In[1]:=
```
SetDirectory["~/KappaLib/"];
<< kappaLib-1.1.m
<< helper.m
```

```
KappaLib v1.1
```

```
Loading helper.m..
```

- ## Metaclass VII:

In[4]:=
```
vars = {x0, x1, x2, x3};
```

In[5]:=

$$\textbf{kappa = emMatrixToKappa}\left[\begin{pmatrix} a1 & 0 & 0 & a4 & 0 & 0 \\ 0 & a2 & 0 & 0 & a5 & 0 \\ 0 & 0 & a3 & 0 & 0 & a6 \\ a4 & 0 & 0 & a1 & 0 & 0 \\ 0 & a5 & 0 & 0 & a2 & 0 \\ 0 & 0 & a6 & 0 & 0 & a3 \end{pmatrix}\right];$$

In[6]:=
```
fr = emKappaToFresnel[kappa, vars];
```

- ## We may assume that a4, a5, a6 != 0

In[7]:=
```
FullSimplify[fr /. {a4 → 0, x2 → 0, x3 → 0}]
FullSimplify[fr /. {a5 → 0, x1 → 0, x3 → 0}]
FullSimplify[fr /. {a6 → 0, x0 → 0, x3 → 0}]
```

Out[7]= 0

Out[8]= 0

Out[9]= 0

- ## Define Fresnel equation using constants D0, D1, D2, D3

In[10]:=

$$\text{frSym} = x0^4 + x1^4 + x2^4 + x3^4 + D0\, x0\, x1\, x2\, x3 - D3\left(x1^2\, x2^2 + x0^2\, x3^2\right) - D2\left(x1^2\, x3^2 + x0^2\, x2^2\right) - D1\left(x2^2\, x3^2 + x0^2\, x1^2\right);$$

In[11]:=

$$\text{subSym} = \Big\{$$

$$D0 \to \frac{1}{a4\, a5\, a6}\, 2\left(a1^2\,(a2 - a3) + a2^2\, a3 + a3\,(a4 - a5)\,(a4 + a5) - a2\left(a3^2 + a4^2 - a6^2\right) + a1\left(-a2^2 + a3^2 + a5^2 - a6^2\right)\right),$$

$$D1 \to \frac{(a2 - a3)^2 - a5^2 - a6^2}{a5\, a6},$$

$$D2 \to \frac{(a1 - a3)^2 - a4^2 - a6^2}{a4\, a6},$$

$$D3 \to \frac{(a1 - a2)^2 - a4^2 - a5^2}{a4\, a5}$$

$$\Big\};$$

In[12]:=
```
Simplify[fr - (a4 a5 a6) frSym /. subSym]
```

Out[12]= 0

- ## Implicit expression for D0

In[13]:=

$$\text{D0sq} = 4\left(-4 + D1^2 + D2^2 + D3^2 + D1\, D2\, D3\right);$$

In[14]:=
```
Simplify[D0^2 - D0sq /. subSym]
```

Out[14]= 0

- **We assume that the Fresnel polynomial factorises:**

In[15]:=
```
A = Table[ToExpression["A" <> ToString[Min[{i, j}]] <> ToString[Max[{i, j}]]],
    {i, 0, 3}, {j, 0, 3}];
B = Table[ToExpression["B" <> ToString[Min[{i, j}]] <> ToString[Max[{i, j}]]],
    {i, 0, 3}, {j, 0, 3}];
A // MatrixForm
B // MatrixForm
factorised = (vars.A.vars) (vars.B.vars);
```

Out[17]//MatrixForm=

A00	A01	A02	A03
A01	A11	A12	A13
A02	A12	A22	A23
A03	A13	A23	A33

Out[18]//MatrixForm=

B00	B01	B02	B03
B01	B11	B12	B13
B02	B12	B22	B23
B03	B13	B23	B33

In[20]:=
```
cons = Union[Flatten[CoefficientList[frSym - factorised, vars]]];
```

