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A STRATIFICATION OF THE SET OF NORMAL MATRICES

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Abstract: *In this paper we consider the set of normal matrices* $\mathcal{N} \subset \mathbb{C}^{n\times n}$ *as a stratified submanifold of* $\mathbb{R}^{2n^2}$. *We construct a stratification of* $\mathcal{N}$ *in such a way that we obtain a very simple parametrization for the stratum having maximal dimension* n^2+n. *More precisely, using the Toeplitz decomposition, a generic normal matrix is of the form* $H + ip(H)$ *for a Hermitian matrix* H *and for a polynomial* p *with real coefficients. Consequently, this parametrization allows to approach several computational problems involving generic normal matrices in a new way at same time giving arise to various iterative schemes of the Arnoldi type. To give an example, we consider the problem of finding the eigenvalues, or some, of a generic normal matrix. It turns out that the degree of* p *becomes an important factor.*

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1 Introduction

The set of normal matrices, denoted by $\mathcal{N} \subset \mathbb{C}^{n\times n}$, is a rich class of matrices well-suited for numerical computations. To give an example of the computational well-behavior, extreme sensitivity of eigenvalues and eigenvectors does not occur among the set of normal matrices. The ways to characterize normality is a "rich set" as well. So far there exists about ninety equivalent conditions for a matrix to be normal collected in [9] by Grone, Johnson, Sa and Wolkowicz and in [4] by Elsner and Ikramov. The most standard definition of normality for a matrix $Z \in \mathbb{C}^{n\times n}$, or condition 0 as taken in [9], deals with the equation given by the self-commutator $[Z, Z^*] = ZZ^* - Z^*Z = 0$ for Z. Instead of considering different equivalent definitions of normality, in this paper we study $\mathcal{N}$ as a set. More precisely, we a view the set of normal matrices as a stratified submanifold of $\mathbb{R}^{2n^2}$.

It is not difficult to verify that $\mathcal{N}$ is a stratified submanifold of $\mathbb{R}^{2n^2}$. Instead, it is more difficult to construct a stratification of $\mathcal{N}$ that would be structure revealing as well as concrete enough to be useful in practical problems. Consequently, the purpose of this paper is to try to introduce a stratification that would, at least to some extent, have these properties. For this purpose we take as a starting point the Toeplitz decomposition, also called the Cartesian decomposition,

$$Z = H + iK, \text{ where } H = \frac{1}{2}(Z + Z^*) \text{ and } K = \frac{1}{2i}(Z - Z^*), \tag{1}$$

of a matrix $Z \in \mathbb{C}^{n\times n}$. Clearly H and K are both Hermitian matrices. As it is well-known, Z is normal if and only if H and K commute, see e.g. condition 21 in [9]. Thus, normality makes H and K in the Toeplitz decomposition strongly interdependent. Using this property, a way to achieve a simple stratification for the set of normal matrices is to let the Hermitian part H vary first and only thereafter consider all the possible K that commute with H. This is achieved as follows. For all possible sets of positive integers $k_l > 1$, for $l = 1, ..., j$, with the property $\sum_{l=1}^{j} k_l \leq n$ we constrain H to have exactly j eigenvalues with multiplicities $k_1, .., k_j$. Restricting H in such a way, we obtain a stratification of $\mathcal{N}$ with the strata of dimension $n^2+n-\sum_{l=1}^{j}(k_l-1)$. Thus, the maximal dimension $n^2 + n$ corresponds to the case when this set of integers is empty and H varies among the set of nondegoratory Hermitian matrices. The smallest dimension n^2+1 occurs when there is just one integer $k_1 = n$, i.e., $H = sI$ for $s \in \mathbb{R}$.

What makes the described stratification to be of potential use is the property that for the stratum of maximal dimension we obtain a very simple parametrization using the Toeplitz decomposition as follows. Denoting by $\mathcal{H}_0$ the set of nondegoratory Hermitian matrices, the stratum having the maximal dimension in the constructed stratification is given by the mapping

$$(H, \alpha_0, ..., \alpha_{n-1}) \to H + i\sum_{j=0}^{n-1} \alpha_j H^j \tag{2}$$

from $\mathcal{H}_0 \times \mathbb{R}^n$ to $\mathcal{N}$. We will demonstrate that, if the image of this mapping is denoted by $\mathcal{N}_0$, then $\mathcal{N}\backslash\mathcal{N}_0$ is a closed nowhere dense set in $\mathcal{N}$ in the norm topology inherited from $\mathbb{R}^{2n^2}$. Thus, a generic normal matrix Z is of the form $Z = H + ip(H)$ for $H \in \mathcal{H}_0$ and a polynomial p with real coefficients. In particular, the set of normal matrices can thus be characterized as

$$\operatorname{clos}\{H + ip(H) : H \in \mathcal{H},\ p \text{ is a polynomial with real coefficients}\}, \tag{3}$$

where $\mathcal{H}$ denotes the set of Hermitian matrices.

