In[1]:=
```
SetDirectory["~/writing/WIP/KappaLib/"];
<< kappaLib.m
<< helper.m
```

```
KappaLib v1.1

Loading helper.m..
```

- **Define the medium**

In[4]:=
```
  (* We assume kappa = C1 ast_g  + C2 Id where
    g = (-1,-1,1,1)
*)
g = DiagonalMatrix[{-1, -1, 1, 1}];
kappa = C1 emHodge[g] + C2 emIdentityKappa[];
```

In[6]:=
```
FullSimplify[emDet[kappa]]
```

Out[6]= $-\left(C1^2 - C2^2\right)^3$

- **Compute Fresnel polynomial**

In[7]:=
```
xi = {x0, x1, x2, x3};

fresnel = FullSimplify[emKappaToFresnel[kappa, xi]]
```

Out[8]= $C1^3 \left(x0^2 + x1^2 - x2^2 - x3^2\right)^2$

- **Find non-zero points on the Fresnel surface of kappa.**

  Note: We do not need to trust the below code. We will verify the output in the below.

In[9]:=
```
coordValues = {1, Sqrt[2], Sqrt[3], 0};

LL = Length[coordValues];
```

In[11]:=
```
xiList = {};

For[i0 = 1, i0 ≤ LL, i0++,
 For[i1 = 1, i1 ≤ LL, i1++,
  For[i2 = 1, i2 ≤ LL, i2++,
   For[i3 = 1, i3 ≤ LL, i3++,

    frSub = {
      x0 → coordValues[[i0]],
      x1 → coordValues[[i1]],
      x2 → coordValues[[i2]],
      x3 → coordValues[[i3]]};
    frPoint = xi /. frSub;

    (* if point is non-zero and belongs
      to the Fresnel surface add it to list. *)
    If[Simplify[frPoint.frPoint] ≠ 0,
     If[Simplify[fresnel /. frSub] == 0,
       xiList = Append[xiList, frSub];
      ];
    ];

   ]
  ]
 ]
]
```

In[13]:=
```
xiList
```

Out[13]=
$\{\{x0\to 1, x1\to 1, x2\to 1, x3\to 1\},$
$\{x0\to 1, x1\to 1, x2\to \sqrt{2}, x3\to 0\}, \{x0\to 1, x1\to 1, x2\to 0, x3\to \sqrt{2}\},$
$\{x0\to 1, x1\to \sqrt{2}, x2\to 1, x3\to \sqrt{2}\}, \{x0\to 1, x1\to \sqrt{2}, x2\to \sqrt{2}, x3\to 1\},$
$\{x0\to 1, x1\to \sqrt{2}, x2\to \sqrt{3}, x3\to 0\}, \{x0\to 1, x1\to \sqrt{2}, x2\to 0, x3\to \sqrt{3}\},$
$\{x0\to 1, x1\to \sqrt{3}, x2\to 1, x3\to \sqrt{3}\}, \{x0\to 1, x1\to \sqrt{3}, x2\to \sqrt{2}, x3\to \sqrt{2}\},$
$\{x0\to 1, x1\to \sqrt{3}, x2\to \sqrt{3}, x3\to 1\}, \{x0\to 1, x1\to 0, x2\to 1, x3\to 0\},$
$\{x0\to 1, x1\to 0, x2\to 0, x3\to 1\}, \{x0\to \sqrt{2}, x1\to 1, x2\to 1, x3\to \sqrt{2}\},$
$\{x0\to \sqrt{2}, x1\to 1, x2\to \sqrt{2}, x3\to 1\}, \{x0\to \sqrt{2}, x1\to 1, x2\to \sqrt{3}, x3\to 0\},$
$\{x0\to \sqrt{2}, x1\to 1, x2\to 0, x3\to \sqrt{3}\}, \{x0\to \sqrt{2}, x1\to \sqrt{2}, x2\to 1, x3\to \sqrt{3}\},$
$\{x0\to \sqrt{2}, x1\to \sqrt{2}, x2\to \sqrt{2}, x3\to \sqrt{2}\}, \{x0\to \sqrt{2}, x1\to \sqrt{2}, x2\to \sqrt{3}, x3\to 1\},$
$\{x0\to \sqrt{2}, x1\to \sqrt{3}, x2\to \sqrt{2}, x3\to \sqrt{3}\}, \{x0\to \sqrt{2}, x1\to \sqrt{3}, x2\to \sqrt{3}, x3\to \sqrt{2}\},$
$\{x0\to \sqrt{2}, x1\to 0, x2\to 1, x3\to 1\}, \{x0\to \sqrt{2}, x1\to 0, x2\to \sqrt{2}, x3\to 0\},$
$\{x0\to \sqrt{2}, x1\to 0, x2\to 0, x3\to \sqrt{2}\}, \{x0\to \sqrt{3}, x1\to 1, x2\to 1, x3\to \sqrt{3}\},$
$\{x0\to \sqrt{3}, x1\to 1, x2\to \sqrt{2}, x3\to \sqrt{2}\}, \{x0\to \sqrt{3}, x1\to 1, x2\to \sqrt{3}, x3\to 1\},$
$\{x0\to \sqrt{3}, x1\to \sqrt{2}, x2\to \sqrt{2}, x3\to \sqrt{3}\}, \{x0\to \sqrt{3}, x1\to \sqrt{2}, x2\to \sqrt{3}, x3\to \sqrt{2}\},$
$\{x0\to \sqrt{3}, x1\to \sqrt{3}, x2\to \sqrt{3}, x3\to \sqrt{3}\}, \{x0\to \sqrt{3}, x1\to 0, x2\to 1, x3\to \sqrt{2}\},$
$\{x0\to \sqrt{3}, x1\to 0, x2\to \sqrt{2}, x3\to 1\}, \{x0\to \sqrt{3}, x1\to 0, x2\to \sqrt{3}, x3\to 0\},$
$\{x0\to \sqrt{3}, x1\to 0, x2\to 0, x3\to \sqrt{3}\}, \{x0\to 0, x1\to 1, x2\to 1, x3\to 0\},$
$\{x0\to 0, x1\to 1, x2\to 0, x3\to 1\}, \{x0\to 0, x1\to \sqrt{2}, x2\to 1, x3\to 1\},$
$\{x0\to 0, x1\to \sqrt{2}, x2\to \sqrt{2}, x3\to 0\}, \{x0\to 0, x1\to \sqrt{2}, x2\to 0, x3\to \sqrt{2}\},$
$\{x0\to 0, x1\to \sqrt{3}, x2\to 1, x3\to \sqrt{2}\}, \{x0\to 0, x1\to \sqrt{3}, x2\to \sqrt{2}, x3\to 1\},$
$\{x0\to 0, x1\to \sqrt{3}, x2\to \sqrt{3}, x3\to 0\}, \{x0\to 0, x1\to \sqrt{3}, x2\to 0, x3\to \sqrt{3}\}\}$