In[21]:=
```
cons = simp[cons];
show[cons] // MatrixForm
```

Out[22]//MatrixForm=

1	:	0
2	:	1 - A00 B00
3	:	1 - A11 B11
4	:	1 - A22 B22
5	:	1 - A33 B33
6	:	-2 (A01 B00 + A00 B01)
7	:	-2 (A02 B00 + A00 B02)
8	:	-2 (A03 B00 + A00 B03)
9	:	-2 (A11 B01 + A01 B11)
10	:	-2 (A12 B11 + A11 B12)
11	:	-2 (A13 B11 + A11 B13)
12	:	-2 (A22 B02 + A02 B22)
13	:	-2 (A22 B12 + A12 B22)
14	:	-2 (A23 B22 + A22 B23)
15	:	-2 (A33 B03 + A03 B33)
16	:	-2 (A33 B13 + A13 B33)
17	:	-2 (A33 B23 + A23 B33)
18	:	-A11 B00 - 4 A01 B01 - A00 B11 - D1
19	:	-A33 B22 - 4 A23 B23 - A22 B33 - D1
20	:	-A22 B00 - 4 A02 B02 - A00 B22 - D2
21	:	-A33 B11 - 4 A13 B13 - A11 B33 - D2
22	:	-A22 B11 - 4 A12 B12 - A11 B22 - D3
23	:	-A33 B00 - 4 A03 B03 - A00 B33 - D3
24	:	-2 (A12 B00 + 2 A02 B01 + 2 A01 B02 + A00 B12)
25	:	-2 (2 A12 B01 + A11 B02 + A02 B11 + 2 A01 B12)
26	:	-2 (A13 B00 + 2 A03 B01 + 2 A01 B03 + A00 B13)
27	:	-2 (2 A13 B01 + A11 B03 + A03 B11 + 2 A01 B13)
28	:	-2 (A22 B01 + 2 A12 B02 + 2 A02 B12 + A01 B22)
29	:	-2 (A23 B00 + 2 A03 B02 + 2 A02 B03 + A00 B23)
30	:	-2 (2 A23 B02 + A22 B03 + A03 B22 + 2 A02 B23)
31	:	-2 (A23 B11 + 2 A13 B12 + 2 A12 B13 + A11 B23)
32	:	-2 (2 A23 B12 + A22 B13 + A13 B22 + 2 A12 B23)
33	:	-2 (A33 B01 + 2 A13 B03 + 2 A03 B13 + A01 B33)
34	:	-2 (A33 B02 + 2 A23 B03 + 2 A03 B23 + A02 B33)
35	:	-2 (A33 B12 + 2 A23 B13 + 2 A13 B23 + A12 B33)
36	:	-4 (A23 B01 + A13 B02 + A12 B03 + A03 B12 + A02 B13 + A01 B23) + D0

- **Equation (2): By renaming and scaling, we may assume that A00 = 1.**

In[23]:=
```
sub = {A00 -> 1, B00 → 1};
cons = simp[cons //. sub];
show[cons]
```

Out[25]//MatrixForm=

1	:	0
2	:	1 - A11 B11
3	:	1 - A22 B22
4	:	1 - A33 B33
5	:	-2 (A01 + B01)
6	:	-2 (A02 + B02)
7	:	-2 (A03 + B03)
8	:	-2 (A11 B01 + A01 B11)
9	:	-2 (A12 B11 + A11 B12)
10	:	-2 (A13 B11 + A11 B13)
11	:	-2 (A22 B02 + A02 B22)
12	:	-2 (A22 B12 + A12 B22)
13	:	-2 (A23 B22 + A22 B23)
14	:	-2 (A33 B03 + A03 B33)
15	:	-2 (A33 B13 + A13 B33)
16	:	-2 (A33 B23 + A23 B33)
17	:	-A11 - 4 A01 B01 - B11 - D1
18	:	-A22 - 4 A02 B02 - B22 - D2
19	:	-A33 - 4 A03 B03 - B33 - D3
20	:	-A33 B22 - 4 A23 B23 - A22 B33 - D1
21	:	-A33 B11 - 4 A13 B13 - A11 B33 - D2
22	:	-A22 B11 - 4 A12 B12 - A11 B22 - D3
23	:	-2 (A12 + 2 A02 B01 + 2 A01 B02 + B12)
24	:	-2 (A13 + 2 A03 B01 + 2 A01 B03 + B13)
25	:	-2 (A23 + 2 A03 B02 + 2 A02 B03 + B23)
26	:	-2 (2 A12 B01 + A11 B02 + A02 B11 + 2 A01 B12)
27	:	-2 (2 A13 B01 + A11 B03 + A03 B11 + 2 A01 B13)
28	:	-2 (A22 B01 + 2 A12 B02 + 2 A02 B12 + A01 B22)
29	:	-2 (2 A23 B02 + A22 B03 + A03 B22 + 2 A02 B23)
30	:	-2 (A23 B11 + 2 A13 B12 + 2 A12 B13 + A11 B23)
31	:	-2 (2 A23 B12 + A22 B13 + A13 B22 + 2 A12 B23)
32	:	-2 (A33 B01 + 2 A13 B03 + 2 A03 B13 + A01 B33)
33	:	-2 (A33 B02 + 2 A23 B03 + 2 A03 B23 + A02 B33)
34	:	-2 (A33 B12 + 2 A23 B13 + 2 A13 B23 + A12 B33)
35	:	-4 (A23 B01 + A13 B02 + A12 B03 + A03 B12 + A02 B13 + A01 B23) + D0

