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GENERALIZED ELECTROMAGNETIC SCATTERING IN A COMPLEX GEOMETRY

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Abstract: *We consider generalized time-harmonic Maxwell's equations on a real manifold of arbitrary dimension. Since the field tensors have complex coefficients the manifold is endowed with complex tangent and cotangent bundles and a complex valued pseudo-Riemannian metric. The lack of geodesics in general forces us to a restricted and careful use of standard differential geometric methods. We apply our machinery to scattering by a bounded body. As the main result we prove that the existence and uniqueness of a solution to an exterior boundary value problem is independent of the metric. This study originates from the Perfectly Matched Layer or PML technique in computational electromagnetics.*

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1 Introduction

Time-harmonic Maxwell's equations are fundamental tools in electrical engineering and they have an ample variety of applications, for instance, in communication technology and geophysical exploration. This work has its roots in computational electromagnetics and, on the other hand, in a theoretical work by Hermann Weyl in the early 1950's.

We are considering electromagnetic scattering by a bounded obstacle. Field computation using finite elements gives rise to the problem of mesh termination with minimal reflections. In 1994 Bérenger (see [1]) introduced a material absorbing boundary condition called perfectly matched layer (PML): the computational domain is surrounded by a layer of imaginary PML material. Indeed, the PML layer can be regarded as complex stretching of the spatial coordinates in the imaginary direction (see [13]). Technically speaking, around the computational domain the originally Cartesian metric tensor is stretched to a complex valued pseudo-Riemannian metric. A sophisticated analysis of the coordinate stretching is found in [6] for scalar waves.

Another remarkable improvement within computational electromagnetics was carried out by Bossavit in the late eighties (see [2]). He described the so called Whitney elements which are vector elements of various dimensions. In fact, Whitney elements can be thought of as discretized differential forms and as such they are most applicable when building accurate computational models for electromagnetic fields.

Summa summarum, there is demand for a differential geometric electromagnetic scattering theory in a complex geometry. In [7] we develop such a theory in $\mathbb{R}^3$ equipped with a complex metric. This paper is devoted to an arbitrary dimensional generalization based on Weyl's research in 1952 (see [14]) which was continued by Picard in 1985 (see [10]). As Picard writes, the multidimensional theory "reveals the structural beauty of Maxwell's equations".

2 Maxwell's Operator in a Complex Geometry

Let M be an n-dimensional real C^∞-manifold endowed with a complex tangent bundle TM and a complex valued pseudo-Riemannian metric g_{jl}. It is also required that there exists a global relative scalar $\sqrt{g}$ for the determinant $g := \det(g_{jl})$ (see Appendix).

From now on

$$\begin{aligned} p, q &\in \{0, 1, \ldots, n-1\}, \\ n &= p+q+1, \\ \bar{r} &:= n - r, \\ (-)^r &:= (-1)^r. \end{aligned}$$

When we introduce a p-form τ on M by

$$\tau \ := \ \frac{1}{p!}\tau_{j_1\ldots j_p}\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_p}$$

we implicitly assume that $\tau_{j_1\ldots j_p}$ is totally antisymmetric. Hence

$$\tau \ = \ \tau_{j_1\ldots j_p}\mathrm{d}x^{j_1}\otimes\ldots\otimes\mathrm{d}x^{j_p}.$$

Our purpose is not to plunge into the depths of Sobolev spaces; therefore every tensor field is presumed to be of class C^∞ unless explicitly stated otherwise.

We define a covariant p-curl operator by

$$\mathrm{Curl}_p : X_{j_1\ldots j_p} \mapsto \frac{1}{p!}\,\varepsilon^{uj_1\ldots j_p l_1\ldots l_q}X_{j_1\ldots j_p;u}$$

and seek out its connection with the exterior derivative operator d. Note that $\mathrm{Curl}_p X$ is totally antisymmetric for all p-covectors X.

Lemma 2.1 *Let $A^{j_1\ldots j_{p+1}}$ be a totally antisymmetric array, B^h_{jl} an array symmetric in the lower indices j, l, and $C_{j_1\ldots j_p}$ an arbitrary array. Then*

$$A^{uj_1\ldots j_p}\sum_{r=1}^{p}B^{h_r}_{j_r u}C_{j_1\ldots h_r\ldots j_p} = 0. \tag{1}$$

Proof: The claim is obvious for $p=0$ and $p=1$. Assume that (1) holds for some $p\geq 1$. Then

$$A^{uj_1\ldots j_p j_{p+1}}\sum_{r=1}^{p+1}B^{h_r}_{j_r u}C_{j_1\ldots h_r\ldots j_{p+1}} \ =$$
$$A^{uj_1\ldots j_p j_{p+1}}\sum_{r=1}^{p}B^{h_r}_{j_r u}C_{j_1\ldots h_r\ldots j_p j_{p+1}} + A^{uj_1\ldots j_p j_{p+1}}B^{h_{p+1}}_{j_{p+1}u}C_{j_1\ldots j_p h_{p+1}}.$$

The former term vanishes according to (1) (consider a fixed j_{p+1}). The latter term vanishes since

$$A^{uj_1\ldots j_p j_{p+1}}B^{h_{p+1}}_{j_{p+1}u} = -A^{j_{p+1}j_1\ldots j_p u}B^{h_{p+1}}_{uj_{p+1}}.$$

□

Corollary 2.2 *For a covariant p-tensor $X_{j_1\ldots j_p}$ on M*

$$\varepsilon^{uj_1\ldots j_p l_1\ldots l_q}X_{j_1\ldots j_p;u} = \varepsilon^{uj_1\ldots j_p l_1\ldots l_q}\frac{\partial}{\partial x^u}X_{j_1\ldots j_p}. \tag{2}$$

Proof: By definition the covariant derivative is

$$X_{j_1\ldots j_p;u}=\frac{\partial}{\partial x^u}X_{j_1\ldots j_p}-\sum_{r=1}^{p}\left\{\begin{matrix}h_r\\ j_r\ u\end{matrix}\right\}X_{j_1\ldots h_r\ldots j_p}.$$

Choose

$$\begin{aligned}A^{uj_1\ldots j_p}&=\varepsilon^{uj_1\ldots j_pl_1\ldots l_q},\\ B^{h_r}_{j_ru}&=\left\{\begin{matrix}h_r\\ j_r\ u\end{matrix}\right\},\\ C_{j_1\ldots h_r\ldots j_p}&=X_{j_1\ldots h_r\ldots j_p}\end{aligned}$$

in (1) to get

$$\varepsilon^{uj_1\ldots j_pl_1\ldots l_q}\sum_{r=1}^{p}\left\{\begin{matrix}h_r\\ j_r\ u\end{matrix}\right\}X_{j_1\ldots h_r\ldots j_p}=0.$$

□

Lemma 2.3 *For p-forms τ on M*

$$*G\mathrm{Curl}_p\tau=(-)^{p(n-p)}\mathrm{d}\tau. \tag{3}$$

Proof: Let

$$X:=X_{j_1\ldots j_p}\mathrm{d}x^{j_1}\otimes\ldots\otimes\mathrm{d}x^{j_p}.$$

From the definition of Curl_p and (2) we obtain

$$\mathrm{Curl}_p(p!X)=\varepsilon^{uj_1\ldots j_pl_1\ldots l_q}\frac{\partial}{\partial x^u}X_{j_1\ldots j_p}\frac{\partial}{\partial x^{l_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{l_q}}.$$

After lowering indices we have

$$\begin{aligned}G\mathrm{Curl}_p(p!X)&=\varepsilon^{uj_1\ldots j_p}{}_{l_1\ldots l_q}\frac{\partial}{\partial x^u}X_{j_1\ldots j_p}\mathrm{d}x^{l_1}\otimes\ldots\otimes\mathrm{d}x^{l_q}\\ &=\frac{\partial}{\partial x^u}X_{j_1\ldots j_p}\frac{1}{q!}\varepsilon^{uj_1\ldots j_p}{}_{l_1\ldots l_q}\mathrm{d}x^{l_1}\wedge\ldots\wedge\mathrm{d}x^{l_q}\\ &=\frac{\partial}{\partial x^u}X_{j_1\ldots j_p}*(\mathrm{d}x^u\wedge\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_p})\\ &=*\mathrm{d}(X_{j_1\ldots j_p}\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_p}).\end{aligned}$$

Let

$$\tau\ :=\ \frac{1}{p!}\tau_{j_1\ldots j_p}\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_p}\ =\ \tau_{j_1\ldots j_p}\mathrm{d}x^{j_1}\otimes\ldots\otimes\mathrm{d}x^{j_p}$$

be an arbitrary p-form. As we just saw

$$G\mathrm{Curl}_p(p!\tau)=*\mathrm{d}(p!\tau).$$

Applying $(p!)^{-1}*$ to both sides we conclude that

$$*G\mathrm{Curl}_p\tau\ =\ **\mathrm{d}\tau\ =\ (-)^{p(n-p)}\mathrm{d}\tau.$$

□

In the following

$$\begin{aligned} J &= (j_1, \ldots, j_p), && 1 \leq j_1 < \ldots < j_p \leq n, \\ K &= (k_1, \ldots, k_{p+1}), && 1 \leq k_1 < \ldots < k_{p+1} \leq n, \end{aligned}$$

are ordered multi-indices of lengths $\#J = p$ and $\#K = p+1$. For instance,

$$\begin{aligned} \delta_K^{uJ} &:= \delta_{k_1 \ldots k_{p+1}}^{uj_1 \ldots j_p}, \\ \mathrm{d}x^K &:= \mathrm{d}x^{k_1} \wedge \ldots \wedge \mathrm{d}x^{k_{p+1}}. \end{aligned}$$

Corollary 2.4 *For p-forms τ on M*

$$\mathrm{d}(\tau_J \mathrm{d}x^J) = (-)^{n-1} q! \delta_K^{uJ} \tau_{J;u} \mathrm{d}x^K. \tag{4}$$

Proof: Straightforward from Lemma 2.3. □

Remark 2.5 Corollary 2.4 ensures that $\mathrm{d}\tau = 0$ for all parallel forms τ. In classical differential geometry this can be seen by choosing a geodetic coordinate system at a point x. Then the Christoffel symbols all vanish at x. In the case of a complex valued metric it is usual that there are no geodesics.

The operators $*\mathrm{d}*\mathrm{d}$ and $\mathrm{d}*\mathrm{d}*$ can be expressed by means of second covariant derivatives:

Lemma 2.6 *Let*

$$\tau := \frac{1}{p!} \tau_{j_1 \ldots j_p} \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_p}$$

be a p-form on M. Then we have

$$\mathrm{d}*\mathrm{d}*\tau = \frac{(-)^{(p-1)(q+1)}}{p!(p-1)!} \delta_{th_1 \ldots h_{p-1}}^{j_1 \ldots j_p} g^{rt} \tau_{j_1 \ldots j_p;rs} \mathrm{d}x^s \wedge \mathrm{d}x^{h_1} \wedge \ldots \wedge \mathrm{d}x^{h_{p-1}} \tag{5}$$

and

$$*\mathrm{d}*\mathrm{d}\tau = \frac{(-)^{pq}}{(p!)^2} \delta_{th_1 \ldots h_p}^{rj_1 \ldots j_p} g^{st} \tau_{j_1 \ldots j_p;rs} \mathrm{d}x^{h_1} \wedge \ldots \wedge \mathrm{d}x^{h_p}. \tag{6}$$

Proof: We begin by proving (5). Denote

$$\begin{aligned} \alpha &:= \frac{1}{p!(n-p)!(p-1)!}, \\ A_{l_1 \ldots l_{n-p}} &:= \tau_{j_1 \ldots j_p} \varepsilon^{j_1 \ldots j_p}{}_{l_1 \ldots l_{n-p}}. \end{aligned}$$

By definitions

$$
\begin{aligned}
*\tau &= \frac{1}{p!(n-p)!} A_{l_1\ldots l_{n-p}} \mathrm{d}x^{l_1} \wedge \ldots \wedge \mathrm{d}x^{l_{n-p}}, \\
\mathrm{d}{*}\tau &= \frac{1}{p!(n-p)!} \frac{\partial}{\partial x^r} A_{l_1\ldots l_{n-p}} \mathrm{d}x^r \wedge \mathrm{d}x^{l_1} \wedge \ldots \wedge \mathrm{d}x^{l_{n-p}}, \\
\mathrm{d}{}\tau &= \alpha \frac{\partial}{\partial x^r} A_{l_1\ldots l_{n-p}} \varepsilon^{r l_1 \ldots l_{n-p}}{}_{h_1\ldots h_{p-1}} \mathrm{d}x^{h_1} \wedge \ldots \wedge \mathrm{d}x^{h_{p-1}}, \\
\mathrm{d}{*}\mathrm{d}{*}\tau &= \alpha \frac{\partial}{\partial x^s} \left(\frac{\partial}{\partial x^r} A_{l_1\ldots l_{n-p}} \varepsilon^{r l_1 \ldots l_{n-p}}{}_{h_1\ldots h_{p-1}} \right) \mathrm{d}x^s \wedge \mathrm{d}x^{h_1} \wedge \ldots \wedge \mathrm{d}x^{h_{p-1}}.
\end{aligned}
$$

Corollary 2.2 implies the universal formula

$$
\varepsilon^{u j_1 \ldots j_p}{}_{l_1\ldots l_q} X_{j_1\ldots j_p;u} = \varepsilon^{u j_1 \ldots j_p}{}_{l_1\ldots l_q} \frac{\partial}{\partial x^u} X_{j_1\ldots j_p}, \tag{7}
$$

from which it follows that

$$
\varepsilon^{r l_1 \ldots l_{n-p}}{}_{h_1\ldots h_{p-1}} \frac{\partial}{\partial x^r} A_{l_1\ldots l_{n-p}} = \varepsilon^{r l_1 \ldots l_{n-p}}{}_{h_1\ldots h_{p-1}} A_{l_1\ldots l_{n-p};r} =: B_{h_1\ldots h_{p-1}}.
$$

Since

$$
\mathrm{d}x^s \wedge \mathrm{d}x^{h_1} \wedge \ldots \wedge \mathrm{d}x^{h_{p-1}} = \varepsilon_{k_1\ldots k_p} \varepsilon^{s h_1 \ldots h_{p-1}} \mathrm{d}x^{k_1} \otimes \ldots \otimes \mathrm{d}x^{k_p}
$$

we have

$$
\mathrm{d}{*}\mathrm{d}{*}\tau = \alpha \varepsilon_{k_1\ldots k_p} \varepsilon^{s h_1 \ldots h_{p-1}} \frac{\partial}{\partial x^s} B_{h_1\ldots h_{p-1}} \mathrm{d}x^{k_1} \otimes \ldots \otimes \mathrm{d}x^{k_p}.
$$

From (7) we get

$$
\mathrm{d}{*}\mathrm{d}{*}\tau = \alpha \varepsilon_{k_1\ldots k_p} \varepsilon^{s h_1 \ldots h_{p-1}} B_{h_1\ldots h_{p-1};s} \mathrm{d}x^{k_1} \otimes \ldots \otimes \mathrm{d}x^{k_p}.
$$

Here

$$
\begin{aligned}
B_{h_1\ldots h_{p-1};s} &= \varepsilon^{r l_1 \ldots l_{n-p}}{}_{h_1\ldots h_{p-1}} A_{l_1\ldots l_{n-p};rs} \\
&= \varepsilon^{r l_1 \ldots l_{n-p}}{}_{h_1\ldots h_{p-1}} \varepsilon^{j_1\ldots j_p}{}_{l_1\ldots l_{n-p}} \tau_{j_1\ldots j_p;rs} \\
&= g^{rt} \varepsilon_{t l_1 \ldots l_{n-p} h_1 \ldots h_{p-1}} \varepsilon^{j_1\ldots j_p l_1 \ldots l_{n-p}} \tau_{j_1\ldots j_p;rs} \\
&= (-)^{(p-1)(n-p)} (n-p)! g^{rt} \delta^{j_1\ldots j_p}_{t h_1\ldots h_{p-1}} \tau_{j_1\ldots j_p;rs}.
\end{aligned}
$$

Hence

$$
\mathrm{d}{*}\mathrm{d}{*}\tau = \frac{(-)^{(p-1)(q+1)}}{p!(p-1)!} \delta^{j_1\ldots j_p}_{t h_1\ldots h_{p-1}} g^{rt} \tau_{j_1\ldots j_p;rs} \delta^{s h_1\ldots h_{p-1}}_{k_1\ldots k_p} \mathrm{d}x^{k_1} \otimes \ldots \otimes \mathrm{d}x^{k_p}
$$

and (5) follows from the definition of exterior product. We continue by proving (6). By Lemma 2.3

$$
\begin{aligned}
\mathrm{d}_p &= (-)^{p(q+1)} *_q G_q \mathrm{Curl}_p, \\
\mathrm{d}_q &= (-)^{q(p+1)} *_p G_p \mathrm{Curl}_q.
\end{aligned}
$$

Since

$$*_{q+1}*_p \;=\; (-)^{p(q+1)} \quad\text{and}\quad *_{p+1}*_q \;=\; (-)^{q(p+1)}$$

it follows that

$$\begin{aligned}
\mathrm{d}\mathrm{d} &= *_{q+1}\mathrm{d}_q *_{p+1}\mathrm{d}_p \\
&= *_{q+1}(-)^{q(p+1)}*_p G_p \mathrm{Curl}_q\, *_{p+1}(-)^{p(q+1)}*_q G_q \mathrm{Curl}_p \\
&= G_p \mathrm{Curl}_q G_q \mathrm{Curl}_p.
\end{aligned}$$

For

$$\tau = \tau_{j_1\ldots j_p}\mathrm{d}x^{j_1}\otimes\ldots\otimes\mathrm{d}x^{j_p}$$

we have (omitting the basis vectors in notation)

$$\begin{aligned}
\mathrm{Curl}_p\tau &= \frac{1}{p!}\varepsilon^{rj_1\ldots j_p l_1\ldots l_q}\tau_{j_1\ldots j_p;r}, \\
G_q\mathrm{Curl}_p\tau &= \frac{1}{p!}\varepsilon^{rj_1\ldots j_p}{}_{l_1\ldots l_q}\tau_{j_1\ldots j_p;r}, \\
\mathrm{Curl}_q G_q \mathrm{Curl}_p\tau &= \frac{1}{q!}\varepsilon^{sl_1\ldots l_q h_1\ldots h_p}\left(\frac{1}{p!}\varepsilon^{rj_1\ldots j_p}{}_{l_1\ldots l_q}\tau_{j_1\ldots j_p;r}\right)_{;s} \\
&= \frac{1}{p!q!}\varepsilon^{sl_1\ldots l_q h_1\ldots h_p}\varepsilon^{rj_1\ldots j_p}{}_{l_1\ldots l_q}\tau_{j_1\ldots j_p;rs}, \\
G_p\mathrm{Curl}_q G_q \mathrm{Curl}_p\tau &= \frac{1}{p!q!}\varepsilon^{sl_1\ldots l_q}{}_{h_1\ldots h_p}\varepsilon^{rj_1\ldots j_p}{}_{l_1\ldots l_q}\tau_{j_1\ldots j_p;rs} \\
&= \frac{1}{p!q!}g^{st}\delta^{rj_1\ldots j_p l_1\ldots l_q}_{tl_1\ldots l_q h_1\ldots h_p}\tau_{j_1\ldots j_p;rs} \\
&= \frac{(-)^{pq}}{p!}g^{st}\delta^{rj_1\ldots j_p}_{th_1\ldots h_p}\tau_{j_1\ldots j_p;rs}.
\end{aligned}$$

The claim (6) follows from

$$\delta^{rj_1\ldots j_p}_{th_1\ldots h_p}\mathrm{d}x^{h_1}\otimes\ldots\otimes\mathrm{d}x^{h_p} = \frac{1}{p!}\delta^{rj_1\ldots j_p}_{th_1\ldots h_p}\mathrm{d}x^{h_1}\wedge\ldots\wedge\mathrm{d}x^{h_p}.$$

□

In the next section we will need a specific expression for the *generalized Laplace operator*

$$\mathrm{d}*\mathrm{d}* + (-)^n *\mathrm{d}*\mathrm{d}$$

consisting of second covariant derivatives. For the sake of brevity we employ the ordered multi-indices H, J and K of lengths $\#H = \#J = p$ and $\#K = p-1$.

