On the Stable H^2 and H^∞ Infinite-Dimensional Regulator Problems and Their Algebraic Riccati Equations

Kalle Mikkola
Institute of Mathematics
Helsinki University of Technology
FIN-02015 Espoo, Finland
Kalle.Mikkola@iki.fi

14th November 1997

Abstract. We obtain (necessary and) sufficient conditions in terms of spectral factorizations for the existence of a unique self-adjoint stabilizing solution to an algebraic Riccati equation, working in the class of stable infinite-dimensional weakly regular well-posed linear systems in the sense of G. Weiss.

Our main contribution is the sufficiency part, which was missing in the earlier theory, whereas our necessary conditions are extensions to the weakly regular case of those found independently by O. Staffans and M. & G. Weiss. These results are applied to the H^2 (LQR) and H^∞ (minimax) full information regulator problems and to the special case where the input/output map belongs to the Wiener class.

AMS Subject Classification 49J27, 93B36, 47A68

Keywords H^2 Control, suboptimal H^∞ control, (J, S)-spectral factorization, algebraic Riccati equation, weakly regular well-posed linear systems

Contents

1 Introduction

We study weakly regular Well-Posed Linear Systems (WPLSs) [Def2.1] presented in Weiss:TransferII and Weiss∗2. The class of weakly regular WPLSs is a generalization of both the class of finite-dimensional systems and the class of Pritchard–Salamon systems, but weak regularity allows them to still be written in the familiar form

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t) \\ y(t) &= C_w x(t) + Du(t) \end{aligned} \tag{1}$$

for almost all t [Weiss:TransferI:Th2.3] [Prop4.3c]; here C_w is (the weak) extension [Def4.2] of the output operator C presented in Salamon:InfDim (cf. Def2.2), where WPLSs were first introduced. The input space U, the state space H and the output space Y are Hilbert spaces.

We use the notation introduced by Staffans (see the symbol lists starting on pages 6 and 10). Thus the equations (1) are written in the integral form

$$\begin{aligned} x &= \mathcal{A}x_0 + \mathcal{B}\tau u \\ y &= \mathcal{C}x_0 + \mathcal{D}u \end{aligned}$$

(see Definition 2.1 for details), where $u, y \in L^2$ and $x_0 \in H$ is the initial state (at $t = 0$). Here A is the infinitesimal generator of the semigroup $\mathcal{A}$, $\mathcal{C}x_0 = C_w\mathcal{A}x_0\ \forall x_0 \in \mathrm{Dom}(A)$ and the equations

$$(\mathcal{B}\tau u)(t) = \int_0^t \mathcal{A}(t-s)Bu(s)\,ds,\ \ (\mathcal{D}u)(t) = C_w\int_0^t \mathcal{A}(t-s)Bu(s)\,ds + Du(t)$$

hold in a certain sense, see St:StQuadr:Sec7, Weiss:TransferI:(2.16)&Th4.6 or Weiss:TransferII:Sec2 for details. The weak regularity of a WPLS is, roughly speaking, equivalent to the assumption that the system has a feed-through operator D.[1]

[1] The equations (1) are true also for systems satisfying an even weaker regularity assumption [M:SubregInfrareg], but the use of that assumption would make the formulation of the results awkwardly complicated, hence they are not presented here.

In the (full information infinite horizon) H^2 or linear quadratic regulator (LQR) problem the aim is usually to choose the control $u_{\min}(x_0)$ for a given initial state x_0 to minimize the cost $Q = \|y\|_{L^2}$ (or a more complex cost function, see Definition 2.3).

The solution for WPLSs with a uniformly positive Popov operator (that is, $\pi_+\mathcal{D}^*\mathcal{D}\pi_+ \geq \epsilon I$) was found independently by Staffans [St:StQuadr:Th27] [St:Quadr:Th4.4] and M. & G. Weiss [Weiss∗2:Lemma8.2], and it was extended to the non-definite but invertible case in St:Crit:Th16, see Theorem 5.4. The uniform positivity of the Popov operator is equivalent to the existence of a spectral factorization [Def5.1] of $\mathcal{D}^*\mathcal{D}$ [St:Quadr:Lemma2.4i].

In the (full information infinite horizon) minimax H^∞ problem the input (now $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ instead of u) is separated into two parts, namely the control u and the disturbance w. The aim is to find a causal, time-invariant control law $u = \mathcal{U}w$ such that the norm of the mapping $w \mapsto y$ becomes less than a given number $\gamma > 0$.

Under a certain coercivity assumption (the minimax J-coercivity of $\mathcal{D}$ [Def8.2]) the optimal control u_{crit} and the worst disturbance w_{crit} form a minimax solution that is a saddle point of a two player dynamic game [St:StHinf].

The existence of a solution that can be represented in state feedback form is equivalent to the existence a spectral factorization of $\mathcal{D}^*J\mathcal{D}$ with a sensitivity operator compatible with the minimax form of the problem [Prop8.5&Th8.7].

For (weakly) regular systems having a regular [Def4.2] spectral factorization with an invertible feed-through operator, the Riccati operator corresponding to the optimal or minimax solution was shown to satisfy an algebraic Riccati equation, namely

$$A^\times \Pi + \Pi A + C^\times JC = (B_w^*\Pi + D^*JC)^\times (X^*SX)^{-1}(B_w^*\Pi + D^*JC)$$

(this is explained in Theorem 5.5), see St:Quadr:Th6.1v and Weiss∗2:Th12.8 for the H^2 problem and St:StHinf:Th12 for the H^∞ problem.

Here we extend these results to the class of weakly regular WPLSs having a weakly regular spectral factorization with a right-invertible feed-through operator [Th5.5].

After that we proceed with the main result of this work, Theorem 6.7, which addresses the converse question: in Section 6 we assume Π to be a self-adjoint stabilizing solution [Def6.3] of the Riccati equation corresponding to a weakly regular WPLS[2] and prove that the system has a spectral factorization which is weakly regular and has an invertible feed-through operator (these properties are implicitly contained the definition of the solution). We also prove the uniqueness of the solution [Prop6.9] and examine it further.

In Section 7 we summarize the above results on the equivalence of the existence of a self-adjoint stabilizing solution of the Riccati equation and the existence of a weakly regular spectral factorization [Th7.1]. Then we apply Theorem 7.1 to the standard and nonstandard H^2 problems, i.e., we establish the equivalence of the existence of a solution of the H^2 problem and the existence of a solution of the corresponding Riccati equation, under the regularity conditions mentioned above.

In Section 8 we review some H^∞ results from St:StHinf and apply Theorem 7.1 to establish the equivalence of the existence of a solution of the H^∞ minimax problem and the existence of a solution of the corresponding Riccati equation, this again under certain restricting assumptions [Th8.7].

For general (weakly) regular WPLSs, the coefficient S (the sensitivity operator) in the Riccati equation need no longer be equal to D^*JD (where $J = I$ in the standard

[2]We need some assumptions on the system to be able to define the coefficients in the equation; these assumptions are true in the case mentioned above [Lemma6.4]).

H^2 problem), but S has to be calculated in a more complex way, as noted by Staffans [St:Quadr:Cor7.2] and M. & G. Weiss [Weiss∗2:Rem12.9] (originally in St:SSQuadr). This forces us to make some extra assumptions that complicate the formulation of a stabilizing solution [Def6.3]. There is a slightly less general class of systems, the Wiener class[3], for which several aspects of the theory simplify as shown in Theorems 7.2 and 8.10.

For the Wiener class, the coefficients in the Riccati equation (see Theorems 7.2 and 8.10 and, in particular, their footnotes) are the same as those in the finite-dimensional case [GreenLim:(5.2.29)&p.251], or those in the case of WPLSs with bounded input and output operators ($B \in \mathcal{L}(U, H)$ and $C_w = C \in \mathcal{L}(H, Y)$) [CurtOostv(1)] (cf. Remark 2.5), or those in the Pritchard–Salamon case [Keulen:(3.60)&Rem3.13&(4.11)],[4] (except that for the Wiener class we still have to use the adjoint of the input operator B in the extended form $(B^*{}_w)$), hence Theorems 7.2 and 8.10 are, formally, straight-forward extensions of the classical results.

For a theory and results, similar to those of this paper, for discrete systems, see Malinen:WPD and Malinen:NDT. For a different approach to WPLSs and their Riccati equations, see FlandLasTrig (where the assumption of weak regularity has been replaced by the assumption of the boundedness of the output operator C, and, according to the authors, this assumption can easily be weakened).

When writing this paper, we tried to make it readable *per se*, although readers unfamiliar with WPLSs may find it hard to follow. Hence, on the first few pages, we mainly recall definitions, results and figures presented in papers by Staffans. Of course, the proofs still contain quite a few references to other papers. These references often point not to the original results but to papers that have the needed results in a form that best suits our needs.

We are very grateful to professor Staffans, who encouraged us to study this "converse direction", and whose suggestions have been very valuable to this work. We also want to thank professor G. Weiss, who gave us access to his unpublished study on weak regularity [Weiss:TransferII].

[3]With this we mean that the input/output map $\mathcal{D}$ of the system belongs to the Wiener class, i.e., $\mathcal{D}u = (L\delta_0 + f) * u$, where $L \in \mathcal{L}(U, Y) \land f \in L^1(\mathbb{R}_+, \mathcal{L}(U, Y))$.

[4]The three cases mentioned above are special cases of regular WPLSs in an increasing order of generality.

The class of regular WPLSs contains the class of all possible discrete time systems, and, as noted by Staffans [St:DCRicc], this fact forces the Riccati equation to take a form similar to the classical finite-dimensional discrete time Riccati equation (see also Proposition 5.6 and its footnote).

The same phenomenon can be seen in our results too, e.g., Theorem 8.7 has the standard features of the discrete case [GreenLim:ThB.2.2] not visible when the system belongs to the smooth Pritchard–Salamon class [Keulen:Th4.20].

1.1 Notation

We use the following notation (for the correspondence with the notation used by G. and M. Weiss (among others), see p. 10):

[Def2.1]: A reference to Definition 2.1 of this text; sometimes we write just [2.1]. [St:StQuadr:Sec7] is a reference to Section 7 of St:StQuadr (see "References" at the end of this text). [Weiss:TransferI:p.831] is a reference to page 831 of Weiss:Transfer. (4) is a reference to equation (4) of this text etc. When the reference is a part of a sentence, the brackets are removed (e.g., "see Weiss:Repr for more").

$A := B$: "A is equal to B by the definition of A".

$\mathcal{L}(U,Y)$, $\mathcal{L}(U)$: The set of bounded linear operators from U into Y or from U into itself, respectively.

I: The identity operator.

$(s-A)$: $(s-A) := sI - A$, when $s \in \mathbb{C}$.

$\exists A^{-1}$: There exists a bounded inverse of the operator A (in particular, A is one-to-one and onto).

$\sigma(A)$: The spectrum $\{s \in \mathbb{C} \,|\, \not\exists (s-A)^{-1}\}$ of A.

ω_A: The growth rate $\omega_A := \inf_{t>0}[t^{-1}\log\|\mathcal{A}(t)\|]$, when A is the infinitesimal generator of a semigroup $\mathcal{A}$ (cf. Pazy:Th1.5.3).

A^*, $A^\times$: A^* is the (Hilbert space) adjoint of the operator A; see Definition 2.2 for $A^\times$.

$\mathrm{Dom}(A)$: The domain of the (unbounded) operator A.

$\mathrm{Ran}(A)$: The range of the operator A.

$\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_-$: $\mathbb{R} := (-\infty,\infty)$, $\mathbb{R}_+ := [0,\infty)$, and $\mathbb{R}_- := (-\infty, 0]$.

$\mathbb{C}^+_\omega$, $\mathbb{C}_+$: $\mathbb{C}^+_\omega := \{s \in \mathbb{C} \,|\, \operatorname{Re} s > \omega\}$, $\mathbb{C}_+ := \mathbb{C}^+_0$.

$H^\infty(\mathbb{C}_+, X)$: bounded holomorphic functions $F : \mathbb{C}_+ \to X$.

$\hat{u}$: The (bilateral) Laplace transform of u, i.e., $\hat{u}(s) := \int_{\mathbb{R}} e^{-st}u(t)\,dt$.

$P(J;\mathcal{L}(U,Y))$: The set of functions $F : J \to \mathcal{L}(U,Y)$ for which $Fu_0 : J \to Y$ is strongly measurable $\forall u_0 \in U$.

$P^p(J;\mathcal{L}(U,Y))$: The space of (equivalence classes of) functions $F \in P(J;\mathcal{L}(U,Y))$ for which $\|F\|_{P^p} < \infty$, where $\|F\|_{P^p} := \sup_{u\in U}\|Fu\|_{L^p}$.

$L^2(J;U)$: The set of U-valued L^2-functions on J.

$L^2_\omega(J;U)$: The set $\{\, u \in L^2_{\mathrm{loc}}(J;U) \;\big|\; (t \mapsto \mathrm{e}^{-\omega t}u(t)) \in L^2(J;U) \,\}$.

$W^{1,2}(J;U)$: The set of functions in $L^2(J;U)$ with a (distribution) derivative in $L^2(J;U)$.

$\mathcal{C}(J,U)$: The set of continuous functions $J \to U$.

$\mathcal{C}^1(J,U)$: The set of continuously differentiable functions $J \to U$.

$\mathcal{C}_c(J,U)$: The set of compactly supported functions $f \in \mathcal{C}(J,U)$.

$\langle\cdot,\cdot\rangle_H$: The inner product in the Hilbert space H.

$\tau(t)$: The bilateral time shift operator $\tau(t)u(s) = u(t+s)$ (this is a left-shift when $t>0$ and a right-shift when $t<0$).

π_J: $(\pi_J u)(s) := u(s)$ if $s \in J$ and $(\pi_J u)(s) := 0$ if $s \notin J$. Here J is a subset of $\mathbb{R}$. This operator is used both as a projection operator $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ and as an embedding operator $L^2(J) \to L^2(\mathbb{R})$.

π_+, π_-: $\pi_+ := \pi_{\mathbb{R}_+}$ and $\pi_- := \pi_{\mathbb{R}_-}$.

$A \ge B$: iff $B \le A$ iff $\langle (A-B)x, x\rangle \ge 0\ \forall x$.

$A >> B$: iff $B << A$ iff for some $\epsilon > 0$ we have $\langle (A-B)x, x\rangle \ge \epsilon\|x\|^2\ \forall x$.

TI(U,Y): The set of operators $\mathcal{D} \in \mathcal{L}(L^2(\mathbb{R},U); L^2(\mathbb{R},Y))$ that are time-invariant (i.e., $\tau(t)\mathcal{D} = \mathcal{D}\tau(t)\ \forall t \in \mathbb{R}$).
TIC(U,Y): The set of operators $\mathcal{D} \in$ TI(U,Y) that are causal ($\pi_-\mathcal{D}\pi_+ = 0$).
$\mathcal{W}_+(U,Y)$: The set of measures of the form $F + L\delta_0$, where $F \in L^1(\mathbb{R}_+, \mathcal{L}(U,Y)) \wedge L \in \mathcal{L}(U,Y)$ and $\delta_0 * u = u\ \forall u \in L^2$ (the delta distribution).[5] Note that, by the Riemann–Lebesgue lemma, $\widehat{F + L\delta_0}(s) \to 0$, when $\overline{\mathbb{C}_+} \ni s \to \infty$.
$\mathcal{W}_+ * (U,Y)$: $\mathcal{D} \in \mathcal{W}_+ *$ iff $\mathcal{D} \in$ TIC is such that for some $\gamma \in \mathcal{W}_+$ we have $\mathcal{D}u = \gamma * u\ \forall u \in L^2$.
WPLS, CWPLS, OSCWPLS: See Definition 2.1.
U, H, Y; $\mathcal{U}, \mathcal{Y}$: Hilbert spaces of arbitrary dimension [Def2.1&Rem2.5].
$\Psi, \Psi^*, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$: See Definition 2.1.
A, B, C; W, V, V^*, W^*; W_B, V_C^*; $A^\times$: See Definition 2.2.
D: The feed-through operator of $\mathcal{D}$. See Definition 4.2.
Q, J: The cost function and the cost operator, respectively. See Definition 2.3.
$\Psi_{\text{ext}}, \mathcal{K}, \mathcal{F}$: See Definition 3.3 and Theorem 5.4.
$S, \mathcal{X}$: The sensitivity operator and the spectral factor, respectively [Def5.1].
Π; $\mathcal{N}, \mathcal{M}$; $\Psi_{\circlearrowleft}, \mathcal{A}_{\circlearrowleft}, \mathcal{B}_{\circlearrowleft}, \ldots$: See Lemma 2.4 and Theorem 5.4.
C_w, K_w, B_w^*: The weak Weiss extensions of C, K and B^*, respectively [Prop4.3].
s.t.: "such that" or "so that".
iff: "if and only if".
admissible: See Definitions 3.1 and 3.3.

We extend an L^2-function u defined on a subinterval J of $\mathbb{R}$ to the whole real line by requiring u to be zero outside of J, and we denote the extended function by $\pi_J u$. Thus, we use the same symbol π_J both for the embedding operator $L^2(J) \to L^2(\mathbb{R})$ and for the corresponding orthogonal projection operator $L^2(\mathbb{R}) \to \text{Ran}(\pi_J)$. With this interpretation, $\pi_+ L^2(\mathbb{R};U) = L^2(\mathbb{R}_+;U) \subset L^2(\mathbb{R};U)$ and $\pi_- L^2(\mathbb{R};U) = L^2(\mathbb{R}_-;U) \subset L^2(\mathbb{R};U)$.

2 A Review of Well-Posed Linear Systems

We recall the following basic definition from St:Crit:

Definition 2.1 *Let U, H, and Y be Hilbert spaces, and let $\omega \in \mathbb{R}$. A* Causal ω-stable Well-Posed Linear System on (U,H,Y) *(*CWPLS$_\omega(U,H,Y)$*) is a quadruple $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$, where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are bounded linear operators of the following type:*

1. *$\mathcal{A}(t)\colon H \to H$ is a strongly continuous semigroup of bounded linear operators on H satisfying $\sup_{t\in\mathbb{R}_+} \|e^{-\omega t}\mathcal{A}(t)\| < \infty$;*
2. *$\mathcal{B}\colon L^2_\omega(\mathbb{R};U) \to H$ satisfies $\mathcal{A}(t)\mathcal{B}u = \mathcal{B}\tau(t)\pi_- u$ for all $u \in L^2_\omega(\mathbb{R};U)$ and $t \in \mathbb{R}_+$;*
3. *$\mathcal{C}\colon H \to L^2_\omega(\mathbb{R};Y)$ satisfies $\mathcal{C}\mathcal{A}(t)x = \pi_+\tau(t)\mathcal{C}x$ for all $x \in H$ and $t \in \mathbb{R}_+$;*
4. *$\mathcal{D}\colon L^2_\omega(\mathbb{R};U) \to L^2_\omega(\mathbb{R};Y)$ satisfies $\tau(t)\mathcal{D}u = \mathcal{D}\tau(t)u$, $\pi_-\mathcal{D}\pi_+ u = 0$, and $\pi_+\mathcal{D}\pi_- u = \mathcal{C}\mathcal{B}u$ for all $u \in L^2_\omega(\mathbb{R};U)$ and $t \in \mathbb{R}$.*

[5] We could have, equivalently, defined the Wiener class to be $P^1 + \mathcal{L}\delta_0$ instead of $L^1 + \mathcal{L}\delta_0$, because $n := \dim U < \infty$ implies that $\mathcal{L}(U,Y) \cong Y^n$ and hence $P^1(J, \mathcal{L}(U,Y)) = L^1(J, \mathcal{L}(U,Y))$.

