Determination of electromagnetic medium from the Fresnel surface

Matias F. Dahl

Aalto University, Mathematics, P.O. Box 11100, FI-00076 Aalto, Finland

AMS classification scheme numbers: 78A25, 83C50, 53C50, 78A02, 78A05

E-mail: `matias.dahl@aalto.fi`

Abstract. We study Maxwell's equations on a 4-manifold where the electromagnetic medium is described by a suitable antisymmetric $\binom{2}{2}$-tensor κ with real components. In this setting, the Tamm-Rubilar tensor density determines a polynomial surface of fourth order in each cotangent space. This surface is called the Fresnel surface and acts as a generalisation of the null cone determined by a Lorentz metric; the Fresnel surface parameterises electromagnetic wavespeed as a function of direction. We show that if *(a)* κ has no skewon and no axion component, *(b)* κ is invertible and *(c)* the Fresnel surface is pointwise a Lorentz null cone, then the medium is isotropic, that is, the medium is proportional to a Hodge star operator of a Lorentz metric. In other words, in a suitable class of media one can recognise isotropic media from wavespeed alone. What is more, we study the nonunique dependence between the medium tensor κ, its Tamm-Rubilar tensor density and Fresnel polynomial. For example, we show that if κ is invertible then κ and κ^{-1} have the same Fresnel surfaces.

Submitted to:

1. Introduction

The purpose of this work is to study properties of propagating electromagnetic waves on a 4-manifold N in the premetric setting [1]. Then the electromagnetic medium is represented by a suitable antisymmetric $\binom{2}{2}$-tensor κ, whence the medium is pointwise determined by 36 real parameters. For the study of wave propagation in this setting, a key object is the *Fresnel surface*, which can be seen a generalisation of the null cone [1, 2, 3]. In Lorentz geometry, the null cone is always a polynomial surface of second order in each cotangent space. The Fresnel surface, in turn, is a polynomial surface of fourth order. For example, the Fresnel surface can be the union of two Lorentz null cones. This allows the Fresnel surface to describe the wavespeed behaviour also in birefringent medium. That is, in medium where differently polarised waves can propagate with different wavespeeds. In more detail, the Fresnel surface is determined by the *Tamm-Rubilar tensor density* and we have following dependence:

$$\begin{array}{ccccc} \text{Medium} & \to & \text{Tamm-Rubilar tensor density} & \to & \text{Fresnel surface} \\ \kappa & \to & \mathscr{G}^{ijkl} & \to & \{\xi : \mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = 0\}. \end{array}$$

In Lorentz geometry, we know that the null cone of a Lorentz metric g uniquely determines the metric g up to a conformal factor [4, Theorem 3]. In this work we will study the analogue relation between a general electromagnetic medium tensor κ and its Fresnel surface. By scaling invariance we can never uniquely determine the medium from the Fresnel surface. However, we may still ask how much information about the medium κ is contained in the Fresnel surface. Namely:

Question 1.1. Suppose κ is an electromagnetic medium on a 4-manifold N, and suppose we know the Fresnel surface of κ at a point $p \in N$. How much can we say about the coefficients in κ at p?

In terms of physics, Question 1.1 asks how much of the anisotropic structure of an electromagnetic medium can be recovered from pointwise wavespeed information alone. A proper understanding of this question is not only of theoretical interest. Since wavespeed is a physical observable, the question is also of interest in possible engineering applications like electromagnetic traveltime tomography. Question 1.1 is also similar is spirit to a question in general relativity, where one would like to understand when the the conformal class of a Lorentz metric can be determined from the five dimensional manifold of null-geodesics [5].

We know that in *isotropic medium* the Fresnel surface is one Lorentz null cone at each point of N. That is, in isotropic medium wave propagation is described using Lorentz geometry. A main result of this paper is to show that isotropic media is the only class of medium with this property (under suitable assumptions). More precisely we will show that if κ is a medium tensor with real coefficients and

(a) κ has no skewon and no axion component,

(b) κ is invertible,

(c) the Fresnel surface is pointwise a Lorentz null cone,

then κ must be isotropic, that is, κ must be proportional to the Hodge star operator of a Lorentz metric. Thus, in media that satisfy *(a)* and *(b)*, isotropic media can be characterised by the behaviour of wavespeed alone. Below, this result is implication *(iii)* $\Rightarrow$ *(ii)* in Theorem 4.3. Here, the assumption that the medium has no skewon and no axion component essentially means that the medium is non-dissipative.

Apart from implications *(iii)* $\Rightarrow$ *(i)* and *(iii)* $\Rightarrow$ *(ii)*, the other implications in Theorem 4.3 are known. For a discussion, see Section 4. In particular, equivalence *(iii)* $\Leftrightarrow$ *(i)* is closely related to the following statement: *if* κ *has no skewon component and no axion component then* κ *satisfies the* closure condition *(*$\kappa^2 = -\lambda Id$ *for a positive function* λ*) if and only if the Fresnel surface is pointwise a Lorentz null cone.* This is a conjecture that has been formulated and studied in a number of papers [2, 6, 7, 8, 9]. See also the book [1] by Hehl and Obukhov. The conjecture has been proven in a number of different settings: in the absence of magneto-electric effects (that is, for $\mathscr{C} = 0$ where $\mathscr{C}$ is as in Section 2.5) by Obukhov, Fukui and Rubilar [8], and in a special class of non-linear media by Obukhov and Rubilar [9]. On the level of the Tamm-Rubilar tensor density, Favaro and Bergamin have shown that if $\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = \sigma\,(g^{ij}\xi_i\xi_j)^2$ for a factor σ and a Lorentz metric g, then the medium must be isotropic [10]. See also [10] for a discussion about the analogous problem for non-Lorentzian g. For further results and discussions, see also [2, 10, 11, 12, 13, 14, 15]. Implication *(iii)* $\Rightarrow$ *(i)* in Theorem 4.3 shows that the above conjecture holds under the assumption that κ is invertible.

The proof of implication *(iii)* $\Rightarrow$ *(i)* in Theorem 4.3 is a slight modification of the argument used in [15] to describe all invertible skewon-free medium tensors where the Fresnel surface is the union of two distinct Lorentz null cones. This result is closely related to characterising medium tensors with only one Lorentz null cone, but there is also a small difference. With two distinct Lorentz null cones, the Fresnel surface uniquely determines the two Lorentz metrics up to scaling [15, Proposition 1.3 and 1.4]. However, with only one Lorentz null cone, one needs to rule out a possible positive definite factor in $\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l$. See Example 4.2 and Lemma 4.6 below. The proof of the latter lemma is a slight modification of the argument in [15]. Hence we will only indicate how the argument changes. Let us note that the argument in [15] relies on two main tools: first, the classification of skewon-free medium tensors into 23 normal forms by Schuller, Witte, and Wohlfarth [13] and second, the computer algebra technique of *Gröbner bases* for eliminating variables from polynomial equations [16].

A second contribution of this paper is given in Section 5, which studies the non-unique dependence of κ, its Tamm-Rubilar tensor density and the Fresnel surface. For example, in Theorem 5.1 *(iv)* we show that if κ is invertible, then κ and κ^{-1} have the same Fresnel surfaces. Also, in Example 5.3 we construct a κ with complex coefficients on $\mathbb{R}^4$. At each $p \in \mathbb{R}^4$, this medium is determined by one arbitrary complex number, and hence the medium can depend on both time and space. However, at each point, the Fresnel surface of κ coincides with the usual null cone of the flat Minkowski metric $g = \operatorname{diag}(-1, 1, 1, 1)$. Let us note that the use of complex coefficient medium is well developed in time-harmonic fields [17, 18]. However, their use in a premetric setting does not seem to be as well developed. For example, currently there does not seem to exist a homogenous premetric description of *chiral medium* (which typically is modelled using complex coefficients [18, 19]).

The paper is organised as follows. In Section 2 we review Maxwell's equations and linear electromagnetic medium on a 4-manifold. In Section 3 we describe how the Tamm-Rubilar tensor density and Fresnel surface is related to wave propagation. To derive these objects we use the approach of geometric optics. In Section 4 we prove the main result Theorem 4.3, and in Section 5 we study non-uniqueness in Question 1.1. That is, we describe general results and examples where the Fresnel surface does not determined the conformal class of the medium tensor.

This paper relies on a number of computations done with computer algebra.

Mathematica notebooks for these can be found on the author's homepage.

2. Premetric electrodynamics

By a *manifold* M we mean a second countable topological Hausdorff space that is locally homeomorphic to $\mathbb{R}^n$ with C^∞-smooth transition maps. All objects are assumed to be smooth where defined. Let TM and T^*M be the tangent and cotangent bundles, respectively, and for $k \geq 1$, let $\Lambda^k(M)$ be the set of p-covectors, so that $\Lambda^1(N) = T^*N$. Let $\Omega^k_l(M)$ be $\binom{k}{l}$-tensors that are antisymmetric in their k upper indices and l lower indices. In particular, let $\Omega^k(M)$ be the set of k-forms. Let also $\mathfrak{X}(M)$ be the set of vector fields, and let $C^\infty(M)$ be the set of functions (that is, $\binom{0}{0}$-tensors). By $\Omega^k(M) \times \mathbb{R}$ we denote the set of k-forms that depend smoothly on a parameter $t \in \mathbb{R}$. By $T(M, \mathbb{C})$, $T^*(M, \mathbb{C})$, $\Lambda^p(M, \mathbb{C})$, $\Omega^k_l(M, \mathbb{C})$ and $\mathfrak{X}(M, \mathbb{C})$ we denote the complexification of the above spaces where component may also take complex values. Smooth complex valued functions are denoted by $C^\infty(M, \mathbb{C})$. The Einstein summing convention is used throughout. When writing tensors in local coordinates we assume that the components satisfy the same symmetries as the tensor.

To formulate Maxwell's equations we will also need *twisted tensors* [1, Section A.2.6], [20, Supplement 7.2A]. We will denoted these by a tilde over the tensor space. For example, by $\widetilde{\Omega}^2(N)$ we denote the space of twisted 2-forms on N. Let also $\widetilde{C}^\infty(N)$ be the set of twisted $\binom{0}{0}$-tensors on N. If M is orientable and oriented, then the set of twisted tensors coincide with their normal (or untwisted) counterparts. Say, $\widetilde{\Omega}^2_2(N) = \Omega^2_2(N)$. For the explicit transformation rules for elements in $\widetilde{\Omega}^2_2(N)$, see equation (10) below.

The *Levi-Civita permutation symbols* are denoted by ε_{ijkl} and ε^{ijkl}. Even if these coincide as combinatorial functions so that $\varepsilon_{ijkl} = \varepsilon^{ijkl}$, they are also different as they globally define different objects on a manifold. Namely, if $\varepsilon_{ijkl}, \varepsilon^{ijkl}$ and $\widetilde{\varepsilon}_{ijkl}, \widetilde{\varepsilon}^{ijkl}$ are defined on overlapping coordinate charts (U, x^i) and $(\widetilde{U}, \widetilde{x}^i)$, respectively, then

$$\widetilde{\varepsilon}_{abcd} = \det\left(\frac{\partial \widetilde{x}^i}{\partial x^j}\right) \varepsilon_{pqrs} \frac{\partial x^p}{\partial \widetilde{x}^a} \frac{\partial x^q}{\partial \widetilde{x}^b} \frac{\partial x^r}{\partial \widetilde{x}^c} \frac{\partial x^s}{\partial \widetilde{x}^d}, \tag{1}$$

$$\widetilde{\varepsilon}^{abcd} = \det\left(\frac{\partial x^i}{\partial \widetilde{x}^j}\right) \varepsilon^{pqrs} \frac{\partial \widetilde{x}^a}{\partial x^p} \frac{\partial \widetilde{x}^b}{\partial x^q} \frac{\partial \widetilde{x}^c}{\partial x^r} \frac{\partial \widetilde{x}^d}{\partial x^s}. \tag{2}$$

That is, ε_{ijkl} defines a $\binom{4}{0}$-tensor density of weight -1 on N and ε^{ijkl} defines a $\binom{0}{4}$-tensor density of weight 1.

