In[1]:=
```
SetDirectory["~/KappaLib"];
<< kappaLib-1.2.m
<< helper.m
```

```
Loading KappaLib v1.2

Loading helper.m..
```

- **Define Metaclass I with parameters:**

alpha_i in R, beta_i in R\0, and beta_i all have same sign.

In[4]:=
```
kappa = emMatrixToKappa[
  a1  0   0   -b1  0    0
  0   a2  0   0    -b2  0
  0   0   a3  0    0    -b3
  b1  0   0   a1   0    0
  0   b2  0   0    a2   0
  0   0   b3  0    0    a3
];
```

Parameters can be permutated by a coordinate change

In[5]:=
```
L12 =
  1  0  0   0
  0  0  -1  0
  0  1  0   0
  0  0  0   1
;

L23 =
  1  0  0  0
  0  1  0  0
  0  0  0  -1
  0  0  1  0
;
```

In[7]:=
```
(* Exchange 1 <-> 2 *)
Sign[Det[L12]] emKappaToMatrix[emCoordinateChange[kappa, L12]] // MatrixForm

(* Exchange 2 <-> 3 *)
Sign[Det[L23]] emKappaToMatrix[emCoordinateChange[kappa, L23]] // MatrixForm
```

Out[7]//MatrixForm=

$$\begin{pmatrix} a2 & 0 & 0 & -b2 & 0 & 0 \\ 0 & a1 & 0 & 0 & -b1 & 0 \\ 0 & 0 & a3 & 0 & 0 & -b3 \\ b2 & 0 & 0 & a2 & 0 & 0 \\ 0 & b1 & 0 & 0 & a1 & 0 \\ 0 & 0 & b3 & 0 & 0 & a3 \end{pmatrix}$$

Out[8]//MatrixForm=

$$\begin{pmatrix} a1 & 0 & 0 & -b1 & 0 & 0 \\ 0 & a3 & 0 & 0 & -b3 & 0 \\ 0 & 0 & a2 & 0 & 0 & -b2 \\ b1 & 0 & 0 & a1 & 0 & 0 \\ 0 & b3 & 0 & 0 & a3 & 0 \\ 0 & 0 & b2 & 0 & 0 & a2 \end{pmatrix}$$

- **Note: in KappaLib, emCoordinateChange computes coordinate change for an (untwisted) tensor. Since kappa is a twisted tensor, we manually need to add the extra Sign(Det(..)) in the transformation rule.**

Write out algebraic equations that kappa satisfies and eliminate variables for A and B

In[9]:=
```
eta = kappa + mu emIdentityKappa[];
LHS = emCompose[eta, eta];
AA = emMatrix["A", 4, Structure → "AntiSymmetric"];
BB = emMatrix["B", 4, Structure → "AntiSymmetric"];
RHS = - lambda emIdentityKappa[] + emBiProduct[rho, AA, BB] + emBiProduct[rho, BB, AA];
```

