```
In[1]:= SetDirectory["~/KappaLib/"];
        << kappaLib-1.1.m
        << helper.m

        KappaLib v1.1

        Loading helper.m..
```

■ **Metaclass I:**

```
In[4]:= vars = {x0, x1, x2, x3};
```

$$\texttt{kappa = emMatrixToKappa}\left[\begin{pmatrix} a1 & 0 & 0 & -b1 & 0 & 0 \\ 0 & a2 & 0 & 0 & -b2 & 0 \\ 0 & 0 & a3 & 0 & 0 & -b3 \\ b1 & 0 & 0 & a1 & 0 & 0 \\ 0 & b2 & 0 & 0 & a2 & 0 \\ 0 & 0 & b3 & 0 & 0 & a3 \end{pmatrix}\right];$$

```
        fr = emKappaToFresnel[kappa, vars];

In[7]:= FullSimplify[emDet[kappa]]
```

Out[7]= $\left(a1^2 + b1^2\right) \left(a2^2 + b2^2\right) \left(a3^2 + b3^2\right)$

■ **Write Fresnel polynomial using D0, D1, D2, D3.**

In[8]:= subSym = {

$$D1 \;\text{->}\; \frac{\left((a2 - a3)^2 + b2^2 + b3^2\right)}{b2\, b3},$$

$$D3 \;\text{->}\; \frac{\left((a1 - a2)^2 + b1^2 + b2^2\right)}{b1\, b2},$$

$$D2 \;\text{->}\; \frac{\left((a1 - a3)^2 + b1^2 + b3^2\right)}{b1\, b3},$$

$$D0 \to 2\left(\left(a1\,(b2\hat{\ }2 - b3\hat{\ }2) + a2\left(b3^2 - b1^2\right) + a3\,(b1\hat{\ }2 - b2\hat{\ }2)\right) - (a1 - a2)\,(a1 - a3)\,(a2 - a3)\right) / (b1\, b2\, b3)$$

};

In[9]:= $\texttt{frSym} = x0^4 + x1^4 + x2^4 + x3^4 - D0\, x0\, x1\, x2\, x3 + D1\left(x2^2\, x3^2 - x0^2\, x1^2\right) + D2\left(x1^2\, x3^2 - x0^2\, x2^2\right) + D3\left(x1^2\, x2^2 - x0^2\, x3^2\right)$

Out[9]= $x0^4 + x1^4 + x2^4 - D0\, x0\, x1\, x2\, x3 + x3^4 + D3\left(x1^2\, x2^2 - x0^2\, x3^2\right) + D2\left(-x0^2\, x2^2 + x1^2\, x3^2\right) + D1\left(-x0^2\, x1^2 + x2^2\, x3^2\right)$

```
In[10]:= Simplify[fr - b1 b2 b3 frSym /. subSym]

Out[10]= 0
```

■ **Verify: If D1 = D2 = D3 = 2 then the Fresnel surface is a single light cone**

```
In[11]:= FullSimplify[frSym /. {D1 → 2, D2 → 2, D3 → 2, D0 → 0}]
```

Out[11]= $\left(-x0^2 + x1^2 + x2^2 + x3^2\right)^2$

■ **Implicit equation for D0**

In[12]:= $\texttt{Simplify}\left[D0^2 + 4\left(-4 + D1^2 + D2^2 + D3^2 - D1\, D2\, D3\right) \;\text{/.}\; \texttt{subSym}\right]$

Out[12]= 0

■ We assume that the Fresnel polynomial factorises:

In[13]:=
```
A = Table[ToExpression["A" <> ToString[Min[{i, j}]] <> ToString[Max[{i, j}]]],
    {i, 0, 3}, {j, 0, 3}];
B = Table[ToExpression["B" <> ToString[Min[{i, j}]] <> ToString[Max[{i, j}]]],
    {i, 0, 3}, {j, 0, 3}];
A // MatrixForm
B // MatrixForm
factorised = (vars.A.vars) (vars.B.vars);
```

Out[15]//MatrixForm=

$$\begin{pmatrix} A00 & A01 & A02 & A03 \\ A01 & A11 & A12 & A13 \\ A02 & A12 & A22 & A23 \\ A03 & A13 & A23 & A33 \end{pmatrix}$$

Out[16]//MatrixForm=

$$\begin{pmatrix} B00 & B01 & B02 & B03 \\ B01 & B11 & B12 & B13 \\ B02 & B12 & B22 & B23 \\ B03 & B13 & B23 & B33 \end{pmatrix}$$

In[18]:=
```
cons = Union[Flatten[CoefficientList[factorised - frSym, vars]]];
cons = simp[cons];
show[cons]
```