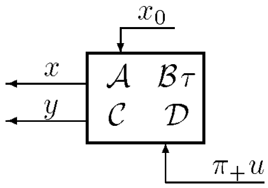

Figure 1: Input/State/Output Diagram of Ψ

If, moreover, $\mathrm{e}^{-\omega t}\mathcal{A}(t)x \to 0$ *as* $t \to \infty$ *for all* $x \in H$*, then* Ψ *is* strongly ω-stable*. If conditions (i) and (ii) hold for some* ω *and conditions (iii) and (iv) hold for* $\omega = 0$*, then* Ψ *is an* Output Stable Causal Well-Posed Linear System *(*$\mathrm{OSCWPLS}_0$*).*[6]

The different components of Ψ *are named as follows:* U *is the input space,* H *the state space,* Y *the output space,* $\mathcal{A}$ *the semigroup,* $\mathcal{B}$ *the controllability map,* $\mathcal{C}$ *the observability map, and* $\mathcal{D}$ *the input/output map of* Ψ*. In the initial value setting with initial time zero, initial value* $x_0 \in H$*, and control* $u \in L^2_\omega(\mathbb{R}_+, U)$*, the controlled state* $x(t) \in H$ *at time* $t \in \mathbb{R}_+$ *and the observation* $y \in L^2_\omega(\mathbb{R}_+, Y)$ *of* Ψ *are given by (cf. Figure 1)*

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{A}(t) & \mathcal{B}\tau(t) \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} x_0 \\ \pi_+ u \end{bmatrix} = \begin{bmatrix} \mathcal{A}(t)x_0 + \mathcal{B}\tau(t)\pi_+ u \\ \mathcal{C}x_0 + \mathcal{D}\pi_+ u \end{bmatrix}. \tag{2}$$

In the time-invariant setting, the controlled state $x(t) \in H$ *at time* $t \in \mathbb{R}$ *and the output* $y \in L^2_\omega(\mathbb{R}, Y)$ *of* Ψ *with control* $u \in L^2_\omega(\mathbb{R}, U)$ *are given by*

$$\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{B}\tau(t)u \\ \mathcal{D}u \end{bmatrix}.$$

We call Ψ *a causal well-posed linear system on* (U, H, Y) *(*$\mathrm{CWPLS}(U, H, Y)$*) iff it is an* ω*-stable causal well-posed linear system on* (U, H, Y) *for some* $\omega \in \mathbb{R}$*. We also use* CWPLS *to denote the class of such operators, similarly for* OSCWPLS *etc.*

We call $\Psi^* := \left[\begin{smallmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & \mathcal{D}^* \end{smallmatrix}\right]$ *(the adjoint or the anti-causal dual of* Ψ*) an anti-causal well-posed linear system iff* $\Psi \in \mathrm{CWPLS}$*.*[7] *A* WPLS *is a causal or anti-causal Well-Posed Linear System.*

Intuitively, the controllability map $\mathcal{B}$ maps past controls into the present state, the observability map $\mathcal{C}$ maps the present state into future observations, and the input/output map $\mathcal{D}$ maps inputs into outputs in a causal way. The condition "4."

[6]These definitions are as in St:Crit except for the concept of output stability, which seems the weakest reasonable assumption for the purposes of this paper. Output stability is equivalent to the condition that the system maps the initial state $x_0 \in H$ and input $u \in L^2$ continuously to the output $y = \mathcal{C}x_0 + \mathcal{D}\tau u$, i.e., it is equivalent to the stability assumption used in Weiss∗2:Sec2.

Staffans' results for stable systems [St:StQuadr&Crit&Hinf] hold also in this output stable case, because the stability of $\mathcal{A}$ and $\mathcal{B}$ is not used in the proofs (and in St:StQuadr $\mathcal{A}$ is not even assumed to be stable), as pointed out to us by Staffans (cf. St:Quadr:Rem2.7).

[7]The state $x^*(t)$ and observation $u^*(t)$ of Ψ^* are defined as for Ψ but with reversed time-axis, i.e., $x^*(s) := \mathcal{A}^*(-s)x_0 + \mathcal{C}^*\tau(s)y^* \wedge u^* = \mathcal{B}^*x_0 + \mathcal{D}^*y^*\ \forall s < 0$ in the initial value setting and $x^* := \mathcal{C}^*\tau y^* \wedge u^* = \mathcal{D}^*y^*$ in the time-invariant setting.

We need Ψ^* just to simplify some proofs, hence we do not present it more deeply. For more information on Ψ^* see St:Coprime:Def2.13 (or St:StQuadr:Def8&Def9) and note that the adjoints $\mathcal{B}^*$, $\mathcal{C}^*$ and $\mathcal{D}^*$ are taken with respect to L^2 (i.e., $\langle L^2_\omega, L^2_{-\omega}\rangle$, not $\langle L^2_\omega, L^2_\omega\rangle$) inner product [St:Coprime].

M. & G. Weiss use the causal dual Ψ^d instead of Ψ^* [Weiss∗2:Prop6.1]. All though they have to use it in the anti-causal way [Weiss∗2:Sec8] as Staffans does, the causal definition removes the need to duplicate their definitions results and for the anti-causal case.

imposed on $\mathcal{D}$ with $\omega = 0$ requires that $\mathcal{D} \in TIC(U;Y)$ and that the Hankel operator [St:StQuadr:Def3] induced by $\mathcal{D}$ is equal to $\mathcal{CB}$. The definitions of this section are more widely explained in St:StQuadr, St:Crit, St:Coprime and St:Quadr.

Definition 2.2 *As in St:StQuadr&Quadr&Crit&StHinf, we denote the generators (and feed-through operators) with the same letters as corresponding maps, e.g., we denote the generators of* $\Psi := \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] \in \mathrm{CWPLS}_\omega$ *by* $\left[\begin{smallmatrix} A & B \\ C & ? \end{smallmatrix}\right]$.[8]

Choose $\alpha \in \sigma(A)^c$. *We define* $W := \mathrm{Dom}(A)$ *(with the graph norm) and* $V :=$ *the completion of* H *with under the norm* $\|(\alpha - A)^{-1} \cdot \|_H$ *(thus* $W \subset H \subset V$*). Similarly* $V^* := \mathrm{Dom}(A^*) \subset H \subset W^* := \mathrm{cl}_{\|(\bar{\alpha}-A^*)^{-1}\cdot\|_H}(H)$. *Note that the unique extension* $\mathcal{A}_V$ *of* $\mathcal{A}$ *onto* V *is a semigroup isomorphic to the original* $\mathcal{A}$ *and the generator of* $\mathcal{A}_V$ *is an extension of* A *(cf. St:StQuadr:Sec7 or Weiss:TransferI:p.831 etc.) which will be denoted by* $\bar{A}$.

We extend $\langle w, x\rangle_{\langle W,W^*\rangle} := \langle w, x\rangle_H \ \forall w \in W \ \forall x \in H$ *continuously to* $W \times W^*$ *to get an interpretation of* W^* *as the dual of* W*, and we do the same for* $\langle x, v^*\rangle_{\langle V,V^*\rangle} := \langle v, x\rangle_H \ \forall x \in H \ \forall v \in V^*$ *[St:StQuadr:Sec7] [Keulen:Subsec2.5]. As above,* $A^* : \mathrm{Dom}(A^*) \to H$ *means the adjoint of the unbounded operator* A. *By* $A^\times \in \mathcal{L}(H, W^*)$ *we denote the "adjoint" of the bounded operator* $A \in \mathcal{L}(W, H)$*, defined by* $\langle Aw, x\rangle_H = \langle w, A^\times x\rangle_{\langle W,W^*\rangle} \ \forall w \in W \ \forall x \in H$.[9]

Finally, we define $W_B := (\alpha - A)^{-1}[H + BU] = \{x_0 \in H \mid \exists u_0 \in U \ [Ax_0 + Bu_0 \in H]\} \subset H$ *with* $\|z\|_{W_B} := \inf\{\|(x,u)\|_{H\times U} \mid (\alpha - A)^{-1}(x + Bu) = z\}$*, similarly* $V_C^* := (\bar{\alpha} - A^*)^{-1}[H + C^*Y] \subset H$ *with* $\|z\|_{V_C^*} := \inf\{\|(x,y)\|_{H\times Y} \mid (\bar{\alpha} - A^*)^{-1}(x + C^*y) = z\}$ *[St:StQuadr:Lemma32].*

Next we recall some further definitions and results from St:Crit.

Definition 2.3 *[St:Crit:Def2&Def4] Let* $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a stable well-posed linear system on* (U, H, Y)*, and let* $J = J^* \in \mathcal{L}(Y)$. *Define the* cost function Q *by (cf. Remark 2.5)*

$$Q(x_0, u) = \int_{\mathbb{R}_+} \langle y(s), Jy(s)\rangle_Y \ ds,$$

where $y = \mathcal{C}x_0 + \mathcal{D}\pi_+ u$ *is the observation of* Ψ *with initial value* $x_0 \in H$ *and control* $u \in L^2(\mathbb{R}_+; U)$. *A control* $u_{\mathrm{crit}}(x_0)$ *is* J-critical *if the (real) Fréchet derivative of* Q *with respect to* u *vanishes at* $(x_0, u_{\mathrm{crit}}(x_0))$.

The system Ψ *is* J-coercive *iff its input/output map* $\mathcal{D} \in \mathrm{TIC}(U,Y)$ *is* J-coercive *iff the Toeplitz operator ("Popov operator")* $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+$ *is invertible in* $\mathcal{L}(L^2(\mathbb{R}_+; U))$.

If Q is strictly convex, then there is a unique J-critical control u_{crit}, namely the unique Q-minimizing control (cf. St:StQuadr:Lemma13ii). In the minimax H^∞ control problem treated in Section 8 we seek a saddle point of Q, i.e., the J-coercivity of the control is again a necessary condition for the solution of the problem [St:Crit:Sec1]&[St:StHinf:Sec1&Lemma19

[8] This means that A is the infinitesimal generator of $\mathcal{A}$, and $B \in \mathcal{L}(U, V)$ and $C \in \mathcal{L}(W, Y)$ are such that $x_0 \in H \wedge u \in L^2_\omega \wedge x = \mathcal{A}x_0 + \mathcal{B}\tau u \implies x' = \bar{A}x + Bu$ in V a.e. on $\mathbb{R}_+$ and $x_0 \in W \wedge y = \mathcal{C}x_0 \implies y = Cx$ on $\mathbb{R}_+$; see, e.g., St:StQuadr:Prop29 or Weiss:TransferI for more details. Note that $\left[\begin{smallmatrix} A & B \\ C & ? \end{smallmatrix}\right]$ generates infinitely many different systems, since it does not determine $\mathcal{D}$ uniquely. However, if Ψ is weakly regular [Def4.2] with feed-through operator D, then $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ determines Ψ uniquely (and vice versa), e.g., $\widehat{\mathcal{D}} = C_w(s - A)^{-1}B + D$ [Prop4.3], and, in that case, by generators of Ψ we mean $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.

[9] Note that all the inner products and other pairings here are sesquilinear, as usually. We need $A^\times$ only to simplify Riccati equations [Th5.5b].

Lemma 2.4 *[St:Crit:Lemma5&Def6] Let* $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right] \in \mathrm{OSCWPLS}_0(U,H,Y)$ *be J-coercive, where* $J = J^* \in \mathcal{L}(Y)$*, and define*

$$\begin{aligned}
\mathcal{A}_{\circlearrowleft} &= \mathcal{A} - \mathcal{B}\tau\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\mathcal{C},\\
\mathcal{C}_{\circlearrowleft} &= \left(I - \mathcal{D}\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\right)\mathcal{C},\\
\mathcal{K}_{\circlearrowleft} &= -(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\mathcal{C},\\
\Pi &= \mathcal{C}^*\left(J - J\mathcal{D}\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J\right)\mathcal{C}.
\end{aligned}$$

Then, for every $x_0 \in H$*, there is a unique J-critical control* $u_{\mathrm{crit}}(x_0)$*, and it is given by* $u_{\mathrm{crit}}(x_0) = \mathcal{K}_{\circlearrowleft}x_0$*.*

The corresponding critical state $x_{\mathrm{crit}}(x_0)$*, the critical observation* $y_{\mathrm{crit}}(x_0)$*, and the critical value of Q are given by*

$x_{\mathrm{crit}}(x_0) = \mathcal{A}_{\circlearrowleft}x_0$*,* $y_{\mathrm{crit}}(x_0) = \mathcal{C}_{\circlearrowleft}x_0$*, and* $Q(x_0, u_{\mathrm{crit}}(x_0)) = \langle x_0, \Pi x_0\rangle_H$*.*

We call Π *the* Riccati operator *of* Ψ *(with cost operator J).*

Another (equivalent) way to define a (causal) well-posed linear system (alias an abstract linear system) is to replace the controllability map $\mathcal{B}$ with a family of operators $(\Phi_t)_{t\ge 0}$ (with $\Phi_t = \mathcal{B}\tau(t)\pi_{[0,t]}$) and define the state to be $x(t) = \mathcal{A}(t)x_0 + \Phi_t u$ [Weiss:TransferI:Def2.1]. However, we have chosen the above definitions mainly borrowed from St:Crit, because this makes the formulation of some arguments more fluent and simplifies the references to the papers by Staffans. As noted in St:Coprime:Lemma2.6, an abstract linear system (as defined by Weiss) is always a CWPLS_ω for any ω greater than the exponential growth rate of the semigroup $\mathcal{A}$, hence the definitions used by Weiss and Staffans are equivalent.

Thus, the notation used by Staffans (in, e.g., in St:StQuadr) and this paper (with the exception that we use C_Λ and C_w instead of $\bar{C}$ and C_{Λ_w} etc.) relates to the G. Weiss' and M. Weiss' notation (in, e.g., Weiss:TransferI and [Weiss*2]) in the following way (Staffans' notations $\doteq$ Weiss' notation):

$H \doteq X$ (the state space), $U \doteq U$ (the input space), $Y \doteq Y$ (the output space) (complex Hilbert spaces of any dimension).
$\mathcal{A}(t) \doteq \mathbb{T}_t \in \mathcal{L}(X)$ $\forall t \ge 0$ (the semigroup), $A \doteq A$ (the infinitesimal generator of $\mathcal{A}$).
$\mathcal{B} \in \mathcal{L}(\pi_- L^2_\omega(\mathbb{R},U);X))$ (the controllability map) is the operator for which we have $\mathcal{B}\tau(t)\pi_{[0,t]} \doteq \Phi_t$,
$\mathcal{C} \doteq \Psi_\infty \in \mathcal{L}(H, \pi_+ L^2_\omega(\mathbb{R},Y))$ (the observability map) ($\pi_{[0,t]}\mathcal{C} \doteq \Psi_t$),
$\mathcal{D} \in \mathcal{L}(L^2_\omega(\mathbb{R},U); L^2_\omega(\mathbb{R},Y))$ (the input/output map) is the causal, time-invariant operator for which $\mathcal{D}\pi_+ \doteq \mathbb{F}_\infty$ (and hence $\pi_{[0,t]}\mathcal{D}\pi_+ \doteq \mathbb{F}_t$).[10]
$\left[\begin{smallmatrix}A & B\\ C & ?\end{smallmatrix}\right] \doteq \left[\begin{smallmatrix}A & B\\ C & ?\end{smallmatrix}\right]$ (the generating operators of Ψ), see Def2.2 or Weiss:TransferI.
$W \doteq X_1 := \mathrm{Dom}(A)$, $V \doteq X_{-1} := \mathrm{cl}_{\|(\alpha-A)^{-1}\cdot\|}(H)$ [Weiss:TransferI:p.831], in particular, $W \subset H \subset V$;
$V^* \doteq Z_1 := \mathrm{Dom}(A^*)$, $W^* \doteq Z_{-1} := \mathrm{cl}_{\|(\bar{\alpha}-A^*)^{-1}\cdot\|}(H)$ similarly [Weiss*2:Sec6].
$W_B \doteq \{x_0 \in H \mid \exists u_0 \in U\ [Ax_0 + Bu_0 \in H]\}$ [Def2.2] [Prop4.3];
$V_C^* \doteq \{x_0 \in H \mid \exists y_0 \in Y\ [A^*x_0 + C^*y_0 \in H]\}$.
$\overline{B^*} \doteq (B^*)_\Lambda = B^*_\Lambda = \lim_{s\to+\infty} B^*s(s-A^*)^{-1}$, $\overline{C} \doteq C_\Lambda$, $\overline{K} \doteq K_\Lambda$ [Prop4.3].[11]
$\widehat{\mathcal{D}} \doteq \mathbb{H}$ (the transfer function) [Prop4.1].
$\pi_+\widehat{\mathcal{D}^*J\mathcal{D}}\pi_+ \doteq \Pi$ (the Popov function), $\Pi \doteq X$ (the Riccati operator), $\mathcal{X} \doteq \Xi$ (the

[10]Note that $\mathcal{D}$ is uniquely determined by $\mathcal{D}\pi_+ \doteq \mathbb{F}_\infty$, and that $\|\mathcal{D}\|_{\mathcal{L}} = \|\mathcal{D}\pi_+\|_{\mathcal{L}}$.

[11]In this paper we prefer C_Λ to $\bar{C}$ and we mainly use the weak extensions $B^*_w := B^*_{\Lambda_w}$, $C_w := C_{\Lambda_w}$, $K_w := K_{\Lambda_w}$, where $C \subset C_\Lambda \subset C_w$ etc., cf. Proposition 4.3.

spectral factor [5.1]) $X \doteq D$ (its feed-through operator) (in particular, $X^*SX \doteq D^*D$ for $S >> 0$), $K \doteq F$ (the state feedback operator) $D^*JD \doteq R$ (see Remark 2.5).

Remark 2.5 *We get the formulation of the Riccati equation used by Weiss [Weiss*2:Th12.8]*

$$A^\times \Pi + \Pi A + C^\times QC = (B_w^*\Pi + NC)^\times (X^*X)^{-1}(B_w^*\Pi + NC)$$

from Staffans' formulation [Th5.5b], if we set $Y := \mathcal{Y} \times \mathcal{U}$, $U := \mathcal{U}$, $J := \left[\begin{smallmatrix} Q & N^* \\ N & R \end{smallmatrix}\right]$ $\mathcal{D} := \left[\begin{smallmatrix} \mathcal{D}_1 \\ I \end{smallmatrix}\right]$, $\mathcal{C} := \left[\begin{smallmatrix} \Psi_\infty \\ 0 \end{smallmatrix}\right]$, *where* $\mathcal{D}_1 \in \mathrm{TIC}(U, \mathcal{Y})$ *is the unique* TIC*-extension of* $\mathbb{F}$, *normalize the feed-through operator of* $\mathcal{D}$ *to be to* $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right] \in \mathcal{L}(\mathcal{Y} \times \mathcal{U})$, *and suppose that* $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+ >> 0$ *as in Weiss*2:Sec2 (cf. St:Quadr:Cor8.1).*[12] *In this case, the cost function [Def2.3] can be written in the form [Weiss*2:(2.8)]*

$$Q(x_0, u) = \int_0^\infty \left\langle \begin{bmatrix} Q & N^* \\ N & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}, \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}, \right\rangle dt.$$

The simple proof of the following often used lemma is left to the reader.

Lemma 2.6 *Let* A, B, C, D, S, T, U, V, x, y *and* z *be continuous linear operators between some topological vector spaces such that the following formulas are well defined (e.g.,* $A, S, z \in \mathcal{L}(X) \wedge C, U, x \in \mathcal{L}(X, Y) \wedge B, T, y \in \mathcal{L}(Y, X) \wedge D, V \in \mathcal{L}(Y)$ *where* X *and* Y *are TVSs). Then the following claims hold (here, as elsewhere, the inverses are required to be continuous and everywhere defined):*[13]

(a1) $z(I + z)^{-1} = (I + z)^{-1}z = I - (I + z)^{-1}$, *if* $\exists (I + z)^{-1}$.

(a2) $\exists (I - xy)^{-1} \implies \exists (I - yx)^{-1} = I + y(I - xy)^{-1}x \wedge y(I - xy)^{-1} = (I - yx)^{-1}y$.

(b1) Let $\exists \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^{-1} =: \left[\begin{smallmatrix} S & T \\ U & V \end{smallmatrix}\right]$.
Then $\exists S^{-1} \iff \exists D^{-1}$. *Moreover, if* $\exists S^{-1}$, *then* $D^{-1} = V - US^{-1}T$.

(b2) Let $\exists A^{-1}$. *Then* $\exists \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^{-1}$ *iff* $\exists (D - CA^{-1}B)^{-1}$. *Moreover, if* $\exists (D - CA^{-1}B)^{-1}$, *then* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{smallmatrix}\right]$.

3 Output Feedback

We formulate the solutions to the H^2 and H^∞ problems by using output feedback, hence in this short section we recall shortly a part of St:StQuadr:Sec5 adapted for output stable systems (as noted in a footnote to Definition 2.1, the same proofs apply).

In the output feedback we feed a part $z = Ly$ of the output y of a well-posed linear system Ψ back into the input, as Figure 2 shows. Here L is a bounded linear operator from the output space into the input space. Then, in the initial value setting with initial value x_0 and input v, we find that the effective input u, the state $x(t)$ at time $t \geq 0$, the output y, and the feedback control signal z satisfy the equations

$$\begin{array}{rclrcl} u & = & z + \pi_+ v, & x(t) & = & \mathcal{A}(t)x_0 + \mathcal{B}\tau(t)u, \\ y & = & \mathcal{C}x_0 + \mathcal{D}u, & z & = & Ly, \end{array} \tag{3}$$

which can be uniquely solved in terms of x_0 and $\pi_+ v$ iff $(I - \mathcal{D}L)$ is invertible (cf. Weiss:Feedback:Prop3.6). We call such an L admissible:

[12] The assumption $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$ does not reduce generality, because one can set R to be D^*JD.