2.1. The sourceless Maxwell's equations on a 4-manifold

Suppose E, D, B, H are forms that depend smoothly on a parameter t, $E \in \Omega^1(M) \times \mathbb{R}$ $H \in \widetilde{\Omega}^1(M) \times \mathbb{R}$, $D \in \widetilde{\Omega}^2(M) \times \mathbb{R}$ and $B \in \Omega^2(M) \times \mathbb{R}$. If N is the 4-manifold $N = \mathbb{R} \times M$, let $F \in \Omega^2(N)$, $G \in \widetilde{\Omega}^2(N)$ be forms

$$F = B + E \wedge dt, \tag{3}$$

$$G = D - H \wedge dt. \tag{4}$$

It follows that E, D, B, H solve the usual sourceless Maxwell's equations if and only if

$$dF = 0, \tag{5}$$

$$dG = 0, \tag{6}$$

where d is the exterior derivative on N. More generally, if N is a 4-manifold, and $F \in \Omega^2(N)$, $G \in \widetilde{\Omega}^2(N)$ then we say that F, G solve the *sourceless Maxwell's equations* when equations (5)–(6) hold. Since we are only interested in wave propagation away from possible sources, we will work with the sourceless Maxwell's equations. By an *electromagnetic medium* on N we mean a map

$$\kappa \colon \Omega^2(N) \to \widetilde{\Omega}^2(N).$$

We then say that 2-forms $F \in \Omega^2(N)$ and $G \in \widetilde{\Omega}^2(N)$ *solve Maxwell's equations in medium* κ if F and G satisfy equations (5)–(6) and

$$G = \kappa(F). \tag{7}$$

Equation (7) is known as the *constitutive equation.* If κ is invertible, it follows that one can eliminate half of the free variables in Maxwell's equations (5)–(6). We assume that κ is linear and local so that we can represent κ by an antisymmetric $\binom{2}{2}$-tensor $\kappa \in \widetilde{\Omega}^2_2(N)$. If in coordinates $\{x^i\}_{i=0}^3$ for N we have

$$\kappa = \frac{1}{8}\kappa^{ij}_{lm} dx^l \wedge dx^m \otimes \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \tag{8}$$

and $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$ and $G = \frac{1}{2}G_{ij}dx^i \wedge dx^j$, then constitutive equation (7) reads

$$G_{ij} = \frac{1}{2}\kappa^{rs}_{ij} F_{rs}. \tag{9}$$

Suppose $\kappa \in \widetilde{\Omega}^2_2(N)$ and κ^{ij}_{lm} and $\widetilde{\kappa}^{ij}_{lm}$ represent κ in overlapping coordinates x^i and $\widetilde{x}^i$, respectively. Then we have the transformation rule

$$\widetilde{\kappa}^{ij}_{lm} = \operatorname{sgn} \det \left(\frac{\partial \widetilde{x}^p}{\partial x^q} \right) \kappa^{rs}_{ab} \frac{\partial x^a}{\partial \widetilde{x}^l} \frac{\partial x^b}{\partial \widetilde{x}^m} \frac{\partial \widetilde{x}^i}{\partial x^r} \frac{\partial \widetilde{x}^j}{\partial x^s}, \tag{10}$$

where sgn is the *sign function*, $\operatorname{sgn} x = x/|x|$ for $x \neq 0$ and $\operatorname{sgn} x = 0$ for $x = 0$.

Equations (5)—(7) form the basis of the *premetric* formulation for electromagnetics on a 4-manifold without source terms. Let us emphasise that these equations do not depend on any metric. For a systematic presentation, see [1, 2].

2.2. Operations on medium tensors

An element in $\Omega^2_2(N)$ defines a linear map $\Omega^2_p(N) \to \Omega^2_p(N)$ for each $p \in N$. Hence we can define the determinant and trace of κ and these are smooth functions $\det \kappa, \operatorname{trace} \kappa \in C^\infty(N)$. Moreover, if κ is invertible we can define the inverse $\kappa^{-1} \in \Omega^2_2(N)$. Next, we describe how these operations generalise to elements in $\widetilde{\Omega}^2_2(N)$.

Suppose $\kappa \in \widetilde{\Omega}^2_2(N)$ on a 4-manifold N. It is clear that in each chart (U, x^i) on N we can restrict κ to an element $\kappa|_U \in \Omega^2_2(U)$, and for each $p \in U$ we can treat $\kappa|_U$ as a linear map $\kappa|_U \colon \Omega^2_p(N) \to \Omega^2_p(N)$. In each chart on N we can then apply the above argument. Locally, we can define the determinant and trace and these are smooth functions $\det \kappa|_U, \operatorname{trace}|_U \in C^\infty(U)$. Moreover, if $\kappa|_U$ is invertible, we can define $(\kappa|_U)^{-1} \in \Omega^2_2(U)$. The next proposition shows how these local definitions give rise to global objects on N.

Proposition 2.1. *If $\kappa \in \widetilde{\Omega}^2_2(N)$, and $\det \kappa$ and $\operatorname{trace} \kappa$ are defined as above, then*

$$\det \kappa \in C^\infty(N), \quad \operatorname{trace} \kappa \in \widetilde{C}^\infty(N).$$

Moreover, if κ is invertible (that is, $\kappa|_U$ is invertible in each chart U) and κ^{-1} is defined as above, then $\kappa^{-1} \in \widetilde{\Omega}^2_2(N)$.

Proof. Let O be the ordered set of index pairs $\{01, 02, 03, 23, 31, 12\}$ [1, Section A.1.10], [10]. If $I \in O$, let I_1 and I_2 denote the individual indices. Say, if $I = 31$ then $I_2 = 1$. Suppose (U, x^i) are local coordinates for N. For $J \in O$ we define $dx^J = dx^{J_1} \wedge dx^{J_2}$. Locally, a basis for $\Omega^2(N)$ is then given by $\{dx^J : J \in O\}$, that is, by

$$\{dx^0 \wedge dx^1, dx^0 \wedge dx^2, dx^0 \wedge dx^3, dx^2 \wedge dx^3, dx^3 \wedge dx^1, dx^1 \wedge dx^2\}. \tag{11}$$

If $\kappa \in \widetilde{\Omega}^2_2(N)$ is written as in equation (8) and $J \in O$, it follows that

$$\kappa(dx^J) = \sum_{I \in O} \kappa_I^J dx^I, \quad J \in O, \tag{12}$$

where $\kappa_I^J = \kappa_{I_1 I_2}^{J_1 J_2}$. Thus, κ is locally determined by components $\{\kappa_I^J : I, J \in O\}$, and we identify these components with the 6×6 matrix $A = (\kappa_I^J)_{IJ}$. That is, if b is the natural bijection $b: O \to \{1, \ldots, 6\}$, then $A = (\kappa_{b^{-1}(i)}^{b^{-1}(j)})_{ij}$. The motivation for this identification is that for each $p \in U$, matrix $A|_p$ is the matrix representation of the linear map $\kappa|_U: \Omega^2_p(U) \to \Omega^2_p(U)$ with respect to the basis (11). Thus

$$\det(\kappa|_U) = \det A, \quad \operatorname{trace}(\kappa|_U) = \operatorname{trace} A, \quad ((\kappa|_U^{-1})_I^J)_{IJ} = A^{-1}. \tag{13}$$

Next, suppose $\{x^i\}_{i=0}^3$ and $\{\widetilde{x}^i\}_{i=0}^3$ are overlapping coordinates, and $A = (\kappa_I^J)_{IJ}$ and $\widetilde{A} = (\widetilde{\kappa}_I^J)_{IJ}$ are matrices that represent tensor κ in these coordinates. For $I, J \in O$ let

$$\frac{\partial x^J}{\partial \widetilde{x}^I} = \frac{\partial x^{J_1}}{\partial \widetilde{x}^{I_1}} \frac{\partial x^{J_2}}{\partial \widetilde{x}^{I_2}} - \frac{\partial x^{J_2}}{\partial \widetilde{x}^{I_1}} \frac{\partial x^{J_1}}{\partial \widetilde{x}^{I_2}},$$

and similarly, define $\frac{\partial \widetilde{x}^J}{\partial x^I}$ by exchanging x and $\widetilde{x}$. Then equation (10) reads

$$\widetilde{\kappa}_I^J = \operatorname{sgn} \det \left(\frac{\partial \widetilde{x}^i}{\partial x^j} \right) \sum_{K, L \in O} \frac{\partial x^K}{\partial \widetilde{x}^I} \kappa_K^L \frac{\partial \widetilde{x}^J}{\partial x^L}, \quad I, J \in O.$$

For matrices $T = (\frac{\partial x^J}{\partial \widetilde{x}^I})_{IJ}$ and $S = (\frac{\partial \widetilde{x}^J}{\partial x^I})_{IJ}$, we have $T = S^{-1}$. In matrix form, equation (10) then reads

$$\widetilde{A} = \operatorname{sgn} \det \left(\frac{\partial \widetilde{x}^i}{\partial x^j} \right) TAT^{-1}. \tag{14}$$

The claim follows by equations (13) and (14). $\square$

Let us make two comments regarding Proposition 2.1. First, if $\kappa \in \widetilde{\Omega}^2_2(N)$ and κ is locally given by equation (8), then from equation (13) in the proof we see that

$$\operatorname{trace} \kappa = \frac{1}{2} \kappa_{ij}^{ij}.$$

Second, it turns out that the global behaviour of elements in $\widetilde{C}^\infty(N)$ is coupled to the orientability of the underlying manifold N. This phenomenon will be described in Proposition 3.7 below.

2.3. Decomposition of electromagnetic medium

At each point of a 4-manifold N, an element of $\widetilde{\Omega}^2_2(N)$ depends on 36 parameters. Pointwise, such $\binom{2}{2}$-tensors canonically decompose into three linear subspaces. The motivation for this decomposition is that different components in the decomposition enter in different parts of electromagnetics. See [1, Section D.1.3]. The below formulation is based on [21].

If Id is the identity map $\mathrm{Id} \in \Omega^2_2(N)$, then writing Id as in equation (8) gives $\mathrm{Id}\,^{ij}_{rs} = \delta^i_r \delta^j_s - \delta^i_s \delta^j_r$, where δ^i_j is the *Kronecker delta symbol.*

Proposition 2.2. *Let N be a 4-manifold, and let*

$$\begin{aligned}
Z &= \{\kappa \in \widetilde{\Omega}^2_2(N) : u \wedge \kappa(v) = \kappa(u) \wedge v \textit{ for all } u, v \in \Omega^2(N), \\
&\qquad\qquad \mathit{trace}\, \kappa = 0\}, \\
W &= \{\kappa \in \widetilde{\Omega}^2_2(N) : u \wedge \kappa(v) = -\kappa(u) \wedge v \textit{ for all } u, v \in \Omega^2(N)\} \\
&= \{\kappa \in \widetilde{\Omega}^2_2(N) : u \wedge \kappa(v) = -\kappa(u) \wedge v \textit{ for all } u, v \in \Omega^2(N), \\
&\qquad\qquad \mathit{trace}\, \kappa = 0\}, \\
U &= \{f Id \in \widetilde{\Omega}^2_2(N) : f \in \widetilde{C}^\infty(N)\} \\
&= \{\kappa \in \widetilde{\Omega}^2_2(N) : u \wedge \kappa(v) = \frac{1}{6} \mathit{trace}\,(\kappa) u \wedge v \textit{ for all } u, v \in \Omega^2(N)\}.
\end{aligned}$$

Then

$$\widetilde{\Omega}^2_2(N) = Z \,\oplus\, W \,\oplus\, U, \tag{15}$$

and pointwise, $\dim Z = 20$, $\dim W = 15$ *and* $\dim U = 1$.

If we write a $\kappa \in \widetilde{\Omega}^2_2(N)$ as

$$\kappa = {}^{(1)}\kappa \,+\, {}^{(2)}\kappa \,+\, {}^{(3)}\kappa \tag{16}$$

with ${}^{(1)}\kappa \in Z$, ${}^{(2)}\kappa \in W$, ${}^{(3)}\kappa \in U$, then we say that ${}^{(1)}\kappa$ is the *principal part*, ${}^{(2)}\kappa$ is the *skewon part*, ${}^{(3)}\kappa$ is the *axion part* of κ [1].