Out[20]//MatrixForm=

1	:	0
2	:	-1 + A00 B00
3	:	-1 + A11 B11
4	:	-1 + A22 B22
5	:	-1 + A33 B33
6	:	2 (A01 B00 + A00 B01)
7	:	2 (A02 B00 + A00 B02)
8	:	2 (A03 B00 + A00 B03)
9	:	2 (A11 B01 + A01 B11)
10	:	2 (A12 B11 + A11 B12)
11	:	2 (A13 B11 + A11 B13)
12	:	2 (A22 B02 + A02 B22)
13	:	2 (A22 B12 + A12 B22)
14	:	2 (A23 B22 + A22 B23)
15	:	2 (A33 B03 + A03 B33)
16	:	2 (A33 B13 + A13 B33)
17	:	2 (A33 B23 + A23 B33)
18	:	A33 B22 + 4 A23 B23 + A22 B33 - D1
19	:	A11 B00 + 4 A01 B01 + A00 B11 + D1
20	:	A33 B11 + 4 A13 B13 + A11 B33 - D2
21	:	A22 B00 + 4 A02 B02 + A00 B22 + D2
22	:	A22 B11 + 4 A12 B12 + A11 B22 - D3
23	:	A33 B00 + 4 A03 B03 + A00 B33 + D3
24	:	2 (A12 B00 + 2 A02 B01 + 2 A01 B02 + A00 B12)
25	:	2 (2 A12 B01 + A11 B02 + A02 B11 + 2 A01 B12)
26	:	2 (A13 B00 + 2 A03 B01 + 2 A01 B03 + A00 B13)
27	:	2 (2 A13 B01 + A11 B03 + A03 B11 + 2 A01 B13)
28	:	2 (A22 B01 + 2 A12 B02 + 2 A02 B12 + A01 B22)
29	:	2 (A23 B00 + 2 A03 B02 + 2 A02 B03 + A00 B23)
30	:	2 (2 A23 B02 + A22 B03 + A03 B22 + 2 A02 B23)
31	:	2 (A23 B11 + 2 A13 B12 + 2 A12 B13 + A11 B23)
32	:	2 (2 A23 B12 + A22 B13 + A13 B22 + 2 A12 B23)
33	:	2 (A33 B01 + 2 A13 B03 + 2 A03 B13 + A01 B33)
34	:	2 (A33 B02 + 2 A23 B03 + 2 A03 B23 + A02 B33)
35	:	2 (A33 B12 + 2 A23 B13 + 2 A13 B23 + A12 B33)
36	:	4 (A23 B01 + A13 B02 + A12 B03 + A03 B12 + A02 B13 + A01 B23) + D0

■ Equation (2): By rescaling we may assume that A00 = 1.

In[21]:=
```
sub = {A00 → 1, B00 → 1};
cons = simp[cons //. sub];
show[cons]
```

Out[23]//MatrixForm=
```
1  :  0
2  :  - 1 + A11 B11
3  :  - 1 + A22 B22
4  :  - 1 + A33 B33
5  :  2 (A01 + B01)
6  :  2 (A02 + B02)
7  :  2 (A03 + B03)
8  :  2 (A11 B01 + A01 B11)
9  :  2 (A12 B11 + A11 B12)
10 :  2 (A13 B11 + A11 B13)
11 :  2 (A22 B02 + A02 B22)
12 :  2 (A22 B12 + A12 B22)
13 :  2 (A23 B22 + A22 B23)
14 :  2 (A33 B03 + A03 B33)
15 :  2 (A33 B13 + A13 B33)
16 :  2 (A33 B23 + A23 B33)
17 :  A11 + 4 A01 B01 + B11 + D1
18 :  A22 + 4 A02 B02 + B22 + D2
19 :  A33 + 4 A03 B03 + B33 + D3
20 :  A33 B22 + 4 A23 B23 + A22 B33 - D1
21 :  A33 B11 + 4 A13 B13 + A11 B33 - D2
22 :  A22 B11 + 4 A12 B12 + A11 B22 - D3
23 :  2 (A12 + 2 A02 B01 + 2 A01 B02 + B12)
24 :  2 (A13 + 2 A03 B01 + 2 A01 B03 + B13)
25 :  2 (A23 + 2 A03 B02 + 2 A02 B03 + B23)
26 :  2 (2 A12 B01 + A11 B02 + A02 B11 + 2 A01 B12)
27 :  2 (2 A13 B01 + A11 B03 + A03 B11 + 2 A01 B13)
28 :  2 (A22 B01 + 2 A12 B02 + 2 A02 B12 + A01 B22)
29 :  2 (2 A23 B02 + A22 B03 + A03 B22 + 2 A02 B23)
30 :  2 (A23 B11 + 2 A13 B12 + 2 A12 B13 + A11 B23)
31 :  2 (2 A23 B12 + A22 B13 + A13 B22 + 2 A12 B23)
32 :  2 (A33 B01 + 2 A13 B03 + 2 A03 B13 + A01 B33)
33 :  2 (A33 B02 + 2 A23 B03 + 2 A03 B23 + A02 B33)
34 :  2 (A33 B12 + 2 A23 B13 + 2 A13 B23 + A12 B33)
35 :  4 (A23 B01 + A13 B02 + A12 B03 + A03 B12 + A02 B13 + A01 B23) + D0
```

