In[1]:=
```
SetDirectory["~/writing/WIP/KappaLib/"];
<< kappaLib.m
<< helper.m
```

KappaLib v1.1

Loading helper.m..

■ Define the medium

In[4]:=
```
  (* We assume kappa = C1 ast_g  + C2 Id where

     g = (1,-1,-1,-1) or g=diag(-1,1,1,1)

     However, since the Hodge operator is conformally
   invariant, we assume that g=(-1,1,1,1).
*)
g = DiagonalMatrix[{-1, 1, 1, 1}];
kappa = C1 emHodge[g] + C2 emIdentityKappa[];
```

In[6]:=
```
FullSimplify[emDet[kappa]]
```

Out[6]= $\left(C1^{2}+C2^{2}\right)^{3}$

■ Compute Fresnel polynomial

In[7]:=
```
xi = {x0, x1, x2, x3};

fresnel = FullSimplify[emKappaToFresnel[kappa, xi]]
```

Out[8]= $-C1^{3}\left(-x0^{2}+x1^{2}+x2^{2}+x3^{2}\right)^{2}$

■ Find non-zero points on the Fresnel surface of kappa.

Note: We do not need to trust the below code. We will verify the output in the below.

In[9]:=
```
coordValues = {1, Sqrt[2], Sqrt[3], 0};

LL = Length[coordValues];
```

In[11]:=
```
xiList = {};

For[i0 = 1, i0 ≤ LL, i0++,
 For[i1 = 1, i1 ≤ LL, i1++,
  For[i2 = 1, i2 ≤ LL, i2++,
   For[i3 = 1, i3 ≤ LL, i3++,

     frSub = {
       x0 → coordValues[[i0]],
       x1 → coordValues[[i1]],
       x2 → coordValues[[i2]],
       x3 → coordValues[[i3]]};
     frPoint = xi /. frSub;

     (* if point is non-zero and belongs
       to the Fresnel surface add it to list. *)
     If[Simplify[frPoint.frPoint] ≠ 0,
      If[Simplify[fresnel /. frSub] == 0,
        xiList = Append[xiList, frSub];
       ];
     ];

    ]
   ]
  ]
 ]
```

In[13]:=
```
xiList
```

Out[13]=
$$\{\{x0\to 1, x1\to 1, x2\to 0, x3\to 0\},\ \{x0\to 1, x1\to 0, x2\to 1, x3\to 0\},\ \{x0\to 1, x1\to 0, x2\to 0, x3\to 1\},\ \{x0\to\sqrt{2}, x1\to 1, x2\to 1, x3\to 0\},\ \{x0\to\sqrt{2}, x1\to 1, x2\to 0, x3\to 1\},\ \{x0\to\sqrt{2}, x1\to\sqrt{2}, x2\to 0, x3\to 0\},\ \{x0\to\sqrt{2}, x1\to 0, x2\to 1, x3\to 1\},\ \{x0\to\sqrt{2}, x1\to 0, x2\to\sqrt{2}, x3\to 0\},\ \{x0\to\sqrt{2}, x1\to 0, x2\to 0, x3\to\sqrt{2}\},\ \{x0\to\sqrt{3}, x1\to 1, x2\to 1, x3\to 1\},\ \{x0\to\sqrt{3}, x1\to 1, x2\to\sqrt{2}, x3\to 0\},\ \{x0\to\sqrt{3}, x1\to 1, x2\to 0, x3\to\sqrt{2}\},\ \{x0\to\sqrt{3}, x1\to\sqrt{2}, x2\to 1, x3\to 0\},\ \{x0\to\sqrt{3}, x1\to\sqrt{2}, x2\to 0, x3\to 1\},\ \{x0\to\sqrt{3}, x1\to\sqrt{3}, x2\to 0, x3\to 0\},\ \{x0\to\sqrt{3}, x1\to 0, x2\to 1, x3\to\sqrt{2}\},\ \{x0\to\sqrt{3}, x1\to 0, x2\to\sqrt{2}, x3\to 1\},\ \{x0\to\sqrt{3}, x1\to 0, x2\to\sqrt{3}, x3\to 0\},\ \{x0\to\sqrt{3}, x1\to 0, x2\to 0, x3\to\sqrt{3}\}\}$$