- **Since rho, A,B are all non-zero, we may scale A and assume that rho = 1**

In[14]:=
```
rho = 1;
```

- **Definition of decomposable medium (with**

In[15]:=
```
eqs = simp[Union[Flatten[LHS - RHS]]];
show[eqs]
```

Out[16]//MatrixForm=

1	:	0
2	:	$2\,(A13\,B12 + A12\,B13)$
3	:	$2\,(A14\,B13 + A13\,B14)$
4	:	$2\,(A23\,B13 + A13\,B23)$
5	:	$2\,(A24\,B12 + A12\,B24)$
6	:	$2\,(A24\,B14 + A14\,B24)$
7	:	$2\,(A24\,B23 + A23\,B24)$
8	:	$2\,(A34\,B13 + A13\,B34)$
9	:	$2\,(A34\,B24 + A24\,B34)$
10	:	$-2\,(A13\,B12 + A12\,B13)$
11	:	$-2\,(A14\,B12 + A12\,B14)$
12	:	$-2\,(A14\,B13 + A13\,B14)$
13	:	$-2\,(A23\,B12 + A12\,B23)$
14	:	$-2\,(A23\,B13 + A13\,B23)$
15	:	$-2\,(A24\,B12 + A12\,B24)$
16	:	$-2\,(A24\,B14 + A14\,B24)$
17	:	$-2\,(A24\,B23 + A23\,B24)$
18	:	$-2\,(A34\,B13 + A13\,B34)$
19	:	$-2\,(A34\,B14 + A14\,B34)$
20	:	$-2\,(A34\,B23 + A23\,B34)$
21	:	$-2\,(A34\,B24 + A24\,B34)$
22	:	$4\,A13\,B13 - 2\,b2\,(a2 + mu)$
23	:	$4\,A24\,B24 + 2\,b2\,(a2 + mu)$
24	:	$-4\,A34\,B34 - 2\,b1\,(a1 + mu)$
25	:	$-4\,A12\,B12 + 2\,b1\,(a1 + mu)$
26	:	$-4\,A23\,B23 - 2\,b3\,(a3 + mu)$
27	:	$-4\,A14\,B14 + 2\,b3\,(a3 + mu)$
28	:	$2\,A24\,B13 - b2^2 + 2\,A13\,B24 + lambda + (a2 + mu)^2$
29	:	$-b1^2 - 2\,A34\,B12 - 2\,A12\,B34 + lambda + (a1 + mu)^2$
30	:	$-2\,A23\,B14 - 2\,A14\,B23 - b3^2 + lambda + (a3 + mu)^2$

In[17]:=
```
elimVars = Variables[AA.BB]
```

Out[17]=
```
{A12, A13, A14, A23, A24, A34, B12, B13, B14, B23, B24, B34}
```

In[18]:=
```
condVars = Join[Variables[kappa], {lambda, mu}]
```

Out[18]=
```
{a1, a2, a3, b1, b2, b3, lambda, mu}
```

■ Eliminate variables using a Gröbner basis

In[19]:=
```
gb = GroebnerBasis[eqs, condVars, elimVars]; // Timing
gb = simp[gb]; // Timing
Length[gb]
```

Out[19]= {64.6377, Null}

Out[20]= {1.21321, Null}

Out[21]= 45

In[22]:=
```
show[Take[gb, 10]]
```

Out[22]//MatrixForm=

1	:	b1 b2 b3 (a1 + mu) (a2 + mu) (a3 + mu)
2	:	b1 b2 b3 $(b3^2 - \text{lambda})$ (a1 + mu) (a2 + mu)
3	:	b1 b2 b3 $(b2^2 - \text{lambda})$ (a1 + mu) (a3 + mu)
4	:	b1 b2 b3 $(b1^2 - \text{lambda})$ (a2 + mu) (a3 + mu)
5	:	b1 b2 b3 $(b2^2 - \text{lambda})$ $(b3^2 - \text{lambda})$ (a1 + mu)
6	:	b1 b2 b3 $(b1^2 - \text{lambda})$ $(b3^2 - \text{lambda})$ (a2 + mu)
7	:	b1 b2 b3 $(b1^2 - \text{lambda})$ $(b2^2 - \text{lambda})$ (a3 + mu)
8	:	b1 b2 b3 $(b1^2 - \text{lambda})$ $(b2^2 - \text{lambda})$ $(b3^2 - \text{lambda})$
9	:	b2 b3 (a2 + mu) (a3 + mu) $(-b1^2 + \text{lambda} + (a1 + mu)^2)$
10	:	b1 b3 (a1 + mu) (a3 + mu) $(-b2^2 + \text{lambda} + (a2 + mu)^2)$

Case 1: b1 = b2 = b3.