In[24]:=
```
tmp = Take[cons, {5, 7}]
```

Out[24]=
```
{2 (A01 + B01), 2 (A02 + B02), 2 (A03 + B03)}
```

In[25]:=
```
Solve[toEqs[tmp], {B01, B02, B03}]
```

Out[25]=
```
{{B01 → - A01, B02 → - A02, B03 → - A03}}
```

In[26]:=
```
sub = Join[sub, %[[1]]]
```

Out[26]=
```
{A00 → 1, B00 → 1, B01 → - A01, B02 → - A02, B03 → - A03}
```

In[27]:=
```
tmp = Join[Take[cons, {17, 19}], Take[cons, {23, 25}]];
tmp // MatrixForm
```

Out[28]//MatrixForm=
```
A11 + 4 A01 B01 + B11 + D1
A22 + 4 A02 B02 + B22 + D2
A33 + 4 A03 B03 + B33 + D3
2 (A12 + 2 A02 B01 + 2 A01 B02 + B12)
2 (A13 + 2 A03 B01 + 2 A01 B03 + B13)
2 (A23 + 2 A03 B02 + 2 A02 B03 + B23)
```

In[29]:=
```
Solve[toEqs[tmp], {B33, B22, B11, B12, B13, B23}]
```

Out[29]=
```
{{B33 → -A33 - 4 A03 B03 - D3, B22 → -A22 - 4 A02 B02 - D2,
  B11 → -A11 - 4 A01 B01 - D1, B12 → -A12 - 2 A02 B01 - 2 A01 B02,
  B13 → -A13 - 2 A03 B01 - 2 A01 B03, B23 → -A23 - 2 A03 B02 - 2 A02 B03}}
```

In[30]:=
```
sub = Join[sub, %[[1]]]
```

Out[30]=
```
{A00 → 1, B00 → 1, B01 → -A01, B02 → -A02, B03 → -A03, B33 → -A33 - 4 A03 B03 - D3,
 B22 → -A22 - 4 A02 B02 - D2, B11 → -A11 - 4 A01 B01 - D1, B12 → -A12 - 2 A02 B01 - 2 A01 B02,
 B13 → -A13 - 2 A03 B01 - 2 A01 B03, B23 → -A23 - 2 A03 B02 - 2 A02 B03}
```