- **If**

  xi = xi_i dx^i,
  alpha = alpha_i dx^i,
  kappa = 1/8 kappa^ij_rs dx^r ∧ dx^s otimes d/dx^i ∧ dx^j

  then xi ∧ kappa (xi ∧ alpha) = 0 holds if and only if

  xi_i xi_a kappa^ab_lm epsilon^ilmk alpha_b = 0, k=0,...,3

  Compute the 4x4 matrix Lxi such that the above equation is equivalent with Lxi.alpha =0.

In[14]:=
```
Lxi = Table[
   Sum[
    xi[[i]] xi[[a]] emReadNormal[kappa, a, b, l, m] Signature[{i, l, m, k}],
    {a, 1, 4}, {i, 1, 4}, {l, 1, 4}, {m, 1, 4}]
   ,
   {k, 1, 4}, {b, 1, 4}
  ];
```

In[15]:=
```
(* matrix is symmetric *)
Union[Flatten[Lxi - Transpose[Lxi]]]
```

Out[15]=
```
{0}
```

In[16]:=
```
Lxi // MatrixForm
```

Out[16]//MatrixForm=

$$\begin{pmatrix} -2\,C1\,x1^2 - 2\,C1\,x2^2 - 2\,C1\,x3^2 & 2\,C1\,x0\,x1 & 2\,C1\,x0\,x2 & \\ 2\,C1\,x0\,x1 & -2\,C1\,x0^2 + 2\,C1\,x2^2 + 2\,C1\,x3^2 & -2\,C1\,x1\,x2 & \\ 2\,C1\,x0\,x2 & -2\,C1\,x1\,x2 & -2\,C1\,x0^2 + 2\,C1\,x1^2 + 2\,C1\,x3^2 & \\ 2\,C1\,x0\,x3 & -2\,C1\,x1\,x3 & -2\,C1\,x2\,x3 & -2\,C1\,x0 \end{pmatrix}$$

- **For each xi find 2 linearly independent alpha such that Lxi. alpha = 0 and alpha ∧ xi != 0.**

  Note: We do not need to trust the below code. We will verify that lists xiList and alphaList have the sought properties in the next step.

In[17]:=
```
alphaList = {};

For[ii = 1, ii ≤ Length[xiList], ii++,

  (*** Get a point on the Fresnel surface    ***)
  iSub = xiList[[ii]];
  frPoint = Simplify[xi /. iSub];

  (*** Find alpha such that g(frPoint,alpha)=0  ***)
  aa = {a0, a1, a2, a3};
  evecs = Eigenvectors[(Lxi /. iSub)];
  evals = Eigenvalues[(Lxi /. iSub)];

  alphas = {};

  For[jj = 1, jj ≤ 4, jj++,
   If[evals[[jj]] == 0,
     ej = evecs[[jj]];

     (* If eigenvector is not proportional to xi, add it to list of
      possible alpha:s *)

     If[Length[alphas] == 0,
      (* First alpha: if alpha is not proportional to xi, then it *)
      propToXi = Length[Solve[ej == Const frPoint, Const]];
      If[propToXi == 0,
       alphas = Append[alphas, ej];
      ];

      ,
      (* Subsequent alphas: add unless new alpha is in the
         span of old alphas and xi.
      *)

      consts =
       Table[ToExpression["Const" <> ToString[co]], {co, 1, Length[alphas] + 1}];

      spanVectors = Append[alphas, frPoint];
      inSpan = Length[Solve[ej == consts.spanVectors, consts]];
      If[inSpan == 0,
       alphas = Append[alphas, evecs[[jj]]];
      ];
     ];
    ];
  ];

  (* collect *)
  alphaList = Append[alphaList, alphas];

 ];
```