In[23]:=
```
subs = {b3 → b2, b2 → b1};
```

In[24]:=
```
show[Take[simp[gb //. subs], 10]]
```

Out[24]//MatrixForm=

1	:	$(b1^3 - b1\ \text{lambda})^3$
2	:	$b1^3\ (b1^2 - \text{lambda})^2$ (a1 + mu)
3	:	$b1^3\ (b1^2 - \text{lambda})^2$ (a2 + mu)
4	:	$b1^3\ (b1^2 - \text{lambda})^2$ (a3 + mu)
5	:	$b1^3$ (a1 + mu) (a2 + mu) (a3 + mu)
6	:	$b1^3\ (b1^2 - \text{lambda})$ (a1 + mu) (a2 + mu)
7	:	$b1^3\ (b1^2 - \text{lambda})$ (a1 + mu) (a3 + mu)
8	:	$b1^3\ (b1^2 - \text{lambda})$ (a2 + mu) (a3 + mu)
9	:	$(b1^3 - b1\ \text{lambda})^2\ (-b1^2 + \text{lambda} + (a1 + mu)^2)$
10	:	$(b1^3 - b1\ \text{lambda})^2\ (-b1^2 + \text{lambda} + (a2 + mu)^2)$

■ Equation (1) implies that lambda = b1^2

In[25]:=
```
subs = Append[subs, lambda → b1^2]
```

Out[25]= $\{b3 \to b2,\ b2 \to b1,\ \text{lambda} \to b1^2\}$

In[26]:=
```
show[simp[gb //. subs]]
```

Out[26]//MatrixForm=

$$\begin{pmatrix}
1 & : & 0 \\
2 & : & b1^3\,(a1+mu)\,(a2+mu)\,(a3+mu) \\
3 & : & (a1+mu)^2\,(a2+mu)^2\,(a3+mu)^2 \\
4 & : & b1^2\,(a1+mu)^2\,(a2+mu)\,(a3+mu) \\
5 & : & b1^2\,(a1+mu)\,(a2+mu)^2\,(a3+mu) \\
6 & : & b1\,(a1+mu)^2\,(a2+mu)^2\,(a3+mu) \\
7 & : & b1^2\,(a1+mu)\,(a2+mu)\,(a3+mu)^2 \\
8 & : & b1\,(a1+mu)^2\,(a2+mu)\,(a3+mu)^2 \\
9 & : & b1\,(a1+mu)\,(a2+mu)^2\,(a3+mu)^2 \\
10 & : & (a1+mu)^2\,(a2+mu)^2\,\left(4\,b1^2+(a1+mu)^2\right) \\
11 & : & (a1+mu)^2\,(a3+mu)^2\,\left(4\,b1^2+(a1+mu)^2\right) \\
12 & : & (a1+mu)^2\,(a2+mu)^2\,\left(4\,b1^2+(a2+mu)^2\right) \\
13 & : & (a2+mu)^2\,(a3+mu)^2\,\left(4\,b1^2+(a2+mu)^2\right) \\
14 & : & (a1+mu)^2\,(a3+mu)^2\,\left(4\,b1^2+(a3+mu)^2\right) \\
15 & : & (a2+mu)^2\,(a3+mu)^2\,\left(4\,b1^2+(a3+mu)^2\right) \\
16 & : & b1\,(a1+mu)^2\,(a2+mu)\,\left(4\,b1^2+(a1+mu)^2\right) \\
17 & : & b1\,(a1+mu)^2\,(a3+mu)\,\left(4\,b1^2+(a1+mu)^2\right) \\
18 & : & b1\,(a1+mu)\,(a2+mu)^2\,\left(4\,b1^2+(a2+mu)^2\right) \\
19 & : & b1\,(a2+mu)^2\,(a3+mu)\,\left(4\,b1^2+(a2+mu)^2\right) \\
20 & : & b1\,(a1+mu)\,(a3+mu)^2\,\left(4\,b1^2+(a3+mu)^2\right) \\
21 & : & b1\,(a2+mu)\,(a3+mu)^2\,\left(4\,b1^2+(a3+mu)^2\right)
\end{pmatrix}$$