Lemma 2.7 *Let*

$$\tau := \frac{1}{p!}\tau_{j_1\ldots j_p}\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_p}$$

be a p-form on M. Then

$$\begin{aligned} \mathrm{d}{*}\mathrm{d}{*}\tau + (-)^n{*}\mathrm{d}{*}\mathrm{d}\tau \;&=\; (-)^{pq+n}\Bigg[g^{rs}\left(\sum_{s\in J}\tau_{J;rs} + \sum_{s\notin J}\tau_{J;sr}\right)\mathrm{d}x^J \\ &\quad + \; g^{rt}\sum_s \delta^{sJ}_{stK}\left(\tau_{J;rs} - \tau_{J;sr}\right)\mathrm{d}x^s\wedge\mathrm{d}x^K\Bigg]. \end{aligned} \tag{8}$$

Proof: Note that $(-)^{(p-1)(q+1)} = (-)^{pq+n}$. By interchanging r and s in (6) we get from Lemma 2.6

$$\begin{aligned} &\mathrm{d}{*}\mathrm{d}{*}\tau + (-)^n{*}\mathrm{d}{*}\mathrm{d}\tau \;= \\ &(-)^{pq+n}\delta^J_{tK}g^{rt}\tau_{J;rs}\mathrm{d}x^s\wedge\mathrm{d}x^K + (-)^{pq+n}\delta^{sJ}_{tH}g^{rt}\tau_{J;sr}\mathrm{d}x^H \;= \\ &(-)^{pq+n}g^{rt}\left(A_{rt}+B_{rt}\right), \end{aligned}$$

where

$$\begin{aligned} A_{rt} &:= \delta^J_{tK}\tau_{J;rs}\mathrm{d}x^s\wedge\mathrm{d}x^K, \\ B_{rt} &:= \delta^{sJ}_{tH}\tau_{J;sr}\mathrm{d}x^H. \end{aligned}$$

Let us fix s for the present. If $s\in K$ the exterior product $\mathrm{d}x^s\wedge\mathrm{d}x^K$ in A_{rt} vanishes. If $s\notin K$ the corresponding term in A_{rt} is

$$\begin{aligned} &\delta^J_{sK}\tau_{J;rs}\mathrm{d}x^s\wedge\mathrm{d}x^K, \qquad s=t, \\ &\delta^{sJ}_{stK}\tau_{J;rs}\mathrm{d}x^s\wedge\mathrm{d}x^K, \qquad s\neq t. \end{aligned}$$

If $s\in H$ in B_{rt}, by changing the order to get s as the first index in H, we see that for a fixed H the corresponding term in B_{rt} is

$$\delta^{sJ}_{tsK}\tau_{J;sr}\mathrm{d}x^s\wedge\mathrm{d}x^K.$$

The multi-index K consists of the remaining indices of H in an ascending order. If $s\notin H$ the index $t=s$ only yields a nonvanishing term

$$\delta^{sJ}_{sH}\tau_{J;sr}\mathrm{d}x^H$$

in B_{rt}. Hence for a fixed s the corresponding term is

$$g^{rs}\delta^J_{sK}\tau_{J;rs}\mathrm{d}x^s\wedge\mathrm{d}x^K + g^{rt}\delta^{sJ}_{stK}\tau_{J;rs}\mathrm{d}x^s\wedge\mathrm{d}x^K$$

in $g^{rt}A_{rt}$ and

$$g^{rt}\delta^{sJ}_{tsK}\tau_{J;sr}\mathrm{d}x^s\wedge\mathrm{d}x^K + g^{rs}\delta^{sJ}_{sH}\tau_{J;sr}\mathrm{d}x^H$$

in $g^{rt}B_{rt}$. After releasing s and summing we obtain

$$\begin{aligned} g^{rt}\left(A_{rt}+B_{rt}\right) \;&=\; g^{rs}\left(\sum_{s\in J}\tau_{J;rs} + \sum_{s\notin J}\tau_{J;sr}\right)\mathrm{d}x^J \\ &\quad + g^{rt}\sum_s \delta^{sJ}_{stK}\left(\tau_{J;rs}-\tau_{J;sr}\right)\mathrm{d}x^s\wedge\mathrm{d}x^K. \end{aligned}$$

□

On an n-dimensional manifold the generalized complex electric and magnetic fields can be regarded as covariant tensors $E_{j_1 \ldots j_p}$ and $H_{l_1 \ldots l_q}$, respectively, satisfying the *generalized time-harmonic Maxwell's equations**

$$
\begin{aligned}
\frac{1}{p!}\, \varepsilon^{u j_1 \ldots j_p l_1 \ldots l_q} E_{j_1 \ldots j_p ; u} &= \mathrm{i}\omega\mu_0 H^{l_1 \ldots l_q}, && (9)\\
\frac{1}{q!}\, \varepsilon^{u l_1 \ldots l_q j_1 \ldots j_p} H_{l_1 \ldots l_q ; u} &= (-)^{pq}\mathrm{i}\omega\epsilon_0 E^{j_1 \ldots j_p}. && (10)
\end{aligned}
$$

The constants $\epsilon_0 > 0$ and $\mu_0 > 0$ are called the *electric permittivity* and the *magnetic permeability*, respectively. For every pair (p, q) we have different Maxwell's equations. From (9) and (10) $E_{j_1 \ldots j_p}$ and $H_{l_1 \ldots l_q}$ are seen to be totally antisymmetric. Hence Lemma 2.3 implies that for

$$
\begin{aligned}
E &= E_{j_1 \ldots j_p} \mathrm{d}x^{j_1} \otimes \ldots \otimes \mathrm{d}x^{j_p} &= \frac{1}{p!} E_{j_1 \ldots j_p} \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_p},\\
H &= H_{l_1 \ldots l_q} \mathrm{d}x^{l_1} \otimes \ldots \otimes \mathrm{d}x^{l_q} &= \frac{1}{q!} H_{l_1 \ldots l_q} \mathrm{d}x^{l_1} \wedge \ldots \wedge \mathrm{d}x^{l_q},
\end{aligned}
$$

remembering the notation $\bar{r} := n - r$, the equations (9) and (10) are equivalent to

$$
\begin{aligned}
(-)^{p\bar{p}} \mathrm{d}E &= \mathrm{i}\omega\mu_0 {*H}, && (11)\\
(-)^{q\bar{q}} \mathrm{d}H &= (-)^{pq}\mathrm{i}\omega\epsilon_0 {*E}. && (12)
\end{aligned}
$$

These equations are, as appropriately interpreted, consistent with those in [10] although Picard has order dependent ϵ_p and μ_p instead of ϵ_0 and μ_0. Since we have fixed p and q the difference is mainly notational. Applying d to both sides of (11) and (12) we obtain the *generalized divergence equations*

$$
\begin{aligned}
\mathrm{d}{*E} &= 0, && (13)\\
\mathrm{d}{*H} &= 0. && (14)
\end{aligned}
$$

Applying $*\mathrm{d}*$ to (11) and (12) yields

$$
\begin{aligned}
\mathrm{d}\mathrm{d}E + (-)^{pq} k^2 E &= 0, && (15)\\
\mathrm{d}\mathrm{d}H + (-)^{pq} k^2 H &= 0. && (16)
\end{aligned}
$$

Conversely, if (15) holds and we define

$$
H := (-)^{n-1} \frac{1}{\mathrm{i}\omega\mu_0} *\mathrm{d}E
$$

*As far as time-harmonic electromagnetic theory is concerned the field tensors are complex. Thus it is quite natural to consider the complexified tangent and cotangent bundles in this connection.

the pair (E, H) fulfils (11) and (12). On the other hand, if (16) holds and

$$E := (-)^{\bar{p}\bar{q}-1} \frac{1}{\mathrm{i}\omega\epsilon_0} {*\mathrm{d}H}$$

we get (11) and (12) as well. These facts are readily verified. For divergence-free fields, i.e. fields that satisfy (13) and (14), the equations (15) and (16) become

$$(\mathrm{d}{*}\mathrm{d}{*} + (-)^n {*}\mathrm{d}{*}\mathrm{d})\, E + (-)^{\bar{p}\bar{q}} k^2 E = 0, \tag{17}$$

$$(\mathrm{d}{*}\mathrm{d}{*} + (-)^n {*}\mathrm{d}{*}\mathrm{d})\, H + (-)^{\bar{p}\bar{q}} k^2 H = 0. \tag{18}$$

The *generalized Helmholtz equations* (17) and (18) have been built on the Laplace operator which was introduced previously in this section.

Next we are going to prove a useful formula which we call the *Maxwell duality* (cf. [8]). Let R, S, T and U be r-, s-, t- and u-forms, respectively, with $r + t = s + u = n$. Define a bilinear product

$$\left\langle \begin{pmatrix} R \\ S \end{pmatrix}, \begin{pmatrix} T \\ U \end{pmatrix} \right\rangle_\Omega = \int_\Omega (R \wedge T + S \wedge U)\,.$$

Here Ω is a *regular submanifold* of M, that is to say, an open n-dimensional submanifold of M whose closure $\overline{\Omega}$ is a compact oriented submanifold with boundary $\partial\Omega$. These assumptions make it possible to use the *Stokes formula* (see [12])

$$\int_\Omega \mathrm{d}\theta = \int_{\partial\Omega} \theta$$

for any $(n-1)$-form θ of class C^1. We define the *Maxwell operator* $\mathcal{M}$ by

$$\mathcal{M} \begin{pmatrix} E \\ H \end{pmatrix} := \begin{pmatrix} (-)^{p\bar{p}}\mathrm{d}E & - & \mathrm{i}\omega\mu_0 {*H} \\ (-)^{q\bar{q}}\mathrm{d}H & - & (-)^{pq}\mathrm{i}\omega\epsilon_0 {*E} \end{pmatrix}$$

and a sort of adjoint $\mathcal{M}^*$ by

$$\mathcal{M}^* \begin{pmatrix} \tilde{H} \\ \tilde{E} \end{pmatrix} := \begin{pmatrix} (-)^{q\bar{q}}\mathrm{d}\tilde{H} & - & (-)^{pq}\mathrm{i}\omega\epsilon_0 {*\tilde{E}} \\ (-)^{p\bar{p}}\mathrm{d}\tilde{E} & - & \mathrm{i}\omega\mu_0 {*\tilde{H}} \end{pmatrix}$$

for p-forms E, $\tilde{E}$ and q-forms H, $\tilde{H}$.

Lemma 2.8 *For p-forms E, $\tilde{E}$ and q-forms H, $\tilde{H}$*

$$\begin{aligned} \left\langle \mathcal{M} \begin{pmatrix} E \\ H \end{pmatrix}, \begin{pmatrix} w\tilde{H} \\ v\tilde{E} \end{pmatrix} \right\rangle_\Omega &= \left\langle \begin{pmatrix} (-)^{p\bar{p}} vE \\ (-)^{q\bar{q}} wH \end{pmatrix}, \mathcal{M}^* \begin{pmatrix} \tilde{H} \\ \tilde{E} \end{pmatrix} \right\rangle_\Omega \\ &\quad + \left\langle \begin{pmatrix} (-)^{p\bar{p}} wE \\ (-)^{q\bar{q}} vH \end{pmatrix}, \begin{pmatrix} \tilde{H} \\ \tilde{E} \end{pmatrix} \right\rangle_{\partial\Omega} \end{aligned} \tag{19}$$

whenever $v = (-)^{\bar{p}\bar{q}} w$.

Proof: The left hand side equals

$$\int_\Omega \Big((-)^{p\bar p} \mathrm{d}E \wedge w\tilde H - \mathrm{i}\omega\mu_0 {*H} \wedge w\tilde H + (-)^{q\bar q} \mathrm{d}H \wedge v\tilde E - (-)^{pq} \mathrm{i}\omega\epsilon_0 {*E} \wedge v\tilde E \Big).$$

By Stokes theorem

$$\begin{aligned}
\int_\Omega \mathrm{d}E \wedge \tilde H &= \int_{\partial\Omega} E \wedge \tilde H - \int_\Omega (-)^p E \wedge \mathrm{d}\tilde H, \\
\int_\Omega \mathrm{d}H \wedge \tilde E &= \int_{\partial\Omega} H \wedge \tilde E - \int_\Omega (-)^q H \wedge \mathrm{d}\tilde E.
\end{aligned}$$

Since

$$*H \wedge \tilde H = (-)^{q\bar q} H \wedge *\tilde H, \quad *E \wedge \tilde E = (-)^{p\bar p} E \wedge *\tilde E,$$

and $(-)^p(-)^{p\bar p} = (-)^q(-)^{q\bar q}$ the left hand side of (19) is equal to

$$\begin{aligned}
&\int_{\partial\Omega} \Big((-)^{p\bar p} wE \wedge \tilde H + (-)^{q\bar q} vH \wedge \tilde E \Big) \\
&- \int_\Omega \Big((-)^q(-)^{q\bar q} wE \wedge \mathrm{d}\tilde H + (-)^p(-)^{p\bar p} vH \wedge \mathrm{d}\tilde E \Big) \\
&+ \int_\Omega \Big(-(-)^{q\bar q} w\, \mathrm{i}\omega\mu_0 H \wedge *\tilde H - (-)^{p\bar p} v (-)^{pq} \mathrm{i}\omega\epsilon_0 E \wedge *\tilde E \Big) = \\
&\int_\Omega \Big[(-)^{q+1} wE \wedge (-)^{q\bar q} \mathrm{d}\tilde H + (-)^{p\bar p} vE \wedge \Big(-(-)^{pq} \mathrm{i}\omega\epsilon_0 * \tilde E \Big) \\
&\quad + (-)^{p+1} vH \wedge (-)^{p\bar p} \mathrm{d}\tilde E + (-)^{q\bar q} wH \wedge (-\mathrm{i}\omega\mu_0 * \tilde H) \Big] \\
&+ \int_{\partial\Omega} \Big((-)^{p\bar p} wE \wedge \tilde H + (-)^{q\bar q} vH \wedge \tilde E \Big).
\end{aligned}$$

If $v = (-)^{\bar p \bar q} w$ then

$$\begin{aligned}
(-)^{p+1} v &= (-)^{p+1}(-)^{\bar p\bar q} w &= (-)^{q\bar q} w, \\
(-)^{q+1} w &= (-)^{q+1}(-)^{\bar p\bar q} v &= (-)^{p\bar p} v,
\end{aligned}$$

and (19) has been proved. □

Remark 2.9 It is obvious from the proof that the constants μ_0 and ϵ_0 can be replaced by smooth functions μ and ϵ in Lemma 2.8.

Proposition 2.10 (Maxwell duality) *For p-forms E, $\tilde E$ and q-forms H, $\tilde H$*

$$\begin{aligned}
&\left\langle \mathcal{M} \begin{pmatrix} E \\ H \end{pmatrix}, \begin{pmatrix} \tilde b \tilde H \\ \tilde a \tilde E \end{pmatrix} \right\rangle_\Omega = \\
&\left\langle \begin{pmatrix} aE \\ bH \end{pmatrix}, \mathcal{M}^* \begin{pmatrix} \tilde H \\ \tilde E \end{pmatrix} \right\rangle_\Omega + \left\langle \begin{pmatrix} \alpha E \\ \beta H \end{pmatrix}, \begin{pmatrix} \tilde H \\ \tilde E \end{pmatrix} \right\rangle_{\partial\Omega}
\end{aligned} \tag{20}$$

whenever

$$a=(-)^{\bar{p}},\ \ b=(-)^{q\bar{q}},\ \ \tilde{a}=(-)^{\bar{p}\bar{q}},\ \ \tilde{b}=1,\ \ \alpha=(-)^{p\bar{p}},\ \ \beta=(-)^{\bar{q}}.$$

Proof: Since $(-)^{p\bar{p}}(-)^{\bar{p}\bar{q}}=(-)^{\bar{p}}$ and $(-)^{q\bar{q}}(-)^{\bar{p}\bar{q}}=(-)^{\bar{q}}$ the claim follows from (19) by setting $w=1$. □

Remark 2.11 If E_0 is a p-form, H_0 is a q-form and $(E_0,H_0)^T$ satisfies the homogeneous Maxwell's equations $\mathcal{M}(E_0,H_0)^T=0$, then by linearity

$$\mathcal{M}\begin{pmatrix} E+E_0 \\ H+H_0 \end{pmatrix}=\mathcal{M}\begin{pmatrix} E \\ H \end{pmatrix}.$$

The conclusion is that adding such a field $(E_0,H_0)^T$ to $(E,H)^T$ in (19) has no effect on either side of (19).

