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OPTIMAL WRAP-AROUND NETWORK SIMULATION

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Abstract: *Applying the mathematical theory of tilings, we develop conditions for a minimal grid to yield adequate results in cellular network simulation, adopting the common criterion of the first tier of interference being correctly modelled. We further demonstrate that such a grid can always be realized with a special wrap-around technique. The theory is applicable to various cellular systems, and it yields the best possible solution when T/FDMA with omnidirectional antennas is simulated. The minimal grid then consists of $7N$ sites, where N is the cluster size.*

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1 Introduction

Simulation of a cellular radio network is a prominent tool in testing new algorithms and optimising the system capacity [1, 2, 3, 4]. Such simulations involve multiple aspects and tend to consume lots of computer time, in particular, if a large enough grid, comparable with a real network, is used. Therefore, it would be of interest to know the minimum size of the grid required to get adequate results. Various aspects of network simulation have been studied, but no definite answer to the above mentioned question has been obtained so far.

We present here a method of constructing a minimal grid adopting a commonly accepted criterion of having the *first tier of interference* correctly simulated [4, 5]. This is achieved by use of a revised form of *periodic boundary conditions*, also known as *wrap-around*. The mathematical machinery to be used is developed rather rigorously, but thereafter the formulation of the ideas in applications becomes straightforward and compact. On the way to the applications we also show that some general formulas involved in network design are readily derived from the somewhat abstract results on *tilings* to be discussed first.

The applicability of the general framework of the theory is rather wide, and similar ideas have previously been exploited in molecular dynamics simulations [6]. However, the cellular network system possesses some rather individual characteristics, with which our method works particularly well. The method has been implemented in the network simulator developed at Nokia Research Center [1, 2], and it has been in use in several macro cell simulations.

The paper is organised as follows. The abstract theory to be used in applications is treated in Ch. 2. Sect.'s 2.1 to 2.2 fix some basic notation, whereas symmetry properties of tilings are discussed in Sect.'s 2.3 to 2.4 and inclusion properties in Sect.'s 2.5 to 2.6. Ch. 3 begins with the principles of cellular network design and simulation in Sect.'s 3.1 and 3.2. The main result on optimal grid size is derived in Sect. 3.3 and a concrete example on applying the theory is given in Sect. 3.4. Some extensions and related ideas are visited in Sect.'s 3.5 to 3.6. We conclude the paper in Ch. 4.

2 Properties of tilings

The mathematical machinery and notation is presented in this chapter.

2.1 Basic definitions

A *tiling* is a collection of non-overlapping open bounded sets, the closures of which cover the plane, and each of which is obtained from any other by mere translation. (This is a standard definition, see e.g. [7], except for the translation property, which is not required in general. For our purposes, however, we are only interested in tilings with this property, so we take it as a definition to simplify notion. The word *tessellation* is also used in a similar context.)

The points covered by the tiling (i.e., all the points of the plane, except the ones on the boundary of a tile) are called *tiling points*. Two points a and b in two tiles of a tiling $\mathcal{T}$, say $a \in A$ and $b \in B$, are *equivalent* in $\mathcal{T}$, if the unique translation mapping A onto B maps a into b. The equivalence class of a is denoted by $[a]$. For any tiling point a, $[a]$ consists of exactly one point in each $A \in \mathcal{T}$. Two tilings are equivalent, denoted by $\mathcal{T} \equiv \mathcal{T}'$, if the equivalence relations of points induced by these tilings coincide (for all points covered by both tilings).

The following *finiteness property* of tilings comes to use: A bounded area can only have points of finitely many different tiles. Indeed, assume there were points of infinitely many tiles. Since the tiles are bounded, the whole tiles would be included in a (possibly larger) bounded domain. But the openness of the tiles implies that they take some (equal) positive area, so infinitely many tiles would take an infinite area, contradicting the boundedness of the domain including them.

The *equivalence proximity* $\mathcal{P}(a)$ of a point a is the convex domain

$$\{x : \forall b \in [a] - \{a\} : |x - a| < |x - b|\}$$

of points closer to a than any equivalent point. (The convexity follows from the fact that this domain is an intersection of half-planes.) Given $[a]$, clearly every x is in the closure of $\mathcal{P}(b)$ for some $b \in [a]$, and $\mathcal{P}(b) \cap \mathcal{P}(c) = \emptyset$ for $b, c \in [a]$, $b \neq c$.