- **The sought equations analysed in the paper are equations (10) -- (15).**

Case 2: Exactly two of b1, b2, b3 coincide

- **By a change of coordinates, may assume that b2 = b3, b1 != b2**

In[27]:=
```
subs = {b3 → b2};
```

In[28]:=
```
show[Take[simp[gb //. subs], 3]]
```

Out[28]//MatrixForm=

$$\begin{pmatrix}
1 & : & b1\,\left(b2^3-b2\,lambda\right)^2\,(a1+mu) \\
2 & : & b1\,b2^2\,(a1+mu)\,(a2+mu)\,(a3+mu) \\
3 & : & b1\,\left(b1^2-lambda\right)\,\left(b2^3-b2\,lambda\right)^2
\end{pmatrix}$$

- **By equation (3) we have lambda = b1^2 or lambda = b2^2**

Case 2: Subcase 1: lambda = b1^2

In[29]:=
```
Case = Append[subs, lambda → b1^2]
```

Out[29]= $\{b3 \to b2,\ lambda \to b1^2\}$

In[30]:=
```
tmp = simp[gb //. Case];
show[Take[tmp, 3]]
```

Out[31]//MatrixForm=

$$\begin{pmatrix}
1 & : & 0 \\
2 & : & b2^2\,\left(b1^2-b2^2\right)^2\,(a1+mu)^2 \\
3 & : & b1\,b2^2\,\left(b1^2-b2^2\right)^2\,(a1+mu)
\end{pmatrix}$$

- **Since b1 != b2, equation (3) implies that mu = -a1**

In[32]:=
```
Case = Append[Case, mu → -a1]
tt = simp[gb //. Case];
show[tt]
```

Out[32]= $\{b3 \to b2,\ lambda \to b1^2,\ mu \to -a1\}$

Out[34]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & -(a1-a3)\left((a1-a2)^2+(b1-b2)^2\right) b2 \left((a1-a2)^2+(b1+b2)^2\right) \\ 3 & : & -(a1-a2)\left((a1-a3)^2+(b1-b2)^2\right) b2 \left((a1-a3)^2+(b1+b2)^2\right) \\ 4 & : & b2\left(-b1^2+b2^2\right)\left(\left((a1-a2)^2+b1^2\right)^2+2\left((a1-a2)^2-b1^2\right) b2^2+b2^4\right) \\ 5 & : & b2\left(-b1^2+b2^2\right)\left(\left((a1-a3)^2+b1^2\right)^2+2\left((a1-a3)^2-b1^2\right) b2^2+b2^4\right) \\ 6 & : & \left((a1-a3)^2+b1^2-b2^2\right)\left(\left((a1-a2)^2+b1^2\right)^2+2\left((a1-a2)^2-b1^2\right) b2^2+b2^4\right) \\ 7 & : & \left((a1-a2)^2+b1^2-b2^2\right)\left(\left((a1-a3)^2+b1^2\right)^2+2\left((a1-a3)^2-b1^2\right) b2^2+b2^4\right) \end{pmatrix}$$

- **Equation (2) implies that a1=a3**
  Equation (3) implies that a1=a2

In[35]:=
```
Case = Append[Case, a3 → a1];
Case = Append[Case, a2 → a1];
```

In[37]:=
```
show[simp[gb //. Case]]
```

Out[37]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & \left(b1^2-b2^2\right)^3 \\ 3 & : & b2\left(-b1^2+b2^2\right)^3 \end{pmatrix}$$

- **Equation (2) implies that b1 = b2. Thus Case 2: Subcase 1 leads to a contradiction**

