

Descending Maps Between Slashed Tangent Bundles

Matias Dahl

University College London, UK

joint work with

Ioan Bucataru

A.I. Cuza University, Romania

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Mathematical setting

- M is a manifold
- $TM \setminus \{0\}$ is the **slashed tangent bundle**

The Problem

- Suppose F is a diffeomorphism

$$F: TM \setminus \{0\} \rightarrow TM \setminus \{0\}.$$

Characterize those F that can be written as

$$F = D\phi|_{TM \setminus \{0\}}.$$

for a diffeomorphism $\phi: M \rightarrow M$.

When ϕ exists, one say that F **descends**.

Canonical involution in the second tangent bundle

Definition: Let M be a manifold. Then the **canonical involution** is the diffeomorphism

$$\kappa: TTM \rightarrow TTM$$

that is locally given by

$$\kappa(x, y, X, Y) = (x, X, y, Y).$$

Note:

- $\kappa^2 = \text{identity}$.

First main theorem:

If M is a manifold and F is a diffeomorphism

$$F: TM \setminus \{0\} \rightarrow TM \setminus \{0\},$$

then the following are equivalent:

(i) There exists a diffeomorphism $\phi: M \rightarrow M$ such that

$$F = D\phi|_{TM \setminus \{0\}}.$$

(ii) $DF = \kappa \circ DF \circ \kappa$

A related result

Theorem [Robbin-Weinstein-Lie]:

Let F be a diffeomorphism

$$F: T^*M \rightarrow T^*M.$$

Then the following are equivalent:

- (i) $F = \phi^*$ for a diffeomorphism $\phi: M \rightarrow M$.
- (ii) $F^*\theta = \theta$.

Here:

- ϕ^* = **pullback of ϕ** , $\phi(x, \xi) = \left((\phi^{-1})^i(x), \frac{\partial(\phi^{-1})^i}{\partial x^a} \xi_i \right)$
- θ = **canonical 1-form** $\theta \in \Omega^1(T^*M)$, $\theta = \xi_i dx^i$

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Claim: There exists a map $\phi: M \rightarrow M$ such that $F = D\phi|_{TM \setminus \{0\}}$.

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Proof: Let locally $F(x, y) = (F_1(x, y), F_2(x, y))$. Then

$$\begin{aligned} DF(x, y, X, Y) \\ = & \left(F_1(x, y), F_2(x, y), \frac{\partial F_1}{\partial x^a}(x, y)X^a + \frac{\partial F_1}{\partial y^a}(x, y)Y^a, \right. \\ & \left. \frac{\partial F_2}{\partial x^a}(x, y)X^a + \frac{\partial F_2}{\partial y^a}(x, y)Y^a \right), \end{aligned}$$

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$$\begin{aligned} \kappa \circ DF \circ \kappa(x, y, X, Y) &= \left(F_1(x, X), \frac{\partial F_1}{\partial x^a}(x, X)y^a + \frac{\partial F_1}{\partial y^a}(x, X)Y^a, F_2(x, X), \right. \\ &\quad \left. \frac{\partial F_2}{\partial x^a}(x, X)y^a + \frac{\partial F_2}{\partial y^a}(x, X)Y^a \right). \end{aligned}$$

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- First components: $F_1(x, y) = F_1(x, X)$.

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- First components: $F_1(x, y) = F_1(x, X)$. Thus $F_1(x, y)$ does not depend on y . Set $\phi(x) = F_1(x, y)$.

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- First components: $F_1(x, y) = F_1(x, X)$. Thus $F_1(x, y)$ does not depend on y . Set $\phi(x) = F_1(x, y)$.
- Second components: $F_2(x, y) = \frac{\partial \phi}{\partial x^a}(x)y^a$. Thus $F = D\phi|_{TM \setminus \{0\}}$.

Second main theorem:

Suppose M is a connected manifold with $\dim M \geq 2$ and F is a diffeomorphism

$$F: TM \setminus \{0\} \rightarrow TM \setminus \{0\}.$$

If M has two complete Riemann metrics g and \tilde{g} such that

- (i) g has a **trapping hypersurface** $\Sigma \subset M \Leftrightarrow$ Every geodesic of g intersects Σ
- (ii) for all $p \in \Sigma$,

$$\begin{aligned}g_{ij}(p) &= \tilde{g}_{ij}(p), & \Gamma_{jk}^i(p) &= \tilde{\Gamma}_{jk}^i(p), \\DF(\xi) &= \xi, & \xi &\in T(T_p M \setminus \{0\})\end{aligned}$$

- (iii) If $J: I \rightarrow TM \setminus \{0\}$ is a Jacobi field for g then $F \circ J: I \rightarrow TM \setminus \{0\}$ is a Jacobi field for \tilde{g} .

Then there exists a diffeomorphism $\phi: M \rightarrow M$ such that $F = D\phi|_{TM \setminus \{0\}}$ and ϕ is an isometry.