```
In[1]:= SetDirectory["~/KappaLib/"];
        << kappaLib-1.2.m
        << helper.m

        Loading KappaLib v1.2

        Loading helper.m..
```

- **Define medium**

  kappa = C1 ast_g + rho overline (A) otimes A + C2 Id

  where g is a Lorentz metric.

- **In suitable coordinates**

```
In[4]:= Metric = k DiagonalMatrix[{-1, 1, 1, 1}];
```

- **Since hodge operator is conformally invariant, we may assume that k=1.**

```
In[5]:= Metric = Metric /. k → 1;

In[6]:= AA = emMatrix["a", 4, Structure → "AntiSymmetric"];

        (* rho = scalar density of weight 1 *)
        (* C1 = non-zero constant *)
        (* C2 = constant *)

In[7]:= kappa = C1 emHodge[Metric] + emBiProduct[rho, AA, AA] + C2 emIdentityKappa[];
```

- **If kappa is non-birefringent then**

  (kappa - AxionComponent)^2 = -lambda Id

  for some lambda.

In[8]:=
```
eta = kappa - emTrace[kappa] / 6 emIdentityKappa[];
emTrace[eta]
closure = emCompose[eta, eta] + lambda emIdentityKappa[];
eqs = FullSimplify[Union[Flatten[closure]]];
show[simp[eqs]]
```

Out[9]= 0

Out[12]//MatrixForm=

```
1  :  0
2  :  4/3 rho ((a13 a24 - a12 a34) (C1 - 4 a23^2 rho) + 2 a14 a23 (C1 + 2 a23^2 rhc
3  :  2/3 rho (3 (a12 a24 - a13 a34) C1 + 8 a12 a13 (a14 a23 - a13 a24 + a12 a34)
4  :  2/3 rho (3 (a13 a23 - a14 a24) C1 - 8 a13 a14 (a14 a23 - a13 a24 + a12 a34)
5  :  2/3 rho (3 (a12 a14 - a23 a34) C1 + 8 a12 a23 (a14 a23 - a13 a24 + a12 a34)
6  :  2/3 rho (3 (a13 a14 + a23 a24) C1 + 8 a13 a23 (a14 a23 - a13 a24 + a12 a34)
7  :  2/3 rho (3 (a12 a13 + a24 a34) C1 - 8 a12 a24 (a14 a23 - a13 a24 + a12 a34)
8  :  2/3 rho (3 (a13 a14 + a23 a24) C1 - 8 a14 a24 (a14 a23 - a13 a24 + a12 a34)
9  :  2/3 rho (3 (a13 a23 - a14 a24) C1 - 8 a23 a24 (a14 a23 - a13 a24 + a12 a34)
10 :  2/3 rho (3 (a12 a13 + a24 a34) C1 + 8 a13 a34 (a14 a23 - a13 a24 + a12 a34)
11 :  2/3 rho (3 (a12 a14 - a23 a34) C1 + 8 a14 a34 (a14 a23 - a13 a24 + a12 a34)
12 :  2/3 rho (3 (a12 a23 + a14 a34) C1 + 8 a23 a34 (a14 a23 - a13 a24 + a12 a34)
13 :  2/3 rho (3 (a12 a24 - a13 a34) C1 + 8 a24 a34 (a14 a23 - a13 a24 + a12 a34)
14 :  2/3 rho (-3 (a12 a23 + a14 a34) C1 + 8 a12 a14 (a14 a23 - a13 a24 + a12 a34)
15 :  2/3 rho (-3 (a13 a14 + a23 a24) C1 - 8 a13 a23 (a14 a23 - a13 a24 + a12 a34)
16 :  2/3 rho (-3 (a12 a13 + a24 a34) C1 + 8 a12 a24 (a14 a23 - a13 a24 + a12 a34)
17 :  2/3 rho (-3 (a13 a14 + a23 a24) C1 + 8 a14 a24 (a14 a23 - a13 a24 + a12 a34)
18 :  2/3 rho (-3 (a12 a13 + a24 a34) C1 - 8 a13 a34 (a14 a23 - a13 a24 + a12 a34)
19 :  2/3 rho (3 (-a12 a24 + a13 a34) C1 - 8 a24 a34 (a14 a23 - a13 a24 + a12 a34)
20 :  2/3 rho (-3 a12 a24 C1 + 3 a13 a34 C1 - 8 a12 a13 (a14 a23 - a13 a24 + a12 a34
21 :  2/3 rho (-3 a13 a23 C1 + 3 a14 a24 C1 + 8 a13 a14 (a14 a23 - a13 a24 + a12 a34
22 :  2/3 rho (-3 a13 a23 C1 + 3 a14 a24 C1 + 8 a23 a24 (a14 a23 - a13 a24 + a12 a34
23 :  4/3 rho ((a14 a23 - a13 a24 - 2 a12 a34) C1 + 4 a12^2 (a14 a23 - a13 a24 + a12 a3
24 :  4/3 rho ((a14 a23 + 2 a13 a24 + a12 a34) C1 - 4 a24^2 (a14 a23 - a13 a24 + a12 a3
25 :  4/3 rho ((-2 a14 a23 - a13 a24 + a12 a34) C1 + 4 a14^2 (a14 a23 - a13 a24 + a12 a
26 :  4/3 rho (-(a14 a23 + 2 a13 a24 + a12 a34) C1 - 4 a13^2 (a14 a23 - a13 a24 + a12 a
27 :  4/3 rho ((-a14 a23 + a13 a24 + 2 a12 a34) C1 + 4 a34^2 (a14 a23 - a13 a24 + a12 a
28 :  -C1^2 + lambda + 2 (a13 - a24) (a13 + a24) C1 rho + 4/9 (a14 a23 - 13 a13 a24 + a12 a34) (a14 a23
29 :  -C1^2 + lambda + 2 (a14 - a23) (a14 + a23) C1 rho + 4/9 (a14 a23 - a13 a24 + a12 a34) (13 a14 a23
30 :  -C1^2 + lambda + 2 (a12 - a34) (a12 + a34) C1 rho + 4/9 (a14 a23 - a13 a24 + a12 a34) (a14 a23 - a
```

