MATH 3305 General Relativity Problem sheet 6
Please hand in your solutions Friday, 27th November 2009

Problem 1 (10 points) Let (t, x, y, z) be an inertial frame in $\mathbb{R}^4$, and let

$$\eta_{ij} = \begin{pmatrix} 1 & & & \\ & -1/c^2 & & \\ & & -1/c^2 & \\ & & & -1/c^2 \end{pmatrix}_{ij} . \tag{1}$$

be the Minkowski metric on $\mathbb{R}^4$. Compute the geodesics of η_{ij}. Which geodesics describe particles that move slower than the speed of light?

Problem 2 (60 points) The energy tensor of a perfect fluid is defined as

$$T_{ij} = \varrho u_i u_j - p(\eta_{ij} - u_i u_j), \tag{2}$$

where η_{ij} is the Minkowski metric in equation (1), u_i is a $\binom{0}{1}$-tensor such that $\eta_{ij} u^i u^j = 1$ and p, ϱ are scalars on $\mathbb{R}^4$ that describe the pressure and energy density of the fluid, respectively. Suppose

$$\nabla^i T_{ij} = 0, \tag{3}$$

where $\nabla^i = \eta^{ij} \frac{\partial}{\partial X^j}$.

1. Starting from $\eta_{ij} u^i u^j = 1$, show that $u_a \nabla^i u^a = u^a \nabla^i u_a = 0$.
2. Insert equation (2) into equation (3), contract by u^j, and use part 1. to show that

$$u_a \nabla^a \varrho + (\varrho + p) \nabla^a u_a = 0. \tag{4}$$

3. Assume that $p \ll \varrho$ and

$$u^a = \begin{pmatrix} 1 \\ v^1 \\ v^2 \\ v^3 \end{pmatrix}, \tag{5}$$

and show that equation (4) then implies the continuity equation

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho v) = 0. \tag{6}$$

4. Lastly, show that when equation (5) holds, equation $\eta_{ij} u^i u^j = 1$ implies that $|v|/c \approx 0$. This means that the fluid flows much slower than the speed of light. That is, in the special case of a slowly moving fluid, equation (3) implies equation (6).

Problem 3 (30 points) (Review Newtonian gravity). Let (t, x, y, z) is an inertial frame, and let $\varrho(x, y, z)$ be a mass density.

(a) How is the gravitational potential defined. How is it related to ϱ?

(b) Describe the path of a small particle in the gravitation field due to mass density ϱ. In particular, write down the ordinary differential equation that determines the path of the particle?

(Keywords: Gauss law of gravitation, Poisson equation)