The above parametrization (2) of an open dense subset of $\mathcal{N}$ allows to approach a number of computational problems involving normal matrices in a new way. Consider, for instance, the problem of finding the spectrum, or a few eigenvalues of a large, possibly sparse, matrix $Z = H+iK = H+ip(H) \in \mathcal{N}$. This problem can be divided into simpler independent parts. One such is that of computing the spectrum of H accompanied with finding p. Once this is accomplished, the spectrum $\sigma(Z)$ of Z is obtained by simply applying the spectral mapping theorem to the spectrum of H. Obviously here the key is that for a Hermitian matrix, for solving its eigenvalues, or some of them, there exists a large variety of techniques and lot of different preconditioning strategies. To approximate p there, in turn, are several routes. The most inexpensive ones are simple variations of the Arnoldi method that involve only matrix-vector products. This approach becomes more attractive the smaller the degree of the polynomial p, or, the better p can be approximated by a low degree polynomial over $\sigma(H)$. Namely, then the problem of finding $\sigma(Z)$ reduces, in essence, to that of computing $\sigma(H)$. Consequently, in this manner the Kaniel-Paige convergence theory can be applied straightforwardly to normal matrices of this particular type. Furthermore, in this approach there is one canonical parameter that can be adjusted and which can chance the convergence drastically. That is a rotation $e^{i\theta}Z$ of Z with $\theta \in [0, 2\pi)$. Except for a finite number of θ, the rotated $e^{i\theta}Z$ has the representation $e^{i\theta}Z = H_\theta + ip_\theta(H_\theta)$ for $H_\theta \in \mathcal{H}$ and p_θ with real coefficients as well. The degrees of p and p_θ can, however, be very different. Consequently, the respective approximation problems for Z and $e^{i\theta}Z$ as just described can be completely dissimilar.

The paper is organized as follows. In Section 2 we present a stratification of $\mathcal{N}$ and construct a simple parametrization for an open dense set of $\mathcal{N}$. In Section 3 we outline examples of computational problems where the presented parametrization can be of use. We show how for a certain type of normal matrices the computation of the spectrum reduces essentially to finding the spectrum of a Hermitian matrix. In particular, with this parametrization many approximation problems involving normal matrices can be approach with modifications of the classical Arnoldi iteration.

2 A stratification of the set of normal matrices

So far there exists about ninety equivalent conditions for a matrix to be normal collected in [9] by Grone, Johnson, Sa and Wolkowicz and in [4] by Elsner and Ikramov. The standard definition of normality for $Z \in \mathbb{C}^{n\times n}$, or the so-called condition 0, deals with the equation given by the self-commutator

$$[Z, Z^*] = ZZ^* - Z^*Z = 0. \tag{4}$$

In particular, starting from this, algebraic as well as differential geometric interpretations come naturally. As to algebraic geometric point of view, obviously the elements of the matrix $[Z, Z^*]$ are not, because of the complex conjugation, polynomials with respect to complex variables $\{Z_{i,j}\}_{i,j=1}^n$ denoting the elements of Z. However, they are polynomials with respect to real and imaginary parts of $\{Z_{i,j}\}_{i,j=1}^n$, and, consequently, it is useful to regard the set of all complex $n\times n$ matrices as the real vector space $\mathbb{R}^{2n^2}$. If $Z = X + iY$, where $X, Y \in \mathbb{R}^{n\times n}$ denote the real and imaginary parts of Z respectively, then

$$[Z, Z^*] = XX^T - X^TX + YY^T - Y^TY + i(YX^T - X^TY + Y^TX - XY^T) \tag{5}$$

Let $f_{\Re}, f_{\Im} : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ be the real and imaginary parts of (5), that is,

$$f_{\Re}(X, Y) = XX^T - X^TX + YY^T - Y^TY \tag{6}$$

and

$$f_{\Im}(X, Y) = YX^T - X^TY + Y^TX - XY^T. \tag{7}$$

Requiring $[Z, Z^*] = 0$, the real part gives, because of symmetry, $(n^2 + n)/2$ polynomial equations of degree 2. Analogously, the imaginary part gives, because of skew-symmetry, $(n^2 - n)/2$ polynomial equations of degree 2. These, in all n^2 homogeneous polynomials, define an affine variety in $\mathbb{R}^{2n^2}$. Obviously $\mathcal{H}$, the set of Hermitian and $\mathcal{S}$, the set of Skew-Hermitian matrices are both n^2 dimensional subspaces of $\mathbb{R}^{2n^2}$ contained in $\mathcal{N}$, the set of normal matrices. These two subspaces are the building blocks of the constructed stratification of $\mathcal{N}$ in this paper.

A stratification Σ of a subset X of a manifold M is a partition of X into submanifolds of M, called the strata, which satisfies the local finiteness condition. That is to say, every point in X has a neighborhood in M that meets only finitely many strata. If $X \subset M$ can be stratified, X is called a stratified submanifold of M. For further definitions and properties of stratified submanifolds, see e.g. [1] or [7].

