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DISCRETE TIME RICCATI EQUATIONS AND INVARIANT SUBSPACES OF LINEAR OPERATORS

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Abstract: *Let U, H and Y be separable Hilbert spaces. Let $A \in \mathcal{L}(H)$, $B \in \mathcal{L}(U;H)$, $C \in \mathcal{L}(H;Y)$, $D \in \mathcal{L}(U;Y)$ be such that the open loop transfer function $\mathcal{D}(z) := D + zC(I - zA)^{-1}B \in H^\infty(\mathcal{L}(U;Y))$. Let $J \geq 0$ be a self-adjoint cost operator. We study a subset of self-adjoint solutions P of the discrete time algebraic Riccati equation (DARE)*

$$\begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \\ \Lambda_P K_P = -D^*JC - B^*PA, \end{cases}$$

where $\Lambda_P, \Lambda_P^{-1} \in \mathcal{L}(U)$ and $K_P \in \mathcal{L}(H;U)$. We further assume that a critical solution P^{crit} of DARE exists, such that $\mathcal{X}(z) := I - zK_{P^{\mathrm{crit}}}(I - z(A + BK_{P^{\mathrm{crit}}})^{-1}B \in H^\infty(\mathcal{L}(U;Y))$ become an outer factor of $\mathcal{D}(z)$.
Under technical assumptions, we study connections of the nonnegative solutions of DARE to the invariant subspace structure of $(A^{\mathrm{crit}})^*$.

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1 Introduction

In this paper, we consider the connection of the solution set of a discrete time Riccati equation (DARE) to the invariant subspaces of a linear operator. Because this paper is not written to be self-contained, we assume that the reader has access (and some understanding) to the our previous works [12], [13], [14], [16], [17], and [18]. All these works are written in discrete time but the references they contain are mostly written in continuous time. A preliminary version of this paper have been presented in MMAR98 conference, Poland, see [15].

Let us first recall some basic notions. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable and output stable *discrete time linear system* (DLS), and $J \in \mathcal{L}(Y)$ a self-adjoint cost operator. The symbol $Ric(\phi, J)$ denotes the associated *discrete time Riccati equation*, given by

$$
(1) \qquad \begin{cases} A^*PA - P + C^*JC = K_P^* \Lambda_P K_P \\ \Lambda_P = D^*JD + B^*PB \\ \Lambda_P K_P = -D^*JC - B^*PA. \end{cases}
$$

If P is a self-adjoint solution of $Ric(\phi, J)$, we write $P \in Ric(\phi, J)$. So, the same symbol is used for DARE and its solutions set. We believe this does not cause any confusion.

We make it a standing assumption that ϕ is both I/O stable and output stable. Then DARE (1) is called H^∞DARE, and write $ric(\phi, J)$ in place for $Ric(\phi, J)$. A reasonable theory for H^∞DARE is given in our previous works [16] and [17]. Several subsets of the solution set $Ric(\phi, J)$ are defined and studied in [16]. The most interesting (and smallest) of them, the set of regular H^∞ solutions $ric_0(\phi, J)$, contains those $P \in Ric(\phi, J)$ whose spectral DLS $\phi_P := \left(\begin{smallmatrix} A & B \\ -K_P & I \end{smallmatrix}\right)$ is both output stable and I/O stable, and, in addition, the residual cost operator

$$
L_{A,P} := \mathrm{s} - \lim_{j \to \infty} A^{*j} P A^j
$$

exists and equals 0.

1.1 Partial ordering and Riccati equation

Our starting point is the following lemma, given in [17, Theorem 95]. It relates, under technical assumptions, the natural partial ordering of the nonnegative solutions $P \in ric_0(\phi, J)$ to the partial ordering of certain chains of (adjoined) partial inner factors of the I/O map $\mathcal{D}_\phi$.

Lemma 1. *Let $J \geq 0$ be a cost operator in $\mathcal{L}(Y)$. Let $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable and output stable DLS, such that $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$. Assume that the input space U and the output space Y are separable, and the input operator B is Hilbert–Schmidt. Assume that the regular critical solution $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ exists.*

For $P_1, P_2 \in ric_0(\phi, J)$, the following are equivalent

(i) $P_1 \leq P_2$.

(ii) $\mathrm{range}\left(\widetilde{\mathcal{N}}_{P_1}\bar{\pi}_+\right) \subset \mathrm{range}\left(\widetilde{\mathcal{N}}_{P_2}\bar{\pi}_+\right)$, *where $\mathcal{N}_P$ is the $(\Lambda_P, \Lambda_{P^{\mathrm{crit}}})$-inner factor of $\mathcal{D}_{\phi_P}$.*

We have to explain what the causal Toeplitz operator $\widetilde{\mathcal{N}}_{P_1}\bar{\pi}_+$, with an adjoint symbol, means in the previous lemma. The adjoint I/O map of $\mathcal{N}_P$ by $\widetilde{\mathcal{N}}_P$ is easiest defined in terms of the transfer functions $\widetilde{\mathcal{N}}_P(z) = \mathcal{N}_P(\bar{z})^*$, for all $z \in \mathbf{D}$. To see what $\mathcal{N}_P$ stands for, consider the following spectral DLS, centered at $P \in ric_0(\phi, J)$

$$\phi_P := \begin{pmatrix} A & B \\ -K_P & I \end{pmatrix}$$

for arbitrary $P \in ric_0(\phi, J)$. Its I/O map $\mathcal{D}_{\phi_P}$ is a stable spectral factor of the Popov operator $\mathcal{D}_\phi^* J \mathcal{D}_\phi$. Under the assumptions of Lemma 1, the I/O map $\mathcal{D}_{\phi_P}$ has a $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner/outer factorization $\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X}$, where the outer factor $\mathcal{X}$ has a bounded inverse. Furthermore, $\mathcal{X}$ is independent of the particular choice of solution $P \in ric_0(\phi, J)$, and it follows that the inner part $\mathcal{N}_P$ alone is responsible for parameterizing different stable spectral factors of the Popov operator. We conclude that the partial ordering of the nonnegative $P \in ric_0(\phi, J)$ becomes important because its connection to the spectral factorization structure of $\mathcal{D}_\phi^* J \mathcal{D}_\phi$, and if $P \geq 0$, to the inner-outer factorization of $\mathcal{D}_\phi$ in an order-preserving way, see [17, Lemma 79].

In operator theory, the notion of partial ordering emerges in connection with the lattice of invariant subspaces of a bounded linear operator. The question arises, whether the natural partial ordering of $ric_0(\phi, J)$, as discussed above, would describe the invariant structure of some linear operator in a fruitful way. We are led to seek answers to the following two main questions:

A. Is there a bounded linear operator T, a model operator, such that the natural partial ordering of the solution set $ric_0(\phi, J)$ (under some restrictive, but technical assumptions) gets encoded into the invariant (or co-invariant) subspace structure of T?

B. If such T exists, can it be expressed in simple and practical terms of the given original data, namely the quadruple $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ together with the cost operator J? Furthermore, can we obtain system theoretic information about the DLS ϕ and the associated H^∞DARE (1), by looking at the structure of such an operator T?

It is well know that several variants of both these question can be (and have been) given a positive answer, under some particular restrictive assumptions that vary from work to work. These lead to several approaches, leading to different descriptions of the partial ordering of the solutions set of DARE. We proceed to make a brief survey of this literature in Subsections 1.2 and 1.3. After that we return to interpret Lemma 1 in Subsection 1.4, and get another candidate for the model operator T.

1.2 Description in terms of invariant subspaces of a Hamiltonian operator

In the case of a matrix-valued DARE, the standard theory, as presented in great detail in the monograph [10], provides us answers to the main questions A. and B. of the previous Subsection. In this theory, the solutions of DARE are in one-to-one correspondence with the family of maximal, j-neutral invariant subspaces of a j-unitary Hamiltonian operator T. Here the Hermitian matrix $j := \left(\begin{smallmatrix} 0 & -iI \\ iI & 0 \end{smallmatrix}\right)$ induces an indefinite scalar product, and the requirement of j-neutrality is related to the requirement that the solution of DARE should be self-adjoint. For a particular construction of T from the data of DARE, see [10, Chapter 12]. See also [9] which contains good references and an account of history.

Analogous operator approaches have been developed for systems with an infinite-dimensional state space, see the continuous time example [2, Ex. 6.25] for Hamiltonians that are Riesz spectral operators, and its application [3, Lemma 3.0.4]. We remark that in the literature, the main emphasis lies on a a less general DARE (its continuous time analogue), arising from the Least Squares type of problems. This LQDARE is given by

$$(2) \qquad \begin{cases} A^*PA - P + C^*JC = A^*PB \cdot \Lambda_P^{-1} \cdot B^*PA \\ \Lambda_P = D^*JD + B^*PB. \end{cases}$$

Further comments and comparisons about the Riccati equations (1) and (2) can be found in the introductory section of [17].

1.3 Description in terms of unobservable, unstable subspaces

The unobservable and unstable subspaces of the semigroup generator A can be used to classify the nonnegative solutions P for LQDARE of type (2). These subspaces coincide with (the essential part of) the null spaces $\ker(P)$. In this direction we refer to finite dimensional results [11], [23], [24], and [25]. A particularly interesting result on the factorization of rational discrete time inner function is [8, Theorem 4.1] and a continuous time result [7, Theorem 4.3]. The results in [1] and [3] are also in this directions but infinite dimensional.

We now consider the discrete time matrix work [25] (Wimmer) as a representative of this genre. The LQDARE considered is a special case of (2), written in our notations as

$$A^*PA - P + C^*C = A^*PB\,(I + B^*PB)^{-1}\,B^*PA. \tag{3}$$

The linear system associated to this LQDARE is assumed to output stabilizable, which is a sufficient and necessary condition for the LQDARE to have a nonnegative solution. The state space $\mathbf{C}^n$ is written as a direct sum of two subspaces $\mathbf{C}^n := U_0 \oplus U_r$, where U_0 is a subspace of $V_=(A, C)$, and the latter is the subspace spanned by unobservable generalized eigenvectors associated to the unimodular eigenvalues of A. In [25, Theorem 1.1], it is shown that any nonnegative solution P can be decomposed according to this direct sum representation. The part corresponding to U_0, say $P_0 \geq 0$, is a solution of a Liapunov equation. As a source of inconvenience, P_0 is essentially forgotten. The other part, say $P_r \geq 0$, solves a reduced Riccati equation, and is interesting enough to be further studied. The nonnegative solutions $P_r \in \mathcal{S}$ of the reduced DARE can now be classified roughly as follows. To this end, we define the family $\mathcal{N}$ of subspaces of $\mathbf{C}^n$

$$\begin{aligned}\mathcal{N} := \{N \subset \mathbf{C}^n \quad | \quad & AN \subset N, \\ & V_{\leq}(A, C) \subset N \subset V(A, C), \quad N + R(A, B) + E_{<}(A) = \mathbf{C}^n\}\end{aligned}$$

where $V(A, C)$ is the unobservable subspace, $V_{\leq}(A, C)$ is the stable unobservable subspace, $R(A, B)$ is the controllable subspace (range of the controllability map) and $E_{<}(A)$ is the stable spectral subspace of the semigroup generator A. The set $\mathcal{N}$ is shown to be in one-to-one order-preserving correspondence with the solutions $P_r \in \mathcal{S}$ of the reduced LQDARE, see [25, Theorem 1.3]. The correspondence is given by the mapping $\gamma : \mathcal{S} \to \mathcal{N}$ is given by $\gamma(P_r) = \ker(P_r)$. We remark that for the class of LQDAREs (3), it is quite easy to show that the null spaces $\ker(P)$ are A-invariant. In fact, this technique is used in the proof of Lemma 9.

1.4 Descriptions in terms of shift-invariant subspaces

There is a completely different candidate for a model operator T, discussed in Subsection 1.1. This approach is based on Lemma 1, and it consequently originates from our previous works [16] and [17]. To be more precise, we first have to interpret Lemma 1 in the sense of Beurling–Lax–Halmos Theorem on the shift-invariant subspaces.

In order to be able to speak about the usual inner transfer functions, we normalize and define the I/O map $\widetilde{\mathcal{N}}_P^\circ := \Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}} \widetilde{\mathcal{N}}_P \Lambda_P^{\frac{1}{2}}$. Now the transfer function $\widetilde{\mathcal{N}}_P^\circ(z)$ is inner $\mathcal{L}(U)$-valued analytic function in $\mathbf{D}$, having unitary nontangential boundary limits $\widetilde{\mathcal{N}}_P^\circ(e^{i\theta})$ a.e. $e^{i\theta} \in \mathbf{T}$. Furthermore, $\widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+$ is the Toeplitz operator with causal symbol, equivalent (via Fourier transform) to the multiplication operator by the (boundary trace of the) transfer function $\widetilde{\mathcal{N}}_P^\circ(e^{i\theta})$ on the Hardy space $H^2(\mathbf{T};U) \subset L^2(\mathbf{T};U)$. So as the range spaces range $\left(\widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+\right)$, the reader will immediately notice that this situation is described by the Beurling–Lax–Halmos Theorem of forward shift-invariant subspaces.

Because the inclusion of the ranges range $\left(\widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+\right)$ obey the partial ordering of $P \in ric_0(\phi, J)$ by Lemma 1, it follows that the orthogonal complement spaces, denoted by

$$K_{\widetilde{\phi^\circ(P)}} := \ell^2(\mathbf{Z}_+;U) \ominus \text{range}\left(\widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+\right), \tag{4}$$

are partially ordered by inclusion, but in a reverse direction. Clearly $K_{\widetilde{\phi^\circ(P)}}$ are (backward unilateral shift) S^*-invariant. We conclude that the restrictions $S^*|K_{\widetilde{\phi^\circ(P)}}$ obey the partial ordering of the solution set $ric_0(\phi, J)$, and it is easy to imagine that each $S^*|K_{\widetilde{\phi^\circ(P)}}$, $P \in ric_0(\phi, J)$, can be seen as part of an associated operator T in its invariant subspace. This T would be a restriction of the backward shift, too.

We have presented a rough outline of an answer to the first main question A. we asked in Subsection 1.1. We now proceed to show that alse the second main question B can answered in a satisfactory manner. In Subsection 1.5, we discuss why the present approach is interesting from operator and system theoretic point of view. In Subsection 1.6 we (quite superficially) compare our approach to the two approaches, reviewed in Subsections 1.2 and 1.3.

1.5 Why is the desription by the shift-invariant subspaces interesting?

From first sight it might seem that the choice of (a truncated version of the) the backward shift S^* on $\ell^2(\mathbf{Z}_+;U)$ as the model operator T would be uninteresting. Such T could have very little to do with the original data, namely the I/O stable and output stable DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and the cost operator $J \geq 0$. Even if there were a connection, it might be techically complicated to describe. Such a connection could be quite intractable, so that actual numerical computations (needed in the applications of the Riccati equation theory) could be impossible. In this description, the model operator T operates generally in a infinite-dimensional sequence space $\ell^2(\mathbf{Z}_+;U)$, even if all the spaces U, H and Y were finite dimensional. In Subsections 1.2 and 1.3, the solutions were parameterized by subspaces $H \times H$ and H, respectively, where H is the finite dimensional state space. At least in the first case, the solution of matrix DARE can be found (even numerically!) by solving a generalized Hamiltonian eigenvalue problem.

If all the bad things were true, the second main question B. might lack a reasonable answer, and the practical significance of our earlier works [16] and [17] would be diminished. The main goal of this paper is to establish a clear and simple connection of the compressed shifts $S^*|K_{\widetilde{\phi^\circ(P)}}$, $P \in ric_0(\phi, J)$, to the original data $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and J. We consider first certain closed loop semigroup (co-)invariant subspaces of the state space.

Let $J \geq 0$ be a cost operator, and $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ an output stable and I/O stable DLS, with $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$. Assume that a regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}_\phi^{\mathrm{crit}}\right)^* J\mathcal{C}_\phi^{\mathrm{crit}}$ exists, and let $P \in Ric(\phi, J)$ be such that $0 \leq P \leq P_0^{\mathrm{crit}}$. Define the subspaces $H^P := \ker\left(P_0^{\mathrm{crit}} - P\right)^\perp = \overline{\operatorname{range}\left(P_0^{\mathrm{crit}} - P\right)} \subset H$ where H is the state space of ϕ. Clearly, the subspaces H^P are ordered (by inclusion) in the same way as are the solutions $0 \leq P \leq P_0^{\mathrm{crit}}$ (by nonnegativity). By a particular case of Corollary 8, each H^P is a co-invariant subspace for the closed loop semigroup generator $A^{\mathrm{crit}} := A_{P_0^{\mathrm{crit}}} = A + BK_{P_0^{\mathrm{crit}}}$. We conclude that the $(A^{\mathrm{crit}})^*$-invariant subspaces H^P, together with the restrictions $(A^{\mathrm{crit}})^*|H^P$, obey the partial ordering of the set

$$\{P \in Ric(\phi, J) \quad | \quad 0 \leq P \leq P_0^{\mathrm{crit}}\} = \{P \in ric_0(\phi, J) \quad | \quad P \geq 0\},$$

where the equality is by [17, Theorem 96], under stronger assumptions.