Case 2: Subcase 2: lambda = b2^2

In[38]:=
```
Case = Append[subs, lambda → b2^2]
```

Out[38]= $\{b3 \to b2,\ lambda \to b2^2\}$

In[39]:=
```
show[simp[gb //. Case]]
```

Out[39]//MatrixForm=

1	:	0
2	:	$b1\ (a1+mu)\ (a2+mu)^2\ (a3+mu)^2$
3	:	$b1\ b2^2\ (a1+mu)\ (a2+mu)\ (a3+mu)$
4	:	$b1\ b2\ (a1+mu)\ (a2+mu)^2\ (a3+mu)$
5	:	$b1\ b2\ (a1+mu)\ (a2+mu)\ (a3+mu)^2$
6	:	$b1\ (b1-b2)\ (b1+b2)\ (a2+mu)^2\ (a3+mu)^2$
7	:	$(a2+mu)^2\ (a3+mu)^2\ \left(4\ b2^2+(a2+mu)^2\right)$
8	:	$(a2+mu)^2\ (a3+mu)^2\ \left(4\ b2^2+(a3+mu)^2\right)$
9	:	$b1\ (b1-b2)\ b2^2\ (b1+b2)\ (a2+mu)\ (a3+mu)$
10	:	$b1\ (b1-b2)\ b2\ (b1+b2)\ (a2+mu)^2\ (a3+mu)$
11	:	$b1\ (b1-b2)\ b2\ (b1+b2)\ (a2+mu)\ (a3+mu)^2$
12	:	$b1\ (a1+mu)\ (a2+mu)^2\ \left(4\ b2^2+(a2+mu)^2\right)$
13	:	$b2\ (a2+mu)^2\ (a3+mu)\ \left(4\ b2^2+(a2+mu)^2\right)$
14	:	$b1\ (a1+mu)\ (a3+mu)^2\ \left(4\ b2^2+(a3+mu)^2\right)$
15	:	$b2\ (a2+mu)\ (a3+mu)^2\ \left(4\ b2^2+(a3+mu)^2\right)$
16	:	$(a2+mu)^2\ (a3+mu)^2\ \left(-b1^2+b2^2+(a1+mu)^2\right)$
17	:	$b2^2\ (a2+mu)\ (a3+mu)\ \left(-b1^2+b2^2+(a1+mu)^2\right)$
18	:	$b2\ (a2+mu)^2\ (a3+mu)\ \left(-b1^2+b2^2+(a1+mu)^2\right)$
19	:	$b2\ (a2+mu)\ (a3+mu)^2\ \left(-b1^2+b2^2+(a1+mu)^2\right)$
20	:	$b1\ (b1-b2)\ (b1+b2)\ (a2+mu)^2\ \left(4\ b2^2+(a2+mu)^2\right)$
21	:	$b1\ (b1-b2)\ (b1+b2)\ (a3+mu)^2\ \left(4\ b2^2+(a3+mu)^2\right)$
22	:	$(a2+mu)^2\ \left(-b1^2+b2^2+(a1+mu)^2\right)\ \left(4\ b2^2+(a2+mu)^2\right)$
23	:	$(a3+mu)^2\ \left(-b1^2+b2^2+(a1+mu)^2\right)\ \left(4\ b2^2+(a3+mu)^2\right)$
24	:	$(a2+mu)^2\ \left((b1-b2)^2+(a1+mu)^2\right)\ \left((b1+b2)^2+(a1+mu)^2\right)$
25	:	$(a3+mu)^2\ \left((b1-b2)^2+(a1+mu)^2\right)\ \left((b1+b2)^2+(a1+mu)^2\right)$
26	:	$b2\ (a2+mu)\ \left((b1-b2)^2+(a1+mu)^2\right)\ \left((b1+b2)^2+(a1+mu)^2\right)$
27	:	$b2\ (a3+mu)\ \left((b1-b2)^2+(a1+mu)^2\right)\ \left((b1+b2)^2+(a1+mu)^2\right)$

- **Since b1, b2 have same sign and b1 != b2, equations (20) and (21) imply that mu=-a2=-a3**

In[40]:=
```
Case = Append[Case, mu → -a2];
Case = Append[Case, a3 → a2];
```

In[42]:=
```
tt = simp[gb //. Case];
show[tt]
```

Out[43]//MatrixForm=

(1 : 0)