In[26]:=
```
tmp = Join[Take[cons, {5, 7}], Take[cons, {17, 19}], Take[cons, {23, 25}]];
tmp // MatrixForm
```

Out[27]//MatrixForm=

-2 (A01 + B01)
-2 (A02 + B02)
-2 (A03 + B03)
-A11 - 4 A01 B01 - B11 - D1
-A22 - 4 A02 B02 - B22 - D2
-A33 - 4 A03 B03 - B33 - D3
-2 (A12 + 2 A02 B01 + 2 A01 B02 + B12)
-2 (A13 + 2 A03 B01 + 2 A01 B03 + B13)
-2 (A23 + 2 A03 B02 + 2 A02 B03 + B23)

In[28]:=
```
Solve[toEqs[%], {B01, B02, B03, B11, B22, B33, B12, B13, B23}]
```

Out[28]= {{B01 → -A01, B02 → -A02, B03 → -A03, B11 → 4 $A01^2$ - A11 - D1, B22 → 4 $A02^2$ - A22 - D2, B33 → 4 $A03^2$ - A33 - D3, B12 → 4 A01 A02 - A12, B13 → 4 A01 A03 - A13, B23 → 4 A02 A03 - A23}}

```
In[29]:= sub = Join[sub, %[[1]]]

Out[29]= {A00 → 1, B00 → 1, B01 → -A01, B02 → -A02, B03 → -A03,
  B11 → 4 A01^2 - A11 - D1, B22 → 4 A02^2 - A22 - D2, B33 → 4 A03^2 - A33 - D3,
  B12 → 4 A01 A02 - A12, B13 → 4 A01 A03 - A13, B23 → 4 A02 A03 - A23}

In[30]:= cons = simp[cons //. sub];
         show[cons]

Out[31]//MatrixForm=
 1  :  0
 2  :  2 A01 (-4 A01^2 + 2 A11 + D1)
 3  :  1 + A11 (-4 A01^2 + A11 + D1)
 4  :  2 A02 (-4 A02^2 + 2 A22 + D2)
 5  :  1 + A22 (-4 A02^2 + A22 + D2)
 6  :  2 A03 (-4 A03^2 + 2 A33 + D3)
 7  :  1 + A33 (-4 A03^2 + A33 + D3)
 8  :  8 A02 A12 + 2 A01 (-12 A02^2 + 2 A22 + D2)
 9  :  8 A03 A13 + 2 A01 (-12 A03^2 + 2 A33 + D3)
 10 :  8 A03 A23 + 2 A02 (-12 A03^2 + 2 A33 + D3)
 11 :  -24 A01^2 A02 + 8 A01 A12 + 2 A02 (2 A11 + D1)
 12 :  -24 A01^2 A03 + 8 A01 A13 + 2 A03 (2 A11 + D1)
 13 :  -24 A02^2 A03 + 8 A02 A23 + 2 A03 (2 A22 + D2)
 14 :  -8 A01 A02 A11 - 8 A01^2 A12 + 2 A12 (2 A11 + D1)
 15 :  -8 A01 A03 A11 - 8 A01^2 A13 + 2 A13 (2 A11 + D1)
 16 :  -8 A02^2 A12 - 8 A01 A02 A22 + 2 A12 (2 A22 + D2)
 17 :  -8 A02 A03 A22 - 8 A02^2 A23 + 2 A23 (2 A22 + D2)
 18 :  -8 A03^2 A13 - 8 A01 A03 A33 + 2 A13 (2 A33 + D3)
 19 :  -8 A03^2 A23 - 8 A02 A03 A33 + 2 A23 (2 A33 + D3)
 20 :  8 A03 A12 + 8 A02 A13 + 8 A01 (-6 A02 A03 + A23) + D0
 21 :  -4 A02^2 A11 - 16 A01 A02 A12 + 4 A12^2 - 4 A01^2 A22 + 2 A11 A22 + A22 D1 + A11 D2 - D3
 22 :  -4 A03^2 A11 - 16 A01 A03 A13 + 4 A13^2 - 4 A01^2 A33 + 2 A11 A33 + A33 D1 - D2 + A11 D3
 23 :  -4 A03^2 A22 - 16 A02 A03 A23 + 4 A23^2 - 4 A02^2 A33 + 2 A22 A33 - D1 + A33 D2 + A22 D3
 24 :  2 (-4 A02^2 A13 - 4 A01 A03 A22 + 2 A13 A22 + 4 A12 A23 - 8 A02 (A03 A12 + A01 A23) + A13 D2)
 25 :  2 (-4 A03^2 A12 + 4 A13 A23 - 8 A03 (A02 A13 + A01 A23) - 4 A01 A02 A33 + 2 A12 A33 + A12 D3)
 26 :  2 (-8 A01 A03 A12 + 4 A12 A13 - 4 A02 (A03 A11 + 2 A01 A13) - 4 A01^2 A23 + 2 A11 A23 + A23 D1)
```