It is the main result of this paper to show that the compressions of the shift $S^*|K_{\widetilde{\phi^\circ(P)}}$ can be connected to restrictions $(A^{\mathrm{crit}})^*|H^P$, for all $P \in ric_0(\phi, J)$, $P \geq 0$. We now explain the outline how this is done. For technical simplicity, it is now assumed that $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner, and the outer factor $\mathcal{X}$ of $\mathcal{D}_\phi$ (and each $\mathcal{D}_{\phi_P}$) equals the shift-invariant identity $\mathcal{I}$. A realization $\widetilde{\phi(P)}$ is constructed for $\widetilde{\mathcal{N}}_P$, such that the semigroup generator $\widetilde{\phi(P)}$ is the restriction $(A^{\mathrm{crit}})^*|H^P$, see Lemma 14. Under stronger technical assumptions, $\widetilde{\phi(P)}$ becomes output stable ($\operatorname{dom}\left(\mathcal{C}_{\widetilde{\phi(P)}}\right) = H^P$) and observable ($\ker\left(\mathcal{C}_{\widetilde{\phi(P)}}\right) = \{0\}$), see claims (ii) and (iii) of Lemma 22. Now we have the commutant equation

$$S^*\mathcal{C}_{\widetilde{\phi^\circ(P)}} = \bar{\pi}_+\tau^*\mathcal{C}_{\widetilde{\phi^\circ(P)}} = \mathcal{C}_{\widetilde{\phi^\circ(P)}}(A^{\mathrm{crit}*}|H^P), \tag{5}$$

which connects $(A^{\mathrm{crit}})^*|H^P$ to a compression of the backward shift $S^*|\operatorname{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$. Here $\widetilde{\phi^\circ(P)}$ is a normalized version of $\widetilde{\phi(P)}$. Furthermore, it appears that $\operatorname{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$ is closed, and equals the co-invariant subspace $K_{\widetilde{\phi^\circ(P)}}$, defined in equation (4).

This shows that the two descriptions of the set $ric_0(\phi, J)$, the former by restricted operators $(A^{\mathrm{crit}})^*|H^P$ and the latter by restricted shifts $S^*|\operatorname{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$, are connected by a similarity equivalence, induced by a bounded linear bijection. This connection is analogous to the connection of the zeroes and poles of a rational inner function to the generalized eigenvectors and eigenvalues of the semigroup generator of its matrix-valued realization. However, we use neither the notion of zeroes, nor the generalized eigenspaces of the semigroups. In this sense, our results are "genuinely" infinite dimensional.

1.6 Comparison to similar existing theories

In Subsection 1.3, it was indicated how to parameterize the solution of LQ-DARE by A-invariant null spaces $\ker(P)$. In our approach, we seem to have turned everything upside down; we parameterize the solutions of DARE by A^{crit} -co-invariant subspaces $\ker\left(P_0^{\mathrm{crit}} - P\right)$. We now explain why this is done.

For all $P \in ric_0(\phi, J)$, $P \geq 0$, we have the stable factorization

$$J^{\frac{1}{2}}\mathcal{D}_\phi = J^{\frac{1}{2}}\mathcal{D}_{\phi^P} \cdot \mathcal{D}_{\phi_P}, \tag{6}$$

assuming that the technical assumptions of [17, Lemma 79] are satisfied. In principle, each of the factors $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ and $\mathcal{D}_{\phi_P}$ could be used to associate chains of inner factors and shift-invariant subspaces to chains in $ric_0(\phi, J)$. In [17], we have chosen to use spectral DLS ϕ_P because it is an easier object to handle than the I/O map of $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$. The first reason for this is that the input space U and the output space Y of $J^{\frac{1}{2}}\mathcal{D}_\phi$ are generally different. We have the additional trouble that for noncoercive $J \geq 0$, we can only conclude the output stability and I/O stability of $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ in [17, Lemma 79], but not that of $\mathcal{D}_{\phi^P}$; thus $Ric(\phi^P, J)$ is not generally a H^∞DARE. Finally, if we make the requirement (and we always do!) that a solution $P \in Ric(\phi, J)$ should have an invertible indicator Λ_P, it then follows that each of the spectral DLSs ϕ_P can be normalized to have a boundedly invertible feed-through operator; in our case it is the indentity. Thus the inconvenient nonsquareness and possible "zero" of the transfer function $\mathcal{D}_\phi(z)$ at $z = 0$ will always be included in the left factors $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ in the factorization (6).

We now explain why the choise of ϕ_P over ϕ^P "turns everything upside down". By $\mathcal{D}_{\phi_P} = \mathcal{N}_P\mathcal{X}$ denote the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization. Because the inner factor in $\mathcal{D}_{\phi_P}$ "decomposes" from the left in factorization (6), and it should "decompose" from the right in order to be in harmony with the Beurling–Lax–Halmos Theorem, we have to adjoin once and use $\widetilde{\mathcal{N}}_P$ instead of $\mathcal{N}_P$ in Lemma 1. This is the reason why A^{crit}-co-invariant subspaces H^P must be used, instead of some A^{crit}-invariant subspaces. An analogous comment can be made why the spaces $\ker\left(P_0^{\mathrm{crit}} - P\right)$ rather than $\ker(P)$ are used.

We also remark that, under technical assumptions, the approaches presented in Subsections 1.2 and 1.3 give a full classification of the solution sets of the DARE. Our corresponding results work only in one direction: to each reasonable solution of DARE, a restricted backward shift is associated, but not conversely. Much of this apparent weakness could be fixed if we a practical form of a state space isomorphism theorem were available. Unfortunately, this is not possible in the full generality that we are considering.

1.7 The technical outline of this work

In this subsection, we give an outline and a technical battle plan of this paper. The following standing assumptions are used throughout the paper:

(i) The basic DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is I/O stable and output stable, such that $\mathrm{dom}\,(\mathcal{C}_\phi) := \{x \in H \mid \mathcal{C}x \in \ell^2(\mathbf{Z}_+; Y)\}$ is all of the state space H. Furthermore, ϕ is assumed to be approximately controllable in the sense $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$, where $\mathrm{range}\,(\mathcal{B}_\phi) := \mathcal{B}_\phi \mathrm{dom}\,(\mathcal{B}_\phi)$ and $\mathrm{dom}\,(\mathcal{B}_\phi) := Seq_-(U)$.

(ii) The input space U, the state space H, and the output space Y are separable Hilbert spaces.

(iii) H^∞DARE (1) has the unique regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}_\phi^{\mathrm{crit}}\right)^* J\mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ whose indicator satisfies $\Lambda_{P_0^{\mathrm{crit}}} > 0$. Here

$$\mathcal{C}_\phi^{\mathrm{crit}} := (\mathcal{I} - \bar{\pi}_+\mathcal{D}_\phi(\bar{\pi}_+\mathcal{D}_\phi^* J\mathcal{D}_\phi\bar{\pi}_+)^{-1}\bar{\pi}_+\mathcal{D}_\phi^* J)\mathcal{C}_\phi$$

is the critical closed loop observability map, see [16, Definition 28 and Proposition 29].

We also assume that the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner, but this technical assumption is lifted in the final Section 7. To obtain the full results of this paper, the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is assumed input stable, the input operator B is Hilbert–Schmidt, and the cost operator J is nonnegative. In this case, the regular critical solution P_0^{crit} is nonnegative, and its indicator is definitely positive.

In Section 2, we give basic result for DLSs ϕ whose I/O map $\mathcal{D}_\phi$ is (J, S)-inner, i.e.

$$\mathcal{D}_\phi^* J\mathcal{D}_\phi = S \cdot \mathcal{I}$$

for some self-adjoint, boundedly invertible $S \in \mathcal{L}(U)$. It appears that the H^∞DARE $ric(\phi, J)$ always has the critical regular solution P_0^{crit}, and in fact $\Lambda_{P_0^{\mathrm{crit}}} = S$, see Proposition 2. In claim (iii) of Lemma 6, we show that $P_0^{\mathrm{crit}} = \mathcal{C}_\phi^* J\mathcal{C}_\phi$. In claim (iv) of Lemma 6, we show that the null space $\ker\left(P_0^{\mathrm{crit}} - P\right)$ is A-invariant, for $P \in ric_0(\phi, J)$ with a positive indicator. The rest of Section 2 is devoted to proving that the null spaces of type $\ker\left(\tilde{P} - P\right)$ are $A_{\tilde{P}}$-invariant, provided that $P, \tilde{P} \in ric_0(\phi, J)$ are comparable to each other, see Lemma 9 and Corollary 10.

The reason to study a DLS with a $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner I/O map is the following. If we consider the cost optimization problem in the sense of [12], associated to the pair (ϕ, J), many proofs and formulae will simplify. Same comment holds also for the H^∞DARE theory, as presented in [16] and [17]. This is due to the fact that the outer factor $\mathcal{X}$ in the $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization $\mathcal{D} = \mathcal{N}\mathcal{X}$ is identity, because we normalize $S = \Lambda_{P_0^{\mathrm{crit}}}$ and $\pi_0\mathcal{X}\pi_0 = I$. We take the full advantage of all this triviality. In the final

Section 7, we generalize the results to DLSs having a nontrivial outer factor $\mathcal{X} \neq \mathcal{I}$, by using the results of [17, Section 15]. It is the price of this additional generality that stronger technical assumptions must be made, see Theorems 23 and 27.

In Proposition 12, the null space of the observability map $\mathcal{C}_\phi$ is "divided away" from the state space H, to obtain an observable DLS ϕ^{red} that has the same I/O map as ϕ but a smaller state space. We remark that $\mathcal{D}_\phi$ is not required to be $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner in Proposition 12. In Definition 13, we associate the characteristic DLS $\phi(P)$ to each $P \in ric_0(\phi, J)$. The characteristic DLS $\phi(P)$ is simply a reduced, observable version of the spectral DLS ϕ_P in the sense of Proposition 12. The basic properties of $\phi(P)$ are given in Lemma 14. In particular, $\mathcal{D}_{\phi(P)} = \mathcal{D}_{\phi_P} = \mathcal{N}_P$, where $\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X} = \mathcal{N}_P \mathcal{I}$ is the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization, see [16, Proposition 55].

The semigroup generator of $\phi(P)$ is the compression $\Pi_P A | H^P$, where Π_P is the orthogonal projection of H onto $\ker \left(P_0^{\mathrm{crit}} - P\right)^\perp$, and $H^P := \mathrm{range}\,(\Pi_P)$ is the state space of $\phi(P)$. Because $\Pi_P A = \Pi_P A \Pi_P$ by Lemma 6, $\left(\Pi_P A | H^P\right)^*$ equals the restriction $A^* | H^P$. Trivially, if $P_0^{\mathrm{crit}} \geq P_1 \geq P_2$ for $P_1, P_2 \in ric_0(\phi, J)$, then $\{0\} = H^{P_0^{\mathrm{crit}}} \subset H^{P_1} \subset H^{P_2} \subset H$. This connects the partial ordering of the solution set $ric_0(\phi, J)$ to the partial ordering of the A^*-invariant subspaces H^P, for the DLS ϕ with a (J, S)-inner I/O map.

In Section 4, an orthogonality result is given for DLSs whose transfer functions are inner. In claim (iii) of Proposition 15, it is shown that $\mathrm{range}\,(\mathcal{C}_\phi) = \mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)$ if $\mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)$ is closed and proper technical assumptions hold. An application of this result is Lemma 17, where the orthogonal direct sum decomposition

$$\ell^2(\mathbf{Z}_+; U) = \mathrm{range}\left(\widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+\right) \oplus \mathrm{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) \tag{7}$$

is proved for DLSs ϕ whose I/O map is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner and $P \in ric_0(\phi, J)$ is arbitrary. We remark that $\mathrm{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$ is closed as a conclusion, not as an assumption of Lemma 17. The operator $\widetilde{\mathcal{N}}_P^\circ$ and the DLS $\widetilde{\phi^\circ(P)}$ are connected to the characteristic DLS $\phi(P)$ by equations (13) and (14).

In Section 5, we give a brief overview about a particular case of the Sz.Nagy–Foias shift operator model. The inner characteristic functions for class C_{00}-contractions are introduced, and necessary results from the spectral function theory are presented. Some work is done to translate the frequency space notions, commonly used in the literature, to the time domain notions used in our Riccati equation work.

In Section 6 we give our first main results. The battle plan here is roughly as follows. For arbitrary $P \in ric_0(\phi, J)$, we study the normalized and adjoint version of the characteristic DLS $\phi(P)$, denoted by $\widetilde{\phi^\circ(P)}$ and defined in equation (14). The inner transfer function $\mathcal{D}_{\widetilde{\phi^\circ(P)}}(z) = \widetilde{\mathcal{N}}_P^\circ(z)$ is the characteristic function of the truncated shift operator $S^* | K_{\widetilde{\phi^\circ(P)}}$ in the sense of Sz.Nagy–Foias. Here $K_{\widetilde{\phi^\circ(P)}} := \ell^2(\mathbf{Z}_+; U) \ominus \mathrm{range}\left(\mathcal{D}_{\widetilde{\phi^\circ(P)}}\right)$ is the S^*-invariant subspace, as given in Definition 21. The spectral function theory, presented in Section 5, connects effectively the operator theoretic properties of the C_{00}-contraction $S^*|K_{\widetilde{\phi^\circ(P)}}$ to the function theory of the normalized transfer function $\widetilde{\mathcal{N}}_P^\circ(z)$, without assuming any finite dimensionality in any of the spaces or the operators.

Because $\mathcal{D}_{\phi(P)} = \mathcal{D}_{\phi_P} = \mathcal{N}_P$ by our (practically) standing assumption on the outer factor $\mathcal{X} = \mathcal{I}$, we conclude, by Lemma 17, the equality $K_{\widetilde{\phi^\circ(P)}} = \text{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$ from equation (7). We have obtained the similarity transform

$$\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right)\mathcal{C}_{\widetilde{\phi^\circ(P)}} = \bar{\pi}_+\tau^*\mathcal{C}_{\widetilde{\phi^\circ(P)}} = \mathcal{C}_{\widetilde{\phi^\circ(P)}}\left(A^*|H^P\right)$$

by the basic formula $\bar{\pi}_+\tau^*\mathcal{C}_\phi = \mathcal{C}_\phi A$ that decribes the interaction of the backward time shift and the semigroup generator A for any DLS ϕ. It is clear that such a similarity transform gives us quite strong results about the restricted adjoint semigroups $A^*|H^P$ for $P \in ric_0(\phi, J)$. Of course, the strongest results are obtained when the similarity transform $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is a bounded bijection with a bounded inverse, see Lemma 22 and Theorem 23. Then the restrictions $A^*|H^P$ are similar to a C_{00}-contractions, whose characteristic functions are causal, shift-invariant and stable partial inner factors of the I/O map $\mathcal{D}_\phi$, see [17, Theorems 81 and 83].

So far we have considered only DLSs $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ whose I/O maps are $(J, \Lambda_{P_0^{\text{crit}}})$-inner. The general case, when $\mathcal{D}_\phi$ is only assumed to be I/O stable, is considered in Section 7. Instead of requiring an inner I/O map, we now require only that the regular critical solution $P_0^{\text{crit}} \in ric_0(\phi, J)$ exists. It is shown in [17, Section 15], that the structure of the H^∞DARE $ric(\phi, J)$ remains unchanged, if a preliminary critical feedback associated to $P_0^{\text{crit}} \in ric_0(\phi, J)$ is applied. The resulting (closed loop) inner DLS has a $(J, \Lambda_{P_0^{\text{crit}}})$ -inner I/O map, and the results of the previous sections can be applied on the pair $(\phi^{P_0^{\text{crit}}}, J)$ instead of the original pair (ϕ, J). In order to have the equality $ric_0(\phi, J) = ric_0(\phi^{P_0^{\text{crit}}}, J)$ for the regular H^∞ solution sets. we must assume, in addition to the assumptions of Theorem 23, that the input operator B is Hilbert–Schmidt, and the cost operator J is nonnegative. For details, see Theorem 27. Clearly, now the co-invariant subspace results are for the critical closed loop semigroup generator $A^{\text{crit}} = A_{P_0^{\text{crit}}}$ of $\phi^{P_0^{\text{crit}}}$, rather than the open loop semigroup generator A of ϕ.