A *neighbour* of a is an equivalent point $b \in [a] - \{a\}$ at the closest distance, i.e., $\forall c \in [a] - \{a\} : |c - a| \geq |b - a|$. This closest distance is called the *neighbour distance* $\mathcal{D}(a)$. The *set of neigbours* of a is denoted by $\mathcal{N}(a)$. Any tiling point in any tiling has at least one neighbour, i.e., $\mathcal{N}(a) \neq \emptyset$, since certainly there are some equivalent points, and in a bounded region there are only finitely many, of which some is at the smallest distance.

An open set or a collection of open sets which can be used as tiles in a tiling are said to have the *tiling property*.

Several concepts introduced above are illustrated in Fig. 1. In this figure, $\mathcal{P}(a)$ is surrounded by the dashed polygon, $\mathcal{N}(a)$ consists of a single point b and $\mathcal{D}(a) = D$. If the bottom row of tiles were moved slightly to the left, we would have $\mathcal{N}(a) = \{b, c\}$, and similarly we could have $\mathcal{N}(a) = \{b, d\}$ by shifting the bottom row to the right.

2.2 Homogeneity

A tiling is *homogeneous*, if it has the following property: If b is equivalent to a, then for every tiling point c, $c+(b-a)$ is equivalent to c. In a homogeneous tiling the equivalence of a and b clearly follows, if the translation taking a into b is the same as any of the translations of a tile onto another. Also, if $b \in \mathcal{N}(a)$, then $a' + (b - a)$ is equivalent to a' and we have that $\mathcal{D}(a') \leq |b-a| = \mathcal{D}(a)$, and since the choice of points was quite arbitrary we conclude

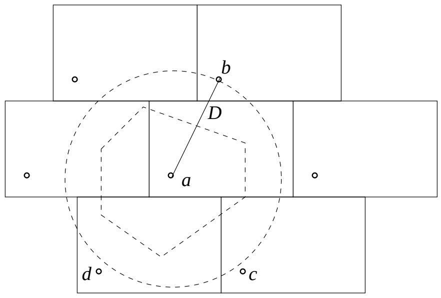

Figure 1: Part of a typical tiling.

that $\mathcal{D}(a) = \mathcal{D}(a')$ is independent of the point a. Thus we denote by $\mathcal{D}(\mathcal{T})$ the common neighbour distance of all the tiling points of a homogeneous tiling $\mathcal{T}$. In particular, if $a \in \mathcal{N}(b)$ then a and b are equivalent and $|a - b| = \mathcal{D}(b)$, but since $\mathcal{D}(b) = \mathcal{D}(a)$ we find that also $b \in \mathcal{N}(a)$, and the relation "is a neighbour of" is a symmetric relation in a homogeneous tiling.

From the homogeneity it also follows, by a similar straightforward argument, that $\mathcal{P}(a)$ and $\mathcal{P}(a')$ only differ by translation for any tiling points a, a'. Consequently, $\mathcal{T}' = \{\mathcal{P}(b) : b \in [a]\}$ is a tiling for any $[a]$. This tiling is equivalent to the original homogeneous tiling $\mathcal{T}$, since $c_1 \in \mathcal{P}(a_1), c_2 \in \mathcal{P}(a_2)$ (where $a_i \in [a]$) are equivalent in $\mathcal{T}'$ iff the translation of c_1 into c_2 maps $\mathcal{P}(a_1)$ onto $\mathcal{P}(a_2)$ iff $c_1 - c_2 = a_1 - a_2$ iff c_1, c_2 are equivalent in $\mathcal{T}$, since a_1, a_2 are equivalent and $\mathcal{T}$ is homogeneous.

As a final property of homogeneous tilings at this stage, we show that $b \in \mathcal{P}(a)$ iff $a \in \mathcal{P}(b)$, i.e., "is in the proximity of" is also a symmetric relation in a homogeneous tiling. Indeed, $b \in \mathcal{P}(a)$ iff the closed disc of centre b and radius $|b - a|$ does not contain any elements of $[a] - \{a\}$. Assume then that $a \notin \mathcal{P}(b)$, i.e., the closed disc of centre a and radius $|a - b|$ does contain $b' \in [b] - \{b\}$. By homogeneity, $a + (b - b') \in [a] - \{a\}$ and $a + (b - b') = b + (a - b')$ is in the closed disc of centre b and radius $|a - b|$, which is a contradiction.