[13] This lemma is clearly true, if all the operators (and inverses) are required to belong to TIC.

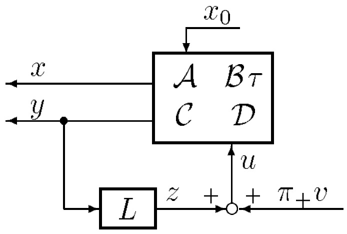

Figure 2: Static Output Feedback

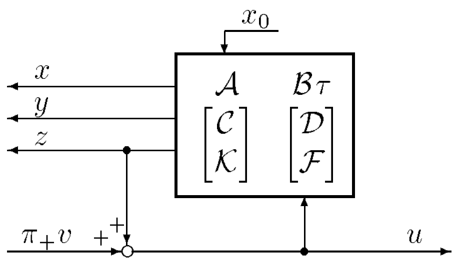

Figure 3: State Feedback Connection

Definition 3.1 *[St:StQuadr:Def19] Let* $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right] \in \mathrm{OSCWPLS}_0(U,H,Y)$*. The operator* $L \in \mathcal{L}(Y,U)$ *is called an* admissible stable output feedback operator *for* Ψ *iff* $\exists (I - L\mathcal{D})^{-1} \in \mathrm{TIC}(U)$*, i.e., iff* $\exists (I - \mathcal{D}L)^{-1} \in \mathrm{TIC}(Y)$*.*

As proved in Weiss:Feedback:Sec6, x and y in (3) can be interpreted as the state and output of another well-posed linear system:

Proposition 3.2 *[St:StQuadr:Prop20] Let* $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right] \in \mathrm{OSCWPLS}_0(U,H,Y)$ *and let* $L \in \mathcal{L}(Y;U)$ *be an admissible stable output feedback operator for* Ψ*. Then* $\Psi_L \in \mathrm{OSCWPLS}_0(U,H,Y)$*, where*

$$\Psi_L = \begin{bmatrix}\mathcal{A}_L & \mathcal{B}_L\tau\\ \mathcal{C}_L & \mathcal{D}_L\end{bmatrix} = \begin{bmatrix}\mathcal{A} + \mathcal{B}\tau L\,(I - \mathcal{D}L)^{-1}\,\mathcal{C} & \mathcal{B}\,(I - L\mathcal{D})^{-1}\,\tau\\ (I - \mathcal{D}L)^{-1}\,\mathcal{C} & \mathcal{D}\,(I - L\mathcal{D})^{-1}\end{bmatrix}.$$

We call Ψ_L *the closed loop system with output feedback operator* L*. In the initial value setting with initial time zero, initial value* x_0*, and control* v*, the controlled state* $x(t)$ *at time* t *and the output* y *of* Ψ_L *form the unique solution of equations (3).*

A state feedback can be reduced to an output feedback as follows. The appropriate connection has been drawn in Figure 3.

Definition 3.3 *Let* $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right] \in \mathrm{OSCWPLS}_0(U,H,Y)$ *and let* $L \in \mathcal{L}(Y;U)$ *be an admissible stable output feedback operator for* Ψ*. The pair* $\begin{pmatrix}\mathcal{K} & \mathcal{F}\end{pmatrix}$ *is called an admissible stable state feedback pair for* Ψ *iff the extended system*

$$\Psi_{\mathrm{ext}} := \begin{bmatrix}\mathcal{A} & \mathcal{B}\\ \begin{pmatrix}\mathcal{C}\\ \mathcal{K}\end{pmatrix} & \begin{pmatrix}\mathcal{D}\\ \mathcal{F}\end{pmatrix}\end{bmatrix} \in \mathrm{OSCWPLS}_0(U,H,Y)$$

and $L := \begin{pmatrix}0 & I\end{pmatrix}$ *is an admissible stable output feedback operator for* Ψ_{ext}*, i.e.,* $\exists (I - \mathcal{F})^{-1} \in \mathrm{TIC}(U)$*.*

4 Strong and Weak Regularity

To formulate Riccati equations [Th5.5], we need certain feed-through operators. The existence of a feed-through operator of some $\mathcal{D} \in \mathrm{TIC}$ is, roughly speaking, equivalent to (weak) regularity of $\mathcal{D}$. Hence we recall some basic facts about weak and strong regularity in this section.

The transfer functions in the infinite-dimensional setting are similar to those in the finite-dimensional case:

Proposition 4.1 *[Weiss:TransferI:Th3.1] For each $\mathcal{D} \in \mathrm{TIC}(U,Y)$ there is a unique function $\widehat{\mathcal{D}} \in H^\infty(\mathbb{C}_+, \mathcal{L}(U,Y))$, called the* transfer function *of $\mathcal{D}$, s.t. $\widehat{\mathcal{D}u} = \widehat{\mathcal{D}}\hat{u}$ on $\mathbb{C}_+$ for all $u \in L^2(\mathbb{R}_+, U)$. The mapping $\mathcal{D} \mapsto \widehat{\mathcal{D}}$ is an isometric isomorphism of* TIC *onto H^∞.*[14]

Weiss [Weiss:TransferI:Th5.8] gives eight equivalent characterizations of regularity and does the same for weak regularity in Weiss:TransferII. We recall some of his results in Definition 4.2 and Proposition 4.3:

Definition 4.2 *$\mathcal{D} \in \mathrm{TIC}(U,Y)$ is called* weakly regular *iff*

$$\exists Du_0 := \widehat{\mathcal{D}}(+\infty)u_0 := \operatorname*{w-lim}_{s\to+\infty} \widehat{\mathcal{D}}(s)u_0 \ \forall u_0 \in U,$$

i.e., iff the transfer function $\mathcal{D}$ has a weak limit in infinity along the positive real axis. D is called the feed-through operator *of $\mathcal{D}$. Clearly in this case (and only this case) we have $\exists D^* y_0 = \text{w-lim}_{s\to+\infty} \widehat{\mathcal{D}}(s)^* y_0 \ \forall y_0 \in Y$, hence then we also say that $\mathcal{D}^*$ is weakly regular.*

$\mathcal{D} \in \mathrm{TIC}(U,Y)$ is called (strongly) regular *iff $\exists \lim_{s\to+\infty} \widehat{\mathcal{D}}(s)u_0 \ \forall u_0 \in U$ (the strong limit of $\widehat{\mathcal{D}}$ along the positive real axis). $\mathcal{D}^*$ is called* (strongly) regular *iff $\exists \lim_{s\to+\infty} \widehat{\mathcal{D}}^*(s)u_0 \ \forall u_0 \in U$.*

We call $\Psi = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right] \in \mathrm{CWPLS}$ regular (resp. weakly regular) iff its input/output map $\mathcal{D}$ is regular (resp. weakly regular); similarly for Ψ^* and $\mathcal{D}^*$. When Ψ is weakly regular, we mean by the generators *of Ψ the operators $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ (cf. Definition 2.2).*

We shall denote feed-through operators by the same letter as corresponding weakly regular operators, as above.

Note that each $\mathcal{D} \in \mathcal{W}_+*$ is regular, by the Riemann–Lebesgue lemma. Note also, that $\alpha\mathcal{D} + \beta\widetilde{\mathcal{D}}$, $L\mathcal{D}$ and $\mathcal{D}\mathcal{E}$ are regular (resp. weakly regular) for any $\alpha, \beta \in \mathbb{C}$, $L \in \mathcal{L}(Y,H)$, if $\mathcal{D} \in \mathrm{TIC}$ and $\widetilde{\mathcal{D}} \in \mathrm{TIC}$ are regular (resp. weakly regular) and $\mathcal{E} \in \mathrm{TIC}$ is regular.

Proposition 4.3 *Let $\Psi \in \mathrm{CWPLS}(U,H,Y)$ have generating operators $\left[\begin{smallmatrix} A & B \\ C & ?\end{smallmatrix}\right]$ [Def2.2].*

(a) $W_B \subset \mathrm{Dom}(C_\Lambda) := \{x \in H \,|\, \exists C_\Lambda x := \lim_{s\to+\infty} Cs(s-A)^{-1}x\}$ iff Ψ is regular. In this case, $C_\Lambda \in \mathcal{L}(W_B, Y)$.
$V_C^* \subset \mathrm{Dom}(B_\Lambda^*) := \{x \in H \,|\, \exists B_\Lambda^* x := \lim_{s\to+\infty} B^* s(s-A^*)^{-1}x\}$ iff Ψ^* is regular. In this case, $B_\Lambda^* \in \mathcal{L}(V_C^*, U)$.

(b) $W_B \subset \mathrm{Dom}(C_w) := \{x \in H \,|\, \exists C_w x := \text{w-lim}_{s\to+\infty} Cs(s-A)^{-1}x\}$ iff Ψ is weakly regular. In this case, $C_w \in \mathcal{L}(W_B, Y)$.
$V_C^* \subset \mathrm{Dom}(B_w^*) := \{x \in H \,|\, \exists B_w^* x := \text{w-lim}_{s\to+\infty} B^* s(s-A^*)^{-1}x\}$ iff Ψ^* is weakly regular. In this case, $B_w^* \in \mathcal{L}(V_C^*, U)$.

[14] A similar (non-bijective, non-isometric) result is true for any Banach spaces U and Y and any L^p with $1 \le p < \infty$, see Weiss:Repr:Th2.3.

(c) If Ψ is weakly regular and either $y = \mathcal{C}x_0 + \mathcal{D}u \wedge x_0 \in H \wedge u \in L^2_{loc}(\mathbb{R}_+, U)$ or $y = \mathcal{D}u \wedge u \in L^2(\mathbb{R}, U)$, then $y(t) = C_w x(t) + Du(t)$ a.e. t and in all points t, where u and y are right-continuous (similarly $u^* = \mathcal{B}^* x_0 + \mathcal{D}^* y^* \vee u^* = \mathcal{D}^* y^* \implies u^*(t) = B^*{}_w x^*(t) + D^* y^*(t)$ a.e.).

(d) If $\mathcal{D}$ is weakly regular, then $\widehat{\mathcal{D}} = C_w(s - A)^{-1}B + D$.

(e) Let $\mathcal{X} \in \mathrm{TIC}$ be regular and let $\exists \mathcal{X}^{-1} \in \mathrm{TIC}$. The feed-through operator X of $\mathcal{X}$ is invertible iff $\mathcal{X}^{-1}$ is regular.

(f) If $\mathcal{X} \in \mathrm{TIC}$ and $\mathcal{X}^*$ are regular, then $\exists X^{-1}$.

If Ψ is regular, then it is weakly regular; if Ψ is weakly regular, then Ψ^* is weakly regular. For any $\Psi \in \mathrm{CWPLS}$ we have $C \subset C_\Lambda \subset C_w \wedge B^* \subset B^*_\Lambda \subset B^*_w$.

For more information on the (strong) *Weiss extension* C_Λ, see Weiss:AdmObs and Weiss:Feedback:Sec5; for more information on the *weak Weiss extension* C_w, see Weiss:TransferII or Weiss∗2:Sec2&4.

Proof: For the definitions of $W_B := (\alpha - A)^{-1}[H + BU]$ and V^*_C see Definition 2.2. Here, as elsewhere in the text, "$s \to +\infty$" means that $s \to \infty$ along $\mathbb{R}_+$ (along $\mathbb{R}_+ \cap \sigma(A)^c$, to be more exact).

Except for the results $C_w \in \mathcal{L}(W_B, Y) \wedge B^*_w \in \mathcal{L}(V^*_C, U)$, which follow directly from the Banach–Steinhaus theorem [RudinFA:Th2.8] (or from a direct calculation of the norm of $\|C_w\|_{\mathcal{L}(W_B,Y)}$ and $\|B^*_w\|_{\mathcal{L}(V^*_C,U)}$), part (a) is St:StQuadr:Prop36 and parts (b) and (c) are contained in Weiss:TransferII, (see Weiss∗2:Th4.4&Th4.5) because the case $y = \mathcal{D}u$ follows easily from the case $y = \mathcal{C}x_0 + \mathcal{D}u$ contained in Weiss∗2.

Part (d) is Weiss∗2:Th4.4, and (e) and (f) follow from Weiss:Feedback: Th4.7&Th4.8 (with $\mathbf{H} := I - \widehat{\mathcal{X}} \wedge K := I$). The observations at the end of the proposition are trivial. □

5 From Spectral Factorization to Riccati Equation

Staffans [St:Quadr&St:Crit:Th17] has shown that if a regular CWPLS Ψ with a regular adjoint Ψ^* has a spectral factorization, then the corresponding Riccati operator is a solution of a Riccati equation [Th5.5b] (the converse result (and uniqueness) will be proved in Section 6).

M. Weiss and G. Weiss [Weiss∗2:Sec12] have proved independently the same result allowing the input/output map $\mathcal{D}$ of the original system to be only weakly regular but assuming still that the spectral factor $\mathcal{X}$ is regular and that its feed-through operator X is invertible (X is invertible in St:Quadr:Th6.1v too).

In this section we slightly generalize the results using weak regularity instead of regularity[15]. At first we need the definition of a spectral factor.

Definition 5.1 *$\mathcal{Y}^*\mathcal{X}$ is a* spectral factorization *of $\mathcal{E} \in \mathrm{TI}(U)$, if $\mathcal{Y}, \mathcal{X}, \mathcal{Y}^{-1}, \mathcal{X}^{-1} \in \mathrm{TI}(U)$. If, moreover, $\mathcal{E} = \mathcal{E}^*$, then $\mathcal{Y} = S\mathcal{X}$ for some invertible $S = S^* \in \mathcal{L}(U)$ [St:Crit:Lemma11ii]. In this case we call $\mathcal{X}$ an S-*spectral factor *of $\mathcal{E}$ and S the corresponding* sensitivity operator. *An I-spectral factor is called merely a* spectral factor.

[15] We do not know whether the closed loop system $\Psi_{\circlearrowleft}$ is always weakly regular in this case. In the cases mentioned above $\Psi_{\circlearrowleft}$ is weakly regular.

Let us recall the following result from St:Crit:Lemma11.

Lemma 5.2 *Let $\widetilde{\mathcal{X}}^*\widetilde{\mathcal{S}}\widetilde{\mathcal{X}}$ be a spectral factorization of $\mathcal{E} = \mathcal{E}^* \in \mathrm{TI}(U)$. Then all spectral factorizations of $\mathcal{E}$ can be parametrized as $\mathcal{X} = E\widetilde{\mathcal{X}}, \mathcal{S} = (E^*)^{-1}\widetilde{S}E^{-1}$, where E varies over the set of all invertible operators in $\mathcal{L}(U)$.*

The following lemma follows from a straight-forward calculation.

Lemma 5.3 *If $\mathcal{E} = \mathcal{E}^* \in \mathrm{TI}(U)$ has a spectral factorization $\mathcal{X}^*S\mathcal{X}$, then the Toeplitz operator $\pi_+\mathcal{E}\pi_+$ is invertible in $L^2(\mathbb{R}_+, U)$, and its inverse is $\pi_+\mathcal{X}^{-1}S^{-1}\pi_+(\mathcal{X}^*)^{-1}\pi_+$. In particular, if $\mathcal{D}^*J\mathcal{D}$ has a spectral factorization, then $\mathcal{D}$ is J-coercive [Def2.3].*

Next we recall some of the main results of St:Crit.[16]

Theorem 5.4 *[St:Crit:Lemma5&Th16] Let $\Psi = \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right] \in \mathrm{OSCWPLS}_0(U, H, Y)$ and let $J = J^* \in \mathcal{L}(Y)$. Assume that there is[17] a spectral factorization $\mathcal{X}^*S\mathcal{X}$ of $\mathcal{D}^*J\mathcal{D}$. Define*

$$\begin{bmatrix}\mathcal{K} & \mathcal{F}\end{bmatrix} := \begin{bmatrix}-S^{-1}\pi_+\mathcal{N}^*J\mathcal{C} & (I - \mathcal{X})\end{bmatrix}, \qquad \mathcal{N} := \mathcal{D}\mathcal{X}^{-1}, \qquad \mathcal{M} := \mathcal{X}^{-1}.$$

Then $\Psi_{\mathrm{ext}} := \left[\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\\ \mathcal{K} & \mathcal{F}\end{smallmatrix}\right] \in \mathrm{OSCWPLS}_0(U, H, Y \times U)$, $\mathcal{D}$ is J-coercive, and

$$\begin{bmatrix}x_{\mathrm{crit}}(t, x_0)\\ y_{\mathrm{crit}}(x_0)\\ u_{\mathrm{crit}}(x_0)\end{bmatrix} = \begin{bmatrix}\mathcal{A}_\circlearrowleft(t)\\ \mathcal{C}_\circlearrowleft\\ \mathcal{K}_\circlearrowleft\end{bmatrix} x_0 = \begin{bmatrix}\mathcal{A}(t) + \mathcal{B}\mathcal{M}\tau(t)\mathcal{K}\\ \mathcal{C} + \mathcal{N}\mathcal{K}\\ \mathcal{M}\mathcal{K}\end{bmatrix} x_0$$

is equal to the state and output of the closed loop system $\Psi_\circlearrowleft \in \mathrm{OSCWPLS}_0(U, H, Y \times U)$ defined by

$$\Psi_\circlearrowleft = \begin{bmatrix}\mathcal{A}_\circlearrowleft & \mathcal{B}_\circlearrowleft\\ \begin{bmatrix}\mathcal{C}_\circlearrowleft\\ \mathcal{K}_\circlearrowleft\end{bmatrix} & \begin{bmatrix}\mathcal{D}_\circlearrowleft\\ \mathcal{F}_\circlearrowleft\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\mathcal{A} + \mathcal{B}\mathcal{M}\tau\mathcal{K} & \mathcal{B}\mathcal{M}\\ \begin{bmatrix}\mathcal{C} + \mathcal{N}\mathcal{K}\\ \mathcal{M}\mathcal{K}\end{bmatrix} & \begin{bmatrix}\mathcal{N}\\ \mathcal{M} - I\end{bmatrix}\end{bmatrix}$$

with initial value x_0, initial time zero, and zero control $u_\circlearrowleft$ (see Figure 4).[18] *The closed loop cost function $Q_\circlearrowleft(x_0, u_\circlearrowleft)$ for $y = \mathcal{C}_\circlearrowleft x_0 + \mathcal{D}_\circlearrowleft\pi_+u_\circlearrowleft$ is given by*

$$Q_\circlearrowleft(x_0, u_\circlearrowleft) := \langle y, Jy\rangle_{L^2(R_+,Y)} = \langle x_0, \Pi x_0\rangle_H + \langle u_\circlearrowleft, Su_\circlearrowleft\rangle_{L^2(\mathbb{R}_+,U)},$$

in particular, u_{crit} is minimizing iff $S >> 0$ iff $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+ >> 0$.

If $\mathcal{X}$ is regular and $\exists X^{-1}$, then $u_{\mathrm{crit}}(t, x_0) = (\mathcal{K}_\circlearrowleft x_0)(t) = MK_\Lambda x(t)$ a.e.

The Riccati operator Π [Lemma2.4] of Ψ (with cost operator J) can be written in the following alternative forms:

$$\Pi = \mathcal{C}^*J\mathcal{C} - \mathcal{K}^*S\mathcal{K} = \mathcal{C}^*J\mathcal{C}_\circlearrowleft = \mathcal{C}_\circlearrowleft^*J\mathcal{C}_\circlearrowleft = \mathcal{C}_\circlearrowleft^*J\mathcal{C},$$

and we have $\langle\mathcal{A}(t)x_0, \Pi\mathcal{A}(t)x_1\rangle \overset{t\to+\infty}{\to} 0 \ \forall x_0, x_1 \in H$.

Finally, the system $\Psi_\circlearrowleft$ is stable (resp. strongly stable) iff Ψ is stable (resp. strongly stable).

---

[16] Still replacing the word "stable" with "output stable", see the first footnote to Definition 2.1.

[17] If $\mathcal{D}^*J\mathcal{D} >> 0$ and U is separable, then a spectral factor $\mathcal{X}$ always exists [St:Crit:Lemma18ii].

[18] Note that $[\mathcal{K}\ \mathcal{F}]$ is an admissible stable state feedback pair for Ψ and $\Psi_\circlearrowleft$ is the corresponding closed loop system, as in Definition 3.3.