Proposition 1 *$\mathcal{N}$ is a connected star-shaped stratified submanifold of $\mathbb{R}^{2n^2}$.*

Proof. It is obvious that $\mathcal{N}$ is connected as each of its element is linearly path connected to zero matrix. By this argument $\mathcal{N}$ is obviously star-shaped. The set of normal matrices is the image of the mapping $(U, D) \to UDU^*$ (or the equivalent real mapping constructed in an obvious way). By Stratification Theorem [10], $\mathcal{N}$ is a stratified submanifold of $\mathbb{R}^{2n^2}$ as this mapping is real analytic and proper (compact sets have compact preimages). □

As to how actually construct a stratification of $\mathcal{N}$, there are several routes. For practical purposes we have chosen to start from the Toeplitz decomposition since it turns out that in this way we obtain a very simple parametrization for the stratum having maximal dimension. For that purpose, let $Z = H + iK$ denote the Toeplitz decomposition of a matrix Z as defined in (1). We call H and iK the Hermitian and Skew-Hermitian parts of Z respectively. Let $C(Z) = \{B \in \mathbb{C}^{n\times n} : BZ = ZB\}$ denote the centralizer of Z. It is obvious that $C(Z)$ is a subspace of $\mathbb{C}^{n\times n}$.

Lemma 2 *Assume $M \in \mathcal{H}$ and $B = H + iK$ commutes with M. Then $M + iK$ is normal. Conversely, every normal matrix with $M \in \mathcal{H}$ as its Hermitian part is of this form for some B commuting with M.*

Proof. Take a B from the centralizer of M. Since M is Hermitian, B^* commutes with M as well and, consequently, M commutes with $iK = (B - B^*)/2$. Thus $M + iK$ is normal by condition 21 in [9]. The converse holds trivially by the equivalence of the definition of normality. □

In what follows we will either consider real or complex dimension. This will be indicated by an inclusion either in $\mathbb{R}^{2n^2}$ or in $\mathbb{C}^{n\times n}$ respectively.

Lemma 3 *Assume $M \in \mathcal{H}$ and the dimension of $C(M) \subset \mathbb{C}^{n\times n}$ is l. Then the dimension of $\mathcal{S} \bigcap C(M) \subset \mathbb{R}^{2n^2}$ is l.*

Proof. It is obvious that $\mathcal{S} \bigcap C(M)$ is a subspace of $\mathbb{R}^{2n^2}$ so assume first that the dimension of $C(M) \subset \mathbb{C}^{n\times n}$ is l. Then suppose for $S_j \in \mathcal{S} \bigcap C(M) \subset \mathbb{R}^{2n^2}$, $j = 1, ..., l+1$, there do not exist $\alpha_j \in \mathbb{R}$ for $j = 1, ..., l+1$, of which at least one is nonzero such that $\sum_{j=1}^{l+1} \alpha_j S_j = 0$. Clearly, then there cannot exist any $\beta_j \in \mathbb{C}$, $j = 1, ..., l+1$, of which at least one is nonzero such that $\sum_{j=1}^{l+1} \beta_j S_j = 0$ either. This contradicts the assumption. Thus, the dimension of $\mathcal{S} \bigcap C(M) \subset \mathbb{R}^{2n^2}$ is at most l. Assume then that S_j, for $j = 1, ..., l-1$ is a basis of $\mathcal{S} \bigcap C(M) \subset \mathbb{R}^{2n^2}$ and $Z = H + iK \in C(M) \subset \mathbb{C}^{n\times n}$. Then for some $\alpha_j \in \mathbb{R}$ holds $\sum_{j=1}^{l-1} \alpha_j S_j = iH$ and for some $\beta_j \in \mathbb{R}$ holds $\sum_{j=1}^{l-1} \beta_j S_j = iK$ as by Lemma 2 both $iH, iK \in C(M)$. Thus, $\sum_{j=1}^{l-1} (-i\alpha_j + \beta_j) S_j = Z$ and, consequently, S_j, for $j = 1, ..., l-1$ is a basis of $C(M) \subset \mathbb{C}^{n\times n}$ as well. This, however, contradicts the assumption that the dimension of $C(M) \subset \mathbb{C}^{n\times n}$ equals l. The converse claim follows by the same reasoning. □

Let $\mathcal{H}(k_1, ..., k_j) \subset \mathcal{H}$ denote those Hermitian matrices that have *exactly* j eigenvalues with multiplicities $k_1, ..., k_j$ all strictly larger that 1. Further, let $\mathcal{H}_0$ denote the set of nondegoratory Hermitian matrices.

Proposition 4 *$\mathcal{H}(k_1, ..., k_j) \subset \mathbb{R}^{2n^2}$ is a smooth manifold of dimension $n^2 + \sum_{l=1}^{j} (1 - k_l^2)$.*

Proof. The dimension of the set of unitary matrices $\mathcal{U}$ as a real smooth manifold is n^2. Fix an element $U \in \mathcal{U}$ and consider those unitary matrices

V obtained via

$$V = U \begin{bmatrix} V_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & V_j & 0 \\ 0 & \dots & 0 & \Phi \end{bmatrix} \tag{8}$$

for $V_l \in \mathbb{C}^{k_l \times k_l}$, for $1 \leq l \leq j$, unitary and $\Phi = \operatorname{diag}(\theta_1, ..., \theta_{n-p})$, where $p = \sum_{l=1}^{j} k_l$ and $|\theta_l| = 1$ for $1 \leq l \leq p$. This set is a smooth manifold of real dimension $\sum_{l=1}^{j} k_l^2 + (n-p)$ since the set of unitary matrices in each $\mathbb{C}^{k_l \times k_l}$ is k_l^2-dimensional and the product space of $n-p$ unit circles is $(n-p)$-dimensional manifold.