- **Points on Fresnel surface:**

In[19]:= **xiList**

Out[19]=

$$\{\{x0 \to 1, x1 \to 1, x2 \to 0, x3 \to 0\}, \{x0 \to 1, x1 \to 0, x2 \to 1, x3 \to 0\}, \{x0 \to 1, x1 \to 0, x2 \to 0, x3 \to 1\}, \{x0 \to \sqrt{2}, x1 \to 1, x2 \to 1, x3 \to 0\}, \{x0 \to \sqrt{2}, x1 \to 1, x2 \to 0, x3 \to 1\}, \{x0 \to \sqrt{2}, x1 \to \sqrt{2}, x2 \to 0, x3 \to 0\}, \{x0 \to \sqrt{2}, x1 \to 0, x2 \to 1, x3 \to 1\}, \{x0 \to \sqrt{2}, x1 \to 0, x2 \to \sqrt{2}, x3 \to 0\}, \{x0 \to \sqrt{2}, x1 \to 0, x2 \to 0, x3 \to \sqrt{2}\}, \{x0 \to \sqrt{3}, x1 \to 1, x2 \to 1, x3 \to 1\}, \{x0 \to \sqrt{3}, x1 \to 1, x2 \to \sqrt{2}, x3 \to 0\}, \{x0 \to \sqrt{3}, x1 \to 1, x2 \to 0, x3 \to \sqrt{2}\}, \{x0 \to \sqrt{3}, x1 \to \sqrt{2}, x2 \to 1, x3 \to 0\}, \{x0 \to \sqrt{3}, x1 \to \sqrt{2}, x2 \to 0, x3 \to 1\}, \{x0 \to \sqrt{3}, x1 \to \sqrt{3}, x2 \to 0, x3 \to 0\}, \{x0 \to \sqrt{3}, x1 \to 0, x2 \to 1, x3 \to \sqrt{2}\}, \{x0 \to \sqrt{3}, x1 \to 0, x2 \to \sqrt{2}, x3 \to 1\}, \{x0 \to \sqrt{3}, x1 \to 0, x2 \to \sqrt{3}, x3 \to 0\}, \{x0 \to \sqrt{3}, x1 \to 0, x2 \to 0, x3 \to \sqrt{3}\}\}$$

- **For each point, a list of three alphas:**

In[20]:= **alphaList // MatrixForm**

Out[20]//MatrixForm=

$$\begin{pmatrix}
\begin{pmatrix}0\\0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\1\\0\end{pmatrix} & \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\1\end{pmatrix} & \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\1\\0\end{pmatrix} \\
\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \\
\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \\
\begin{pmatrix}0\\0\\1\\0\end{pmatrix} & \begin{pmatrix}0\\1\\0\\0\end{pmatrix}
\end{pmatrix}$$

$$\left.\begin{matrix}
\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\end{matrix}\right)$$

- **Check that xiList and alphaList satisfy the sought properties:**

In[21]:=
```
vecNorm[v_] := Simplify[v.v]
```

In[22]:=
```
frVal = {};
alphasForXi = {};
C123Solutions = {};
isInKernel = {};

For[ii = 1, ii ≤ Length[xiList], ii++,
  xiSub = xiList[[ii]];
  alphas = alphaList[[ii]];

  (* Check that xi is on Fresnel surface *)
  (* should be zero *)
  frVal = Append[frVal, Simplify[fresnel /. xiSub]];

  (* check that there are two alpha:s associated to each xi. *)
  (* should be 2 *)
  alphasForXi = Append[alphasForXi, Length[alphas]];

  (* check that alpha:s and xi are linearly independent *)
  (* only solution should be C1=C2=C3=0 *)
  eqs = C1 alphas[[1]] + C2 alphas[[2]] + C3 (xi /. xiSub);
  C123Solutions = Append[C123Solutions, Solve[toEqs[eqs], {C1, C2, C3}]];

  (* check that xi /\ kappa(xi /\ alpha) = 0 *)
  (* should output (0) *)
  cond = {};
  For[k = 1, k ≤ 2, k++,
   LL = Lxi /. xiSub;
   isInKernel = Append[isInKernel, vecNorm[LL.alphas[[k]]]];

  ];

 ];
```