- **We have found a solution.**

In[44]:=
```
emKappaToMatrix[kappa //. Case] // MatrixForm
```

Out[44]//MatrixForm=

$$\begin{pmatrix} a1 & 0 & 0 & -b1 & 0 & 0 \\ 0 & a2 & 0 & 0 & -b2 & 0 \\ 0 & 0 & a2 & 0 & 0 & -b2 \\ b1 & 0 & 0 & a1 & 0 & 0 \\ 0 & b2 & 0 & 0 & a2 & 0 \\ 0 & 0 & b2 & 0 & 0 & a2 \end{pmatrix}$$

- **For this medium we know that the Fresnel surface is a double light cone.**

Case 3: All of b1, b2, b3 are distinct

In[45]:=
```
subs = {};
show[Take[simp[gb], 8]]
```

Out[46]//MatrixForm=

$$\begin{pmatrix} 1 & : & b1\,b2\,b3\,(a1+mu)\,(a2+mu)\,(a3+mu) \\ 2 & : & b1\,b2\,b3\,(b3^2-lambda)\,(a1+mu)\,(a2+mu) \\ 3 & : & b1\,b2\,b3\,(b2^2-lambda)\,(a1+mu)\,(a3+mu) \\ 4 & : & b1\,b2\,b3\,(b1^2-lambda)\,(a2+mu)\,(a3+mu) \\ 5 & : & b1\,b2\,b3\,(b2^2-lambda)\,(b3^2-lambda)\,(a1+mu) \\ 6 & : & b1\,b2\,b3\,(b1^2-lambda)\,(b3^2-lambda)\,(a2+mu) \\ 7 & : & b1\,b2\,b3\,(b1^2-lambda)\,(b2^2-lambda)\,(a3+mu) \\ 8 & : & b1\,b2\,b3\,(b1^2-lambda)\,(b2^2-lambda)\,(b3^2-lambda) \end{pmatrix}$$

- **Equation (8) and (5)--(7) imply that lambda = bi^2 and mu = -ai for some i=1,2,3**

Case 3: Subcase 1: i=1

In[47]:=
```
subs = {lambda → b1^2, mu → -a1};
show[simp[gb //. subs]]
```

Out[48]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & -(a1-a2)\,b2\,\left((a1-a3)^2+(b1-b3)^2\right)\left((a1-a3)^2+(b1+b3)^2\right) \\ 3 & : & (-a1+a3)\left(\left((a1-a2)^2+b1^2\right)^2+2\left((a1-a2)^2-b1^2\right)b2^2+b2^4\right)b3 \\ 4 & : & \left(\left((a1-a2)^2+b1^2\right)^2+2\left((a1-a2)^2-b1^2\right)b2^2+b2^4\right)b3\,\left(-b1^2+b3^2\right) \\ 5 & : & b2\,\left(-b1^2+b2^2\right)\left(\left((a1-a3)^2+b1^2\right)^2+2\left((a1-a3)^2-b1^2\right)b3^2+b3^4\right) \\ 6 & : & \left(\left((a1-a2)^2+b1^2\right)^2+2\left((a1-a2)^2-b1^2\right)b2^2+b2^4\right)\left((a1-a3)^2+b1^2-b3^2\right) \\ 7 & : & \left((a1-a2)^2+b1^2-b2^2\right)\left(\left((a1-a3)^2+b1^2\right)^2+2\left((a1-a3)^2-b1^2\right)b3^2+b3^4\right) \end{pmatrix}$$

- **Since b1 != b3, equation (2) implies that a1 = a2**

In[49]:=
```
subs = Append[subs, a2 → a1];
show[simp[gb //. subs]]
```

Out[50]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & -(a1-a3)\,\left(b1^2-b2^2\right)^2 b3 \\ 3 & : & -(b1-b2)^2\,(b1+b2)^2\,(b1-b3)\,b3\,(b1+b3) \\ 4 & : & (b1-b2)^2\,(b1+b2)^2\left((a1-a3)^2+(b1-b3)\,(b1+b3)\right) \\ 5 & : & b2\,\left(-b1^2+b2^2\right)\left(\left((a1-a3)^2+b1^2\right)^2+2\left((a1-a3)^2-b1^2\right)b3^2+b3^4\right) \\ 6 & : & (b1-b2)\,(b1+b2)\left(\left((a1-a3)^2+b1^2\right)^2+2\left((a1-a3)^2-b1^2\right)b3^2+b3^4\right) \end{pmatrix}$$

