MATH 3305 General Relativity Problem sheet 2

Please hand in your solutions Friday, 23 October 2009

Problem 1 (10 points) Some 3-vector identities and index gymnastics. In index notation show that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}). \quad (1)$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \quad (2)$$

$$\nabla \times (f\,\mathbf{a}) = \nabla f \times \mathbf{a} + f \nabla \times \mathbf{a}, \quad (3)$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are 3-vectors and f is a smooth function.

Problem 2 (30 points) Let the metric of Euclidean three-dimensional space be given by

$$ds^2 = dx^2 + dy^2 + dz^2. \quad (4)$$

What are the components of the metric in the new coordinate system

$$r = \sqrt{x^2 + y^2 + z^2} \quad (5)$$

$$y/x = \tan\phi \quad (6)$$

$$z/r = \cos\theta. \quad (7)$$

(Hint: three-dimensional spherical coordinates)

Problem 3 (30 points) In (x, y) coordinates for the plane, let us consider the $\binom{1}{0}$-tensor W^i with components

$$W^1(x, y) = 0, \quad W^2(x, y) = 1.$$

Compute $\widetilde{W}^i$ in polar coordinates (r, θ), and show that

$$g_{ij} W^i W^j = \widetilde{g}_{ij} \widetilde{W}^i \widetilde{W}^j = 1,$$

when g_{ij} is the Euclidean metric in coordinates (x, y) and $\widetilde{g}_{ij}$ is the metric in polar coordinates.

Problem 4 (30 points) Suppose $A^i(X^1, \ldots, X^n)$ are components for a $\binom{1}{0}$-tensor. How does

$$\frac{\partial A^i}{\partial X^j} \quad (8)$$

transform under coordinate transformations? (Hint: compute $\frac{\partial \widetilde{A}^i}{\partial \widetilde{X}^j}$) Is this quantity a tensor? Do you think the partial derivative is a good differential operator in tensor analysis? What property should a good operator have?