1.8 Notations

We use the following notations throughout the paper: $\mathbf{Z}$ is the set of integers. $\mathbf{Z}_+ := \{j \in \mathbf{Z} \mid j \geq 0\}$. $\mathbf{Z}_- := \{j \in \mathbf{Z} \mid j < 0\}$. $\mathbf{T}$ is the unit circle and $\mathbf{D}$ is the open unit disk of the complex plane $\mathbf{C}$. If H is a Hilbert space, then $\mathcal{L}(H)$ denotes the bounded and $\mathcal{LC}(H)$ the compact linear operators in H. Elements of a Hilbert space are denoted by upper case letters; for example $u \in U$. Sequences in Hilbert spaces are denoted by $\tilde{u} = \{u_i\}_{i \in I} \subset U$, where I is the index set. Usually $I = \mathbf{Z}$ or $I = \mathbf{Z}_+$. Given a Hilbert space Z, we define the sequence spaces

$$
\begin{aligned}
&Seq(Z) := \big\{\{z_i\}_{i \in \mathbf{Z}} \mid z_i \in Z \quad \text{and} \quad \exists I \in \mathbf{Z} \quad \forall i \leq I : z_i = 0\big\}, \\
&Seq_+(Z) := \big\{\{z_i\}_{i \in \mathbf{Z}} \mid z_i \in Z \quad \text{and} \qquad \forall i < 0 : z_i = 0\big\}, \\
&Seq_-(Z) := \big\{\{z_i\}_{i \in \mathbf{Z}} \in Seq(Z) \mid z_i \in Z \quad \text{and} \qquad \forall i \geq 0 : z_i = 0\big\}, \\
&\ell^p(\mathbf{Z}; Z) := \Big\{\{z_i\}_{i \in \mathbf{Z}} \subset Z \mid \sum_{i \in \mathbf{Z}} ||z_i||_Z^p < \infty\Big\} \quad \text{for} \quad 1 \leq p < \infty, \\
&\ell^p(\mathbf{Z}_+; Z) := \Big\{\{z_i\}_{i \in \mathbf{Z}_+} \subset Z \mid \sum_{i \in \mathbf{Z}_+} ||z_i||_Z^p < \infty\Big\} \quad \text{for} \quad 1 \leq p < \infty, \\
&\ell^\infty(\mathbf{Z}; Z) := \big\{\{z_i\}_{i \in \mathbf{Z}} \subset Z \mid \sup_{i \in \mathbf{Z}} ||z_i||_Z < \infty\big\}.
\end{aligned}
$$

The following linear operators are defined for $\tilde{z} \in Seq(Z)$:

- the projections for $j, k \in \mathbf{Z} \cup \{\pm\infty\}$

$$
\begin{aligned}
&\pi_{[j,k]}\tilde{z} := \{w_j\}; \quad w_i = z_i \quad \text{for} \quad j \leq i \leq k, \quad w_i = 0 \quad \text{otherwise}, \\
&\pi_j := \pi_{[j,j]}, \quad \pi_+ := \pi_{[1,\infty]}, \quad \pi_- := \pi_{[-\infty,-1]}, \\
&\bar{\pi}_+ := \pi_0 + \pi_+, \quad \bar{\pi}_- := \pi_0 + \pi_-,
\end{aligned}
$$

- the bilateral forward time shift τ and its inverse, the backward time shift τ^*

$$
\begin{aligned}
\tau\tilde{u} &:= \{w_j\} \quad \text{where} \quad w_j = u_{j-1}, \\
\tau^*\tilde{u} &:= \{w_j\} \quad \text{where} \quad w_j = u_{j+1}.
\end{aligned}
$$

Other notations are introduced when they are needed.

2 DLSs with inner I/O maps

As discussed in Section 1, we start this paper by considering first DLSs $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ whose I/O map $\mathcal{D}_\phi$ is (J,S)-inner for two self-adjoint operators $J \in \mathcal{L}(Y)$ and $S \in \mathcal{L}(U)$. Basic results for such DLSs are given in this section. In particular, we are interested in the invariant subspaces of the semigroup generator A that are of the form $\ker\left(P_0^{\rm crit} - P\right)$. Here $P_0^{\rm crit} := \left(\mathcal{C}_\phi^{\rm crit}\right)^* J\mathcal{C}_\phi^{\rm crit} \in ric_0(\phi, J)$ is a regular critical solution, the closed loop critical observability map is given by

$$\mathcal{C}_\phi^{\rm crit} := (\mathcal{I} - \bar{\pi}_+\mathcal{D}_\phi(\bar{\pi}_+\mathcal{D}_\phi^* J\mathcal{D}_\phi\bar{\pi}_+)^{-1}\bar{\pi}_+\mathcal{D}_\phi^* J)\mathcal{C}_\phi,$$

and $P \in ric_0(\phi, J)$ is another solution that is comparable to $P_0^{\rm crit}$. Such invariant subspaces are considered in Corollary 10. The A-co-invariant orthogonal complements $H^P := \ker\left(P_0^{\rm crit} - P\right)^\perp$ in H are central in the later developments of this work.

In order to be able to speak about the spaces $\ker\left(P_0^{\rm crit} - P\right)$, the regular critical solution $P_0^{\rm crit}$ must, of course, exist. Clearly, for an (J,S)-inner I/O map $\mathcal{D}_\phi$, the Popov operator is a static constant: $\mathcal{D}_\phi^* J\mathcal{D}_\phi = S$. Then the sufficient and necessary conditions for the existence of a critical solution of DARE are easy to give. The following result is a consequence of [16, Theorem 27 and Proposition 29].

Proposition 2. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator, and $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ an output stable and I/O stable DLS, such that $\mathcal{D}_\phi$ is (J,S)-inner.*

Then S has a bounded inverse if and only if a regular critical solution $P_0^{\rm crit} \in ric_0(\phi, J)$ exists. When this equivalence holds, $S = \Lambda_{P_0^{\rm crit}}$ and $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\rm crit}})$-inner.

For later reference, we give somewhat trivial and technical results about DLSs with an inner I/O map. If a DLS has an inner I/O map, so has its adjoint DLS:

Proposition 3. *Assume that $S_1, S_2 \in \mathcal{L}(U)$ are boundedly invertible, $S_1 > 0$, $S_2 > 0$, where U is separable Hilbert. Suppose that $\mathcal{N}$ is a (S_1, S_2)-inner I/O map of an I/O stable DLS with input space U, such that the static part satisfies $\mathcal{N}(0) = I$. Then the adjoint I/O map $\widetilde{\mathcal{N}}$ is (S_2^{-1}, S_1^{-1})-inner.*

Proof. By normalizing $\mathcal{N}^\circ := S_1^{\frac{1}{2}}\mathcal{N}S_2^{-\frac{1}{2}}$, we get the transfer function $\mathcal{N}^\circ(z)$ be inner from the left. Because $\mathcal{N}^\circ(0) = S_1^{\frac{1}{2}}S_2^{-\frac{1}{2}}$ has a bounded inverse, it follows by [16, Proposition 34] that $\mathcal{N}^\circ(z)$ is inner inner from both sides. The nontangential boundary trace $\mathcal{N}^\circ(e^{i\theta})$ is unitary a.e. $e^{i\theta} \in \mathbf{T}$. So the nontangential boundary trace of the adjoint function is $\widetilde{\mathcal{N}}^\circ(e^{i\theta}) := S_2^{-\frac{1}{2}}\widetilde{\mathcal{N}}(e^{i\theta})S_1^{\frac{1}{2}} = \mathcal{N}^\circ(e^{i\theta})^*$. But now $\widetilde{\mathcal{N}}$ is (S_2^{-1}, S_1^{-1})-inner. $\square$

The following corollary is about the I/O map $\widetilde{\mathcal{N}}_P$ whose Toeplitz operator appears in Lemma 1.

Corollary 4. *Let $J \in \mathcal{L}(Y)$ a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, with a separable input space U. Assume that a critical $P_0^{\rm crit} \in ric_0(\phi, J)$ exists, such that $\Lambda_{P_0^{\rm crit}} > 0$. For any $P \in ric_0(\phi, J)$, let $\mathcal{N}_P$ denote the $(\Lambda_P, \Lambda_{P_0^{\rm crit}})$-inner factor of $\mathcal{D}_{\phi_P}$. Then the adjoint I/O map $\widetilde{\mathcal{N}}_P$ is $(\Lambda_{P_0^{\rm crit}}^{-1}, \Lambda_P^{-1})$-inner.*

Proof. By [16, claim (i) of Proposition 55], $\mathcal{D}_{\phi_P}$ has the $(\Lambda_P, \Lambda_{P_0^{\rm crit}})$-inner factor $\mathcal{N}_P$. The static part of $\mathcal{N}_P$ is identity, by [16, claim (ii) of Proposition 55]. The inertia result [16, Lemma 53] implies that $\Lambda_P > 0$ for all $P \in ric_0(\phi, J)$. An application of Proposition 3 completes the proof. □

If $J \geq 0$, there are plenty of examples of DLS with (J, S)-inner I/O maps. If the conditions of [17, claim (iii) of Lemma 79] are satisfied, the (normalized) inner DLS $J^{\frac{1}{2}}\phi^P$ has a (I, Λ_P)-inner I/O map, for each nonnegative $P \in ric_0(\phi, J)$. We also remark that, under restrictive assumptions, the family of DLSs with inner I/O maps is sufficiently rich to carry the structure of all H^∞DAREs that have a critical solution, in the sense of [17, Theorem 105]. This will be exploited in Section 7 where the results of this paper are extended to the general DLSs that do not have an inner I/O map.

The rest of this section is devoted to the study the Riccati equation, and semigroup invariant subspaces of the state space. We start with a technical proposition that only marginally depends on the structure of DARE.

Proposition 5. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS and J a self-adjoint cost operator. Let $P_1, P_2 \in Ric(\phi, J)$. Then $K_{P_2} - K_{P_1} = \Lambda_{P_2}^{-1} B^*(P_2 - P_1) A_{P_1}$ and $\Lambda_{P_1}^{-1} B^*(P_2 - P_1) A_{P_1} = \Lambda_{P_2}^{-1} B^*(P_2 - P_1) A_{P_2}$.*

Proof. To prove the first equation, we calculate

$$K_{P_1} - K_{P_2} = \Lambda_{P_1}^{-1} Q_{P_1} - \Lambda_{P_2}^{-1} Q_{P_2} = (\Lambda_{P_1}^{-1} - \Lambda_{P_2}^{-1}) Q_{P_1} + \Lambda_{P_2}^{-1}(Q_{P_1} - Q_{P_2}),$$

where $Q_P := -D^*JC - B^*PA$. Because $x^{-1} - y^{-1} = y^{-1}(y - x)x^{-1}$, we have $\Lambda_{P_1}^{-1} - \Lambda_{P_2}^{-1} = \Lambda_{P_2}^{-1} B^*(P_2 - P_1) B \Lambda_{P_1}^{-1}$. Now we obtain, because $Q_{P_1} - Q_{P_2} = B^*(P_2 - P_1)A$

$$\begin{aligned} K_{P_1} - K_{P_2} &= \Lambda_{P_2}^{-1} \left(B^*(P_2 - P_1) B\, K_{P_1} + B^*(P_2 - P_1) A\right) \\ &= \Lambda_{P_2}^{-1} B^*(P_2 - P_1)(A + BK_{P_1}). \end{aligned}$$

This gives the first equation of the claim. The second equation is obtained by interchanging P_1 and P_2 in the first equation, and comparing these two equations. □

Basic properties of DLSs with $(J, \Lambda_{P^{\rm crit}})$-inner I/O map are given below.

Lemma 6. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$. Assume that the regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}_\phi^{\mathrm{crit}}\right)^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ exists, and the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner.*

Then for any $P \in Ric(\phi, J)$ the following holds:

(i) The feedback operators satisfy $K_{P_0^{\mathrm{crit}}} = 0$ and $K_P = -\Lambda_P^{-1} B^*(P_0^{\mathrm{crit}} - P)A$. Furthermore, $A_{P_0^{\mathrm{crit}}} = A$ and $C_{P_0^{\mathrm{crit}}} = C$. The operator $Q = P_0^{\mathrm{crit}} - P$ satisfies the following Riccati equation

$$
\begin{cases} A^*QA - Q + A^*QB \cdot \Lambda_P^{-1} \cdot B^*QA = 0, \\ \Lambda_P = D^*JD + B^*PB. \end{cases} \tag{8}
$$

(ii) The spectral DLS ϕ_P can be written in the following equivalent forms:

$$
\phi_P = \begin{pmatrix} A & B \\ -K_P & I \end{pmatrix} = \begin{pmatrix} A_{P_0^{\mathrm{crit}}} & B \\ K_{P_0^{\mathrm{crit}}} - K_P & I \end{pmatrix} = \begin{pmatrix} A & B \\ \Lambda_P^{-1} B^*(P_0^{\mathrm{crit}} - P)A & I \end{pmatrix}. \tag{9}
$$

(iii) We have $\mathcal{C}_\phi = \mathcal{C}_{\phi^{P_0^{\mathrm{crit}}}} = \mathcal{C}_\phi^{\mathrm{crit}}$ and $P_0^{\mathrm{crit}} = \mathcal{C}_\phi^* J \mathcal{C}_\phi$.

(iv) Assume, in addition, that $P \in ric_0(\phi, J)$ and $\Lambda_P > 0$. Then $\ker\left(P_0^{\mathrm{crit}} - P\right) = \ker\left(\mathcal{C}_{\phi_P}\right)$. In particular, $\ker\left(P_0^{\mathrm{crit}} - P\right)$ is A-invariant.

Proof. Because $\mathcal{D}_\phi$ is assumed to be $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner, the outer factor $\mathcal{X}$ in the unique $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$ equals the identity $\mathcal{I}$. The outer factor $\mathcal{X} = \mathcal{I}$ is the I/O map of the spectral DLS $\phi_{P_0^{\mathrm{crit}}} = \left(\begin{smallmatrix} A & B \\ -K_{P_0^{\mathrm{crit}}} & I \end{smallmatrix}\right)$, whence we conclude that $-K_{P_0^{\mathrm{crit}}} | \operatorname{range}(\mathcal{B}_\phi) = 0$. Because $K_{P_0^{\mathrm{crit}}}$ is a bounded operator and $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$, by explicit assumption, it follows that the critical feedback operator $K_{P_0^{\mathrm{crit}}} = 0$. Immediately $A_{P_0^{\mathrm{crit}}} = A + BK_{P_0^{\mathrm{crit}}} = A$, $C_{P_0^{\mathrm{crit}}} = C + DK_{P_0^{\mathrm{crit}}} = C$, and the second equality in (9) is proved.

By applying Proposition 5 to $K_P = K_P - K_{P_0^{\mathrm{crit}}}$ we obtain $K_P = -\Lambda_P^{-1} B^*(P_0^{\mathrm{crit}} - P)A$, for any $P \in Ric(\phi, J)$. This gives the third equality in (9), and completes the proof of claim (ii).

To complete the proof of claim (i), the Riccati equation (8) must be verified. Because $A_{P_0^{\mathrm{crit}}}^* P_0^{\mathrm{crit}} A_{P_0^{\mathrm{crit}}} - P_0^{\mathrm{crit}} + C_{P_0^{\mathrm{crit}}}^* J C_{P_0^{\mathrm{crit}}} = 0$ by [17, Proposition 68], and $A_{P_0^{\mathrm{crit}}} = A$, $C_{P_0^{\mathrm{crit}}} = C$, we have

$$
A^* P_0^{\mathrm{crit}} A - P_0^{\mathrm{crit}} + C^* JC = 0.
$$

By rewriting the original DARE (1) with the aid of the already proved $K_P = -\Lambda_P^{-1} B^*(P_0^{\mathrm{crit}} - P)A$, we obtain for any $P \in Ric(\phi, J)$

$$
A^*PA - P + C^*JC = A^*(P_0^{\mathrm{crit}} - P)B \cdot \Lambda_P^{-1} \cdot B^*(P_0^{\mathrm{crit}} - P)A.
$$

Subtracting these equations will give give the Riccati equation (8).

We now consider claim (iii). Because $K_{P_0^{\mathrm{crit}}} = 0$, the inner DLS at P_0^{crit} satisfies

$$\phi^{P_0^{\mathrm{crit}}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \phi,$$

and so $\mathcal{C}_\phi = \mathcal{C}_{\phi^{P_0^{\mathrm{crit}}}}$. Now claim [16, claim (iv) of Proposition 103] gives $\mathcal{C}_{\phi^{P_0^{\mathrm{crit}}}} = \mathcal{C}\phi^{\mathrm{crit}}$, where

$$\mathcal{C}_\phi^{\mathrm{crit}} := (\mathcal{I} - \bar{\pi}_+ \mathcal{D}_\phi (\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}_\phi^* J) \mathcal{C}_\phi.$$

Thus $P_0^{\mathrm{crit}} := \left(\mathcal{C}_\phi^{\mathrm{crit}}\right)^* J \mathcal{C}_\phi^{\mathrm{crit}} = \mathcal{C}_\phi^* J \mathcal{C}_\phi$, and claim (iii) follows.