Let X be the open set in $\mathbb{R}^{n-p+j}$ defined as the complement of the inverse image of 0 of the function $(\lambda_1, ..., \lambda_{n-p+j}) \to \prod_{j=2}^{n-p+j} (\lambda_j - \lambda_{j-1})$ from $\mathbb{R}^{n-p+j}$ to $\mathbb{R}$. Identify X with those diagonal matrices D that have the first j blocks equaling eigenvalues in each block of size k_l for $1 \leq l \leq j$. The remaining eigenvalues are all different from each as well as from those in the first j blocks. It should be clear that $\mathcal{H}(k_1, ..., k_j)$ equals the image of the mapping $(U, D) \to UDU^*$ from $\mathcal{U} \times X$.

For a fixed D and for arbitrary $U, V \in \mathcal{U}$ there holds $UDU^* = VDV^*$ if and only if U and V are related as in (8). Consequently, the dimension of the image of the mapping $(U, D) \to UDU^*$ is

$$n^2 - \sum_{l=1}^{j} k_l^2 - (n-p) + (n-p+j) = n^2 - \sum_{l=1}^{j} k_l^2 + j$$

and the claim follows as it is obviously smooth as well. □

Proposition 5 *If $H \in \mathcal{H}(k_1, ..., k_j)$, then the dimension of $\mathcal{S} \bigcap C(H) \subset \mathbb{R}^{2n^2}$ is $n + \sum_{l=1}^{j} k_l(k_l - 1)$.*

Proof. Let $H = U\Lambda U^*$ be a diagonalization of H by a unitary similarity. Then $BH = HB$ is equivalent to $U^*BU\Lambda = \Lambda U^*BU$, that is, we can consider the centralizer of Λ and then use this unitary similarity to get the centralizer of H. A block, say, of size k_l has the centralizer of dimension k_l^2 as a subspace of $\mathbb{C}^{k_l \times k_l}$ simply because all the matrices of respective size commute with this block. The corresponding Skew-Hermitian subspace is k_l^2 dimensional in $\mathbb{R}^{2k_l^2}$ by Lemma 3. Thus continuing in this manner we obtain a subspace of dimension $\sum_{l=1}^{j} k_l^2$. Then the remaining eigenvalues are all different, i.e., this block is nondegoratory. Thus, for this block the dimension of the centralizer is $n - \sum_{l=1}^{j} k_l$ [11][p. 275] in $\mathbb{C}^{(n-p)\times(n-p)}$, where $p = \sum_{l=1}^{j} k_l$. Consequently, using again Lemma 3 with this block we obtain the claim after an addition. □

The set of nondegoratory Hermitian matrices $\mathcal{H}_0$ is of interest for the following reason.

Proposition 6 *Assume $H \in \mathcal{H}_0$ and p is a polynomial with real coefficients. Then $H + ip(H)$ is normal. Conversely, every normal matrix with $H \in \mathcal{H}_0$ as its Hermitian part is of this form for a polynomial p with real coefficients.*

Proof. The first claim is obvious. For the converse, let K be Hermitian. By condition 21 in [9] $H+iK$ is normal if and only if $K \in C(H)$. Since H is nondegoratory, $K = p(H)$ for a polynomial p, see e.g. [11][p. 275]. If p is not real, then taking the real part $p_{\Re}$ of p (i.e., $p = p_{\Re} + ip_{\Im}$ where both $p_{\Re}$ and $p_{\Im}$ are polynomials with real coefficients) gives the claim. □

Let $\mathcal{N}_0$ denote the set of normal matrices having nondegoratory Hermitian part.

Theorem 7 *$\mathcal{N}\backslash\mathcal{N}_0$ is a closed nowhere dense set in $\mathcal{N}$ in the norm topology inherited from $\mathbb{R}^{2n^2}$.*

Proof. Let $\lambda_j(A)$ denote the eigenvalues of a $A \in \mathbb{C}^{n\times n}$, counting multiplicities, arranged in decreasing order. The function $A = H + iK \to \prod_{j=1}^{n-1}(\lambda_j(H) - \lambda_{j+1}(H))$ is continuous from $\mathbb{C}^{n\times n}$ to $\mathbb{R}$. Thereby the inverse image of 0 for this function is a closed set in $\mathbb{C}^{n\times n}$. Since we use the inherited topology, its intersection with $\mathcal{N}$, which obviously equals $\mathcal{N}\backslash\mathcal{N}_0$, is a closed set in $\mathcal{N}$.