In[27]:=
```
Union[frVal] (* should be 0*)
Union[alphasForXi ](* should be 3 *)
Union[C123Solutions](* should be C1=C2=C3=0 *)
Union[isInKernel] (* should be 0:s *)
```

Out[27]= {0}

Out[28]= {2}

Out[29]= {{{C1 → 0, C2 → 0, C3 → 0}}}

Out[30]= {0}

In[31]:=
```
Length[xiList]
Dimensions[alphaList]
```

Out[31]= 19

Out[32]= {19, 2, 4}

■ Define A and B bivectors

In[33]:= Abivector = $\begin{pmatrix} 0 & A12 & A13 & A14 \\ -A12 & 0 & A23 & A24 \\ -A13 & -A23 & 0 & A34 \\ -A14 & -A24 & -A34 & 0 \end{pmatrix}$;

Bbivector = $\begin{pmatrix} 0 & B12 & B13 & B14 \\ -B12 & 0 & B23 & B24 \\ -B13 & -B23 & 0 & B34 \\ -B14 & -B24 & -B34 & 0 \end{pmatrix}$;

```
Abivector + Transpose[Abivector]
Bbivector + Transpose[Bbivector]
```

Out[35]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Out[36]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

In[37]:=
```
(* For bivector 'bivector' and 1-forms 'co1' and 'co2' compute

       bivector (co1 /\ co2 )

*)
contract[bivector_, co1_, co2_] := Module[{i, j},
  Sum[
   bivector[[i]][[j]] (co1[[i]] co2[[j]] - co1[[j]] co2[[i]]),
   {i, 1, 4}, {j, 1, 4}
  ]
 ]
```

In[38]:=
```
eqs = {};

For[ii = 1, ii ≤ Length[xiList], ii++,

  (*** Get a point on the Fresnel surface    ***)
  iSub = xiList[[ii]];
  xiVec = Simplify[xi /. iSub];

  (* Go through both alpha:s associated to this xi. *)
  For [k = 1, k ≤ 2, k++,

   alpha = alphaList[[ii]][[k]];

   ee = contract[Abivector, xiVec, alpha] contract[Bbivector, xiVec, alpha];
   ee = Simplify[ee];
   eqs = Append[eqs, ee];
  ];
 ];
```