Proof. Let us start with two observations. First, if $\kappa \in \widetilde{\Omega}^2_2(N)$, then $\kappa \in W$ (with W defined on the first line) if and only if

$$\kappa^{ij}_{lm} \varepsilon^{lmpq} = -\,\kappa^{pq}_{lm} \varepsilon^{lmij} \tag{17}$$

when κ is written as in equation (8). Since $\varepsilon^{lmpq}\varepsilon_{lmij} = 4\delta^p_{[i}\delta^q_{j]}$, it follows that the two expressions for W coincide. Here we use the bracket notation to indicate that indices i, j are antisymmetrised (with scaling 1/2!). The equality of the two expressions for U follows similarly. Second, let W' be defined as

$$W' = \{\eta \in \widetilde{\Omega}^1_1(N) : \mathrm{trace}\, \eta = 0\},$$

where $\mathrm{trace}\,\eta$ is locally defined as $\mathrm{trace}\,\eta = \eta^i_i$. Moreover, let σ be the linear map $\sigma \colon W' \to \widetilde{\Omega}^2_2(N)$ such that if $\eta \in W'$ and locally $\eta = \eta^i_j dx^j \otimes \frac{\partial}{\partial x^i}$ then

$$\sigma(\eta)^{ij}_{lm} = 2\eta^{[i}_{[l}\delta^{j]}_{m]}. \tag{18}$$

Lastly, if $\kappa \in \widetilde{\Omega}^2_2(N)$, let $\not\kappa \in W'$ be the tensor locally defined as $\not\kappa = \not\kappa^i_j dx^j \otimes \frac{\partial}{\partial x^i}$, where $\not\kappa^i_j = \kappa^{im}_{jm} - \frac{1}{2}\mathrm{trace}\,(\kappa)\delta^i_j$ [1, Section D.1.2]. The remaining arguments in this paragraph rely on computer algebra. Equations (17)–(18) show that $\sigma(W') \subset W$ and, moreover, $\kappa = \sigma(\not\kappa)$ for all $\kappa \in W$. Since $\sigma(\eta) = 0$ for $\eta \in W'$ implies that $\eta = 0$, it follows that σ is a linear isomorphism $\sigma\colon W' \to W$.

If $\kappa \in \widetilde{\Omega}^2_2(N)$, then

$$ {}^{(1)}\kappa = \kappa - {}^{(2)}\kappa - {}^{(3)}\kappa, \quad {}^{(2)}\kappa = \sigma(\not\kappa), \quad {}^{(3)}\kappa = \frac{1}{6}\operatorname{trace}(\kappa)\mathrm{Id} $$

satisfy ${}^{(1)}\kappa \in Z$, ${}^{(2)}\kappa \in W$ and ${}^{(3)}\kappa \in U$. This can be seen by computer algebra. Thus $\widetilde{\Omega}^2_2(N) = Z + W + U$. To see that the sum is direct, that is, to see that the decomposition in equation (16) is unique, suppose we have two decompositions

$$ \begin{aligned} \kappa &= {}^{(1)}\kappa \ + \ {}^{(2)}\kappa \ + \ {}^{(3)}\kappa \\ &= {}^{(1)}\kappa' \ + \ {}^{(2)}\kappa' \ + \ {}^{(3)}\kappa'. \end{aligned} $$

Taking trace shows that ${}^{(3)}\kappa = {}^{(3)}\kappa'$, and uniqueness follows since ${}^{(1)}\kappa - {}^{(1)}\kappa' = {}^{(2)}\kappa' - {}^{(2)}\kappa \in Z \cap W = \{0\}$. The pointwise dimensions for Z, W, U follow since W' has dimension 15 and U has dimension 1. $\square$

2.4. The Hodge star operator

By a *pseudo-Riemann metric* on a manifold N we mean a symmetric real $\binom{0}{2}$-tensor g that is non-degenerate. If N is not connected we also assume that g has constant signature. If g is positive definite, we say that g is a *Riemann metric.* A pseudo-Riemann metric g is a *Lorentz metric* if N is 4-dimensional and g has signature $(+---)$ or $(-+++)$.

By $\sharp$ and $\flat$ we denote the isomorphisms $\sharp\colon T^*N \to TN$ and $\flat\colon TN \to T^*N$. By $\mathbb{R}$-linearity we extend g, $\sharp$ and $\flat$ to complex arguments. Moreover, we extend g also to covectors by setting $g(\xi, \eta) = g(\xi^\sharp, \eta^\sharp)$ when $\xi, \eta \in \Lambda^1_p(N, \mathbb{C})$. For a Lorentz metric, we define the *null cone* at p as the set $\{\xi \in \Lambda^1_p(N) : g(\xi, \xi) = 0\}$.

If g is a pseudo-Riemann metric on a 4-manifold N, then the *Hodge star operator* for g is the twisted $\binom{2}{2}$-tensor $\kappa = *_g \in \widetilde{\Omega}^2_2(N)$ defined as follows. If $\kappa = *_g$ is written as in equation (8) for local coordinates x^i and $g = g_{ij} dx^i \otimes dx^j$, then

$$ \kappa^{ij}_{rs} = \sqrt{|\det g|}\; g^{ia} g^{jb} \varepsilon_{abrs}. \tag{19} $$

Here $\det g = \det g_{ij}$ and g^{ij} is the ijth entry of $(g_{ij})^{-1}$. That $*_g$ in equation (19) defines a twisted tensor $*_g \in \widetilde{\Omega}^2_2(N)$ follows by equation (1).

2.5. Decomposition of κ into four 3×3 matrices

Next we show that if N is a 4-manifold, then any tensor $\kappa \in \widetilde{\Omega}^2_2(N)$ is locally determined by four smoothly varying 3×3-matrices. If x^i are coordinates around a $p \in N$, then we can locally decompose N into a product manifold, by treating x^0 is the coordinate for $\mathbb{R}$ and (x^1, x^2, x^3) as coordinates for some 3-manifold M. By writing F, G as in equations (3)–(4), we denote local components for F and G as

$$ F_{i0} = E_i, \quad F_{ij} = B_{ij}, \quad G_{i0} = -H_i, \quad G_{ij} = D_{ij}, $$

where $i, j \in \{1, 2, 3\}$. Equation (9) then reads

$$ H_i \ = \ -\kappa^{r0}_{i0} E_r - \frac{1}{2}\kappa^{rs}_{i0} B_{rs}, \tag{20} $$

$$ D_{ij} = \kappa^{r0}_{ij} E_r + \frac{1}{2}\kappa^{rs}_{ij} B_{rs}, \tag{21} $$

where $i,j\in\{1,2,3\}$ and r,s are summed over $1,2,3$.

Let $\{B^i\}_{i=1}^3$ be defined as $B^i=\frac{1}{2}\sum_{j,k=1}^3\varepsilon^{ijk}B_{jk}$. Then $B_{rs}=\sum_{i=1}^3\varepsilon_{irs}B^i$, and similarly, we also define $\{D^i\}_{i=1}^3$. Then equations (20)–(21) can be rewritten as

$$H_i = \mathscr{C}^r{}_i(-E_r)+\mathscr{B}_{ri}B^r, \tag{22}$$

$$D^i = \mathscr{A}^{ri}(-E_r)+\mathscr{D}_r{}^iB^r, \tag{23}$$

where $i\in\{1,2,3\}$, r is summed over $1,2,3$, and

$$\mathscr{C}^r{}_i=\kappa_{i0}^{r0},\quad \mathscr{B}_{ri}=-\frac{1}{2}\varepsilon_{rab}\kappa_{i0}^{ab},\quad \mathscr{A}^{ri}=-\frac{1}{2}\varepsilon^{iab}\kappa_{ab}^{r0},\quad \mathscr{D}_r{}^i=\frac{1}{4}\varepsilon_{rmn}\varepsilon^{iab}\kappa_{ab}^{mn}. \tag{24}$$

Inverting the relations gives

$$\kappa_{0i}^{0r}=\mathscr{C}^r{}_i,\quad \kappa_{0r}^{ij}=\varepsilon^{kij}\mathscr{B}_{kr},\quad \kappa_{rs}^{0i}=\varepsilon_{krs}\mathscr{A}^{ik},\quad \kappa_{rs}^{ij}=\varepsilon_{krs}\varepsilon^{lij}\mathscr{D}_l{}^k, \tag{25}$$

where $i,j,r,s\in\{1,2,3\}$ and k,l are summed over $1,2,3$.

We have shown that in coordinates x^i, tensor κ is represented by the smoothly varying 3×3 matrices $\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D}$ defined as

$$\mathscr{A}=\left(\mathscr{A}^{ri}\right)_{ri},\quad \mathscr{B}=\left(\mathscr{B}_{ri}\right)_{ri},\quad \mathscr{C}=\left(\mathscr{C}^r{}_i\right)_{ri},\quad \mathscr{D}=\left(\mathscr{D}_r{}^i\right)_{ri}.$$

These matrices coincide with the corresponding matrices in [1, Section D.1.6] and [2]. Since each matrix is only part of tensor κ, it does not transform in a simple way under a general coordinate transformation in N (see equations D.5.28–D.5.30 in [1]). However, suppose $\{x^i\}_{i=0}^3$ and $\{\widetilde{x}^i\}_{i=0}^3$ are overlapping coordinates such that

$$\widetilde{x}^0 = x^0,$$
$$\widetilde{x}^i = \widetilde{x}^i(x^1,x^2,x^3),\quad i\in\{1,2,3\}.$$

Then equations (10), (24), (25) and identity $\varepsilon_{ijk}A_a^iA_b^jA_c^k=\det A\,\varepsilon_{abc}$ for any 3×3 matrix $A=(A_j^i)_{ij}$, yield the following transformation rules

$$\widetilde{\mathscr{C}}^r{}_i = \operatorname{sgn}\det\left(\frac{\partial\widetilde{x}^m}{\partial x^n}\right)\mathscr{C}^a{}_b\frac{\partial x^b}{\partial\widetilde{x}^i}\frac{\partial\widetilde{x}^r}{\partial x^a}, \tag{26}$$

$$\widetilde{\mathscr{B}}_{ri} = \left|\det\left(\frac{\partial\widetilde{x}^m}{\partial x^n}\right)\right|\mathscr{B}_{ab}\frac{\partial x^a}{\partial\widetilde{x}^r}\frac{\partial x^b}{\partial\widetilde{x}^i}, \tag{27}$$

$$\widetilde{\mathscr{A}}^{ri} = \left|\det\left(\frac{\partial x^m}{\partial\widetilde{x}^n}\right)\right|\mathscr{A}^{ab}\frac{\partial\widetilde{x}^r}{\partial x^a}\frac{\partial\widetilde{x}^i}{\partial x^b}, \tag{28}$$

$$\widetilde{\mathscr{D}}_r{}^i = \operatorname{sgn}\det\left(\frac{\partial\widetilde{x}^m}{\partial x^n}\right)\mathscr{D}_a{}^b\frac{\partial\widetilde{x}^i}{\partial x^b}\frac{\partial x^a}{\partial\widetilde{x}^r}, \tag{29}$$

where $i,r\in\{1,2,3\}$ and a,b are summed over $1,2,3$.

If ${}^{(2)}\kappa=0$ then Proposition 2.2 implies that κ is pointwise determined by 21 coefficients. The next proposition shows that these coefficients can pointwise be reduced to 18 when the coordinates are chosen suitably.

Proposition 2.3. *Suppose N is a 4-manifold and $\kappa\in\widetilde{\Omega}_2^2(N)$. Then*

(i) *κ has no skewon component if and only if locally*

$$\mathscr{A}=\mathscr{A}^T,\quad \mathscr{B}=\mathscr{B}^T,\quad \mathscr{C}=\mathscr{D}^T,$$

where ^{T} is the matrix transpose, and $\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D}$ are defined as above.