The operators $\mathcal{A}_\circlearrowleft$, $\mathcal{C}_\circlearrowleft$, $\mathcal{K}_\circlearrowleft$ and Π are, of course, the ones given in Lemma 2.4.

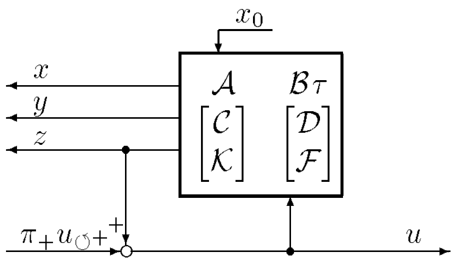

Figure 4: Optimal state feedback connection

Proof: The result is contained in St:Crit:Lemma5&Th16 (because the proofs do not require the stability of $\mathcal{A}$ and $\mathcal{B}$, as noted earlier) except for the three claims proved below.

If $\mathcal{X}$ is regular and $\exists X^{-1}$, then $\mathrm{Dom}(K_\Lambda) = \mathrm{Dom}((K_\circlearrowleft)_\Lambda) \wedge K_\Lambda = MK$ [Weiss:Feedback:Prop7.1] On the other hand, always $(\mathcal{K}_\circlearrowleft x_0)(t) = (K_\circlearrowleft)_\Lambda \mathcal{A}_\circlearrowleft(t) x_0$ a.e. [Weiss:AdmObs:Th4.5], whence follows $u_{\mathrm{crit}}(t, x_0) = (\mathcal{K}_\circlearrowleft x_0)(t) = MK_\Lambda \mathcal{A}_\circlearrowleft(t) x_0$ a.e.

$\Pi = \mathcal{C}^* J \mathcal{C} - \mathcal{K}^* S \mathcal{K}$ [St:Crit:Th16i] and $\Pi = \mathcal{A}^*(t) \Pi \mathcal{A}(t) + \mathcal{C}^* J \pi_{[0,t]} \mathcal{C} - \mathcal{K}^* S \pi_{[0,t]} \mathcal{K}$ [St:Quadr:Lemma5.5] [St:Crit:Th17], hence $\langle \mathcal{A}(t) x_0, \Pi \mathcal{A}(t) x_1 \rangle = \langle \mathcal{C} x_0, J \mathcal{C} x_1 \rangle - \langle \mathcal{C} x_0, \pi_{[0,t]} J \mathcal{C} x_1 \rangle - \langle \mathcal{K} x_0, S \mathcal{K} x_1 \rangle + \langle \mathcal{K} x_0, \pi_{[0,t]} S \mathcal{K} x_1 \rangle \overset{t \to +\infty}{\to} 0$ for all $x_0, x_1 \in H$.

The equivalence of the stabilities of Ψ and $\Psi_\circlearrowleft$ follow from Lemma3.5 & Prop3.2 of St:Coprime □

Now we can present St:Quadr:Th6.1 in a slightly more general form. Note that here, as elsewhere, we denote the generators of a system (and feed-through operators of weakly regular operators) with the same letters as the corresponding operators (cf. Def2.2).

Theorem 5.5 *Make the same assumptions and definitions as in Theorem 5.4. Suppose, in addition, that Ψ_{ext} is weakly regular and that the feed-through operator X of $\mathcal{X}$ is right-invertible, i.e., $\exists M^* := (I - F^*)^{-1}_{\mathrm{left}} := (X^*)^{-1}_{\mathrm{left}} \in \mathcal{L}(U)$. Then we have the following:*

(a) $SKx = -M^*(B_w^* \Pi + D^* JC)x \ \forall x \in W := \mathrm{Dom}(A)$.

(b) Π *satisfies the Riccati equation*

$$\begin{aligned} &\langle Ax_0, \Pi x_1 \rangle_H + \langle x_0, \Pi A x_1 \rangle_H + \langle C x_0, J C x_1 \rangle_Y \\ &\quad = \langle S^{-1} M^* (B_w^* \Pi + D^* JC) x_0, M^* (B_w^* \Pi + D^* JC) x_1 \rangle_U \ \forall x_0, x_1 \in W, \end{aligned}$$

which can be written in the form (see Definition 2.2 for $A^\times$)[19]

$$A^\times \Pi + \Pi A + C^\times JC = (B_w^* \Pi + D^* JC)^\times (X^* S X)^{-1} (B_w^* \Pi + D^* JC).$$

(c) If $\mathcal{M} := (I - \mathcal{F})^{-1}$ is weakly regular and one of $\mathcal{D}^*$ and $\mathcal{M}$ is regular, then $\Psi_\circlearrowleft$ is weakly regular.

[19] This form is the same as in Weiss*2:Th12.8, i.e., all the terms are in $\mathcal{L}(W, W^*)$. Note that $\Pi A x_1 \in H$ implies that $\langle x_0, \Pi A x_1 \rangle_{\langle W, W^* \rangle} = \langle x_0, \Pi A x_1 \rangle_H$. Note also, that $X^* S X$ is independent of the particular spectral factorization chosen [5.2].

Proof: According to St:Crit:Th33, all the results in St:Quadr:Th6.1 are valid for any $S = S^* \in \mathcal{L}(U)$, hence, to prove (a) and (b), we only have to weaken the regularity assumptions in St:Quadr:Th6.1(iv)&(v), which contain the strongly regular versions of (a) and (b).

In the proof of St:Quadr:Th6.1iv we have, by the weak regularity of Ψ^* and the continuity of u^* and $\left[\begin{smallmatrix} y^* \\ u^* \end{smallmatrix}\right]$, that for any $s \in [0,t]$ $u^*(s) = B_w^* x^*(s) + \left[\begin{smallmatrix} D^* & F^* \end{smallmatrix}\right] \left[\begin{smallmatrix} y^* \\ u^* \end{smallmatrix}\right](s) = B_w^* x^*(s) + D^* y^*(s) + F^* u^*(s)$ [4.3c]. This implies that $u^*(s) = M^*[B_w^* x^*(s) + D^* y^*(s)] = M^*[B_w^* x^*(s) + D^* y^*(s)]$, where $M^* := (I - F^*)_{\text{left}}^{-1}$. Hence $-SKx(s) = u^*(s) = M^*[B_w^* \Pi x(s) + D^* JCx(s)]$. The rest of the proof goes as in St:Quadr:Th6.1.[20]

(c) Clearly $\langle y(s), z(s) \rangle \to \langle y, z \rangle$ whenever $y(s) \to y$ strongly and $z(s) \to z$ weakly. Hence $\mathcal{D}\mathcal{M}$ is weakly regular whenever one of $\mathcal{D}^*$ and $\mathcal{M}$ is regular and the other one is weakly regular. □

We need also St:Quadr:Sec7 in a slightly more general form:

Proposition 5.6 *Make the same assumptions and definitions as in Theorem 5.4. Suppose, in addition, that Ψ_{ext} is weakly regular.*

Then the conclusions in St:Quadr:Th7.1&Cor7.2 hold, except that we have to replace the strong limit in St:Quadr:Cor7.2i by a weak limit (part (b) below), and

(a) If $X = I$, then for all $x_0 \in H$ and $u_0 \in U$, for which $Ax_0 + Bu_0 \in H$, we have

$$(B_w^* \Pi + D^* JC_w + SK_w)x_0 = (S - D^* JD)u_0 = \operatorname*{w-lim}_{\alpha \to +\infty} B_w^* \Pi (\alpha - A)^{-1} B u_0.$$

(b) If $X = I$, then for all $u_0 \in U$ we have[21]

$$Su_0 := D^* JDu_0 + \operatorname*{w-lim}_{\alpha \to +\infty} B_w^* \Pi (\alpha - A)^{-1} B u_0.$$

(c) Let Ψ_{ext} be regular and let $X = I - F = 0$. Then $\Pi \geq 0 \implies S \geq D^* JD$ and $\Pi \leq 0 \implies S \leq D^* JD$.

Proof: As noted in St:Crit:Th33, the strongly regular versions of these results [St:Quadr:Th7.1&Cor7.2&Rem7.3] hold for non-positive (invertible) $S = S^*$ too.

Part St:Quadr:Th7.1i is already stated with no regularity assumptions. One clearly sees from the proof of St:Quadr:Th7.1 that in St:Quadr:Th7.1 the formula (7.5) holds iff $\mathcal{D}$ is weakly regular; (7.6) holds iff $\mathcal{F}$ is weakly regular; (7.7), (7.8), (7.9) and St:Quadr:Cor7.2 hold if $\mathcal{D}$ and $\mathcal{F}$ are weakly regular. The proof of St:Quadr:Cor7.2 does not need any chances (except that the limit must be taken in the weak sense) (note that, e.g., $C_w(\alpha - A)^{-1} Bu_0 \to 0$ weakly as $\alpha \to +\infty$ [Weiss∗2:4.3&4.4]).

---

[20] Note that in St:Quadr:Th6.1 the systems Ψ and Ψ^* are regular, which implies that $I - F$ is invertible and $\Psi_{\circlearrowleft}$ is regular [Prop4.3ef]. In our case we do not know, whether $\Psi_{\circlearrowleft}$ is weakly regular.

[21] The generality of regular WPLSs allows a wide range of discontinuities, in particular, all discrete systems can be written in the form of a WPLS. Thus, in the formula for S, we must add this "$B^*\Pi B$-term" as in the (classical) discrete case (see, e.g., GreenLim:(B.2.27)); this can be seen as a result of the fact that the assumptions on the transfer function $\widehat{\mathcal{D}}$ are mild [Def4.2]. For a further discussion on this phenomenon and for an example where $S \neq D^* JD$, see St:DCRicc.

This phenomenon is visible also in several other results concerning the sensitivity operator S, confer, e.g., Theorem 8.7 and GreenLim:ThB.2.2.

Now it only remains to prove (c). By the assumptions and Prop4.3e, $\mathcal{X}^{-1}$ is now regular, hence so is $\mathcal{N} = \mathcal{D}\mathcal{X}^{-1}$. This implies that

$$\langle \mathcal{N}(s)u_0, J\mathcal{N}(s)u_0\rangle_Y \overset{s\to+\infty}{\to} \langle DX^{-1}u_0, JDX^{-1}u_0\rangle_Y \ \forall u_0 \in U,$$

hence (c) can be proved in the same way as St:Crit:Th26. □

6 From Riccati Equation to Spectral factorization

There are certain assumptions [Def6.3] we have to make in order to formulate the Riccati equation [5.5b]. We must be able to define the operator S, hence we assume that $\mathcal{D}$ is weakly regular ("D exists") and that the weak limit in Prop5.6b exists (in particular, we assume that $\forall u_0 \in U\ \exists r > \omega_A$ s.t. $\Pi(\alpha - A)^{-1}Bu_0 \in \mathrm{Dom}(B_w^*)\ \forall \alpha > r$). As Lemma 6.2 shows, these assumptions imply the weak regularity of the system Ψ_{ext} in Definition 6.3.

Lemma 6.1 *Let A be an infinitesimal generator of a C_0-semigroup on a complex Hilbert space H. Then, for all $x \in H$, we have*

(a) $\mathrm{Dom}(A) \ni s(s-A)^{-1}x \overset{s\to+\infty}{\to} x$ *in* H,

(b) $H \ni A(s-A)^{-1}x \overset{s\to+\infty}{\to} 0$ *in* H,

(c) $\mathrm{Dom}(A) \ni (s-A)^{-1}x \overset{s\to+\infty}{\to} 0$ *in* $\mathrm{Dom}(A)$.

Note that, in our case, $H = \mathrm{Dom}(\bar{A}) \ni \overline{s(s-A)^{-1}}x \overset{s\to+\infty}{\to} x\ \forall x \in V$ too.

Proof: (a) Choose some $r > \omega_A$. By the Hille–Yosida Theorem [Pazy:Th1.5.3], $\|(s-A)^{-1}\|_{\mathcal{L}} \le M/(\mathrm{Re}\, s - \omega_A) \ \Leftarrow \mathrm{Re}\, s > \omega_A$ (where $\mathcal{L}$ may be $\mathcal{L}(A)$ or $\mathcal{L}(\mathrm{Dom}(A))$ (in the latter case $(s-A)^{-1}$ must be interpreted with $A = A|_{\mathrm{Dom}(A^2)}$), because ω_A is the same in both of them). Thus

$$\|s(s-A)^{-1}\| \le M'[1 + \omega_A/(s-\omega_A)] < M'[1 + \omega_A/(r-\omega_A)] =: M \quad \forall s > r.$$

Define $r_{x,s} := \|x - s(s-A)^{-1}x\|_H = \|A(s-A)^{-1}x\|_H$. For $x \in \mathrm{Dom}(A)$ we have $r_{x,s} = \|(s-A)^{-1}Ax\| < M\|Ax\|/s \to 0$, hence $r_{x,s} \to 0$ for all $x \in H$, by the uniform boundedness of $s(s-A)^{-1}$ and the density of $\mathrm{Dom}(A)$ in H. Thus (a) and (b) are true.

(c) By (b), $\|(s-A)^{-1}x\|_{\mathrm{Dom}(A)} := \|(s-A)^{-1}x\|_H + \|A(s-A)^{-1}x\|_H \to 0 + 0 = 0$. □

Lemma 6.2 *Let Ψ be a weakly regular CWPLS. Let the limit $Su_0 := D^*JDu_0 + \text{w-lim}_{\alpha\to+\infty} B_w^*\Pi(\alpha-A)^{-1}Bu_0$ exist $\forall u_0 \in U$ (as in 5.6b) and let S be an invertible element of $\mathcal{L}(U)$. Let $K := -S^{-1}(B_w^*\Pi + D^*JC) \in \mathcal{L}(W,U)$ (i.e., $B_w^*\Pi \in \mathcal{L}(W,U)$).*

Then $W_B \subset \mathrm{Dom}(K_w) \cap \mathrm{Dom}(B_w^*\Pi)$ and $(B_w^*\Pi + D^*JC_w + SK_w)x_0 = (S - D^*JD)u_0 = \text{w-lim}_{\alpha\to+\infty} B_w^*\Pi(\alpha-A)^{-1}Bu_0$ whenever $Ax_0 + Bu_0 \in H$ (as in 5.6a).

In particular, if $\left[\begin{smallmatrix} A & B \\ C & D \\ K & F\end{smallmatrix}\right]$ generates a CWPLS Ψ_{ext}, then Ψ_{ext} is weakly regular.

Ψ_{ext} is regular iff Ψ is regular and the limit in S is a strong limit.

Proof: Let $x_0 \in W_B$ and let $u_0 \in U$ be s.t. $z_0 := Ax_0 + Bu_0 \in H$ [2.2].

1° "$\Pi x_0 \in \mathrm{Dom}(B_w^*)$": Take α s.t. $\Pi(\alpha - A)^{-1}Bu_0 \in \mathrm{Dom}(B_w^*)$.[22] $x_0' := (\alpha - A)^{-1}Bu_0$. Now

$$Ax_0' - Ax_0 = A(\alpha - A)^{-1}Bu_0 + Bu_0 - z_0 = \alpha(\alpha - A)^{-1}Bu_0 - z_0 \in H,$$

i.e., $x_0' - x_0 \in \mathrm{Dom}(A) =: W$. On the other hand, $W \subset \mathrm{Dom}(B_w^*\Pi)$, because $B_w^*\Pi \in \mathcal{L}(W, U)$. Thus $x_0 = x_0' - (x_0' - x_0) \in \mathrm{Dom}(B_w^*\Pi) + W = \mathrm{Dom}(B_w^*\Pi)$.

2° "$x_0 \in \mathrm{Dom}(K_w)$": Define $x_s := s(s-A)^{-1}x_0 \in W$ so that $K_w x_0 := \text{w-lim}_{s\to+\infty} Kx_s$ [4.3b] (which exists, as we shall prove below). Using $K := -S^{-1}(B_w^*\Pi + D^*JC)$, we have

$$\begin{aligned} SKx_s &= -D^*JCx_s - B_w^*\Pi x_s \\ &\to -D^*JC_w x_0 - B_w^*\Pi x_0 + \underset{s\to+\infty}{\text{w-lim}}\, B_w^*\Pi(s-A)^{-1}Bu_0, \end{aligned}$$

because

$$\begin{aligned} B_w^*\Pi x_s &= B_w^*\Pi s(s-A)^{-1}x_0 = B_w^*\Pi(I + A(s-A)^{-1})x_0 \\ &= B_w^*\Pi x_0 + B_w^*\Pi(s-A)^{-1}(z_0 - Bu_0) \\ &\to B_w^*\Pi x_0 + 0 - \underset{s\to+\infty}{\text{w-lim}}\, B_w^*\Pi(s-A)^{-1}Bu_0, \end{aligned}$$

because $(s-A)^{-1}z_0 \underset{W}{\to} 0$ [6.1c] and $B_w^*\Pi \in \mathcal{L}(W, U)$.

3° Ψ_{ext} is weakly regular iff $W_B \subset \mathrm{Dom}([\begin{smallmatrix} C \\ K \end{smallmatrix}]_w) = \mathrm{Dom}(C_w) \cap \mathrm{Dom}(K_w)$ [4.3b]. If $x_0 \in W_B$, then $x_0 \in \mathrm{Dom}(C_w)$ by the weak regularity of Ψ and $x_0 \in \mathrm{Dom}(K_w)$ by 1°. [23]

4° $W_B \subset \mathrm{Dom}(C_\Lambda)$ iff Ψ is regular, and in that case, $W_B \subset \mathrm{Dom}(K_\Lambda)$ iff the limit in S exists strongly, as we see from 1° and 2°. □

Now we are able to state a mild set of conditions (the weak regularity of the system and the condition (1.) below) that makes it possible to define the Riccati equation (4). As is often done in the classical case too, we require the solutions to be stabilizing (condition (2.)). Condition (3.) is often replaced with the stronger assumption that $\mathcal{A}$ is strongly stable (cf. Remark 6.5).

Definition 6.3 *Let $\Psi = [\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}] \in$ OSCWPLS$_0$ be weakly regular and let $J = J^* \in \mathcal{L}(Y)$. We call $\Pi = \Pi^* \in \mathcal{L}(H)$* a self-adjoint stabilizing solution of the Riccati equation *induced by Ψ and J iff (1.) $\wedge$ (2.) $\wedge$ (3.) $\wedge$ (4.), where*

(1.) The weak limit $Su_0 := D^*JDu_0 + \text{w-lim}_{\alpha\to+\infty} B_w^*\Pi(\alpha - A)^{-1}Bu_0$ exists $\forall u_0 \in U$ and S is an invertible element of $\mathcal{L}(U)$.[24]

[22] Such an α exists by the S-assumption. On the other hand, $(\alpha - A)^{-1}BU \subset W_B$ [2.2], hence the part 1^o of this proof shows that $\Pi(\alpha - A)^{-1}Bu_0 \in \mathrm{Dom}(B_w^*)$ $\forall \alpha \in \sigma(A)^c$.

[23] In the proof of 2° we saw that when $B_w^*\Pi \in \mathcal{L}(W, U)$ and Ψ is weakly regular, we have $\exists (B_w^*\Pi + D^*JC)_w x_0$ $\forall x_0 \in W_B \iff \exists\, \text{w-lim}_{\alpha\to\infty} B_w^*\Pi(\alpha - A)^{-1}Bu_0$ $\forall u_0 \in U$. Thus the weak regularity assumption on K was hidden in the assumption on the existence of S (which was necessary for the formulation of the Riccati equation).

Note also, that even though for any Π satisfying the requirements of Def6.3 we have $B_w^*\Pi \in \mathcal{L}(W_B, U)$, because $\Pi \in \mathcal{L}(W_B, V_{(C;K)}^*)$ (by [St:Quadr:7.1] and [6.9]), and $B_w^* \in \mathcal{L}(V_{(C;K)}^*, U)$ (by the regularity of Ψ_{ext} [Prop4.3b]), $B_w^*\Pi$ is not continuous in the weaker topology of W_B inherited from $\mathrm{Dom}([\begin{smallmatrix} C_w \\ K_w \end{smallmatrix}])$ [Weiss:TransferII] (because W is dense in W_B in that topology and hence we would have $(S - D^*JD)u = (B_w^*\Pi + D^*JC_w + SK_w)x_u = 0$ $\forall u \in U$), except for systems for which $\text{w-lim}_{\alpha\to\infty} B_w^*\Pi(\alpha - A)^{-1}Bu_0 = 0$ $\forall u_0 \in U$.

[24] In particular, we suppose that $\Pi(\alpha - A)^{-1}Bu_0 \in \mathrm{Dom}(B_w^*)$ $\forall \alpha > r$ for some $r > 0$. Cf. the part 1^o of the proof of Lemma 6.2.