Analysis of constraints

In[40]:=
```
Length[eqs]
```

Out[40]= 38

In[41]:=
```
eqs = simp[eqs]; // Timing
```

Out[41]= {0.892016, Null}

In[42]:=
```
show[eqs]
```

Out[42]//MatrixForm=

1	:	$4\,(A12 - A23)\,(B12 - B23)$
2	:	$8\,(A12 - A23)\,(B12 - B23)$
3	:	$4\,(A13 + A23)\,(B13 + B23)$
4	:	$8\,(A13 + A23)\,(B13 + B23)$
5	:	$4\,(A12 - A24)\,(B12 - B24)$
6	:	$8\,(A12 - A24)\,(B12 - B24)$
7	:	$4\,(A14 + A24)\,(B14 + B24)$
8	:	$8\,(A14 + A24)\,(B14 + B24)$
9	:	$4\,(A13 - A34)\,(B13 - B34)$
10	:	$8\,(A13 - A34)\,(B13 - B34)$
11	:	$4\,(A14 + A34)\,(B14 + B34)$
12	:	$8\,(A14 + A34)\,(B14 + B34)$
13	:	$12\,(A12 - A23)\,(B12 - B23)$
14	:	$12\,(A13 + A23)\,(B13 + B23)$
15	:	$12\,(A12 - A24)\,(B12 - B24)$
16	:	$12\,(A14 + A24)\,(B14 + B24)$
17	:	$12\,(A13 - A34)\,(B13 - B34)$
18	:	$12\,(A14 + A34)\,(B14 + B34)$
19	:	$4\left(\sqrt{2}\ A12 - A23 - A24\right)\left(\sqrt{2}\ B12 - B23 - B24\right)$
20	:	$4\left(\sqrt{2}\ A13 + A23 - A34\right)\left(\sqrt{2}\ B13 + B23 - B34\right)$
21	:	$4\left(\sqrt{2}\ A14 + A24 + A34\right)\left(\sqrt{2}\ B14 + B24 + B34\right)$
22	:	$4\left(\sqrt{3}\ A12 - \sqrt{2}\ A23 - A24\right)\left(\sqrt{3}\ B12 - \sqrt{2}\ B23 - B24\right)$
23	:	$4\left(\sqrt{3}\ A12 - A23 - \sqrt{2}\ A24\right)\left(\sqrt{3}\ B12 - B23 - \sqrt{2}\ B24\right)$
24	:	$4\left(\sqrt{3}\ A13 + \sqrt{2}\ A23 - A34\right)\left(\sqrt{3}\ B13 + \sqrt{2}\ B23 - B34\right)$
25	:	$4\left(\sqrt{3}\ A14 + \sqrt{2}\ A24 + A34\right)\left(\sqrt{3}\ B14 + \sqrt{2}\ B24 + B34\right)$
26	:	$4\left(\sqrt{3}\ A13 + A23 - \sqrt{2}\ A34\right)\left(\sqrt{3}\ B13 + B23 - \sqrt{2}\ B34\right)$
27	:	$4\left(\sqrt{3}\ A14 + A24 + \sqrt{2}\ A34\right)\left(\sqrt{3}\ B14 + B24 + \sqrt{2}\ B34\right)$
28	:	$\left(\sqrt{2}\ A12 - \sqrt{2}\ A13 - 2\ A23\right)\left(\sqrt{2}\ B12 - \sqrt{2}\ B13 - 2\ B23\right)$
29	:	$\left(\sqrt{2}\ A12 - \sqrt{2}\ A14 - 2\ A24\right)\left(\sqrt{2}\ B12 - \sqrt{2}\ B14 - 2\ B24\right)$
30	:	$\left(\sqrt{2}\ A13 - \sqrt{2}\ A14 - 2\ A34\right)\left(\sqrt{2}\ B13 - \sqrt{2}\ B14 - 2\ B34\right)$
31	:	$\frac{4}{9}\left(\sqrt{6}\ A12 - \sqrt{3}\ A13 - 3\ A23\right)\left(\sqrt{6}\ B12 - \sqrt{3}\ B13 - 3\ B23\right)$
32	:	$\frac{4}{9}\left(\sqrt{6}\ A12 - \sqrt{3}\ A14 - 3\ A24\right)\left(\sqrt{6}\ B12 - \sqrt{3}\ B14 - 3\ B24\right)$
33	:	$\frac{4}{9}\left(\sqrt{6}\ A13 - \sqrt{3}\ A14 - 3\ A34\right)\left(\sqrt{6}\ B13 - \sqrt{3}\ B14 - 3\ B34\right)$
34	:	$\frac{4}{9}\left(\sqrt{6}\ A12 - 2\sqrt{3}\ A13 - 3\sqrt{2}\ A23\right)\left(\sqrt{6}\ B12 - 2\sqrt{3}\ B13 - 3\sqrt{2}\ B23\right)$
35	:	$\frac{4}{9}\left(\sqrt{6}\ A12 - 2\sqrt{3}\ A14 - 3\sqrt{2}\ A24\right)\left(\sqrt{6}\ B12 - 2\sqrt{3}\ B14 - 3\sqrt{2}\ B24\right)$
36	:	$\frac{4}{9}\left(\sqrt{6}\ A13 - 2\sqrt{3}\ A14 - 3\sqrt{2}\ A34\right)\left(\sqrt{6}\ B13 - 2\sqrt{3}\ B14 - 3\sqrt{2}\ B34\right)$
37	:	$\frac{4}{9}\left(\sqrt{3}\ A12 - 2\sqrt{3}\ A13 + \sqrt{3}\ A14 - 3\ A23 + 3\ A34\right)\left(\sqrt{3}\ B12 - 2\sqrt{3}\ B13 + \sqrt{3}\ B14 - 3\ B23 + 3\ B3\right.$
38	:	$\frac{4}{9}\left(\sqrt{3}\ A12 + \sqrt{3}\ A13 - 2\sqrt{3}\ A14 - 3\ (A24 + A34)\right)\left(\sqrt{3}\ B12 + \sqrt{3}\ B13 - 2\sqrt{3}\ B14 - 3\ (B24 + B34\right.$