(ii) *Let $p\in N$. If κ has no skewon component, then there are local coordinates around p such that $\mathscr{A}$ is diagonal at p.*

Proof. Part *(i)* follows by [1, Equation D.1.100]. Since any symmetric matrix can be diagonalised by an orthogonal matrix part *(ii)* follows by part *(i)* and equation (28). □

3. Geometric optics solutions

Suppose $\kappa \in \widetilde{\Omega}^2_2(N)$ on a 4-manifold N. To study wave propagation in the medium κ we will use the technique of *geometric optics*. We then assume that field quantities F and G in Maxwell's equations can be written as asymptotic sums [22]

$$F = \operatorname{Re}\left\{ e^{iP\Phi} \sum_{k=0}^{\infty} \frac{A_k}{(iP)^k} \right\}, \quad G = \operatorname{Re}\left\{ e^{iP\Phi} \sum_{k=0}^{\infty} \frac{B_k}{(iP)^k} \right\}, \tag{30}$$

where $P > 0$ is the asymptotic parameter, $\Phi \in C^\infty(N)$, $A_k \in \Omega^2(N, \mathbb{C})$ and $B_k \in \widetilde{\Omega}^2(N, \mathbb{C})$. In this setting, function Φ is called the *phase function*, and forms A_k, B_k are called *amplitudes*. Let us emphasise that we will treat the above sums as formal sums and will not consider convergence questions. For an analysis, see [23, 24]. Let us also note that there are other approaches for studying propagation in premetric electromagnetics [1, 13, 25].

Substituting F and G into the sourceless Maxwell equations and differentiating termwise shows that F and G form an asymptotic solution provided that

$$d\Phi \wedge A_0 = 0, \tag{31}$$

$$d\Phi \wedge B_0 = 0, \tag{32}$$

$$B_k = \kappa A_k, \tag{33}$$

$$d\Phi \wedge A_{k+1} + dA_k = 0, \tag{34}$$

$$d\Phi \wedge B_{k+1} + dB_k = 0, \quad k = 0, 1, \ldots. \tag{35}$$

In equation (33) we treat κ as a linear map $\kappa\colon \Omega^2(N, \mathbb{C}) \to \widetilde{\Omega}^2(N, \mathbb{C})$.

Let us assume that $d\Phi$ is never zero. Then we can find an $X \in \mathfrak{X}(N)$ such that $d\Phi(X) = 1$ and contracting equation (31) yields a 1-form $a_0 \in \Omega^1(N, \mathbb{C})$ with $A_0 = d\Phi \wedge a_0$, whence

$$d\Phi \wedge \kappa(d\Phi \wedge a_0) = 0. \tag{36}$$

Since equation (36) is a linear in a_0, we may study the dimension of the the solution space for a_0. To do this, let $\xi \in \Lambda^1_p(N)$ for some $p \in N$ and for ξ let L_ξ be the linear map $L_\xi\colon \Lambda^1_p(N) \to \widetilde{\Lambda}^3_p(N)$,

$$L_\xi(\alpha) = \xi \wedge \kappa(\xi \wedge \alpha), \quad \alpha \in \Lambda^1_p(N). \tag{37}$$

We always have $\xi \in \ker L_\xi$. For all $\xi \in \Lambda^1_p N \backslash \{0\}$ we can then find a (non-unique) vector subspace $V_\xi \subset \Lambda^1_p N$ such that

$$\ker L_\xi = V_\xi \oplus \operatorname{span} \xi. \tag{38}$$

Let $\xi = d\Phi|_p$ be non-zero. Then $V_\xi \backslash \{0\}$ parameterises possible a_0 that solve equation (36) and for which $A_0 = d\Phi \wedge a_0$ is non-zero. For a general $\kappa \in \widetilde{\Omega}^2_2(N)$ and $\xi \in \Lambda^1(N) \backslash \{0\}$ we can have $\dim V_\xi \in \{0, 1, 2, 3\}$: Proposition 3.5 will show that $\dim V_\xi$ can be 0 or 2. In Example 3.6 we will see that $\dim V_\xi = 1$ is possible, say, in a biaxial crystal. The next proposition characterises those $\kappa|_p$ such that $\dim V_\xi = 3$ for all $\xi \in \Lambda^1_p(N) \backslash \{0\}$.

Proposition 3.1. *Let $\kappa \in \widetilde{\Omega}^2_2(N)$ on a 4-manifold N and let $p \in N$. Then the following are equivalent:*

(i) *$\kappa|_p$ is of axion type.*

(ii) *$\dim V_\xi = 3$ for all $\xi \in \Lambda^1_p(N) \backslash \{0\}$.*

Proof. Implication *(i)* $\Rightarrow$ *(ii)* is clear. For the converse direction suppose that *(ii)* holds and $\{x^i\}_{i=0}^3$ are local coordinates around p. It follows that

$$\zeta \wedge \xi \wedge \kappa(\xi \wedge \alpha) = 0, \quad \alpha, \xi, \zeta \in \Lambda^1_p(N).$$

If locally $\xi = \xi_i dx^i|_p$ then $\xi_i \xi_j \kappa^{ir}_{ab} \varepsilon^{jsab} = 0$. Differentiating with respect to ξ_c and ξ_d gives $\kappa^{cr}_{ab}\varepsilon^{dsab} + \kappa^{dr}_{ab}\varepsilon^{csab} = 0$. Contracting both sides by ε_{dsij} using identities

$$\varepsilon^{srab}\varepsilon_{srij} = 2\delta^a_{[i}\delta^b_{j]}, \quad \varepsilon^{sabc}\varepsilon_{sijk} = 3!\delta^a_{[i}\delta^b_j\delta^c_{k]} \tag{39}$$

gives

$$3\kappa^{cr}_{ij} = \kappa^{sr}_{sj}\delta^c_i - \kappa^{sr}_{si}\delta^c_j. \tag{40}$$

Setting $i = r$ and summing r over $0, 1, 2, 3$ gives $\kappa^{si}_{sj} = \frac{1}{2}\mathrm{trace}\,\kappa\,\delta^i_j$ whence equation (40) yields $\kappa = \frac{1}{6}\mathrm{trace}\,\kappa \;\mathrm{Id}$ and *(i)* follows. $\square$

3.1. The Fresnel surface

Let $\kappa \in \widetilde{\Omega}^2_2(N)$ on a 4-manifold N. If κ is locally given by equation (8) in coordinates $\{x^i\}$, let

$$\mathscr{G}_0^{ijkl} = \frac{1}{48}\kappa^{a_1a_2}_{b_1b_2}\kappa^{a_3i}_{b_3b_4}\kappa^{a_4j}_{b_5b_6}\varepsilon^{b_1b_2b_5k}\varepsilon^{b_3b_4b_6l}\varepsilon_{a_1a_2a_3a_4}. \tag{41}$$

If $\{\widetilde{x}^i\}$ are overlapping coordinates, then equations (10), (1) and (2) imply that components $\mathscr{G}_0^{ijkl}$ satisfy transformation rules

$$\widetilde{\mathscr{G}_0^{ijkl}} = \left|\det\left(\frac{\partial x^r}{\partial \widetilde{x}^s}\right)\right| \mathscr{G}_0^{abcd}\frac{\partial \widetilde{x}^i}{\partial x^a}\frac{\partial \widetilde{x}^j}{\partial x^b}\frac{\partial \widetilde{x}^k}{\partial x^c}\frac{\partial \widetilde{x}^l}{\partial x^d}. \tag{42}$$

Equation (42) states that components $\mathscr{G}_0^{ijkl}$ define a twisted $\binom{4}{0}$-tensor density $\mathscr{G}_0$ on N of weight 1. The *Tamm-Rubilar tensor density* [1, 2] is the symmetric part of $\mathscr{G}_0$ and we denote this twisted tensor density by $\mathscr{G}$. In coordinates, $\mathscr{G}^{ijkl} = \mathscr{G}_0^{(ijkl)}$, where parenthesis indicate that indices $ijkl$ are symmetrised with scaling $1/4!$. For $\xi = \xi_i dx^i$ in local coordinates let us also write $\mathscr{G}(\xi,\xi,\xi,\xi) = \mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = \mathscr{G}_0^{ijkl}\xi_i\xi_j\xi_k\xi_l$. Using $\mathscr{G}$, the *Fresnel surface* at a point $p \in N$ is defined as

$$F_p = \{\xi \in \Lambda^1_p(N) : \mathscr{G}(\xi,\xi,\xi,\xi) = 0\}. \tag{43}$$

By equation (42), the definition of F_p does not depend on local coordinates. Let F be the disjoint union of all Fresnel surfaces, $F = \coprod_{p\in N} F_p$. To indicate that F_p and F depend on κ we also write $F_p(\kappa)$ and $F(\kappa)$.

If $\xi \in F_p$ then $\lambda\xi \in F_p$ for all $\lambda \in \mathbb{R}$. In particular $0 \in F_p$ for each $p \in N$. When $\mathscr{G}|_p$ is non-zero, equation (43) shows that F_p is a fourth order surface in $\Lambda^1_p(N)$, so F_p may contain non-smooth self intersections.

If Φ is a phase function as in equation (30) and $\xi = d\Phi|_p$, then V_ξ in equation (38) is a vector space that parameterises possible polarisations for the chosen phase function. For example, if $\dim V_\xi = 0$, then are no propagating waves. The importance of the next theorem is that it characterises when $\dim V_\xi \geq 1$ using the Fresnel surface. Thus, the Fresnel surface can be seen as a premetric (and tensorial) analogue to the classical dispersion equation. This characterisation is due to Obukhov, Fukui and Rubilar [8]. For further results regarding the applicability and derivation of the Fresnel surface, see [1, 2, 9, 12, 13, 14, 24].

Theorem 3.2. *Suppose N is a 4-manifold and $\kappa \in \widetilde{\Omega}^2_2(N)$. If $\xi \in \Lambda^1_p(N)$ is non-zero, then the following are equivalent:*

(i) $\dim V_\xi \geq 1$ *where V_ξ are defined as in equation (38).*

(ii) ξ *belongs to the Fresnel surface $F_p \subset \Lambda^1_p(N)$.*

Proof. Let $\{x^i\}_{i=0}^3$ be coordinates around p such that $dx^0|_p = \xi$. By the the second identity in equation (39), we obtain

$$\begin{aligned} L_\xi(\alpha) &= \frac{1}{2}\alpha_r \kappa^{0r}_{jk} dx^0 \wedge dx^j \wedge dx^k \\ &= \frac{1}{12}\alpha_r \kappa^{0r}_{jk} \varepsilon^{s0jk} \varepsilon_{suvw} dx^u \wedge dx^v \wedge dx^w, \end{aligned}$$

where $\alpha = \alpha_r dx^r|_p$ and κ^{ij}_{ab} are defined as in equation (8). Thus the matrix representing L_ξ between bases $\{dx^i|_p\}_{i=0}^3$ and $\{\frac{1}{3!}\varepsilon_{suvw} dx^u \wedge dx^v \wedge dx^w|_p\}_{s=0}^3$ is the 4×4 matrix $P = (P^{rs})_{r,s=0}^3$,

$$P^{rs} = \frac{1}{2}\kappa^{0r}_{jk}\varepsilon^{s0jk}. \tag{44}$$

It is clear that P has the form $P = \operatorname{diag}(0, Q)$ for the 3×3 matrix $Q = (P^{ij})_{i,j=1}^3$. By equation (38), $\dim V_\xi \geq 1$ is then equivalent with $\dim\ker P \geq 2$, which is equivalent with $\det Q = 0$. We know that

$$\det Q = \frac{1}{3!}\varepsilon_{0abc}\varepsilon_{0ijk} P^{ai} P^{bj} P^{ck}, \tag{45}$$

where all variables are summed over $0,\ldots,3$. However, due to the Levi-Civita permutation symbols, only terms where all variables are in $1,2,3$ can be non-zero. Using antisymmetry and the second identity in equation (39), it follows that

$$\sum_{k=1}^3 \varepsilon_{0ijk} P^{ck} = \begin{cases} -\kappa^{0c}_{ij}, & \text{when } i,j \in \{1,2,3\}, \\ 0, & \text{when } i = 0 \text{ or } j = 0, \end{cases} \tag{46}$$

$$\sum_{a=1}^3 \varepsilon_{0abc} P^{ai} = \begin{cases} \frac{1}{4}\varepsilon_{efbc}\kappa^{ef}_{uv}\varepsilon^{i0uv}, & \text{when } b,c \in \{1,2,3\}, \\ 0, & \text{when } b = 0 \text{ or } c = 0. \end{cases} \tag{47}$$

Then equations (41) and (44)–(47) imply that $\det Q = -\mathscr{G}^{0000}$, whence $\det Q = 0$ and $\xi \in F_p$ are equivalent. The result follows. □

A key property of symmetric $\binom{p}{0}$-tensors is that they are completely determined by their values on the diagonal [3]. For symmetric $\binom{4}{0}$-tensors on a 4-manifold, the precise statement is contained in the following polarisation identity.