Because $P \in ric(\phi, J)$, both ϕ and ϕ_P are output stable. As in [16, proof of Proposition 23], we conclude from DARE $A^*PA - P + C^*JC = K_P^* \Lambda_P K_P$ that

$$P = P - L_{A,P} = \mathcal{C}_\phi^* J \mathcal{C}_\phi - \mathcal{C}_{\phi_P}^* \Lambda_P \mathcal{C}_{\phi_P}, \tag{10}$$

where the residual cost $L_{A,P} = s - lim_{n \to \infty} A^*PA$ exists and vanishes because $P \in ric_0(\phi, J)$, by assumption. Inserting $P_0^{\mathrm{crit}} = \mathcal{C}_\phi^* J \mathcal{C}_\phi$ into equation (10) gives

$$P_0^{\mathrm{crit}} - P = \mathcal{C}_{\phi_P}^* \Lambda_P \mathcal{C}_{\phi_P}$$

where $P \in ric_0(\phi, J)$ is arbitrary. Because $\Lambda_P > 0$, claim (iv) immediately follows because $\ker(\mathcal{C}_{\phi_P})$ is A-invariant. $\square$

Actually, we now have all the results on invariant subspaces of the semigroup that we need to complete this work. For academic interest, we continue to study the subspaces $\ker\left(P_0^{\mathrm{crit}} - P\right)$. We begin with another variant for the result of claim (iv) of Lemma 6 is the following:

Corollary 7. *Make the same assumptions as in Lemma 6. Let $P \in Ric(\phi, J)$ be arbitrary, such that $\Lambda_P > 0$ and $P \leq P_0^{\mathrm{crit}}$.*

Then $A \ker\left(P_0^{\mathrm{crit}} - P\right) \subset \ker\left(P_0^{\mathrm{crit}} - P\right)$.

Proof. Now $Q := P_0^{\mathrm{crit}} - P \geq 0$ satisfies DARE (8). Furthermore, this equation can be put into form

$$A^* Q^{\frac{1}{2}} \cdot R \cdot Q^{\frac{1}{2}} A = Q, \quad R = I + Q^{\frac{1}{2}} B \Lambda_P^{-1} B^* Q^{\frac{1}{2}}.$$

Now, because $\Lambda_P > 0$ and the indicator is always invertible, $\Lambda_P^{-1} > 0$. It now follows that $R \geq I$. For any $x \in H$ we can now write the balance equation

$$||R^{\frac{1}{2}} \cdot Q^{\frac{1}{2}} Ax|| = ||Q^{\frac{1}{2}} x||.$$

Because $\ker\left(Q^{\frac{1}{2}}\right) = \ker(Q) = \ker\left(P_0^{\mathrm{crit}} - P\right)$, and $R^{\frac{1}{2}}$ has a bounded inverse, the claim follows. $\square$

The case when $P_0^{\rm crit} \leq P$ instead of $P_0^{\rm crit} \geq P$ is investigated similarly:

Corollary 8. *Make the same assumptions as in Lemma 6, but assume, in addition, that* $0 \in Ric(\phi, J)$*,* $\Lambda_0 > 0$*, and* $P_0^{\rm crit} \geq 0$*. Let* $P \in Ric(\phi, J)$ *be arbitrary, such that* $\Lambda_P > 0$ *and* $P_0^{\rm crit} \leq P$*.*

Then $A \ker\left(P_0^{\rm crit} - P\right) \subset \ker\left(P_0^{\rm crit} - P\right)$*.*

Proof. Again, we use the DARE (8). This time we write $Q := P - P_0^{\rm crit} \geq 0$. By claim (i) of Lemma 6, Q satisfies

$$A^* Q^{\frac{1}{2}} \cdot R \cdot Q^{\frac{1}{2}} A = Q, \quad R = I - Q^{\frac{1}{2}} B \Lambda_P^{-1} B^* Q^{\frac{1}{2}}.$$

This is exactly the same as the corresponding equation in Corollary 7, except that one $+$ has changed into $-$. The claim is proved when we can show, under the additional assumption, that nevertheless $R > 0$ is boundedly invertible.

Because $P_0^{\rm crit} \geq 0$, we have $0 < \Lambda_{P - P_0^{\rm crit}} = \Lambda_P - B^* P_0^{\rm crit} B \leq \Lambda_P$. Because the indicator operator always has a bounded inverse, it follows that $0 < \Lambda_P^{-1} \leq \Lambda_{P - P_0^{\rm crit}}^{-1} = \Lambda_Q^{-1}$. Now, clearly $R > 0$ has a bounded inverse, if in equation

$$R \geq I - Q^{\frac{1}{2}} B \Lambda_{P - P_0^{\rm crit}}^{-1} B^* Q^{\frac{1}{2}} = I - Q^{\frac{1}{2}} B \Lambda_Q^{-1} B^* Q^{\frac{1}{2}}$$

the right hand side is strictly positive. Because $0 \in Ric(\phi, J)$, is follows that $\Lambda_0 = D^* J D > 0$ has a bounded inverse. We have

$$\begin{aligned} &Q^{\frac{1}{2}} B \Lambda_Q^{-1} B^* Q^{\frac{1}{2}} = Q^{\frac{1}{2}} B \left(\Lambda_0 + B^* Q B\right)^{-1} B^* Q^{\frac{1}{2}} \\ &= Q^{\frac{1}{2}} B \Lambda_0^{-\frac{1}{2}} \left(I + \Lambda_0^{-\frac{1}{2}} B^* Q B \Lambda_0^{-\frac{1}{2}}\right)^{-1} \Lambda_0^{-\frac{1}{2}} B^* Q^{\frac{1}{2}} = Q^{\frac{1}{2}} \tilde{B} \left(I + \tilde{B}^* Q \tilde{B}\right)^{-1} \tilde{B}^* Q^{\frac{1}{2}}, \end{aligned}$$

where $\tilde{B} := B \Lambda_0^{-\frac{1}{2}}$. Now, by a straightforward calculation (e.g. with the aid of the Neumann series),

$$(I + Q^{\frac{1}{2}} \tilde{B} \tilde{B}^* Q^{\frac{1}{2}})^{-1} = I - Q^{\frac{1}{2}} \tilde{B} (I + \tilde{B}^* Q \tilde{B})^{-1} \tilde{B}^* Q^{\frac{1}{2}} = R,$$

because $Q^{\frac{1}{2}} \tilde{B} \tilde{B}^* Q^{\frac{1}{2}} \geq 0$ and thus $I + Q^{\frac{1}{2}} \tilde{B} \tilde{B}^* Q^{\frac{1}{2}}$ is boundedly invertible. It follows that $R > 0$ with a bounded inverse, and the claim is proved. □

An immediate consequence of Corollaries 7 and 8 is the following:

Lemma 9. *Let* $J \in \mathcal{L}(Y)$ *be a self-adjoint cost operator. Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be an output stable and I/O stable DLS, such that* $\overline{\text{range}\,(\mathcal{B}_\phi)} = H$*. Assume that the regular critical solution* $P_0^{\rm crit} := \left(\mathcal{C}_\phi^{\rm crit}\right)^* J \mathcal{C}_\phi^{\rm crit} \in ric_0(\phi, J)$ *exists, and* $P_0^{\rm crit} \geq 0$*. Assume that the I/O map* $\mathcal{D}_\phi$ *is* $(J, \Lambda_{P_0^{\rm crit}})$*-inner. Assume that* $0 \in Ric(\phi, J)$ *and* $D^* J D = \Lambda_0 > 0$*,*

Let $P \in Ric(\phi, J)$ *be arbitrary, such that* $\Lambda_P > 0$*, and* P *is comparable to* $P_0^{\rm crit}$*. Then* $A \ker\left(P_0^{\rm crit} - P\right) \subset \ker\left(P_0^{\rm crit} - P\right)$*.*

The closed loop semigroup generators $A_{\tilde{P}} = A = B K_{\tilde{P}}$ have the following invariance properties, for $\tilde{P} \in ric_0(\phi, J)$, $\tilde{P} \geq 0$. Recall that these solutions are exactly those that satisfy $0 \leq \tilde{P} \leq P_0^{\rm crit}$, if the conditions of [17, Theorem 96] hold.

Corollary 10. *Let $J > 0$ be a coercive self-adjoint cost operator in $\mathcal{L}(Y)$. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$. Assume that the input space U and the output space Y are separable, and the input operator $B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt. Assume that the regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}_\phi^{\mathrm{crit}}\right)^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ exists, $0 \in Ric(\phi, J)$. Let $\tilde{P} \in ric_0(\phi, J)$, $\tilde{P} \geq 0$, be arbitrary.*

Let $P \in Ric(\phi, J)$ be arbitrary, such that $\Lambda_P > 0$ and P is comparable to $\tilde{P}$. Then $A_{\tilde{P}} \ker\left(\tilde{P} - P\right) \subset \ker\left(\tilde{P} - P\right)$.

Proof. By [17, claim (iii) of Lemma 79] and the assumption that J has a bounded inverse, the inner DLS

$$\phi^{\tilde{P}} = \begin{pmatrix} A_{\tilde{P}} & B \\ C_{\tilde{P}} & D \end{pmatrix}$$

is output stable and I/O stable, and the I/O map $\mathcal{D}_{\phi^{\tilde{P}}}$ is $(J, \Lambda_{\tilde{P}})$-inner. Thus $Ric(\phi^{\tilde{P}}, J)$ is a H^∞DARE. Because $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$, it also follows that $\overline{\mathrm{range}\left(\mathcal{B}_{\phi^{\tilde{P}}}\right)} = H$, as in the proof of [17, Proposition 86]. By Proposition 2, there is a regular critical solution $\tilde{P}_0^{\mathrm{crit}} \in ric_0(\phi^{\tilde{P}}, J)$, and by [17, Lemma 100], $\tilde{P}_0^{\mathrm{crit}} = \tilde{P} \geq 0$. Because the full solution sets of DAREs satisfy $Ric(\phi, J) = Ric(\phi^{\tilde{P}}, J)$ by [17, Lemma 65], it follows that $0 \in Ric(\phi^{\tilde{P}}, J)$. Because $J \geq 0$, it follows that the indicator $\tilde{\Lambda}_0 = \Lambda_0 = D^* J D > 0$. An application of Lemma 9 on DLS $\phi^{\tilde{P}}$ and cost operator J proves the claim. $\square$

3 Characteristic DLS $\phi(P)$

In this section, we first develop tools that are required to "divide" the unobservable subspace $\ker(\mathcal{C}_\phi)$ away from the state space. This gives us a reduced DLS. With the aid of this construction, we define the characteristic DLS $\phi(P)$ for each solution $P \in ric(\phi, J)$, see Definition 13. The basic properties of $\phi(P)$ are given in Lemma 14.

Proposition 11. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable DLS. Then $\widetilde{\phi} = \left(\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right)$ is input stable, and $\mathcal{C}_\phi^* = \mathcal{B}_{\widetilde{\phi}} \cdot \mathrm{flip}$. Here $\mathrm{flip} = \mathrm{flip}^2 = \mathrm{flip}^*$ is the unitary mapping on $\tilde{y} \in \ell^2(\mathbf{Z}; Y)$, given by*

$$(\mathrm{flip}\, \tilde{y})_j = y_{-j-1}.$$

Proof. Let $\tilde{y} \in \ell^2(\mathbf{Z}_+; Y)$, $x_0 \in H$ be arbitrary. Then

$$\begin{aligned} \langle \tilde{y}, \mathcal{C}x_0 \rangle &= \sum_{j=0}^{\infty} \left\langle y_j, CA^j x_0 \right\rangle = \sum_{j=0}^{\infty} \left\langle A^{*j} C^* y_j, x_0 \right\rangle \\ &= \sum_{j=0}^{\infty} \left\langle A^{*j} C^* (\mathrm{flip}\, \tilde{y})_{-j-1}, x_0 \right\rangle = \left\langle \mathcal{B}_{\widetilde{\phi}} (\mathrm{flip}\, \tilde{y}), x_0 \right\rangle = \left\langle \mathcal{C}_\phi^* \tilde{y}, x_0 \right\rangle. \end{aligned}$$

Actually the previous is (at first) true only for $\tilde{y}$ with finitely many nonzero components. Only in this case $\mathrm{flip}\, \tilde{y} \in \mathrm{dom}\left(\mathcal{B}_{\widetilde{\phi}}\right)$, but then because $\mathrm{dom}\left(\mathcal{B}_{\widetilde{\phi}}\right) := Seq_-(Y)$ is dense in $\ell^2(\mathbf{Z}_-; Y)$, it follows that $\mathcal{B}_{\widetilde{\phi}} \cdot \mathrm{flip}$ coincides with the bounded operator $\mathcal{C}^*$ in a dense set. Because $flip$ is unitary, it follows that $\mathcal{B}_{\widetilde{\phi}}$ is bounded and $\widetilde{\phi}$ is input stable. Recall that $\mathrm{dom}(\mathcal{B}) := Seq_-(U)$ consist of finitely long input sequences for all controllability maps. The input stable controllability map $\mathcal{B}$ can always be extended by continuity from $\mathrm{dom}(\mathcal{B})$ to all of $\ell^2(\mathbf{Z}_-; U)$. ☐

For a quite general DLS ϕ, the kernel $\ker(\mathcal{C}_\phi)$ can be divided away from the state space, without changing the I/O map $\mathcal{D}_\phi$.

Proposition 12. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, with state space H. Assume that $H_0 := \ker(\mathcal{C}_\phi)$ is nontrivial.*

(i) Then there is a reduced DLS ϕ^{red} with a smaller state space $H^{red} := \ker(\mathcal{C}_\phi)^\perp \subset H$, $H = H_0 \oplus H^{red}$, such that $\mathcal{D}_\phi = \mathcal{D}_{\phi^{red}}$ and $\ker\left(\mathcal{C}_{\phi^{red}}\right) = \{0\}$. The DLS ϕ^{red} is given by

$$\phi^{red} := \begin{pmatrix} \Pi^{red} A | H^{red} & \Pi^{red} B \\ C | H^{red} & D \end{pmatrix},$$

where Π^{red} is the orthogonal projection of H onto H^{red}. In particular, ϕ^{red} is I/O stable and output stable.

(ii) We have $\Pi^{red}A = \Pi^{red}A\Pi^{red}$*,* $\mathcal{B}_{\phi^{red}} = \Pi^{red}\mathcal{B}_\phi$ *and* $\mathcal{C}_{\phi^{red}} = \mathcal{C}_\phi|H^{red}$*. Thus* ϕ^{red} *written in I/O-form is*

$$\Phi^{red} = \begin{bmatrix} (\Pi^{red}A|H^{red})^j & \Pi^{red}\mathcal{B}_\phi\tau^{*j} \\ \mathcal{C}_\phi|H^{red} & \mathcal{D}_\phi \end{bmatrix}.$$

(iii) The adjoint DLS $\widetilde{\phi^{red}}$ *is I/O stable and input stable. Furthermore,* $\text{range}\left(\mathcal{B}_{\widetilde{\phi^{red}}}\right) = H^{red}$*.*

(iv) If, in addition, ϕ *is input stable, then* ϕ^{red} *is input stable and* $\widetilde{\phi^{red}}$ *is output stable.*

Proof. Trivially $H_0 := \ker(\mathcal{C}_\phi) = \cap_{j\geq 0}\ker(CA^j)$ is A-invariant. By Proposition 11, $\mathcal{C}_\phi^* = \mathcal{B}_{\tilde{\phi}} \cdot \text{flip}$, where *flip* is the unitary flip reflecting $\ell^2(\mathbf{Z}_+;Y)$ onto $\ell^2(\mathbf{Z}_-;Y)$. We have $\ker(\mathcal{C}_\phi) = \text{range}\left(\mathcal{C}_\phi^*\right)^\perp = \text{range}\left(\mathcal{B}_{\tilde{\phi}}\right)^\perp$, where $\tilde{\phi} = \left(\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right)$ is the adjoint DLS of ϕ.

Because the semigroup generator of $\tilde{\phi}$ is A^*, it follows that the controllable subspace of $\tilde{\phi}$, given by $H^{red} := \text{range}\left(\mathcal{B}_{\tilde{\phi}}\right) = \ker(\mathcal{C}_\phi)^\perp$ is A^*-invariant, and we have the orthogonal direct sum decomposition $H_0 \oplus H^{red} = H$. If Π^{red} is the orthogonal projection onto H^{red}, then $A^*\Pi^{red} = \Pi^{red}A^*\Pi^{red}$ because the ransge of the observability map is always semigroup invariant.

Define the bounded operators via their adjoints as follows: $(A^{red})^* := A^*|H^{red} : H^{red} \to H^{red}$, $(C^{red})^* := \Pi^{red}C^* : Y \to H^{red}$ and $(B^{red})^* := B^*|H^{red} : H^{red} \to U$. Define the DLSs

$$\phi^{red} := \begin{pmatrix} A^{red} & B^{red} \\ C^{red} & D \end{pmatrix}, \quad \widetilde{\phi^{red}} = \begin{pmatrix} (A^{red})^* & (C^{red})^* \\ (B^{red})^* & D^* \end{pmatrix}.$$

These DLSs are adjoints of each other, and the state space of both ϕ^{red} and $\widetilde{\phi^{red}}$ is, by definition, $H^{red} \subset H$. It is easy to see that ϕ^{red} equals the one given in claim (i).