- **Eliminate**

```
In[32]:= Variables[cons]

Out[32]= {A01, A02, A03, A11, A12, A13, A22, A23, A33, D0, D1, D2, D3}

In[33]:= elimVars = Variables[A]
         condVars = {D0, D1, D2, D3}

Out[33]= {A00, A01, A02, A03, A11, A12, A13, A22, A23, A33}

Out[34]= {D0, D1, D2, D3}

In[35]:= gb = simp[GroebnerBasis[cons, condVars, elimVars]]; // Timing

Out[35]= {10.1926, Null}
```

In[36]:=
```
show[gb]
```

Out[36]//MatrixForm=

$$\begin{pmatrix} 1 & : & D0\left(-4+D1^2\right)\left(-4+D2^2\right) \\ 2 & : & D0\left(-4+D1^2\right)\left(-4+D3^2\right) \\ 3 & : & D0\left(-4+D2^2\right)\left(-4+D3^2\right) \\ 4 & : & \left(-4+D1^2\right)\left(-4+D2^2\right)\left(-4+D3^2\right) \\ 5 & : & \left(-4+D1^2\right)\left(-4+D2^2\right)(2\,D1+D2\,D3) \\ 6 & : & \left(-4+D2^2\right)(D1\,D2+2\,D3)\left(-4+D3^2\right) \\ 7 & : & \left(-4+D2^2\right)(2\,D2+D1\,D3)\left(-4+D3^2\right) \\ 8 & : & \left(-4+D1^2\right)(2\,D1+D2\,D3)\left(-4+D3^2\right) \\ 9 & : & \left(-4+D2^2\right)(D2-D3)(D2+D3)\left(-4+D3^2\right) \\ 10 & : & D0^2-4\left(-4+D1^2+D2^2+D1\,D2\,D3+D3^2\right) \end{pmatrix}$$

- **Case 1: D0 = 0**

In[37]:=
```
show[simp[gb /. {D0 → 0}]]
```

Out[37]//MatrixForm=

$$\begin{pmatrix} 1 & : & 0 \\ 2 & : & \left(-4+D1^2\right)\left(-4+D2^2\right)\left(-4+D3^2\right) \\ 3 & : & \left(-4+D1^2\right)\left(-4+D2^2\right)(2\,D1+D2\,D3) \\ 4 & : & \left(-4+D2^2\right)(D1\,D2+2\,D3)\left(-4+D3^2\right) \\ 5 & : & \left(-4+D2^2\right)(2\,D2+D1\,D3)\left(-4+D3^2\right) \\ 6 & : & \left(-4+D1^2\right)(2\,D1+D2\,D3)\left(-4+D3^2\right) \\ 7 & : & -4\left(-4+D1^2+D2^2+D1\,D2\,D3+D3^2\right) \\ 8 & : & \left(-4+D2^2\right)(D2-D3)(D2+D3)\left(-4+D3^2\right) \end{pmatrix}$$

- **It follows: If D0=0 then there exists an i in {1,2,3} such that**

  D0=0, D_i = -2 sigma, D_i' = sigma D_i''