- **Since b1!= b2 and b1, b2 have same sign, equation (2) implies that a3=a1**

In[51]:=
```
subs = Append[subs, a3 → a1];
show[simp[tmp //. subs]]
```

Out[52]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & \left(b1^2-b2^2\right)^3 \\ 3 & : & b2\,\left(-b1^2+b2^2\right)^3 \end{pmatrix}$$

- **Equation (2) implies that b1=b2. This contradicts that beta_i are all distinct (and of same sign). Hence Case 3: Subcase 1 leads to a contradiction.**

Note: By symmetry, the case i=1 should be enough. However, since cases i=2,3 require some manual simplification (motivated by the i=1 case), these cases are provided below for completeness.

Case 3: Subcase 2: i=2

In[53]:=
```
subs = {lambda → b2^2, mu → -a2};
show[simp[gb //. subs]]
```

Out[54]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & (a1-a2)\, b1\, \left(\left((a2-a3)^2+b2^2\right)^2+2\left((a2-a3)^2-b2^2\right) b3^2+b3^4\right) \\ 3 & : & (-a2+a3)\, \left(\left((a1-a2)^2+b1^2\right)^2+2\left((a1-a2)^2-b1^2\right) b2^2+b2^4\right) b3 \\ 4 & : & \left(\left((a1-a2)^2+b1^2\right)^2+2\left((a1-a2)^2-b1^2\right) b2^2+b2^4\right) b3\, \left(-b2^2+b3^2\right) \\ 5 & : & b1\,(b1-b2)\,(b1+b2)\, \left(\left((a2-a3)^2+b2^2\right)^2+2\left((a2-a3)^2-b2^2\right) b3^2+b3^4\right) \\ 6 & : & \left(\left((a1-a2)^2+b1^2\right)^2+2\left((a1-a2)^2-b1^2\right) b2^2+b2^4\right)\left((a2-a3)^2+b2^2-b3^2\right) \\ 7 & : & \left((a1-a2)^2-b1^2+b2^2\right)\left(\left((a2-a3)^2+b2^2\right)^2+2\left((a2-a3)^2-b2^2\right) b3^2+b3^4\right) \end{pmatrix}$$

- **The factor in equation (2) can be rewritten**

In[55]:=
```
fa = (((a2 - a3)^2 + b2^2)^2 + 2 ((a2 - a3)^2 - b2^2) b3^2 + b3^4);
fa2 = ((a2 - a3)^2 + (b2 - b3)^2) ((a2 - a3)^2 + (b2 + b3)^2);
Simplify[fa - fa2]
```

Out[57]= 0

- **Thus, since b2 != b3, equation (2) implies that a1 = a2**

In[58]:=
```
subs = Append[subs, a2 → a1];
show[simp[gb //. subs]]
```

Out[59]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & -(a1-a3)\left(b1^2-b2^2\right)^2 b3 \\ 3 & : & -(b1-b2)^2\,(b1+b2)^2\,(b2-b3)\, b3\,(b2+b3) \\ 4 & : & (b1-b2)^2\,(b1+b2)^2\left((a1-a3)^2+(b2-b3)\,(b2+b3)\right) \\ 5 & : & (-b1+b2)\,(b1+b2)\left((a1-a3)^2+(b2-b3)^2\right)\left((a1-a3)^2+(b2+b3)^2\right) \\ 6 & : & b1\,(b1-b2)\,(b1+b2)\left(\left((a1-a3)^2+b2^2\right)^2+2\left((a1-a3)^2-b2^2\right) b3^2+b3^4\right) \end{pmatrix}$$