- **If**

  xi = xi_i dx^i,
  alpha = alpha_i dx^i,
  kappa = 1/8 kappa^ij_rs dx^r Λ dx^s otimes d/dx^i Λ dx^j

  then xi Λ kappa (xi Λ alpha) = 0 holds if and only if

  xi_i xi_a kappa^ab_lm epsilon^ilmk alpha_b = 0, k=0,...,3

  Compute the 4x4 matrix Lxi such that the above equation is equivalent with Lxi.alpha =0.

In[14]:=
```
Lxi = Table[
   Sum[
    xi[[i]] xi[[a]] emReadNormal[kappa, a, b, l, m] Signature[{i, l, m, k}],
    {a, 1, 4}, {i, 1, 4}, {l, 1, 4}, {m, 1, 4}]
   ,
   {k, 1, 4}, {b, 1, 4}
  ];
```

In[15]:=
```
(* matrix is symmetric *)
Union[Flatten[Lxi - Transpose[Lxi]]]
```

Out[15]= {0}

In[16]:=
```
Lxi // MatrixForm
```

Out[16]//MatrixForm=

2 C1 x1² − 2 C1 x2² − 2 C1 x3²	− 2 C1 x0 x1	2 C1 x0 x2	2 (
− 2 C1 x0 x1	2 C1 x0² − 2 C1 x2² − 2 C1 x3²	2 C1 x1 x2	2 (
2 C1 x0 x2	2 C1 x1 x2	− 2 C1 x0² − 2 C1 x1² + 2 C1 x3²	− 2
2 C1 x0 x3	2 C1 x1 x3	− 2 C1 x2 x3	− 2 C1 x0² − :

- **For each xi find 2 linearly independent alpha such that Lxi. alpha = 0 and alpha Λ xi != 0.**

  Note: We do not need to trust the below code. We will verify that lists xiList and alphaList have the sought properties in the next step.

In[17]:=
```
alphaList = {};

For[ii = 1, ii ≤ Length[xiList], ii++,

  (*** Get a point on the Fresnel surface    ***)
  iSub = xiList[[ii]];
  frPoint = Simplify[xi /. iSub];

  (*** Find alpha such that g(frPoint,alpha)=0  ***)
  aa = {a0, a1, a2, a3};
  evecs = Eigenvectors[(Lxi /. iSub)];
  evals = Eigenvalues[(Lxi /. iSub)];

  alphas = {};

  For[jj = 1, jj ≤ 4, jj++,
   If[evals[[jj]] == 0,
     ej = evecs[[jj]];

     (* If eigenvector is not proportional to xi, add it to list of
      possible alpha:s *)

     If[Length[alphas] == 0,
      (* First alpha: if alpha is not proportional to xi, then it *)
      propToXi = Length[Solve[ej == Const frPoint, Const]];
      If[propToXi == 0,
       alphas = Append[alphas, ej];
      ];

      ,
      (* Subsequent alphas: add unless new alpha is in the
         span of old alphas and xi.
      *)

      consts =
       Table[ToExpression["Const" <> ToString[co]], {co, 1, Length[alphas] + 1}];

      spanVectors = Append[alphas, frPoint];
      inSpan = Length[Solve[ej == consts.spanVectors, consts]];
      If[inSpan == 0,
       alphas = Append[alphas, evecs[[jj]]];
      ];
     ];
    ];
  ];

  (* collect *)
  alphaList = Append[alphaList, alphas];

 ];
```