2.3 Polygonal symmetry

A tiling possesses *g-gonal symmetry*, if the following condition holds: If b is equivalent to a, then all the points $a + (b - a)e^{in\frac{2\pi}{g}}$, $n \in \mathbb{Z}$, (i.e., the vertices of a regular g-gon with centre a and one corner at b) are equivalent to a.

In a g-gonally symmetric tiling there are clearly *at least* g neighbours for any tiling point a. Let $g \geq 3$ and b be one of the neighbours of a and assume $\mathcal{D}(b) < \mathcal{D}(a)$. (It is obvious that $\mathcal{D}(b) \leq \mathcal{D}(a)$.) It follows from elementary geometry that then at least one of the g neighbours to b is closer to a than $\mathcal{D}(a)$, which is a contradiction, since these are equivalent to a. (We essentially use the fact that for equal circles with centres at the distance of their common radius, the arc of one circle inside the other disc is $\frac{1}{3} \geq \frac{1}{g}$ of the circumference, and should the other circle be smaller, the above mentioned arc is of strictly larger proportion, and thus contains at least one of the g

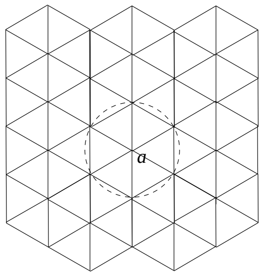

Figure 2: The honeycomb lattice.

neighbours of b.) Thus $\mathcal{D}(b) = \mathcal{D}(a)$ for all $b \in \mathcal{N}(a)$.

If $g > 3$, it follows that there are *exactly* g neighbours for each tiling point a. Indeed, if the number g was exceeded by even one, then the g-gonal symmetry guarantees that there would actually be at least $2g > 6$ neighbours, and the distance between two of these would necessarily be less than $\mathcal{D}(a)$. (This follows from the fact that one edge of a regular hexagon inscribed in a circle is equal to the radius of the circle, and in any n-gon with $n > 6$ the edge is smaller than the radius.)

Now it is easy to inductively extend the pattern of equivalent points to a by using the fact that each such point has g neighbours at the distance $\mathcal{D}(a)$ at the vertices of a regular n-gon. However, the geometry of this procedure readily reveals that, for $g > 3$, only the symmetries with $g = 4, 6$ are possible, since otherwise the extending pattern soon violates the fact $\mathcal{D}(b) = \mathcal{D}(a)$. In these two particular cases, the equivalence class $[a]$ turns out to be the square lattice $\{a + (m + in)(b - a)\}$ or the honeycomb lattice $\{a + (m + e^{i\frac{\pi}{3}}n)(b - a)\}$, respectively, with $m, n \in \mathbb{Z}$. These cases $g = 4, 6$ of particular interest will be referred to as *square symmetry* and *hexagonal symmetry*, respectively The latter is illustrated in Fig. 2, where the lattice points correspond to the crossings of lines. The six neighbours at equal distance from the point a are also shown. Note that while Fig. 2 describes the lattice of one equivalence class of points in any hexagonally symmetric tiling, the shape of the tiles themselves is in no way indicated in the figure.

With $g = 4, 6$, we now know that any point is equivalent to a if and only if it differs from a by $(m + ne^{i\frac{2\pi}{g}})(b - a)$. Now, given any equivalent points c_1, c_2, let $a_1, a_2 \in [a]$ be the unique points of $[a]$ in the same tile with c_1, c_2, respectively. Thus, due to the translation property of a tile onto another, $c_2 - c_1 = a_2 - a_1 = (m + ne^{i\frac{2\pi}{g}})(b - a)$. Conversely, if c_1, c_2 are any two tiling points separated by $(m + ne^{i\frac{2\pi}{g}})(b - a)$, then $a_1 \in [a]$ in the same tile with c_1 is separated from some $a_2 \in [a]$ by this same translation, and since this is then the translation mapping the tile of a_1 onto the tile of a_2, and it also maps c_1 into c_2, we conclude that c_1 and c_2 are equivalent. Thus *any* two points are equivalent in a g-gonal tiling, $g = 4, 6$, if and only if they differ by $(m + e^{i\frac{2\pi}{g}}n)(b - a)$, where a and b are some neighbours. In particular, we find that a g-gonally symmetric tiling, $g = 4, 6$, is homogeneous.

