MATH 3305 General Relativity Problem sheet 5
Please hand in your solutions Friday, 20th November 2009

Problem 1 (30 points) Suppose $\gamma\colon [a,b] \to M$ is a geodesic for a metric tensor g_{ij}. Show that

$$g_{ij} \frac{d\gamma^i}{d\lambda} \frac{d\gamma^j}{d\lambda}$$

is constant with respect to λ.

Problem 2 (10 points) Suppose (y, x, y, z) is an inertial frame A, and let $(\widetilde{t}, \widetilde{x}, \widetilde{y}, \widetilde{z})$ be the intertial frame B defined by

$$\begin{aligned}
\widetilde{t} &= \gamma(t - xv/c^2), && (1)\\
\widetilde{x} &= \gamma(x - vt), && (2)\\
\widetilde{y} &= y, && (3)\\
\widetilde{z} &= z, && (4)
\end{aligned}$$

where $\gamma = 1/\sqrt{1-(v/c)^2}$ for some $v \in (-c, c)$. Show that

$$\begin{aligned}
t &= \gamma(\widetilde{t} + \widetilde{x}v/c^2),\\
x &= \gamma(\widetilde{x} + v\widetilde{t}),\\
y &= \widetilde{y},\\
z &= \widetilde{z}.
\end{aligned}$$

Problem 3 (30 points) Suppose (y, x, y, z) is an inertial frame A, and let $(\widetilde{t}, \widetilde{x}, \widetilde{y}, \widetilde{z})$ be the intertial frame B such that equations (1)–(4) holds for some $v \in (-c, c)$. If $\gamma(t) = (x(t), y(t), z(t))$ is a curve with zero acceleration in A, show that $\widetilde{\gamma}(\widetilde{t}) = (\widetilde{x}(\widetilde{t}), \widetilde{y}(\widetilde{t}), \widetilde{z}(\widetilde{t}))$ also has zero acceleration in B.

Problem 4 (30 points) In the lectures we showed that if $\gamma = (X^1(\lambda), \ldots, X^n(\lambda))$ is a curve $\gamma\colon [a,b] \to M$, then its tangent transforms as

$$\frac{d\widetilde{X}^j}{d\lambda} = \frac{\partial \widetilde{X}^j}{\partial X^r} \frac{dX^r}{d\lambda}. \tag{5}$$

(a) Show that the second derivative of γ transforms as

$$\frac{d^2\widetilde{X}^j}{d\lambda^2} = \frac{\partial^2 \widetilde{X}^j}{\partial X^r \partial X^s} \frac{\partial X^r}{d\lambda} \frac{dX^s}{d\lambda} + \frac{\partial \widetilde{X}^j}{\partial X^r} \frac{d^2 X^r}{d\lambda^2}.$$

Conclude that on a manifold, a curve can have zero acceleration in one coordinate system, but be accelerated in another.

(b) Use the geodesic equation, equation (5) and Problem 4.4 to show that the geodesic equation has the same form in overlapping coordinates. That is, if a curve is a geodesic in one coordinate system, then it is a geodesic in all overlapping coordinate systems.