Proposition 3.3. *Suppose L is a symmetric $\binom{4}{0}$-tensor on a 4-manifold N. If $\eta^{(1)},\ldots,\eta^{(4)} \in \Lambda^1_p(N)$ then*

$$L(\eta^{(1)},\ldots,\eta^{(4)}) = \frac{1}{4!2^4}\sum_{\theta_1,\ldots,\theta_4\in\{\pm 1\}} \theta_1\theta_2\theta_3\theta_4\, L(\sum_{i=1}^4 \theta_i\eta^{(i)},\ldots,\sum_{i=1}^4\theta_i\eta^{(i)}).$$

For an analytic proof of the general case, see [26, Theorem 5.6]. However, since the rank and dimension is here fixed, the proposition can also be verified by computer algebra.

3.2. Electromagnetic medium induced by a Hodge star operator

The next proposition collects known results for the Hodge star operator associated to a pseudo-Riemann g. In particular, the proposition shows that if the Hodge star operator is induced by a metric g with signatures $(++++)$ or $(----)$, then the medium with $\kappa = *_g$ has no asymptotic solutions. That is, if $d\Phi|_p$ is non-zero, then equation (36) implies that $A_0|_p = 0$. The proposition also shows that if $\kappa = *_g$ for an indefinite metric g, then A_0 can be non-zero only when $d\Phi|_p$ is a *null covector*, that is, when $g(d\Phi|_p, d\Phi|_p) = 0$. For generalisations, see [27, 28, 29].

Proposition 3.4. *Suppose g is a pseudo-Riemann metrics on N on a 4-manifold N, and $*_g \in \widetilde{\Omega}_2^2(N)$ is the associated Hodge star operator. Then $*_g$ has only a principal part, and*

$$\mathscr{G}_{*g}(\xi,\xi,\xi,\xi) = sgn\,(\det g)\,\sqrt{|\det g|}\,\left(g(\xi,\xi)\right)^2, \quad \xi \in \Lambda^1(N),$$

and the Fresnel surface induced by $*_g$ is given by

$$F(*_g) = \{\xi \in \Lambda^1(N) : g(\xi,\xi) = 0\}.$$

Proof. To see that $*_g$ has only a principal part we will use Theorem 2.2. Since $u \wedge *_g(v) = *_g(u) \wedge v$ for all $u, v \in \Omega^2(N)$ [20, Proposition 6.2.13], we only need to prove that $\operatorname{trace} \kappa = 0$. Let us fix $p \in N$ and let x^i be coordinates such that $g|_p$ is diagonal. At p, we then need to show that $g^{ia}g^{jb}\varepsilon_{abij} = 0$. However, this follows since g^{ij} is diagonal and ε_{abij} is non-zero only when $abij$ are distinct. For the second claim, a rather lengthy computation using equations (19), (41) and the first identity in equation (39) (or, alternatively, computer algebra) shows that

$$\mathscr{G}_0^{abcd}\xi_a\xi_b\xi_c\xi_d = \operatorname{sgn}(\det g)\,\sqrt{|\det g|}\,\left(g(\xi,\xi)\right)^2,$$

where $\xi = \xi_a dx^a$ in arbitrary coordinates x^i. The result follows. □

A particular example of a Hodge star operator is given by $\kappa = \sqrt{\frac{\epsilon}{\mu}} *_g$ where g is the Lorentz metric $g = \operatorname{diag}(-\frac{1}{\epsilon\mu}, 1, 1, 1)$ on $\mathbb{R}^4$. For this κ the constitutive equation (7) models standard isotropic medium on $\mathbb{R}^4$ with permittivity $\epsilon > 0$ and $\mu > 0$.

We know that a general plane wave in homogeneous isotropic medium in $\mathbb{R}^3$ can be written as a sum of two circularly polarised plane waves with opposite handedness. The *Bohren decomposition* generalise this classical result to electromagnetic fields in homogeneous isotropic chiral medium [18]. The *Moses decomposition*, or *helicity decomposition*, further generalise this decomposition to arbitrary vector fields on $\mathbb{R}^3$, and for Maxwell's equations, see [30, 31]. Part *(i)* in the next proposition proves an analogous result for asymptotic solutions as defined above when the medium is given by the Hodge star operator of an indefinite metric.

Proposition 3.5. *Let N be a 4-dimensional manifold, and let $\kappa \in \widetilde{\Omega}_2^2(N)$ be defined as $\kappa = *_g$ for a pseudo-Riemann metric g on N.*

(i) *If $\xi \in \Lambda^1(N)$ is non-zero, and V_ξ is as in equation (38), then*

$$\dim V_\xi = \begin{cases} 2, & \textit{when } \xi \in F(\kappa), \\ 0, & \textit{when } \xi \notin F(\kappa). \end{cases}$$

(ii) *If $\xi \in F(\kappa)$ is non-zero, and L_ξ is as in equation (37) then*

$$ker\, L_\xi = \xi^\perp,$$

where $\xi^\perp = \{\alpha \in \Lambda^1(N) : g(\alpha,\xi) = 0\}$. Thus, for any choice of V_ξ in equation (38) we have $V_\xi \subset \xi^\perp$.

Proof. Let p be the basepoint of ξ and let $\{x^i\}_{i=0}^3$ be local coordinates for N around p such that $g = g_{ij} dx^i \otimes dx^j$ and $g_{ij}|_p$ is diagonal with entries ± 1. We know that $\kappa^2 = *_g^2 = (-1)^\sigma \mathrm{Id}$, where σ is the *index* of g [20, Proposition 6.2.13]. If $\alpha \in \Lambda_p^1(N)$, equations (37) and (19) imply that

$$\begin{aligned} L_\xi(\alpha) &= \frac{1}{2} \xi_r \xi_s \alpha_i g^{ra} g^{ib} \varepsilon_{abcd} dx^s \wedge dx^c \wedge dx^d \\ &= \det g \, (-1)^\sigma \alpha_i H^{ir} g_{rs} * dx^s, \end{aligned}$$

where $\xi = \xi_i dx^i|_p$ and $\alpha = \alpha_i dx^i|_p$ and

$$H^{ir} = g(\xi, \xi) g^{ir} - \xi_a g^{ai} \xi_b g^{br}. \tag{48}$$

For part *(i)*, equations (48) and (38) imply that $\dim V_\xi = \dim \ker H - 1$ where H is the 4×4 matrix with entries H^{ij}. Let $\sigma(H)$ denote the spectrum of H with eigenvalues repeated according to their algebraic multiplicity. With computer algebra we find that

$$\sigma(H) = \left(0, C_1 g(\xi, \xi), C_2 g(\xi, \xi), C_3 \sum_{i=0}^3 \xi_i^2 \right),$$

where $C_i \in \{\pm 1\}$ are constants that depend only the signature of g. Now part *(i)* follows by Proposition 3.4 and since algebraic and geometric multiplicity of an eigenvalue coincide for symmetric matrices [32, p. 260]. For part *(ii)*, equality $\ker L_\xi = \xi^\perp$ follows from the local representation of L_ξ in equation (48). □

The next example shows that in a *biaxial crystal* [33, Section 15.3.3] we can have $\dim V_\xi = 1$ in equation (38).

Example 3.6. On $N = \mathbb{R} \times \mathbb{R}^3$, let $\kappa \in \Omega_2^2(N)$ be defined by

$$\mathscr{A} = -\mathrm{diag}\,(1, 2, 3), \quad \mathscr{B} = \mathrm{Id}\,, \quad \mathscr{C} = \mathscr{D} = 0. \tag{49}$$

Let S be the projection of the Fresnel surface into $\mathbb{R}^3$ when $\xi_0 = 1$. Then S is mirror symmetric about the $\xi_1\xi_2$, $\xi_1\xi_3$ and $\xi_2\xi_3$ coordinate planes. Figure 1 below illustrates S in the quadrant $\xi_1 \geq 0, \xi_2 \geq 0, \xi_3 \geq 0$, and in this quadrant we see that S has one singular point $\xi_{\text{sing}} \in S$.

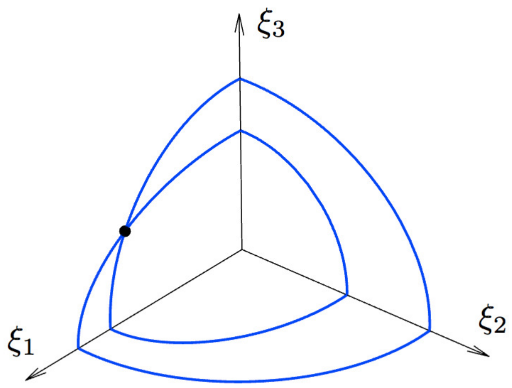

Figure 1. One quadrant in $\mathbb{R}^3$ of a Fresnel surface with a singular point illustrated by a dot.

Surface S is defined implicitly by $f(\xi_1, \xi_2, \xi_3) = 0$ and singular points are characterised by $\nabla f = 0$. (Or, for an alternative way to solve this point, see [31, Lemma 4.2 *(iii)*].) Using computer algebra and the arguments used to prove Theorem 3.2 we find that $\dim V_\xi$ when $\xi_0 = 1$ and S intersects one of the coordinate planes $\{\xi_i = 0\}_{i=1}^3$. In these intersections we obtain $\dim V_\xi = 1$ except at the singular point, where $\dim V_\xi = 2$. □

For a medium $\kappa \in \widetilde{\Omega}_2^2(N)$, the constraint ${}^{(3)}\kappa = 0$ introduced in [34] is known as the *Post constraint*. For many media, this constraint is satisfied. One can also show that the axion component ${}^{(3)}\kappa$ does not contribute to electromagnetic energy nor does it influence propagation in the geometric optics limit [1]. Nevertheless, there are electromagnetic media, that have an axion component. One example is chromium sesquioxide (Cr_2O_3) in a magnetic field [35, 36]. The next proposition shows that an identically non-zero axion field imposes a topological constraint on the underlying manifold. Let us emphasise that this result does not involve Maxwell's equations, but is a mathematical consequence from the definition of twisted antisymmetric $\binom{2}{2}$-tensors [35].

Proposition 3.7. *If N is a 4-manifold, then the following are equivalent:*

(i) *N is orientable.*

(ii) *There exists a $\kappa \in \widetilde{\Omega}_2^2(N)$ with an identically non-zero axion component ${}^{(3)}\kappa$.*

Proof. If part *(i)* holds, then $\widetilde{\Omega}_2^2(N) = \Omega_2^2(N)$ and part *(ii)* follows by taking $\kappa = \mathrm{Id}$. Conversely, if *(ii)* holds, then Proposition 2.1 implies that $\phi = \operatorname{trace}\kappa \in \widetilde{C}^\infty(N)$ is identically non-zero. Let ρ be any twisted $\binom{0}{0}$-tensor density on N of weight 1 that is nowhere zero. Thus, if x^i and $\widetilde{x}^i$ are overlapping coordinates, we have transformation rules

$$\widetilde{\phi} = \operatorname{sgn}\left(\det \frac{\partial x^i}{\partial \widetilde{x}^j}\right)\, \phi, \quad \widetilde{\rho} = \left|\left(\det \frac{\partial x^i}{\partial \widetilde{x}^j}\right)\right| \rho. \tag{50}$$

(To see that such a ρ exists one can for example take $\rho = (\det g)^{1/2}$ for any positive definite Riemann metric g on N.) In each chart (U, x^i) let $\omega = \phi\rho\, dx^0 \wedge \cdots \wedge dx^3$. By equations (50), this assignment defines a global $\omega \in \Omega^4(N)$, and since ω is never zero, N is orientable. □

4. Determining the medium from the Fresnel surface

As described in the introduction, implication *(iii)* ⇒ *(i)* in Theorem 4.3 below is a main result of this paper. Regarding the other implications let us make a few remarks. Implication *(ii)* ⇒ *(i)* is well known. In electromagnetics, the converse implication *(i)* ⇒ *(ii)* seems to first to have been derived by Schönberg [2, 37]. For further derivations and discussions, see [1, 2, 8, 6, 38]. Below we give yet another proof using computer algebra. The proof follows [1] and we use a Schönberg-Urbantke-like formula (equation (59)) to define a metric g from κ. However, the below argument that g transforms as a tensor density in Lemma 4.5 seems to be new. For a different argument, see [1, Section D.5.4].