Assume $Z \in \mathcal{N}\backslash\mathcal{N}_0$. We need to show that there is $Z_\epsilon \in \mathcal{N}_0$ arbitrarily close to Z. Let H be the Hermitian part of Z. Assume Z has j eigenvalues $\lambda_1, ..., \lambda_j$ with multiplicities strictly larger than one. Let $\hat{\Lambda}$ denote the remaining eigenvalues. Let U be a unitary matrix diagonalizing H such that the diagonal blocks corresponding to $\lambda_1, ..., \lambda_j$ come first. As reasoned in the proof of Proposition 5, then all normal matrices with H as their Hermitian part are of the form

$$U \begin{bmatrix} \Lambda_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \Lambda_j & 0 \\ 0 & \dots & 0 & \hat{\Lambda} \end{bmatrix} U^* + iU \begin{bmatrix} S_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & S_j & 0 \\ 0 & \dots & 0 & p(\hat{\Lambda}) \end{bmatrix} U^*,$$

where $S_1, ..., S_j$ are Hermitian matrices and p is a polynomial with real coefficients. Assume Z is put in this form and diagonalize, for $1 \le l \le j$, each S_l (block-wise) using a unitary transformation U_l to get

$$\hat{U} \begin{bmatrix} \Lambda_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \Lambda_j & 0 \\ 0 & 0 & 0 & \hat{\Lambda} \end{bmatrix} \hat{U}^* + i\hat{U} \begin{bmatrix} \hat{S}_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \hat{S}_j & 0 \\ 0 & 0 & 0 & p(\hat{\Lambda}) \end{bmatrix} \hat{U}^*, \tag{9}$$

where $\hat{S}_1, ..., \hat{S}_j$ are diagonal matrices and where $\hat{U} = U\mathrm{blockdiag}(U_1, ..., U_j, I)$ is unitary. Then perturb each diagonal element in each $\Lambda_1, ..., \Lambda_j$ slightly to get a nondegoratory Hermitian matrix $\hat{U}(\Lambda + \epsilon)\hat{U}^*$. This resulting perturbed matrix Z_ϵ of Z remains also normal since its Hermitian part commutes with its Skew-Hermitian part and the claim follows as the perturbation can be made arbitrarily small. The polynomial p_ϵ in $Z_\epsilon = H_\epsilon + ip_\epsilon(H_\epsilon)$ is found via interpolation. □

In particular, the set of normal matrices can be characterized as follows.

Corollary 8 clos$\mathcal{N}_0 = \mathcal{N}$ *and, in particular,*

$$\text{clos}\{H + ip(H) : H \in \mathcal{H},\ p \text{ is a polynomial with real coefficients}\} = \mathcal{N} \tag{10}$$

and $\{p(H) : H \in \mathcal{H},\ p \text{ is a polynomial}\} = \mathcal{N}$.

Proof. Only $\{p(H) : H \in \mathcal{H},\ p$ is a polynomial$\} = \mathcal{N}$ needs to be shown. This follows from Proposition 12. □

An immediate question is, how about $\mathcal{L}(H)$, the set of bounded linear operators on a separable Hilbert space H? Considering a normal operator Z with a hole in its spectrum it can be seen that (10) cannot not hold. The reason is that the spectrum function is uniformly continuous on the set of normal elements (and thus it is not possible to interpolate with a polynomial "over a hole"), see e.g. [3]. This is possibly the reason why we have not encountered this way of viewing normal matrices as almost all concepts for normality stem from operator theory for $\mathcal{L}(H)$. The closure is however needed and, in fact, clos$\{p(H) : H \in \mathcal{H},\ p$ is a polynomial$\} = \mathcal{N}$ can be shown to be true. To not to get distracted we do not include a proof here.

For $\mathcal{N}_0$ we obtain a smooth structure in simple manner from Proposition 6 and Theorem 7.

Corollary 9 $\mathcal{N}_0 \subset \mathbb{R}^{2n^2}$ *is a smooth connected manifold of dimension* $n^2 + n$.

Proof. Form a mapping

$$(H, \alpha_0, ..., \alpha_{n-1}) \to H + i\sum_{j=0}^{n-1} \alpha_j H^j \tag{11}$$

from $\mathcal{H}_0 \times \mathbb{R}^n$ to $\mathcal{N}_0$. This is bijective, since H is nondegoratory and the polynomial is of degree $n-1$ at most. Also it is clearly smooth. As to the connectedness, suppose $N_1, N_2 \in \mathcal{N}_0$. Thus, $N_1 = U\Lambda_1 U^* + iUp(\Lambda_1)U^*$ and $N_2 = V\Lambda_2 V^* + iVq(\Lambda_2)V^*$ for some $U, V \in \mathcal{U}$ and some polynomials p and q with real coefficients. Since $\mathcal{U}$ is path-connected (every unitary Q is of the form $Q = e^{iE}$ for a Hermitian matrix E. Thus $Q_t = e^{itE}$, for $0 \le t \le 1$, connects Q to the identity matrix.), V can be connected with a path to U. Since Λ_1 and Λ_2 are both sets with n distinct elements, they can be transformed smoothly to one another such that the amount of distinct points remains equal to n during the process. And finally, the coefficients of p and q can be smoothly transformed to one another. □

Considering $\mathcal{N}_0$ as the stratum of maximal dimension, let $\mathcal{N}(k_1, ..., k_j) \subset \mathcal{N}$ denote those normal matrices $N = H + iK$ that have $H \in \mathcal{H}(k_1, ..., k_j)$ as their Hermitian part. The union of these sets with $\mathcal{N}_0$ gives the set of normal matrices. The following shows that they provide a stratification of $\mathcal{N}$ as well.