In[43]:=
```
gb = GroebnerBasis[eqs, Variables[eqs]]; // Timing
gb = simp[gb]; // Timing
```

Out[43]= {1.15503, Null}

Out[44]= {0.374502, Null}

In[45]:=
```
Length[gb]
```

Out[45]= 116

In[46]:=
```
(* Routine to extract equations that depend on a given variable  *)
emEqsWithVariable[eqs_, var_] := Module[
  {ii, res, eq},
  res = {};

  For[ii = 1, ii ≤ Length[eqs], ii++,

   eq = eqs[[ii]];
   If[Count[Variables[eq], var] > 0,
    res = Append[res, eq];
   ];

  ];

  res
 ]
```

- **Solve the Gröbner basis equations. We know that the Gröbner basis equations have the same solution as the original equations (in the complex domain). See for example D.Cox, J.Little, D.O'Shea, "Ideals, Varieties, and Algorithms".**

In[47]:=
```
show[Take[simp[emEqsWithVariable[gb, A34]], 6]]
```

Out[47]//MatrixForm=

$$\begin{pmatrix} 1 & : & A34^4\, B12 \\ 2 & : & A34^3\, B13 \\ 3 & : & A34^3\, B14 \\ 4 & : & A34^3\, B23 \\ 5 & : & A34^3\, B24 \\ 6 & : & A34^3\, B34 \end{pmatrix}$$

- **If A34 != 0, then B=0. Thus A34 = 0.**

In[48]:=
```
subs = {A34 → 0}
```

Out[48]= {A34 → 0}

In[49]:=
```
tmp = simp[emEqsWithVariable[gb //. subs, B12]];
tmp = Take[tmp, 7];
show[tmp]
```

Out[51]//MatrixForm=

$$\begin{pmatrix} 1 & : & A13^2\, B12 \\ 2 & : & A23^2\, B12 \\ 3 & : & A14\, B12^2 \\ 4 & : & A23\, B12^2 \\ 5 & : & A24\, B12^2 \\ 6 & : & -A12\, B12 \\ 7 & : & A13\, A14\, B12 \end{pmatrix}$$

- **If B12 != 0, we have A=0. Thus B12 = 0**

In[52]:=
```
subs = Append[subs, B12 → 0]
```

Out[52]= {A34 → 0, B12 → 0}

In[53]:=
```
tmp = simp[emEqsWithVariable[gb //. subs, B13]];
tmp = Take[tmp, 7];
show[tmp]
```

Out[55]//MatrixForm=

1	:	$A12\ B13$
2	:	$A23^2\ B13$
3	:	$A24^2\ B13$
4	:	$A14\ B13^2$
5	:	$A23\ B13^2$
6	:	$A24\ B13^2$
7	:	$-A13\ B13$

- **If B13 != 0, then A = 0. We can therefore assume that B13 = 0.**

In[56]:=
```
subs = Append[subs, B13 → 0]
```

Out[56]= {A34 → 0, B12 → 0, B13 → 0}

In[57]:=
```
tmp = simp[emEqsWithVariable[gb //. subs, B23]];
tmp = Take[tmp, 9];
show[tmp]
```