- **Case i=1:**

In[38]:=
```
dsub = {D1 → -sigma 2, D3 → sigma D2, D0 → 0};
```

In[39]:=

$$\text{LPlus} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \text{sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(-D2+\sqrt{-4+D2^2}\right) & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(-D2+\sqrt{-4+D2^2}\right)\text{sigma} \end{pmatrix};$$

$$\text{LMinus} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \text{sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{2}\left(-D2-\sqrt{-4+D2^2}\right) & 0 \\ 0 & 0 & 0 & \frac{1}{2}\left(-D2-\sqrt{-4+D2^2}\right)\text{sigma} \end{pmatrix};$$

In[40]:=
```
dd = (vars.LPlus.vars) (vars.LMinus.vars);
simp[Flatten[CoefficientList[dd - frSym, vars] //. dsub]]
```

Out[41]= $\{0,\ -1+\text{sigma}^2,\ D2-D2\,\text{sigma}^2\}$

- **Case: i=2**

In[42]:=
```
dsub = {D2 → -sigma 2, D3 → sigma D1, D0 → 0};
```

In[43]:=
```
LPlus = ( 1   0                              0       0
          0   1/2 (-D1 + Sqrt[-4 + D1^2])    0       0
          0   0                              sigma   0
          0   0                              0       1/2 (-D1 + Sqrt[-4 + D1^2]) sigma );

LMinus = ( 1   0                              0       0
           0   1/2 (-D1 - Sqrt[-4 + D1^2])    0       0
           0   0                              sigma   0
           0   0                              0       1/2 (-D1 - Sqrt[-4 + D1^2]) sigma );
```

In[44]:=
```
dd = (vars.LPlus.vars) (vars.LMinus.vars);
simp[Flatten[CoefficientList[dd - frSym, vars]] //. dsub]
```

Out[45]= $\{0, -1 + \text{sigma}^2, \text{D1} - \text{D1}\,\text{sigma}^2\}$

- **Case: i=3**

In[46]:=
```
dsub = {D3 → -sigma 2, D2 → sigma D1, D0 → 0};
```

In[47]:=
```
LPlus = ( 1   0                              0                                   0
          0   1/2 (-D1 + Sqrt[-4 + D1^2])    0                                   0
          0   0                              1/2 (-D1 + Sqrt[-4 + D1^2]) sigma   0
          0   0                              0                                   sigma );

LMinus = ( 1   0                              0                                   0
           0   1/2 (-D1 - Sqrt[-4 + D1^2])    0                                   0
           0   0                              1/2 (-D1 - Sqrt[-4 + D1^2]) sigma   0
           0   0                              0                                   sigma );
```

In[48]:=
```
dd = (vars.LPlus.vars) (vars.LMinus.vars);
simp[Flatten[CoefficientList[dd - frSym, vars] //. dsub]]
```

Out[49]= $\{0, -1 + \text{sigma}^2, \text{D1} - \text{D1}\,\text{sigma}^2\}$

- **Case 2: D0 != 0**

In[50]:=
```
gb // show
```

Out[50]//MatrixForm=

1	:	$\text{D0}\,(-4 + \text{D1}^2)\,(-4 + \text{D2}^2)$
2	:	$\text{D0}\,(-4 + \text{D1}^2)\,(-4 + \text{D3}^2)$
3	:	$\text{D0}\,(-4 + \text{D2}^2)\,(-4 + \text{D3}^2)$
4	:	$(-4 + \text{D1}^2)\,(-4 + \text{D2}^2)\,(-4 + \text{D3}^2)$
5	:	$(-4 + \text{D1}^2)\,(-4 + \text{D2}^2)\,(2\,\text{D1} + \text{D2}\,\text{D3})$
6	:	$(-4 + \text{D2}^2)\,(\text{D1}\,\text{D2} + 2\,\text{D3})\,(-4 + \text{D3}^2)$
7	:	$(-4 + \text{D2}^2)\,(2\,\text{D2} + \text{D1}\,\text{D3})\,(-4 + \text{D3}^2)$
8	:	$(-4 + \text{D1}^2)\,(2\,\text{D1} + \text{D2}\,\text{D3})\,(-4 + \text{D3}^2)$
9	:	$(-4 + \text{D2}^2)\,(\text{D2} - \text{D3})\,(\text{D2} + \text{D3})\,(-4 + \text{D3}^2)$
10	:	$\text{D0}^2 - 4\,(-4 + \text{D1}^2 + \text{D2}^2 + \text{D1}\,\text{D2}\,\text{D3} + \text{D3}^2)$