Theorem 10 *The set $\mathcal{N}(k_1, ..., k_j) \subset \mathbb{R}^{2n^2}$ is a smooth manifold of dimension $n^2 + n - \sum_{l=1}^{j}(k_l - 1)$.*

Proof. $\mathcal{H}(k_1, ..., k_j)$ is a smooth manifold of dimension $n^2 + \sum_{l=1}^{j}(1 - k_l^2)$ by Proposition 4 and in Proposition 5 we demonstrated that for a fixed $H \in \mathcal{H}(k_1, ..., k_j)$ the dimension of $\mathcal{S} \bigcap C(H) \subset \mathbb{R}^{2n^2}$ is $n + \sum_{l=1}^{j} k_l(k_l - 1)$. If $H = U\Lambda U^* \in \mathcal{H}(k_1, ..., k_j)$, then in the proof of Proposition 5 we showed that the centralizer of H was the direct sum of full matrix algebras

$$U(\mathcal{M}_{k_1} \oplus \cdots \oplus \mathcal{M}_{k_l} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C})U^*. \tag{12}$$

In particular, $C(H)$ is independent on the numerical values of the eigenvalues of Λ as long as they are constrained to have fixed multiplicities in this order such that $\Lambda \in \mathcal{H}(k_1, ..., k_j)$. Further, the computation of the Skew-Hermitian part from (12) is a smooth operation. Thus, when U and Λ vary smoothly, the smoothness of the structure follows. □

In particular, $\mathcal{N}(n)$ is connected and consists of matrices $sI + iH$ for $s \in \mathbb{R}$ and $H \in \mathcal{H}$ and is the stratum of the least dimension $n^2 + 1$. A manifold $\mathcal{N}(k_1, ..., k_j)$ is not connected unless $k_1 = k_2 = \cdots = k_j$ and $\sum_{l=1}^{j} k_l = n$. In case there are varying multiplicities, individual components of $\mathcal{N}(k_1, ..., k_j)$ differ only in the ordering of the eigenvalues of the Hermitian part with multiplicities. The reason for that they cannot be connected to one another is that when trying to move from one component to another, i.e., when trying to change the ordering of the multiplicities of the eigenvalues of $H \in \mathcal{H}(k_1, ..., k_j)$ on $\mathbb{R}$, some eigenvalues of will coalesce. This in turn means that the matrix has entered into another manifold $\mathcal{N}(p_1, ..., p_l)$ with the indices $p_1, ..., p_l$ corresponding to the arisen coalescence. While $\text{clos}\mathcal{N}_0 = \mathcal{N}$, the other strata have the following property when taking the closure.

Proposition 11 *Let $\{p_{s_1}\}, \{p_{s_2}\}, ..., \{p_{s_j}\}$ be a partition of $\{p_1, ..., p_l\}$ such that $\sum p_{s_m} \leq k_m$, for $1 \leq m \leq j$. Then $\mathcal{N}(k_1, ..., k_j) \subset \text{clos}\mathcal{N}(p_1, ..., p_l)$.*

Proof. Take $Z \in \mathcal{N}(k_1, ..., k_j)$ and assume Z has been decomposed as in (9). It should be obvious how an element of $\mathcal{N}(p_1, ..., p_l)$ close to Z is now constructed: Slightly vary each block $\Lambda_1, ..., \Lambda_j$ appropriately so as to get the right amount of eigenvalues with multiplicities $p_1, ...p_l$. □

Thus, generically one can view a normal matrix Z to be of the form $H + ip(H)$ for a Hermitian matrix H and a polynomial p with real coefficients. As to applications, the degree of p becomes an important factor. Somewhat overstating, one could say that the smaller the degree of p, the more Hermitian Z is, as far as Krylov methods are concerned. In the following section we describe problems where this approach can be useful.

3 Applications to problems involving normal matrices

With a simple parametrization for an open dense set of $\mathcal{N}$ it is possible to solve approximation problems involving normal matrices in a new way. To demonstrate this we outline an approach for two well-known examples: The eigenvalue problem and the problem of finding a closest normal approximant to a matrix $A \in \mathbb{C}^{n\times n}$.

Instead of starting from the Hermitian part of $Z \in \mathcal{N}$, the computations can, of course, be performed with the Skew-Hermitian part of Z. That is, a generic $Z = H + iK \in \mathcal{N}$ can be presented either as $Z = H + ip(H)$ or as $Z = q(K) + iK$ for polynomials p and q with real coefficients. And more generally, for the rotated $e^{i\theta}Z$ with a $\theta \in [0, 2\pi)$, this type of representation does exist except in very exceptional cases. To illustrate this, let us denote by H_θ the Hermitian part of $e^{i\theta}Z$.