Knowing how the points of $[a]$ are distributed, it is also clear that $\mathcal{P}(a)$ is determined solely by the condition $|x-a| < |x-(a+(b-a)e^{in\frac{2\pi}{g}})|$, $n = 0, \ldots, g-1$, and it is easy to see that the domain determined by such a condition is a regular g-gon. From the results about homogeneous tilings we conclude that a square symmetric tiling is equivalent to a tiling by squares, and a hexagonally symmetric tiling is equivalent to a tiling by regular hexagons. The squares or hexagons in question may be chosen as $\{\mathcal{P}(b) : b \in [a]\}$ for a desired tiling point a.

2.4 Characterization of symmetry

We now have a look at the lower order polygonal symmetries not dealt with in Sect. 2.3, and with the understanding of these, we obtain a complete characterization of polygonally symmetric tilings.

It is possible to show that 3-gonal, i.e., triangular, symmetry always implies hexagonal symmetry. Indeed, if we start from one tiling point and its neighbour in a triangularly symmetric tiling $\mathcal{T}$ and extend the pattern of equivalent points, as above, according to the triangular symmetry, we obtain a honeycomb lattice similar to that with hexagonal symmetry, but with a sublattice removed. The equivalence class of the point 1 (possibly after a change of coordinates) is given by $[1] = L_1 - L_2 = \{m + ne^{i\frac{\pi}{3}}\} - \{(1 + e^{i\frac{\pi}{3}})(k + \ell e^{i\frac{\pi}{3}})\}$. This lattice can also be described in terms of the sublattice as $[1] = (L_2 + 1) \cup (L_2 + e^{i\pi 3})$. (A picture of the situation will clarify the argumentation here.) Should the equivalence class include any of the points L_2, then it would include all of L_2 (as a consequence of triangular symmetry), and we would have hexagonal symmetry. Since the closures of the tiles cover the plane, a given point $c \in L_2$ is in the closure of some $A \in \mathcal{T}$, and we have an $a \in [1] \cap A$. If $a \in L_2 + 1$, then $a + 1 \in [1]$, and the tile $A + 1$ of $a + 1$ contains points arbitrarily close to $c + 1 \in L_2 + 1 \subset [1]$, which is a contradiction, since $c + 1$ is a tiling point and has some open neighbourhood in its own tile. Similar reasoning applies, if $a \in L_2 + e^{i\frac{\pi}{3}}$. Thus triangular symmetry does not exist on its own, but only as part of hexagonal symmetry.

2-gonal, or linear, symmetry simply means that $a + c \in [a]$ implies $a - c \in [a]$, i.e., that the lattice $[a]$ is the same forward and backward in any direction from any point a of the lattice. This can also be described as a reflection property. If $a \in \mathcal{N}(0)$ and $b \in [0]$ is the closest equivalent to 0 outside the line through 0 and a, then reflection arguments can be used to show that $[0]$ at least contains the lattice $\{ma + nb\} - \{ka + \ell b : k, \ell \text{ odd}\}$. Argumentation similar to the proof that triangular implies hexagonal symmetry can then be used to show that even the a priori subtracted lattice must be in $[0]$. Therefore, now that $[0]$ is a simple parallelogram lattice, an argument similar to that showing the homogeneity of square and hexagonally symmetric tilings shows that the linearly symmetric tiling is homogeneous. (The converse of this statement is trivially true.)

We can again apply the property of homogeneous tilings that the set $\{\mathcal{P}(a) : a \in [0]\}$ gives a tiling equivalent to the original, but we can also

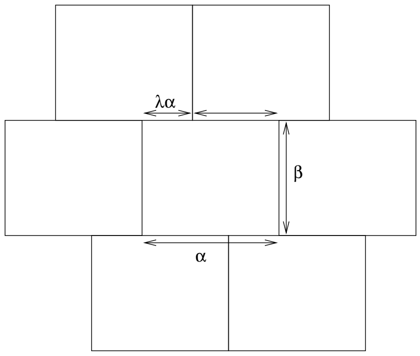

Figure 3: Characteristic parameters of symmetric tilings.

construct explicitly an even more convenient equivalent tiling. Let us fix the coordinate system in such a way that the parallelogram lattice above is of the form $\{m\alpha + n(\lambda\alpha + i\beta)\}$. To each $a \in [0]$ we associate the rectangle with vertices $a \pm \frac{1}{2}\alpha \pm i\frac{1}{2}\beta$. We claim that the set of these rectangles forms a tiling $\widetilde{\mathcal{T}}$ equivalent to the original $\mathcal{T}$. The proof is really quite similar to the proof of the same property for equivalence proximities, so we omit the details.