- **Compute Gröbner basis. Then equations ‘gb’ have the same solution set as equations ‘eqs’.**

In[13]:=
```
gb = simp[GroebnerBasis[eqs, Variables[eqs]]]; // Timing
```

Out[13]= {1.81224, Null}

In[14]:=
```
sub = {};
tmp = simp[Take[gb, 10]];
show[tmp]
```

Out[16]//MatrixForm=

1	:	$(C1^2 - lambda)^3$
2	:	$a34^3 (C1^2 - lambda)$
3	:	$a14 (C1^2 - lambda)^2$
4	:	$a23 (C1^2 - lambda)^2$
5	:	$a24 (C1^2 - lambda)^2$
6	:	$a34 (C1^2 - lambda)^2$
7	:	a14 a24 a34 lambda rho
8	:	a23 a24 a34 lambda rho
9	:	$a12\, a13 (C1^2 - lambda)$
10	:	$a12\, a14 (C1^2 - lambda)$

In[17]:=
```
sub = Append[sub, lambda → C1^2];
tmp = Take[simp[gb //. sub], 20];
show[tmp]
```

Out[19]//MatrixForm=

1	:	0
2	:	$4\, a23^3\, C1^2\, rho$
3	:	$4\, a24^3\, C1^2\, rho$
4	:	$4\, a34^3\, C1^2\, rho$
5	:	$4\, a14\, a23^2\, C1^2\, rho$
6	:	$4\, a23^2\, a24\, C1^2\, rho$
7	:	$4\, a14\, a24^2\, C1^2\, rho$
8	:	$4\, a23\, a24^2\, C1^2\, rho$
9	:	$4\, a23^2\, a34\, C1^2\, rho$
10	:	$a14\, a24\, a34\, C1^2\, rho$
11	:	$a23\, a24\, a34\, C1^2\, rho$
12	:	$4\, a24^2\, a34\, C1^2\, rho$
13	:	$4\, a14\, a34^2\, C1^2\, rho$
14	:	$4\, a23\, a34^2\, C1^2\, rho$
15	:	$4\, a24\, a34^2\, C1^2\, rho$
16	:	$4\, a14\, a23\, a24\, C1^2\, rho$
17	:	$4\, a14\, a23\, a24\, C1^3\, rho$
18	:	$-4\, a14\, a23\, a34\, C1^2\, rho$
19	:	$-4\, a14\, a23\, a34\, C1^3\, rho$
20	:	$a14 (a13^2 + a24^2)\, C1\, rho$

- **If rho = 0 we are done. Let us therefore assume that rho != 0.**
- **Equations (2)--(4) imply that**

a23= a24 = a34 =0

In[20]:=
```
sub = Append[sub, a23 → 0];
sub = Append[sub, a24 → 0];
sub = Append[sub, a34 → 0];
tmp = Take[simp[gb //. sub], 8];
show[tmp]
```

Out[24]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & 4\, a13^3\, C1\, rho \\ 3 & : & 4\, a14^3\, C1\, rho \\ 4 & : & 3\, a12\, a13\, C1\, rho \\ 5 & : & 3\, a12\, a14\, C1\, rho \\ 6 & : & 3\, a13\, a14\, C1\, rho \\ 7 & : & a13^2\, a14\, C1\, rho \\ 8 & : & 2\, a12^2\, C1^2\, rho \end{pmatrix}$$

- **Equations (2) -- (3) imply that (at least) one of the below must hold**

a13 = 0 and a14 = 0

In[25]:=
```
sub = Append[sub, a13 → 0];
sub = Append[sub, a14 → 0];
tmp = simp[gb //. sub];
show[tmp]
```

Out[28]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & a12^2\, C1\, rho \\ 3 & : & 2\, a12^2\, C1^2\, rho \end{pmatrix}$$

- **Equation (2) implies that a12 = 0**

In[29]:=
```
sub = Append[sub, a12 → 0];
tmp = simp[gb //. sub];
show[tmp]
```

Out[31]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \end{pmatrix}$$

In[32]:=
```
AA //. sub // MatrixForm
```

Out[32]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- **It follows that A = 0.**