- **Points on Fresnel surface:**

In[19]:= **xiList**

Out[19]=
```
{{x0 → 1, x1 → 1, x2 → 1, x3 → 1},
 {x0 → 1, x1 → 1, x2 → √2, x3 → 0}, {x0 → 1, x1 → 1, x2 → 0, x3 → √2},
 {x0 → 1, x1 → √2, x2 → 1, x3 → √2}, {x0 → 1, x1 → √2, x2 → √2, x3 → 1},
 {x0 → 1, x1 → √2, x2 → √3, x3 → 0}, {x0 → 1, x1 → √2, x2 → 0, x3 → √3},
 {x0 → 1, x1 → √3, x2 → 1, x3 → √3}, {x0 → 1, x1 → √3, x2 → √2, x3 → √2},
 {x0 → 1, x1 → √3, x2 → √3, x3 → 1}, {x0 → 1, x1 → 0, x2 → 1, x3 → 0},
 {x0 → 1, x1 → 0, x2 → 0, x3 → 1}, {x0 → √2, x1 → 1, x2 → 1, x3 → √2},
 {x0 → √2, x1 → 1, x2 → √2, x3 → 1}, {x0 → √2, x1 → 1, x2 → √3, x3 → 0},
 {x0 → √2, x1 → 1, x2 → 0, x3 → √3}, {x0 → √2, x1 → √2, x2 → 1, x3 → √3},
 {x0 → √2, x1 → √2, x2 → √2, x3 → √2}, {x0 → √2, x1 → √2, x2 → √3, x3 → 1},
 {x0 → √2, x1 → √3, x2 → √2, x3 → √3}, {x0 → √2, x1 → √3, x2 → √3, x3 → √2},
 {x0 → √2, x1 → 0, x2 → 1, x3 → 1}, {x0 → √2, x1 → 0, x2 → √2, x3 → 0},
 {x0 → √2, x1 → 0, x2 → 0, x3 → √2}, {x0 → √3, x1 → 1, x2 → 1, x3 → √3},
 {x0 → √3, x1 → 1, x2 → √2, x3 → √2}, {x0 → √3, x1 → 1, x2 → √3, x3 → 1},
 {x0 → √3, x1 → √2, x2 → √2, x3 → √3}, {x0 → √3, x1 → √2, x2 → √3, x3 → √2},
 {x0 → √3, x1 → √3, x2 → √3, x3 → √3}, {x0 → √3, x1 → 0, x2 → 1, x3 → √2},
 {x0 → √3, x1 → 0, x2 → √2, x3 → 1}, {x0 → √3, x1 → 0, x2 → √3, x3 → 0},
 {x0 → √3, x1 → 0, x2 → 0, x3 → √3}, {x0 → 0, x1 → 1, x2 → 1, x3 → 0},
 {x0 → 0, x1 → 1, x2 → 0, x3 → 1}, {x0 → 0, x1 → √2, x2 → 1, x3 → 1},
 {x0 → 0, x1 → √2, x2 → √2, x3 → 0}, {x0 → 0, x1 → √2, x2 → 0, x3 → √2},
 {x0 → 0, x1 → √3, x2 → 1, x3 → √2}, {x0 → 0, x1 → √3, x2 → √2, x3 → 1},
 {x0 → 0, x1 → √3, x2 → √3, x3 → 0}, {x0 → 0, x1 → √3, x2 → 0, x3 → √3}}
```

- **For each point, a list of three alphas:**

In[20]:= **alphaList // MatrixForm**

Out[20]//MatrixForm=

$$\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} \sqrt{2} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ \ldots \end{pmatrix} & \begin{pmatrix} \sqrt{2} \\ \ldots \end{pmatrix} \end{pmatrix}$$

$\begin{pmatrix}0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\end{pmatrix}$
$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}\sqrt{3}\\0\\1\\0\end{pmatrix}$
$\begin{pmatrix}\sqrt{3}\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$
$\begin{pmatrix}\sqrt{3}\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$
$\begin{pmatrix}\sqrt{2}\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}\sqrt{2}\\0\\1\\0\end{pmatrix}$
$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}\sqrt{3}\\0\\1\\0\end{pmatrix}$
$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$
$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$
$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\1\\0\end{pmatrix}$
$\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$
$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}\sqrt{\frac{3}{2}}\\0\\1\\0\end{pmatrix}$
$\begin{pmatrix}\sqrt{\frac{3}{2}}\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$
$\begin{pmatrix}\sqrt{\frac{3}{2}}\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\1\\0\end{pmatrix}$
$(1$	$(1$

$$
\left|\begin{array}{cc}
\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} \sqrt{\frac{3}{2}} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \sqrt{\frac{3}{2}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} \sqrt{\frac{3}{2}} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{array}\right|
$$

$$
\begin{pmatrix}
\begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ \sqrt{\frac{2}{3}} \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\end{pmatrix}
$$