We now have a complete characterization of all homogeneous plane tilings. Up to equivalence and the choice of coordinate axes, all homogeneous tilings are parametrised by the three parameters α, β and λ as in Fig. 3, where α, β are the lengths of the edges of the rectangle, and λ, $0 \leq \lambda \leq \frac{1}{2}$, indicates how much one row of rectangles has been shifted relative to another. (We can always switch from λ to $1 - \lambda$ after rotating the plane by π radians; this is the reason for $\lambda \leq \frac{1}{2}$ instead of $\lambda < 1$.)

We note in particular that square symmetry is described in terms of these parameters by $\alpha = \beta$, $\lambda = 0$ and hexagonal symmetry by $\alpha : \beta = 2 : \sqrt{3}$, $\lambda = \frac{1}{2}$.

2.5 Inclusions

It is possible to introduce a partial ordering in the set of all tilings in a convenient way, which will be discussed here. Given two tilings, $\mathcal{T}$ and $\mathcal{T}'$, we say that $\mathcal{T}'$ *is included in* $\mathcal{T}$, denoted by $\mathcal{T}' \leq \mathcal{T}$, if the equivalence of any two points in $\mathcal{T}$ implies equivalence also in $\mathcal{T}'$. In particular, this is the case if each of the tiles of $\mathcal{T}$ is composed of N tiles of $\mathcal{T}'$ (similarly located with respect to each other), where $N \in \mathbb{Z}^+$. This gives some justification for the name. It is also obvious that $\mathcal{T}_1 \leq \mathcal{T}_2$ and $\mathcal{T}_2 \leq \mathcal{T}_3$ imply $\mathcal{T}_1 \leq \mathcal{T}_2$.

We also say that $\mathcal{T}' \leq \mathcal{T}$ if equivalence in $\mathcal{T}$ implies equivalence in $\mathcal{T}'$ for all common tiling points (i.e., in the set $(\cup\mathcal{T}) \cap (\cup\mathcal{T}')$). It follows that the two conditions $\mathcal{T}_1 \leq \mathcal{T}_2$ and $\mathcal{T}_2 \leq \mathcal{T}_1$ hold simultaneously precisily when $\mathcal{T}_1 \equiv \mathcal{T}_2$. By using equivalent tilings, where each tile $A \in \mathcal{T}$ is replaced by $A \cap (\cup\mathcal{T}')$ and vice versa, we can assume, without loss of generality, that all tiling points are common to both tilings in any particular inclusion to be considered. This is henceforth understood to be the case here, in order to simplify notion.[1]

Denoting the equivalence class of a in $\mathcal{T}$ by $[a]_{\mathcal{T}}$ etc., we have $\mathcal{T}' \leq \mathcal{T}$ iff $b \in [a]_{\mathcal{T}} \Rightarrow b \in [a]_{\mathcal{T}'}$, i.e., $[a]_{\mathcal{T}} \subset [a]_{\mathcal{T}'}$. If we have $[a]_{\mathcal{T}} = [a]_{\mathcal{T}'}$ then $\mathcal{T} \equiv \mathcal{T}'$. Thus we assume that $[a]_{\mathcal{T}'} \ni b \notin [a]_{\mathcal{T}}$, but then we must also have $[b]_{\mathcal{T}} \subset [a]_{\mathcal{T}'}$. It follows that the equivalence classes of $\mathcal{T}'$ are of the form $[a]_{\mathcal{T}'} = \cup [a_i]_{\mathcal{T}}$.

Let us now take instances $\widetilde{a}_i \in [a_i]_{\mathcal{T}}$ for each i so that $\widetilde{a}_i \in A \in \mathcal{T}$, for the same tile A. (This is possible, since each tile contains exactly one element from each equivalence class.) Now $\widetilde{a}_i \in [a]_{\mathcal{T}'}$ are distinct but equivalent in $\mathcal{T}'$, and thus must lie on different tiles A_i' of $\mathcal{T}'$. Suppose first that the union $[a]_{\mathcal{T}'} = \cup [a_i]_{\mathcal{T}}$ is infinite. But then there are infinitely many tiles $A_i' \in \mathcal{T}'$ at least partly on the tile $A \in \mathcal{T}$, and this contradicts the finiteness property of the tilings. Thus we have a finite union $[a]_{\mathcal{T}'} = \cup_{i=1}^{N} [a_i]_{\mathcal{T}}$, where $[a_i]_{\mathcal{T}}$ are disjoint.

