h$ is a constant.*

2. *If g is a function $g : M \to \mathbb{R}$, then Λ_g locally determines g up to an additive constant.*

Proof. For part 1., let $x \in M$. Then there is a unique $\xi \in \Lambda_h$ such that $\pi(\xi) = x$ where π is the projection $\pi : T^*M \to M$. Also, by assumption, there is a unique $\eta \in \Lambda_g$ such that $\xi = \eta$. Hence $\pi(\xi) = \pi(\eta) = x$, so $\xi = (dh)_x$ and $\eta = (dg)_x$. Thus $d(g - h) = 0$, and $g - h$ is a constant. For part 2., suppose $x \in M$. Then there is a unique $\xi_x \in \Lambda_h$ such that $\pi(\xi_x) = x$. Hence $x \mapsto \xi_x$ defines a 1-form on M, say $\alpha \in \Omega^1 M$. Since $\xi_x = (dh)_x$, it follows that α is closed. Thus, if we fix $x \in M$, we can find an open neighbourhood $U \subset M$ of x and a function $\phi : U \to \mathbb{R}$, such that $\iota^*\alpha = d\phi$, where ι is the inclusion $\iota : U \hookrightarrow M$. It follows that $\Lambda_\phi = \Lambda_{\iota^* h} \subset T^*U$ so by part 1., $\phi - h|_U$ is a constant. $\square$

The next proposition gives a converse to Proposition 4.7; every Lagrangian manifolds $\Lambda \subset T^*M$ for which the projection $p : \Lambda \to M$ is locally a diffeomorphism may locally be written as the gradient of a function $M \to \mathbb{R}$.

Proposition 4.9 *Suppose M is a manifold, and Λ is a Lagrangian submanifold of T^*M. If $\lambda \in \Lambda$ and λ has a contractible neighborhood $U \subset \Lambda$ where the projection $p : \Lambda \to M$ is a diffeomorphism, then there exists a function $h : p(U) \to \mathbb{R}$ such that*

$$\Lambda \cap U = \Lambda_h.$$

Proof. On U, the projection p can be written as $p = \pi \circ \iota$, where

$$\begin{aligned} \iota &: U \hookrightarrow T^*(p(U)), \\ \pi &: T^*(p(U)) \to p(U). \end{aligned}$$

Suppose $\theta \in \Omega^1(T^*M)$ is the Poincaré 1-form on T^*M, and suppose j is the inclusion $j : U \hookrightarrow T^*M$. Then, since Λ is a Lagrangian submanifold, we have $dj^*\theta = 0$. Since U is contractible, we can find a function $\phi : U \to \mathbb{R}$ such that $j^*\theta = d\phi$. For $x \in p(U)$, let $\alpha(x) = (\iota \circ p^{-1})(x)$. Then $\alpha \in \Omega^1(p(U))$. Indeed, as a composition of smooth functions, α is smooth, and it also preserves the basepoint; $\pi \circ \alpha = \mathrm{id}_{p(U)}$. Let $r : T^*(p(U)) \hookrightarrow T^*M$ be the natural inclusion. Thus, if $\theta_{p(U)}$ is the Poincaré 1-form on $T^*(p(U))$, we have $r^*\theta = \theta_{p(U)}$, and $r \circ \iota = j$. Putting all this together using Lemma 2.14 yields

$$\begin{aligned} \alpha &= \alpha^*\theta_{p(U)} \\ &= (\iota \circ p^{-1})^* r^*\theta \\ &= (p^{-1})^*(r \circ \iota)^*\theta \\ &= d(\phi \circ p^{-1}), \end{aligned}$$

and $U = \Lambda_{\phi \circ p^{-1}} = \{d(\phi \circ p^{-1})_x \mid x \in p(U)\} \subset T^*M$. $\square$

5 Lagrangian submanifolds as solutions to the Hamilton-Jacobi equation

In this section, we take another approach to analysing the Hamilton-Jacobi equation. The key observation will be that finding a solution to the Hamilton-Jacobi equation is equivalent to finding a certain Lagrangian submanifold in T^*Q containing the initial condition. Let us establish this equivalence.