Check that xiList and alphaList satisfy the sought properties:

In[21]:=
```
vecNorm[v_] := Simplify[v.v]
```

In[22]:=
```
frVal = {};
alphasForXi = {};
C123Solutions = {};
isInKernel = {};

For[ii = 1, ii ≤ Length[xiList], ii++,
  xiSub = xiList[[ii]];
  alphas = alphaList[[ii]];

  (* Check that xi is on Fresnel surface *)
  (* should be zero *)
  frVal = Append[frVal, Simplify[fresnel /. xiSub]];

  (* check that there are two alpha:s associated to each xi. *)
  (* should be 2 *)
  alphasForXi = Append[alphasForXi, Length[alphas]];

  (* check that alpha:s and xi are linearly independent *)
  (* only solution should be C1=C2=C3=0 *)
  eqs = C1 alphas[[1]] + C2 alphas[[2]] + C3 (xi /. xiSub);
  C123Solutions = Append[C123Solutions, Solve[toEqs[eqs], {C1, C2, C3}]];

  (* check that xi /\ kappa(xi /\ alpha) = 0 *)
  (* should output (0) *)
  cond = {};
  For[k = 1, k ≤ 2, k++,
   LL = Lxi /. xiSub;
   isInKernel = Append[isInKernel, vecNorm[LL.alphas[[k]]]];

  ];

 ];
```

In[27]:=
```
Union[frVal] (* should be 0*)
Union[alphasForXi ](* should be 3 *)
Union[C123Solutions](* should be C1=C2=C3=0 *)
Union[isInKernel] (* should be 0:s *)
```

Out[27]= {0}

Out[28]= {2}

Out[29]= {{{C1 → 0, C2 → 0, C3 → 0}}}

Out[30]= {0}

In[31]:=
```
Length[xiList]
Dimensions[alphaList]
```

Out[31]= 43

Out[32]= {43, 2, 4}

Define A and B bivectors

In[33]:=
```
Abivector = ( 0    A12  A13  A14
             -A12  0    A23  A24
             -A13 -A23  0    A34
             -A14 -A24 -A34  0  );

Bbivector = ( 0    B12  B13  B14
             -B12  0    B23  B24
             -B13 -B23  0    B34
             -B14 -B24 -B34  0  );

Abivector + Transpose[Abivector]
Bbivector + Transpose[Bbivector]
```

Out[35]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[36]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

In[37]:=
```
(* For bivector 'bivector' and 1-forms 'co1' and 'co2' compute

       bivector (co1 /\ co2 )

*)
contract[bivector_, co1_, co2_] := Module[{i, j},
  Sum[
   bivector[[i]][[j]] (co1[[i]] co2[[j]] - co1[[j]] co2[[i]]),
   {i, 1, 4}, {j, 1, 4}
  ]
 ]
```

In[38]:=
```
eqs = {};

For[ii = 1, ii ≤ Length[xiList], ii++,

  (*** Get a point on the Fresnel surface    ***)
  iSub = xiList[[ii]];
  xiVec = Simplify[xi /. iSub];

  (* Go through both alpha:s associated to this xi. *)
  For [k = 1, k ≤ 2, k++,

   alpha = alphaList[[ii]][[k]];

   ee = contract[Abivector, xiVec, alpha] contract[Bbivector, xiVec, alpha];
   ee = Simplify[ee];
   eqs = Append[eqs, ee];
  ];
 ];
```

Analysis of constraints

In[40]:=
```
Length[eqs]
```

Out[40]= 86

In[41]:=
```
eqs = simp[eqs]; // Timing
```

Out[41]= {3.10516, Null}

In[42]:=
```
show[eqs]
```

Out[42]//MatrixForm=

1	:	4 (A12 + A13) (B12 + B13)
2	:	8 (A12 + A13) (B12 + B13)
3	:	4 (A12 + A14) (B12 + B14)
4	:	8 (A12 + A14) (B12 + B14)
5	:	4 (A12 - A23) (B12 - B23)
6	:	8 (A12 - A23) (B12 - B23)
7	:	4 (A12 - A24) (B12 - B24)
8	:	8 (A12 - A24) (B12 - B24)
9	:	4 (A13 - A34) (B13 - B34)