3 Fundamental Solutions

We will give expressions for the fields of generalized electric and magnetic dipoles. For the remainder of this work, the Riemann curvature tensor of M is assumed to vanish everywhere. As an immediate consequence the higher covariant derivatives are symmetric in the following sense:

Lemma 3.1 *For a tensor $X^{j_1\ldots j_r}_{l_1\ldots l_s}$ on M we have*

$$X^{j_1\ldots j_r}_{l_1\ldots l_s;m_1\ldots m_t}=X^{j_1\ldots j_r}_{l_1\ldots l_s;m_{\sigma(1)}\ldots m_{\sigma(t)}},$$

where σ is any permutation of $\{1,\ldots,t\}$.

Proof: In [7] we prove that locally M can be isometrically imbedded in $\mathbb{C}^n$. With respect to the standard $\tilde{x}$-coordinate system of $\mathbb{C}^n$ a covariant derivative $\tilde{X}^{a_1\ldots a_r}_{b_1\ldots b_s;c}$ of a holomorphic tensor field $\tilde{X}^{a_1\ldots a_r}_{b_1\ldots b_s}$ is just the ordinary partial derivative $\partial\tilde{X}^{a_1\ldots a_r}_{b_1\ldots b_s}/\partial\tilde{x}^c$. Hence the covariant derivatives with respect to different indices commute in $\mathbb{C}^n$. The claim follows by pointwise approximation of the imbedding and the tensor by their Taylor polynomials of degree t. Formally, if we denote $\theta=\sigma^{-1}$,

$$\begin{aligned}
X^{j_1\ldots j_r}_{l_1\ldots l_s;m_1\ldots m_t} &= \frac{\partial x^{j_1}}{\partial\tilde{x}^{a_1}}\cdots\frac{\partial x^{j_r}}{\partial\tilde{x}^{a_r}}\frac{\partial\tilde{x}^{b_1}}{\partial x^{l_1}}\cdots\frac{\partial\tilde{x}^{b_s}}{\partial x^{l_s}}\frac{\partial\tilde{x}^{c_1}}{\partial x^{m_1}}\cdots\frac{\partial\tilde{x}^{c_t}}{\partial x^{m_t}}\tilde{X}^{a_1\ldots a_r}_{b_1\ldots b_s;c_1\ldots c_t}\\
&= \frac{\partial x^{j_1}}{\partial\tilde{x}^{a_1}}\cdots\frac{\partial x^{j_r}}{\partial\tilde{x}^{a_r}}\frac{\partial\tilde{x}^{b_1}}{\partial x^{l_1}}\cdots\frac{\partial\tilde{x}^{b_s}}{\partial x^{l_s}}\frac{\partial\tilde{x}^{c_{\theta(1)}}}{\partial x^{m_1}}\cdots\frac{\partial\tilde{x}^{c_{\theta(t)}}}{\partial x^{m_t}}\tilde{X}^{a_1\ldots a_r}_{b_1\ldots b_s;c_1\ldots c_t}\\
&= \frac{\partial x^{j_1}}{\partial\tilde{x}^{a_1}}\cdots\frac{\partial x^{j_r}}{\partial\tilde{x}^{a_r}}\frac{\partial\tilde{x}^{b_1}}{\partial x^{l_1}}\cdots\frac{\partial\tilde{x}^{b_s}}{\partial x^{l_r}}\frac{\partial\tilde{x}^{c_1}}{\partial x^{m_{\sigma(1)}}}\cdots\frac{\partial\tilde{x}^{c_t}}{\partial x^{m_{\sigma(t)}}}\tilde{X}^{a_1\ldots a_r}_{b_1\ldots b_s;c_1\ldots c_t}\\
&= X^{j_1\ldots j_r}_{l_1\ldots l_s;m_{\sigma(1)}\ldots m_{\sigma(t)}}.
\end{aligned}$$

□

Lemma 3.1 simplifies the expression (8) of the Laplace operator remarkably:

Corollary 3.2 *Let*

$$\tau \;:=\; \frac{1}{p!}\tau_{j_1\ldots j_p}\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_p}.$$

Then

$$\mathrm{d}{*}\mathrm{d}{*}\tau + (-)^n{*}\mathrm{d}{*}\mathrm{d}\tau = \frac{(-)^{\bar p\bar q}}{p!}g^{rs}\tau_{j_1\ldots j_p;rs}\mathrm{d}x^{j_1}\wedge\ldots\wedge\mathrm{d}x^{j_p}.$$

Proof: The claim is an immediate implication of Lemma 2.7. □

From now on we assume that there exists a fundamental solution to the scalar Helmholtz operator

$$\varphi\mapsto g^{jl}\varphi_{;jl}+k^2\varphi.$$

In other words, there is a smooth scalar $\Phi=\Phi(x,y)$ such that for all $x,y\in M$

$$g^{jl}(x)\Phi_{;jl}(x,y)+k^2\Phi(x,y)=-\delta_y(x) \tag{21}$$

which is, by Corollary 3.2, equivalent to

$${*}\mathrm{d}{*}\mathrm{d}\Phi(x,y)+k^2\Phi(x,y)=-\delta_y(x). \tag{22}$$

Unless otherwise stated, derivatives are taken with respect to x. Later, when we introduce the radiation conditions, we will expect some extra features from the fundamental solution Φ.

In addition, assume that the geometry of M admits of a global parallel nowhere vanishing tensor field of order q.† Suppose, without loss of generality, that $\hat\pi_{l_1\ldots l_q}$ is a totally antisymmetric parallel tensor:

$$\hat\pi \;:=\; \hat\pi_{l_1\ldots l_q}\mathrm{d}x^{l_1}\otimes\ldots\otimes\mathrm{d}x^{l_q} \;=\; \frac{1}{q!}\hat\pi_{l_1\ldots l_q}\mathrm{d}x^{l_1}\wedge\ldots\wedge\mathrm{d}x^{l_q}.$$

Define

$$\begin{aligned}\hat\eta &:= \Phi\hat\pi,\\ \hat\tau &:= (-)^{\bar p\bar q}\hat\pi.\end{aligned}$$

Proposition 3.3 *The q-form $\hat\eta$ is a fundamental solution to the Helmholtz equation in the following sense:*

$$\left(\mathrm{d}{*}\mathrm{d}{*}+(-)^n{*}\mathrm{d}{*}\mathrm{d}\right)\hat\eta+(-)^{\bar p\bar q}k^2\hat\eta=-\delta_y\hat\tau. \tag{23}$$

Proof: Since $\hat\pi$ is parallel and Φ is a fundamental solution

$$g^{rs}\hat\eta_{l_1\ldots l_q;rs} \;=\; g^{rs}\Phi_{;rs}\hat\pi_{l_1\ldots l_q} \;=\; \left(-k^2\Phi-\delta_y\right)\hat\pi_{l_1\ldots l_q} \;=\; -k^2\hat\eta_{l_1\ldots l_q}-\delta_y\hat\pi_{l_1\ldots l_q}.$$

On the other hand, Corollary 3.2 implies

$$g^{rs}\hat\eta_{l_1\ldots l_q;rs}\mathrm{d}x^{l_1}\otimes\ldots\otimes\mathrm{d}x^{l_q}=(-)^{\bar p\bar q}\left(\mathrm{d}{*}\mathrm{d}{*}\hat\eta+(-)^n{*}\mathrm{d}{*}\mathrm{d}\hat\eta\right).$$

□

†Since the curvature vanishes it is sufficient, but by no means necessary, to the existence of a parallel field that M is simply connected. See later in this section.

Remark 3.4 How should the Dirac measure δ_y be interpreted in expressions like (23)? According to de Rham (see [11]) differential forms with distribution coefficients are called *currents*. If $F = F_J \mathrm{d}x^J$ is a locally integrable p-current its value for a test p-form (i.e. a compactly supported smooth p-form) $\varphi = \varphi_L \mathrm{d}x^L$ is, departing from [11], defined by

$$\langle F, \varphi \rangle \;:=\; \int_M F \wedge *\varphi \;=\; \int_M F_J \varphi^J \sqrt{g}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n. \tag{24}$$

The right hand side identity comes from

$$\begin{aligned} F_J \mathrm{d}x^J \wedge *\varphi_L \mathrm{d}x^L &= F_J \varphi_L \varepsilon^L{}_M \mathrm{d}x^J \wedge \mathrm{d}x^M \\ &= F_J \varphi_L \varepsilon^L{}_M \varepsilon^{JM} \sqrt{g}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \\ &= F_J \varphi^L \delta^J_L \sqrt{g}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n \\ &= F_J \varphi^J \sqrt{g}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n. \end{aligned}$$

Here $\sqrt{g}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ is the coordinate invariant volume element (see [9]). For a scalar test function φ

$$\langle \delta_y, \varphi \rangle := \varphi(y),$$

as usual. Formally

$$\langle \delta_y, \varphi \rangle \;=\; \int_M \delta_y \wedge *\varphi \;=\; \int_M \delta_y \varphi \sqrt{g}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n.$$

With the aid of the fundamental solution to the Helmholtz equation we are able to define certain fields that may be regarded as the fields of *generalized electromagnetic dipoles*. The parallel form $\hat{\pi}$ is referred to as a *generalized dipole moment.*

Proposition 3.5 *Define a* q*-form* ${}^{\epsilon}E$ *and a* p*-form* ${}^{\epsilon}H$ *by*

$$\begin{aligned} {}^{\epsilon}E &= \frac{(-)^{pq}}{\mathrm{i}\omega\epsilon_0}\left((-)^{n-1} *\mathrm{d}*\mathrm{d}\hat{\eta} - \delta_y \hat{\tau}\right), \\ {}^{\epsilon}H &= *\mathrm{d}\hat{\eta}. \end{aligned}$$

The pair $({}^{\epsilon}E,\, {}^{\epsilon}H)^T$ *satisfies the Maxwell's equations*

$$\begin{aligned} (-)^{q\bar{q}} \mathrm{d}\,{}^{\epsilon}E &= \mathrm{i}\omega\mu_0 *{}^{\epsilon}H, \\ (-)^{p\bar{p}} \mathrm{d}\,{}^{\epsilon}H &= (-)^{pq} \mathrm{i}\omega\epsilon_0 *{}^{\epsilon}E + \delta_y *\hat{\tau}, \end{aligned}$$

for a generalized electric dipole.

Proof: This is a straightforward calculation. Note that

$$(-)^{p\bar{p}} = (-)^{n-1}(-)^{q\bar{q}}.$$

□

Proposition 3.6 *Define a p-form* ${}^{\mu}E$ *and a q-form* ${}^{\mu}H$ *by*

$$\begin{aligned} {}^{\mu}E &= *\mathrm{d}\hat{\eta}, \\ {}^{\mu}H &= \frac{1}{\mathrm{i}\omega\mu_0}\left((-)^{n-1}*\mathrm{d}*\mathrm{d}\hat{\eta} - \delta_y\hat{\tau}\right). \end{aligned}$$

The pair $({}^{\mu}E,\, {}^{\mu}H)^T$ *satisfies the Maxwell's equations*

$$\begin{aligned} (-)^{p\bar{p}}\mathrm{d}\,{}^{\mu}E &= \mathrm{i}\omega\mu_0 *{}^{\mu}H + \delta_y*\hat{\tau}, \\ (-)^{q\bar{q}}\mathrm{d}\,{}^{\mu}H &= (-)^{pq}\mathrm{i}\omega\epsilon_0 *{}^{\mu}E, \end{aligned}$$

for a generalized magnetic dipole.

Proof: Straightforward. □

Proposition 3.7 *The dipole fields have expressions*

$$\begin{aligned} {}^{\epsilon}E(x,y) &= \frac{(-)^{n-1}}{\mathrm{i}\omega\epsilon_0}\left(\delta^{rL}_{tK}g^{st}(x)\Phi_{;rs}(x,y) + \delta_y(x)\delta^L_K\right)\hat{\pi}_L(x)\mathrm{d}x^K, \\ {}^{\epsilon}H(x,y) &= (-)^{n-1}\varepsilon^{uL}{}_J(x)\Phi_{;u}(x,y)\hat{\pi}_L(x)\mathrm{d}x^J, \\ {}^{\mu}E(x,y) &= (-)^{n-1}\varepsilon^{uL}{}_J(x)\Phi_{;u}(x,y)\hat{\pi}_L(x)\mathrm{d}x^J, \\ {}^{\mu}H(x,y) &= \frac{(-)^{\bar{p}\bar{q}-1}}{\mathrm{i}\omega\mu_0}\left(\delta^{rL}_{tK}g^{st}(x)\Phi_{;rs}(x,y) + \delta_y(x)\delta^L_K\right)\hat{\pi}_L(x)\mathrm{d}x^K, \end{aligned}$$

where $\#J = p$, $\#L = \#K = q$.

Proof: These formulae are straightforward consequences of Proposition 3.5, Proposition 3.6, Lemma 2.3, Lemma 2.6 and the parallelism of $\hat{\pi}$. □

Next we are going to prove analogues to the reciprocity theorems, Stratton-Chu formulae and the Lippmann-Schwinger equation which is usually called the volume integral equation. For this purpose we begin with studying the existence of a global frame in Lemma 3.8, Lemma 3.9 and Proposition 3.10.

Lemma 3.8 *The tangent space* T_xM *has an orthonormal basis for every* $x \in M$.

Proof: Diagonalization of the symmetric matrix $(g_{jl}(x))$. □

Lemma 3.9 *Let* $X_1, \ldots, X_n$ *be (local or global) vector fields on* M. *Denote*

$$X_{(j_1 \ldots j_p)} := X_{j_1} \wedge \ldots \wedge X_{j_p}.$$

(i) If $X_{j_1}, \ldots, X_{j_p}$ *are parallel then* $X_{(j_1 \ldots j_p)}$ *is parallel.*

(ii) If $X_{j_1}, \ldots, X_{j_p}$ are $\mathbb{C}$-linearly independent then

$$X_{(j_1 .. j_p)}, \quad 1 \leq j_1 < \ldots < j_p \leq n,$$

are $\mathbb{C}$-linearly independent.

(iii) If $X_{j_1}, \ldots, X_{j_p}$ are orthonormal in the sense of the complex metric then

$$X_{(j_1 .. j_p)}, \quad 1 \leq j_1 < \ldots < j_p \leq n,$$

are orthonormal.

Proof:

(i) This is an implication of the Leibniz's rule for covariant derivatives and of the fact that the δ-tensors are parallel.

(ii) A well known algebraic fact.

(iii) Straightforward.

□

Proposition 3.10 *If M is simply connected and the Riemannian curvature tensor vanishes everywhere on M then each orthonormal basis $V_1, \ldots, V_n$ of the tangent space $T_{x_0}M$ of M at $x_0 \in M$ can be uniquely extended to a parallel global smooth orthonormal frame $X_1, \ldots, X_n$ such that*

$$X_1(x_0) = V_1, \ldots, X_n(x_0) = V_n.$$

Proof: The existence and uniqueness of a local extension is proved in [7]. The existence and uniqueness of an extension along a path is proved as in conventional differential geometry (see [3]). Given two paths $\gamma_1, \gamma_2 : [0,1] \to M$ such that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ there is a path homotopy connecting γ_1 and γ_2. The homotopy can be constructed step by step in such a way that it deforms the path within an arbitrarily small neighbourhood at a time. This proves the claim since these elementary deformation steps preserve the orthogonal frame provided that the neighbourhoods are chosen sufficiently small. □

Since the index lowering operator preserves parallelism, linear independence and orthonormality the previous results also apply to cotangent vectors. From now on, let $\pi^{(1)}, \ldots, \pi^{(n)}$ be a parallel orthonormal basis for the $\mathbb{C}$-linear space of 1-forms on M. Lemma 3.9 implies that the exterior products

$$\pi^{(J)} := \pi^{(j_1)} \wedge \ldots \wedge \pi^{(j_p)}, \quad 1 \leq j_1 < \ldots < j_p \leq n,$$

form a parallel basis for the $\mathbb{C}$-linear space of p-forms on M. In Proposition 3.7 the fields are of the form

$$F(x,y) \;=\; \tilde{F}^L{}_M(x,y)\hat{\pi}_L(x)\mathrm{d}x^M \;=\; \hat{\pi}_K(x)\mathrm{d}x^K \rfloor \tilde{F}_{LM}(x,y)\mathrm{d}x^L \otimes \mathrm{d}x^M.$$

By change of basis according to

$$\mathrm{d}x^K \;=\; a^K{}_P(x)\pi^{(P)}(x)$$

we obtain

$$\begin{aligned}\tilde{F}_{LM}(x,y)\mathrm{d}x^L \otimes \mathrm{d}x^M \;&=\; \tilde{F}_{LM}(x,y)a^L{}_R(x)a^M{}_S(x)\pi^{(R)}(x)\otimes\pi^{(S)}(x)\\ &=:\; F_{RS}(x,y)\pi^{(R)}(x)\otimes\pi^{(S)}(x).\end{aligned}$$

It is readily seen that $F_{RS}(x,y)$ is a global invariant for any pair of ordered multi-indices R and S. Choose

$$\hat{\pi}_K(x)\mathrm{d}x^K \;=\; \pi^{(B)}(x)$$

to obtain

$$F(x,y) \;=\; \pi^{(B)}(x)\rfloor F_{RS}(x,y)\pi^{(R)}(x)\otimes\pi^{(S)}(x) \;=\; F_{BS}(x,y)\pi^{(S)}(x).$$

Hence the dipole fields have expressions

$$\begin{aligned}{}^{\epsilon}\!E(x,y) \;&=\; {}^{\epsilon}\!E_{BL}(x,y)\pi^{(L)}(x),\\ {}^{\epsilon}\!H(x,y) \;&=\; {}^{\epsilon}\!H_{BJ}(x,y)\pi^{(J)}(x),\\ {}^{\mu}\!E(x,y) \;&=\; {}^{\mu}\!E_{BJ}(x,y)\pi^{(J)}(x),\\ {}^{\mu}\!H(x,y) \;&=\; {}^{\mu}\!H_{BL}(x,y)\pi^{(L)}(x),\end{aligned} \tag{25}$$

where $\#J = p$ and $\#B = \#L = q$. Note that ${}^{\epsilon}\!E$ and ${}^{\mu}\!H$ are of the same order q as the dipole moment form. The Maxwell's equations become

$$\begin{aligned}\mathcal{M}\begin{pmatrix}{}^{\epsilon}\!E(x,y)\\ {}^{\epsilon}\!H(x,y)\end{pmatrix} \;&=\; \begin{pmatrix}0\\ (-)^{\bar{p}\bar{q}}\delta_y(x) * \pi^{(A)}(x)\end{pmatrix},\\ \mathcal{M}\begin{pmatrix}{}^{\mu}\!E(x,y)\\ {}^{\mu}\!H(x,y)\end{pmatrix} \;&=\; \begin{pmatrix}(-)^{\bar{p}\bar{q}}\delta_y(x) * \pi^{(A)}(x)\\ 0\end{pmatrix}.\end{aligned}$$

Proposition 3.11 (Stratton-Chu) *Assume that E is a p-form and H is a q-form which satisfy the homogeneous Maxwell's equations $\mathcal{M}(E,H)^T = 0$ in Ω. Then we have the following representation formulae for $E(y)$ and $H(y)$, $y \in \Omega$:*

$$\begin{aligned}E_A(y) \;=\; \int_{\partial\Omega} \big((-)^{pq+1}\, {}^{\epsilon}\!H_{AL}(x,y)\pi^{(L)}(x)\wedge E_J(x)\pi^{(J)}(x)\\ +\; {}^{\epsilon}\!E_{AJ}(x,y)\pi^{(J)}(x)\wedge H_L(x)\pi^{(L)}(x)\big)\,,\end{aligned} \tag{26}$$

$$\begin{aligned}H_B(y) \;=\; \int_{\partial\Omega} \big({}^{\mu}\!H_{BL}(x,y)\pi^{(L)}(x)\wedge E_J(x)\pi^{(J)}(x)\\ +\; (-)^{pq+1}\, {}^{\mu}\!E_{BJ}(x,y)\pi^{(J)}(x)\wedge H_L(x)\pi^{(L)}(x)\big)\,.\end{aligned} \tag{27}$$

Here $\#A = \#J = p$ and $\#B = \#L = q$.