(2.) *The operators* $\left[\begin{smallmatrix} A & B \\ C & D \\ K & 0 \end{smallmatrix}\right]$, *where* $K := -S^{-1}(B_w^*\Pi + D^*JC)$, *are the generators*[25] *of some* $\Psi_{\rm ext} = \left[\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \\ \mathcal{K} & \mathcal{F} \end{smallmatrix}\right] \in \mathrm{OSCWPLS}_0$ *for which the feedback* $L = [0\ I]$ *is admissible. We denote the corresponding closed loop system* ($\in \mathrm{OSCWPLS}_0$) *by* $\Psi_{\circlearrowleft}$.

(3.) $\langle \mathcal{A}(t)x_0, \Pi\mathcal{A}(t)x_0\rangle \overset{t\to+\infty}{\to} 0\ \forall x_0 \in H$.[26]

(4.) *The operator* Π *satisfies the Riccati equation (cf. Theorem 5.5)*

$$A^\times\Pi + \Pi A + C^\times JC = (B_w^*\Pi + D^*JC)^\times S^{-1}(B_w^*\Pi + D^*JC). \tag{4}$$

We denote $\mathcal{X} := I - \mathcal{F}$, $\mathcal{M} := \mathcal{X}^{-1}$.[27]

Lemma 6.4 *Make the same assumptions and definitions as in Theorem 5.4. Suppose, in addition, that* $\Psi_{\rm ext}$ *is weakly regular and* $X := I - F$ *is invertible. Then the Riccati operator* Π *of* Ψ *satisfies the requirements of Definition 6.3; in particular, this is true when the assumptions of St:Quadr:Th6.1v or those of Weiss∗2:Th12.8 hold.*[28]

Proof: If the assumptions are satisfied, one can choose X to be I [5.2] and thus see that the requirements of Definition 6.3 hold [5.4&5.5&5.6].

The assumptions of St:Quadr:Th6.1v as well as those of Weiss∗2:Th12.8 imply the assumptions of this lemma. □

Remark 6.5 *If one of* Ψ *and* $\Psi_{\circlearrowleft}$ *is strongly stable and the other one is stable, then both of them are strongly stable [St:Coprime:3.5ii] and hence the assumption* $\langle \mathcal{A}(t)x_0, \Pi\mathcal{A}(t)x_0\rangle \to 0$ *is redundant in that case. Note that instead of checking that the equation (4) holds, it is enough to check its alternate form, the first equation in Th5.5b, for all* $x_0 = x_1 \in W$ *(use the continuity of the operators and RudinFA:Th12.7).*

Now we shall have a closer look at the assumptions of Definition 6.3.

Lemma 6.6 *Let the assumptions of Definition 6.3 hold.*

The system $\Psi_{\rm ext}$ *is weakly regular. If* $\mathcal{M}$ *is weakly regular and at least one of* $\mathcal{D}^*$ *and* $\mathcal{M}$ *is regular (e.g., if* $\mathcal{X}$ *is regular), then* $\Psi_{\circlearrowleft}$ *is weakly regular.*

$\Psi_{\rm ext}$ *is regular iff* $\mathcal{D}$ *is regular and the limit in the definition of* S *exists strongly, in which case* $\Psi_{\circlearrowleft}$ *is regular too.*

We have $K \in \mathcal{L}(W, U)$, $W_B \subset \mathrm{Dom}(K_w)$, *and* $(B_w^*\Pi + D^*JC_w + SK_w)x_0 = (S - D^*JD)u_0 = \text{w-lim}_{\alpha\to\infty} B_w^*\Pi(\alpha - A)^{-1}Bu_0$ *whenever* $Ax_0 + Bu_0 \in H$.

[25]Because Ψ is weakly regular and (1) holds, any extension $\Psi_{\rm ext}$ of Ψ with generators $\left[\begin{smallmatrix} A & B \\ C & ? \\ K & ? \end{smallmatrix}\right]$ is weakly regular, by Lemma 6.2.

However, we do not know, whether it is possible that $X := I - F$ is non-invertible (at least this cannot be the case if $\mathcal{F}$ and $\mathcal{F}^*$ are regular [Prop4.3f]). Hence our assumption "$F = 0$" (i.e., $X = I$) in (2) may be restricting, but it still covers the Riccati equations presented in St:Quadr:Th6.1v and Weiss∗2:Th12.8, because an invertible X can always be normalized to I [Lemma5.2].

[26]Note that $\langle \mathcal{A}_{\circlearrowleft}(t)x_0, \Pi\mathcal{A}_{\circlearrowleft}(t)x_0\rangle \overset{t\to+\infty}{\to} 0\ \forall x_0 \in H$ is a natural condition stating that the remainder cost of an optimal (or critical) control goes to zero. If Ψ is stable, then $[\mathcal{A}_{\circlearrowleft}(t) - \mathcal{A}(t)]x_0 \to 0$ by St:StQuadr:Lemma21.

[27]$\exists\mathcal{X}^{-1} \in \mathrm{TIC}(U)$ by the assumption on the admissibility [Def3.1] of L.

[28]However, it is not clear, whether the same is true for the systems studied in FlandLasTrig.

Moreover, $SK_w = S^*K_w$ on $\mathrm{Dom}(K_w)$ and $S\mathcal{K} = S^*\mathcal{K}$ on H. The Riccati equation (4) can be written in the Liapunov form (as in St:Quadr:Th6.1i)
$A^\times\Pi + \Pi A = -C^\times JC + K^\times SK \in \mathcal{L}(W, W^*)$, *i.e.,*

$$\langle Ax_0, \Pi x_1\rangle_H + \langle x_0, \Pi Ax_1\rangle_H = -\langle Cx_0, JCx_1\rangle_Y + \langle Kx_0, SKx_1\rangle_U \ \forall x_0, x_1 \in W.$$

We will see below [6.7], that Definition 6.3 implies that $S = S^*$ on all of U. As noted in the proof of Lemma 6.2, the weak regularity of Ψ_{ext} was hidden in the (hardly unavoidable) assumption on the existence of S. Note that in general we do not know — nor need to know [7.1] — whether $\Psi_{\circlearrowleft}$ is weakly regular.

Proof: $\Psi_{\mathrm{ext}} \in$ CWPLS implies that $\mathcal{L}(W, U) \ni K := -S^{-1}(B_w^*\Pi + D^*JC)$ (i.e., $B_w^*\Pi \in \mathcal{L}(W, U)$). By lemma 6.2, Ψ_{ext} is weakly regular, $W_B \subset \mathrm{Dom}(K_w)$ and $(B_w^*\Pi + D^*JC_w + SK_w)x_0 = (S - D^*JD)u_0 = \text{w-lim}_{\alpha\to\infty} B_w^*\Pi(\alpha - A)^{-1}Bu_0$ whenever $Ax_0 + Bu_0 \in H$.

If $\mathcal{M}$ is weakly regular and one of $\mathcal{D}^*$ and $\mathcal{M}$ is regular, then $\Psi_{\circlearrowleft}$ is weakly regular as in 5.5c. If $\mathcal{F}$ is regular, $\Psi_{\circlearrowleft}$ is weakly regular by Weiss∗2:Prop12.3. If Ψ_{ext} is regular (i.e., if $\mathcal{D}$ and $\mathcal{F}$ are, which is true iff $\mathcal{D}$ is regular and the limit defining S exists strongly [Lemma6.2]) then $\Psi_{\circlearrowleft}$ is regular by Prop4.3e.

By the Liapunov equation (which follows straight from the Riccati equation (4) and the definition of K) and the fact that $J = J^*$, we have that $\langle Kx_0, SKx_0\rangle = \langle SKx_0, Kx_0\rangle\ \forall x_0 \in W$, hence $\langle u_0, Su_0\rangle = \langle Su_0, u_0\rangle\ \forall u_0 \in U_K := \overline{K[W]} \subset U$, hence $S_{|U_K} = S^*_{|U_K}$ [RudinFA:Th12.7]. On the other hand, clearly $K_wx_0 := \text{w-lim}_{s\to\infty} Ks(s - A)^{-1}x_0 \in U_K\ \forall x_0 \in \mathrm{Dom}(K_w)$ [RudinFA:Th3.12a], hence $SK_w = S^*K_w$. Now $(\mathcal{K}x_0)(t) = K\mathcal{A}(t)x_0\ \forall x_0 \in W$ [St:StQuadr:Prop29ii], hence $S\mathcal{K} = S^*\mathcal{K}$, by the density of W and the continuity of S and $\mathcal{K}$. □

We are now ready to state the main result of the text, i.e., the fact that a self-adjoint stabilizing solution of the Riccati equation gives rise to a spectral factorization of $\mathcal{D}^*J\mathcal{D}$.

Theorem 6.7 *Let $\Pi \in \mathcal{L}(H)$ be a self-adjoint stabilizing solution of the Riccati equation induced by Ψ and J [Def6.3]. Then $S = S^* \wedge \Pi = \Pi^*$ and $\mathcal{X} := I - \mathcal{F}$ is a weakly regular S-spectral factor of $\mathcal{D}^*J\mathcal{D}$. Moreover, $\widehat{\mathcal{X}}(s) = I - K_w(s - A)^{-1}B \in H^\infty(\mathbb{C}_+, \mathcal{L}(U))$.*

Proof: 1° "$\mathcal{D}^*J\mathcal{D} = \mathcal{X}^*S\mathcal{X}$": Let $u \in \mathcal{C}_c^\infty(\mathbb{R}, U) \subset W^{1,2}(\mathbb{R}, U)$. Then $y := \mathcal{D}u \in W^{1,2}(\mathbb{R}, Y) \wedge x = \mathcal{B}\tau u \in \mathcal{C}^1(\mathbb{R}, H) \wedge \bar{A}x + Bu = x' = \mathcal{B}\tau u' \in \mathcal{C}(\mathbb{R}, H)$ [St:StQuadr:Prop29iii]. Moreover, $y(t) = (\mathcal{D}u)(t) = C_wx(t) + Du(t)$, where $x(t) := \mathcal{B}\tau(t)u \in \mathrm{Dom}(C_w) \subset H\ \forall t \in \mathbb{R}$ [Prop4.3c], and similarly $(\mathcal{X}u)(t) = u(t) - K_wx(t)\ \forall t$. Thus

$$\langle \mathcal{D}u, J\mathcal{D}u\rangle_{L^2(Y)} = \langle Du + C_wx, JDu + JC_wx\rangle_{L^2(Y)}$$

$$= \langle u, D^*JDu\rangle_{L^2(U)} + \langle u, D^*JC_wx\rangle_{L^2(U)} + \langle D^*JC_wx, u\rangle_{L^2(U)} + \langle C_wx, JC_wx\rangle_{L^2(Y)}.$$

Because $\mathcal{X}u = (I - \mathcal{F})u \in L^2(\mathbb{R}, U)$ (by the output stability of Ψ_{ext}), we get (because $SK_w = S^*K_w$ [Lemma6.6])

$$\langle \mathcal{X}u, S\mathcal{X}u\rangle_{L^2(U)} = \langle u, Su\rangle_{L^2(U)} - \langle u, SK_wx\rangle_{L^2(U)} - \langle SK_wx, u\rangle_{L^2(U)} + \langle K_wx, SK_wx\rangle_{L^2(U)}$$

$= \int_{\mathbb{R}} f(t)\,dt$, where $f(t) := \langle u(t), Su(t)\rangle_U - \langle u(t), SK_wx(t)\rangle_U - \langle SK_wx(t), u(t)\rangle_U + \langle K_wx(t), SK_wx(t)\rangle_U$ and $f \in L^1$. Let $t \in \mathbb{R}$ be arbitrary. Setting $x_s := s(s -$

$A)^{-1}x(t) \in W$ for $s > \omega_A$ we have (by the definition of K_w [Prop4.3b])

$$\begin{aligned}
f(t) &= \lim_{r\to+\infty}\lim_{s\to+\infty}\left[\langle u(t), Su(t)\rangle_U - \langle u(t), SKx_s\rangle_U - \langle SKx_r, u(t)\rangle_U + \langle Kx_r, SKx_s\rangle_U\right] \\
&\overset{K-def}{=} \lim_{r\to+\infty}\lim_{s\to+\infty}\left[\langle u(t), Su(t)\rangle_U + \langle u(t), (B_w^*\Pi + D^*JC)x_s\rangle_U\right. \\
&\qquad \left.+\langle (B_w^*\Pi + D^*JC)x_r, u(t)\rangle_U + \langle Kx_r, SKx_s\rangle_U\right] \\
&\overset{Riccati}{=} \lim_{r\to+\infty}\lim_{s\to+\infty}\left[\langle u(t), D^*JDu(t)\rangle_U + \langle u(t), (S - D^*JD)u(t)\rangle_U\right. \\
&\qquad +\langle u(t), (B_w^*\Pi + D^*JC)x_s\rangle_U + \langle (B_w^*\Pi + D^*JC)x_r, u(t)\rangle_U \\
&\qquad \left.+\langle Ax_r, \Pi x_s\rangle_H + \langle \Pi x_r, Ax_s\rangle_H + \langle Cx_r, JCx_s\rangle_Y\right] = g(t) + h(t),
\end{aligned}$$

where $g(t) := \lim_{r\to+\infty}\lim_{s\to+\infty}[\langle u(t), D^*JDu(t)\rangle_U + \langle u(t), D^*JCx_s\rangle_U + \langle D^*JCx_r, u(t)\rangle_U + \langle Cx_r, JCx_s\rangle_Y] = \langle (\mathcal{D}u)(t), (J\mathcal{D}u)(t)\rangle_Y$ and

$$\begin{aligned}
h(t) := \lim_{r\to+\infty}\lim_{s\to+\infty}&[\langle u(t), (S - D^*JD)u(t)\rangle_U + \langle u(t), B_w^*\Pi x_s\rangle_U + \langle Ax_r, \Pi x_s\rangle_H \\
&+\langle B_w^*\Pi x_r, u(t)\rangle_U + \langle \Pi x_r, Ax_s\rangle_H].
\end{aligned}$$

On the other hand, $h(t) = h_1(t) + h_2(t)$, where[29]

$$\begin{aligned}
h_2(t) &:= \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle B_w^*\Pi x_r, u(t)\rangle_U + \langle \Pi x_r, Ax_s\rangle_H\right] \\
&= \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle B^*s(s - A^*)^{-1}\Pi x_r, u(t)\rangle_U + \langle \Pi x_r, Ax_s\rangle_H\right] \\
&= \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle \Pi x_r, s\overline{(s - A)^{-1}}Bu(t)\rangle_H + \langle \Pi x_r, s\overline{(s - A)^{-1}}Ax(t)\rangle_H\right] \\
&= \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle \Pi x_r, s(s - A)^{-1}\left[x'(t)\right]\rangle_H\right] \\
&\overset{6.1}{=} \langle \Pi x(t), x'(t)\rangle_H = \langle x(t), \Pi x'(t)\rangle_H
\end{aligned}$$

and

$$h_1(t) := \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle u(t), (S - D^*JD)u(t)\rangle_U + \langle u(t), B_w^*\Pi x_s\rangle_U + \langle Ax_r, \Pi x_s\rangle_H\right].$$

Now

$$\begin{aligned}
&\lim_{s\to\infty}\langle u(t), B_w^*\Pi x_s\rangle_U + \langle u(t), (S - D^*JD)u(t)\rangle_U \\
&\overset{K-def}{=} \lim_{s\to\infty}\langle u(t), -(D^*JC + SK)x_s\rangle_U + \langle u(t), (S - D^*JD)u(t)\rangle_U \\
&= \langle u(t), -(D^*JC_w + SK_w)x(t) + (S - D^*JD)u(t)\rangle_U \\
&\overset{6.2}{=} \langle u(t), B_w^*\Pi x(t)\rangle_U,
\end{aligned}$$

because $\bar{A}x(t) + Bu(t) = x'(t) \in H$ as required in Lemma 6.2. Thus

$$\begin{aligned}
\langle x'(t), \Pi x(t)\rangle_H &= \lim_{r\to\infty}\langle r(r - A)^{-1}\left[x'(t)\right], \Pi x\rangle_H \\
&= \lim_{r\to\infty}\langle r\overline{(r - A)^{-1}}\left[Ax(t) + Bu(t)\right], \Pi x\rangle_H \\
&= \lim_{r\to\infty}\left[\langle r\overline{(r - A)^{-1}}Ax(t), \Pi x\rangle_H + \langle r\overline{(r - A)^{-1}}Bu(t), \Pi x\rangle_H\right] \\
&= \lim_{r\to\infty}\left[\langle Ax_r, \Pi x\rangle_H + \langle u(t), B_w^*\Pi x\rangle_U\right] = h_1(t).
\end{aligned}$$

[29]By $\overline{(s-A)^{-1}} \in \mathcal{L}(V, H)$ we denote the continuous extension of $(s-A)^{-1} \in \mathcal{L}(H, W)$. One can easily verify that $\overline{(s-A)^{-1}}$ is the inverse of $s - \bar{A}$ and $\langle x, \overline{(s-A)^{-1}}v\rangle_H = \langle (s - A^*)^{-1}x, v\rangle_{\langle V^*, V\rangle}$ $\forall x \in H$ $\forall v \in V$.

(N.B. $\lim_{r\to\infty}\langle Ax_r, \Pi x\rangle_H$ exists.)[30] Thus, still for an arbitrary $t \in \mathbb{R}$,

$$h(t) = h_1(t) + h_2(t) = \langle x'(t), \Pi x(t)\rangle_H + \langle x(t), \Pi x'(t)\rangle_H = \langle x, \Pi x\rangle_H'(t).$$

We have $f, g \in L^1$, hence $h = f - g \in L^1$ too. As noted above, $x \in \mathcal{C}^1(\mathbb{R}, H)$, hence $h(t) \in \mathcal{C}(\mathbb{R})$. On the other hand, for some $T > 0$ we have $\operatorname{supp} u \subset [-T, T]$. Thus $x(t) = 0\ \forall t \leq -T$ and $x(t) = \mathcal{A}(t-T)x(T)\ \forall t \geq T$, hence $\langle x(t), \Pi x(t)\rangle_H \stackrel{t\to+\infty}{\to} 0$, by the assumptions [Def6.3]. Now

$$\begin{aligned}
\int_{\mathbb{R}} f(r)\,dr &= \int_{\mathbb{R}} g(r)\,dr + \int_{\mathbb{R}} h(r)\,dr \\
&= \int_{\mathbb{R}} \langle (\mathcal{D}u)(r), (J\mathcal{D}u)(r)\rangle_Y\,dr + \int_{\mathbb{R}} \langle x, \Pi x\rangle_H'(r)\,dr \\
&= \langle \mathcal{D}u, J\mathcal{D}u\rangle_{L^2(Y)} + \lim_{t\to\infty}\int_{-T}^{t} \langle x, \Pi x\rangle_H'(r)\,dr \\
&= \langle \mathcal{D}u, J\mathcal{D}u\rangle_{L^2(Y)} + \lim_{t\to\infty}\langle x(t), \Pi x(t)\rangle_H = \langle \mathcal{D}u, J\mathcal{D}u\rangle_{L^2(Y)}.
\end{aligned}$$

From

$$\begin{aligned}
\langle u, \mathcal{X}^* S \mathcal{X} u\rangle_{L^2(U)} &= \langle \mathcal{X}u, S\mathcal{X}u\rangle_{L^2(U)} = \int_{\mathbb{R}} f(r)\,dr \\
&= \langle \mathcal{D}u, J\mathcal{D}u\rangle_{L^2(Y)} = \langle u, \mathcal{D}^* J\mathcal{D}u\rangle_{L^2(U)}\ \forall u \in \mathcal{C}_c^\infty(\mathbb{R}, U),
\end{aligned}$$

we get, by density and continuity, that $\langle u, \mathcal{X}^* S\mathcal{X}u\rangle_{L^2(U)} = \langle u, \mathcal{D}^* J\mathcal{D}u\rangle_{L^2(Y)}$ $\forall u \in L^2$, hence $\mathcal{X}^* S\mathcal{X} = \mathcal{D}^* J\mathcal{D}$ [RudinFA:Th12.7]. On the other hand, $\mathcal{X} = I - \mathcal{F}$ is invertible in $\mathrm{TIC}(U)$ by the well-posedness and output stability of $\Psi_{\circlearrowleft}$ [Def6.3]. Thus $\mathcal{X}$ is an S-spectral factor of $\mathcal{D}^* J\mathcal{D}$ [Def5.1]. $\widehat{\mathcal{F}}(s) = K_w(s-A)^{-1}B \in H^\infty(\mathbb{C}_+, \mathcal{L}(U))$ [Prop4.3d], hence $\widehat{\mathcal{X}}(s) = I - K_w(s-A)^{-1}B$. The weak regularity of $\mathcal{X}$ follows from that of $\mathcal{F}$ [Lemma6.6].