Proposition 12 *Assume Z is normal. Then for θ belonging to an open dense subset of $[0, 2\pi)$ holds $e^{i\theta}Z = H_\theta + ip_\theta(H_\theta)$ a polynomial p_θ with real coefficients.*

Proof. Draw lines through each pair of eigenvalues of Z. Each rotation for which there are no vertical lines one can construct a polynomial p_θ such that $e^{i\theta}Z = H_\theta + ip_\theta(H_\theta)$ by interpolation as in the proof of Theorem 7. □

Recall that H_θ appears while approximating the field of values of a (not necessarily normal) matrix Z. Namely then, for a finite number of different θ, one computes the largest eigenvalue of H_θ and intersects certain half-planes defined on the basis of these eigenvalues, see e.g. [11][Thm 1.5.12, 1.5.14]. For a normal Z the rotation obviously affects the degree of p_θ in the representation $e^{i\theta}Z = H_\theta + ip_\theta(H_\theta)$. Let us illustrate this with a simple example.

Example 1. Assume $Z = H+iK$ is normal and $\sigma(Z)$ lies on the parabola $x = y^2$ such that $\sigma(K)$ in indefinite. If Z can be presented as $H+ip(H)$ with a polynomial with real coefficients, then the degree of p can be quite high, even n. Whereas for $e^{i\frac{\pi}{2}}Z$ the representation exists and $e^{i\frac{\pi}{2}}Z = H_{\frac{\pi}{2}} + ip_{\frac{\pi}{2}}(H_{\frac{\pi}{2}})$ is obtained for polynomial $p_{\frac{\pi}{2}}$ of degree 2.

Definition 13 *Let $Z \in \mathbb{C}^{n\times n}$ and $\theta \in [0, 2\pi)$. Then $Z = e^{-i\theta}H_\theta + ie^{-i\theta}K_\theta$ is the rotated Toeplitz decomposition of Z by the angle θ.*

Obviously the self-adjointness of the parts are lost in this decomposition.

3.1 Spectral approximation

Consider the problem of computing an approximation to the eigenvalues, or some, of a normal matrix Z. Assuming Z to be generic there holds $Z = H + iK = H + ip(H)$ for $H \in \mathcal{H}$ and for a polynomial p with real coefficients. Or, based on some a priori information, a rotated Toeplitz decomposition of Z can be used so as to have this type of representation. Below we will show how to pick a rotation for this purpose.

Having $Z = H+ip(H)$, it is apparent that H is readily computed whereas p is not available. Proceeding with H and finding an eigenvector, say x, of H yields an eigenvalue of Z after evaluating Zx. This is clearly not very practical approach as it gives eigenvalues of Z almost randomly. To avoid this, some information about p is needed and, consequently, p has to be approximated in some manner. This can be obtained from a straightforward modification of the Arnoldi method [2] for H without any additional cost when the spectrum of H is approximated with the same method.

The well-known classical Arnoldi method for the eigenvalue approximation starts from the construction of a Krylov subspace with $A \in \mathbb{C}^{n\times n}$ and a vector $b \in \mathbb{C}^n$. Then a monic polynomial q_k of degree k is computed with the property that $\|q_k(A)b\|$ is minimized over all monic polynomials of degree k, that is,

$$\|q_k(A)b\| = \min_{\alpha_1,\dots,\alpha_k \in \mathbb{C}} \|(A^k - \sum_{i=1}^{k} \alpha_i A^{k-i})b\|. \tag{13}$$

The roots of the polynomial q_k are then taken as an approximate eigenvalues of A. For more information of the Arnoldi method for the eigenvalue problems, see e.g. [14]. An eigenvalue approximation for $Z = H + ip(H)$ involving the Hermitian part H of Z can be obtained as follows. Compute polynomials q_k and p_k via

$$\|q_k(H)b\| = \min_{\alpha_1,\dots,\alpha_k \in \mathbb{C}} \|(H^k - \sum_{i=1}^{k} \alpha_i H^{k-i})b\| \tag{14}$$

and

$$\|Kb - p_k(H)b\| = \min_{\beta_1,\dots,\beta_k \in \mathbb{R}} \|(K - \sum_{i=1}^{k} \beta_i H^{k-i})b\|. \tag{15}$$

That is, q_k and p_k are constructed to approximate the eigenvalues of H and the polynomial p respectively. Then an eigenvalue approximation for Z is obtained by applying the spectral mapping theorem with polynomial $z + ip_k(z)$ to the eigenvalue approximation obtained with q_k for H in (14). Obviously (15) can be replaced with the approximation

$$\|Zb - \hat{p}_k(H)b\| = \min_{\beta_1,\dots,\beta_k \in \mathbb{C}} \|(Z - \sum_{i=1}^{k} \beta_i H^{k-i})b\| \tag{16}$$

instead. The spectral mapping theorem is then applied with $\hat{p}_k$ to the computed approximation of $\sigma(H)$. Altogether, a single Krylov subspace with H applied to b is generated. Then, from this Krylov subspace, two approximations are constructed, that is, for $H^k b$ as well as Kb (or for Zb).

Note that all the above minimization problems involve commuting normal matrices so that max-min property holds [8]. Thus, in this respect the approximation problems (14) and (16) are equivalent. As opposed to power method, for obvious reasons we are tempted to call this method a *fractional power* Krylov subspace method for Z. Namely, if $Z = H + ip(H)$ for a polynomial p of degree $l \geq 1$, then the fractional power can be considered to be $1/l$.