Proof: Let us prove (26). Application of the Maxwell duality yields

$$\begin{aligned}
&\left\langle \mathcal{M}\begin{pmatrix} E(x) \\ H(x) \end{pmatrix}, \begin{pmatrix} \tilde{b}\,{}^{\epsilon}\!H(x,y) \\ \tilde{a}\,{}^{\epsilon}\!E(x,y) \end{pmatrix} \right\rangle_{\Omega} \\
= &\left\langle \begin{pmatrix} aE(x) \\ bH(x) \end{pmatrix}, \mathcal{M}^{*}\begin{pmatrix} {}^{\epsilon}\!H(x,y) \\ {}^{\epsilon}\!E(x,y) \end{pmatrix} \right\rangle_{\Omega} \\
&+\left\langle \begin{pmatrix} \alpha E(x) \\ \beta H(x) \end{pmatrix}, \begin{pmatrix} {}^{\epsilon}\!H(x,y) \\ {}^{\epsilon}\!E(x,y) \end{pmatrix} \right\rangle_{\partial\Omega}
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\left\langle \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tilde{b}\,{}^{\epsilon}\!H_{AL}(x,y)\pi^{(L)}(x) \\ \tilde{a}\,{}^{\epsilon}\!E_{AJ}(x,y)\pi^{(J)}(x) \end{pmatrix} \right\rangle_{\Omega} \\
= &\left\langle \begin{pmatrix} aE_J(x)\pi^{(J)}(x) \\ bH_L(x)\pi^{(L)}(x) \end{pmatrix}, \begin{pmatrix} (-)^{\bar{p}\bar{q}}\delta_y(x) * \pi^{(A)}(x) \\ 0 \end{pmatrix} \right\rangle_{\Omega} \\
&+\left\langle \begin{pmatrix} \alpha E_J(x)\pi^{(J)}(x) \\ \beta H_L(x)\pi^{(L)}(x) \end{pmatrix}, \begin{pmatrix} {}^{\epsilon}\!H_{AL}(x,y)\pi^{(L)}(x) \\ {}^{\epsilon}\!E_{AJ}(x,y)\pi^{(J)}(x) \end{pmatrix} \right\rangle_{\partial\Omega}.
\end{aligned}$$

By the definition of $< \cdot, \cdot >$

$$\begin{aligned}
0 \;=\; & (-)^{\bar{p}\bar{q}} a \int_{\Omega} E_J(x)\pi^{(J)}(x) \wedge \delta_y(x) * \pi^{(A)}(x) \\
&+ \int_{\partial\Omega} \big(\alpha E_J(x)\pi^{(J)}(x) \wedge {}^{\epsilon}\!H_{AL}(x,y)\pi^{(L)}(x) \\
&\quad + \beta H_L(x)\pi^{(L)}(x) \wedge {}^{\epsilon}\!E_{AJ}(x,y)\pi^{(J)}(x)\big).
\end{aligned}$$

By orthonormality the first term on the right equals

$$(-)^{\bar{p}\bar{q}} aE_J(y)\pi^{(J)}(y) \rfloor \pi^{(A)}(y) = (-)^{\bar{p}\bar{q}} aE_A(y).$$

Hence

$$\begin{aligned}
E_A(y) \;=\; & -(-)^{\bar{p}\bar{q}} a \int_{\partial\Omega} \big(\alpha E_J(x)\pi^{(J)}(x) \wedge {}^{\epsilon}\!H_{AL}(x,y)\pi^{(L)}(x) \\
&+\beta H_L(x)\pi^{(L)}(x) \wedge {}^{\epsilon}\!E_{AJ}(x,y)\pi^{(J)}(x)\big).
\end{aligned}$$

The commutativity rule

$$\pi^{(J)}(x) \wedge \pi^{(L)}(x) = (-)^{pq}\pi^{(L)}(x) \wedge \pi^{(J)}(x)$$

implies then

$$\begin{aligned}
E_A(y) \;=\; & -(-)^{\bar{p}\bar{q}} a \int_{\partial\Omega} \big(\alpha(-)^{pq}\,{}^{\epsilon}\!H_{AL}(x,y)\pi^{(L)}(x) \wedge E_J(x)\pi^{(J)}(x) \\
&+\beta(-)^{pq}\,{}^{\epsilon}\!E_{AJ}(x,y)\pi^{(J)}(x) \wedge H_L(x)\pi^{(L)}(x)\big).
\end{aligned}$$

Now (26) follows from $-(-)^{\bar{p}\bar{q}} a\alpha = (-)^1$ and $-(-)^{\bar{p}\bar{q}} a\beta = (-)^{pq}$. The proof of (27) is essentially the same. The field $({}^{\epsilon}\!E,\, {}^{\epsilon}\!H)^T$ should be replaced by $({}^{\mu}\!E,\, {}^{\mu}\!H)^T$. $\square$

At this stage we are ready to employ appropriate radiation conditions that have control over the electromagnetic field far away. These conditions are only relevant for noncompact oriented manifolds M having an *exhaustion*, i.e., a one-parameter family of regular submanifolds B_r, $r > 0$, for which the following hold:

(i) If $r < s$ then $\overline{B_r} \subset B_s$,

(ii) $\bigcup_{r>0} B_r = M$.

Definition 3.12 A field $(E, H)^T$ satisfies the *electric radiation condition* if there exists

$$\lim_{r\to\infty} \left\langle \begin{pmatrix} \alpha E(x) \\ \beta H(x) \end{pmatrix}, \begin{pmatrix} {}^{\epsilon}\!H(x,y) \\ {}^{\epsilon}\!E(x,y) \end{pmatrix} \right\rangle_{\partial B_r} = 0 \tag{28}$$

for all $y \in M$. A field $(E, H)^T$ satisfies the *magnetic radiation condition* if there exists

$$\lim_{r\to\infty} \left\langle \begin{pmatrix} \alpha E(x) \\ \beta H(x) \end{pmatrix}, \begin{pmatrix} {}^{\mu}\!H(x,y) \\ {}^{\mu}\!E(x,y) \end{pmatrix} \right\rangle_{\partial B_r} = 0 \tag{29}$$

for all $y \in M$.[‡]

We assume the scalar Helmholtz fundamental solution Φ to be chosen such that $({}^{\epsilon}\!E(\cdot,z),\, {}^{\epsilon}\!H(\cdot,z))^T$ and $({}^{\mu}\!E(\cdot,z),\, {}^{\mu}\!H(\cdot,z))^T$ satisfy both of the conditions (28) and (29) for all $z \in M$ in the following strong sense:

For any compact $(n-1)$-dimensional submanifold K of M there exists

$$\lim_{r\to\infty} \sup_{y\in K} \left| \left\langle \begin{pmatrix} \alpha E(x) \\ \beta H(x) \end{pmatrix}, \begin{pmatrix} {}^{\epsilon}\!H(x,y) \\ {}^{\epsilon}\!E(x,y) \end{pmatrix} \right\rangle_{\partial B_r} \right| = 0 \tag{30}$$

and

$$\lim_{r\to\infty} \sup_{y\in K} \left| \left\langle \begin{pmatrix} \alpha E(x) \\ \beta H(x) \end{pmatrix}, \begin{pmatrix} {}^{\mu}\!H(x,y) \\ {}^{\mu}\!E(x,y) \end{pmatrix} \right\rangle_{\partial B_r} \right| = 0. \tag{31}$$

Remark 3.13 If a locally integrable n-current θ is not Lebesgue-integrable in M but there exists a finite

$$\lim_{r\to\infty} \int_{B_r} \theta$$

we still denote

$$\lim_{r\to\infty} \int_{B_r} \theta =: \int_M \theta.$$

This is a sort of *principal value integral.*

[‡]Note that the radiation conditions are associated with a fixed exhaustion.

Proposition 3.14 (Reciprocity) *Dipole fields obey the following reciprocity rules:*

$$\begin{aligned}
{}^{\epsilon}\!E_{BA}(y,z) &= {}^{\epsilon}\!E_{AB}(z,y), &(32)\\
{}^{\mu}\!H_{BA}(y,z) &= {}^{\mu}\!H_{AB}(z,y), &(33)\\
{}^{\mu}\!E_{BA}(y,z) &= (-)^{pq+1}\,{}^{\epsilon}\!H_{AB}(z,y). &(34)
\end{aligned}$$

Proof: All of the three equations are proved from the Maxwell duality in a similar manner. We restrict ourselves to the proof of (34). Let ${}^{\epsilon}\!E$, ${}^{\mu}\!E$ be p-forms and ${}^{\epsilon}\!H$, ${}^{\mu}\!H$ q-forms. Then we have the duality

$$\begin{aligned}
&\left\langle \mathcal{M}\begin{pmatrix} {}^{\epsilon}\!E(x,y)\\ {}^{\epsilon}\!H(x,y)\end{pmatrix}, \begin{pmatrix} \tilde{b}\,{}^{\mu}\!H(x,z)\\ \tilde{a}\,{}^{\mu}\!E(x,z)\end{pmatrix}\right\rangle_M\\
=\ &\left\langle \begin{pmatrix} a\,{}^{\epsilon}\!E(x,y)\\ b\,{}^{\epsilon}\!H(x,y)\end{pmatrix}, \mathcal{M}^*\begin{pmatrix} {}^{\mu}\!H(x,z)\\ {}^{\mu}\!E(x,z)\end{pmatrix}\right\rangle_M
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\left\langle \begin{pmatrix} 0\\ (-)^{\bar{p}\bar{q}}\delta_y(x)*\pi^{(A)}(x)\end{pmatrix}, \begin{pmatrix} \tilde{b}\,{}^{\mu}\!H_{BL}(x,z)\pi^{(L)}(x)\\ \tilde{a}\,{}^{\mu}\!E_{BJ}(x,z)\pi^{(J)}(x)\end{pmatrix}\right\rangle_M\\
=\ &\left\langle \begin{pmatrix} a\,{}^{\epsilon}\!E_{AJ}(x,y)\pi^{(J)}(x)\\ b\,{}^{\epsilon}\!H_{AL}(x,y)\pi^{(L)}(x)\end{pmatrix}, \begin{pmatrix} 0\\ (-)^{\bar{p}\bar{q}}\delta_z(x)*\pi^{(B)}(x)\end{pmatrix}\right\rangle_M
\end{aligned}$$

with $\#A=\#J=p$ and $\#B=\#L=q$. From the definition of $<\cdot,\cdot>$ it is obtained

$$\begin{aligned}
&(-)^{\bar{p}\bar{q}}\tilde{a}\int_M \delta_y(x)*\pi^{(A)}(x)\wedge {}^{\mu}\!E_{BJ}(x,z)\pi^{(J)}(x)\\
=\ &(-)^{\bar{p}\bar{q}}b\int_M {}^{\epsilon}\!H_{AL}(x,y)\pi^{(L)}(x)\wedge\delta_z(x)*\pi^{(B)}(x).
\end{aligned}$$

Since

$$*\pi^{(A)}(x)\wedge\pi^{(J)}(x) = (-)^{p\bar{p}}\pi^{(J)}(x)\wedge *\pi^{(A)}(x)$$

it follows that

$$\begin{aligned}
&(-)^{p\bar{p}}\tilde{a}\int_M \delta_y(x)\,{}^{\mu}\!E_{BJ}(x,z)\pi^{(J)}(x)\wedge *\pi^{(A)}(x)\\
=\ &b\int_M \delta_z(x)\,{}^{\epsilon}\!H_{AL}(x,y)\pi^{(L)}(x)\wedge *\pi^{(B)}(x).
\end{aligned}$$

which is equivalent to

$$(-)^{p\bar{p}}\tilde{a}\,{}^{\mu}\!E_{BJ}(y,z)\pi^{(J)}(y)\rfloor\pi^{(A)}(y) = b\,{}^{\epsilon}\!H_{AL}(z,y)\pi^{(L)}(z)\rfloor\pi^{(B)}(z).$$

As a consequence of the orthonormality

$$(-)^{p\bar{p}}\tilde{a}\,{}^{\mu}\!E_{BA}(y,z) = b\,{}^{\epsilon}\!H_{AB}(z,y).$$

In Proposition 2.10 we denoted $\tilde{a} = (-)^{\bar{p}\bar{q}}$ and $b = (-)^{q\bar{q}}$. Hence $(-)^{p\bar{p}}\tilde{a} = (-)^{\bar{p}}$. Multiplying both sides by $(-)^{\bar{p}}$ and noting that $(-)^{\bar{p}}b = (-)^{pq+1}$ we finally get (34). The proof of (32) is based on the duality

$$\begin{aligned} &\left\langle \mathcal{M}\begin{pmatrix} {}^{\epsilon}\!E(x,y) \\ {}^{\epsilon}\!H(x,y) \end{pmatrix}, \begin{pmatrix} \tilde{b}\,{}^{\epsilon}\!H(x,z) \\ \tilde{a}\,{}^{\epsilon}\!E(x,z) \end{pmatrix} \right\rangle_M \\ = &\left\langle \begin{pmatrix} a\,{}^{\epsilon}\!E(x,y) \\ b\,{}^{\epsilon}\!H(x,y) \end{pmatrix}, \mathcal{M}^* \begin{pmatrix} {}^{\epsilon}\!H(x,z) \\ {}^{\epsilon}\!E(x,z) \end{pmatrix} \right\rangle_M \end{aligned}$$

and (33) follows from

$$\begin{aligned} &\left\langle \mathcal{M}\begin{pmatrix} {}^{\mu}\!E(x,y) \\ {}^{\mu}\!H(x,y) \end{pmatrix}, \begin{pmatrix} \tilde{b}\,{}^{\mu}\!H(x,z) \\ \tilde{a}\,{}^{\mu}\!E(x,z) \end{pmatrix} \right\rangle_M \\ = &\left\langle \begin{pmatrix} a\,{}^{\mu}\!E(x,y) \\ b\,{}^{\mu}\!H(x,y) \end{pmatrix}, \mathcal{M}^* \begin{pmatrix} {}^{\mu}\!H(x,z) \\ {}^{\mu}\!E(x,z) \end{pmatrix} \right\rangle_M . \end{aligned}$$

□

Corollary 3.15 *Assume that E is a p-form and H is a q-form which satisfy the homogeneous Maxwell's equations $\mathcal{M}(E,H)^T = 0$ in Ω. Then we have the following representation formulae for $E(y)$ and $H(y)$, $y \in \Omega$:*

$$E_A(y) = \int_{\partial\Omega} \left({}^{\mu}\!E_{LA}(y,x)\pi^{(L)}(x) \wedge E_J(x)\pi^{(J)}(x) + {}^{\epsilon}\!E_{JA}(y,x)\pi^{(J)}(x) \wedge H_L(x)\pi^{(L)}(x) \right), \tag{35}$$

$$H_B(y) = \int_{\partial\Omega} \left({}^{\mu}\!H_{LB}(y,x)\pi^{(L)}(x) \wedge E_J(x)\pi^{(J)}(x) + {}^{\epsilon}\!H_{JB}(y,x)\pi^{(J)}(x) \wedge H_L(x)\pi^{(L)}(x) \right). \tag{36}$$

Here $\#A = \#J = p$ and $\#B = \#L = q$.