In[31]:=
```
cons = simp[cons //. sub];
show[cons]
```

Out[32]//MatrixForm=
```
1  :  0
2  :  8 A01^3 - 2 A01 (2 A11 + D1)
3  :  8 A02^3 - 2 A02 (2 A22 + D2)
4  :  8 A03^3 - 2 A03 (2 A33 + D3)
5  :  -1 + 4 A01^2 A11 - A11 (A11 + D1)
6  :  -1 + 4 A02^2 A22 - A22 (A22 + D2)
7  :  -1 + 4 A03^2 A33 - A33 (A33 + D3)
8  :  -8 A02 A12 + A01 (24 A02^2 - 2 (2 A22 + D2))
9  :  -8 A03 A13 + A01 (24 A03^2 - 2 (2 A33 + D3))
10 :  -8 A03 A23 + A02 (24 A03^2 - 2 (2 A33 + D3))
11 :  8 A01 A02 A11 + 8 A01^2 A12 - 2 A12 (2 A11 + D1)
12 :  8 A01 A03 A11 + 8 A01^2 A13 - 2 A13 (2 A11 + D1)
13 :  8 A02^2 A12 + 8 A01 A02 A22 - 2 A12 (2 A22 + D2)
14 :  8 A02 A03 A22 + 8 A02^2 A23 - 2 A23 (2 A22 + D2)
15 :  8 A03^2 A13 + 8 A01 A03 A33 - 2 A13 (2 A33 + D3)
16 :  8 A03^2 A23 + 8 A02 A03 A33 - 2 A23 (2 A33 + D3)
17 :  -2 (-12 A01^2 A02 + 4 A01 A12 + A02 (2 A11 + D1))
18 :  -2 (-12 A01^2 A03 + 4 A01 A13 + A03 (2 A11 + D1))
19 :  -2 (-12 A02^2 A03 + 4 A02 A23 + A03 (2 A22 + D2))
20 :  48 A01 A02 A03 - 8 A03 A12 - 8 A02 A13 - 8 A01 A23 + D0
21 :  4 A02^2 A11 + 16 A01 A02 A12 - 4 A12^2 + 4 A01^2 A22 - A22 (2 A11 + D1) - A11 D2 - D3
22 :  4 A03^2 A11 + 16 A01 A03 A13 - 4 A13^2 + 4 A01^2 A33 - A33 (2 A11 + D1) - D2 - A11 D3
23 :  4 A03^2 A22 + 16 A02 A03 A23 - 4 A23^2 + 4 A02^2 A33 - D1 - A33 (2 A22 + D2) - A22 D3
24 :  2 (4 A02^2 A13 + 4 A01 A03 A22 - 4 A12 A23 + 8 A02 (A03 A12 + A01 A23) - A13 (2 A22 + D2))
25 :  2 (4 A03^2 A12 - 4 A13 A23 + 8 A03 (A02 A13 + A01 A23) + 4 A01 A02 A33 - A12 (2 A33 + D3))
26 :  2 (8 A01 A03 A12 - 4 A12 A13 + 4 A02 (A03 A11 + 2 A01 A13) + 4 A01^2 A23 - A23 (2 A11 + D1))
```

In[33]:=
```
Solve[cons[[20]] == 0, D0]
```

Out[33]=
```
{{D0 → -8 (6 A01 A02 A03 - A03 A12 - A02 A13 - A01 A23)}}
```

In[34]:=
```
sub = Join[sub, %[[1]]]
```

Out[34]=
```
{A00 → 1, B00 → 1, B01 → -A01, B02 → -A02, B03 → -A03,
 B33 → -A33 - 4 A03 B03 - D3, B22 → -A22 - 4 A02 B02 - D2, B11 → -A11 - 4 A01 B01 - D1,
 B12 → -A12 - 2 A02 B01 - 2 A01 B02, B13 → -A13 - 2 A03 B01 - 2 A01 B03,
 B23 → -A23 - 2 A03 B02 - 2 A02 B03, D0 → -8 (6 A01 A02 A03 - A03 A12 - A02 A13 - A01 A23)}
```

In[35]:=
```
cons = simp[cons //. sub];
show[FullSimplify[cons]]
```

Out[36]//MatrixForm=

1	:	0
2	:	$8\,A01^3 - 2\,A01\,(2\,A11 + D1)$
3	:	$8\,A02^3 - 2\,A02\,(2\,A22 + D2)$
4	:	$8\,A03^3 - 2\,A03\,(2\,A33 + D3)$
5	:	$-1 + 4\,A01^2\,A11 - A11\,(A11 + D1)$
6	:	$-1 + 4\,A02^2\,A22 - A22\,(A22 + D2)$
7	:	$-1 + 4\,A03^2\,A33 - A33\,(A33 + D3)$
8	:	$-8\,A02\,A12 + A01\,\left(24\,A02^2 - 2\,(2\,A22 + D2)\right)$
9	:	$-8\,A03\,A13 + A01\,\left(24\,A03^2 - 2\,(2\,A33 + D3)\right)$
10	:	$-8\,A03\,A23 + A02\,\left(24\,A03^2 - 2\,(2\,A33 + D3)\right)$
11	:	$8\,A01\,A02\,A11 + 8\,A01^2\,A12 - 2\,A12\,(2\,A11 + D1)$
12	:	$8\,A01\,A03\,A11 + 8\,A01^2\,A13 - 2\,A13\,(2\,A11 + D1)$
13	:	$8\,A02^2\,A12 + 8\,A01\,A02\,A22 - 2\,A12\,(2\,A22 + D2)$
14	:	$8\,A02\,A03\,A22 + 8\,A02^2\,A23 - 2\,A23\,(2\,A22 + D2)$
15	:	$8\,A03^2\,A13 + 8\,A01\,A03\,A33 - 2\,A13\,(2\,A33 + D3)$
16	:	$8\,A03^2\,A23 + 8\,A02\,A03\,A33 - 2\,A23\,(2\,A33 + D3)$
17	:	$-2\,\left(-12\,A01^2\,A02 + 4\,A01\,A12 + A02\,(2\,A11 + D1)\right)$
18	:	$-2\,\left(-12\,A01^2\,A03 + 4\,A01\,A13 + A03\,(2\,A11 + D1)\right)$
19	:	$-2\,\left(-12\,A02^2\,A03 + 4\,A02\,A23 + A03\,(2\,A22 + D2)\right)$
20	:	$4\,A02^2\,A11 + 16\,A01\,A02\,A12 - 4\,A12^2 + 4\,A01^2\,A22 - A22\,(2\,A11 + D1) - A11\,D2 - D3$
21	:	$4\,A03^2\,A11 + 16\,A01\,A03\,A13 - 4\,A13^2 + 4\,A01^2\,A33 - A33\,(2\,A11 + D1) - D2 - A11\,D3$
22	:	$4\,A03^2\,A22 + 16\,A02\,A03\,A23 - 4\,A23^2 + 4\,A02^2\,A33 - D1 - A33\,(2\,A22 + D2) - A22\,D3$
23	:	$2\,\left(4\,A02^2\,A13 + 4\,A01\,A03\,A22 - 4\,A12\,A23 + 8\,A02\,(A03\,A12 + A01\,A23) - A13\,(2\,A22 + D2)\right)$
24	:	$2\,\left(4\,A03^2\,A12 - 4\,A13\,A23 + 8\,A03\,(A02\,A13 + A01\,A23) + 4\,A01\,A02\,A33 - A12\,(2\,A33 + D3)\right)$
25	:	$2\,\left(8\,A01\,A03\,A12 - 4\,A12\,A13 + 4\,A02\,(A03\,A11 + 2\,A01\,A13) + 4\,A01^2\,A23 - A23\,(2\,A11 + D1)\right)$