Because $A^*\Pi^{red} = \Pi^{red}A^*\Pi^{red}$, it follows that $(A^{red})^{*j}(C^{red})^* = (A^*)^j\Pi^{red}C^*$. Now, because C^* is the input operator of $\tilde{\phi}$, we have $\text{range}(C^*) \subset \text{range}\left(\mathcal{B}_{\tilde{\phi}}\right)$, and thus $\Pi^{red}C^* = C^*$. This shows that $\mathcal{B}_{\widetilde{\phi^{red}}} = \mathcal{B}_{\tilde{\phi}} = \Pi^{red}\mathcal{B}_{\tilde{\phi}}$ where H^{red} is regarded as a subspace of H and the projection Π^{red} serves only as a reminder of this. In particular, because ϕ is output stable, then $\tilde{\phi}$ is input stable together with $\widetilde{\phi^{red}}$. But then, ϕ^{red} is output stable. From definition of H^{red}, it immediately follows that $\text{range}\left(\mathcal{B}_{\widetilde{\phi^{red}}}\right)$ is dense in H^{red}, and then $\ker\left(\mathcal{C}_{\phi^{red}}\right) = \{0\}$, where $\mathcal{C}_{\phi^{red}} : H^{red} \to \ell^2(\mathbf{Z}_+;Y)$.

Claim (i) is proved, once we show that the I/O maps coincide $\mathcal{D}_{\tilde{\phi}} = \mathcal{D}_{\widetilde{\phi^{red}}}$. Because $A^*\Pi^{red} = \Pi^{red}A^*\Pi^{red}$, then $(A|H^{red})^j = A^j|H^{red}$. Now

$$(B^{red})^*\left(A^{red}\right)^{*j}(C^{red})^* = B^*(A^*)^j|H^{red} \cdot \Pi^{red}C^*$$

As above, from the inclusion range $(C^*) \subset$ range $\left(\mathcal{B}_{\tilde{\phi}}\right)$ it follows that $(B^{red})^* \left(A^{red}\right)^{*j} (C^{red})^* = B^*(A^*)^j C^*$ for all $j \geq 0$. Because also the static parts coincide, we have $\mathcal{D}_{\tilde{\phi}} = \mathcal{D}_{\widetilde{\phi^{red}}}$, and equivalently $\mathcal{D}_\phi = \mathcal{D}_{\phi^{red}}$.

We consider the second claim (ii). The claim about the semigroup is already settled. We have already shown $\mathcal{B}_{\widetilde{\phi^{red}}} = \Pi^{red}\mathcal{B}_{\tilde{\phi}}$, and adjoining this gives flip $\cdot\, \mathcal{C}_\phi \Pi^{red} =$ flip $\cdot\, \mathcal{C}_{\phi^{red}}$, or $\mathcal{C}_\phi | H^{red} = \mathcal{C}_{\phi^{red}}$, because flip is unitary.

It remains to consider the controllability map of ϕ^{red}. Because $\Pi^{red} A = \Pi^{red} A \Pi^{red}$, $(A^{red})^j B^{red} = (\Pi^{red} A \Pi^{red})^j \Pi^{red} B = \Pi^{red} A^j B$. Thus $\mathcal{B}_{\phi^{red}} \tilde{u} = \Pi^{red} \mathcal{B}_\phi \tilde{u}$ for all $\tilde{u} \in \mathrm{dom}\,(\mathcal{B}_\phi)$. Consequently, if ϕ is input stable, so is ϕ^{red}. This proves claims (ii) and (iv). The claim (iii) follows by adjoining the previous results. $\square$

We make an additional remark on the controllability properties of ϕ^{red}. Because $\mathcal{B}_{\phi^{red}} = \Pi^{red}\mathcal{B}_\phi$, it follows from the boundedness of the orthogonal projection that $\Pi^{red}\,\overline{\mathrm{range}\,(\mathcal{B}_\phi)} \subset \overline{\Pi^{red}\mathrm{range}\,(\mathcal{B}_\phi)} = \overline{\mathrm{range}\,\left(\mathcal{B}_{\phi^{red}}\right)}$. Because the range of the projection $\Pi^{red} : H \to H^{red}$ is of the second category in H^{red}, $\Pi^{red}\,\overline{\mathrm{range}\,(\mathcal{B}_\phi)}$ is, by the Open Mapping Theorem, a closed subspace of $\overline{\mathrm{range}\,\left(\mathcal{B}_{\phi^{red}}\right)}$, in the norm of H^P. If ϕ is approximately controllable, then $\Pi^{red}\,\mathrm{range}\,(\mathcal{B}_\phi)$ is dense in H^{red}, because a continuous surjective mapping maps dense sets onto dense sets. It then follows that $\overline{\mathrm{range}\,\left(\mathcal{B}_{\phi^{red}}\right)} = H^{red}$; i.e. ϕ^{red} is approximately controllable.

Similar results as Proposition 12 for continuous time well-posed linear systems are given in [21]. There, the state space of the reduced system is a factor space of type $H/\ker\,(\mathcal{C}_\phi)$. If H is a Hilbert space, we can identify this with the Hilbert subspace $\ker\,(\mathcal{C}_\phi)^\perp$.

We are ready to define the main object of this section, namely the characteristic DLS $\phi(P)$, for $P \in ric(\phi, J)$.

Definition 13. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS. Assume that there exists a regular critical solution $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ and the I/O map $\mathcal{D}$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner. Let $P \in ric(\phi, J)$ be arbitrary.*

(i) Define the closed subspaces

$$H_P := \ker\,(\mathcal{C}_{\phi_P})\,, \quad H^P := \ker\,(\mathcal{C}_{\phi_P})^\perp\,,$$

of the state space H. By Π_P denote the orthogonal projection onto H^P.

(ii) The reduced DLS $(\phi_P)^{red}$ of ϕ_P (as given in Proposition 12) is denoted by

$$\phi(P) := \begin{pmatrix} \Pi_P A | H^P & \Pi_P B \\ -K_P | H^P & I \end{pmatrix},$$

The DLS $\phi(P)$ is called the characteristic DLS (of pair (ϕ, J)), centered at P

The following lemma collects the results we have obtained in a useful form.

Lemma 14. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS. Assume that there exists a regular critical solution $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$, and the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner. Let $P \in ric_0(\phi, J)$ be arbitrary. Then the following holds:*

(i) The state space of $\phi(P)$ is H^P. The DLS $\phi(P)$ is I/O stable, output stable, and $\ker\left(\mathcal{C}_{\phi(P)}\right) = \{0\}$. *The I/O map of $\phi(P)$ satisfies $\mathcal{D}_{\phi(P)} = \mathcal{D}_{\phi_P}$.*

The adjoint DLS $\widetilde{\phi(P)}$ is input stable and approximately controllable: $\overline{\mathrm{range}\left(\mathcal{B}_{\widetilde{\phi(P)}}\right)} = H^P$.

(ii) If, in addition, ϕ is input stable, then $\phi(P)$ is input stable and $\widetilde{\phi(P)}$ is output stable.

(iii) Assume, in addition, that $\overline{\mathrm{range}\left(\mathcal{B}_\phi\right)} = H$, *and $\Lambda_P > 0$. Then* $H_P = \ker\left(P_0^{\mathrm{crit}} - P\right)$, *where* $P_0^{\mathrm{crit}} := \left(\mathcal{C}_\phi^{\mathrm{crit}}\right)^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ *is the unique regular critical solution.*

Proof. Claim (i) follows from claims (i) and (iii) of Proposition 12. If ϕ is input stable, so are all spectral DLSs ϕ_P, $P \in ric(\phi, J)$ because they have the same controllability map. Claim (ii) follows now from claim (iv) of Proposition 12. Claim (iii) is a consequence of claim (iv) of Lemma 6. □

We remark that only the last claim (iii) required the I/O map of ϕ to be $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner. Because we can write H_P in terms of the solutions P and P_0^{crit}, we can actually calculate the projection Π_P and also the operators appearing in $\phi(P)$.

4 Hankel and Toeplitz operators, and the characteristic DLS $\phi(P)$

Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator, and ϕ be an I/O stable and output stable DLS, such that a regular critical $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ exists. Furthermore, assume that ϕ has a $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner I/O map. In Definition 13 and Lemma 14, we associate to each $P \in ric_0(\phi, J)$ the characteristic DLS $\phi(P)$. The I/O map $\mathcal{D}_{\phi(P)}$ equals the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner operator $\mathcal{N}_P$, where $\mathcal{N}_P$ is the inner factor in the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization of the spectral factor $\mathcal{D}_{\phi_P} = \mathcal{N}_P \mathcal{X}$. If $\mathcal{D}_\phi$ itself is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner, then $\mathcal{D}_{\phi_P} = \mathcal{N}_P$ and the outer factor is trivially $\mathcal{X} = \mathcal{I}$, see [16, Proposition 55]. However, we use the symbol $\mathcal{N}_P$ in place for $\mathcal{D}_{\phi_P}$, because in the final Section 7, we allow $\mathcal{D}_{\phi_P}$ to have a nontrivial outer factor $\mathcal{X}$.

In the main result of this section, Lemma 17, we consider the ranges of the observability map $\mathcal{C}_{\widetilde{\phi(P)}}$ and the Hankel operator $\bar{\pi}_+ \widetilde{\mathcal{N}}_P \pi_-$ of the adjoint characteristic DLS given by

$$\widetilde{\phi(P)} := \begin{pmatrix} A^*|H^P & -\Pi_P K_P^* \\ B^*|H^P & I \end{pmatrix}.$$

Naturally, the I/O map of $\widetilde{\phi(P)}$ equals $\widetilde{\mathcal{N}}_P$. Because $\mathcal{N}_P$ is $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner, $\widetilde{\mathcal{N}}_P$ is $(\Lambda_{P_0^{\mathrm{crit}}}^{-1}, \Lambda_P^{-1})$-inner, by Corollary 4.

The DLS $\widetilde{\phi(P)}$ is interesting because the ranges of the Toeplitz operators $\widetilde{\mathcal{N}}_P \bar{\pi}_+$ code the partial ordering of the solution set $ric_0(\phi, J)$, even if $\mathcal{D}_{\phi_P}$ contains a nontrivial outer factor. For details, see Lemma 1 and the discussion associated to it. We remark that because Lemma 1 deals with the adjoint operators $\widetilde{\mathcal{N}}_P$ rather than the original $\mathcal{N}_P$, the adjoint DLS $\widetilde{\phi(P)}$ must be considered instead of $\phi(P)$.

In order to prove Lemma 17, we again need auxiliary Propositions 15 and 16 that have some interest in themselves. Let ϕ be a quite general I/O stable and output stable DLS. In Proposition 15, we consider the inclusions of the ranges $\mathrm{range}\,(\mathcal{C}_\phi)$ and $\mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)$. In the particular case, when the $\mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)$ is closed, equality of the ranges appears.

Proposition 15. *Let $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, with input space U, state space H and output space Y. Define the domains and ranges as follows:* $\mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-) := \bar{\pi}_+ \mathcal{D}_\phi \, \ell^2(\mathbf{Z}_-; U)$, $\mathrm{dom}\,(\mathcal{B}_\phi) := Seq_-(U)$, $\mathrm{range}\,(\mathcal{B}_\phi) := \mathcal{B}_\phi \, \mathrm{dom}\,(\mathcal{B}_\phi)$, *and* $\mathrm{range}\,(\mathcal{C}_\phi) := \mathcal{C}_\phi \, H$.

(i) If ϕ is input stable, then

$$\mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-) \subset \mathrm{range}\,(\mathcal{C})\,.$$

(ii) If ϕ is approximately controllable, i.e. $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$, *then*

$$\mathrm{range}\,(\mathcal{C}_\phi) \subset \overline{\mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)}.$$

(iii) If ϕ is input stable and approximately controllable, and the Hankel operator $\bar{\pi}_+ \mathcal{D}_\phi \pi_-$ has closed range, then

$$\mathrm{range}\,(\mathcal{C}_\phi) = \mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \pi_-)\,.$$

Proof. We start by establishing claim (i). Let $\tilde{y} \in \mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \bar{\pi}_-)$ be arbitrary. Then there exists a (possibly nonunique) $\tilde{u} \in \ell^2(\mathbf{Z}_-; U)$ such that $\tilde{y} = \bar{\pi}_+ \mathcal{D}_\phi \pi_- \tilde{u}$. Because $\mathrm{dom}\,(\mathcal{B}_\phi) := Seq_-(U)$ is dense in $\ell^2(\mathbf{Z}_-; U)$, we can choose a sequence $\{\tilde{u}_j\}_{j \geq 0} \subset \mathrm{dom}\,(\mathcal{B}_\phi)$ such that $\tilde{u}_j \to \tilde{u}$ in the norm of $\ell^2(\mathbf{Z}_-; U)$. Then, because $\mathcal{D}_\phi$ is bounded,

$$\bar{\pi}_+ \mathcal{D}_\phi \pi_- \tilde{u}_j \to \tilde{y} \quad \text{as} \quad j \to \infty, \tag{11}$$

in the norm of $\ell^2(\mathbf{Z}_+; Y)$. Because $\mathcal{B}_\phi$ is bounded, there is $x \in H$, such that $\mathcal{B}_\phi \pi_- \tilde{u}_j \to x$. Because $\mathcal{C}_\phi$ is bounded,

$$\mathcal{C}_\phi \mathcal{B}_\phi \pi_- \tilde{u}_j \to \mathcal{C}_\phi x \quad \text{as} \quad j \to \infty, \tag{12}$$

in the norm of $\ell^2(\mathbf{Z}_+; Y)$. Because $\bar{\pi}_+ \mathcal{D}_\phi \pi_- = \mathcal{C}_\phi \mathcal{B}_\phi$ on $\mathrm{dom}\,(\mathcal{B}_\phi)$, we have $\mathcal{C}_\phi x = \tilde{y}$ and $\tilde{y} \in \mathrm{range}\,(\mathcal{C}_\phi)$, by equations (11), (12), and the uniqueness of the limit. Because $\tilde{y} \in \mathrm{range}\,(\bar{\pi}_+ \mathcal{D}_\phi \bar{\pi}_-)$ was arbitrary, claim (i) follows.

The proof of claim (ii) is straightforward. Trivially $\mathcal{C}_\phi\,\mathrm{range}\,(\mathcal{B}_\phi) \subset \mathrm{range}\,(\bar{\pi}_- \mathcal{D} \pi_-)$. But then, the continuity of $\mathcal{C}_\phi$ implies the inclusions

$$\mathrm{range}\,(\mathcal{C}) := \mathcal{C}\,H = \mathcal{C}\,\overline{\mathrm{range}\,(\mathcal{B}_\phi)} \subset \overline{\mathcal{C}\,\mathrm{range}\,(\mathcal{B}_\phi)} \subset \overline{\mathrm{range}\,(\bar{\pi}_- \mathcal{D} \pi_-)},$$

because $H = \overline{\mathrm{range}\,(\mathcal{B}_\phi)}$ as claimed. The last claim (iii) is an easy consequence of the previous claims. □

Proposition 16. *Let H be a Hilbert Space, and H_1 its closed subspace. Let H_2 be a (possibly nonclosed) vector subspace of H, such that $H_1 \perp H_2$ and $H = H_1 + H_2$.*

Then H_2 is closed, and we have the orthogonal direct sum decomposition $H = H_1 \oplus H_2$.