In[51]:=
```
show[simp[gb /. D1 → s1 2 /. D2 → s2 2]]
```

Out[51]//MatrixForm=

$$\begin{pmatrix} 1 & : & 4\,D0\,(-4+D3^2)\,(-1+s1^2) \\ 2 & : & 4\,D0\,(-4+D3^2)\,(-1+s2^2) \\ 3 & : & 16\,D0\,(-1+s1^2)\,(-1+s2^2) \\ 4 & : & 16\,(-4+D3^2)\,(-1+s1^2)\,(-1+s2^2) \\ 5 & : & 8\,(-4+D3^2)\,(-1+s1^2)\,(2\,s1+D3\,s2) \\ 6 & : & 8\,(-4+D3^2)\,(D3\,s1+2\,s2)\,(-1+s2^2) \\ 7 & : & 8\,(-4+D3^2)\,(D3+2\,s1\,s2)\,(-1+s2^2) \\ 8 & : & 32\,(-1+s1^2)\,(2\,s1+D3\,s2)\,(-1+s2^2) \\ 9 & : & -4\,(-4+D3^2)\,(D3^2-4\,s2^2)\,(-1+s2^2) \\ 10 & : & D0^2-4\,(D3^2+4\,D3\,s1\,s2+4\,(-1+s1^2+s2^2)) \end{pmatrix}$$

In[52]:=
```
DB12 = {D1 → s1 2, D2 → s2 2, D3 → 1/2 (-4 s1 s2 + s3 D0)};

DB23 = {D2 → s1 2, D3 → s2 2, D1 → 1/2 (-4 s1 s2 + s3 D0)};

DB13 = {D1 → s1 2, D3 → s2 2, D2 → 1/2 (-4 s1 s2 + s3 D0)};
```

In[55]:=
```
For[i = 0, i ≤ 1, i++,
 For[j = 0, j ≤ 1, j++,
  For[k = 0, k ≤ 1, k++,
   loopSub = {s1 → (-1)^i, s2 → (-1)^j, s3 → (-1)^k};
   t13 = Simplify[D0sq /. DB13 /. loopSub];
   t23 = Simplify[D0sq /. DB23 /. loopSub];
   t12 = Simplify[D0sq /. DB12 /. loopSub];
   Print[{t13, t23, t12}]
  ]
 ]
]
```

$\{D0^2, D0^2, D0^2\}$

$\{D0^2, D0^2, D0^2\}$

$\{D0^2, D0^2, D0^2\}$

$\{D0^2, D0^2, D0^2\}$

$\{D0^2, D0^2, D0^2\}$

$\{D0^2, D0^2, D0^2\}$

$\{D0^2, D0^2, D0^2\}$

$\{D0^2, D0^2, D0^2\}$

Decomposition for (i,j)= (1,2)

In[56]:=

$$\text{LPlus} = \begin{pmatrix} 1 & 0 & 0 & \frac{\sqrt{D0}}{2\sqrt{2}\sqrt{s3}} \\ 0 & -s1 & -\frac{\sqrt{D0}\sqrt{s3}}{2\sqrt{2}} & 0 \\ 0 & -\frac{\sqrt{D0}\sqrt{s3}}{2\sqrt{2}} & -s2 & 0 \\ \frac{\sqrt{D0}}{2\sqrt{2}\sqrt{s3}} & 0 & 0 & s1\,s2 \end{pmatrix};$$

$$\text{LMinus} = \begin{pmatrix} 1 & 0 & 0 & -\frac{\sqrt{D0}}{2\sqrt{2}\sqrt{s3}} \\ 0 & -s1 & \frac{\sqrt{D0}\sqrt{s3}}{2\sqrt{2}} & 0 \\ 0 & \frac{\sqrt{D0}\sqrt{s3}}{2\sqrt{2}} & -s2 & 0 \\ -\frac{\sqrt{D0}}{2\sqrt{2}\sqrt{s3}} & 0 & 0 & s1\,s2 \end{pmatrix};$$

```
dd = (vars.LPlus.vars) (vars.LMinus.vars);
t = Union[Flatten[CoefficientList[Simplify[(dd - frSym)], vars]]];
FullSimplify[t //. DB12]
```