We claim that the N in the expression above is independent of the choice of a. Indeed, assume $[a]_{\mathcal{T}'} = \cup_{i=1}^{N} [a_i]_{\mathcal{T}}$ and $[b]_{\mathcal{T}'} = \cup_{i=1}^{M} [b_i]_{\mathcal{T}}$. Consider then a number of tiles of $\mathcal{T}$. Each of these tiles contains a unique element from each $[a_i]_{\mathcal{T}}$ and each $[b_i]_{\mathcal{T}}$, thus in total N elements of $[a]_{\mathcal{T}'}$ and M of $[b]_{\mathcal{T}'}$. We then take a disc and let the radius tend to infinity, so that the ratio of the number of elements of $[a]_{\mathcal{T}'}$ inside the disc to those of $[b]_{\mathcal{T}'}$ will tend to $\frac{N}{M}$. Now this ratio must be 1, since each tile of $\mathcal{T}'$ contains exactly one element of $[a]_{\mathcal{T}'}$ and one of $[b]_{\mathcal{T}'}$. (We use the fact that both $\mathcal{T}$ and $\mathcal{T}'$ are tilings, in particular essential covers, of the plane.)

Consider further the $\widetilde{a}_i \in A \in \mathcal{T}$ and let $A_i' \in \mathcal{T}'$ be the tile of $\widetilde{a}_i$ in $\mathcal{T}'$. Now each tile of $\mathcal{T}'$ is uniquely related to exactly one element of $[a]_{\mathcal{T}'} = \cup_{i=1}^{N} [a_i]_{\mathcal{T}}$, i.e., to exactly one element of one of the $[a_i]_{\mathcal{T}}$, $i = 1, \ldots, N$. The elements $\widetilde{a}_i \in [a_i]_{\mathcal{T}}$ are, in turn, uniquely related to the tiles A_i' constructed above for $A \in \mathcal{T}$. Thus taking all the tiles $A_i' \in \mathcal{T}'$ for each $i = 1, \ldots, N$ and $A \in \mathcal{T}$, constructed as above, we get exactly once *all* tiles of $\mathcal{T}'$.

For each $A \in \mathcal{T}$, we then take the union $\cup_{i=1}^{N} A_i'$. From the previous remark it follows in particular that all these unions form an essential cover of the plane. We claim that these unions form a tiling $\widetilde{\mathcal{T}}$. Indeed, if $B, C \in \mathcal{T}$, then the translation of B onto C takes in particular each $b_i \in [a_i]_{\mathcal{T}} \cap B$ into $c_i \in [a_i]_{\mathcal{T}} \cap C$. Thus this one translation takes *each* B_i' onto the corresponding C_i', where these are constructed as above. Therefore, the translation of B onto C takes $\cup_{i=1}^{N} B_i'$ onto $\cup_{i=1}^{N} C_i'$. This means in particular that each $\cup_{i=1}^{N} A_i'$ is obtained from any other by mere translation, and $\widetilde{\mathcal{T}}$ is a tiling.

Assume further that $\mathcal{T}$ is homogeneous. Then $c \in A_k', d \in B_j'$ are equivalent in $\widetilde{\mathcal{T}}$ iff the translation of c into d maps $\cup A_i'$ onto $\cup B_i'$ iff $c - d = \widetilde{a}_k - \widetilde{b}_j$ and $k = j$ iff c, d are equivalent in $\mathcal{T}$, since $\widetilde{a}_k, \widetilde{b}_k$ are equivalent in $\mathcal{T}$, and $\mathcal{T}$ is homogeneous. Thus $\widetilde{\mathcal{T}} \equiv \mathcal{T}$. (Note the similarity with the argument which was used to prove the equivalence $\{\mathcal{P}(b) : b \in [a]_{\mathcal{T}}\} \equiv \mathcal{T}$ when $\mathcal{T}$ is homogeneous.)