10 : $8\ (A13 - A34)\ (B13 - B34)$
11 : $4\ (A23 - A34)\ (B23 - B34)$
12 : $8\ (A23 - A34)\ (B23 - B34)$
13 : $4\ (A14 + A34)\ (B14 + B34)$
14 : $8\ (A14 + A34)\ (B14 + B34)$
15 : $4\ (A24 + A34)\ (B24 + B34)$
16 : $8\ (A24 + A34)\ (B24 + B34)$
17 : $12\ (A12 + A13)\ (B12 + B13)$
18 : $12\ (A12 + A14)\ (B12 + B14)$
19 : $12\ (A12 - A23)\ (B12 - B23)$
20 : $12\ (A12 - A24)\ (B12 - B24)$
21 : $12\ (A13 - A34)\ (B13 - B34)$
22 : $12\ (A23 - A34)\ (B23 - B34)$
23 : $12\ (A14 + A34)\ (B14 + B34)$
24 : $12\ (A24 + A34)\ (B24 + B34)$
25 : $4\ (A12 + A13 - A24 - A34)\ (B12 + B13 - B24 - B34)$
26 : $8\ (A12 + A13 - A24 - A34)\ (B12 + B13 - B24 - B34)$
27 : $4\ (A12 + A14 - A23 + A34)\ (B12 + B14 - B23 + B34)$
28 : $8\ (A12 + A14 - A23 + A34)\ (B12 + B14 - B23 + B34)$
29 : $12\ (A12 + A13 - A24 - A34)\ (B12 + B13 - B24 - B34)$
30 : $12\ (A12 + A14 - A23 + A34)\ (B12 + B14 - B23 + B34)$
31 : $4\ (\sqrt{2}\ A12 + A13 + A14)\ (\sqrt{2}\ B12 + B13 + B14)$
32 : $4\ (\sqrt{2}\ A12 + A13 - A23)\ (\sqrt{2}\ B12 + B13 - B23)$
33 : $4\ (\sqrt{2}\ A12 + A14 - A24)\ (\sqrt{2}\ B12 + B14 - B24)$
34 : $4\ (\sqrt{2}\ A12 - A23 - A24)\ (\sqrt{2}\ B12 - B23 - B24)$
35 : $4\ (A13 + A23 - \sqrt{2}\ A34)\ (B13 + B23 - \sqrt{2}\ B34)$
36 : $4\ (A14 + A24 + \sqrt{2}\ A34)\ (B14 + B24 + \sqrt{2}\ B34)$
37 : $4\ (\sqrt{3}\ A12 + \sqrt{2}\ A13 + A14)\ (\sqrt{3}\ B12 + \sqrt{2}\ B13 + B14)$
38 : $4\ (\sqrt{3}\ A12 + A13 + \sqrt{2}\ A14)\ (\sqrt{3}\ B12 + B13 + \sqrt{2}\ B14)$
39 : $4\ (\sqrt{3}\ A12 - \sqrt{2}\ A23 - A24)\ (\sqrt{3}\ B12 - \sqrt{2}\ B23 - B24)$
40 : $4\ (\sqrt{3}\ A12 - A23 - \sqrt{2}\ A24)\ (\sqrt{3}\ B12 - B23 - \sqrt{2}\ B24)$
41 : $4\ (\sqrt{2}\ A13 + A23 - \sqrt{3}\ A34)\ (\sqrt{2}\ B13 + B23 - \sqrt{3}\ B34)$
42 : $4\ (A13 + \sqrt{2}\ A23 - \sqrt{3}\ A34)\ (B13 + \sqrt{2}\ B23 - \sqrt{3}\ B34)$
43 : $4\ (\sqrt{2}\ A14 + A24 + \sqrt{3}\ A34)\ (\sqrt{2}\ B14 + B24 + \sqrt{3}\ B34)$
44 : $4\ (A14 + \sqrt{2}\ A24 + \sqrt{3}\ A34)\ (B14 + \sqrt{2}\ B24 + \sqrt{3}\ B34)$
45 : $(\sqrt{6}\ A12 + \sqrt{2}\ A13 - 2\ A23)\ (\sqrt{6}\ B12 + \sqrt{2}\ B13 - 2\ B23)$
46 : $(\sqrt{6}\ A12 + \sqrt{2}\ A14 - 2\ A24)\ (\sqrt{6}\ B12 + \sqrt{2}\ B14 - 2\ B24)$
47 : $(\sqrt{2}\ A13 - \sqrt{2}\ A14 - 2\ A34)\ (\sqrt{2}\ B13 - \sqrt{2}\ B14 - 2\ B34)$
48 : $(\sqrt{2}\ A23 - \sqrt{2}\ A24 - 2\ A34)\ (\sqrt{2}\ B23 - \sqrt{2}\ B24 - 2\ B34)$