Out[60]= $\left\{0,\ 0,\ -1+s1^2,\ 0,\ -2\left(-1+s1^2\right)s2,\ -1+s2^2,\ -2\,s1\left(-1+s2^2\right),\ -1+s1^2\,s2^2,\ \frac{D0\left(-1+s3^2\right)}{2\,s3},\ 0\right\}$

In[61]:= `qDet = FullSimplify[Det[LPlus]]`

Out[61]= $\frac{(-8\,s1\,s2 + D0\,s3)\,(D0 - 8\,s1\,s2\,s3)}{64\,s3}$

In[62]:= `FullSimplify[Det[LMinus] - qDet]`

Out[62]= 0

D13

In[63]:=

$$\text{LPlus} = \begin{pmatrix} 1 & 0 & \frac{\sqrt{D0}\sqrt{s3}}{2\sqrt{2}} & 0 \\ 0 & -s1 & 0 & -\frac{\sqrt{D0}}{2\sqrt{2}\sqrt{s3}} \\ \frac{\sqrt{D0}\sqrt{s3}}{2\sqrt{2}} & 0 & s1\,s2 & 0 \\ 0 & -\frac{\sqrt{D0}}{2\sqrt{2}\sqrt{s3}} & 0 & -s2 \end{pmatrix};$$

$$\text{LMinus} = \begin{pmatrix} 1 & 0 & -\frac{\sqrt{D0}\sqrt{s3}}{2\sqrt{2}} & 0 \\ 0 & -s1 & 0 & \frac{\sqrt{D0}}{2\sqrt{2}\sqrt{s3}} \\ -\frac{\sqrt{D0}\sqrt{s3}}{2\sqrt{2}} & 0 & s1\,s2 & 0 \\ 0 & \frac{\sqrt{D0}}{2\sqrt{2}\sqrt{s3}} & 0 & -s2 \end{pmatrix};$$

```
dd = (vars.LPlus.vars) (vars.LMinus.vars);
t = Union[Flatten[CoefficientList[Simplify[(dd - frSym)], vars]]];
FullSimplify[t //. DB13]
```

Out[67]= $\left\{0,\ 0,\ -1+s1^2,\ 0,\ -2\left(-1+s1^2\right)s2,\ -1+s2^2,\ -2\,s1\left(-1+s2^2\right),\ -1+s1^2\,s2^2,\ \frac{D0\left(-1+s3^2\right)}{2\,s3},\ 0\right\}$

In[68]:=
```
FullSimplify[qDet - Det[LPlus]]
FullSimplify[qDet - Det[LMinus]]
```

Out[68]= 0

Out[69]= 0

- **D23**

In[70]:= LPlus = $\begin{pmatrix} 1 & \frac{\sqrt{D0}}{2\sqrt{2}\sqrt{s3}} & 0 & 0 \\ \frac{\sqrt{D0}}{2\sqrt{2}\sqrt{s3}} & s1\,s2 & 0 & 0 \\ 0 & 0 & -s1 & -\frac{\sqrt{D0}\sqrt{s3}}{2\sqrt{2}} \\ 0 & 0 & -\frac{\sqrt{D0}\sqrt{s3}}{2\sqrt{2}} & -s2 \end{pmatrix}$;

LMinus = $\begin{pmatrix} 1 & -\frac{\sqrt{D0}}{2\sqrt{2}\sqrt{s3}} & 0 & 0 \\ -\frac{\sqrt{D0}}{2\sqrt{2}\sqrt{s3}} & s1\,s2 & 0 & 0 \\ 0 & 0 & -s1 & \frac{\sqrt{D0}\sqrt{s3}}{2\sqrt{2}} \\ 0 & 0 & \frac{\sqrt{D0}\sqrt{s3}}{2\sqrt{2}} & -s2 \end{pmatrix}$;

```
dd = (vars.LPlus.vars) (vars.LMinus.vars);
t = Union[Flatten[CoefficientList[Simplify[(dd - frSym)], vars]]];
FullSimplify[t //. DB23]
```

Out[74]= $\left\{0,\ 0,\ -1+s1^2,\ 0,\ -2\left(-1+s1^2\right)s2,\ -1+s2^2,\ -2\,s1\left(-1+s2^2\right),\ -1+s1^2\,s2^2,\ \frac{D0\left(-1+s3^2\right)}{2\,s3},\ 0\right\}$

In[75]:=
```
FullSimplify[qDet - Det[LMinus]]
FullSimplify[qDet - Det[LPlus]]
```

Out[75]= 0

Out[76]= 0