When a general $\kappa \in \widetilde{\Omega}_2^2(N)$ on a 4-manifold N satisfies $\kappa^2 = -f\mathrm{Id}$ as in condition *(i)* one says that κ satisfies the *closure condition*. For physical motivation, see [1, Section D.3.1]. For a study of more general closure relations, and in particular, for κ with a possible skewon component, see [29, 39]. Let us emphasise that Theorem 4.3

is a global result. The result gives criteria for the existence of a Lorentz metric on a 4-manifold. In general, we know that a connected manifold N has a Lorentz metric if and only if N is non-compact, or if N is compact and the Euler number $\chi(N)$ is zero [40, Theorem 2.4]. Therefore, the closure condition does impose a constraint on the global topology of N. Let us also note that if J is an *almost complex structure* on a manifold M, that is, J is a $\binom{1}{1}$-tensor on M with $J^2 = -\mathrm{Id}$ and $\dim M \geq 2$, then M is orientable [41, p. 77]. The next example shows that the closure condition for twisted $\binom{2}{2}$-tensors does not imply orientability.

Example 4.1. Let $N = M_1 \times M_2$ be the smooth 4-manifold with the Lorentz metric $g = g_1 \times g_2$, where M_1 is a 2-dimensional non-orientable manifold with a positive definite Riemann metric g_1, and $M_2 = \mathbb{R}^2$ with the pseudo-Riemann metric $g_2 = \mathrm{diag}\,(+1, -1)$. Then N is not orientable [42, Remark 16.21.9.3], but $\kappa = *_g \in \widetilde{\Omega}^2_2(N)$ satisfies $\kappa^2 = -\mathrm{Id}$. □

The next example illustrates the possible difference between the full Tamm-Rubilar tensor density $\mathscr{G}^{ijkl}$ and the Fresnel surface $F_p(\kappa)$ which only contains the real roots to the equation $\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = 0$. When equivalence holds in Theorem 4.3, the implication is that both objects contain the same information (up to scaling). In Example 5.4 we will see that in general this need not be the case.

Example 4.2. Suppose κ and $\widetilde{\kappa}$ are invertible and skewon-free medium tensors on $\mathbb{R}^4$ with constant coefficients and with Tamm-Rubilar tensor densities

$$\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = (\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2)^2, \tag{51}$$

$$\widetilde{\mathscr{G}}^{ijkl}\xi_i\xi_j\xi_k\xi_l = (\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2)(\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2), \tag{52}$$

respectively. By equation (43), the Fresnel surfaces of κ and $\widetilde{\kappa}$ coincide. Thus the two media can not be distinguished from their wavespeed behaviours for propagating plane waves. However, if one can also observe evanescent waves, then one can distinguish the two medium tensors. Namely, tensor $\widetilde{\kappa}$ has evanescent waves (that is, solutions that correspond to complex solutions to $\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 0$) that are not present in κ. However, Proposition 3.4 and implication *(iii)* $\Rightarrow$ *(ii)* in Theorem 4.3 show that this is not necessary; there is no invertible and skewon-free medium tensor $\widetilde{\kappa}$ for which the Tamm-Rubilar tensor density factors as in equation (52). □

Theorem 4.3. *Suppose N is a 4-manifold. If $\kappa \in \widetilde{\Omega}^2_2(N)$ satisfies ${}^{(2)}\kappa = 0$, then the following conditions are equivalent:*

(i) $\kappa^2 = -f Id$ *for some function* $f \in C^\infty(N)$ *with* $f > 0$.

(ii) *there exists a Lorentz metric g and a nonvanishing function $f \in C^\infty(N)$ such that*

$$\kappa = f *_g . \tag{53}$$

(iii) ${}^{(3)}\kappa = 0$, $\det \kappa \neq 0$ *and there exists a Lorentz metric g such that*

$$F(\kappa) = F(*_g),$$

where $F(\kappa)$ is the Fresnel surface for κ and $F(*_g)$ is the Fresnel surface for the Hodge star operator $*_g \in \widetilde{\Omega}^2_2(N)$ associated to g.

Moreover, when equivalence holds, then metrics g in conditions (ii) and (iii) are conformally related.

Proof. For implication *(i)* $\Rightarrow$ *(ii)*, suppose *(i)* holds. By scaling, we may assume that $f = 1$. Let $\mathscr{T}$ be an atlas as in Lemma 4.4, whence in each chart $(U, x^i) \in \mathscr{T}$ there is a Lorentz metric h. Moreover, let ρ be a strictly positive odd $\binom{0}{0}$-tensor density on N of weight $1/2$. That is, in each chart U, there is a function $\rho\colon U \to (0,\infty)$ and if ρ and $\widetilde{\rho}$ are functions in overlapping charts (U, x^i) and $(\widetilde{U}, \widetilde{x}^i)$ we have

$$\widetilde{\rho} = \left|\det\left(\frac{\partial x^i}{\partial \widetilde{x}^j}\right)\right|^{1/2} \rho \text{ on } U \cap \widetilde{U}. \tag{54}$$

(To see that such a density exists one can for example take $\rho = (\det k)^{1/4}$ for a positive definite Riemann metric k on N.) In each chart $(U, x^i) \in \mathscr{T}$, let g be Lorentz metric $g = \rho h$. Then equations (54) and (60) imply that g is a global Lorentz metric on N, and Lemma 4.5 *(i)* implies that $-\operatorname{sgn}\det \mathscr{A}\, *_g \in \widetilde{\Omega}^2_2(N)$. Lastly, since the Hodge operator is conformally invariant, implication *(i)* $\Rightarrow$ *(ii)* follows. Implication *(ii)* $\Rightarrow$ *(iii)* follows by Proposition 3.4 and equation (43). For implication *(iii)* $\Rightarrow$ *(i)*, Lemma 4.6 shows that there exists a (possibly non-continuous) function $\phi\colon N \to (0,\infty)$ such that $\kappa^2 = -\phi \mathrm{Id}$ To see that ϕ is a smooth function on N it suffices to note that $\phi = -\frac{1}{6}\operatorname{trace}\kappa^2$. This completes the proof of implication *(iii)* $\Rightarrow$ *(i)*.

We know that two Lorentz metrics are conformally related if their null cones coincide [4, Theorem 3]. Thus, Proposition 3.4 implies that the Lorentz metrics in conditions *(ii)* and *(iii)* are conformally related when equivalence holds. □

The next two lemmas was used to prove implication *(i)* $\Rightarrow$ *(ii)* in Theorem 4.3. The lemmas collect results from [1, Sections D.3–D.5].

Lemma 4.4. *Suppose N is a 4-manifold and $\kappa \in \widetilde{\Omega}^2_2(N)$. If κ has no skewon component and $\kappa^2 = -Id$, then N has an atlas $\mathscr{T}$ with the following property: Each $p \in N$ can be covered with a connected chart $(U, x^i) \in \mathscr{T}$ such that if $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D}$ represent κ in U, then*

(i) *$\mathscr{A}$ is invertible in U.*

(ii) *In U there exists a smoothly varying antisymmetric 3×3 matrix $\mathscr{K}$ such that*

$$\mathscr{B} = -\mathscr{A}^{-1}\left(Id + \left(\mathscr{K}\mathscr{A}^{-1}\right)^2\right), \quad \mathscr{C} = \mathscr{K}\mathscr{A}^{-1}, \quad \mathscr{D} = -\mathscr{A}^{-1}\mathscr{K}.$$

(iii) *In U there is a Lorentz metric $h = h_{ij}dx^i \otimes dx^j$ with signature $(+---)$ such that*

$$\det h = -1, \quad \kappa|_U = -sgn\, det \mathscr{A}\ *_h\,. \tag{55}$$

Proof. Let us first make an observations: Suppose $\{x^i\}_{i=0}^3$ are arbitrary coordinates for N and $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ are 3×3 matrices that represent κ in these coordinates. Then Proposition 2.3 *(i)* implies that $\kappa^2 = -\mathrm{Id}$ is equivalent with

$$\mathscr{C}^2 + \mathscr{A}\mathscr{B} \quad = -\mathrm{Id}\,, \tag{56}$$

$$\mathscr{B}\mathscr{C} + \mathscr{C}^T\mathscr{B} = 0, \tag{57}$$

$$\mathscr{C}\mathscr{A} + \mathscr{A}\mathscr{C}^T = 0. \tag{58}$$

Let $\mathscr{T}_0$ is a maximal atlas for N. The proof is divided into two subclaims, Claim 1 and Claim 2.

Claim 1. For each $p \in N$ there exists a connected chart $(U, x^i) \in \mathscr{T}_0$ that satisfies condition *(i)*.

By Proposition 2.3 *(ii)* we can find a connected chart (U, x^i) that contains p and where matrix $\mathscr{A}$ for κ is diagonal at p. The rest of Claim 1 is divided into four cases depending on the eigenvalues of $\mathscr{A}|_p$.

Case A. Suppose $\mathscr{A}|_p$ has three non-zero eigenvalues. Since eigenvalues depend continuously on the matrix entries [43], we can shrink U and Claim 1 follows.

Case B. Suppose $\mathscr{A}|_p$ has two non-zero eigenvalues. By permutating the coordinates (see equation (28)) we may assume that $\mathscr{A}|_p = \operatorname{diag}(a_1, a_2, 0)$ for some $a_1, a_2 \neq 0$. Writing out equations (56)–(58) with computer algebra gives

$$\mathscr{C}^1{}_1 = \mathscr{C}^2{}_2 = \mathscr{C}^3{}_1 = \mathscr{C}^3{}_2 = 0, \quad (\mathscr{C}^3{}_3)^2 = -1$$

at p. The last equation contradicts that $\mathscr{C}$ is real. Case B is therefore not possible.

Case C. Suppose $\mathscr{A}|_p$ has one non-zero eigenvalue. As in Case B, we can find a chart (U, x^i) for which $\mathscr{A}|_p = \operatorname{diag}(a_1, 0, 0)$ for some $a_1 \neq 0$. Writing out equations (56)–(58) as in Case B gives

$$\mathscr{C}^1{}_1 = \mathscr{C}^2{}_1 = \mathscr{C}^3{}_1 = 0, \quad \mathscr{B}_{11} \neq 0, \quad \mathscr{C}^2{}_3 \neq 0, \quad \mathscr{C}^3{}_2 \neq 0$$

at p. Let $\{\widetilde{x}^i\}_{i=0}^3$ be coordinates around p defined as

$$\begin{aligned}\widetilde{x}^0 &= x^0 + x^3, \\ \widetilde{x}^i &= x^i, \quad i \in \{1, 2, 3\}.\end{aligned}$$

In these coordinates, we have $\widetilde{\mathscr{A}}|_p = -\mathscr{B}_{11}(\mathscr{C}^3{}_2)^2 \neq 0$ and Claim 1 follows.

Case D. Suppose all eigenvalues of $\mathscr{A}|_p$ are zero. Then $\mathscr{A}|_p = 0$ and equation (56) implies that $(\det \mathscr{C}|_p)^2 = -1$. This contradicts that $\mathscr{C}|_p$ is a real matrix. Case D is therefore not possible.

Claim 2. Let $\mathscr{T}$ be the collection of all charts (U, x^i) as in Step 1 when p ranges over all points in N. Then $\mathscr{T}$ is an atlas for N that satisfies properties *(i)*, *(ii)* and *(iii)*.