49 : $4\ (\sqrt{6}\ A12 + 2\ A13 - \sqrt{2}\ A23)\ (\sqrt{6}\ B12 + 2\ B13 - \sqrt{2}\ B23)$
50 : $4\ (\sqrt{6}\ A12 + 2\ A14 - \sqrt{2}\ A24)\ (\sqrt{6}\ B12 + 2\ B14 - \sqrt{2}\ B24)$
51 : $\frac{4}{9}\ (\sqrt{6}\ A13 - \sqrt{3}\ A14 - 3\ A34)\ (\sqrt{6}\ B13 - \sqrt{3}\ B14 - 3\ B34)$
52 : $\frac{4}{9}\ (\sqrt{6}\ A23 - \sqrt{3}\ A24 - 3\ A34)\ (\sqrt{6}\ B23 - \sqrt{3}\ B24 - 3\ B34)$
53 : $4\ (2\ A12 + \sqrt{2}\ A13 + A14 - \sqrt{2}\ A24 - A34)\ (2\ B12 + \sqrt{2}\ B13 + B14 - \sqrt{2}\ B24 - B$
54 : $4\ (2\ A12 + A13 + \sqrt{2}\ A14 - \sqrt{2}\ A23 + A34)\ (2\ B12 + B13 + \sqrt{2}\ B14 - \sqrt{2}\ B23 + B$
55 : $\frac{4}{9}\ (\sqrt{6}\ A13 - 2\sqrt{3}\ A14 - 3\sqrt{2}\ A34)\ (\sqrt{6}\ B13 - 2\sqrt{3}\ B14 - 3\sqrt{2}\ B34)$

```
56 :   4/9 (√6 A23 - 2 √3 A24 - 3 √2 A34) (√6 B23 - 2 √3 B24 - 3 √2 B34)
57 :   4 (3 A12 + √3 A13 + 2 A14 - √3 A24 - A34) (3 B12 + √3 B13 + 2 B14 - √3 B24 -
58 :   4 (3 A12 + 2 A13 + √3 A14 - √3 A23 + A34) (3 B12 + 2 B13 + √3 B14 - √3 B23 +
59 :   4 (√6 A12 + 2 A13 + A14 - √3 A24 - √2 A34) (√6 B12 + 2 B13 + B14 - √3 B24 -
60 :   4 (√6 A12 + A13 + 2 A14 - √3 A23 + √2 A34) (√6 B12 + B13 + 2 B14 - √3 B23 +
61 :   (√2 A12 - √2 A13 + 2 (A14 - A23 + √2 A34)) (√2 B12 - √2 B13 + 2 (B14 - B23 +
62 :   (√2 A12 + 2 A13 - √2 A14 - 2 (A24 + √2 A34)) (√2 B12 + 2 B13 - √2 B14 - 2 (B24
63 :   4/9 (√3 A12 - 2 √3 A13 + 3 (A14 - A23 + √3 A34)) (√3 B12 - 2 √3 B13 + 3 (B14 - B23
64 :   4/9 (√3 A12 + 3 A13 - 2 √3 A14 - 3 (A24 + √3 A34)) (√3 B12 + 3 B13 - 2 √3 B14 - 3 (B2
65 :   (2 √3 A12 + √2 A13 + √6 A14 - 2 √2 A23 + 2 A34) (2 √3 B12 + √2 B13 + √6 B14 - 2 √
66 :   (2 A12 + √6 A13 - √2 A14 - 2 √2 A24 - 2 √3 A34) (2 B12 + √6 B13 - √2 B14 - 2 √2 B
67 :   (2 A12 - √2 A13 + √6 A14 - 2 √2 A23 + 2 √3 A34) (2 B12 - √2 B13 + √6 B14 - 2 √2 B
68 :   (2 √3 A12 + √6 A13 + √2 A14 - 2 (√2 A24 + A34)) (2 √3 B12 + √6 B13 + √2 B14 - 2 (
69 :   4/9 (√6 A12 - √3 A13 + 2 √3 A14 - 3 A23 + 3 √2 A34) (√6 B12 - √3 B13 + 2 √3 B14 - 3 I
70 :   4/9 (√6 A12 + 2 √3 A13 - √3 A14 - 3 (A24 + √2 A34)) (√6 B12 + 2 √3 B13 - √3 B14 - 3
71 :   (3 √2 A12 + 2 √3 A13 + √2 A14 - 2 √3 A24 - 2 √2 A34) (3 √2 B12 + 2 √3 B13 + √2 B14 - 2
72 :   (3 √2 A12 + √2 A13 + 2 √3 A14 - 2 √3 A23 + 2 √2 A34) (3 √2 B12 + √2 B13 + 2 √3 B14 - 2
73 :   4/9 (2 √3 A12 + 3 √2 A13 - √3 A14 - 3 √2 A24 - 3 √3 A34) (2 √3 B12 + 3 √2 B13 - √3 B14 -
74 :   4/9 (2 √3 A12 - √3 A13 + 3 √2 A14 - 3 √2 A23 + 3 √3 A34) (2 √3 B12 - √3 B13 + 3 √2 B14 -
```