2° In 1° we used the fact that $S^* K_w = SK_w$ [6.6]. Knowing now that $\mathcal{D}^* J\mathcal{D} = \mathcal{X}^* S\mathcal{X}$, we can deduce that $S = (\mathcal{X}^*)^{-1}\mathcal{D}^* J\mathcal{D}\mathcal{X}^{-1} = S^*$. □

Lemma 6.8 *Let* $\left[\begin{smallmatrix} A & B \\ C & D \\ K & F \end{smallmatrix}\right]$ *be the generators a CWPLS* Ψ_{ext} *on* $(U, H, Y \times U)$ *and let* $S \in \mathcal{L}(U) \wedge J \in \mathcal{L}(Y)$. *Let* $\Pi \in \mathcal{L}(H)$ *satisfy the Lyapunov equation* $A^\times \Pi + \Pi A = -C^\times JC + K^\times SK \in \mathcal{L}(W, W^*)$, *i.e.,*

$$\langle Ax_0, \Pi x_1\rangle + \langle x_0, \Pi A x_1\rangle = -\langle Cx_0, JCx_1\rangle + \langle Kx_0, SKx_1\rangle\ \forall x_0, x_1 \in W$$

as in Lemma 6.6. Then

$$\Pi = \mathcal{A}^*(t)\Pi\mathcal{A}(t) + \mathcal{C}^* J\pi_{[0,t]}\mathcal{C} - \mathcal{K}^* S\pi_{[0,t]}\mathcal{K} \in \mathcal{L}(H)\ \forall t \geq 0$$

as in St:Quadr:Lemma5.5. In particular, $\Pi = \mathcal{C}^* J\mathcal{C} - \mathcal{K}^* S\mathcal{K} \in \mathcal{L}(H)$, *if* $\Psi_{\mathrm{ext}} \in \mathrm{OSCWPLS}_0$ *and* $\langle \mathcal{A}(t)x_0, \Pi\mathcal{A}(t)x_0\rangle \stackrel{t\to+\infty}{\to} 0\ \forall x_0 \in H$ *as in Definition 6.3 (e.g., if* Ψ_{ext} *is strongly stable).*

---

[30] Note that the commutator $(\lim_s \lim_r - \lim_r \lim_s)\,(\langle u(t), B_w^* \Pi x_s\rangle_U + \langle Ax_r, \Pi x_s\rangle_H)$ of the expression in $h_1(t)$ was equal to the term $\langle u(t), (S - D^* JD)u(t)\rangle_U$ (which is frequently zero, e.g., when $\mathcal{D}, \mathcal{X} \in \mathcal{W}_+*$). When this commutator is zero, we can calculate $h_1(t)$ in the same way as $h_2(t)$.

If $\mathcal{D}$ and $\mathcal{F}$ are regular, the limits in f and g exist also as $r, s \to +\infty$ independently, as one can see by slightly altering the above proof.

Proof: Let $a, b \in W$ and $t \in \mathbb{R}_+$. Set $x_0 := \mathcal{A}(t)a \in W$, $x_1 := \mathcal{A}(t)b \in W$ [Pazy:Th2.4c]. Then, because $\mathcal{C}x = C\mathcal{A}(\cdot)x \wedge \mathcal{K}x = K\mathcal{A}(\cdot)x \wedge \exists \frac{d}{dt}[\mathcal{A}(t)x] = A\mathcal{A}(t)x \quad \forall x \in W := \mathrm{Dom}(A)$ [Weiss:TransferI:(2.10)] [Pazy: Th2.5c], by substituting these to the Liapunov equation we get

$$\begin{aligned}
0 &= \langle \mathcal{A}'(t)a, \Pi\mathcal{A}(t)b\rangle + \langle \mathcal{A}(t)a, \Pi\mathcal{A}'(t)b\rangle + \langle C\mathcal{A}(t)a, JC\mathcal{A}(t)b\rangle - \langle K\mathcal{A}(t)a, SK\mathcal{A}(t)b\rangle \\
&= \frac{d}{dt}\left[\langle \mathcal{A}(t)a, \Pi\mathcal{A}(t)b\rangle_H + \int_0^t \langle C\mathcal{A}(t)a, JC\mathcal{A}(t)b\rangle_Y - \int_0^t \langle K\mathcal{A}(t)a, SK\mathcal{A}(t)b\rangle_U\right] \\
&= \frac{d}{dt}\left[\langle a, \mathcal{A}(t)^*\Pi\mathcal{A}(t)b\rangle_H + \langle a, \mathcal{C}^*J\pi_{[0,t]}\mathcal{C}b\rangle_H - \langle a, \mathcal{K}^*S\pi_{[0,t]}\mathcal{K}b\rangle_H\right].
\end{aligned}$$

Thus

$$\begin{aligned}
&\langle a, \mathcal{A}(t)^*\Pi\mathcal{A}(t)b\rangle + \langle a, \mathcal{C}^*J\pi_{[0,t]}\mathcal{C}b\rangle - \langle a, \mathcal{K}^*S\pi_{[0,t]}\mathcal{K}b\rangle \\
&\quad = \langle a, \mathcal{A}(0)^*\Pi\mathcal{A}(0)b\rangle + \langle a, \mathcal{C}^*J\pi_{[0,0]}\mathcal{C}b\rangle - \langle a, \mathcal{K}^*S\pi_{[0,0]}\mathcal{K}b\rangle = \langle a, \Pi b\rangle.
\end{aligned}$$

The same holds for $a, b \in H \times H$ too, because $W \times W$ is dense in $H \times H$ [Pazy:Cor2.5].

We have $\pi_{[0,t]}\mathcal{C}a \to \mathcal{C}a$ and $\pi_{[0,t]}\mathcal{K}a \to \mathcal{K}a$ in L^2 as $t \to +\infty$, if $\mathcal{C}, \mathcal{K} \in \mathcal{L}(H, L^2)$ (i.e., they are stable). If, in addition, $\langle \mathcal{A}(t)a, \Pi\mathcal{A}(t)a\rangle \to 0 \ \forall a \in H$ (or $\forall a \in \mathrm{Dom}(A)$), then we get from the above formula with $b = a$ that $\langle a, \Pi a\rangle = \langle a, (\mathcal{C}^*J\mathcal{C} - \mathcal{K}^*S\mathcal{K})a\rangle \ \forall a$, which implies that $\Pi = \mathcal{C}^*J\mathcal{C} - \mathcal{K}^*S\mathcal{K} \in \mathcal{L}(H)$ [RudinFA:Th12.7]. □

Now we are able to proof the uniqueness of Π:

Proposition 6.9 *There is at most one self-adjoint stabilizing solution Π of the Riccati equation induced by Ψ and J [Def6.3].*

If such a solution exists, then the assumptions and conclusions of 2.4&5.4&5.5 hold and Π and Ψ_{ext} are the ones given in 2.4&5.4 (in particular, S, K, $\mathcal{X}$ and $\mathcal{K}$ are unique, because we must have $X = I$).

Proof: Let Π and Π' be as in Definition 6.3. Let $\mathcal{X} = I - \mathcal{F}$ and $\mathcal{X}' = I - \mathcal{F}'$ be the corresponding spectral factors and $\mathcal{K}$ and $\mathcal{K}'$ the corresponding state feedback observability maps. Then, by Theorem 6.7, $\mathcal{X}^*S\mathcal{X}$ and $\mathcal{X}'^*S'\mathcal{X}'$ are spectral factorizations of $\mathcal{D}^*J\mathcal{D}$. Thus $\mathcal{X}' = E^{-1}\mathcal{X}$ for some invertible $E \in \mathcal{L}(U)$ [5.2]. On the other hand, $I = \widehat{\mathcal{X}'}(\infty) = E^{-1}\widehat{\mathcal{X}}(\infty) = E^{-1}$, hence $\mathcal{X} = \mathcal{X}' \wedge S = S' \wedge \mathcal{F} = I - \mathcal{X} = I - \mathcal{X}' = \mathcal{F}'$. Now $\mathcal{K}'\mathcal{B} = \pi_+\mathcal{F}'\pi_- = \pi_+\mathcal{F}\pi_- = \mathcal{K}\mathcal{B}$, hence we have that $\forall u, v \in \mathcal{C}_c^\infty(\mathbb{R}, U)$

$$\begin{aligned}
\langle \Pi'\mathcal{B}u, \mathcal{B}v\rangle_H &\overset{6.8}{=} \langle \mathcal{C}\mathcal{B}u, J\mathcal{C}\mathcal{B}v\rangle_{L^2} - \langle \mathcal{K}'\mathcal{B}u, S'\mathcal{K}'\mathcal{B}v\rangle_{L^2} \\
&= \langle \mathcal{C}\mathcal{B}u, J\mathcal{C}\mathcal{B}v\rangle_{L^2} - \langle \mathcal{K}\mathcal{B}u, S\mathcal{K}\mathcal{B}v\rangle_{L^2} = \langle \Pi\mathcal{B}u, \mathcal{B}v\rangle_H.
\end{aligned}$$

We thus have $\Pi' = \Pi$ on the reachable subspace $H_B := \overline{\{\mathcal{B}u \mid u \in \mathcal{C}_c^\infty(\mathbb{R}, U)\}} \subset H$. By Weiss:AdmContr:Rem3.12 we have

$$H_B \ni \widehat{\mathcal{B}u}(s) = (s - A)^{-1}B\hat{u}(s) \quad \forall u \in \mathcal{C}_c^\infty(\mathbb{R}_+, U) \ \forall s > \omega_A,$$

hence $(s - A)^{-1}BU \subset H_B \ \forall s > \omega_A$. Thus, for $x_0 \in W$, we get

$$\begin{aligned}
\langle B_w^*\Pi'x_0, u_0\rangle_U &= \lim_{s \to +\infty} \langle B^*s(s - A^*)^{-1}\Pi'x_0, u_0\rangle_U \\
&= \lim_{s \to +\infty} \langle x_0, \Pi's(s - A)^{-1}Bu_0\rangle_H = \langle B_w^*\Pi x_0, u_0\rangle_U \ \forall u_0 \in U.
\end{aligned}$$

Thus $K'x_0 = -S^{-1}(B_w^*\Pi' + D^*JC)x_0 = Kx_0\ \forall x_0 \in W$.[31] This gives

$$(\mathcal{K}x_0)(t) \overset{S3:Prop28ii}{=} K\mathcal{A}(t)x_0 = (\mathcal{K}'x_0)(t)\ \forall x_0 \in W,$$

from which we get, by density, $\mathcal{K} = \mathcal{K}'$ and, finally, $\Pi' = \Pi$ [6.8].[32]

By the existence of the spectral factorizations, the Π defined in Lemma 2.4 exists and satisfies Theorem 5.4; we shall denote that Π by $\widetilde{\Pi}$. By Lemma 5.2, we must have $\widetilde{\mathcal{X}} = E\mathcal{X}$ for some invertible E; choose $E = I$ to see that $\widetilde{\Pi}$ satisfies the same Riccati equation [5.5b] as Π. Thus, by Lemma 6.4, $\widetilde{\Pi}$ also satisfies the requirements of Definition 6.3 and is therefore equal to the unique Π. □

7 The Riccati Equation: Summary and Applications to Quadratic Control

Here we summarize the results of Sections 5 and 6:

Theorem 7.1 *Let $\Psi \in \mathrm{OSCWPLS}_0(U, H, Y)$ be weakly regular and let $J = J^* \in \mathcal{L}(Y)$.*

$\mathcal{D}^*J\mathcal{D}$ has a weakly regular S-spectral factor $\mathcal{X}$ with an invertible feed-through operator $X := \widehat{\mathcal{X}}(+\infty)$ for some invertible $S \in \mathcal{L}(U)$ iff there exists a self-adjoint stabilizing solution Π of the Riccati equation induced by Ψ and J.

Moreover, if Π is such a solution and we normalize X to I [Lemma5.2], we also have the following: The assumptions and conclusions in 5.4&5.5&5.6&6.9 hold. The invertible S corresponding to the factor $\mathcal{X}$ with $\widehat{\mathcal{X}}(\infty) = I$ is the one appearing in Definition 6.3, $S = S^*$, $\widehat{\mathcal{X}}(s) = I - K_w(s-A)^{-1}B \in H^\infty(\mathbb{C}_+, \mathcal{L}(U))$, $\mathcal{D}$ is J-coercive, and $\Pi = \mathcal{C}^*(J - J\mathcal{D}\pi_+(\pi_+\mathcal{D}^*J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^*J)\mathcal{C}$; in particular, Π is unique. If $\mathcal{X}$ is regular, then the critical control u_{crit} is given by $u_{\mathrm{crit}}(t) = K_\Lambda x(t)$ a.e., as in Theorem 5.4.

Proof: The iff-claim is true by Lemma 6.4 and Theorem 6.7. The rest of the claims follow easily from Proposition 6.9 and Theorem 6.7. Note that, with $X = I$, we now have $(K_{\circlearrowleft})_\Lambda = K_\Lambda$ [Th5.4]. □

To get Theorem 7.1 into a simpler form we restrict to systems whose input/output map belongs to the Wiener class $\mathcal{W}_+*$ [p.7] with a finite-dimensional input space U.

Theorem 7.2 *Let $\Psi \in \mathrm{OSCWPLS}_0(U, H, Y)$ be such that $\mathcal{D} \in \mathcal{W}_+ * (U, Y)$, let $J = J^* \in \mathcal{L}(Y)$ and let $\dim U < \infty$. Then the following conditions are equivalent:*

(i) $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+$ is invertible in $\mathcal{L}(L^2(\mathbb{R}, U))$.

(i') $\mathcal{D}^*J\mathcal{D}$ has a spectral factorization.

(ii) There exists a self-adjoint stabilizing solution Π of the Riccati equation induced by Ψ and J.

[31] The signals in the system Ψ_{ext} give as information on $\mathcal{K}|_{H_B}$ only, thus we needed the definition of K to get further.

[32] The standard uniqueness proof (e.g., M.Weiss:Thesis:Lemma2.13) is now hard to apply, because $\mathrm{Dom}(A_K)$ and $\mathrm{Dom}(A_{K'})$ need not have to have much in common *a priori*.

(ii') D^*JD is invertible in $\mathcal{L}(U)$ and there exists an operator $\Pi \in \mathcal{L}(H)$ that satisfies the Riccati equation[33] *(here $B_w^*\Pi = B_\Lambda^*\Pi$)*

$$A^\times\Pi + \Pi A + C^\times JC = (B_w^*\Pi + D^*JC)^\times(D^*JD)^{-1}(B_w^*\Pi + D^*JC).$$

The operators $\left[\begin{smallmatrix} A & B \\ C & D \\ K & 0 \end{smallmatrix}\right]$, where $K := -(D^*JD)^{-1}(B_w^*\Pi + D^*JC)$, are the generators of an OSCWPLS$_0$ Ψ_{ext} for which the feedback $L = [0\ I]$ is admissible creating a closed loop system $\Psi_{\circlearrowleft} \in$ OSCWPLS$_0$. Moreover, $\langle \mathcal{A}(t)x_0, \Pi\mathcal{A}(t)x_0\rangle \overset{t\to+\infty}{\to} 0\ \forall x_0 \in H$.

If the the equivalent conditions stated above hold, then $\mathcal{D}^*J\mathcal{D}$ has a D^*JD-spectral factor $\mathcal{X} \in \mathcal{W}_+ * (U)$, $X := \widehat{\mathcal{X}}(+\infty) = I$, $\mathcal{X}^{-1} \in \mathcal{W}_+ * (U)$, $\Pi = \Pi^*$, and all the assumptions and conclusions in Theorem 7.1 hold; in particular, Π is unique.

Proof: This proof is written in a short form; we shall study the Wiener class in greater detail in M:GRPRicc.

1° (i") $\Longrightarrow$ (i') $\Longrightarrow$ (i) [Lemma5.3], where (i") is defined by

(i") $\mathcal{D}^*J\mathcal{D}$ has a weakly regular spectral factorization $\mathcal{X}^*S\mathcal{X}$ with $X = I$ (as in Theorem 7.1).

2° (i) $\Longrightarrow$ (i"): Because the Toeplitz operator $\pi_+\mathcal{D}^*J\mathcal{D}\pi_+$ is invertible, $\mathcal{D}^*J\mathcal{D}$ has a spectral factorization $\mathcal{X}^*S\mathcal{X}$ with $\mathcal{X}, \mathcal{X}^{-1} \in \mathcal{W}_+*(U)$, by ClancGohb:ThII.6.3&CorIII.1.1&PropII

The fact, that $\mathcal{X}$ and $\mathcal{X}^{-1}$ are regular, implies that $\exists X^{-1}$ [Prop4.3e], hence X can be normalized to I [Lemma5.2].

3° (i) $\Longleftrightarrow$ (ii): This follows from Theorem 7.1.

4° (ii') $\Longrightarrow$ (ii): We have $\Pi = \mathcal{C}^*J\mathcal{C} - \mathcal{K}^*(D^*JD)\mathcal{K}$ [Lemma6.8], hence $\Pi = \Pi^*$. The rest of the requirements of Definition 6.3 are clearly satisfied. On the other hand, this and the regularity of $\mathcal{F}$ [Lemma6.6] imply that $\Pi \in \mathcal{L}(W_B, V^*_{[C;K]}) \subset \mathcal{L}(W, \mathrm{Dom}(B_\Lambda^*))$ [St:Quadr:Th7.1], hence (ii') is true also for B_Λ^* [Prop4.3a].

5° (ii) $\Longrightarrow$ (ii'): We suppose that (ii) and hence (i), (i') and (i") are true. Then $(\mathcal{D}u_0)(i\omega) \overset{\omega\to+\infty}{\to} Du_0$, by the Riemann–Lebesgue lemma, and hence $D^*JD = X^*SX$, in particular, D^*JD is the invertible S corresponding to $X = I$. The rest of (ii') is contained in (ii) [Def6.3].

6° The rest of the claims follow from Theorem 7.1 and 1° − −5°. □

The applications of Theorem 7.1 to the nonstandard [Cor7.3] and standard [Cor7.4] quadratic minimization problems are straight-forward:

Corollary 7.3 *Let $\Psi \in$ OSCWPLS$_0(U, H, Y)$ be weakly regular and let $J = J^* \in \mathcal{L}(Y)$.*

$\mathcal{D}^*J\mathcal{D}$ has a weakly regular spectral factor $\mathcal{X}$ with an invertible feed-through operator $X := \widehat{\mathcal{X}}(+\infty)$ iff there exists a self-adjoint stabilizing solution Π of the Riccati equation induced by Ψ and J such that the corresponding S [6.3] is positive.[34]

[33] Note that, with the standard assumptions $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right] \wedge C = \left[\begin{smallmatrix} C_1 \\ 0 \end{smallmatrix}\right] \wedge J = I$ (as in GreenLim:p.182 and in Keulen:Rem3.13; cf. Rem2.5), we get the Riccati equation to the form

$$A^\times\Pi + \Pi A + C_1^\times C_1 = (B_w^*\Pi)^\times B_w^*\Pi$$

as in Keulen:Rem3.13&Th3.10 and in GreenLim:(5.2.29).

[34] If $\mathcal{D}^*J\mathcal{D} >> 0$ and U is separable, then a spectral factor $\mathcal{X}$ always exists [St:Crit:Lemma18ii], though it does not have to be weakly regular.

If Π is such a solution and we normalize X to I, then the assumptions and conclusions in Theorem 7.1 hold, Π is unique, and the minimizing control is equal to the critical state feedback $\mathcal{K}_{\circlearrowleft}x_0 = K_\Lambda x(t)$ a.e., as in Theorem 5.4.

See Remark 2.5 for a different form of the Riccati equation.

Proof: This follows from Theorem 7.1, because one may always scale a positive S to I by choosing $E := S^{-1/2}$ [Lemma5.2]; The critical state feedback is now minimizing [Th5.4]&[St:Quadr:Lemma2.5]. □

Corollary 7.4 *Let $\Psi \in \mathrm{OSCWPLS}_0(U, H, Y)$ be weakly regular and let $0 << J \in \mathcal{L}(Y)$.*

$\mathcal{D}^* J \mathcal{D}$ has a weakly regular spectral factor $\mathcal{X}$ with an invertible feed-through operator $X := \widehat{\mathcal{X}}(+\infty)$ iff there exists a self-adjoint stabilizing solution Π of the Riccati equation induced by Ψ and J.