Proof. If $x \in H_1 \cap H_2$, then the orthogonality of H_1 and H_2 implies that $0 = \langle x, x \rangle = ||x||^2$, whence $x = 0$. Thus $H_1 \cap H_2 = \{0\}$, and $H = H_1 + H_2$ is an algebraic direct sum. Assume $x \in \overline{H_2}$, and let $H_2 \ni x_j \to x$ in the norm of H. Then $x = \tilde{x}_1 + \tilde{x}_2$ for unique $\tilde{x}_1 \in H_1$ and $\tilde{x}_2 \in H_2$. Let P be the orthogonal projection onto H_1. Then $Px_j = 0$ for all j because $x_j \in H_2 \subset H_1^\perp$. Now we can estimate

$$||Px|| = ||Px - Px_j|| \leq ||x - x_j|| \to 0 \quad \text{as} \quad j \to \infty.$$

It follows that $0 = Px = P\tilde{x}_1 + P\tilde{x}_2$. Because $\tilde{x}_1 \in H_1$, then $P\tilde{x}_1 = \tilde{x}_1$. Because $\tilde{x}_2 \in H_2 \subset H_1^\perp$, then $P\tilde{x}_2 = 0$. Thus $\tilde{x}_1 = 0$ and $x = \tilde{x}_2 \in H_2$. This implies that H_2 is (sequentially) closed. □

Now we have obtained necessary preliminary results, and it remains to apply Propositions 15 and 16 to the adjoint characteristic DLS $\widetilde{\phi(P)}$. We work under the assumption that a regular critical $P_0^{\rm crit} \in ric_0(\phi, J)$ exists, and the indicator $\Lambda_{P_0^{\rm crit}} > 0$. Then, as in Corollary 4, all indicators Λ_P, $P \in ric_0(\phi, J)$ are positive because it is assumed that the input space U is separable. So we can define the normalized I/O maps

$$\mathcal{N}_P^\circ := \Lambda_P^{\frac{1}{2}} \mathcal{N}_P \Lambda_{P_0^{\rm crit}}^{-\frac{1}{2}}, \quad \widetilde{\mathcal{N}}_P^\circ := \Lambda_{P_0^{\rm crit}}^{-\frac{1}{2}} \widetilde{\mathcal{N}}_P \Lambda_P^{\frac{1}{2}} \tag{13}$$

where $\mathcal{N}_P^\circ$ is inner from the left, (i.e. (I, I)-inner). In fact, the transfer functions of both these normalized DLSs are inner (from both sides). If the input space U is finite dimensional, this is a trivial fact because all isometries are unitary in a finite dimensional space. The general case, when U is just a separable Hilbert space, is related to the fact that the evaluation of the transfer function $\mathcal{N}_P(0)$ is identity, and thus $\mathcal{N}_{P_0^\circ}(0) = \Lambda_P^{\frac{1}{2}} \mathcal{N}_P(0) \Lambda_{P_0^{\rm crit}}^{\frac{1}{2}}$ has a bounded inverse. For details, see [16, Proposition 34]. The normalized DLSs are defined analogously:

$$\phi^\circ(P) := \Lambda_P^{\frac{1}{2}} \phi(P) \Lambda_{P_0^{\rm crit}}^{-\frac{1}{2}}, \quad \text{and} \quad \widetilde{\phi^\circ(P)} := \Lambda_{P_0^{\rm crit}}^{-\frac{1}{2}} \widetilde{\phi(P)} \Lambda_P^{\frac{1}{2}}. \tag{14}$$

In the following lemma, we consider the adjoint characteristic DLS $\widetilde{\phi^\circ(P)}$. We show that the range of the Toeplitz operator $\widetilde{\mathcal{N}}_P \bar{\pi}_+$ is "complemented" in $\ell^2(\mathbf{Z}_+; U)$ by the state space H^P of $\widetilde{\phi(P)}$, through the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$.

Lemma 17. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let ϕ be an input stable, output stable and I/O stable DLS, such that the input space U is separable. Assume that a regular critical $P_0^{\rm crit} \in ric_0(\phi, J)$ exists, and $\Lambda_{P_0^{\rm crit}} > 0$. Assume that the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\rm crit}})$-inner.*

For all $P \in ric_0(\phi, J)$, we have an orthogonal direct sum decomposition

$$\ell^2(\mathbf{Z}_+; U) = \operatorname{range}\left(\widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+\right) \oplus \operatorname{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right),$$

where the symbols are defined as in equations (13) and (14). In fact, $\operatorname{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) = \operatorname{range}\left(\bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \pi_-\right)$, *where both subspaces are closed.*

Proof. We first show that

$$\ell^2(\mathbf{Z}_+; U) = \operatorname{range}\left(\widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+\right) \oplus \operatorname{range}\left(\bar{\pi}_+ \widetilde{\mathcal{N}}_P \bar{\pi}_-\right), \tag{15}$$

where both the spaces are closed in $\ell^2(\mathbf{Z}_+; U)$. Because $\mathcal{N}_P^\circ(e^{i\theta})$ is inner from both sides, also $\widetilde{\mathcal{N}}_P^\circ(e^{i\theta})$ is inner from both sides.

Thus $\widetilde{\mathcal{N}}_P^\circ : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ is a bounded bijection, with $\operatorname{range}\left(\widetilde{\mathcal{N}}_P^\circ\right) = \ell^2(\mathbf{Z}; U)$ and a bounded, shift-invariant (but noncausal) inverse. Thus, for each $\tilde{w} \in \ell^2(\mathbf{Z}_+; U)$, there is a $\tilde{u} \in \ell^2(\mathbf{Z}; U)$ such that

$$\tilde{w} = \bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \tilde{u} = \bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \bar{\pi}_+ \tilde{u} + \bar{\pi}_+ \widetilde{\mathcal{N}}_P^\circ \pi_- \tilde{u}.$$

So the algebraic direct sum of the (yet possibly nonclosed) vector spaces range $\left(\bar{\pi}_+\widetilde{\mathcal{N}}_P^\circ\bar{\pi}_+\right)$ and range $\left(\bar{\pi}_+\widetilde{\mathcal{N}}_P^\circ\pi_-\right)$ is all of $\ell^2(\mathbf{Z}_+;U)$.

We prove the orthogonality of these spaces. $\widetilde{\mathcal{N}}_P^\circ$ is a causal isometry on $\ell^2(\mathbf{Z};U)$, by [5, part (a) Theorem 1.1]; here we have used the fact that $\mathcal{N}_P^\circ(e^{i\theta})$ is unitary a.e. $e^{i\theta}\in\mathbf{T}$, as discussed before this lemma. We have

$$
\begin{aligned}
&(\bar{\pi}_+\widetilde{\mathcal{N}}_P^\circ\bar{\pi}_+)^*\cdot\bar{\pi}_+\widetilde{\mathcal{N}}_P^\circ\pi_- = \bar{\pi}_+(\widetilde{\mathcal{N}}_P^\circ)^*\bar{\pi}_+\cdot\bar{\pi}_+\widetilde{\mathcal{N}}_P^\circ\pi_-\\
&= \bar{\pi}_+(\widetilde{\mathcal{N}}_P^\circ)^*\widetilde{\mathcal{N}}_P^\circ\pi_- - (\pi_-\widetilde{\mathcal{N}}_P^\circ\bar{\pi}_+)^*\pi_-\widetilde{\mathcal{N}}_P^\circ\pi_-\\
&= \bar{\pi}_+\pi_- - (\pi_-\widetilde{\mathcal{N}}_P^\circ\bar{\pi}_+)^*\pi_-\widetilde{\mathcal{N}}_P^\circ\pi_- = 0,
\end{aligned}
$$

because $\pi_-\widetilde{\mathcal{N}}_P^\circ\bar{\pi}_+=0$ by causality. The range of the Toeplitz operator $\widetilde{\mathcal{N}}_P^\circ\bar{\pi}_+$ is closed, because its symbol is inner from both sides. The range of the Hankel operator range $\left(\bar{\pi}_+\widetilde{\mathcal{N}}_P^\circ\pi_-\right)$ is closed, by Proposition 16 where the spaces are $H=\ell^2(\mathbf{Z}_+;U)$, $H_1=\text{range}\left(\widetilde{\mathcal{N}}_P^\circ\right)$ and $H_2=\text{range}\left(\bar{\pi}_+\widetilde{\mathcal{N}}_P^\circ\pi_-\right)$. This verifies that we have the orthogonal direct sum decomposition (15), and it remains to show that the same is essentially true when the Hankel operator is replaced by the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$.

As discussed before the statement of this Lemma, $\Lambda_P>0$ for all $P\in ric_0(\phi,J)$, and the adjoint charcteristic DLS is described by Lemma 14. Clearly $\widetilde{\phi^\circ(P)}$ is I/O stable, because its I/O map is even inner. By claim (i) of Lemma 14, $\widetilde{\phi^\circ(P)}$ is input stable, and approximately controllable range $\left(\mathcal{B}_{\widetilde{\phi^\circ(P)}}\right)=H^P$. Finally, by claim (ii) of Lemma 14, $\widetilde{\phi^\circ(P)}$ is output stable, because ϕ is assumed to be input stable. Now, claim (iii) of Proposition 15 implies that range $\left(\bar{\pi}_+\widetilde{\mathcal{N}}_P^\circ\pi_-\right)=\text{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$, and, in particular, they are closed subspaces. The proof is now complete. □

For the closedness of the range of a Hankel operator, see [6, p. 258-259]. In Theorem 23 it is important that the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is coercive. To have this under the conditions of Lemma 17, it is enough to establish injectivity.

5 Truncated shifts and operator models

In this section, we recall some notions from the Sz.Nagy–Foias operator model for later use in Section 6. Good references are e.g. [5, Chapter IX, Section 5], [19], and [22]. In this section, all Hilbert spaces are assumed to be separable. This makes it possible to work in terms of the boundary traces because our transfer functions are always of bounded type. As before, if Θ denotes an I/O map, then $\Theta(z)$ is its transfer function, and $\Theta(e^{i\theta})$ is the nontangential boundary trace. We identify the spaces $H^2(\mathbf{T};U)$, ($L^2(\mathbf{T};U)$) and $\ell^2(\mathbf{Z}_+;U)$, ($\ell^2(\mathbf{Z};U)$, respectively), by Fourier transform. With this identification, the unilateral shift operator $S = \tau\bar{\pi}_+$ denotes the forward shift on $\ell^2(\mathbf{Z}_+;U)$ as well as multiplication by $e^{i\theta}$ on $H^2(\mathbf{T};U)$. The adjoint backward shift $S^* = \bar{\pi}_+\tau^*$ is understood in the analogous way. Finally, the symbol Θ denotes the multiplication operator by $\Theta(e^{i\theta})$ on $L^2(U)$, as well as the corresponding I/O map on $\ell^2(\mathbf{Z};U)$.

As before, an analytic function $\Theta(z) \in H^\infty(\mathcal{L}(U))$ is called inner (inner from the left), if the boundary trace function $\Theta(e^{i\theta})$ is unitary (isometry, respectively) a.e. $e^{i\theta} \in \mathbf{T}$. If $\Theta(z)$ is an inner from the left, the closed subspace is defined by

$$K_\Theta := H^2(\mathbf{T};U) \ominus \Theta H^2(\mathbf{T};U). \tag{16}$$

By P_Θ we denote the orthogonal projection onto K_Θ. Because $\Theta H^2(\mathbf{T};U)$ is S-invariant, K_Θ is S^*-invariant, or equivalently, S-co-invariant. By the Beurling–Lax–Halmos Theorem, all S^*-invariant subspaces of $H^2(\mathbf{T};U)$ are of the form $H^2(\mathbf{T};U) \ominus \Theta H^2(\mathbf{T};U')$, where $\Theta(z) \in H^\infty(\mathcal{L}(U;U'))$ is inner from the left, and $U' \subset U$ is a Hilbert subspace.

We now consider the restriction $S^*|K_\Theta$ and its adjoint, the compression $P_\Theta S|K_\Theta$. The restriction $S^*|K_\Theta$ is a contractive linear operator on the Hilbert subspace $K_\Theta \subset H^2(\mathbf{T};U)$. It is well known that various properties of $S^*|K_\Theta$ are coded into the function $\Theta(e^{i\theta})$; for this reasion it is called the characteristic function of $S^*|K_\Theta$. In a more general case, the characteristic function $\Theta(e^{i\theta}) \in H^\infty(\mathbf{T};U)$ can be allowed to be just contractive in the sense that $||\Theta(e^{i\theta})|| \leq 1$ a.e. $e^{i\theta} \in \mathbf{T}$. In this case, the set of operators $\{S^*|K_\Theta\}$ is rich enough to model all contractive linear operators. This is the famous Sz.Nagy–Foias operator model of contractions. For a lucid introduction, see [5, Chapter IX, Section 5]. The special case, appropriate to this work, is when the characteristic function $\Theta(e^{i\theta})$ is inner. Then the contraction $S^*|K_\Theta$ has a number of interesting properties and we now look at some of them. The following proposition is [19, Corollary, p. 43]:

Proposition 18. *Let $\Theta(e^{i\theta})$ be a contractive analytic function. Then $\Theta(e^{i\theta})$ is inner (from both sides) if and only if $S^*|K_\Theta \in C_{00}$. Here C_{00} denotes the class of contractions T on a Hilbert space, such that*

$$s - lim_{j\to\infty} T^j = 0, \quad s - lim_{j\to\infty} T^{*j} = 0.$$

We clearly see that class of C_{00}-contractions is invariant under unitary similarity, and closed under taking the Hilbert space adjoint. Actually [19, Corollary on p. 43] says more than Proposition 18: all C_{00}-contractions are unitarily equivalent to some $S^*|K_\Theta$, for some inner $\Theta(z)$. The adjoint $(S^*|K_\Theta)^* = P_\Theta S|K_\Theta$ is a C_{00}-contraction, and it is unitarily equivalent to $S^*|K_{\widetilde{\Theta}}$, where $\widetilde{\Theta}(z) = \Theta(\bar{z})^*$ is the adjoint inner function. For proof, see [19, Lemma on p. 75].

The spectrum of $S^*|K_\Theta \in C_{00}$ is studied in Lemma 20 with the aid of spectrum of the function $\Theta(z)$, defined as follows:

Definition 19. *Let $\Theta(z)$ be an inner function. Its spectrum $\sigma(\Theta)$ is defined to be the complement of the set of $z \in \overline{\mathbf{D}}$, such that an open neighborhood $N_z \subset \mathbf{C}$ of z exists with*

(i) $\Theta(z)^{-1}$ exists in $N_z \cap \overline{\mathbf{D}}$,

(ii) $\Theta(z)^{-1}$ can be analytically continued to a full neighborhood N_z.

For the proof of the following Livsic-Möller -type result, [19, Theorem on p. 75].

Lemma 20. *Let U be a separable Hilbert space, and $\Theta(z) \in H^\infty(\mathcal{L}(U))$ be inner. Define $T_\Theta := P_\Theta S|K_\Theta \in \mathcal{L}(K_\Theta)$. Then*

(i) $\sigma(T_\Theta) = \sigma(\Theta)$, where $\sigma(\Theta) \subset \overline{\mathbf{D}}$ is the spectrum of the characteristic function $\Theta(z)$.

(ii) The point spectrum of T_Θ and $T_\Theta^* = S^*|K_\Theta$ satisfies

$$\begin{aligned}\sigma_p(T_\Theta) &= \{z \in \mathbf{D} \quad | \quad \ker(\Theta(z)) \neq \{0\}\} \\ \sigma_p(T_\Theta^*) &= \{z \in \mathbf{D} \quad | \quad \ker\left(\widetilde{\Theta}(z)\right) \neq \{0\}\}\end{aligned}$$

We remark that $\sigma_P(T_\Theta) \subset \sigma(T_\Theta)$, and the inclusion can be proper. The dimension $\dim U$ is the multiplicity of the shift that models T_Θ. If $\dim U < \infty$, then $\sigma_p(T_\Theta^*) = \overline{\sigma_p(T_\Theta)}$, by dimension counting. Also, $\dim\ker(z - T_\Theta) \leq \dim U$ for all $z \in D$. Much more is known about the truncated shift $S^*|K_\Theta$ if we know its characteristic function $\Theta(z)$, and conversely. For example, the invariant subspace structure of $S^*|K_\Theta$ and the left inner factors of $\Theta(z)$ are connected. To apply these descriptions to DARE, we need to translate these notions into the time domain and state space language.

Definition 21. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable and output stable DLS. We define the following subspaces*

$$\begin{aligned}K_\phi &:= \ell^2(\mathbf{Z}_+; Y) \ominus \operatorname{range}(\mathcal{D}_\phi \bar{\pi}_+) \\ \widetilde{K}_\phi &:= \operatorname{range}(\mathcal{C}_\phi) \subset \ell^2(\mathbf{Z}_+; Y).\end{aligned}$$

Both K_ϕ and $\widetilde{K}_\phi$ are S^*-invariant. If the transfer function $\mathcal{D}_\phi(z)$ is inner, we see that the closed subspace K_ϕ corresponds, via Fourier transform, to the co-invariant subspace $K_{\mathcal{D}_\phi} \subset H^2(\mathbf{T};Y)$, as defined in equation (16). In this paper, the spaces $K_{\widetilde{\phi^\circ(P)}}$ is investigated. Under the assumptions of Lemma 17, we have the equality of the spaces range $\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) = \widetilde{K}_{\widetilde{\phi^\circ(P)}} = K_{\widetilde{\phi^\circ(P)}}$, where $\mathcal{D}_{\widetilde{\phi^\circ(P)}} = \widetilde{\mathcal{N}}_P^\circ$. The model operator $S^*|K_{\widetilde{\mathcal{N}}_P^\circ}$ is the truncated unilateral shift $(\bar{\pi}_+\tau^*)\,|K_{\widetilde{\phi^\circ(P)}}$ in space $\ell^2(\mathbf{Z}_+;U)$. Actually, we shall write S^* instead of $\bar{\pi}_+\tau^*$ also in the time domain. Stated in other words, the backward shift $S^* = \bar{\pi}_+\tau^*$, restricted to $K_{\widetilde{\phi^\circ(P)}} = \text{range}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$ is a contractive linear operator whose characteristic function is $\widetilde{\mathcal{N}}_P^\circ(z) \in H^\infty(\mathcal{L}(U))$. In the next section, we shall make a connection to the state space and semigroup of $\widetilde{\phi^\circ(P)}$.