Proof: This is an obvious implication of Proposition 3.11 and Proposition 3.14. □

Corollary 3.16 *Assume that E is a p-form and H is a q-form both of which satisfy the radiation conditions (28), (29) and the homogeneous Maxwell's equations $\mathcal{M}(E,H)^T = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. Then we have the following representation formulae for $E(y)$ and $H(y)$, $y \in \mathbb{R}^3 \setminus \overline{\Omega}$:*

$$E_A(y) = \int_{\partial\Omega} \left({}^{\mu}\!E_{LA}(y,x)\pi^{(L)}(x) \wedge E_J(x)\pi^{(J)}(x) + {}^{\epsilon}\!E_{JA}(y,x)\pi^{(J)}(x) \wedge H_L(x)\pi^{(L)}(x) \right), \tag{37}$$

$$H_B(y) = \int_{\partial\Omega} \left({}^{\mu}\!H_{LB}(y,x)\pi^{(L)}(x) \wedge E_J(x)\pi^{(J)}(x) + {}^{\epsilon}\!H_{JB}(y,x)\pi^{(J)}(x) \wedge H_L(x)\pi^{(L)}(x) \right). \tag{38}$$

Here $\#A = \#J = p$ and $\#B = \#L = q$.

Proof: The claim follows immediately from Proposition 3.11 and Proposition 3.14 when Ω is replaced by $B_r \setminus \overline{\Omega}$ with $r > 0$ large enough so that $\overline{\Omega} \subset B_r$. Finally, let r tend to infinity. □

Remark 3.17 The signs of the integrals in (37) and (38) compared with those in (35) and (36) do not differ from each other since the orientation of $\partial\Omega$ is reversed.

Remark 3.18 Assume that E is a p-form, H is a q-form and $\mathcal{M}(E, H)^T = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. Then $(E, H)^T$ satisfies the electric radiation condition (28) if and only if E has the representation (37). Likewise, $(E, H)^T$ satisfies the magnetic radiation condition (29) if and only if H has the representation (38). From the homogeneous Maxwell's equations it is clear that E has the representation (37) if and only if H has the representation (38).

Proposition 3.19 *Let $\tau_J \pi^{(J)}$ be a p-form and $\eta_L \pi^{(L)}$ a q-form on $\partial\Omega$. Define*

$$\begin{aligned} E_A(y) &:= \int_{\partial\Omega} \big({}^{\mu}\!\mathcal{E}_{LA}(y,x)\pi^{(L)}(x) \wedge \tau_J(x)\pi^{(J)}(x) \\ &\qquad + {}^{\epsilon}\!\mathcal{E}_{JA}(y,x)\pi^{(J)}(x) \wedge \eta_L(x)\pi^{(L)}(x)\big)\,, \end{aligned} \tag{39}$$

$$\begin{aligned} H_B(y) &:= \int_{\partial\Omega} \big({}^{\mu}\!\mathcal{H}_{LB}(y,x)\pi^{(L)}(x) \wedge \tau_J(x)\pi^{(J)}(x) \\ &\qquad + {}^{\epsilon}\!\mathcal{H}_{JB}(y,x)\pi^{(J)}(x) \wedge \eta_L(x)\pi^{(L)}(x)\big)\,. \end{aligned} \tag{40}$$

Then $(E, H)^T = (E_A\pi^{(A)}, H_B\pi^{(B)})^T$ satisfies the homogeneous Maxwell's equations (11), (12) in Ω and in $M \setminus \overline{\Omega}$ as well as the radiation conditions (28), (29).

Proof: The Maxwell's equations follow from

$$\mathrm{d}_y \int_{\partial\Omega(x)} = \int_{\partial\Omega(x)} \mathrm{d}_y \quad \text{and} \quad *_y \int_{\partial\Omega(x)} = \int_{\partial\Omega(x)} *_y$$

since

$$({}^{\epsilon}\!\mathcal{E}_{JA}(\cdot,x)\pi^{(A)}, {}^{\epsilon}\!\mathcal{H}_{JB}(\cdot,x)\pi^{(B)})^T \quad \text{and} \quad ({}^{\mu}\!\mathcal{E}_{LA}(\cdot,x)\pi^{(A)}, {}^{\mu}\!\mathcal{H}_{LB}(\cdot,x)\pi^{(B)})^T$$

satisfy (11), (12) for all $x \in \partial\Omega$. The radiation conditions follow from

$$\int_{B_r(y)} \int_{\partial\Omega(x)} = \int_{\partial\Omega(x)} \int_{B_r(y)}$$

since

$$({}^{\epsilon}\!\mathcal{E}(\cdot,z), {}^{\epsilon}\!\mathcal{H}(\cdot,z))^T \quad \text{and} \quad ({}^{\mu}\!\mathcal{E}(\cdot,z), {}^{\mu}\!\mathcal{H}(\cdot,z))^T$$

satisfy the strong radiation conditions (30), (31) for all $z \in M$. □

In order to prove the Lippmann-Schwinger equation we assume that the submanifold Ω contains a scatterer, i.e., $\epsilon = \epsilon(x)$ and $\mu = \mu(x)$ are not constant in Ω. To be exact, we assume that

$$\mathrm{supp}(\epsilon - \epsilon_0) \cup \mathrm{supp}(\mu - \mu_0) \subset \Omega.$$

Let E, $\tilde{E}$ be p-forms and H, $\tilde{H}$ q-forms. A Maxwell operator for the non-constant ϵ and μ is

$$\mathcal{M}_{\epsilon\mu}\begin{pmatrix} E \\ H \end{pmatrix} := \begin{pmatrix} (-)^{p\bar{p}}\mathrm{d}E & - & \mathrm{i}\omega\mu{*}H \\ (-)^{q\bar{q}}\mathrm{d}H & - & (-)^{pq}\mathrm{i}\omega\epsilon{*}E \end{pmatrix}.$$

Given a dipole moment $\pi^{(A)}$, suppose that there are solutions $({}^{\epsilon}\!E^{\mathrm{tot}}, {}^{\epsilon}\!H^{\mathrm{tot}})$ and $({}^{\mu}\!E^{\mathrm{tot}}, {}^{\mu}\!H^{\mathrm{tot}})$ to the equations

$$\mathcal{M}_{\epsilon\mu}\begin{pmatrix} {}^{\epsilon}\!E^{\mathrm{tot}}(x,y) \\ {}^{\epsilon}\!H^{\mathrm{tot}}(x,y) \end{pmatrix} = \begin{pmatrix} 0 \\ (-)^{\bar{p}\bar{q}}\delta_y(x) * \pi^{(A)}(x) \end{pmatrix}, \tag{41}$$

$$\mathcal{M}_{\epsilon\mu}\begin{pmatrix} {}^{\mu}\!E^{\mathrm{tot}}(x,y) \\ {}^{\mu}\!H^{\mathrm{tot}}(x,y) \end{pmatrix} = \begin{pmatrix} (-)^{\bar{p}\bar{q}}\delta_y(x) * \pi^{(A)}(x) \\ 0 \end{pmatrix}, \tag{42}$$

in M and that these solutions satisfy the radiation conditions. For the sake of consistence we introduce an alternative notation for the fields in (25) with $\pi^{(A)}$ as a dipole moment:

$$\begin{aligned} ({}^{\epsilon}\!E^{\mathrm{in}}, {}^{\epsilon}\!H^{\mathrm{in}}) &:= ({}^{\epsilon}\!E, {}^{\epsilon}\!H), \\ ({}^{\mu}\!E^{\mathrm{in}}, {}^{\mu}\!H^{\mathrm{in}}) &:= ({}^{\mu}\!E, {}^{\mu}\!H). \end{aligned}$$

Then we have the Maxwell's equations

$$\mathcal{M}\begin{pmatrix} {}^{\epsilon}\!E^{\mathrm{in}}(x,y) \\ {}^{\epsilon}\!H^{\mathrm{in}}(x,y) \end{pmatrix} = \begin{pmatrix} 0 \\ (-)^{\bar{p}\bar{q}}\delta_y(x) * \pi^{(A)}(x) \end{pmatrix}, \tag{43}$$

$$\mathcal{M}\begin{pmatrix} {}^{\mu}\!E^{\mathrm{in}}(x,y) \\ {}^{\mu}\!H^{\mathrm{in}}(x,y) \end{pmatrix} = \begin{pmatrix} (-)^{\bar{p}\bar{q}}\delta_y(x) * \pi^{(A)}(x) \\ 0 \end{pmatrix}, \tag{44}$$

in M. Define

$$\begin{aligned} ({}^{\epsilon}\!E^{\mathrm{sc}}, {}^{\epsilon}\!H^{\mathrm{sc}}) &:= ({}^{\epsilon}\!E^{\mathrm{tot}} - {}^{\epsilon}\!E^{\mathrm{in}}, {}^{\epsilon}\!H^{\mathrm{tot}} - {}^{\epsilon}\!H^{\mathrm{in}}), \\ ({}^{\mu}\!E^{\mathrm{sc}}, {}^{\mu}\!H^{\mathrm{sc}}) &:= ({}^{\mu}\!E^{\mathrm{tot}} - {}^{\mu}\!E^{\mathrm{in}}, {}^{\mu}\!H^{\mathrm{tot}} - {}^{\mu}\!H^{\mathrm{in}}). \end{aligned}$$

By subtracting (43) from (41) and (44) from (42) we see that

$$\begin{aligned} \mathcal{M}\begin{pmatrix} {}^{\epsilon}\!E^{\mathrm{sc}}(x,y) \\ {}^{\epsilon}\!H^{\mathrm{sc}}(x,y) \end{pmatrix} &= \begin{pmatrix} \mathrm{i}\omega(\mu(x) - \mu_0) * {}^{\epsilon}\!H^{\mathrm{tot}}(x,y) \\ (-)^{pq}\mathrm{i}\omega(\epsilon(x) - \epsilon_0) * {}^{\epsilon}\!E^{\mathrm{tot}}(x,y) \end{pmatrix}, \\ \mathcal{M}\begin{pmatrix} {}^{\mu}\!E^{\mathrm{sc}}(x,y) \\ {}^{\mu}\!H^{\mathrm{sc}}(x,y) \end{pmatrix} &= \begin{pmatrix} \mathrm{i}\omega(\mu(x) - \mu_0) * {}^{\mu}\!H^{\mathrm{tot}}(x,y) \\ (-)^{pq}\mathrm{i}\omega(\epsilon(x) - \epsilon_0) * {}^{\mu}\!E^{\mathrm{tot}}(x,y) \end{pmatrix}, \end{aligned}$$

in M. Thus we have defined the *total field of an electric dipole*

$$\begin{aligned} {}^{\epsilon}\!E^{\mathrm{tot}}(x,y) &= {}^{\epsilon}\!E^{\mathrm{tot}}_{AL}(x,y)\pi^{(L)}(x), \\ {}^{\epsilon}\!H^{\mathrm{tot}}(x,y) &= {}^{\epsilon}\!H^{\mathrm{tot}}_{AJ}(x,y)\pi^{(J)}(x), \end{aligned}$$

the *total field of a magnetic dipole*

$$\begin{aligned} {}^{\mu}\!E^{\mathrm{tot}}(x,y) &= {}^{\mu}\!E^{\mathrm{tot}}_{AJ}(x,y)\pi^{(J)}(x), \\ {}^{\mu}\!H^{\mathrm{tot}}(x,y) &= {}^{\mu}\!H^{\mathrm{tot}}_{AL}(x,y)\pi^{(L)}(x), \end{aligned}$$

the *incident field of an electric dipole*

$$\begin{aligned} {}^{\epsilon}\!E^{\mathrm{in}}(x,y) &= {}^{\epsilon}\!E^{\mathrm{in}}_{AL}(x,y)\pi^{(L)}(x), \\ {}^{\epsilon}\!H^{\mathrm{in}}(x,y) &= {}^{\epsilon}\!H^{\mathrm{in}}_{AJ}(x,y)\pi^{(J)}(x), \end{aligned}$$

the *incident field of a magnetic dipole*

$$\begin{aligned} {}^{\mu}\!E^{\mathrm{in}}(x,y) &= {}^{\mu}\!E^{\mathrm{in}}_{AJ}(x,y)\pi^{(J)}(x), \\ {}^{\mu}\!H^{\mathrm{in}}(x,y) &= {}^{\mu}\!H^{\mathrm{in}}_{AL}(x,y)\pi^{(L)}(x), \end{aligned}$$

the *scattered field of an electric dipole*

$$\begin{aligned} {}^{\epsilon}\!E^{\mathrm{sc}}(x,y) &= {}^{\epsilon}\!E^{\mathrm{sc}}_{AL}(x,y)\pi^{(L)}(x), \\ {}^{\epsilon}\!H^{\mathrm{sc}}(x,y) &= {}^{\epsilon}\!H^{\mathrm{sc}}_{AJ}(x,y)\pi^{(J)}(x), \end{aligned}$$

and the *scattered field of a magnetic dipole*

$$\begin{aligned} {}^{\mu}\!E^{\mathrm{sc}}(x,y) &= {}^{\mu}\!E^{\mathrm{sc}}_{AJ}(x,y)\pi^{(J)}(x), \\ {}^{\mu}\!H^{\mathrm{sc}}(x,y) &= {}^{\mu}\!H^{\mathrm{sc}}_{AL}(x,y)\pi^{(L)}(x). \end{aligned}$$

In Proposition 3.20 we write the total field as a sum of the incident field and a volume integral which stands for the scattered field.

Proposition 3.20 (Lippmann-Schwinger) *The field of an electric dipole is*

$$\begin{aligned} &{}^{\epsilon}\!E^{\mathrm{tot}}_{BD}(x,y) = \\ &{}^{\epsilon}\!E^{\mathrm{in}}_{BD}(x,y) \quad + \quad (-)^{n}\mathrm{i}\omega\int_{\Omega}(\epsilon(z)-\epsilon_0)\,{}^{\epsilon}\!E^{\mathrm{in}}(z,x)\wedge *\,{}^{\epsilon}\!E^{\mathrm{tot}}(z,y) \\ &\qquad\qquad\qquad + \quad (-)^{n+1}\mathrm{i}\omega\int_{\Omega}(\mu(z)-\mu_0)\,{}^{\epsilon}\!H^{\mathrm{in}}(z,x)\wedge *\,{}^{\epsilon}\!H^{\mathrm{tot}}(z,y), \end{aligned} \tag{45}$$

$$\begin{aligned} &{}^{\epsilon}\!H^{\mathrm{tot}}_{BA}(x,y) = \\ &{}^{\epsilon}\!H^{\mathrm{in}}_{BA}(x,y) \quad + \quad (-)^{n\bar{q}+1}\mathrm{i}\omega\int_{\Omega}(\epsilon(z)-\epsilon_0)\,{}^{\mu}\!E^{\mathrm{in}}(z,x)\wedge *\,{}^{\epsilon}\!E^{\mathrm{tot}}(z,y) \\ &\qquad\qquad\qquad + \quad (-)^{\bar{p}\bar{q}}\mathrm{i}\omega\int_{\Omega}(\mu(z)-\mu_0)\,{}^{\mu}\!H^{\mathrm{in}}(z,x)\wedge *\,{}^{\epsilon}\!H^{\mathrm{tot}}(z,y). \end{aligned} \tag{46}$$

The field of a magnetic dipole is

$$\begin{aligned} &{}^{\mu}\!E^{\mathrm{tot}}_{BA}(x,y) = \\ &{}^{\mu}\!E^{\mathrm{in}}_{BA}(x,y) \quad + \quad (-)^{n}\mathrm{i}\omega\int_{\Omega}(\epsilon(z)-\epsilon_0)\,{}^{\epsilon}\!E^{\mathrm{in}}(z,x)\wedge *\,{}^{\mu}\!E^{\mathrm{tot}}(z,y) \\ &\qquad\qquad\qquad + \quad (-)^{n+1}\mathrm{i}\omega\int_{\Omega}(\mu(z)-\mu_0)\,{}^{\epsilon}\!H^{\mathrm{in}}(z,x)\wedge *\,{}^{\mu}\!H^{\mathrm{tot}}(z,y), \end{aligned} \tag{47}$$

$$
\begin{aligned}
{}^{\mu}\!H^{\rm tot}_{BD}(x,y) \;=&\\
{}^{\mu}\!H^{\rm in}_{BD}(x,y) \quad &+\; (-)^{n\bar q+1}\mathrm{i}\omega\int\limits_{\Omega}(\epsilon(z)-\epsilon_0)\,{}^{\mu}\!E^{\rm in}(z,x)\wedge *\,{}^{\mu}\!E^{\rm tot}(z,y)\\
&+\; (-)^{\bar p\bar q}\mathrm{i}\omega\int\limits_{\Omega}(\mu(z)-\mu_0)\,{}^{\mu}\!H^{\rm in}(z,x)\wedge *\,{}^{\mu}\!H^{\rm tot}(z,y).
\end{aligned}
\tag{48}
$$

Here $\#A=\#C=\#J=p$, $\#B=\#D=\#L=q$, $x,y\in M\setminus\overline{\Omega}$ *and* $x\neq y$.