The outlined approach to the computation of an approximation to the spectrum depends strongly on the polynomial p in $Z = H + ip(H)$. If p is of low degree, then the computation of $\sigma(Z)$ reduces essentially to that of finding $\sigma(H)$. For instance, if the degree of p is one, then the eigenvalues lie on a line and only two dimensional Krylov subspace is needed for finding p. Or, if $\sigma(Z)$ lies on a parabola $y = ax^2+bx+c$ with $a, b, c \in \mathbb{R}$, then the needed dimension is 3. Or, more generally, if p can be approximated well with a low degree polynomial over an interval containing $\sigma(H)$, then the computation of $\sigma(Z)$ reduces essentially to finding the eigenvalues of H. In particular, finding a good rotation parameter $\theta \in [0, 2\pi)$ for a rotated Toeplitz decomposition can make a big difference as demonstrated in Example 1. A way to find one is to test with a few rotations, with a small k, how the corresponding minimization problem (15) (or (16)) does behave. Based on this, the rotation giving the smallest value will then be chosen.

If only certain, like interior, eigenvalues of Z are being computed, then the computation should be divided into two parts. First, one needs to find an approximation to p. This happens as just described above. When an approximative p is computed, then with H it is possible to use preconditioning techniques in order to locate interior eigenvalues. Clearly these tasks can be performed completely independently and parallel. It is also apparent that for finding a good "local" approximation to p, that is, an approximation that is good over a certain part of $\sigma(H)$, it is possible to use preconditioning. The most elementary approach is to use inverse iteration type of algorithms with a translated Z. Then, obviously, a priori knowledge of the location of searched eigenvalues must be available.

Another way of computing an approximation to $\sigma(Z)$ can be based on finding an approximation to the eigenvalues of H and K separately. Then computing p will reveal how $\sigma(H)$ and $\sigma(K)$ are connected. This is based on the fact that if $\sigma(H) = \{\alpha_1, ..., \alpha_n\}$ and $\sigma(K) = \{\beta_1, ..., \beta_n\}$, then there exists a permutation $\sigma \in S_n$ such that $\sigma(Z) = \{\alpha_1 + i\beta_{\sigma(j)} : j = 1, ..., n\}$, see condition 34 in [9]. A possible advantage of this approach is that if the eigenvalues of Z are well separated, then the accuracy requirement for an approximation to p need not be high.

Finally, a related problem to finding the spectrum is that of finding an appropriate polynomial preconditioner for a linear system involving $Z = H + ip(H)$. If p is well-approximated by a low degree polynomial over the spectrum of H, then with a very small amount of work based on using (14) or (15) one obtains a good overview of the spectrum of Z. Namely knowing that $\sigma(Z)$ lies almost on a curve defined by a computed polynomial, it is straightforward to construct a polynomial preconditioner for Z.

3.2 Closeness problems

Inexpensive approximative solutions to departure from normality in the sense of Henrici have been derived by S. Lee [12]. Another way to measure of normality is to find a closest normal matrix to $A \in \mathbb{C}^{n\times n}$. In the Frobenius norm this was solved by R. Gabriel [6] and A. Ruhe [13]. Though, a computation of an approximation is fairly expensive. As we have demonstrated, this problem can be stated as

$$\inf_{H\in\mathcal{H},\alpha_0,\dots,\alpha_{n-1}\in\mathbb{R}} \|A - H - i\sum_{j=0}^{n-1}\alpha_j H^j\|_{\mathcal{F}} \tag{17}$$

or as

$$\min_{H\in\mathcal{H},\alpha_0,\dots,\alpha_{n-1}\in\mathbb{C}} \|A - \sum_{j=0}^{n-1}\alpha_j H^j\|_{\mathcal{F}} \tag{18}$$

instead of introducing a minimization problem with constraints. In particular, approaching the problem with the parametrization (11) allows to derive inexpensive approximative solutions to the problem of finding a closest normal matrix to A. One such is obtained by making an "initial guess" $H \in \mathcal{H}$ and then applying an Arnoldi type of iteration, that is, using matrix-vector products only. More precisely, one looks for

$$\min_{\alpha_1,\dots,\alpha_k\in\mathbb{C}} \|(A - \sum_{i=1}^{k}\alpha_i H^{k-i})b\| \tag{19}$$

with a vector $b \in \mathbb{C}$. Thus, generated approximations are of the form $p(H)$ for polynomials p of degree k at most. An interesting problem is, how to choose an H to start with. After choosing an H the computation of a normal approximant with (19) is relatively inexpensive and, consequently, testing with a number of different initial guesses becomes feasible. Again, "natural" choices are, perhaps, H_θ from rotated Toeplitz decompositions for A with a few values of $\theta \in [0, 2\pi)$.

4 Conclusions

In this paper we have presented a stratification of the set of normal matrices. The stratification is constructed in such way that the parametrization for the stratum having maximal dimension is readily available. The parametrization is simple enough to be of interest also in computational problems involving normal matrices. In particular, we have described how it is possible to approximate eigenvalues of a generic normal matrix Z by solving two approximation problems in a Krylov subspace for the Hermitian part of Z. This and other problems involving approximations with normal matrices lead to different kind of iterations of the Arnoldi type.

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