6 Invariant subspaces of the semigroup

It is now time to combine the results of previous sections, and produce the first of our main results. We start by reminding the main lines of previous sections. Let $J \in \mathcal{L}(Y)$ be a cost operator, and $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that $\overline{\text{range}\,(\mathcal{B}_\phi)} = H$. We assume that the regular critical solution $P_0^{\text{crit}} := (\mathcal{C}_\phi^{\text{crit}})^* J \mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ exists and $\Lambda_{P_0^{\text{crit}}} > 0$. It then follows that all $P \in ric_0(\phi, J)$ have a positive indicator, see [16, Corollary 54]. In this section, we still make the technical assumption that the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\text{crit}}})$-inner, as in Lemma 14. This assumption will be removed in the final Section 7 of this work.

Under these assumptions, we associate two mutually orthogonal subspaces $H_P := \ker\,(\mathcal{C}_{\phi_P}) \subset H$ and $H^P := H \ominus H_P$ to each solution $P \in ric_0(\phi, J)$. Here, as always before, $\phi_P := \left(\begin{smallmatrix} A & B \\ -K_P & I \end{smallmatrix}\right)$ denotes the spectral DLS, centered at P. In claim (iv) Lemma 6 it is shown that H_P is A-invariant. By the same lemma, the subspace H_P is related to the solution $P \in ric_0(\phi, J)$ in the following simple way: Because $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\text{crit}}})$-inner, $H_P = \ker\,(\mathcal{C}_{\phi_P}) = \ker\left(P_0^{\text{crit}} - P\right)$. Now we see that the solutions $P \in ric_0(\phi, J)$ are immediately associated to a family $\{H^P\}$ of A^*-invariant subspaces. This makes it possible to define the restricted operators $A^*|H^P$ and their adjoints, the compressions $\Pi_P A|H^P$ of the semigroup generator.

In this section, we study the structure of the restriction $A^*|H^P \in \mathcal{L}(H^P)$ in terms of the characteristic (transfer) function $\widetilde{\mathcal{N}}_P^\circ(z)$, for arbitrary $P \in ric_0(\phi, J)$. This is done with the aid of the (normalized) adjoint characteristic DLS $\widetilde{\phi^\circ(P)}$ whose semigroup generator is $A^*|H^P$, and I/O maps is $\mathcal{D}_{\widetilde{\phi^\circ(P)}}(z) = \widetilde{\mathcal{N}}_P^\circ(z)$. The DLS $\widetilde{\phi^\circ(P)}$ is the conveniently normalized adjoint DLS of $\phi(P)$ which has been introduced in the following way: By Proposition 12, the null space $H_P := \ker\,(\mathcal{C}_{\phi_P}) \subset H$ is divided away from the state space H of the spectral DLS ϕ_P. We obtain another DLS, the characteristic

$\phi(P) := (\phi_P)^{red}$ whose state space is H^P — it is the reduced DLS whose I/O map equals that of the spectral DLS ϕ_P. Furthermore, the DLS $\phi(P)$ is output stable and observable: $\ker\left(\mathcal{C}_{\phi(P)}\right) = \{0\}$. The adjoint DLS $\widetilde{\phi(P)}$ is input stable and approximately controllable: $\overline{\text{range}\left(\mathcal{B}_{\widetilde{\phi(P)}}\right)} = H^P$. A simple normalization is now required to turn $\widetilde{\phi(P)}$ into $\widetilde{\phi^\circ(P)}$.

Under the above assumptions, the I/O map $\mathcal{N}_P$ of $\phi(P)$ is $(\Lambda_P, \Lambda_{P_0^{\rm crit}})$-inner, where both Λ_P and $\Lambda_{P_0^{\rm crit}}$ are positive. The normalization, as done in formulae(13) and (14), gives us $\phi^\circ(P)$ and its adjoint DLS $\widetilde{\phi^\circ(P)}$. The latter is particularly interesting to us, and already considered in Section 4. The DLS $\phi(P)$ and its normalized version $\phi^\circ(P)$ is given by

$$\phi(P) := \begin{pmatrix} \Pi_P A|H^P & \Pi_P B \\ -K_P|H^P & I \end{pmatrix}, \quad \phi^\circ(P) := \begin{pmatrix} \Pi_P A|H^P & \Pi_P B \Lambda_{P^{\rm crit}}^{-\frac{1}{2}} \\ -\Lambda_P^{\frac{1}{2}} K_P|H^P & \Lambda_P^{\frac{1}{2}} \Lambda_{P^{\rm crit}}^{-\frac{1}{2}} \end{pmatrix}.$$

The state space of the DLSs $\phi(P)$, $\phi^\circ(P)$, $\widetilde{\phi(P)}$ and $\widetilde{\phi^\circ(P)}$ is H^P, which is regarded as a subspace of H. The properties of $\widetilde{\phi^\circ(P)}$ and its semigroup generator $A^*|H^P$ are described in the following.

Lemma 22. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that the input space U is separable. Assume that the regular critical solution $P_0^{\rm crit} := (\mathcal{C}_\phi^{\rm crit})^* J \mathcal{C}_\phi^{\rm crit} \in ric_0(\phi, J)$ exists, and $\Lambda_{P_0^{\rm crit}} > 0$. Assume that the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\rm crit}})$-inner.*

For arbitrary $P \in ric_0(\phi, J)$, the following holds:

(i) The normalized adjoint characteristic DLS $\widetilde{\phi^\circ(P)}$ is input stable and $\overline{\text{range}\left(\mathcal{B}_{\widetilde{\phi^\circ(P)}}\right)} = H^P$. The observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is densely defined in H^P, and closed. We have the commutant equation

$$(17) \qquad \left(S^*|\tilde{K}_{\widetilde{\phi^\circ(P)}}\right) \cdot \mathcal{C}_{\widetilde{\phi^\circ(P)}}\, x_0 = \mathcal{C}_{\widetilde{\phi^\circ(P)}} \cdot (A^*|H^P)\, x_0, \quad S^* := \bar{\pi}_+ \tau^*,$$

for all $x_0 \in \text{dom}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$, where the possibly nonclosed subspace $\tilde{K}_{\widetilde{\phi^\circ(P)}} \subset \ell^2(\mathbf{Z}_+; U)$ is given in Definition 21.

(ii) Assume, in addition, that ϕ is input stable. Then the DLS $\widetilde{\phi^\circ(P)}$ is output stable and $\text{dom}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) = H^P$. The range of $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is closed, and equals $K_{\widetilde{\phi^\circ(P)}}$, given in Definition 21. The following similarity transform holds

$$(18) \qquad \left(S^*|K_{\widetilde{\phi^\circ(P)}}\right) \cdot \mathcal{C}_{\widetilde{\phi^\circ(P)}} = \mathcal{C}_{\widetilde{\phi^\circ(P)}} \cdot (A^*|H^P),$$

where all the operators are bounded.

(iii) Assume, in addition, that ϕ is input stable and approximately controllable: $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$. *Then* $\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) = \{0\}$, *and the observability map* $\mathcal{C}_{\widetilde{\phi^\circ(P)}} : H^P \to K_{\widetilde{\phi^\circ(P)}}$ *is a bounded bijection with a bounded inverse.*

Proof. We start with claim (i). The DLS $\widetilde{\phi^\circ(P)}$ is input stable and approximately controllable, by claim (i) of Lemma 14, because the normalization by the boundedly invertible indicator operators $\Lambda_{P_0^{\mathrm{crit}}}$ and Λ_P plays no essential role. For any I/O stable DLS ϕ, $\operatorname{range}(\mathcal{B}_\phi) \subset \operatorname{dom}(\mathcal{C}_\phi)$, by [14, Lemmas 49 and 40]. It follows that the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is densely defined in H^P, because $\overline{\operatorname{range}\left(\widetilde{\phi^\circ(P)}\right)} = H^P$. The closedness of $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is dealed in [14, Lemma 27]. Equation (17) is a basic property of the DLS, and claim (i) is now proved.

We proceed to prove claim (ii). Claim (ii) of Lemma 14 implies the output stability of $\widetilde{\phi^\circ(P)}$, if it is assumed that ϕ is input stable. By the Closed Graph theorem, we see that $\operatorname{dom}\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) = H^P$. The range of $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is closed, and equals $K_{\widetilde{\phi^\circ(P)}}$, by Lemma 17. Now the similarity transform (18) follows now from equation (17).

To prove the final claim (iii), we show that approximately controllability $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$ implies the injectivity of the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$. We first show that if $\overline{\operatorname{range}(\mathcal{B}_\phi)} = \overline{\operatorname{range}(\mathcal{B}_{\phi_P})} = H$, then $\overline{\operatorname{range}\left(\mathcal{B}_{\phi(P)}\right)} = H^P = \operatorname{range}(\Pi_P)$. For contradiction, assume that $x_0 \in \operatorname{range}(\Pi_P) \ominus \overline{\operatorname{range}\left(\mathcal{B}_{\phi(P)}\right)}$. Because $\mathcal{B}_{\phi(P)} = \Pi_P \mathcal{B}_{\phi_P} = \Pi_P \mathcal{B}_\phi$ by claim (ii) of Proposition 12, we would have for such x_0 and all $\tilde{u} \in \ell^2(\mathbf{Z}_-; U)$:

$$0 = \langle x_0, \Pi_P \mathcal{B}_\phi \tilde{u} \rangle = \langle \Pi_P x_0, \mathcal{B}_\phi \tilde{u} \rangle = \langle x_0, \mathcal{B}_\phi \tilde{u} \rangle .$$

But then $x_0 = 0$ because $\operatorname{range}(\mathcal{B}_\phi)$ is dense in H. So $\overline{\operatorname{range}\left(\mathcal{B}_{\phi(P)}\right)} = H^P$, or equivalently, $\ker\left(\mathcal{C}_{\widetilde{\phi(P)}}\right) = \{0\}$, by Proposition 11. The proof is completed, by recalling the well known functional analytic fact that a bounded bijection between Hilbert spaces has a bounded inverse. $\square$

We conclude from claim (iii) of Lemma 22 that if the observability map $\mathcal{C}_{\widetilde{\phi^\circ(P)}}$ is injective, then the similarity transform (18) effectively combines the properties of $A^*|H^P$ to the properties of the restricted shift $S^*|K_{\widetilde{\phi^\circ(P)}}$. By using the theory of shift operator models as outlined in Section 5, the properties of $S^*|K_{\widetilde{\phi^\circ(P)}}$ and its characteristic function $\mathcal{D}_{\widetilde{\phi^\circ(P)}}(z) = \widetilde{\mathcal{N}}_P^\circ(z)$ are tied together in a very strong manner.

Theorem 23. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an input stable, output stable and I/O stable DLS, such that the input space U is separable and* $\overline{\operatorname{range}(\mathcal{B}_\phi)} = H$. *Assume that the regular critical $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ exists, and $\Lambda_{P_0^{\mathrm{crit}}} > 0$. Assume that the I/O map $\mathcal{D}_\phi$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner.*

Then for arbitrary $P \in ric_0(\phi, J)$ the following holds:

(i) The restriction $A^*|H^P$ is similar to a C_{00}-contraction, whose inner characteristic function is $\widetilde{\mathcal{N}}_P^\circ(z) \in H^\infty(\mathcal{L}(U))$. The similarity transform is given by

$$\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right) \cdot \mathcal{C}_{\widetilde{\phi^\circ(P)}} = \mathcal{C}_{\widetilde{\phi^\circ(P)}} \cdot (A^*|H^P) \tag{19}$$

where $\mathcal{C}_{\widetilde{\phi^\circ(P)}} : H^P \to K_{\widetilde{\phi^\circ(P)}}$ is a bounded bijection, and the S^*-invariant subspace $K_{\widetilde{\phi^\circ(P)}}$ is given in Definition 21.

(ii) The spectra satisfy $\sigma(\Pi_P A|H^P) = \sigma(\widetilde{\mathcal{N}}_P^\circ) = \overline{\sigma(A^*|H^P)}$, where the bar denotes complex conjugation, and the spectrum of the inner function is given in Definition 19.

In particular, both $\sigma(\Pi_P A|H^P)$ and $\sigma(A^*|H^P)$ are subsets of the closed unit disk $\overline{\mathbf{D}}$.

(iii) The point spectra satisfy

$$\sigma_p(A^*|H^P) = \{z \in \mathbf{D} \mid \ker(\mathcal{N}_P(z)) \neq \{0\}\} \tag{20}$$

and

$$\sigma_p(\Pi_P A|H^P) = \{z \in \mathbf{D} \mid \ker\left(\widetilde{\mathcal{N}}_P(z)\right) \neq \{0\}\}. \tag{21}$$

In particular, if $A^*|H^P$ is compact, then it is power stable (i.e. $\rho(A^*|H^P) < 1$).

(iv) Both $A^*|H^P$ and its adjoint $\Pi_P A|H^P$ are strongly stable.

Proof. The first claim (i) follows from the similarity transform in equation (18) of Lemma 22, together with the discussion in Section 5.

Let us look at claim (ii) of the spectrum. Let $\lambda \in \mathbf{C}$ be arbitrary. Then we have

$$\left(\lambda - S^*|K_{\widetilde{\phi^\circ(P)}}\right) = \mathcal{C}_{\widetilde{\phi^\circ(P)}}\left(\lambda - A^*|H^P\right)\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^{-1}. \tag{22}$$

where $\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^{-1} : K_{\widetilde{\phi^\circ(P)}} \to H^P$ is the bounded inverse of the bounded bijection. Immediately, $\sigma\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right) = \sigma(A^*|H^P)$. By adjoining

$$\sigma\left(P_{\widetilde{\phi^\circ(P)}} S|K_{\widetilde{\phi^\circ(P)}}\right) = \sigma(\left(A^*|H^P\right)^*) = \sigma(\Pi_P A|H^P),$$

where $P_{\widetilde{\phi^\circ(P)}}$ is the orthogonal projection of $\ell^2(\mathbf{Z}_+; U)$ onto $K_{\widetilde{\phi^\circ(P)}}$. Lemma 20 implies now that $\overline{\sigma(A^*|H^P)} = \sigma(\Pi_P A|H^P) = \sigma(\widetilde{\mathcal{N}}_P^\circ)$. This proves claim (ii). Claim (iii) about the point spectra follows similarly from equation (22) and the latter claim of Lemma 20. We just remark that if $A^*|H^P$ is compact, then $\sigma(A^*|H^P) \subset D$ because the origin is the only accumulation point that a spectrum of a compact operator can have.

To verify claim (iv), note first that $\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right)$ is a C_{00}-contraction, see Proposition 18. Then we have

$$|| \left(A^*|H^P\right)^j x_0|| \leq || \left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^{-1} || \cdot || \left(S^*|K_{\widetilde{\phi^\circ(P)}}\right)^j \mathcal{C}_{\widetilde{\phi^\circ(P)}} x_0|| \to 0,$$

as $j \to \infty$. The adjoint part is similar, and the proof is complete. $\square$

Corollary 24. *Make the same assumptions as in Theorem 23, but assume, in addition, that* $\dim U < \infty$. *Then for arbitrary* $P \in ric_0(\phi, J)$

$$\sigma(A^*|H^P) \cap \mathbf{D} = \sigma_p(A^*|H^P) = \overline{\sigma_p(\Pi_P A|H^P)}, \tag{23}$$

where the bar denotes complex conjugation. If $\{\lambda_j(A^*|H^P)\}_{j\geq 1}$ *is the enumeration of the eigenvalues* $\sigma_p(A^*|H^P)$ *in the nondecreasing order of absolute values, then the following Blaschke condition is satisfied*

$$\sum_{j\geq 1} (1 - |\lambda_j(A^*|H^P)|) < \infty. \tag{24}$$

In particular, both $A^*|H^P$ *and* $\Pi_P A|H^P$ *are injective.*

Proof. From claim (iii) of Theorem 23 we conclude that $\sigma_p(A^*|H^P) = \overline{\sigma_p(\Pi_P A|H^P)}$ because for each $z \in \mathbf{D}$, $\ker\left(\widetilde{\mathcal{N}}_P^\circ(z)\right) \neq \{0\}$ is equivalent to $\ker\left(\widetilde{\mathcal{N}}_P^\circ(z)^*\right) = \ker\left(\mathcal{N}_P^\circ(\bar{z})\right) \neq \{0\}$, by dimension counting in the finite dimensional space U. Because $\sigma_p(A^*|H^P) \subset \sigma(A^*|H^P) \cap \mathbf{D}$ by claim (iii) of Theorem 23, the equality (23) is proved once we establish $\sigma(A^*|H^P) \cap \mathbf{D} \subset \sigma_p(A^*|H^P)$.

