MATH 3305 General Relativity Problem sheet 9
Please hand in your solutions Friday, 18th December 2009

This is the last problem sheet and all problems can be done using the theory presented in class (as of December 11th).

Problem 1 (40 points) The Schwarzschild metric on $\mathbb{R}^4$ with coordinates (t, r, θ, ϕ) is the metric tensor g_{ij} with line element

$$ds^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{1}{c^2}\left(\left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right). \tag{1}$$

Here $r_s \geq 0$ is a constant.

(a) Write down the Lagrangian function for ds^2.

(b) Using the Lagrangian method, write down the geodesic equation for a curve
$$(t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda)).$$

Problem 2 (20 points) Print out the Mathematica notebook SC.pdf on the course homepage. For each command In[1], ..., In[14] give a short description of what it does, and show that the Einstein tensor of the metric given by equation (1) is zero.

Problem 3 (40 points) (Constant in Einstein's field equations) Suppose g_{ij} is a Lorentz metric on a 4-manifold.

(a) Show that the Einstein field equations
$$R_{ij} - \frac{1}{2}Rg_{ij} = \kappa T_{ij} \tag{2}$$
can be rewritten as
$$R_{ij} = \kappa(T_{ij} - \frac{1}{2}g_{ij}T), \tag{3}$$
where $T = T_{ij}g^{ij}$.

(b) Let us now assume that $(X^0 = t, X^1 = x, X^2 = y, X^3 = z)$ is an inertial frame for $M = \mathbb{R}^4$ and $\Phi(x, y, z)$ is a gravitational potential determined by a mass density $\varrho(x, y, z)$ so that
$$\nabla^2\Phi = 4\pi G\varrho. \tag{4}$$
Let g_{ij} be the metric
$$g_{ij} = \eta_{ij} - \begin{pmatrix} -\frac{2\Phi}{c^2} & & & \\ & -\frac{2\Phi}{c^4} & & \\ & & -\frac{2\Phi}{c^4} & \\ & & & -\frac{2\Phi}{c^4} \end{pmatrix}.$$

We assume that products $\Phi^2, \Phi\frac{\partial\Phi}{\partial X^r}$ and $\frac{\partial\Phi}{\partial X^r}\frac{\partial\Phi}{\partial X^s}$ are so small that they can be neglected whenever they appear. In the lectures we then proved that geodesics of g_{ij} coincide with the Newton equation of motion. Moreover, from the lectures

$$\begin{aligned}
\Gamma^i_{00} &= \frac{\partial\Phi}{\partial X^i}, \quad \Gamma^0_{0i} = \frac{1}{c^2}\frac{\partial\Phi}{\partial X^i}, \quad \Gamma^i_{j0} = \Gamma^0_{ij} = \Gamma^0_{00} = 0, \\
\Gamma^i_{jk} &= \frac{1}{c^2} \times \text{linear combination of } \left\{\tfrac{\partial\Phi}{\partial x}, \tfrac{\partial\Phi}{\partial y}, \tfrac{\partial\Phi}{\partial z}\right\} .
\end{aligned}$$

for all $i, j, k = 1, 2, 3$. Using this show that

$$R_{00} = \sum_{s=1}^{3} \frac{\partial\Gamma^s_{00}}{\partial X^s} = \nabla^2\Phi.$$

(c) Let T_{ij} be the energy tensor for ϱ given by

$$T_{ij} = \begin{pmatrix} c^2\varrho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}_{ij} .$$

Show that $T \approx c^2\varrho$.

(d) How should the constant κ in the Einstein field equations be chosen such that equation (3) holds for $i = j = 0$.