Proof: Let us first prove (47). From Maxwell duality we obtain

$$
\left\langle \mathcal{M}\begin{pmatrix} {}^{\mu}\!E^{\rm in}(z,y)\\ {}^{\mu}\!H^{\rm in}(z,y)\end{pmatrix}, \begin{pmatrix} \tilde b\,{}^{\epsilon}\!H^{\rm in}(z,x)\\ \tilde a\,{}^{\epsilon}\!E^{\rm in}(z,x)\end{pmatrix}\right\rangle_M + \left\langle \mathcal{M}\begin{pmatrix} {}^{\mu}\!E^{\rm sc}(z,y)\\ {}^{\mu}\!H^{\rm sc}(z,y)\end{pmatrix}, \begin{pmatrix} \tilde b\,{}^{\epsilon}\!H^{\rm in}(z,x)\\ \tilde a\,{}^{\epsilon}\!E^{\rm in}(z,x)\end{pmatrix}\right\rangle_M
$$
$$
= \left\langle \begin{pmatrix} a\,{}^{\mu}\!E^{\rm tot}(z,y)\\ b\,{}^{\mu}\!H^{\rm tot}(z,y)\end{pmatrix}, \mathcal{M}^*\begin{pmatrix} {}^{\epsilon}\!H^{\rm in}(z,x)\\ {}^{\epsilon}\!E^{\rm in}(z,x)\end{pmatrix}\right\rangle_M .
$$

or equivalently

$$
\begin{aligned}
&\int\limits_M \tilde b(-)^{\bar p\bar q}\delta_y(z) *\pi^{(B)}(z)\wedge\, {}^{\epsilon}\!H^{\rm in}_{AL}(z,x)\pi^{(L)}(z) \;+\\
&\int\limits_M \tilde b\mathrm{i}\omega(\mu(z)-\mu_0) *\, {}^{\mu}\!H^{\rm tot}(z,y)\wedge\, {}^{\epsilon}\!H^{\rm in}(z,x) \;+\\
&\int\limits_M \tilde a(-)^{pq}\mathrm{i}\omega(\epsilon(z)-\epsilon_0) *\, {}^{\mu}\!E^{\rm tot}(z,y)\wedge\, {}^{\epsilon}\!E^{\rm in}(z,x)\\
=\;&\int\limits_M a\,{}^{\mu}\!E^{\rm tot}_{BJ}(z,y)\pi^{(J)}(y)\wedge(-)^{\bar p\bar q}\delta_x(z)*\pi^{(A)}(z).
\end{aligned}
$$

Here

$$
\begin{aligned}
*\pi^{(B)}(z)\wedge\pi^{(L)}(z) &= (-)^{q\bar q}\pi^{(L)}(z)\wedge *\pi^{(B)}(z),\\
*\,{}^{\mu}\!H^{\rm tot}(z,y)\wedge\, {}^{\epsilon}\!H^{\rm in}(z,x) &= (-)^{q\bar q}\,{}^{\epsilon}\!H^{\rm in}(z,x)\wedge *\,{}^{\mu}\!H^{\rm tot}(z,y),\\
*\,{}^{\mu}\!E^{\rm tot}(z,y)\wedge\, {}^{\epsilon}\!E^{\rm in}(z,x) &= (-)^{p\bar p}\,{}^{\epsilon}\!E^{\rm in}(z,x)\wedge *\,{}^{\mu}\!E^{\rm tot}(z,y).
\end{aligned}
$$

and by (34)

$$
{}^{\epsilon}\!H^{\rm in}_{AB}(y,x) = (-)^{pq+1}\,{}^{\mu}\!E^{\rm in}_{BA}(x,y).
$$

Hence

$$
\begin{aligned}
&a(-)^{\bar p\bar q}\,{}^{\mu}\!E^{\rm tot}_{BA}(x,y)\\
=\;&\tilde b(-)^{\bar p\bar q}(-)^{q\bar q}(-)^{pq+1}\,{}^{\mu}\!E^{\rm in}_{BA}(x,y)\\
&+(-)^{pq}(-)^{p\bar p}\tilde a\mathrm{i}\omega\int\limits_M(\epsilon(z)-\epsilon_0)\,{}^{\epsilon}\!E^{\rm in}(z,x)\wedge *\,{}^{\mu}\!E^{\rm tot}(z,y)\\
&+(-)^{q\bar q}\tilde b\mathrm{i}\omega\int\limits_M(\mu(z)-\mu_0)\,{}^{\epsilon}\!H^{\rm in}(z,x)\wedge *\,{}^{\mu}\!H^{\rm tot}(z,y)
\end{aligned}
$$

which is equivalent to (47). The identity (45) follows from the duality

$$\left\langle \mathcal{M}\begin{pmatrix} {}^{\epsilon}\!E^{\rm in}(z,y) \\ {}^{\epsilon}\!H^{\rm in}(z,y) \end{pmatrix}, \begin{pmatrix} \tilde{b}\,{}^{\epsilon}\!H^{\rm in}(z,x) \\ \tilde{a}\,{}^{\epsilon}\!E^{\rm in}(z,x) \end{pmatrix} \right\rangle_M + \left\langle \mathcal{M}\begin{pmatrix} {}^{\epsilon}\!E^{\rm sc}(z,y) \\ {}^{\epsilon}\!H^{\rm sc}(z,y) \end{pmatrix}, \begin{pmatrix} \tilde{b}\,{}^{\epsilon}\!H^{\rm in}(z,x) \\ \tilde{a}\,{}^{\epsilon}\!E^{\rm in}(z,x) \end{pmatrix} \right\rangle_M$$
$$= \left\langle \begin{pmatrix} a\,{}^{\epsilon}\!E^{\rm tot}(z,y) \\ b\,{}^{\epsilon}\!H^{\rm tot}(z,y) \end{pmatrix}, \mathcal{M}^*\begin{pmatrix} {}^{\epsilon}\!H^{\rm in}(z,x) \\ {}^{\epsilon}\!E^{\rm in}(z,x) \end{pmatrix} \right\rangle_M,$$

in the same manner, (46) follows from the duality

$$\left\langle \mathcal{M}\begin{pmatrix} {}^{\epsilon}\!E^{\rm in}(z,y) \\ {}^{\epsilon}\!H^{\rm in}(z,y) \end{pmatrix}, \begin{pmatrix} \tilde{b}\,{}^{\mu}\!H^{\rm in}(z,x) \\ \tilde{a}\,{}^{\mu}\!E^{\rm in}(z,x) \end{pmatrix} \right\rangle_M + \left\langle \mathcal{M}\begin{pmatrix} {}^{\epsilon}\!E^{\rm sc}(z,y) \\ {}^{\epsilon}\!H^{\rm sc}(z,y) \end{pmatrix}, \begin{pmatrix} \tilde{b}\,{}^{\mu}\!H^{\rm in}(z,x) \\ \tilde{a}\,{}^{\mu}\!E^{\rm in}(z,x) \end{pmatrix} \right\rangle_M$$
$$= \left\langle \begin{pmatrix} a\,{}^{\epsilon}\!E^{\rm tot}(z,y) \\ b\,{}^{\epsilon}\!H^{\rm tot}(z,y) \end{pmatrix}, \mathcal{M}^*\begin{pmatrix} {}^{\mu}\!H^{\rm in}(z,x) \\ {}^{\mu}\!E^{\rm in}(z,x) \end{pmatrix} \right\rangle_M,$$

and (48) follows from the duality

$$\left\langle \mathcal{M}\begin{pmatrix} {}^{\mu}\!E^{\rm in}(z,y) \\ {}^{\mu}\!H^{\rm in}(z,y) \end{pmatrix}, \begin{pmatrix} \tilde{b}\,{}^{\mu}\!H^{\rm in}(z,x) \\ \tilde{a}\,{}^{\mu}\!E^{\rm in}(z,x) \end{pmatrix} \right\rangle_M + \left\langle \mathcal{M}\begin{pmatrix} {}^{\mu}\!E^{\rm sc}(z,y) \\ {}^{\mu}\!H^{\rm sc}(z,y) \end{pmatrix}, \begin{pmatrix} \tilde{b}\,{}^{\mu}\!H^{\rm in}(z,x) \\ \tilde{a}\,{}^{\mu}\!E^{\rm in}(z,x) \end{pmatrix} \right\rangle_M$$
$$= \left\langle \begin{pmatrix} a\,{}^{\mu}\!E^{\rm tot}(z,y) \\ b\,{}^{\mu}\!H^{\rm tot}(z,y) \end{pmatrix}, \mathcal{M}^*\begin{pmatrix} {}^{\mu}\!H^{\rm in}(z,x) \\ {}^{\mu}\!E^{\rm in}(z,x) \end{pmatrix} \right\rangle_M.$$

□

4 The Scattering Problem

We are going to prove our main result which is the metric independence of the existence and uniqueness of a solution to an exterior boundary value problem. We review the assumptions that we have made up till now:

(i) M is an n-dimensional real oriented C^∞-manifold equipped with a complexified tangent bundle TM and an exhaustion $(B_r)_{r>0}$.

(ii) M has a complex pseudo-Riemannian metric g_{jl} such that there exists a global $\sqrt{g}$ and the Riemann curvature tensor vanishes everywhere.

(iii) Each orthonormal basis $V_1, \ldots, V_n$ of the tangent space T_xM of M at an arbitrary point $x \in M$ can be uniquely extended to a parallel global smooth orthonormal frame $X_1, \ldots, X_n$ such that

$$X_1(x) = V_1, \ldots, X_n(x) = V_n.$$

(iv) There exists a fundamental solution Φ of the scalar Helmholtz operator

$$\varphi \mapsto g^{jl}\varphi_{;jl} + k^2\varphi$$

for which $({}^{\epsilon}\!E(\cdot,z),\, {}^{\epsilon}\!H(\cdot,z))^T$ and $({}^{\mu}\!E(\cdot,z),\, {}^{\mu}\!H(\cdot,z))^T$ satisfy both of the strong radiation conditions (30) and (31) for all $z \in M$.

For a simply connected M there always is a global $\sqrt{g}$ and (iii) is automatically satisfied. From now on Ω and D are supposed to be regular submanifolds of M such that $\overline{\Omega} \subset D$ and $\breve{g}_{jl}$ is another metric tensor for which the assumptions (ii)-(iv) hold. Moreover, it is assumed that

$$g_{jl}|_D = \breve{g}_{jl}|_D. \tag{49}$$

We are considering the following *scattering problem* or *exterior Maxwell boundary value problem* as it is called in [4]:

Find a p-form E and a q-form H both of which are of class C^∞ in M such that

(SC1) E and H satisfy the homogeneous Maxwell's equations (11), (12) in $M \setminus \overline{\Omega}$,

(SC2) E and H satisfy both the electric and magnetic radiation conditions (28), (29) and

(SC3) $E|_{\partial\Omega} = \eta$ for a given p-form η of class C^∞ on $\partial\Omega$.

Here $E|_{\partial\Omega} := i^*E$ is the pull-back of E with respect to the inclusion $i : \partial\Omega \hookrightarrow M$.

Theorem 4.1 *The problem (SC1)-(SC3) for g_{jl} has a unique solution if and only if (SC1)-(SC3) for $\breve{g}_{jl}$ has a unique solution.*

Proof: Assume that (SC1)-(SC3) for g_{jl} has a unique solution (E, H). According to Corollary 3.16 we have the representations (37), (38) for $E(y)$ and $H(y)$, $y \in \mathbb{R}^3 \setminus \overline{\Omega}$. By extending the orthogonal bases $(\pi^{(J)}|_D)$ and $(\pi^{(L)}|_D)$ to $(\breve{\pi}^{(J)})$ and $(\breve{\pi}^{(L)})$ in M (see Lemma 3.10) and replacing g_{jl} by $\breve{g}_{jl}$ in the expressions for $({}^{\epsilon}\!E, {}^{\epsilon}\!H)$, $({}^{\mu}\!E, {}^{\mu}\!H)$ we get the fields

$$\begin{aligned} \breve{E}_A(y)\breve{\pi}^{(A)}(y) &= \int_{\partial\Omega} \Big({}^{\mu}\breve{E}_{LA}(y,x)\pi^{(L)}(x) \wedge E_J(x)\pi^{(J)}(x) \\ &\qquad + {}^{\epsilon}\breve{E}_{JA}(y,x)\pi^{(J)}(x) \wedge H_L(x)\pi^{(L)}(x)\Big)\, \breve{\pi}^{(A)}(y), \end{aligned} \tag{50}$$

$$\begin{aligned} \breve{H}_B(y)\breve{\pi}^{(B)}(y) &= \int_{\partial\Omega} \Big({}^{\mu}\breve{H}_{LB}(y,x)\pi^{(L)}(x) \wedge E_J(x)\pi^{(J)}(x) \\ &\qquad + {}^{\epsilon}\breve{H}_{JB}(y,x)\pi^{(J)}(x) \wedge H_L(x)\pi^{(L)}(x)\Big)\, \breve{\pi}^{(B)}(y). \end{aligned} \tag{51}$$

The conditions (SC1) and (SC2) for $(\breve{E}, \breve{H})^T$ follow from Proposition 3.19. Since $\breve{E}_A|_D = E_A|_D$ and $\breve{H}_B|_D = H_B|_D$ (SC3) is satisfied. Hence $(\breve{E}, \breve{H})^T$ is a solution.

In order to show the uniqueness we assume that $(\breve{E}, \breve{H})^T$ is a solution. Just as above we find a solution (E, H) for g_{jl} the uniqueness of which determines the boundary value $H|_D = \breve{H}|_D$ uniquely. The uniqueness of $(\breve{E}, \breve{H})^T$ is obtained from the representation (50), (51). $\square$

Our goal is to apply Theorem 4.1 in $M = \mathbb{R}^n$ with g_{jl} the standard Cartesian metric and $\breve{g}_{jl}$ a PML-metric arising from a special kind of a smooth stretching function $F_s : \mathbb{R}^n \to \mathbb{R}^n$,

$$s \in \mathbb{C}^{++} = \{a + ib \in \mathbb{C} \mid a, b \geq 0\},$$

for which $F_s|_D = \mathrm{id}_D$. Actually, we have a one-parameter family of stretching functions such that

$$\tilde{x} := F_s(x)$$

is C^∞ in $x \in \mathbb{R}^n$ ([6] assumes C^2) and analytic in $s \in \mathbb{C}^{++}$. Details are found in [6]. For clarity, we assume that $|F_s(x)| \leq p(|x|)$ for some polynomial p ([6] considers more general stretching functions). The metric is defined as in Appendix by

$$\breve{g}_{jl}(x; s) = \sum_{u=1}^{n} \frac{\partial \tilde{x}^u}{\partial x^j} \frac{\partial \tilde{x}^u}{\partial x^l}.$$

Hence we can choose

$$\sqrt{\breve{g}} = \det(\frac{\partial \tilde{x}^j}{\partial x^l}).$$

Since F_s is a $\mathbb{C}$-imbedding $\breve{g}_{jl}(\cdot, s)$ is a complex valued pseudo-Riemannian metric. The metric turns out to be analytic in s and thus is the curvature tensor. For $s \geq 0$ the curvature vanishes which implies that it vanishes for all $s \in \mathbb{C}^{++}$ by analytic continuation. Since $\mathbb{R}^n$ is simply connected an orthonormal frame at $x \in \mathbb{R}^n$ has a unique parallel extension by Proposition 3.10. Moreover, if we choose

$$B_r = \{x \in \mathbb{R}^n \mid |x| < r\}, \quad r > 0,$$

the conditions (i)-(iii) are fulfilled for both g_{jl} and $\breve{g}_{jl}$.

It has been proven in [6] that the standard fundamental solution

$$\Phi(x, y) = \frac{i}{4} \left(\frac{k}{2\pi |x - y|} \right)^{n/2-1} H^{(1)}_{n/2-1}(k |x - y|)$$

of the scalar Helmholtz operator

$$\varphi \mapsto g^{jl} \varphi_{;jl} + k^2 \varphi$$

can be analytically continued to a fundamental solution $\breve{\Phi}$ of the *Bérenger operator*

$$\varphi \mapsto \breve{g}^{jl} \varphi_{;jl} + k^2 \varphi$$

by simply replacing (x, y) by $(\tilde{x}, \tilde{y})$:

$$\breve{\Phi}(x, y) = \frac{i}{4} \left(\frac{k}{2\pi \rho(\tilde{x} - \tilde{y})} \right)^{n/2-1} H^{(1)}_{n/2-1}(k \rho(\tilde{x} - \tilde{y})).$$

The "complex distance" ρ has the property that

$$(\rho(\tilde{x}-\tilde{y}))^2 = \sum_{j=1}^{n} (\tilde{x}^j - \tilde{y}^j)^2.$$

In order to use Theorem 4.1 in $\mathbb{R}^n$ the following Proposition for $\breve{g}_{jl}$ and its counterpart for g_{jl} have to be proved.

Proposition 4.2 *Assume that* $\operatorname{Im} s > 0$*. The dipole fields* $({}^{\epsilon}\breve{E}(\cdot,z),\, {}^{\epsilon}\breve{H}(\cdot,z))^T$ *and* $({}^{\mu}\breve{E}(\cdot,z),\, {}^{\mu}\breve{H}(\cdot,z))^T$ *corresponding* $\breve{g}_{jl}$ *satisfy both of the strong radiation conditions (30) and (31) for all* $z \in M$*.*

Proof: From [6] (Lemma 3.2) we see that for all compact subsets $K \subset \mathbb{R}^n$ there exist constants $C > 0$ and $R > 0$ such that $\operatorname{Im}\rho(\tilde{x}-\tilde{y}) \geq C|x|$ whenever $x \in \mathbb{R}^n$, $|x| > R$ and $y \in K$. Here $\tilde{x} = F_s(x)$ and $\tilde{y} = F_s(y)$. Hence the Hankel function $H^{(1)}_{n/2-1}(k\rho(\tilde{x}-\tilde{y}))$ decays exponentially as $|x| \to \infty$, uniformly with respect to $y \in K$. Since the stretching and thus also the metric have at most a polynomial growth it follows that the dipole fields decay exponentially in the above mentioned sense. Here we used the fact that the derivatives of Hankel functions are linear combinations of Hankel functions. In addition to that the measure of ∂B_r only increases polynomially. □

The case of g_{jl} is much more complicated since the fields $({}^{\epsilon}E(\cdot,z),\, {}^{\epsilon}H(\cdot,z))^T$ and $({}^{\mu}E(\cdot,z),\, {}^{\mu}H(\cdot,z))^T$ do not decay exponentially.

Lemma 4.3 *For a p-form* $\tau = \tau_J \mathrm{d}x^J$*, a q-form* η *and 1-forms* $\gamma = \gamma_j \mathrm{d}x^j$ *and* δ

$$\begin{aligned}
\tau \lfloor \gamma &= \tau_{Ij}\gamma^j \mathrm{d}x^I, & (52)\\
(\tau \wedge \gamma) &= (-)^q {\tau} \lfloor \gamma, & (53)\\
(\tau \lfloor \gamma) &= (-)^{\bar{p}} {\tau} \wedge \gamma, & (54)\\
(\gamma \lfloor \delta)\tau &= (\tau \wedge \gamma) \lfloor \delta + (\tau \lfloor \delta) \wedge \gamma, & (55)\\
*\tau \rfloor (\eta \wedge \gamma) &= \gamma \rfloor *(\tau \wedge \eta). & (56)
\end{aligned}$$

Proof: The first three formulae are readily proved from top to bottom. The last two ones are straightforward. □

The expression (56) can be considered as *the scalar triple product.* If $\gamma = \delta$ and $\gamma \lfloor \gamma = 1$ then (55) becomes

$$\tau = (\tau \wedge \gamma) \lfloor \gamma + (\tau \lfloor \gamma) \wedge \gamma. \qquad (57)$$

In [14] this is called the *Pythagorean theorem.* If $\gamma = \hat{n}$ is a unit normal 1-form of a hypersurface then $\tau_t := (\tau \wedge \hat{n}) \lfloor \hat{n}$ is called the *tangential component* of τ and $\tau_n := (\tau \lfloor \hat{n}) \wedge \hat{n}$ is the *normal component* of τ.