- **Since b1!= b2 and b1, b2 have same sign, equation (2) implies that a3=a1**

In[60]:=
```
subs = Append[subs, a3 → a1];
show[simp[tmp //. subs]]
```

Out[61]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & \left(b1^2-b2^2\right)^3 \\ 3 & : & b2\left(-b1^2+b2^2\right)^3 \end{pmatrix}$$

- **Equation (2) implies that b1=b2. This contradicts that beta_i are all distinct (and of same sign). Hence Case 3: Subcase 2 leads to a contradiction.**

Case 3: Subcase 3: i=3

In[62]:=
```
subs = {lambda → b3^2, mu → -a3};
show[simp[gb //. subs]]
```

Out[63]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & (a2-a3)\,b2\,\left(\left((a1-a3)^2+b1^2\right)^2+2\left((a1-a3)^2-b1^2\right)b3^2+b3^4\right) \\ 3 & : & (a1-a3)\,b1\,\left(\left((a2-a3)^2+b2^2\right)^2+2\left((a2-a3)^2-b2^2\right)b3^2+b3^4\right) \\ 4 & : & b2\,(b2-b3)\,(b2+b3)\,\left(\left((a1-a3)^2+b1^2\right)^2+2\left((a1-a3)^2-b1^2\right)b3^2+b3^4\right) \\ 5 & : & b1\,(b1-b3)\,(b1+b3)\,\left(\left((a2-a3)^2+b2^2\right)^2+2\left((a2-a3)^2-b2^2\right)b3^2+b3^4\right) \\ 6 & : & \left((a2-a3)^2-b2^2+b3^2\right)\left(\left((a1-a3)^2+b1^2\right)^2+2\left((a1-a3)^2-b1^2\right)b3^2+b3^4\right) \\ 7 & : & \left((a1-a3)^2-b1^2+b3^2\right)\left(\left((a2-a3)^2+b2^2\right)^2+2\left((a2-a3)^2-b2^2\right)b3^2+b3^4\right) \end{pmatrix}$$

- **The factor in equation (2) can be rewritten**

In[64]:=
```
fa = (((a1 - a3)^2 + b1^2)^2 + 2 ((a1 - a3)^2 - b1^2) b3^2 + b3^4);
fa2 = ((a1 - a3)^2 + (b1 - b3)^2) ((a1 - a3)^2 + (b1 + b3)^2);
Simplify[fa - fa2]
```

Out[66]= 0

- **Thus, since b1 != b3, equation (2) implies that a2 = a3**

In[67]:=
```
subs = Append[subs, a3 → a2];
show[simp[gb //. subs]]
```

Out[68]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & (a1-a2)\,b1\,\left(b2^2-b3^2\right)^2 \\ 3 & : & b1\,(b1-b3)\,(b1+b3)\,\left(b2^2-b3^2\right)^2 \\ 4 & : & (b2-b3)^2\,(b2+b3)^2\,\left((a1-a2)^2-b1^2+b3^2\right) \\ 5 & : & \left(-b2^2+b3^2\right)\left(\left((a1-a2)^2+b1^2\right)^2+2\left((a1-a2)^2-b1^2\right)b3^2+b3^4\right) \\ 6 & : & b2\,(b2-b3)\,(b2+b3)\,\left(\left((a1-a2)^2+b1^2\right)^2+2\left((a1-a2)^2-b1^2\right)b3^2+b3^4\right) \end{pmatrix}$$

- **Since b3!= b2 and b3, b2 have same sign, equation (2) implies that a2=a1**

In[69]:=
```
subs = Append[subs, a2 → a1];
show[simp[tmp //. subs]]
```

Out[70]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & \left(b1^2-b2^2\right)^3 \\ 3 & : & b2\left(-b1^2+b2^2\right)^3 \end{pmatrix}$$

- **Equation (2) implies that b1=b2. This contradicts that beta_i are all distinct (and of same sign). Hence Case 3: Subcase 3 leads to a contradiction.**