Out[59]//MatrixForm=

1	:	$A12\ B23$
2	:	$A13\ B23$
3	:	$A14^2\ B23$
4	:	$A24^2\ B23$
5	:	$-A23\ B23$
6	:	$A24\ B14\ B23$
7	:	$-A14\ B23^2$
8	:	$-A24\ B23^2$
9	:	$-A14\ A24\ B23$

- **If B23 != 0, then A = 0. Thus B23 = 0**

In[60]:=
```
subs = Append[subs, B23 → 0]
```

Out[60]= {A34 → 0, B12 → 0, B13 → 0, B23 → 0}

In[61]:=
```
tmp = simp[emEqsWithVariable[gb //. subs, B14]];
show[tmp]
```

Out[62]//MatrixForm=

1	:	$A12\ B14$
2	:	$A13\ B14$
3	:	$A23^2\ B14$
4	:	$A24^2\ B14$
5	:	$-A14\ B14$
6	:	$A23\ A24\ B14$
7	:	$-A23\ B14^2$
8	:	$-A24\ B14^2$
9	:	$A23\ B14 + A13\ B24$
10	:	$A24\ B14 + A14\ B24$
11	:	$-A23\ B14 + A12\ B34$

- **If B14 != 0, then A = 0. Thus B14 = 0.**

In[63]:=
```
subs = Append[subs, B14 → 0]
```

Out[63]= {A34 → 0, B12 → 0, B13 → 0, B23 → 0, B14 → 0}

In[64]:=
```
tmp = simp[emEqsWithVariable[gb //. subs, B34]];
show[tmp]
```

Out[65]//MatrixForm=

1	:	A12 B34
2	:	A13 B34
3	:	A14 B34
4	:	A23 B34
5	:	A24 B34

- **If B34 != 0, then A = 0. Thus B34 = 0.**

In[66]:=
```
subs = Append[subs, B34 → 0]
```

Out[66]= {A34 → 0, B12 → 0, B13 → 0, B23 → 0, B14 → 0, B34 → 0}

In[67]:=
```
show[simp[gb //. subs]]
```

Out[67]//MatrixForm=

1	:	0
2	:	A12 B24
3	:	A13 B24
4	:	A14 B24
5	:	A23 B24
6	:	-A24 B24

- **If B24 != 0, then A = 0. Thus B24 = 0**

In[68]:=
```
subs = Append[subs, B24 → 0]
```

Out[68]= {A34 → 0, B12 → 0, B13 → 0, B23 → 0, B14 → 0, B34 → 0, B24 → 0}

In[69]:=
```
Abivector /. subs // MatrixForm
Bbivector /. subs // MatrixForm
```

Out[69]//MatrixForm=

0	A12	A13	A14
-A12	0	A23	A24
-A13	-A23	0	0
-A14	-A24	0	0

Out[70]//MatrixForm=

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

- **In conclusion: Either A = 0 or B = 0.**
- **Extra: Doublecheck result with Mathematica's internal Solve routine**

In[71]:=
```
sol = Solve[toEqs[eqs], Variables[eqs]]; // Timing
```

Solve::svars : Equations may not give solutions for all "solve" variables. ≫

Out[71]= {45.2502, Null}

In[72]:=
```
matrixNorm[mat_] := Simplify[1 / 2 Tr[mat.Transpose[mat]]]
```

In[73]:=
```
Table[
  {
   matrixNorm[Abivector] //. sol[[i]],
   matrixNorm[Bbivector] //. sol[[i]]
  }
  ,
  {i, 1, Length[sol]}
 ] // MatrixForm
```

Out[73]//MatrixForm=

0	$B12^2 + B13^2 + B14^2 + B23^2 + B24^2 + B34^2$
$A12^2 + A13^2 + A14^2 + A23^2 + A24^2 + A34^2$	0
0	$2\ B12^2 + B13^2 + B14^2 + B23^2 + B34^2$
0	$B12^2 + B13^2 + 2\ B14^2 + B23^2 + B34^2$