In that case, Π is unique, $\Pi \geq 0$, and the assumptions and conclusions in Corollary 7.3 hold.

Note that here we could normalize Ψ so that $J = I$.

Proof: This follows from Theorem 7.3, because now

$$S = (\mathcal{X}^*)^{-1}\mathcal{D}^* J \mathcal{D}\mathcal{X}^{-1} \geq 0 \ \wedge \ \Pi = \mathcal{C}_{\circlearrowleft}^* J \mathcal{C}_{\circlearrowleft} \geq 0 \quad \text{[Th5.4]}.$$

□

8 The Riccati Equation and the H^∞ Full Information Control Problem

We adapt some definitions and results from St:StHinf into this section to prepare for the application of our results to the H^∞ full information control problem [Th8.7&Cor8.9&Th8.10].

The motivation behind Hypothesis 8.1 and Definition 8.2 will be explained below.

Hypothesis 8.1 *Throughout this section we shall assume the following (cf. Figure 5):*

Let U, W, H and Y be Hilbert spaces. Let $\Psi \in \mathrm{OSCWPLS}_0(U \times W, H, Y \times W)$, $\pi_+\mathcal{D}_{11}^*\mathcal{D}_{11}\pi_+ >> 0$ on $L^2(\mathbb{R}_+, U)$ and ($\mathcal{B} = [\mathcal{B}_1 \ \mathcal{B}_2]$ and)

$$J = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 \end{bmatrix} \in \mathcal{L}(Y \times W), \quad \mathcal{D} := \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_2 \end{bmatrix} := \begin{bmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} \\ 0 & I \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} \mathcal{C}_1 \\ 0 \end{bmatrix}.$$

Definition 8.2 *Let $y := \left[\begin{smallmatrix} y_1 \\ w \end{smallmatrix}\right] := \mathcal{C}x_0 + \mathcal{D}_1 u + \mathcal{D}_2 w$ be the output of system Ψ with initial state $x_0 \in H$, control $u \in L^2(\mathbb{R}_+, U)$ and disturbance $w \in L^2(\mathbb{R}_+, W)$. The cost function Q [Def2.3] becomes now*

$$Q(x_0, u, w) := \langle \mathcal{D}y, J\mathcal{D}y \rangle = \|y_1\|_2^2 - \gamma^2 \|w\|_2^2.$$

If there is a causal control law $\mathcal{U} \in \mathrm{TIC}(W, U)$ s.t. for some $\epsilon > 0$ we have $Q(0, \mathcal{U}w, w) \leq -\epsilon\|w\|_2^2 \ \forall w \in L^2(\mathbb{R}_+, W)$ (i.e., $\|\mathcal{D}_{11}\mathcal{U} + \mathcal{D}_{12}\| < \gamma$), we call $\mathcal{U}$ a uniformly suboptimal controller.

We call $\mathcal{D}$ minimax J-coercive *iff*

$$\pi_+\mathcal{D}_{12}^*[I - \mathcal{D}_{11}\pi_+(\pi_+\mathcal{D}_{11}^*\mathcal{D}_{11}\pi_+)^{-1}\pi_+\mathcal{D}_{11}^*]\mathcal{D}_{12}\pi_+ << \gamma^2$$

on $L^2(\mathbb{R}_+, W)$ (cf. Lemma 8.4).

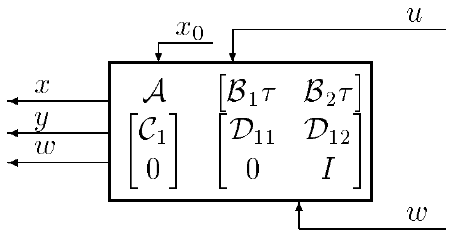

Figure 5: Input-state-output diagram for Ψ

Thus we will use the notation of St:StHinf for the minimax setting, i.e., the input space is now $U \times W$ instead of U, and now also W may be a separate Hilbert space (not necessarily $\mathrm{Dom}(A)$ as in previous sections). We call W the disturbance space and U is the control space of the system Ψ as in St:StHinf.

Note that we have here $y_1 = \mathcal{C}_1 x_0 + \mathcal{D}_{11} u + \mathcal{D}_{12} w$ (the actual output), and the lower part of the output y is a copy of the disturbance signal w, as in St:StHinf:Sec1. This allows us to write the formulas in a more compact form (and with a wider generality, which will not be used here). Note also, that the J-minimax coercivity condition is equal to $\exists\, [(\pi_+ \mathcal{D}^* J \mathcal{D} \pi_+)^{-1}]_{22} << 0$ [Lemma2.6b2].

In the control problems motivating this formulation the aim of the control engineer is to find a controller $\mathcal{U}$ s.t. the norm $\|w \to y_1\|_{\mathcal{L}(L^2,L^2)}$ is minimized. Because it is very difficult to do this directly, the engineer usually searches an approximation of the minimal norm by using a binary search. This requires a method for finding, for each $\gamma > 0$, whether there is any $\mathcal{U}$ that makes $\|w \to y_1\| < \gamma$, i.e., whether there exists a uniformly suboptimal controller $\mathcal{U}$ for that value of γ. Such a method is studied here and in St:StHinf, the main reference of the section.

Below we shall show that under certain conditions finding a uniformly suboptimal $\mathcal{U}$ is equal to finding the stabilizing solution of the Riccati equation studied in the previous sections.

Remark 8.3 *The assumption $\pi_+ \mathcal{D}_{11}^* \mathcal{D}_{11} \pi_+ >> 0$ [Hyp8.1] is equivalent to the (a.e.) left-invertibility of $\widehat{\mathcal{D}}_{11}$ as a member of $\mathrm{P}^\infty(i\mathbb{R}; \mathcal{L}(U,Y))$, i.e., it is a natural infinite-dimensional extension of the standard "full row rank on the imaginary axis" assumption.*

Lemma 8.4 *Let Hypothesis 8.1 hold. For each $w \in L^2(\mathbb{R}_+, W)$ and $x_0 \in H$ the control $u \in L^2(\mathbb{R}_+, U)$ minimizing $\|\mathcal{C}_1 x_0 + \mathcal{D}_{11} u + \mathcal{D}_{12} w\|$ is equal to*

$$-\pi_+ (\pi_+ \mathcal{D}_{11}^* \mathcal{D}_{11} \pi_+)^{-1} \pi_+ \mathcal{D}_{11}^* (\mathcal{C}_1 x_0 + \mathcal{D}_{12} w) =: u_{\min}(x_0, w).$$

Let $\mathcal{U}_0 := -\pi_+ (\pi_+ \mathcal{D}_{11}^* \mathcal{D}_{11} \pi_+)^{-1} \pi_+ \mathcal{D}_{11}^* \mathcal{D}_{12}$.

The Riccati operator Π of Ψ [2.4] can now be simplified to $\Pi = \mathcal{C}_1^* (I - P_1) \mathcal{C}_1$, where $P_1 = \mathcal{D}_{11} \pi_+ (\pi_+ \mathcal{D}_{11}^* \mathcal{D}_{11} \pi_+)^{-1} \pi_+ \mathcal{D}_{11}^* = P_1^* = P_1^2$, hence $\Pi \ge 0$.

There exists an $\mathcal{U} \in \mathcal{L}(L^2(\mathbb{R}_+, W); L^2(\mathbb{R}_+, U))$ s.t. $\|\mathcal{D}_{11} \mathcal{U} + \mathcal{D}_{12}\| < \gamma$ iff $\|\mathcal{D}_{11} \mathcal{U}_0 + \mathcal{D}_{12}\| < \gamma$ iff $\mathcal{D}$ is minimax J-coercive iff $Q(x_0, u_{\min}(x_0, w), w)$ is uniformly concave in $w \in L^2(\mathbb{R}_+, W)$ for each $x_0 \in H$.

Suppose that the equivalent conditions stated above hold. Then there is a unique function $w_{\mathrm{crit}}(x_0)$ that maximizes $Q(x_0, u_{\min}(x_0, w), w)$ with respect to w for a fixed $x_0 \in H$. Let $u_{\mathrm{crit}}(x_0) := u_{\min}(x_0, w_{\mathrm{crit}}(x_0))$ be the corresponding control and $y_{\mathrm{crit}}(x_0) := \mathcal{C} x_0 + \mathcal{D}_1 u_{\mathrm{crit}} + \mathcal{D}_2 w_{\mathrm{crit}}$ be the corresponding output. Then the minimax cost *satisfies*

$$\max_{w \in L^2(\mathbb{R}_+, W)} \min_{u \in L^2(\mathbb{R}_+, U)} Q(x_0, u, w) = Q(x_0, u_{\mathrm{crit}}(x_0), w_{\mathrm{crit}}(x_0)) = \langle x_0, \Pi x_0 \rangle \ \forall x_0 \in H$$

and the critical control and disturbance are given by

$$\begin{bmatrix} u_{\text{crit}}(x_0) \\ w_{\text{crit}}(x_0) \end{bmatrix} = -\pi_+(\pi_+\mathcal{D}^* J\mathcal{D}\pi_+)^{-1}\pi_+\mathcal{D}^* J\mathcal{C}x_0.$$

Proof: The formula for $u_{\min}$ is derived by Fréchet differentiation in the proof of St:StHinf:Lemma19i. It shows that if $\|\mathcal{D}_{11}\mathcal{U} + \mathcal{D}_{12}\| < \gamma$ has any solutions, $\mathcal{U}_0$ is among those. On the other hand, $\gamma > \|\mathcal{D}_{11}\mathcal{U}_0 + \mathcal{D}_{12}\|$ iff

$$\begin{aligned} 0 \;>>\; & \pi_+[\mathcal{D}_{11}\mathcal{U}_0 + \mathcal{D}_{12}]^*\pi_+[\mathcal{D}_{11}\mathcal{U}_0 + \mathcal{D}_{12}]\pi_+ - \gamma^2\pi_+ \\ = \;& \pi_+\mathcal{D}_{12}^*[I - \mathcal{D}_{11}\pi_+(\pi_+\mathcal{D}_{11}^*\mathcal{D}_{11}\pi_+)^{-1}\pi_+\mathcal{D}_{11}^*]\mathcal{D}_{12}\pi_+ - \gamma^2\pi_+, \end{aligned}$$

i.e., iff $\mathcal{D}$ is minimax J-coercive.

Also the condition for Q and the formulas for (and existence of) u_{crit}, w_{crit} and Π follow from a straight-forward calculation [St:StHinf:Lemma19ii]. □

Proposition 8.5 *Let Hypothesis 8.1 hold and let $\mathcal{D}$ be minimax J-coercive. The following conditions are equivalent:*

(i) $\mathcal{D}^* J\mathcal{D} \in \mathrm{TI}(U \times W)$ has a spectral factorization $\mathcal{X}^* S\mathcal{X}$.

(ii) There is an admissible stable state feedback pair $(\mathcal{K}\ \mathcal{F})$ s.t. $\begin{bmatrix} y_{\text{crit}}(x_0) \\ u_{\text{crit}}(x_0) \\ w_{\text{crit}}(x_0) \end{bmatrix}$ is equal to the output of $\Psi_\circlearrowleft$ with $u_\circlearrowleft = 0 \wedge w_\circlearrowleft = 0$ for any $x_0 \in H$ (cf. Th5.4).

If (ii) holds, then $\mathcal{X}$ in (i) can be chosen to be $I - \mathcal{F}$ and the factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$, where $\mathcal{N} := \mathcal{D}\mathcal{X}^{-1} \in \mathrm{TIC}(U \times W, Y)$, is a (J, S)-lossless-outer [St:Crit:Def19iv] factorization of $\mathcal{D}$.

Proof: The equivalence of (i) and (ii) is contained in St:StHinf:Th5. Note that the set of all possible operators $\mathcal{X}$ in (i) is $\{E(I - \mathcal{F}) \mid E, E^{-1} \in \mathcal{L}(U)\}$ [5.2]. In particular, they are all strongly (resp. weakly) regular iff one of them is strongly (resp. weakly) regular.

The fact that $\Pi \geq 0$ implies that $\mathcal{N}$ is (J, S)-lossless [St:Crit:Th22(v)⇒(iv)]. This and St:Crit:Lemma14i imply that the factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ is (J, S)-lossless-outer ((J, S)-losslessness [St:Crit:Def19] is not needed in this paper, it is just mentioned for the interested reader). □

The existence of a spectral factorization [Prop8.5ii] is not enough; we must require more about $\mathcal{X}$ and S (cf. St:StHinf). However, if the disturbance space W is finite-dimensional, these extra requirements can always be fulfilled:

Proposition 8.6 *Let the assumptions and the equivalent conditions in 8.5 hold, let* $\dim W < \infty$ *and set $\mathcal{X} := I - \mathcal{F}$.*

The operator $\mathcal{F}$ can be chosen so that $S = \left[\begin{smallmatrix} S_{11} & 0 \\ 0 & S_{22} \end{smallmatrix}\right]$, $S_{11} >> 0$, $S_{22} << 0$, $\exists \mathcal{X}_{11}^{-1} \in$ $\mathrm{TIC}(U)$, and $\mathcal{U} := -\mathcal{X}_{11}^{-1}\mathcal{X}_{12} \in \mathrm{TIC}(W, U)$ is a uniformly suboptimal controller. We call $\mathcal{U}$ the (uniformly suboptimal) central controller induced by $\mathcal{X}$.[35]

The set of all uniformly suboptimal controllers is

$$\left\{(\mathcal{X}_{11} - \mathcal{V}\mathcal{X}_{21})^{-1}(-\mathcal{X}_{12} + \mathcal{V}\mathcal{X}_{22}) \,\middle|\, \mathcal{V} \in \mathrm{TIC}(W, U) \wedge \|S_{11}^{1/2}\mathcal{V}(-S_{22})^{-1/2}\| < 1\right\}$$

[35]See St:StHinf:Th50&Fig14 for an explanation of in which sense the controller $\mathcal{U}$ is central.

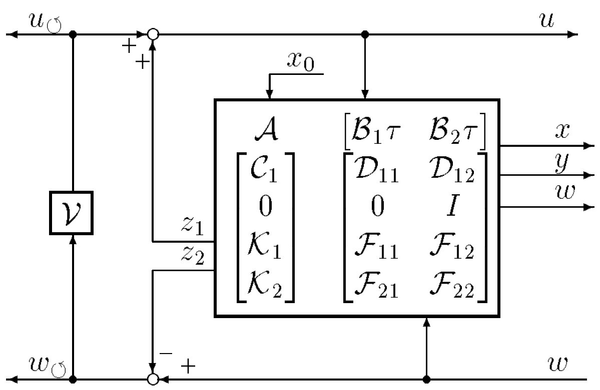

Figure 6: Parameterization of all suboptimal controllers

(i.e., the map $w \mapsto u$ in Figure 6). The operator $[\begin{smallmatrix}0 & I & 0\\0 & 0 & 0\end{smallmatrix}]$ is an admissible stable output feedback operator [Def3.3] for the extended open loop system

$$\Psi_{\mathrm{ext}} = \begin{bmatrix} \mathcal{A} & \begin{bmatrix}\mathcal{B}_1 & \mathcal{B}_2\end{bmatrix} \\ \begin{bmatrix}\mathcal{C}_1\\0\\\mathcal{K}_1\\\mathcal{K}_2\end{bmatrix} & \begin{bmatrix}\mathcal{D}_{11} & \mathcal{D}_{12}\\0 & I\\\mathcal{F}_{11} & \mathcal{F}_{12}\\\mathcal{F}_{21} & \mathcal{F}_{22}\end{bmatrix}\end{bmatrix}.$$

The resulting closed loop system $\Psi^{\frown}$ with the initial state $x_0 = 0$ and input $[\begin{smallmatrix}0\\w\end{smallmatrix}]$ produces the uniformly suboptimal output $z_1 = \mathcal{U}w$ as the third component of its output (see Figure 6 with $\mathcal{V} = 0$), i.e., the corresponding component of the input/output map of $\Psi^{\frown}$ is $\mathcal{F}_{12}^{\frown} = \mathcal{U} = -\mathcal{X}_{11}^{-1}\mathcal{X}_{12}$.

If Ψ_{ext} is weakly regular and the feed-through operator $X = I - F$ is right invertible in $\mathcal{L}(U \times W)$, then the Riccati operator Π of Ψ satisfies the corresponding Riccati equation as in Theorem 5.5.

The above proposition is a summary of St:StHinf:Lemma7&Cor59&Th50&Th31 and Theorem 5.5 (in that order). The system $\Psi^{\frown}$ can also be created by applying the feedback operator $[\begin{smallmatrix}0 & 0 & 0\\0 & 0 & -I\end{smallmatrix}]$ to the system $\Psi_{\circlearrowleft}$, i.e., by opening the loop for the disturbance w [St:StHinf:Th31]. See St:StHinf for more information on the these systems.

Theorem 8.7 states the connection between the Riccati equation and the minimax H^∞ problem. Because of the generality of WPLSs (e.g., we need not have $S = D^*JD$), the theorem is more complicated than the classical ones, but for Wiener class systems [Th8.10] the result again reduces to one similar to that in the finite-dimensional case.

Theorem 8.7 *Let Hypothesis 8.1 hold and let $\dim W < \infty$. Then the following conditions are equivalent:*

(i) $\mathcal{D}$ is minimax J-coercive, the two equivalent conditions in Proposition 8.5 hold, the feed-through operator X of $\mathcal{X}$ is invertible[36] and the sensitivity operator $\widetilde{S}$ corresponding to $X = I$ satisfies $\widetilde{S}_{11} >> 0$.

(ii) There exists a self-adjoint stabilizing solution Π of the Riccati equation induced by Ψ and J satisfying $S_{11} >> 0 \wedge S_{22} - S_{21}S_{11}^{-1}S_{12} << 0$.

[36]This ($\exists X^{-1}$) is always true, when $\mathcal{X}$ and $\mathcal{X}^*$ are regular [Prop4.3f]. We do not know, whether this is always true for weakly regular $\mathcal{X}$.

Suppose that the equivalent conditions stated above hold. Then we have the following:

The assumptions and conclusions in 7.1, 8.4 and 8.5 hold, in particular, the Π in (ii) is unique and it is the Riccati operator of Ψ [Lemma8.4]. A uniformly suboptimal central controller feedback formula for u is (here $\tilde{S}$ corresponds to $X = I$)

$$u(t) = (K_1)_w x(t) + F_{12} w(t) = -\tilde{S}_{11}^{-1}[((B_1^*)_w \Pi)_w + D_{11}^*(C_1)_w]x(t) - \tilde{S}_{11}^{-1}\tilde{S}_{12} w(t).$$

Moreover, if Ψ and $\mathcal{F}$ are regular (i.e., $\mathcal{D}$ is regular and the limit defining S exists strongly [Lemma6.6]), then Ψ_{ext}, $\Psi_{\circlearrowleft}$ and $\Psi^{\frown}$ are regular, in particular, then the central controller is regular.

The assumptions are discussed in Lemma 8.8.

Proof: If (i) holds, then Prop8.6 and St:StHinf:Th65ii imply that the S corresponding to $X = I$ satisfies the conditions given in (ii). The rest of (ii) follows from Lemma 6.4. We shall now assume that (ii) holds and prove (i) and the rest of the claims.

1° Now the assumptions and conclusions in Theorem 7.1 hold, hence Π is the same operator as in Lemma 8.4. In particular, $\Pi \geq 0$.

2° We shall go on by proving that $\mathcal{D}$ is minimax J-coercive. Let $\widetilde{S}$ be the sensitivity operator with $\widetilde{X} := \widehat{\widetilde{\mathcal{X}}}(\infty) = I$. Take $S' := E'^*\widetilde{S}E' = \begin{bmatrix} S_{11}' & 0 \\ 0 & S_{22}' \end{bmatrix}$, where $E' := \begin{bmatrix} I & -\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \\ 0 & I \end{bmatrix} \wedge S_{11}' := S_{11} >> 0 \wedge S_{22}' := S_{22} - S_{21}S_{11}^{-1}S_{12} << 0$. Take now $S := E^*S'E$, where $E = \begin{bmatrix} (S_{11}')^{-1/2} & 0 \\ 0 & (-S_{22}')^{-1/2} \end{bmatrix}$, to get $S = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

Let $\mathcal{N} := \mathcal{D}\mathcal{X}^{-1}$ so that $\mathcal{N}_{22} = \mathcal{M}_{22}$, where $\mathcal{M} := \mathcal{X}^{-1} \in \mathrm{TIC}(U \times W)$. The fact that $\Pi \geq 0$ implies that $\mathcal{N}$ is (J, S)-lossless [St:Crit:Th22(v)$\Rightarrow$(iv)], which in turn implies the invertibility of $\mathcal{N}_{22}$ in TIC as in St:StHinf:Lemma57.