We assume that Q is a manifold, and $H : T^*Q \to \mathbb{R}$ is a function that satisfies $dH \neq 0$, that is, at each point $x \in Q$, there is a v such that $dH(v) \neq 0$. Then the Hamilton-Jacobi equation reads

$$\begin{aligned} (H \circ dS)(x) &= 0, \\ dS_{\pi(\lambda)} = \lambda, & \end{aligned} \tag{5}$$

where $\lambda \in T^*Q$ is a fixed initial condition such that $H(\lambda) = 0$, and π is the projection $\pi : T^*Q \to Q$. Let us emphasize that our goal is only to solve this equation locally. We try to find a function S defined in some open set $U \subset Q$ such that $(dS)_{\pi(\lambda)} = \lambda$ and equation 5 holds.

A first observation is that a function $S : U \to \mathbb{R}$ is a local solution to 5, if and only if

$$\lambda \in \Lambda_S \subset H^{-1}(0).$$

If this is the case, then by Proposition 4.7, Λ_S is a Lagrangian submanifold. What is more, Λ_S is diffeomorphic to U. On the other hand, suppose Λ is a Lagrangian submanifold of T^*Q such that $\lambda \in \Lambda \subset H^{-1}(0)$ and $p : \Lambda \to Q$ is a diffeomorphism near λ. Then, near λ, we can write $\Lambda = \Lambda_S$ for some function S defined in a neighborhood of $\pi(\lambda)$. The conclusion is that solving equation 5 is equivalent to finding a Lagrangian submanifold Λ such that

$$\lambda \in \Lambda \subset H^{-1}(0),$$

and $p : \Lambda \to Q$ is a diffeomorphism near λ.

In the below, Propositions 5.4 and 5.5 show how to construct such a Lagrangian submanifold. Roughly, the idea is that we first construct a $(n-1)$-submanifold of $H^{-1}(0)$ that contains the given initial condition. In addition, the submanifold will be transverse to X_H and isotropic (see below). Then, by letting this submanifold flow using the flow of X_H, we obtain a n-submanifold of T^*Q, and this will be the sought Lagrangian submanifold.

Example 5.1 (Non-physical Lagrangian submanifold) Let us try to interpret the condition that the projection $p : \Lambda \to Q$ is locally a diffeomorphism. As an example where this fails, we have the cotangent space T_x^*Q in T^*Q. It is a Lagrangian submanifold (one can think of T_x^*Q is the conormal bundle of $\{x\}$), but the projection is just the constant map $p : T_x^*Q \to \{x\}$ which is not a diffeomorphism. Let us next describe why this Lagrangian submanifold is not very physical. First, from the above description of the construction of Λ, we see that the flow of X_H carries points in Λ to points in Λ. We also know that the flow of X_H describes the dynamical behaviour of a particle as time progresses. It is thus intuitively rather clear that a curve in T_x^*Q is non-physical; it describes a particle with constant location, but with changing momentum. $\square$

Definition 5.2 (Transverse vector field) *Suppose Σ is a submanifold of a manifold M. Then a vector field $X \in \mathfrak{X}(M)$ is* transverse *to Σ if X is nowhere tangent to Σ, that is, if $\sigma \in \Sigma$ then $X(\sigma) \notin T_\sigma(\Sigma)$.*

Definition 5.3 (Isotropic submanifold) *Suppose Σ is a submanifold of a symplectic manifold (M, ω). Then Σ is an* isotropic submanifold *if ω vanishes on Σ, that is, if ι is the inclusion $\iota : \Sigma \hookrightarrow M$, then $\iota^*\omega = 0$.*

Proposition 5.4 (Existence of initial conditions) *Suppose Q is an n-manifold, $H : T^*Q \to \mathbb{R}$ is a function that satisfies $dH \neq 0$, and X_H is the corresponding Hamiltonian vector field $X_H \in \mathfrak{X}(T^*Q)$.*

Then, if $\lambda \in H^{-1}(0)$, there exists a $(n-1)$-submanifold Λ' of $H^{-1}(0)$ such that

- $\lambda \in \Lambda' \subset H^{-1}(0)$,
- *Λ' is isotropic,*
- *X_H is transverse to Λ'.*