Because $n := \dim U < \infty$, we can consider the complex function $\det \widetilde{\mathcal{N}}_P^\circ(z)$, for $z \in \mathbf{D}$. By recalling the definition of the determinant as a finite sum of products of the matrix elements, we see that $\det \widetilde{\mathcal{N}}_P^\circ(z)$ is an analytic function. For any $n \times n$ matrix M we have by

$$|\det M| = \prod_{j=1}^{n} |\lambda_j(M)| \leq \prod_{j=1}^{n} \sigma_j(M) \leq ||M||^n$$

where $\lambda_j(M)$ are the eigenvalues of H, $\sigma_j(M)$ are the singular values of M, and their inequality is by H. Weyl, see [4, p. 1092]. This makes is possible to conclude that $\det \widetilde{\mathcal{N}}_P^\circ(z) \in H^\infty(\mathbf{D}; \mathbf{C})$, and because $|\det(U)| = 1$ for unitary U, we conclude that $\det \widetilde{\mathcal{N}}_P^\circ(z)$ is an inner function. Of course, the same is true for $\det \mathcal{N}_P^\circ(z)$, too.

We proceed to show that

$$\sigma(\widetilde{\mathcal{N}}_P^\circ) \cap \mathbf{D} = \{z \in \mathbf{D} \mid \det \widetilde{\mathcal{N}}_P^\circ(z) = 0\}. \tag{25}$$

By the basic property of the determinant, the open set $E := \mathbf{D} \setminus \{z \in \mathbf{D} \mid \det \widetilde{\mathcal{N}}_P^{\circ}(z) = 0\}$ is exactly the set of $z \in \mathbf{D}$ where $\widetilde{\mathcal{N}}_P^{\circ}(z)$ is invertible. To show (25), we must additionally show that the mapping $z \mapsto \widetilde{\mathcal{N}}_P^{\circ}(z)^{-1}$ is analytic in the set $E \subset \mathbf{D}$. This follows from the following outline of an argument: Assume $f(z)$ is a matrix-valued analytic function in $E \subset \mathbf{C}$, such that $\det f(z_0) \neq 0$ for some $z_0 \in E$. Then $f(z_0)$ has an inverse, and we can assume that $f(z_0) = I$ without any loss of generality. By developing $f(z)$ into its power series at z_0, we have $||I - f(z)|| \leq 1/2$ if $|z - z_0| < \delta$ for some $\delta > 0$. It then follows that the von Neumann series

$$f(z)^{-1} = (I - (I - f(z)))^{-1} = \sum_{j \geq 0} (I - f(z))^j$$

converges for all $|z - z_0| < \delta$. In fact, the convergence is uniform on the compact subsets of $\{z \mid |z - z_0| < \delta\}$. Because the limit of such a sequence of analytic functions is analytic, $f(z)^{-1}$ is analytic for $|z - z_0| < \delta$. Equation (25) follows from this consideration and Definition 19 of $\sigma(\widetilde{\mathcal{N}}_P^{\circ})$.

From equality (25), we conclude that $\overline{\sigma(A^*|H^P)} \cap \mathbf{D} = \{z \in \mathbf{D} \mid \det \widetilde{\mathcal{N}}_P^{\circ}(z) = 0\}$, by claim (ii) of Theorem 23. Let $z \in \overline{\sigma(A^*|H^P)} \cap \mathbf{D}$ be arbitrary. Then $\det \widetilde{\mathcal{N}}_P^{\circ}(z) = 0$, and the matrix $\widetilde{\mathcal{N}}_P^{\circ}(z)$ fails to be injective. The same is true for $\mathcal{N}_P^{\circ}(\bar{z}) = \widetilde{\mathcal{N}}_P^{\circ}(z)^*$ because $\dim U < \infty$. Now claim (iv) of Theorem 23 shows that $\bar{z} \in \sigma_P(A^*|H^P)$, and the converse inclusion $\sigma(A^*|H^P) \cap \mathbf{D} \subset \sigma_p(A^*|H^P)$ follows.

We have now proved that

$$\sigma(A^*|H^P) \cap \mathbf{D} = \{z \in \mathbf{D} \mid \det \mathcal{N}_P^{\circ}(z) = 0\} = \sigma_P(A^*|H^P),$$

where $\det \mathcal{N}_P^{\circ}(z)$ is an inner function. By e.g. [20, Theorem 17.9], the zeroes of an inner function can be factorized away by a Blaschke product. Because the zeroes of the Blaschke product satisfy the Blaschke condition, equation (24) follows. The final claim about the injectivity of $A^*|H^P$ and $\Pi_P A|H^P$ follows because $\widetilde{\mathcal{N}}_P(0) = I$ is invertible. □

Under particular conditions, we can make conclusions of the unrestricted semigroup generator A itself. The proof of the following corollary is based on Lemma 22 and Corollary 24.

Corollary 25. *Make the same assumptions as in Theorem 23. Assume that there exists a $P \in ric_0(\phi, J)$ such that $H^P = H$. Then A is similar to a C_{00}-contraction, and is strongly stable together with its adjoint A^*. If A is compact, then it is power stable $\rho(A) < \infty$. If $\dim U < \infty$, then the eigenvalues $\lambda_j(A)_{j \geq 0} = \sigma(A) \cap \mathbf{D}$ satisfy the Blaschke condition*

$$\sum_{j \geq 1} (1 - |\lambda_j|) < \infty.$$

In particular, if $P_0^{\mathrm{crit}} > 0$ and there exists a $P \in ric_0(\phi, J)$ such that $P \leq 0$, it follows that $H^P = H$.

We complete this section by considering what happens if the approximate controllability condition in claim (iii) of Lemma 22 is not satisfied, but all the other conditions of the preceeding claim (ii) are satisfied. Then all the operators are bounded in the commutant equation

$$\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right)\cdot\mathcal{C}_{\widetilde{\phi^\circ(P)}}=\mathcal{C}_{\widetilde{\phi^\circ(P)}}\cdot(A^*|H^P),$$

and even range $\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right) = K_{\widetilde{\phi^\circ(P)}}$ is closed. However, $\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$ can be nontrivial. If we make the decomposition of the state space $H^P = \ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp \oplus \ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)$ and use the fact the null space of the observability map is semigroup invariant, the commutant equation takes now the form

$$\begin{aligned}
&\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right)\cdot\left[\mathcal{C}_{\widetilde{\phi^\circ(P)}}|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp \quad 0\right]\\
&=\left[\mathcal{C}_{\widetilde{\phi^\circ(P)}}|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp \quad 0\right]\cdot\\
&\cdot\begin{bmatrix}\Pi_1A^*|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp & 0\\ (I-\Pi_1)A^*|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp & (I-\Pi_1)A^*|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)\end{bmatrix}
\end{aligned}$$

or

$$\begin{aligned}
&\left(S^*|K_{\widetilde{\phi^\circ(P)}}\right)\cdot\mathcal{C}_{\widetilde{\phi^\circ(P)}}|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp\\
&=\mathcal{C}_{\widetilde{\phi^\circ(P)}}|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp\cdot\left(\Pi_1A^*|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp\right),
\end{aligned}$$

where Π_1 is the orthogonal projection of H^P onto $\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp$, and $\mathcal{C}_{\widetilde{\phi^\circ(P)}}|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp$ is now a bounded bijection. What has already been stated about $A^*|H^P$ under the approximate controllability of ϕ, can now be generally stated about the compression $\Pi_1A^*|\ker\left(\mathcal{C}_{\widetilde{\phi^\circ(P)}}\right)^\perp$, at the cost of increased notational burden.

7 Generalization

In this section, we use extensively the tools developed in [17, Section 15], and in particular [17, Proposition 104 and Theorem 105].

The general goal of this section is to translate the results of previous sections (valid for DLSs ϕ having a (J,S)-inner I/O map) to general output stable and I/O stable DLS ϕ without this restriction. For this to be possible, we must require that a regular critical solution $P_0^{\text{crit}} := (\mathcal{C}_\phi^{\text{crit}})^* J \mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ exists, where

$$\mathcal{C}_\phi^{\text{crit}} := (\mathcal{I} - \bar{\pi}_+ \mathcal{D}_\phi (\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}_\phi^* J) \mathcal{C}_\phi. \tag{26}$$

Furthermore, we make it a standing hypothesis that both $J \geq 0$ and $\overline{\text{range}\,(\mathcal{B}_\phi)} = H$. This implies that P_0^{crit} is the unique critical solution in set $ric(\phi, J) \supset ric_0(\phi, J)$.

We first make the preliminary state feedback, associated to the solution P_0^{crit}. This gives the closed loop system

$$\phi^{P_0^{\text{crit}}} = \begin{pmatrix} A_{P_0^{\text{crit}}} & B \\ C_{P_0^{\text{crit}}} & D \end{pmatrix}.$$

This is the inner DLS of ϕ, centered at the regular critical solution $P_0^{\text{crit}} \in ric_0(\phi, J)$. The DLS $\phi^{P_0^{\text{crit}}}$ carries much of the interesting structure of the original DLS ϕ, see [17, Proposition 104], Even the structure H^∞DAREs $ric(\phi, J)$ and $ric(\phi^{P_0^{\text{crit}}}, J)$ is quite similar, see [17, Theorem 105]. However, the I/O map of $\phi^{P_0^{\text{crit}}}$ is $(J, \Lambda_{P_0^{\text{crit}}})$-inner, by [17, Lemma 79]. To the inner DLS $\phi^{P_0^{\text{crit}}}$ and inner DARE $ric(\phi^{P_0^{\text{crit}}}, J)$, we can apply the theory of Section 6. The results are then translated back to the original data, namely the DLS ϕ, cost operator J and H^∞DARE $ric(\phi, J)$. This trick gives us information about the invariant and co-invariant subspace structure of the closed loop semigroup generator $A_{P_0^{\text{crit}}}$, rather than the open loop semigroup generator A.

The full solution sets of the DAREs $Ric(\phi, J)$ and $Ric(\phi^{P_0^{\text{crit}}}, J)$ are equal by [17, Lemma 65]. Thus the spectral DLS $(\phi^{P_0^{\text{crit}}})_P$ makes sense, for all $P \in Ric(\phi, J)$. It is given by

$$(\phi^{P_0^{\text{crit}}})_P = \begin{pmatrix} A_{P_0^{\text{crit}}} & B \\ K_{P_0^{\text{crit}}} - K_P & I \end{pmatrix}. \tag{27}$$

by [17, equation (59) of Proposition 59] With the aid of formula (27), we enlarge the definition of the characteristic DLS $\phi(P)$ (see Definition 13) to DLSs whose I/O map need not be $(J, \Lambda_{P_0^{\text{crit}}})$-inner.

Definition 26. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that the input space U is separable. Assume that the regular critical solution $P_0^{\text{crit}} := (\mathcal{C}_\phi^{\text{crit}})^* J \mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ exists.*

For $P \in ric(\phi, J)$, the characteristic DLS $\phi(P)$ of P is the reduced DLS (in the sense of Proposition 12) of the spectral DLS $(\phi^{P_0^{\mathrm{crit}}})_P$. It is given by

$$\phi(P) = \begin{pmatrix} \Pi_P A_{P_0^{\mathrm{crit}}} | H^P & \Pi_P B \\ (K_{P_0^{\mathrm{crit}}} - K_P) | H^P & I \end{pmatrix}.$$

where $H^P := \ker \left(P_0^{\mathrm{crit}} - P \right)^\perp$, Π_P *is the orthogonal projection of* H *onto* H^P.

If $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$ and ϕ itself has an $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner I/O map, then $K_{P_0^{\mathrm{crit}}} = 0$, $A_{P_0^{\mathrm{crit}}} = A$ and immediately $\phi^{P_0^{\mathrm{crit}}} = \phi$, see the proof of Lemma 6. In this case, the characteristic DLS $\phi(P)$ coincides with the one given in Definition 13, for DLSs with $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner I/O map. We now consider restrictions of $A_{P_0^{\mathrm{crit}}}$ to its certain invariant subspaces, for each $P \in ric_0(\phi, J)$.

Theorem 27. *Let $J \geq 0$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be an input stable, output stable and I/O stable DLS, such that the input space U and output space Y are separable. Assume that $\overline{\mathrm{range}\,(\mathcal{B}_\phi)} = H$ and the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt. Assume that the regular critical solution $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ exists.*

Let $P \in ric_0(\phi, J)$ be arbitrary. By $\phi(P)$ denote its characteristic DLS, given by Definition 26. By $\mathcal{N}_P$ denote the $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor of $\mathcal{D}_{\phi_P}$. Then the following holds:

(i) The restriction of $\Pi_P A^*_{P_0^{\mathrm{crit}}} | H^P$ is similar to a C_{00}-contraction, whose characteristic function is $\widetilde{\mathcal{N}}_P^\circ(z)$. The similarity transform is given by

$$\left(S^* | K_{\widetilde{\phi^\circ(P)}} \right) \mathcal{C}_{\widetilde{\phi^\circ(P)}} = \mathcal{C}_{\widetilde{\phi^\circ(P)}} \cdot \left(A^{\mathrm{crit}*} | H^P \right) \tag{28}$$

where $\mathcal{C}_{\widetilde{\phi^\circ(P)}} : H^P \to K_{\widetilde{\phi^\circ(P)}}$ is a bounded bijection, and the S^*-invariant subspace $K_{\widetilde{\phi^\circ(P)}}$ is given in Definition 21.

(ii) The spectra satisfy $\sigma(\Pi_P A_{P_0^{\mathrm{crit}}} | H^P) = \sigma(\widetilde{\mathcal{N}}_P^\circ) = \overline{\sigma(A^*_{P_0^{\mathrm{crit}}} | H^P)}$, where the bar denotes complex conjugation, and the spectrum of the inner function is given in Definition 19. In particular, both $\sigma(\Pi_P A_{P_0^{\mathrm{crit}}} | H^P)$ and $\sigma(A^*_{P_0^{\mathrm{crit}}} | H^P)$ are subsets of the closed unit disk $\overline{\mathbf{D}}$.

(iii) The point spectra satisfy

$$\sigma_p(A^*_{P_0^{\mathrm{crit}}} | H^P) = \{ z \in \mathbf{D} \mid \ker \left(\mathcal{N}_P(z) \right) \neq \{0\} \}$$

and

$$\sigma_p(\Pi_P A_{P_0^{\mathrm{crit}}} | H^P) = \{ z \in \mathbf{D} \mid \ker \left(\widetilde{\mathcal{N}}_P(z) \right) \neq \{0\} \}.$$

In particular, if $A^*_{P_0^{\mathrm{crit}}} | H^P$ is compact, then it is power stable (i.e. $\rho(A^*_{P_0^{\mathrm{crit}}} | H^P) < 1$).

(iv) Both $A^*_{P_0^{\mathrm{crit}}}|H^P$ *and its adjoint* $\Pi_P A_{P_0^{\mathrm{crit}}}|H^P$ *are strongly stable.*

Proof. We reduce this theorem to Theorem 23 by making a preliminary feedback, associated to the solution $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}}$. This amounts to replacing the original pair (ϕ, J) by the pair $(\phi^{P_0^{\mathrm{crit}}}, J)$. By [17, claims (i) and (iii) of Proposition 104], the inner DLS $\phi^{P_0^{\mathrm{crit}}}$ is input stable, output stable, I/O stable and approximately controllable $\overline{\text{range}\left(\mathcal{B}_{\phi^{P_0^{\mathrm{crit}}}}\right)} = H$. Also, the I/O map of $\phi^{P_0^{\mathrm{crit}}}$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner. The input and output spaces of ϕ and $\phi^{P_0^{\mathrm{crit}}}$ coincide, and are thus separable.

By [17, claim (ii) of Proposition 104], P_0^{crit} is the unique regular critical solution of its inner DARE $ric(\phi^{P_0^{\mathrm{crit}}}, J)$, too. Because $J \geq 0$, it follows that $P_0^{\mathrm{crit}} \geq 0$ and its indicator (equalling $\Lambda_{P_0^{\mathrm{crit}}}$) is positive. We conclude that the inner DLS $\phi^{P_0^{\mathrm{crit}}}$, together with the cost operator J, satisfies the conditions of Theorem 23.

An application of Theorem 23 to the DLS $\phi^{P_0^{\mathrm{crit}}}$, the cost operator J and the H^∞DARE $ric(\phi^{P_0^{\mathrm{crit}}}, J)$ proves all claims (i), (ii), (iii) and (iv) for arbitrary $P \in ric_0(\phi^{P_0^{\mathrm{crit}}}, J)$. But $ric_0(\phi^{P_0^{\mathrm{crit}}}, J) = ric_0(\phi, J)$, by [17, claim (ii) of Theorem 105] and the fact that the input operator B, common to both ϕ and $\phi^{P_0^{\mathrm{crit}}}$, is Hilbert–Schmidt. This completes the proof. $\square$

Under the assumptions of Theorem 27, also the analogous results to Corollaries 24 and 25 hold, if the open loop semigroup generator A is replaced by the closed loop semigroup generator $A_{P_0^{\mathrm{crit}}}$. In particular, Corollary 25 gives a stabilization result for the critical closed loop semigroup. We remark that the Hilbert–Schmidt compactness assumption of the input operator B in Theorem 27 is required only to obtain the equality of the solution sets $ric_0(\phi^{P_0^{\mathrm{crit}}}, J) = ric_0(\phi, J)$. In particular, if $\dim U < \infty$, this assumption is trivially satisfied.

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