Each chart in $\mathscr{T}$ satisfies property *(i)*, and property *(ii)* follows by defining $\mathscr{K} = \mathscr{C}\mathscr{A}$. Indeed, $\mathscr{K}$ is antisymmetric by equation (58), and the expression for $\mathscr{B}$ follows by equation (56). For property *(iii)*, let $G = (G^{ab})_{ab}$ is the 4×4 matrix in U with entries

$$G = \frac{1}{\sqrt{|\det \mathscr{A}|}} \left(\begin{array}{c|c} \det \mathscr{A} & k^i \\ \hline k^j & -\mathscr{A}^{ij} + (\det \mathscr{A})^{-1} k^i k^j \end{array} \right), \tag{59}$$

where $k^i = \mathscr{A}^{ib} \frac{1}{2} \varepsilon_{bcd} \mathscr{K}^{cd}$ for $i \in \{1, 2, 3\}$. Using a Shur complement [44, Theorem 3.1.1] we find that $\det G = -1$. Hence $\det G < 0$, so matrix G is invertible and has constant signature $(-+++)$ or $(+---)$ at each point in U. Let G_{ij} be the ijth entry of the inverse of G. In U we define

$$h = \sigma_U G_{ij} dx^i \otimes dx^j,$$

where constant $\sigma_U \in \{-1, 1\}$ is chosen such that h has signature $(+---)$. Then h is a smooth symmetric $\binom{0}{2}$-tensor in U with signature $(+---)$ and $\det h = -1$. Let $G_0^{ij} = \sqrt{|\det \mathscr{A}|}\, G^{ij}$. Then $h^{ij} = \frac{\sigma_U}{\sqrt{|\det \mathscr{A}|}} G_0^{ij}$ and expanding $*_h$ using equation (19) gives

$$\begin{aligned}-\operatorname{sgn} \det \mathscr{A}\ (*_h)^{ij}_{rs} &= \frac{-1}{\det \mathscr{A}}\ G_0^{ia} G_0^{jb} \varepsilon_{abrs} \\ &= \kappa^{ij}_{rs},\end{aligned}$$

where the last equality follows by computer algebra. Thus $\kappa|_U = -\text{sgn}\det\mathscr{A} *_h$ and part *(iii)* follows. $\square$

The next lemma describe the transformation rules for the objects in Lemma 4.4. Part *(i)* shows that the local functions sgn $\det\widetilde{\mathscr{A}}$ in Lemma 4.4 determine a smooth function on N. Moreover, part *(ii)* shows that the local Lorentz metrics h in Lemma 4.4 determine an odd tensor density of weight $-1/2$. Here we follow the even/odd convention in [45, p. 134]. Namely, a tensor density is odd when the scaling is a power of $|\det\frac{\partial x^a}{\partial\widetilde{x}^b}|$ as in equation (60), and even when the scaling is a power of $\det\frac{\partial x^a}{\partial\widetilde{x}^b}$. For a direct proof of equation (60) based on equation (59), see [1, Section D.5.4].

Lemma 4.5. *Suppose N is a 4-manifold and $\kappa\in\widetilde{\Omega}^2_2(N)$ is as in Lemma 4.4.*

(i) *If $\mathscr{A}$ and $\widetilde{\mathscr{A}}$ are smoothly varying matrices determined by Lemma 4.4 in overlapping charts (U, x^i) and $(\widetilde{U}, \widetilde{x}^i)$, then sgn $\det\mathscr{A}$ = sgn $\det\widetilde{\mathscr{A}}$.*

(ii) *If $h = h_{ij}dx^i\otimes dx^j$ and $\widetilde{h} = \widetilde{h}_{ij}d\widetilde{x}^i\otimes d\widetilde{x}^j$ are Lorentz metrics determined by Lemma 4.4 in overlapping charts (U, x^i) and $(\widetilde{U}, \widetilde{x}^i)$, then*

$$\widetilde{h}_{ij} = \left|\det\frac{\partial x^a}{\partial\widetilde{x}^b}\right|^{-1/2} h_{rs}\frac{\partial x^r}{\partial\widetilde{x}^i}\frac{\partial x^s}{\partial\widetilde{x}^j}. \tag{60}$$

Proof. Let $\mathscr{A}, \widetilde{\mathscr{A}}, h, \widetilde{h}$ be as in parts *(i)* and *(ii)*. Then Proposition 3.4 implies that

$$\begin{aligned}\mathscr{G}^{ijkl}_\kappa\xi_i\xi_j\xi_k\xi_l &= \text{sgn}\det\mathscr{A}\ (h^{ab}\xi_a\xi_b)^2,\\ \widetilde{\mathscr{G}}^{ijkl}_\kappa\widetilde{\xi}_i\widetilde{\xi}_j\widetilde{\xi}_k\widetilde{\xi}_l &= \text{sgn}\det\widetilde{\mathscr{A}}\ (\widetilde{h}^{ab}\widetilde{\xi}_a\widetilde{\xi}_b)^2\end{aligned}$$

for all $\xi = \xi_i dx^i = \widetilde{\xi}_i d\widetilde{x}^i \in \Lambda^1(U\cap\widetilde{U})$. Contracting equation (42) by $\widetilde{\xi}_i\widetilde{\xi}_j\widetilde{\xi}_k\widetilde{\xi}_l$ yields

$$\text{sgn}\det\mathscr{A}\ (h^{ij}\xi_i\xi_j)^2 = \text{sgn}\det\widetilde{\mathscr{A}}\left(\left|\det\frac{\partial\widetilde{x}^a}{\partial x^b}\right|^{1/2}\widetilde{h}^{rs}\frac{\partial x^i}{\partial\widetilde{x}^r}\frac{\partial x^j}{\partial\widetilde{x}^s}\xi_i\xi_j\right)^2 \tag{61}$$

for all $\xi = \xi_i dx^i\in\Lambda^1(U\cap\widetilde{U})$. In each $\Lambda^1_p(U\cap\widetilde{U})$ we can find a ξ such that $h^{ij}\xi_i\xi_j\neq 0$. Thus sgn $\det\mathscr{A}$ = sgn $\det\widetilde{\mathscr{A}}$ in $U\cap\widetilde{U}$ and part *(i)* follows. Since metrics with the same null cone are conformally related [4, Theorem 3], there exists a smooth nonvanishing function $\lambda\colon U\cap\widetilde{U}\to\mathbb{R}$ such that

$$h^{ij} = \lambda\left|\det\frac{\partial\widetilde{x}^a}{\partial x^b}\right|^{1/2}\widetilde{h}^{rs}\frac{\partial x^i}{\partial\widetilde{x}^r}\frac{\partial x^j}{\partial\widetilde{x}^s}.$$

Equation (61) implies that function λ can only take values $\{-1, +1\}$. Since $(h^{ij})_{ij}$ and $(\widetilde{h}^{ij})_{ij}$ have signatures $(+---)$, we have $\lambda = 1$ on $U\cap\widetilde{U}$ and part *(ii)* follows. $\square$

Next we prove Lemma 4.6 which is the key results leading to implication *(iii)* $\Rightarrow$ *(i)* in Theorem 4.3. Let us emphasise that the proof of this proposition closely follows the proof of the main result in [15], which characterises skewon-free media with two Lorentz null cones. Below we will therefore only indicate how the argument in [15] changes to the present setting. In the proof we will need the normal form theorem of Schuller, Witte and Wohlfarth [13] which pointwise divides skewon-free media into 23 metaclasses and gives simple coordinate expressions for each metaclass. For example, if $\kappa|_p$ is in Metaclass I, then there are coordinates around p such that the 6×6 matrix (see proof of Proposition 2.1) that represents $\kappa|_p$ can be written as

$$\begin{pmatrix} \alpha_1 & 0 & 0 & -\beta_1 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & -\beta_2 & 0 \\ 0 & 0 & \alpha_3 & 0 & 0 & -\beta_3 \\ \beta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \beta_3 & 0 & 0 & \alpha_3 \end{pmatrix} \tag{62}$$

for constants $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ and $\beta_1, \beta_2, \beta_3 \in \mathbb{R}\backslash\{0\}$ and all β_i are of the same sign. This normal form result is essentially based on the Jordan normal form. Since $\kappa|_p$ can be represented by 6×6 matrix, it can be transformed into a Jordan normal form by a 6×6 matrix. A main result of [13] is that such a 6×6 transformation matrix (which *a priori* has 36 degrees of freedom) can for skewon-free media essentially be realised using a coordinate transformation on N (which has only 16 degrees of freedom). See equation (14) and for a further discussion see [13, 46]. Let us also note that Lemma 4.6 and [15, Theorem 2.1] are pointwise results, but Theorem 4.3 is a global result on a possibly non-orientable manifold.

Lemma 4.6. *Suppose N is a 4-manifold, $\kappa \in \widetilde{\Omega}^2_2(N)$, ${}^{(2)}\kappa = 0$ and condition* (iii) *holds in Theorem 4.3. If $p \in N$, then there exists a $\lambda > 0$ such that*

$$\kappa^2 = -\lambda Id \quad at\, p. \tag{63}$$

Proof. Let x^i be coordinates around p, and let $\mathscr{G}^{ijkl}$ be components of the Tamm-Rubilar tensor density at p. Let $\gamma\colon \mathbb{R}^4 \to \mathbb{R}$ be the polynomial $\gamma(\xi) = \mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l$. By Claim $(*)$ in the proof of Proposition 1.3 in [15], condition *(iii)* in Theorem 4.3 implies that γ has $g^{ij}\xi_i\xi_j$ as a factor when g^{ij} are components of g in condition *(iii)*. Hence there exists a symmetric $\binom{2}{0}$-tensor $A = A^{ij}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial x^j}$ at p with real coefficients such that

$$\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = (g^{ij}\xi_i\xi_j)(A^{kl}\xi_k\xi_l), \quad \{\xi_i\}_{i=0}^3 \in \mathbb{R}^4. \tag{64}$$

Since $F_p(\kappa) \neq T^*_pN$ we can conclude that $A \neq 0$. Let us also note that if $(B^{ij})_{i,j\in 0}^3 \in \mathbb{C}^{4\times 4}$ then $B^{ij}\xi_i\xi_j$ is irreducible over $\mathbb{C}$ if and only if $\operatorname{adj} B$ is nonzero where $\operatorname{adj} B$ is the *adjugate matrix* of $(B^{ij})_{i,j=0}^3$. Also, if B is real and $B^{ij}\xi_i\xi_j$ is irreducible over $\mathbb{C}$ then $B^{ij}\xi_i\xi_j$ is irreducible over $\mathbb{R}$. See for example [15, Proposition A.1]. By Proposition 3.3, equation (64) holds for all $\xi \in \mathbb{C}^4$.

Since $F_p(\kappa)$ is one Lorentz null cone, [47, Proposition 2] implies that $F_p(\kappa)$ does not contain a 2-dimensional subspace. Hence $\det \kappa \neq 0$ and [13, Lemma 5.1] implies that $\kappa|_p$ is in Metaclasses I, …, VII in the classification of Schuller *et al.* [13]. That the conjugation operator in [13] is not necessary in this case follows by [46, Theorem 6]. For a repeat of this argument in the present setting, see [15, Theorem 1.5].

If $\kappa|_p$ is in Metaclass I, then the factorisation in equation (64) holds also in the coordinates where the normal form for $\kappa|_p$ is valid. In the notation of [15, Theorem 2.1], polynomial γ can be written using constants $D_0, D_1, D_2, D_3 \in \mathbb{R}$. If $D_1 = D_2 = D_3 = 2$ then the expressions for D_1, D_2, D_3 in [15] show that $\kappa|_p$ is proportional to the Hodge star operator of a Lorentz metric, and equation (63) follows. On the other hand, if $D_1 = D_2 = D_3 = 2$ does not hold, then the argument in [15, Theorem 2.1] implies that

$$\mathscr{G}^{ijkl}\xi_i\xi_j\xi_k\xi_l = (g_+^{ij}\xi_i\xi_j)(g_-^{kl}\xi_k\xi_l), \quad \{\xi_i\}_{i=0}^3 \in \mathbb{R}^4 \tag{65}$$

for non-proportional Lorentz metrics $g_\pm$. Thus

$$N_p(g) = N_p(g_+) \cup N_p(g_-). \tag{66}$$

We know that if $N_p(u) \subset N_p(v)$ for Lorentz metrics u, v then $u = \lambda v$ for some $\lambda \neq 0$. See for example [48, Section II.4]. Then equation (66) implies the contradiction that $g_\pm$ are proportional. We have shown that if $\kappa|_p$ is in Metaclass I then necessarily $D_1 = D_2 = D_3 = 2$.

If $\kappa|_p$ is in Metaclass II or IV, then the argument in [15, Theorem 2.1] implies that equation (65) holds for non-proportional Lorentz metrics $g_\pm$. Then equation (66) holds and the argument in Metaclass I shows that Metaclasses II and IV are not possible.

If $\kappa|_p$ is in Metaclass III or V, then the argument in [15, Theorem 2.1] shows that γ can not factor as in equation (64). Hence Metaclasses III and V are not possible.