In[43]:=
```
gb = GroebnerBasis[eqs, Variables[eqs]]; // Timing
gb = simp[gb]; // Timing
```

Out[43]= {1.46458, Null}

Out[44]= {0.370314, Null}

In[45]:=
```
Length[gb]
```

Out[45]= 116

In[46]:=
```
(* Routine to extract equations that depend on a given variable  *)
emEqsWithVariable[eqs_, var_] := Module[
  {ii, res, eq},
  res = {};

  For[ii = 1, ii ≤ Length[eqs], ii++,

   eq = eqs[[ii]];
   If[Count[Variables[eq], var] > 0,
    res = Append[res, eq];
   ];

  ];

  res
 ]
```

- **Solve the Gröbner basis equations. We know that the Gröbner basis equations have the same solution as the original equations (in the complex domain). See for example D.Cox, J.Little, D.O'Shea, "Ideals, Varieties, and Algorithms".**

In[47]:=
```
show[Take[simp[emEqsWithVariable[gb, A34]], 6]]
```

Out[47]//MatrixForm=

$$\begin{pmatrix} 1 & : & A34^4\, B12 \\ 2 & : & A34^3\, B13 \\ 3 & : & A34^3\, B14 \\ 4 & : & A34^3\, B23 \\ 5 & : & A34^3\, B24 \\ 6 & : & A34^3\, B34 \end{pmatrix}$$

- **If A34 != 0, then B=0. Thus A34 = 0.**

In[48]:=
```
subs = {A34 → 0}
```

Out[48]= {A34 → 0}

In[49]:=
```
tmp = simp[emEqsWithVariable[gb //. subs, B12]];
tmp = Take[tmp, 13];
show[tmp]
```

Out[51]//MatrixForm=

$$\begin{pmatrix} 1 & : & A13^2\, B12 \\ 2 & : & A23^2\, B12 \\ 3 & : & A24\, B12^2 \\ 4 & : & -A12\, B12 \\ 5 & : & A13\, A14\, B12 \\ 6 & : & A13\, A23\, B12 \\ 7 & : & A14\, A24\, B12 \\ 8 & : & A23\, A24\, B12 \\ 9 & : & A14\, B12\, B13 \\ 10 & : & A23\, B12\, B13 \\ 11 & : & A24\, B12\, B14 \\ 12 & : & A24\, B12\, B23 \\ 13 & : & -A14^2\, B12 \end{pmatrix}$$

- **If B12 != 0, we have A=0. Thus B12 = 0**

In[52]:=
```
subs = Append[subs, B12 → 0]
```

Out[52]= {A34 → 0, B12 → 0}

In[53]:=
```
tmp = simp[emEqsWithVariable[gb //. subs, B13]];
tmp = Take[tmp, 7];
show[tmp]
```

Out[55]//MatrixForm=

$$\begin{pmatrix} 1 & : & A12\, B13 \\ 2 & : & A13\, B13 \\ 3 & : & A23^2\, B13 \\ 4 & : & A24^2\, B13 \\ 5 & : & A14\, B13^2 \\ 6 & : & A23\, B13^2 \\ 7 & : & A24\, B13^2 \end{pmatrix}$$

- **If B13 != 0, then A = 0. We can therefore assume that B13 = 0.**

In[56]:=
```
subs = Append[subs, B13 → 0]
```

Out[56]= {A34 → 0, B12 → 0, B13 → 0}

In[57]:=
```
tmp = simp[emEqsWithVariable[gb //. subs, B23]];
tmp = Take[tmp, 9];
show[tmp]
```

Out[59]//MatrixForm=

$$\begin{pmatrix} 1 & : & A12\,B23 \\ 2 & : & A13\,B23 \\ 3 & : & A23\,B23 \\ 4 & : & A14^2\,B23 \\ 5 & : & A14\,B23^2 \\ 6 & : & A14\,A24\,B23 \\ 7 & : & -A24^2\,B23 \\ 8 & : & -A24\,B23^2 \\ 9 & : & -A24\,B14\,B23 \end{pmatrix}$$

- **If B23 != 0, then A = 0. Thus B23 = 0**

In[60]:=
```
subs = Append[subs, B23 → 0]
```

Out[60]= {A34 → 0, B12 → 0, B13 → 0, B23 → 0}

In[61]:=
```
tmp = simp[emEqsWithVariable[gb //. subs, B14]];
show[tmp]
```

Out[62]//MatrixForm=

$$\begin{pmatrix} 1 & : & A12\,B14 \\ 2 & : & A13\,B14 \\ 3 & : & A14\,B14 \\ 4 & : & A23^2\,B14 \\ 5 & : & A23\,B14^2 \\ 6 & : & A23\,A24\,B14 \\ 7 & : & -A24^2\,B14 \\ 8 & : & -A24\,B14^2 \\ 9 & : & A23\,B14 + A13\,B24 \\ 10 & : & A24\,B14 + A14\,B24 \\ 11 & : & -A23\,B14 + A12\,B34 \end{pmatrix}$$

- **If B14 != 0, then A = 0. Thus B14 = 0.**

In[63]:=
```
subs = Append[subs, B14 → 0]
```

Out[63]= {A34 → 0, B12 → 0, B13 → 0, B23 → 0, B14 → 0}

In[64]:=
```
tmp = simp[emEqsWithVariable[gb //. subs, B34]];
show[tmp]
```

Out[65]//MatrixForm=

$$\begin{pmatrix} 1 & : & A12\,B34 \\ 2 & : & A13\,B34 \\ 3 & : & A14\,B34 \\ 4 & : & A23\,B34 \\ 5 & : & A24\,B34 \end{pmatrix}$$