In[37]:=
```
Variables[cons]
```

Out[37]= {A01, A02, A03, A11, A12, A13, A22, A23, A33, D1, D2, D3}

In[38]:=
```
elimVars = Variables[A]
condVars = {D1, D2, D3}
```

Out[38]= {A00, A01, A02, A03, A11, A12, A13, A22, A23, A33}

Out[39]= {D1, D2, D3}

In[40]:=
```
gb = simp[GroebnerBasis[cons, condVars, elimVars]]; // Timing
```

Out[40]= {5.26705, Null}

In[41]:=
```
show[gb]
```

Out[41]//MatrixForm=

1	:	$\left(-4 + D1^2\right)\left(-4 + D2^2\right)\left(-4 + D3^2\right)$
2	:	$\left(-4 + D1^2\right)\left(-4 + D2^2\right)(2\,D1 - D2\,D3)$
3	:	$\left(-4 + D2^2\right)(D1\,D2 - 2\,D3)\left(-4 + D3^2\right)$
4	:	$\left(-4 + D1^2\right)(2\,D1 - D2\,D3)\left(-4 + D3^2\right)$
5	:	$-\left(-4 + D2^2\right)(2\,D2 - D1\,D3)\left(-4 + D3^2\right)$
6	:	$\left(-4 + D2^2\right)(D2 - D3)(D2 + D3)\left(-4 + D3^2\right)$

- **Since we assume that 2 <= D_1 <= D_2 <= D_3 and D_2, D_3>2 the first equation implies that D_1 = 2**

In[42]:=
```
show[simp[gb /. D1 → 2]]
```

Out[42]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & 2\left(-4+D2^2\right)(D2-D3)\left(-4+D3^2\right) \\ 3 & : & -2\left(-4+D2^2\right)(D2-D3)\left(-4+D3^2\right) \\ 4 & : & \left(-4+D2^2\right)(D2-D3)(D2+D3)\left(-4+D3^2\right) \end{pmatrix}$$

- **.. and since D_2, D_3>2, equation (2) implies that D_2 = D_3. We have shown that D1 = 2 and D2=D3>2.**
- **D0 vanishes**

In[43]:=
```
Simplify[4 (-4 + D1^2 + D2^2 + D3^2 - D1 D2 D3) /. {D1 → 2, D2 → D3}]
```

Out[43]= 0

- **Verify decomposition**

In[44]:=

$$AA = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(-D3+\sqrt{-4+D3^2}\right) & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(-D3+\sqrt{-4+D3^2}\right) \end{pmatrix};$$

$$BB = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(-D3-\sqrt{-4+D3^2}\right) & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(-D3-\sqrt{-4+D3^2}\right) \end{pmatrix};$$

In[46]:=
```
verify = (vars.AA.vars) (vars.BB.vars);
```

In[47]:=
```
Simplify[verify - frSym /. {D0 → 0, D1 → 2, D2 → D3}]
```

Out[47]= 0