Lemma 4.4 (Stokes theorem) *Let Ω be a regular domain in the standard $\mathbb{R}^n$. If $\hat{n}$ is the exterior unit normal 1-form of Ω on $\partial\Omega$ and τ is an $(n-1)$-form then*

$$\int\limits_{\partial\Omega} *\tau \lfloor \hat{n} = \int\limits_{\Omega} \mathrm{d}\tau.$$

Proof: Choose a tangent-normal coordinate system and use the standard Stokes theorem of differential geometry. $\square$

Proposition 4.5 *In the standard $\mathbb{R}^n$ outside the source point y dipole fields have the following expressions:*

$$\begin{aligned}
{}^{\epsilon}\!E(x,y) &= (-)^n \mathrm{i}\omega\mu_0 \Phi(x,y)\Big((\hat{\pi}\lfloor w)\wedge w - \hat{\pi}\Big) + \mathcal{O}\left(|x|^{-(n+1)/2}\right), \\
{}^{\epsilon}\!H(x,y) &= \mathrm{i}k\Phi(x,y) * \hat{\pi}\lfloor w + \mathcal{O}\left(|x|^{-(n+1)/2}\right), \\
{}^{\mu}\!E(x,y) &= \mathrm{i}k\Phi(x,y) * \hat{\pi}\lfloor w + \mathcal{O}\left(|x|^{-(n+1)/2}\right), \\
{}^{\mu}\!H(x,y) &= (-)^{\bar{p}\bar{q}} \mathrm{i}\omega\epsilon_0 \Phi(x,y)\Big((\hat{\pi}\lfloor w)\wedge w - \hat{\pi}\Big) + \mathcal{O}\left(|x|^{-(n+1)/2}\right).
\end{aligned}$$

Here

$$w := \frac{x^j - y^j}{|x-y|}\,\mathrm{d}x^j.$$

Proof: Let $z, \nu \in \mathbb{C}$ and denote

$$\begin{aligned}
(\nu, m) &:= \frac{[4\nu^2 - 1^2]\,[4\nu^2 - 3^2]\cdots[4\nu^2 - (2m-1)^2]}{2^{2m} m!}, \qquad m = 1,2,3,\ldots, \\
(\nu, 0) &:= 1.
\end{aligned}$$

If $-\pi < \arg z < 2\pi$, $|z| >> |v|$ and $|z| >> 1$ we have the expansion (see [5])

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}}\, e^{\mathrm{i}\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)} \left(\sum_{m=0}^{M-1} \frac{(\nu,m)}{(-2\mathrm{i}z)^m} + \mathcal{O}\left(|z|^{-M}\right)\right).$$

Furthermore

$$\frac{\mathrm{d}H_\nu^{(1)}(z)}{\mathrm{d}z} = \frac{1}{2}\left(H_{\nu-1}^{(1)}(z) - H_{\nu+1}^{(1)}(z)\right).$$

Hence

$$\begin{aligned}
& H_\nu^{(1)}(k|x-y|) \\
=\; & \frac{e^{\mathrm{i}k|x-y|}}{\sqrt{|x-y|}}\sqrt{\frac{2}{\pi k}}\, e^{-\mathrm{i}\frac{\pi}{4}(2\nu+1)} + \mathcal{O}\left(|x|^{-3/2}\right), \\
& \frac{\partial}{\partial x^a}\left(H_\nu^{(1)}(k|x-y|)\right)
\end{aligned}$$

$$
= \quad (\mathrm{i}+1)\sqrt{\frac{k}{\pi}}\; e^{-\mathrm{i}\frac{\pi}{2}\nu}\frac{e^{\mathrm{i}k|x-y|}}{\sqrt{|x-y|}}\frac{x^a-y^a}{|x-y|}+\mathcal{O}\left(|x|^{-3/2}\right),
$$

$$
\frac{\partial^2}{\partial x^b\partial x^a}\left(H_\nu^{(1)}(k|x-y|)\right)
$$

$$
= \quad (\mathrm{i}-1)k\sqrt{\frac{k}{\pi}}\; e^{-\mathrm{i}\frac{\pi}{2}\nu}\frac{e^{\mathrm{i}k|x-y|}}{\sqrt{|x-y|}}\frac{x^a-y^a}{|x-y|}\frac{x^b-y^b}{|x-y|}+\mathcal{O}\left(|x|^{-3/2}\right).
$$

For the fundamental solution Φ we have then

$$
\Phi(x,y) = \mathcal{O}\left(|x|^{-(n-1)/2}\right), \tag{58}
$$

$$
\frac{\partial}{\partial x^a}\Phi(x,y) = \mathrm{i}k\Phi(x,y)\frac{x^a-y^a}{|x-y|}+\mathcal{O}\left(|x|^{-(n+1)/2}\right), \tag{59}
$$

$$
\frac{\partial^2}{\partial x^b\partial x^a}\Phi(x,y) = -k^2\Phi(x,y)\frac{x^a-y^a}{|x-y|}\frac{x^b-y^b}{|x-y|}+\mathcal{O}\left(|x|^{-(n+1)/2}\right). \tag{60}
$$

After replacement in the formulae for dipole fields in Proposition 3.7 we obtain the desired expansions by the aid of Lemma 4.3. □

Let us introduce the *Silver-Müller radiation conditions*:

$$
*E(x)+(-)^n\eta_0 H(x)\wedge\hat{x} = o\left(|x|^{-(n-1)/2}\right), \tag{61}
$$

$$
*H(x)+(-)^{\bar{p}\bar{q}}\eta_0^{-1}E(x)\wedge\hat{x} = o\left(|x|^{-(n-1)/2}\right). \tag{62}
$$

Here $\eta_0 := \sqrt{\mu_0/\epsilon_0}$ is the *wave impedance* and $\hat{x} := x^j/|x|\,\mathrm{d}x^j$.

Proposition 4.6 *In the standard* $\mathbb{R}^n$ *the dipole fields* $({}^{\epsilon}E,\,{}^{\epsilon}H)^T$ *and* $({}^{\mu}E,\,{}^{\mu}H)^T$ *satisfy the following* strong Silver-Müller radiation conditions*: for every compact* $K\subset\mathbb{R}^n$

$$
\sup_{y\in K}\left|*E(x,y)+(-)^n\eta_0 H(x,y)\wedge\hat{x}\right| = o\left(|x|^{-(n-1)/2}\right), \tag{63}
$$

$$
\sup_{y\in K}\left|*H(x,y)+(-)^{\bar{p}\bar{q}}\eta_0^{-1}E(x,y)\wedge\hat{x}\right| = o\left(|x|^{-(n-1)/2}\right). \tag{64}
$$

Proof: The claim follows from Proposition 4.5. □

Corollary 4.7 *Let* $(E,H)^T$ *satisfy the homogeneous Maxwell's equations far from the origin in the standard* $\mathbb{R}^n$*. If* $(E,H)^T$ *satisfies either the electric radiation condition (28) or the magnetic radiation condition (29) then* $(E,H)^T$ *satisfies both of the Silver-Müller conditions (61) and (62).*

Proof: According to Remark 3.18 there are the Stratton-Chu representations (37) and (38) for $(E,H)^T$. It is then straightforward to see that Proposition 4.6 implies the claim. □

Proposition 4.8 *In the standard $\mathbb{R}^n$ the field of an electric dipole satisfies the strong electric radiation condition (30) and the field of a magnetic dipole satisfies the strong magnetic radiation condition (31) for all n.*

Proof: Let K be a compact subset of $\mathbb{R}^n$ and $z \in \mathbb{R}^n$. We have to show that

$$\sup_{y\in K}\left| \int\limits_{S_r^{n-1}} \left(\alpha\, {}^{\epsilon}\!E(x,z) \wedge {}^{\epsilon}\!H(x,y) + \beta\, {}^{\epsilon}\!H(x,z) \wedge {}^{\epsilon}\!E(x,y)\right)\right| \to 0 \qquad (65)$$

and

$$\sup_{y\in K}\left| \int\limits_{S_r^{n-1}} \left(\alpha\, {}^{\mu}\!E(x,z) \wedge {}^{\mu}\!H(x,y) + \beta\, {}^{\mu}\!H(x,z) \wedge {}^{\mu}\!E(x,y)\right)\right| \to 0 \qquad (66)$$

as $r \to \infty$. Let us prove (66). Let the magnetic dipole moment be the q-form $\hat{\pi} = \hat{\pi}_L \mathrm{d}x^L$. Denote

$$\Phi_y := \Phi(\cdot, y) \quad \text{and} \quad \Phi_z := \Phi(\cdot, z).$$

Denote the integrand in (66) by X. After straightforward calculations we obtain from Proposition 3.7

$$\begin{aligned}&(-)^{n-1}\alpha \mathrm{i}\omega\mu_0 X = \\ &(-)^{\bar{p}\bar{q}-1}\delta^{tM}_{sK}\varepsilon^{rL}{}_J\left(\frac{\partial\Phi_z}{\partial x^r}\frac{\partial^2\Phi_y}{\partial x^s\partial x^t} - \frac{\partial\Phi_y}{\partial x^r}\frac{\partial^2\Phi_z}{\partial x^s\partial x^t}\right)\hat{\pi}_L\hat{\pi}_M \mathrm{d}x^J \wedge \mathrm{d}x^K.\end{aligned}$$

Replacement of the derivatives by the expressions (59), (60) yields

$$\begin{aligned}&Y(x,y,z) := \frac{\partial\Phi_z}{\partial x^r}\frac{\partial^2\Phi_y}{\partial x^s\partial x^t} - \frac{\partial\Phi_y}{\partial x^r}\frac{\partial^2\Phi_z}{\partial x^s\partial x^t} = \\ &\mathrm{i}k^3\Phi_y\Phi_z\left(\frac{x^r-y^r}{|x-y|}\frac{x^s-z^s}{|x-z|}\frac{x^t-z^t}{|x-z|} - \frac{x^r-z^r}{|x-z|}\frac{x^s-y^s}{|x-y|}\frac{x^t-y^t}{|x-y|}\right) + \mathcal{O}\left(|x|^{-n}\right).\end{aligned}$$

It is clear that

$$\sup_{y\in K}|Y(x,y,z)| = \mathcal{O}\left(|x|^{1-n}\right)o(1) = o\left(|x|^{1-n}\right)$$

and the claim is proved.

The proof of (65) goes in the same manner. □

Proposition 4.9 *In the standard $\mathbb{R}^n$ the field of an electric dipole satisfies the strong magnetic radiation condition (31) and the field of a magnetic dipole satisfies the strong electric radiation condition (30) provided that n is odd.*

Proof: Let K be a compact subset of $\mathbb{R}^n$ and $z \in \mathbb{R}^n$. We have to show that

$$\sup_{y\in K}\Big|\int\limits_{S_r^{n-1}} \left(\alpha\,{}^{\mu}\!E(x,z)\wedge {}^{\epsilon}\!H(x,y)+\beta\,{}^{\mu}\!H(x,z)\wedge {}^{\epsilon}\!E(x,y)\right)\Big| \to 0 \tag{67}$$

and

$$\sup_{y\in K}\Big|\int\limits_{S_r^{n-1}} \left(\alpha\,{}^{\epsilon}\!E(x,z)\wedge {}^{\mu}\!H(x,y)+\beta\,{}^{\epsilon}\!H(x,z)\wedge {}^{\mu}\!E(x,y)\right)\Big| \to 0 \tag{68}$$

as $r \to \infty$. Let us prove (67). We will imitate the proof of Stratton-Chu formula for exterior domains in [4]. Let $K \subset \mathbb{R}^n$ be a compact set. As in [4] we first prove that

$$\int\limits_{S_r^{n-1}} |\,{}^{\mu}\!E(x,z)|^2 = \mathcal{O}(1).$$

This is a consequence of the radiation condition (61) which holds for $(E,H)^T = ({}^{\mu}\!E(\cdot,z),\, {}^{\mu}\!H(\cdot,z))^T$. Then we represent the integral in (67) as

$$\int\limits_{S_r^{n-1}} {}^{\mu}\!E \rfloor \Big((-)^{p\bar{p}}\alpha{*}({*}\mathrm{d}(\Phi\hat{\pi})\wedge\hat{x}) - (-)^{p}\beta{*}\mathrm{d}{*}(\Phi\hat{\pi})\wedge\hat{x} + (-)^{pq}\beta\mathrm{i}k\Phi\hat{\pi}\Big)$$
$$-(-)^{\bar{p}}\beta\mathrm{i}\omega\mu_0 \int\limits_{S_r^{n-1}} \hat{\pi} \rfloor \Big({*}\,({}^{\mu}\!H\wedge\hat{x}) - (-)^{q\bar{q}}\eta^{-1}\,{}^{\mu}\!E\Big)\Phi.$$

We apply Schwarz inequality to the first integral. Denote

$$u_j := \frac{x^j}{|x|} \quad \text{and} \quad w_j := \frac{x^j-y^j}{|x-y|}.$$

Since $\alpha = (-)^{p\bar{p}}$ and $\beta = (-)^{\bar{q}}$ the right hand side factor of the scalar product is

$$\begin{aligned} Z &:= {*}({*}\mathrm{d}(\Phi\hat{\pi})\wedge\hat{x}) + {*}\mathrm{d}{*}(\Phi\hat{\pi})\wedge\hat{x} - (-)^{p\bar{p}}\mathrm{i}k\Phi\hat{\pi} \\ &= (-)^{q\bar{q}}\delta^{jH}_{lJ}\frac{\partial\Phi}{\partial x^l}\hat{\pi}_J u_j \mathrm{d}x^H + (-)^{q\bar{q}}\delta^{lN}_{J}\frac{\partial\Phi}{\partial x^l}\hat{\pi}_J u_j \mathrm{d}x^j\wedge\mathrm{d}x^N \\ &\quad -(-)^{p\bar{p}}\mathrm{i}k\Phi\hat{\pi}_J\mathrm{d}x^J. \end{aligned}$$

The formula (59) yields

$$\begin{aligned} Z &= (-)^{q\bar{q}}\mathrm{i}k\Phi\Big(\delta^{jH}_{lJ}u_j w_l\hat{\pi}_J\mathrm{d}x^H + \delta^{lN}_{J}u_j w_l\hat{\pi}_J\mathrm{d}x^j\wedge\mathrm{d}x^N + (-)^n\hat{\pi}_J\mathrm{d}x^J\Big) \\ &\quad +\mathcal{O}\left(|x|^{-(n+1)/2}\right) \\ &= (-)^{q\bar{q}}\mathrm{i}k\Phi\Big(\sum_j (u_j w_j + (-)^n)\hat{\pi}_J\mathrm{d}x^J + \sum_{j\neq l}(w_j u_l - u_j w_l)\hat{\pi}_{j\tilde{J}}\mathrm{d}x^{l\tilde{J}}\Big) \\ &\quad +\mathcal{O}\left(|x|^{-(n+1)/2}\right). \end{aligned}$$

If n is odd $u_j w_j + (-)^n = o(1)$. Since always $w_j u_l - u_j w_l = o(1)$ we conclude from (58) that

$$Z = o\left(|x|^{-(n-1)/2}\right).$$

Hence by Schwarz inequality the first integral tends to 0 as $r \to \infty$ provided that n is odd. The second integral tends to 0 for all n because of the radiation condition (63). From the proof it is obvious that the limits are strong with respect to $y \in K$.

The proof of (68) goes in the same manner. □

According to Propositions 4.2, 4.8 and 4.9 we are able to use Theorem 4.1 in $\mathbb{R}^n$ provided that n is odd.

Finally, we want to present some relations between various radiation conditions.

Proposition 4.10 *Let $(E, H)^T$ satisfy the homogeneous Maxwell's equations far from the origin in the standard $\mathbb{R}^n$. If n is odd the radiation conditions (28), (29), (61), (62) are equivalent.*

Proof: From the proof of Proposition 4.9 we see that (61) implies the electric radiation condition (28) and (62) implies the magnetic radiation condition (29). The claim is then a consequence of Remark 3.18 and Corollary 4.7. □

The *Weyl radiation conditions* (see [14]; note that Weyl has a reversed time dependence compared with ours) for a p-form F are defined by

$$(-)^{p\bar{p}}((*\mathrm{d}*F) \wedge \hat{x}) \lfloor \hat{x} - \mathrm{i}kF \lfloor \hat{x} = o\left(|x|^{-(n-1)/2}\right), \tag{69}$$

$$(-)^p \mathrm{d}F \lfloor \hat{x} - \mathrm{i}k(F \wedge \hat{x}) \lfloor \hat{x} = o\left(|x|^{-(n-1)/2}\right). \tag{70}$$

Lemma 4.11 *Let $(E, H)^T$ satisfy the homogeneous Maxwell's equations far from the origin in the standard $\mathbb{R}^n$.*

(i) The Silver-Müller condition (61) for $(E, H)^T$ implies the Weyl radiation conditions (69), (70) for E.

(ii) The Silver-Müller condition (62) for $(E, H)^T$ implies the Weyl radiation conditions (69), (70) for H.