0	$2 B12^2 + B13^2 + B14^2 + B24^2 + B34^2$
0	$B12^2 + 2 B13^2 + B14^2 + B24^2 + B34^2$
0	$B12^2 + B14^2 + B23^2 + B24^2 + 2 B34^2$
0	$B12^2 + B13^2 + B23^2 + B24^2 + 2 B34^2$
0	$2 B12^2 + B14^2 + B23^2 + 2 B34^2$
0	$B12^2 + 2 B14^2 + B23^2 + 2 B34^2$
0	$3 B12^2 + B13^2 + B23^2 + B34^2$
0	$2 B12^2 + B13^2 + B23^2 + 2 B34^2$
0	$B12^2 + B13^2 + B23^2 + 3 B34^2$
0	$3 B12^2 + B13^2 + B14^2 + B34^2$
0	$2 B12^2 + B13^2 + 2 B14^2 + B34^2$
0	$2 B12^2 + 2 B13^2 + B14^2 + B34^2$
0	$B12^2 + 2 B13^2 + 2 B14^2 + B34^2$
0	$3 B12^2 + B14^2 + B24^2 + B34^2$
0	$2 B12^2 + B14^2 + B24^2 + 2 B34^2$
0	$B12^2 + B14^2 + B24^2 + 3 B34^2$
0	$2 B12^2 + B13^2 + B24^2 + 2 B34^2$
0	$B12^2 + 2 B13^2 + B24^2 + 2 B34^2$
0	$B12^2 + B23^2 + B24^2 + 3 B34^2$
0	$B13^2 + B23^2 + 4 B34^2$
0	$3 B12^2 + B23^2 + 2 B34^2$
0	$2 B12^2 + B23^2 + 3 B34^2$
0	$B12^2 + B23^2 + 4 B34^2$
0	$B14^2 + B24^2 + 4 B34^2$
0	$3 B12^2 + B14^2 + 2 B34^2$
0	$2 B12^2 + 2 B14^2 + 2 B34^2$
0	$2 B12^2 + B14^2 + 3 B34^2$
0	$B12^2 + 2 B14^2 + 3 B34^2$
0	$3 B12^2 + B13^2 + 2 B34^2$
0	$2 B12^2 + B13^2 + 3 B34^2$
0	$2 B12^2 + 2 B13^2 + 2 B34^2$
0	$B12^2 + 2 B13^2 + 3 B34^2$
0	$3 B12^2 + B24^2 + 2 B34^2$
0	$2 B12^2 + B24^2 + 3 B34^2$
0	$B12^2 + B24^2 + 4 B34^2$
0	$4 B12^2 + B14^2 + B34^2$
0	$3 B12^2 + 2 B14^2 + B34^2$
0	$4 B12^2 + B13^2 + B34^2$
0	$3 B12^2 + 2 B13^2 + B34^2$
0	$B23^2 + 5 B34^2$
0	$B24^2 + 5 B34^2$
0	$3 B12^2 + 3 B34^2$
0	$2 B12^2 + 4 B34^2$
0	$2 B12^2 + 4 B34^2$
0	$B12^2 + 5 B34^2$
0	$2 B14^2 + 4 B34^2$
0	$B14^2 + 5 B34^2$
0	$2 B13^2 + 4 B34^2$
0	$B13^2 + 5 B34^2$
0	$4 B12^2 + 2 B34^2$
0	$3 B12^2 + 3 B34^2$
0	$5 B12^2 + B34^2$

0	$4\ B12^2 + 2\ B34^2$
0	$3\ B12^2 + 3\ B34^2$
0	$6\ B34^2$
0	$6\ B34^2$
0	$6\ B34^2$
0	$6\ B34^2$
0	$6\ B34^2$
0	$6\ B34^2$

- **We get the same result: Either A=0 or B=0.**