$\mathcal{G} := \pi_+\mathcal{D}^*J\mathcal{D}\pi_+ = \pi_+\mathcal{X}^*S\mathcal{X}\pi_+ \in \mathcal{L}(L^2(\mathbb{R}_+, U \times W))$. By Lemma 5.3, $\exists \mathcal{G}^{-1} = \pi_+\mathcal{M}\pi_+S^{-1}\mathcal{M}^*\pi_+$. The operator $\mathcal{G}_{11} = \pi_+\mathcal{D}_{11}^*\mathcal{D}_{11}\pi_+$ was assumed to be invertible [Hyp8.1], hence from Lemma 2.6b1 we get that $\mathcal{T} := \mathcal{G}_{22} - \mathcal{G}_{21}\mathcal{G}_{11}^{-1}\mathcal{G}_{12}$ must be invertible too and $\mathcal{T}^{-1} = (\mathcal{G}^{-1})_{22} = (\pi_+\mathcal{M}\pi_+S^{-1}\mathcal{M}^*\pi_+)_{22} = \mathcal{M}_{21}\mathcal{M}_{21}^* - \mathcal{M}_{22}\mathcal{M}_{22}^*$. The invertibility of $\mathcal{M}_{22}$ implies that $\exists \mathcal{X}_{11}^{-1} \in \mathrm{TIC}(U)$ [2.6b1].

Now we know that $\mathcal{T} := \mathcal{G}_{22} - \mathcal{G}_{21}\mathcal{G}_{11}^{-1}\mathcal{G}_{12} = -\pi_+\gamma^2 + \pi_+\mathcal{D}_{12}^*\mathcal{D}_{12}\pi_+ - \pi_+\mathcal{D}_{12}^*\mathcal{D}_{11}(\pi_+\mathcal{D}_{11}^*\mathcal{D}_{11}\pi_+)^{-1}\mathcal{D}_{11}^*$ is invertible, so to prove the minimax J-coercivity of $\mathcal{D}$, we only need to show that $\mathcal{T}^{-1} \leq 0$. From $\mathcal{X}\mathcal{M} = I = \mathcal{M}\mathcal{X}$ we get that $\mathcal{X}_{22}\mathcal{M}_{21} = -\mathcal{X}_{21}\mathcal{M}_{11} \wedge \mathcal{X}_{12}\mathcal{M}_{21} = I - \mathcal{X}_{11}\mathcal{M}_{11}$, hence

$$\begin{aligned} \mathcal{M}_{22}^{-1}\mathcal{M}_{21} &= (\mathcal{X}_{22} - \mathcal{X}_{21}\mathcal{X}_{11}^{-1}\mathcal{X}_{12})\mathcal{M}_{21} \\ &= -\mathcal{X}_{21}\mathcal{M}_{11} - \mathcal{X}_{21}\mathcal{X}_{11}^{-1}(I - \mathcal{X}_{11}\mathcal{M}_{11}) = -\mathcal{X}_{21}\mathcal{X}_{11}^{-1}. \end{aligned}$$

Thus $\mathcal{T}^{-1} = \mathcal{M}_{21}\mathcal{M}_{21}^* - \mathcal{M}_{22}\mathcal{M}_{22}^* = -\mathcal{M}_{22}[I - \mathcal{M}_{22}^{-1}\mathcal{M}_{21}\mathcal{M}_{21}^*(\mathcal{M}_{22}^*)^{-1}]\mathcal{M}_{22}^*$ $<< 0$, because $1 > \|\mathcal{M}_{21}^*(\mathcal{M}_{22}^*)^{-1}\| = \|\mathcal{M}_{22}^{-1}\mathcal{M}_{21}\| = \|\mathcal{X}_{21}\mathcal{X}_{11}^{-1}\|$, which can be seen as in St:StHinf:Lemma36:

$$\begin{aligned} 0 &<< \mathcal{D}_{11}^*\mathcal{D}_{11} = (\mathcal{D}^*J\mathcal{D})_{11} = (\mathcal{X}^*S\mathcal{X})_{11} \\ &= \mathcal{X}_{11}^*\mathcal{X}_{11} - \mathcal{X}_{21}^*\mathcal{X}_{21} = \mathcal{X}_{11}^*[I - (\mathcal{X}_{11}^*)^{-1}\mathcal{X}_{21}^*\mathcal{X}_{21}\mathcal{X}_{11}^{-1}]\mathcal{X}_{11}, \end{aligned}$$

i.e., $1 > \|\mathcal{X}_{21}\mathcal{X}_{11}^{-1}\|$.

Now we have shown that $\mathcal{D}$ is minimax J-coercive and that $\mathcal{D}^*J\mathcal{D}$ has a spectral factorization. This implies (i) and the other claims in the theorem (the feedback formula for u is written out in St:StHinf:Sec2, where it is shown that if there are any uniformly suboptimal central controllers, then the one given by the feedback formula is among them) excluding the regularity results, which we prove below.

3° Assume, that Ψ and $\mathcal{F}$ are regular (i.e., Ψ_{ext} is regular). By Lemma 6.6, also $\Psi_{\circlearrowleft}$ is regular. Because $\widetilde{X}_{11} = I$ and $X = EE'\widetilde{X}$, we have that $X_{11} = (S_{11})^{-1/2}$, in particular, $\exists X_{11}^{-1} \in \mathcal{L}(U)$. Thus we can see that also the semi-closed system $\Psi^{\frown}$ [Prop8.6] is regular, by Weiss:Feedback:Th4.7, because the operator "$(I-KD)$" in Weiss:Feedback:Th4.7 is now the invertible operator $\left[\begin{smallmatrix} X_{11} & X_{12} \\ 0 & I \end{smallmatrix}\right]$ (and hence "$(I-DK)$" is invertible too). Hence the central controller $\mathcal{U} = \mathcal{F}_{12}^{\frown}$ is regular too. □

Lemma 8.8 *The assumption $S_{11} >> 0 \wedge S_{22} - S_{21}S_{11}^{-1}S_{12} << 0$ is equivalent to the standard assumption*

$$D_{11}^*D_{11} >> 0 \wedge \gamma^2 - D_{12}^*(I - D_{11}(D_{11}^*D_{11})^{-1}D_{11}^*)D_{12} >> 0,$$

when the system is so smooth that necessarily $S = D^*JD$, e.g., when $B \in \mathcal{L}(U,H)$ (in particular, in the finite-dimensional case), and for Pritchard–Salamon systems and Wiener class systems.

In Th8.7i, instead of $\exists S_{11}^{-1} \wedge S_{22} - S_{21}S_{11}^{-1}S_{12} << 0$, we could have, equivalently, assumed that $(S^{-1})_{22} << 0$ or that $\exists S_{11}^{-1} \wedge S_{22} - S_{21}S_{11}^{-1}S_{12} \leq 0$.

Proof: If $B \in \mathcal{L}(U,H)$, then $B_w^* = B^* \in \mathcal{L}(H,U)$ and hence $\text{w-lim}_{s\to+\infty} B_w^*\Pi(s-A)^{-1}Bu_0 = 0 \; \forall u_0 \in U$ [Lemma6.1c], i.e., $S = D^*JD$ [Prop5.6]. If $\dim H < \infty$, then necessarily $A \in \mathcal{L}(H)$ and hence $B \in \mathcal{L}(U,H)$.

Because the S in (ii) of the theorem is assumed to be invertible [Def6.3], the assumption $(S^{-1})_{22} << 0$ implies the invertibility of S_{11} [Lemma2.6b1]. which in turn implies that $(S^{-1})_{22} = S_{22} - S_{21}S_{11}^{-1}S_{12}$ [Lemma2.6b2]. Assuming that, we can define S' and $\mathcal{X}'$ as in the proof of Theorem 8.7, and we get $0 << \widehat{\mathcal{D}}_{11}^*\widehat{\mathcal{D}}_{11} = \mathcal{X}_{11}^*S_{11}\mathcal{X}_{11} + \mathcal{X}_{21}^*S_{22}'\mathcal{X}_{21}$, which implies that $S_{11} \geq 0$ (because $S_{22}' = S_{22} - S_{21}S_{11}^{-1}S_{12} << 0$), hence $S_{11} >> 0$.

If, instead, the assumption on S is written in the form $\exists S_{11}^{-1} \wedge S_{22} - S_{21}S_{11}^{-1}S_{12} \leq 0$, then $S_{22} - S_{21}S_{11}^{-1}S_{12}$ can be seen to be invertible (hence $<< 0$) by Lemma2.6b2 and S_{11} can be seen positive as above.

□

With the standard assumption $D_{11}^*D_{11} >> 0$ we get the regular case of Theorem 8.7 into a slightly nicer form.

Corollary 8.9 *Let Hypothesis 8.1 hold, let Ψ be regular, let $D_{11}^*D_{11} >> 0$ and let $\dim W < \infty$. Then the following conditions are equivalent:*

(i) $\mathcal{D}$ is minimax J-coercive, the two equivalent conditions in Proposition 8.5 hold, $\mathcal{X}$ is regular and the feed-through operator X of $\mathcal{X}$ is invertible.

(ii) There exists a self-adjoint stabilizing solution Π of the Riccati equation induced by Ψ and J satisfying $S_{11} >> 0 \wedge S_{22} - S_{21}S_{11}^{-1}S_{12} << 0$ such that the limit defining S exists strongly.

Suppose that the equivalent conditions stated above hold. Then we have the following:

The assumptions and conclusions in 7.1, 8.4 and 8.5 hold, in particular, the Π in (ii) is unique and it is the Riccati operator of Ψ [8.4].

Moreover, Ψ_{ext}, $\Psi_{\circlearrowleft}$ and $\Psi^{\frown}$ are regular, in particular, the central controller is regular.

Proof: 1° Assume that (i) holds and normalize $\mathcal{X}$ by $\widetilde{\mathcal{X}} := X^{-1}\mathcal{X}$. The fact that $D_{11}^* D_{11} >> 0$ implies $\widetilde{S}_{11} >> 0$, because $\widetilde{S} \geq D^* J D$ [5.6c][37]; thus $\widetilde{S}_{11} \geq D_{11}^* D_{11} >> 0$ and, by Theorem 8.7 and St:Quadr:Cor7.2i (& St:Crit:Th17), (ii) is true.

2° Assume that (ii) holds. $\mathcal{X}$ is regular [Lemma6.6], hence (i) and the rest of the claims are true [Th8.7]. □

To be able to see through the technical complexities of Theorem 8.7, we shall finish this study by showing how also that theorem can be considerably simplified by restricting it to Wiener class systems with a finite-dimensional input space $U \times W$:

Theorem 8.10 *Let Hypothesis 8.1 hold, $\mathcal{D} \in \mathcal{W}_+ * (U \times W, Y)$, $\dim U < \infty$ and $\dim W < \infty$. Then the following conditions are equivalent:*

(i) $\mathcal{D}$ is minimax J-coercive.

(ii) $\gamma^2 - D_{12}^*(I - D_{11}(D_{11}^* D_{11})^{-1} D_{11}^*) D_{12} >> 0$ and there exists a self-adjoint stabilizing solution Π of the Riccati equation induced by Ψ and J.

(ii') $\gamma^2 - D_{12}^*(I - D_{11}(D_{11}^* D_{11})^{-1} D_{11}^*) D_{12} >> 0$ and there exists an operator $\Pi \in \mathcal{L}(H)$ that satisfies the Riccati equation (here $B_w^*\Pi = B_\Lambda^*\Pi$)[38]

$$A^\times \Pi + \Pi A + C^\times C = (B_w^* \Pi + D^* J C)^\times (D^* J D)^{-1} (B_w^* \Pi + D^* J C).$$

The operators $\begin{bmatrix} A & B \\ C & D \\ K & F \end{bmatrix}$, where $K := -S^{-1}(B_w^*\Pi + D^* J C)$, are the generators of an OSCWPLS$_0$ Ψ_{ext} for which the feedback $L = [0\ I]$ is admissible creating a closed loop system $\Psi_\circlearrowleft \in \text{OSCWPLS}_0$. Finally, $\langle \mathcal{A}(t)x_0, \Pi \mathcal{A}(t)x_0 \rangle \overset{t\to+\infty}{\to} 0\ \forall x_0 \in H$.

(iii) There is a $\mathcal{U} \in \text{TIC}(W, U)$ s.t. $\|\mathcal{D}_{11}\mathcal{U} + \mathcal{D}_{12}\| < \gamma$.

Suppose that the equivalent conditions stated above hold. Then we have the following:

All the assumptions and conclusions in Th7.1, Th8.7, Cor8.9, Lemma8.4 and Prop8.5 hold, in particular, the operator Π in (ii) is unique and it is the Riccati operator of Ψ [Lemma8.4]. The feedback formula for u can be written in the form

$$u(t) = -(D_{11}^* D_{11})^{-1} [((B_1^*)_\Lambda \Pi)_\Lambda + D_{11}^* (C_1)_\Lambda] x(t) - (D_{11}^* D_{11})^{-1} D_{11}^* D_{12} w(t).$$

Moreover, the input/output maps of Ψ_{ext}, $\Psi_\circlearrowleft$ and $\Psi^\frown$ all belong to $\mathcal{W}_+*$, in particular, $\mathcal{X}, \mathcal{U} \in \mathcal{W}_+*$, where $\mathcal{U} := \mathcal{F}_{12}^\frown$ is the central controller.

Proof: Note first that now with $\mathcal{D} \in \mathcal{W}_+*$, by the Riemann–Lebesgue lemma, $\mathcal{X}^* S \mathcal{X} = \mathcal{D}^* J \mathcal{D}$ implies $S = D^* J D$, and the standing assumption $\pi_+ \mathcal{D}_{11}^* \mathcal{D}_{11} \pi_+ >> 0$ implies $D_{11}^* D_{11} >> 0$.

1° We have (i) $\iff$ (iii), by Lemma 8.4, hence each of (i), (ii), (ii') and (iii) implies that the assumptions and conclusions of Theorem 7.2 are true (because (i) and standing hypothesis $\pi_+ \mathcal{D}_{11}^* J \mathcal{D}_{11}^* \pi_+ >> 0$ [Hyp8.1] together imply that $\pi_+ \mathcal{D}^* J \mathcal{D} \pi_+$

[37] We have $\Pi \geq 0$, because, by Theorem 7.1, Π is the one in Lemma 8.4.

[38] Note that, with the standard additional assumption $D_{11}^* D_{11} = I \wedge D_{12} = 0 \wedge D_{11}^* C_1 = 0$, we get the Riccati equation to the standard form (cf. Th7.2)

$$A^\times \Pi + \Pi A + C^\times C = \gamma^2 \left[(B_2^*)_w \Pi\right]^\times (B_2^*)_w \Pi - \left[(B_1^*)_w \Pi\right]^\times (B_1^*)_w \Pi$$

as in GreenLim:p.251 and in Keulen:Rem4.6 (see also Keulen:p.103).

is invertible [Lemma2.6]). In particular, there is a D^*JD-spectral factorization $\mathcal{X}^*(D^*JD)\mathcal{X}$ of $\mathcal{D}^*J\mathcal{D}$ with $X = I$, and (ii) $\Longleftrightarrow$ (ii').

2° (i) $\Longrightarrow$ (ii): As noted above, $S_{11} = D_{11}^* D_{11} >> 0$. By $S_{11} >> 0$ and 1^o, in this case we can deduce Th8.10(i) $\Longrightarrow$ Cor8.9(i) $\Longrightarrow$ Cor8.9(ii) $\Longrightarrow$ Th8.10(ii).

3° (ii) $\Longrightarrow$ (i): The first assumption in (ii) is equal to $S_{22} - S_{21}S_{11}^{-1}S_{12} << 0$, hence (ii) $\Longrightarrow$ (i), via Corollary 8.9 as in 2^o. $\square$

References

[ClancGohb] — K. F. Clancey and I. C. Gohberg. *Factorizations of Matrix Functions and Singular Integral Operators.* Birkhäuser Verlag, Basel Boston Stuttgart, 1981.

[CurtOostv] — R. F. Curtain and J. C. Oostveen. Riccati Equations for Strongly Stabilizable Bounded Linear Systems. Prerprint, 1996.

[FlandLasTrig] — F. Flandoli, I. Lasiecka and R. Triggiani. Algebraic Riccati equations with non-smoothing observation arising in hyperbolic and Euler–Bernoulli boundary control problems. *Annali Mat. Pura Appl.*, 153:307–382, 1988.

[GreenLim] — M. Green and D. J. N. Limebeer. *Linear Robust Control.* Prentice Hall, Englewood Cliffs, New Jersey, 1995.

[Keulen] — B. van Keulen. *H_∞-Control for Distributed Parameter Systems: A State Space Approach.* Birkhäuser Verlag, Basel Boston Berlin, 1993.

[Malinen:WPD] — J. Malinen. Well Posed Discrete Time Linear Systems and Their Feedbacks. Technical Report A384, Institute of mathematics, Helsinki University of Technology, Espoo, Finland, 1997.

[Malinen:NDT] — J. Malinen. Nonstandard Discrete Time Cost Optimization Problem: The Spectral Factorization Approach. Technical Report A385, Institute of mathematics, Helsinki University of Technology, Espoo, Finland, 1997.

[M:SubregInfrareg] — K. M. Mikkola. Subregularity and Infraregularity of Well-Posed Linear Systems. Unpublished, 1997.

[M:GRPRicc] — K. M. Mikkola. On the H^2 and H^∞ Infinite-Dimensional General Regulator Problems and Their Algebraic Riccati Equations. In preparation, 1997.

[Pazy] — A. Pazy. *Semi-Groups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, Berlin, 1983.

[RudinFA] — W. Rudin. *Functional Analysis.* McGraw–Hill, New York, 1973.

[SalamonInfDim] — D. Salamon. Infinite dimensional linear systems with unbounded control and observation: a functional analytic approach. *Transactions of the American Mathematical Society*, 300:383–431, 1987.

[St:SSQuadr] — O. J. Staffans. Quadratic optimal control of stable systems through spectral factorization. *Mathematics of Control, Signals, and Systems*, 8:167–197, 1995.

[St:DCRicc] — O. J. Staffans. On the discrete and continuous time infinite-dimensional algebraic Riccati equations. *Systems and Control Letters*, 29:131–138, 1996.

[St:StQuadr] — O. J. Staffans. Quadratic optimal control of stable well-posed linear systems. To appear in Transactions of American Mathematical Society, 1997.

[St:Coprime] — O. J. Staffans. Coprime factorizations and well-posed linear systems. To appear in SIAM Journal on Control and Optimization, 1997.

[St:Quadr] — O. J. Staffans. Quadratic optimal control of well-posed linear systems. To appear in SIAM Journal on Control and Optimization, 1997.

[St:Crit] — O. J. Staffans. Feedback representations of critical controls for well-posed linear systems. Submitted, 1997.

[St:StHinf] — O. J. Staffans. On the distributed stable full information H^∞ minimax problem. Submitted, 1997.

[Weiss:AdmContr] — G. Weiss. Admissibility of unbounded control operators. *SIAM Journal on Control and Optimization*, 27:527–545, 1989.

[Weiss:AdmObs] — G. Weiss. Admissible observation operators for linear semigroups. *Israel Journal of Mathematics*, 65:17–43, 1989.

[Weiss:Repr] — G. Weiss. Representation of shift-invariant operators on L^2 by H^∞ transfer functions: an elementary proof, a generalization to L^2, and a counterexample for L^∞. *Mathematics of Control, Signals and Systems*, 4:193–203, 1991.

[Weiss:TransferI] — G. Weiss. Transfer functions of regular linear systems. Part I: Characterizations of regularity. *Transactions of American Mathematical Society*, 342:827–854, 1994.

[Weiss:TransferII] — G. Weiss. Transfer functions of regular linear systems. Part II: Duality. In preparation, 1997.

[Weiss:Feedback] — G. Weiss. Regular linear systems with feedback. *Mathematics of Control, Signals, and Systems*, 7:23–57, 1994.

[Weiss∗2] — M. Weiss and G. Weiss. Optimal control of stable weakly regular linear systems. To appear in Mathematics of Control, Signals, and Systems, 1997.

[M.Weiss:Thesis] — M. Weiss. Riccati Equations in Hilbert Space: A Popov Function Approach. Doctoral Dissertation, Rijksuniversiteit Gronigen, 1994.