Proof: The proofs of (i) and (ii) are essentially the same. Let us prove (i). Application of Hodge duality, (53) and (11) to (61) yields

$$\mathrm{i}kE + (-)^{\bar{p}} \mathrm{d}E \lfloor \hat{x} = o\left(|x|^{-(n-1)/2}\right). \tag{71}$$

The identity $(\tau \lfloor \gamma) \lfloor \gamma = 0$ implies then

$$\mathrm{i}kE \lfloor \hat{x} = \mathrm{i}kE \lfloor \hat{x} + (-)^{\bar{p}} (\mathrm{d}E \lfloor \hat{x}) \lfloor \hat{x} = o\left(|x|^{-(n-1)/2}\right).$$

From (13) it follows that in (69) for $F = E$ the divergence $\mathrm{d}{*}E = 0$; hence we have proved (69) in the form

$$E \lfloor \hat{x} = o\left(|x|^{-(n-1)/2}\right).$$

According to (57) we obtain from (71)

$$\begin{aligned} (-)^q \mathrm{d}E \lfloor \hat{x} - \mathrm{i}k(E \wedge \hat{x}) \lfloor \hat{x} &= \mathrm{i}k(E \lfloor \hat{x}) \wedge \hat{x} + o\left(|x|^{-(n-1)/2}\right) = \\ o\left(|x|^{-(n-1)/2}\right) + o\left(|x|^{-(n-1)/2}\right) &= o\left(|x|^{-(n-1)/2}\right). \end{aligned}$$

If n is odd $(-)^q = (-)^p$. □

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Appendix. Some Tensor Calculus

In order to fix the notation we briefly refresh the basic concepts of tensor calculus. As in the body of this paper M is an n-dimensional real C^∞-manifold with a complex tangent bundle TM and a complex valued pseudo-Riemannian metric g_{jl}. Thus at every point $x \in M$ the tangent space T_xM consists of sums $U + \mathrm{i}V$ where U and V are real tangent vectors at x. The cotangent space T_x^*M is the complex dual of T_xM (see [9], CC. 7,8).

In a coordinate neighbourhood T_xM and T_x^*M have the coordinate bases

$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \quad \text{and} \quad \mathrm{d}x^1, \ldots, \mathrm{d}x^n,$$

respectively. A generic element of T_xM is a $\mathbb{C}$-linear combination of the form[§]

$$X^j(x)\frac{\partial}{\partial x^j} := \sum_{j=1}^n X^j(x)\frac{\partial}{\partial x^j}, \quad X^j(x) \in \mathbb{C}.$$

It is called a *vector*, a *tangent vector* or a *contravariant vector*. Elements of T_x^*M are called *covectors*, *cotangent vectors* or *covariant vectors*. They appear as

$$X_j(x)\mathrm{d}x^j := \sum_{j=1}^n X_j(x)\mathrm{d}x^j, \quad X_j(x) \in \mathbb{C}.$$

[§]We are following the *Einstein's summation convention* for repeated indices.

A tensor of type (p,q) is a $\mathbb{C}$-linear combination

$$X^{j_1\ldots j_p}{}_{l_1\ldots l_q}(x)\frac{\partial}{\partial x^{j_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{j_p}}\otimes \mathrm{d}x^{l_1}\otimes\ldots\otimes\mathrm{d}x^{l_q}.$$

It is usual to omit the basis vectors in notation and say, e.g., that $X^{j_1\ldots j_p}{}_{l_1\ldots l_q}$ is a (p,q)-tensor. In this Appendix, contrary to what was said in the beginning of the section 2, p and q may be arbitrary nonnegative integers. A special kind of a (p,p)-tensor is the *δ-tensor*

$$\delta^{j_1\ldots j_p}_{l_1\ldots l_p}=\begin{cases}0, & \#\{j_1\ldots j_p\}\neq p \ \text{ or } \ \{j_1\ldots j_p\}\neq\{l_1\ldots l_p\},\\ (-)^\sigma, & \text{otherwise,}\end{cases}$$

where $(-)^\sigma$ is the sign of the permutation σ for which $\sigma(j_\nu)=l_\nu$, $\nu=1,\ldots,p$. For an even permutation the sign is 1 and for an odd permutation it is -1. If S is a finite set $\#S$ stands for the cardinality of S. By means of δ-tensors we define the *e-symbols*

$$\begin{aligned}e_{l_1\ldots l_n} &= \delta^{1\ldots n}_{l_1\ldots l_n},\\ e^{j_1\ldots j_n} &= \delta^{j_1\ldots j_n}_{1\ldots n}.\end{aligned}$$

They are not tensors, i.e., they do not transform like tensors in coordinate changes. Note that

$$e^{j_1\ldots j_p j_{p+1}\ldots j_n}e_{l_1\ldots l_p j_{p+1}\ldots j_n}=(n-p)!\,\delta^{j_1\ldots j_p}_{l_1\ldots l_p}.$$

If $q\le p$ we define an *inner product* of a covariant and a contravariant basis tensor by

$$\begin{aligned}(\mathrm{d}x^{l_1}\otimes\ldots\otimes\mathrm{d}x^{l_q})(\frac{\partial}{\partial x^{j_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{j_p}}) &= \delta^{l_1}_{j_1}\ldots\delta^{l_q}_{j_q}\frac{\partial}{\partial x^{j_{q+1}}}\otimes\ldots\otimes\frac{\partial}{\partial x^{j_p}},\\ (\frac{\partial}{\partial x^{j_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{j_q}})(\mathrm{d}x^{l_1}\otimes\ldots\otimes\mathrm{d}x^{l_p}) &= \delta^{l_1}_{j_1}\ldots\delta^{l_q}_{j_q}\mathrm{d}x^{l_{q+1}}\otimes\ldots\otimes\mathrm{d}x^{l_p},\\ (\mathrm{d}x^{l_1}\otimes\ldots\otimes\mathrm{d}x^{l_p})(\frac{\partial}{\partial x^{j_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{j_q}}) &= \delta^{l_{p-q+1}}_{j_1}\ldots\delta^{l_p}_{j_q}\mathrm{d}x^{l_1}\otimes\ldots\otimes\mathrm{d}x^{l_{p-q}},\\ (\frac{\partial}{\partial x^{j_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{j_p}})(\mathrm{d}x^{l_1}\otimes\ldots\otimes\mathrm{d}x^{l_q}) &= \delta^{l_1}_{j_{p-q+1}}\ldots\delta^{l_q}_{j_p}\frac{\partial}{\partial x^{j_1}}\otimes\ldots\otimes\frac{\partial}{\partial x^{j_{p-q}}}.\end{aligned}$$

The inner product extends for all tensors of the same types by linearity. If $q\le p$ and either X is a $(p,0)$-tensor and Y is a $(0,q)$-tensor or X is a $(0,p)$-tensor and Y is a $(q,0)$-tensor we denote

$$X\lfloor Y:=\frac{1}{q!}XY \quad\text{and}\quad Y\rfloor X:=\frac{1}{q!}YX.$$

Covariant tensors can be transformed into contravariant ones and vice versa by means of the metric tensor $g_{jl}\mathrm{d}x^j\otimes\mathrm{d}x^l$. We define a $\mathbb{C}$-linear map

$$G=G_x:T_xM\to T_x^*M,\ \frac{\partial}{\partial x^l}\mapsto g_{jl}\mathrm{d}x^j,$$

so that (g_{jl}) is the matrix of G with respect to the coordinate basis. By definition the tensor g_{jl} is said to be a *pseudo-Riemannian metric* if it is symmetric and G_x is an isomorphism for all $x \in M$. If we denote by (g^{jl}) the matrix of G^{-1} and define for a vector X and a covector Y

$$X_j := g_{jl} X^l \quad \text{and} \quad Y^j := g^{jl} Y_l$$

then we have

$$G(X^l \frac{\partial}{\partial x^l}) = X_j \mathrm{d}x^j \quad \text{and} \quad G^{-1}(Y_l \mathrm{d}x^l) = Y^j \mathrm{d}x^j.$$

G is called the *index lowering* or *flat* operator (X^l becomes X_j) and its inverse G^{-1} is the *index raising* or *sharp* operator (Y_l becomes Y^j). They are generalized for tensors of types $(p,0)$ and $(0,p)$ by

$$\begin{aligned} G(\frac{\partial}{\partial x^{l_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{l_p}}) &= G(\frac{\partial}{\partial x^{l_1}}) \otimes \ldots \otimes G(\frac{\partial}{\partial x^{l_p}}) \\ &= g_{j_1 l_1} \ldots g_{j_p l_p} \mathrm{d}x^{j_1} \otimes \ldots \otimes \mathrm{d}x^{j_p}, \\ G^{-1}(\mathrm{d}x^{l_1} \otimes \ldots \otimes \mathrm{d}x^{l_p}) &= G^{-1}(\mathrm{d}x^{l_1}) \otimes \ldots \otimes G^{-1}(\mathrm{d}x^{l_p}) \\ &= g^{j_1 l_1} \ldots g^{j_p l_p} \frac{\partial}{\partial x^{j_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{j_p}}. \end{aligned}$$

Moreover,

$$\begin{aligned} X_{j_1 \ldots j_p} &:= g_{j_1 l_1} \ldots g_{j_p l_p} X^{l_1 \ldots l_p}, \\ X^{j_1 \ldots j_p} &:= g^{j_1 l_1} \ldots g^{j_p l_p} X_{l_1 \ldots l_p}, \\ X_{j_1 \ldots j_p}{}^{l_1 \ldots l_q} &:= g_{j_1 r_1} \ldots g_{j_p r_p} g^{l_1 s_1} \ldots g^{l_q s_q} X^{r_1 \ldots r_p}{}_{s_1 \ldots s_q}, \end{aligned}$$

an so on.

Let $q \leq p$. If X is a $(p,0)$-tensor and Y is a $(q,0)$-tensor we define the *scalar product* of X and Y by

$$X \lfloor Y := \frac{1}{q!} X G(Y) \quad \text{or} \quad Y \rfloor X := \frac{1}{q!} G(Y) X.$$

If X is a $(0,p)$-tensor and Y is a $(0,q)$-tensor the scalar product is defined by

$$X \lfloor Y := \frac{1}{q!} X G^{-1}(Y) \quad \text{or} \quad Y \rfloor X := \frac{1}{q!} G^{-1}(Y) X.$$

For two $(0,p)$-tensors

$$X = X_{j_1 \ldots j_p} \mathrm{d}x^{j_1} \otimes \ldots \otimes \mathrm{d}x^{j_p} \quad \text{and} \quad Y = Y_{j_1 \ldots j_p} \mathrm{d}x^{j_1} \otimes \ldots \otimes \mathrm{d}x^{j_p}$$

we have

$$X \lfloor Y \;=\; X \rfloor Y \;=\; Y \lfloor X \;=\; Y \rfloor X \;=\; \frac{1}{q!} X^{j_1 \ldots j_p} Y_{j_1 \ldots j_p} \;=\; \frac{1}{q!} X_{j_1 \ldots j_p} Y^{j_1 \ldots j_p}.$$

By definition an array $A_{j_1 \ldots j_p}$ is said to be *symmetric* if for any permutation σ of the set $\{1, ..., p\}$

$$A_{j_{\sigma(1)} \cdots j_{\sigma(p)}} = A_{j_1 \ldots j_p}.$$

An array $A_{j_1 \ldots j_p}$ is called *totally antisymmetric* if for any permutation σ

$$A_{j_{\sigma(1)} \cdots j_{\sigma(p)}} = (-)^{\sigma} A_{j_1 \ldots j_p}.$$

A covariant tensor

$$X = X_{j_1 \ldots j_p} \mathrm{d}x^{j_1} \otimes \ldots \otimes \mathrm{d}x^{j_p}$$

is symmetric or totally antisymmetric if $X_{j_1 \ldots j_p}$ is symmetric or totally antisymmetric, respectively. The total antisymmetry of an array $A^{j_1 \cdots j_p}$ or a contravariant tensor is defined in the same way. Usual examples of totally antisymmetric tensors are

$$\begin{aligned} \frac{\partial}{\partial x^{j_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{j_p}} &:= \delta^{l_1 \ldots l_p}_{j_1 \ldots j_p} \frac{\partial}{\partial x^{l_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{l_p}}, \\ \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_p} &:= \delta^{j_1 \ldots j_p}_{l_1 \ldots l_p} \mathrm{d}x^{l_1} \otimes \ldots \otimes \mathrm{d}x^{l_p}. \end{aligned}$$

If $X_{j_1 \ldots j_p}$ is totally antisymmetric then

$$\frac{1}{p!} X_{j_1 \ldots j_p} \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_p} = X_{j_1 \ldots j_p} \mathrm{d}x^{j_1} \otimes \ldots \otimes \mathrm{d}x^{j_p}.$$

Ordered multi-indices are often handy tools when manipulating totally antisymmetric tensors. A sequence $J = (j_1, \ldots, j_p)$ is an *ordered multi-index* if $1 \leq j_1 < \ldots < j_p \leq n$. When an index appears as a capital letter it is by default an ordered multi-index. For example, if $X_{j_1 \ldots j_p}$ is totally antisymmetric

$$X_J \mathrm{d}x^J \;:=\; \sum_J X_J \mathrm{d}x^J \;=\; \frac{1}{p!} X_{j_1 \ldots j_p} \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_p}.$$

The sum is taken over all ordered multi-indices J of length p. The scalar product of two totally antisymmetric tensors $X_{j_1 \ldots j_p}$ and $Y_{j_1 \ldots j_p}$ (or $X^{j_1 \cdots j_p}$ and $Y^{j_1 \cdots j_p}$ as well) is

$$X \rfloor Y \;=\; X_J Y^J \;=\; X^J Y_J.$$

If the metric is Riemannian and X has real coefficients the *length* of X is

$$|X| := \sqrt{X \rfloor X}.$$

Totally antisymmetric $(0, p)$-tensors are called *p-forms* or *differential forms of order p* and denoted by Greek letters τ, η and so on.

Let us define

$$g := \det(G).$$

In this work we assume that there exists a global relative scalar h for which $h^2 = g$ and denote $h =: \sqrt{g}$. In a simply connected neighbourhood the existence of $\sqrt{g}$ is always guaranteed. On the other hand, suppose there are tensors $R^{(j)}_l$ and $T^l_{(j)}$, $j = 1, \ldots, n$, such that

$$R^{(j)}_l T^l_{(m)} \;=\; \delta^j_m \;=\; T^j_{(l)} R^{(l)}_m.$$

If we define

$$g_{jl} \;:=\; \sum_{u=1}^{n} R^{(u)}_j R^{(u)}_l \quad \text{and} \quad g^{jl} \;:=\; \sum_{u=1}^{n} T^j_{(u)} T^l_{(u)}$$

then g_{jl} is a pseudo-Riemannian metric, g^{jl} is the inverse metric and we can choose $\sqrt{g} = \det(R^{(j)}_l)$. If $F : \mathbb{R}^n \to \mathbb{C}^n$, $x \mapsto \tilde{x}$, is a $\mathbb{C}$-immersion, i.e., the *velocity vectors* $\partial\tilde{x}/\partial x^1, \ldots, \partial\tilde{x}/\partial x^n$ are $\mathbb{C}$-linearly independent then we can define

$$R^{(j)}_l \;:=\; \frac{\partial \tilde{x}^j}{\partial x^l} \quad \text{and} \quad T^l_{(j)} \;:=\; \frac{\partial x^l}{\partial \tilde{x}^j}.$$

Of course the latter derivative is just a formal notation for the matrix element of the inverse of the matrix $(R^{(j)}_l)$ unless F is real-analytic.

After defining the *ε-tensors* by

$$\begin{aligned} \varepsilon^{j_1 \ldots j_n} &:= \frac{1}{\sqrt{g}}\, e^{j_1 \ldots j_n}, \\ \varepsilon_{j_1 \ldots j_n} &:= \sqrt{g}\, e_{j_1 \ldots j_n}. \end{aligned}$$

we have enough machinery to define the *Hodge star* $(*)$ operator. Let τ be a p-form. We define $*\tau$ as an $(n-p)$-form such that for all p-forms η

$$\eta \wedge *\tau = (\eta \rfloor \tau)\, \sqrt{g}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n.$$

Here $\sqrt{g}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ is the *coordinate invariant volume element* (see [9]). It is clear that

$$\eta \wedge *\tau \;=\; \tau \wedge *\eta \;=\; (-)^{p(n-p)} *\eta \wedge \tau \;=\; (-)^{p(n-p)} *\tau \wedge \eta.$$

There also is an explicit expression for $*\tau$:

$$*(\tau_J \mathrm{d}x^J) = \tau_J \varepsilon^J{}_L \mathrm{d}x^L$$

or equivalently

$$*(\frac{1}{p!}\tau_{j_1 \ldots j_p} \mathrm{d}x^{j_1} \wedge \ldots \wedge \mathrm{d}x^{j_p}) = \frac{1}{p!(n-p)!} \tau_{j_1 \ldots j_p} \varepsilon^{j_1 \ldots j_p}{}_{l_{p+1} \ldots l_n} \mathrm{d}x^{l_{p+1}} \wedge \ldots \wedge \mathrm{d}x^{l_n}.$$

By straightforward calculations we see that for p-forms τ and η

$$\begin{aligned} *\tau &= \tau \rfloor \sqrt{g}\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n, \\ **\tau &= (-)^{p(n-p)} \tau, \\ *\tau \rfloor *\eta &= \tau \rfloor \eta. \end{aligned}$$

For a (p,q)-tensor $X^{j_1\dots j_p}_{l_1\dots l_q}$ we define the *covariant derivative* with respect to an index r by

$$X^{j_1\dots j_p}_{l_1\dots l_q;r} := \frac{\partial}{\partial x^r} X^{j_1\dots j_p}_{l_1\dots l_q} + \sum_{\nu=1}^{p} \begin{Bmatrix} j_\nu \\ h_\nu \; r \end{Bmatrix} X^{j_1\dots\dots h_\nu \dots j_p}_{l_1\dots l_q} - \sum_{\nu=1}^{q} \begin{Bmatrix} h_\nu \\ l_\nu \; r \end{Bmatrix} X^{j_1\dots j_p}_{l_1\dots h_\nu \dots l_q}.$$

Here

$$\begin{Bmatrix} j \\ h \; r \end{Bmatrix} := g^{jk}\,[hr,k]$$

is a *Christoffel symbol of the second kind* written by means of the *Christoffel symbols of the first kind*

$$[hr,k] := \frac{1}{2}\left(\frac{\partial g_{kh}}{\partial x^r} + \frac{\partial g_{rk}}{\partial x^h} - \frac{\partial g_{hr}}{\partial x^k}\right).$$

It is quite easy to verify that

$$g_{jl;r} = 0, \quad \varepsilon^{j_1\dots j_p}{}_{l_{p+1}\dots l_n;r} = 0, \quad \text{and} \quad \delta^{j_1\dots j_p}_{l_1\dots l_p;r} = 0.$$

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