If $\kappa|_p$ is in Metaclass VI then the argument in [15, Theorem 2.1] shows that γ can be written as in equation (65) for some quadratic forms $g_\pm = g_\pm^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ with real coefficients and $\det g_\pm > 0$. Hence $g_\pm^{ij} \xi_i \xi_j$ are irreducible over $\mathbb{R}$, and equations (64) and (65) imply that $g_\mu = Cg$ for some $C \in \mathbb{R}\backslash\{0\}$ and $\mu \in \{\pm\}$. However, then $0 < \det g_\mu = C^4 \det g < 0$, so $\kappa|_p$ can not be in Metaclass VI.

If $\kappa|_p$ is in Metaclass VII, then the argument in [15, Theorem 2.1] shows that there are quadratic forms $g_\pm = g_\pm^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$ with possibly complex coefficients such that equation (65) holds for all $\xi \in \mathbb{C}^4$. We will not need the explicit expressions for all possible $g_\pm$. However, we will need the following two properties: *(i)* if g_μ for $\mu \in \{\pm\}$ is real then $\det g_\mu \geq 0$, and *(ii)* g_+ and g_- both have at least one nonzero real entry. Since $g^{ij}\xi_i\xi_j$ is irreducible over $\mathbb{C}$, the right hand side of equation (65) should have at least one irreducible factor of degree 2 by unique factorisation [16, Theorem 5 in Section 3.5]. Hence $g_\mu = Cg$ for some $C \in \mathbb{C}\backslash\{0\}$ and $\mu \in \{\pm\}$. By *(ii)*, it follows that $C \in \mathbb{R}\backslash\{0\}$. Hence g_μ is real and repeating the last argument for Metaclass VI shows that Metaclass VII is not possible. $\square$

5. Non-injectivity results

Implication *(iii)* $\Rightarrow$ *(ii)* in Theorem 4.3 shows that for a special class of medium, the Fresnel surface determines the medium up to a conformal factor. In the below we will see that there are various non-uniquenesses that prevents this for a general κ. We will separately study the non-injectivity of the two maps in the below diagram:

$$\kappa \quad \to \quad \mathscr{G}(\kappa) \quad \to \quad F(\kappa), \qquad \kappa \in \widetilde{\Omega}^2_2(N), \tag{67}$$

where $\mathscr{G}(\kappa)$ is the Tamm-Rubilar tensor density for κ and $F(\kappa)$ is the Fresnel surface for κ.

5.1. Non-injectivity of leftmost map

Let us first study the non-injectivity of the leftmost map in diagram (67), that is, the map

$$\kappa \quad \to \quad \mathscr{G}(\kappa), \qquad \kappa \in \widetilde{\Omega}^2_2(N). \tag{68}$$

Parts *(ii)–(iv)* in the next theorem describes three invariances that make map (68) non-injective. Parts *(i)–(iii)* are well known [1, Section 2.2]. However, let us make four remarks regarding part *(iv)*, which describes the Tamm-Rubilar tensor densities for κ

and κ^{-1}. First, let us note that if $\kappa \in \widetilde{\Omega}^2_2(N)$ is invertible, then Proposition 2.1 implies that $\kappa^{-1} \in \widetilde{\Omega}^2_2(N)$. Thus, κ and κ^{-1} both define linear maps $\Omega^2(N) \to \widetilde{\Omega}^2(N)$ and both κ and κ^{-1} can be treated as electromagnetic media. Second, if N is orientable, part *(iv)* has the following interpretation: If $F, G \in \Omega^2(N)$ solve the sourceless Maxwell equations in medium κ, then $G, F \in \Omega^2(N)$ solve the sourceless Maxwell equations in medium κ^{-1}. In this setting, part *(iv)* states that both media have the same Fresnel surfaces. Third, suppose $*_g$ is the twisted $\binom{2}{2}$-tensor induced by a pseudo-Riemann metric g. Then $*_g^2 = \pm \mathrm{Id}$, so $*_g^{-1} = \pm *_g$, whence $\mathscr{G}(*_g)$ and $\mathscr{G}(*_g^{-1})$ are conformally related. Part *(iv)* states that this is not only a result for the Hodge star operator, but a general result for *all* $\kappa \in \widetilde{\Omega}^2_2(N)$. Fourth, the proof of part *(iv)* is based on computer algebra. Of all the proofs in this paper, this computation is algebraically most involved. For example, if we write out equation (69) as a text string, it requires almost 13 megabytes of memory.

Theorem 5.1. *Suppose $\kappa \in \widetilde{\Omega}^2_2(N)$ where N is a 4-manifold. Then*

(i) $\mathscr{G}(f\kappa) = f^3 \mathscr{G}(\kappa)$ *for all* $f \in C^\infty(N)$,

(ii) $\mathscr{G}({}^{(2)}\kappa) = 0$, *where* ${}^{(2)}\kappa$ *is as in Section 2.3,*

(iii) $\mathscr{G}(\kappa) = \mathscr{G}(\kappa + f Id)$ *for all* $f \in \widetilde{C}^\infty(N)$,

(iv) $\mathscr{G}(\kappa^{-1}) = \mathscr{G}(-(\det \kappa)^{-1/3}\kappa)$ *when κ is invertible.*

Proof. Part *(i)* follows by the definition, and parts *(ii)–(iii)* are proven in [1, Section 2.2]. Therefore we only need to prove part *(iv)*. Let $\operatorname{adj}\kappa = \det\kappa\,\kappa^{-1}$ be the *adjugate* of κ. By part *(i)* and Proposition 2.1, it suffices to show that

$$(\det \kappa)^2 \mathscr{G}^{ijkl}_{\kappa} + \mathscr{G}^{ijkl}_{\operatorname{adj}\kappa} = 0, \quad 0 \le i \le j \le k \le l \le 3, \tag{69}$$

where $\mathscr{G}^{ijkl}_{\kappa}$ and $\mathscr{G}^{ijkl}_{\operatorname{adj}\kappa}$ are components of the Tamm-Rubilar tensor densities of κ and $\operatorname{adj}\kappa$, respectively. The motivation for rewriting the claim as in equation (69) is that now both terms are polynomials. Mathematica was not able to verify these polynomials in reasonable time. However, to prove that a multivariable polynomial is the zero polynomial it suffices to make a Taylor expansion with respect to one variable and prove that all Taylor coefficients are zero. By recursively eliminating variables in this way, we can decompose the verification process into smaller polynomials that Mathematica can simplify. In this way, we see that equations (69) hold and part *(iv)* follows. $\square$

Theorem 5.1 *(ii)* shows that if we restrict the map in equation (68) to purely skewon tensors, we do not obtain an injection. The next example shows that the same map is neither an injection when restricted to tensors of purely principal type.

Example 5.2. On $N = \mathbb{R} \times \mathbb{R}^3$ with coordinates $\{x^i\}_{i=0}^3$, let κ be the $\binom{2}{2}$-tensor defined by 3×3-matrices

$$\mathscr{A} = 0_{3\times 3}, \quad \mathscr{B} = \begin{pmatrix} 0 & 0 & \lambda_1 \\ 0 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}, \quad \mathscr{C} = \begin{pmatrix} -2^{-1/3} & 0 & \lambda_4 \\ 0 & -2^{-1/3} & \lambda_5 \\ 0 & 0 & 2^{2/3} \end{pmatrix}, \quad \mathscr{D} = \mathscr{C}^T,$$

where parameters $\lambda_1, \ldots, \lambda_5 \in \mathbb{R}$ are arbitrary. Then κ has only a principal part, $\det \kappa = 1$, and $\mathscr{G}_\kappa = 0$. $\square$

When proving implication *(iii)* $\Rightarrow$ *(i)* in Theorem 4.3 we need to assume that κ has real coefficients. In fact, for $\binom{2}{2}$-tensors with complex coefficients a decomposition into principal-, skewon-, and axion components does not seem to have been developed. The next example shows that there are non-trivial complex tensors whose Fresnel surface everywhere coincides with the Fresnel surface for the standard Minkowski metric $g_0 = \operatorname{diag}(1,-1,-1,-1)$. (For $\kappa \in \widetilde{\Omega}^2_2(N,\mathbb{C})$ we define the Fresnel surface using the same formulas as for real κ.)

Example 5.3. On $N = \mathbb{R}\times\mathbb{R}^3$ with coordinates $\{x^i\}_{i=0}^3$, let κ be the $\binom{2}{2}$-tensor with complex coefficients defined by 3×3-matrices

$$\mathscr{A} = -\begin{pmatrix} \frac{1}{2z^2} & 0 & 0 \\ 0 & 2z & 0 \\ 0 & 0 & z \end{pmatrix}, \qquad \mathscr{B} = -\mathscr{A},$$

$$\mathscr{C} = i\begin{pmatrix} \frac{1}{3z^2} - z & 0 & 0 \\ 0 & -\frac{1}{6z^2} + z & 0 \\ 0 & 0 & -\frac{1}{6z^2} \end{pmatrix}, \qquad \mathscr{D} = \mathscr{C},$$

where z is an arbitrary function $z\colon N \to \mathbb{C}\backslash\{0\}$ and i is the complex unit. At each $p \in N$ the Fresnel surface is then determined by

$$\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0,$$

where $\xi_i dx^i \in \Lambda^1_p(N)$, and

$$\operatorname{trace}\kappa = 0, \quad \det\kappa = \frac{\left(1+6z^3\right)^3\,\left(5 - 126z^3 + 684z^6 - 648z^9\right)}{46656\; z^{12}}.$$

From the latter equation we see that for specific values of z, tensor κ can be non-invertible as a linear map $\square$

5.2. Non-injectivity of rightmost map

The next example shows that there are $\kappa_1, \kappa_2 \in \Omega^2_2(\mathbb{R}\times\mathbb{R}^3)$ with no skewon components that have the same Fresnel surfaces, but their Tamm-Rubilar tensors are not proportional to each other. This shows that the rightmost map in equation (67) is not injective. Let us point out that this contradicts the first proposition in [3] whose proof does not analyse multiplicities of roots to the Fresnel equation.

Example 5.4. On $N = \mathbb{R}\times\mathbb{R}^3$ with coordinates $\{x^i\}_{i=0}^3$, let κ_1 be the $\binom{2}{2}$-tensor defined by 3×3-matrices

$$\mathscr{A}_1 = \begin{pmatrix} 0 & -1 & 1 \\ -1 & -2 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \; \mathscr{B}_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \mathscr{C}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}, \; \mathscr{D}_1 = \mathscr{C}_1^T.$$

Then

$$\mathscr{G}_{\kappa_1}(\xi,\xi,\xi,\xi) = (\xi_0 - \xi_1)(\xi_0 - \xi_2)^3, \quad \xi \in \Lambda^1(N).$$

To exchange the role of ξ_1 and ξ_2, we perform a coordinate change $x_0 \mapsto x_0$, $x_1 \mapsto x_2$, $x_2 \mapsto x_1$, $x_3 \mapsto x_3$. With this as motivation we define κ_2 as the $\binom{2}{2}$-tensor defined by 3×3-matrices

$$\mathscr{A}_2 = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \; \mathscr{B}_2 = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \mathscr{C}_2 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}, \; \mathscr{D}_2 = \mathscr{C}_2^T.$$

Then

$$\mathscr{G}_{\kappa_2}(\xi,\xi,\xi,\xi) = -(\xi_0-\xi_1)^3(\xi_0-\xi_2), \quad \xi\in\Lambda^1(N).$$

Here κ_1 and κ_2 are not proportional, their Tamm-Rubilar tensor densities are not proportional, but their Fresnel surfaces coincide.

Both κ_1 and κ_2 have 1 has an eigenvalue of algebraic multiplicity 6. Hence

$$\det\kappa_1 = \det\kappa_2 = 1, \quad \operatorname{trace}\kappa_1 = \operatorname{trace}\kappa_2 = 6,$$

and for the trace-free components $\widetilde{\kappa}_i = \kappa_i - \mathrm{Id}$ we have $\det\widetilde{\kappa}_i = 0$. $\square$

Acknowledgements. I would like to thank Luzi Bergamin, Alberto Favaro, Yuri Obukhov and Ricardo Gallego Torromé for helpful discussions. I would also like to thank an anonymous reviewer for valuable comments which have improved the paper.

The author gratefully appreciates financial support by the Academy of Finland (project 13132527 and Centre of Excellence in Inverse Problems Research), and by the Institute of Mathematics at Aalto University.

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