Proof. If (x^i, y_i) are standard coordinates near λ, we can, by possibly permuting the x^i coordinates, assume that either $\frac{\partial H}{\partial x^n}(\lambda) \neq 0$, or $\frac{\partial H}{\partial y_n}(\lambda) \neq 0$. Let us only consider the case $\frac{\partial H}{\partial x^n}(\lambda) \neq 0$ since the other case is completely analogous. Also, let us write $x' = (x^1, \ldots, x^{n-1})$ and $y = (y_1, \ldots, y_n)$. Then, by the implicit function theorem, we can solve $x^n = x^n(x', y)$ such that

$$(x', y) \mapsto (x', x^n(x', y), y)$$

parametrize $H^{-1}(0)$ near λ. Therefore, if we reparametrize (x', x^n, y) with coordinates (u', u^n, v) defined as

$$\begin{aligned} u'(x', x^n, y) &= x', \\ u^n(x', x^n, y) &= x^n - x^n(x', y), \\ v(x', x^n, y) &= y \end{aligned}$$

then

$$(u', v) \mapsto (u', 0, v)$$

parametrize $H^{-1}(0)$ near λ, and $H^{-1}(0)$ is a $(2n-1)-$submanifold of T^*M. For $H^{-1}(0)$ we introduce coordinates

$$\begin{aligned}\tilde{u}'(u', v) &= u' - u'(\lambda), \\ \tilde{v}(u', v) &= v - v(\lambda).\end{aligned}$$

In these coordinate, we define Λ' as the set parametrized as

$$\tilde{u}' \mapsto (\tilde{u}', 0).$$

It follows that Λ' is a $(n-1)$-submanifold of $H^{-1}(0)$. Also, in the standard coordinates (x^i, y_i), Λ' is parametrized by

$$x' \mapsto \big(x', x^n(x', y(\lambda)), y(\lambda)\big).$$

Thus, if $\eta \in \Lambda'$, then

$$T_\eta \Lambda' = \operatorname{span}\Big\{ \frac{\partial}{\partial \tilde{u}^i}\Big|_\eta \Big\}_{i=1}^{n-1} = \operatorname{span}\Big\{ \frac{\partial}{\partial x^i}\Big|_\eta \Big\}_{i=1}^{n-1}.$$

To check that Λ' is isotropic, let $a, b \in T_\eta \Lambda'$. Then, from the local expression for ω, we have $\omega_\eta(a, b) = 0$. To see that X_H is transverse to Λ', let us first possibly shrink Λ' such that $\frac{\partial H}{\partial x^n} \neq 0$ on all Λ'. Then, from Example 2.12, we see that $X_H(\eta)$ has a $\frac{\partial}{\partial y_n}$ component for all $\eta \in \Lambda'$, so $X_H \notin T_\eta \Lambda'$. □

Proposition 5.5 (Existence of Lagrangian submanifolds) *Suppose Q is an manifold, and suppose:*

> *$H : T^*Q \to \mathbb{R}$ is a function that satisfies $dH \neq 0$, and $X_H \in \mathfrak{X}(T^*Q)$ is the corresponding Hamiltonian vector field.*

Then, if $\lambda \in H^{-1}(0)$, there exists a Lagrangian submanifold Λ of T^*Q such that

$$\lambda \in \Lambda \subset H^{-1}(0).$$

Proof. Let Λ' be the submanifold given by Proposition 5.4. Then we can find a neighbourhood $U \subset \Lambda'$ of λ, and an interval I containing zero, such that the flow of X_H (from Λ') is a smooth injection $\Phi : I \times U \to T^*Q$. Let us show that the set $\Lambda = \Phi(I \times U)$ satisfies the given properties. First, by Proposition 2.7, we see that $\Lambda \subset H^{-1}(0)$ as sets. Let us show that this also holds in the sense of submanifolds. By possibly shrinking U, we can introduce submanifold coordinates $(x^1, \dots, x^{2n-1})$ for $H^{-1}(0)$ such that $x' = (x^1, \dots, x^{n-1})$ parametrize U when $x^n = \cdots = x^{2n-1} = 0$. Further, by possibly shrinking I, we can assume that $\Phi(\tau, x')$ is always in these coordinates. Then, since

