MATH 3305 General Relativity Problem sheet 8
Please hand in your solutions Friday, 11th December 2009

Problem 1 (40 points) Let R_{ij} be the Ricci curvature of a metric tensor g_{ij} on an n-dimensional manifold, let $R = R_{ab}g^{ab}$ be the scalar curvature, and let G_{ij} be the Einstein tensor

$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R.$$

(a) Starting with $g_{ij}g^{jk} = \delta_i^k$ show that $\nabla_r g^{ab} = 0$. Show how this implies that

$$g^{id}\nabla_k R_{abcd} = \nabla_k R_{abc}{}^{i}, \quad g_{id}\nabla_k R_{abc}{}^{i} = \nabla_k R_{abcd},$$

That is, taking the covariant derivative and raising (and lowering) an index can be done in any order.

(b) Contract the Bianchi identity

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0$$

by the $\binom{4}{0}$-tensor $g^{ac}g^{bd}$ and show that

$$\nabla^i G_{ij} = 0.$$

Problem 2 (60 points) The energy tensor of a moving mass density (fluid) on a 4-manifold M with a Lorentz metric g_{ij} is given by

$$T_{ij} = \varrho u_i u_j, \tag{1}$$

where $\varrho > 0$ is a scalar on M that describes the energy density, u^i is a $\binom{1}{0}$-tensor that describes the 4-velocity of the fluid with $g_{ij}u^i u^j = 1$, and

$$\nabla^i T_{ij} = 0. \tag{2}$$

(a) Using $\nabla^i g^{ab} = 0$ from Problem 8.1 and the argument in Problem 6.2 show that

$$u^a \nabla^i u_a = 0.$$

(b) Insert equation (1) into equation (2) and deduce that

$$\nabla^i(\varrho u_i)u_j + \varrho u_i \nabla^i u_j = 0.$$

(c) Using $g_{ij}u^i u^j = 1$, equation (3) and $\varrho > 0$ deduce that

$$u_i \nabla^i u_j = 0.$$

(d) If a small particle in the fluid moves along a curve $X^i(\lambda)$, then by the definition of u^i we have

$$\frac{dX^i}{d\lambda}(\lambda) = u^i(X^1(\lambda), \ldots, X^4(\lambda)).$$

Show that particles in the fluid move along geodesics of g_{ij}.