- **If B34 != 0, then A = 0. Thus B34 = 0.**

In[66]:=
```
subs = Append[subs, B34 → 0]
```

Out[66]= {A34 → 0, B12 → 0, B13 → 0, B23 → 0, B14 → 0, B34 → 0}

In[67]:=
```
show[simp[gb //. subs]]
```

Out[67]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & A12\,B24 \\ 3 & : & A13\,B24 \\ 4 & : & A14\,B24 \\ 5 & : & A23\,B24 \\ 6 & : & -A24\,B24 \end{pmatrix}$$

- **If B24 != 0, then A = 0. Thus B24 = 0**

In[68]:=
```
subs = Append[subs, B24 → 0]
```

Out[68]=
```
{A34 → 0, B12 → 0, B13 → 0, B23 → 0, B14 → 0, B34 → 0, B24 → 0}
```

In[69]:=
```
Abivector /. subs // MatrixForm
Bbivector /. subs // MatrixForm
```

Out[69]//MatrixForm=

$$\begin{pmatrix} 0 & A12 & A13 & A14 \\ -A12 & 0 & A23 & A24 \\ -A13 & -A23 & 0 & 0 \\ -A14 & -A24 & 0 & 0 \end{pmatrix}$$

Out[70]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- **In conclusion: Either A = 0 or B = 0.**
- **Extra: Doublecheck result with Mathematica's internal Solve routine**

In[71]:=
```
sol = Solve[toEqs[gb], Variables[gb]]; // Timing
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[71]=
```
{120.811, Null}
```

In[72]:=
```
matrixNorm[mat_] := Simplify[1 / 2 Tr[mat.Transpose[mat]]]
```

In[73]:=
```
Table[
  {
   matrixNorm[Abivector] //. sol[[i]],
   matrixNorm[Bbivector] //. sol[[i]]
  }
  ,
  {i, 1, Length[sol]}
 ] // MatrixForm
```

Out[73]//MatrixForm=

$$\begin{pmatrix}
0 & B12^2 + B13^2 + B14^2 + B23^2 + B24^2 + B34^2 \\
A12^2 + A13^2 + A14^2 + A23^2 + A24^2 + A34^2 & 0 \\
0 & B12^2 + B13^2 + B14^2 + B24^2 + \frac{B12^6}{9\,B14^2\,B34^2} + B34^2 \\
0 & B13^2 + B23^2 + B24^2 + B34^2 \\
0 & B13^2 + B14^2 + B24^2 + B34^2 \\
0 & B13^2 + B14^2 + B23^2 + B24^2 \\
0 & \frac{4\,B12^2}{3} + B13^2 + \frac{B12^4}{3\,B14^2} + B14^2 + B24^2 \\
0 & \frac{4\,B12^2}{3} + B13^2 + \frac{B12^4}{3\,B14^2} + B14^2 + B24^2 \\
0 & B13^2 + B23^2 + 2\,B34^2 \\
0 & B13^2 + B23^2 + 2\,B34^2 \\
0 & B13^2 + B14^2 + 2\,B34^2 \\
0 & B13^2 + B14^2 + 2\,B34^2 \\
0 & \frac{5\,B12^2}{3} + B13^2 + \frac{B12^4}{3\,B14^2} + B14^2 \\
0 & \frac{5\,B12^2}{3} + B13^2 + \frac{B12^4}{3\,B14^2} + B14^2 \\
0 & \frac{5\,B12^2}{3} + B13^2 + \frac{B12^4}{3\,B14^2} + B14^2 \\
0 & \frac{5\,B12^2}{3} + B13^2 + \frac{B12^4}{3\,B14^2} + B14^2 \\
0 & B13^2 + 3\,B34^2 \\
0 & B13^2 + 3\,B34^2 \\
0 & B13^2 + 3\,B34^2 \\
0 & B13^2 + 3\,B34^2
\end{pmatrix}$$

0	$B13^2 + B14^2$
0	$3\ B12^2 + B13^2$
0	$3\ B12^2 + B13^2$
0	$3\ B12^2 + B13^2$
0	$3\ B12^2 + B13^2$
0	$3\ B12^2 + B13^2$
0	$3\ B12^2 + B13^2$
0	$3\ B12^2 + B13^2$
0	$3\ B12^2 + B13^2$
0	$4\ B34^2$
0	$4\ B34^2$
0	$4\ B34^2$
0	$4\ B34^2$
0	$4\ B34^2$
0	$4\ B34^2$
0	$4\ B34^2$
0	$4\ B34^2$
0	$B14^2$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$
0	$\frac{10\ B12^2}{3}$

- **We get the same result: Either A=0 or B=0.**