$$(\tau, x') \mapsto \Phi(\tau, x')$$

has rank n, it follows from the constant rank theorem that Λ is a $n-$submanifold of $H^{-1}(0)$, and Λ has coordinates (τ, x') such that $x' \mapsto (0, x')$ parametrizes U, and $\tau \mapsto (\tau, x')$ parametrizes the integral curve of X_H trough $x' \in \Lambda'$. To prove that Λ is a Lagrangian submanifold, we need to check that ω vanishes on Λ. For this, let $\mu \in \Lambda$, that is, $\mu = \Phi(t, \mu')$ for some $t \in I$ and $\mu' \in \Lambda'$. Further, let $(\tau, x^1, \dots, x^{n-1})$ be coordinates for Λ adopted to Λ' and the flow of X_H as above. In these coordinates, let us define $\Psi_t(s, x) = (s + t, x)$. Suppose $A, B \in T_\mu \Lambda$. Then, since Ψ_t is a diffeomorphism between some suitable neighbourhoods of $\mu' = (0, x)$ and $\mu = (t, x)$, we have $A = (D\Psi_t)(a)$ and $B = (D\Psi_t)(b)$ for some $a, b \in T_{\mu'}\Lambda$. If we write $a = \sum_{i=1}^{n-1} a^i \frac{\partial}{\partial x^i}\big|_{(0,x)} + a^* \frac{\partial}{\partial \tau}\big|_{(0,x)}$, then $A = \sum_{i=1}^{n-1} a^i \frac{\partial}{\partial x^i}\big|_{\Phi_t(x)} + a^* \frac{\partial}{\partial \tau}\big|_{\Phi_t(x)}$, so for some $a', b' \in T_{\mu'}\Lambda'$ and $\alpha, \beta \in \mathbb{R}$, we have $A = (D\Phi_t)(a') + \alpha X_H(\mu)$ and $B = (D\Phi_t)(b') + \beta X_H(\mu)$. Now, since H is constant on $\Lambda \subset H^{-1}(0)$ and since $(D\Phi_t)(a') \in T_\lambda \Lambda$, we have $\omega_\mu\big((D\Phi_t)(a'), X_H(\mu)\big) = dH_\mu\big((D\Phi_t)(a')\big) = 0$. Using this and Proposition 2.6, we have

$$\begin{aligned}
\omega_\mu(A, B) &= \omega_{\Phi_t(\mu')}\big((D\Phi_t)(a') + \alpha X_H(\mu), (D\Phi_t)(b') + \beta X_H(\mu)\big) \\
&= \omega_{\Phi_t(\mu')}\big((D\Phi_t)(a'), (D\Phi_t)(b')\big) \\
&= \big(\Phi_t^* \omega\big)_{\mu'}\big(a', b'\big) \\
&= \omega_{\mu'}\big(a', b'\big) \\
&= 0,
\end{aligned}$$

where the last step follows since Λ' is isotropic. We have shown that Λ is a Lagrangian submanifold. $\square$

5.1 Projection onto Q

In this section, we show that under suitable conditions on H, the Lagrangian submanifold constructed in Proposition 5.5 is locally diffeomorphic to Q via the projection $p = \pi \circ \iota : \Lambda \to Q$ [4]. This is an important result since it shows that the Lagrangian submanifold constructed in Proposition 5.5 indeed corresponds to a solution to the Hamilton-Jacobi equation as described on page 13.

Suppose Q, H, λ are as in Proposition 5.5, (x^i, y_i) are local coordinates for T^*Q near λ, and Λ' has been constructed as above, that is, by solving $x^n = x^n(x', y)$.

Then we have

$$(Dp)(T_\lambda \Lambda) = \text{span}\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{n-1}}, (Dp)(X_H)\}.$$

Thus

$$(Dp)(T_\lambda \Lambda) = T_{\pi(\lambda)} Q.$$

if and only if $(Dp)(X_H)$ contains an $\frac{\partial}{\partial x^n}$ differential, that is, $\frac{\partial H}{\partial y^n} \neq 0$. If this is the case, then (Dp) is surjective, and by the rank-nullity theorem, it is therefore also injective. Then Dp is a linear isomorphism, and by the inverse function theorem, Dp is a diffeomorphism near λ.

From the discussion in the introduction on 13, we find:

Proposition 5.6 *Suppose Q is an n-manifold, H is a function $H : T^*Q \to \mathbb{R}$, and $\lambda \in T^*Q$. Further, suppose (x^i, y_i) are local coordinates around λ and there exists a $k = 1, \dots, n$ such that*

$$H(\lambda) = 0, \qquad \frac{\partial H}{\partial x^k}(\lambda) \neq 0, \qquad \frac{\partial H}{\partial y_k}(\lambda) \neq 0.$$

Then the Hamilton-Jacobi equation

$$\begin{aligned} H \circ dS &= 0, \\ dS_{\pi(\xi)} &= \lambda. \end{aligned}$$

has a local solution S around $\pi(\lambda)$.

One can also show that Λ is uniquely determined by Λ' [2, 4]. However, at least in [2, 4], there seem to be no remarks considering the unique dependence on the initial condition λ.

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