Summarising, if $\mathcal{T}' \leq \mathcal{T}$, then there exists a tiling $\widetilde{\mathcal{T}} \geq \mathcal{T}'$ such that each

[1]A reader familiar with measure theory should observe the analogy between equivalence of tilings and almost everywhere equality of measurable functions.

tile of $\widetilde{\mathcal{T}}$ consists of N tiles of $\mathcal{T}'$, and the area of a tile of $\widetilde{\mathcal{T}}$ is equal to the area of a tile of $\mathcal{T}$. The integer N will be referred to as the *tiling ratio*. If $\mathcal{T}$ is homogeneous, then $\widetilde{\mathcal{T}} \equiv \mathcal{T}$.

2.6 Symmetric inclusions

Let $\mathcal{T}, \mathcal{T}'$ be g-gonally symmetric and $\mathcal{T}' \leq \mathcal{T}$. We call this situation a *g-gonal(ly symmetric) inclusion.* It is assumed throughout this section that $g = 4, 6$. Since g-gonal symmetry implies homogeneity, $\mathcal{T} \equiv \widetilde{\mathcal{T}}$, where each tile of $\widetilde{\mathcal{T}}$ is a union of N tiles of $\mathcal{T}'$. Since we are concerned with the equivalence relations induced by the tilings, we assume without loss of generality that $\widetilde{\mathcal{T}} = \mathcal{T}$, i.e., that $\mathcal{T}$ has the above mentioned property. Furthermore, we assume that 0 and 1 are neighbours in $\mathcal{T}'$ (possibly after a new choice of coordinates), whence any two points are equivalent in $\mathcal{T}'$ if and only if they differ by $m + ne^{i\frac{2\pi}{g}}$. We can finally assume that tiles of $\mathcal{T}'$ are regular g-gons, since in any case $\mathcal{T}'$ is equivalent to such a tiling.

Since equivalence in $\mathcal{T}$ implies equivalence in $\mathcal{T}'$, neighbours in $\mathcal{T}$ are separated by $m + ne^{i\frac{2\pi}{g}}$ for some $m, n \in \mathbb{Z}$, i.e., we obtain the following *neighbour ratio*:

$$\frac{\mathcal{D}(\mathcal{T})}{\mathcal{D}(\mathcal{T}')} = |m + ne^{i\frac{2\pi}{g}}| = \sqrt{s_g(m,n)},$$

where $s_4(m,n) = m^2 + n^2, s_6(m,n) = m^2 + mn + n^2$.

It follows from elementary geometry that one tile of $\mathcal{T}$ has the area d^2 in the square symmetry and $\frac{\sqrt{3}}{2}d^2$ in the hexagonal symmetry, where $d = \mathcal{D}(\mathcal{T})$. We denote $\alpha_4 = 1, \alpha_6 = \frac{\sqrt{3}}{2}$.

Now each tile of $\mathcal{T}$ is essentially a union of N tiles of $\mathcal{T}'$, so N is the ratio of the areas of tiles in the two tilings. From above, we have another way of expressing this ratio, and we obtain

$$N = \frac{\alpha_g d^2}{\alpha_g d'^2} = s_g(m,n).$$

Thus only tiling ratios of certain form are possible.

Conversely, given $N = s_g(m,n)$ and a g-gonally symmetric tiling $\mathcal{T}'$ (with points 0 and 1 neighbours after a change of coordinates if necessary), take the g-gonal lattice of points

$$\{(k + \ell e^{i\frac{2\pi}{g}})(m + ne^{i\frac{2\pi}{g}}) : k, \ell \in \mathbb{Z}\}$$

and form the equivalence proximities of these points as if they were an equivalence class of a tiling. These indeed give us a g-gonally symmetric tiling $\mathcal{T}$, and it follows rather readily that $\mathcal{T}' \leq \mathcal{T}$. Thus g-gonal inclusions are possible if and only if the tiling ratio is $s_g(m,n)$ for some $m, n \in \mathbb{Z}$. In fact, it is easy to see that all possible ratios can be obtained with non-negative m, n.

By considering successive inclusions, i.e., $\mathcal{T}_1 \leq \mathcal{T}_2 \leq \mathcal{T}_3$, it follows immediately that the product $s_g(m,n)s_g(k,\ell)$ is a proper tiling ratio. Using the previous results, this can also be verified algebraically; in fact, $s_4(m,n)s_4(k,\ell) = s_4(mk - n\ell, m\ell + nk)$ and $s_6(m,n)s_6(k,\ell) = s_6(mk - n\ell, m\ell + nk + n\ell)$.

3 Applications

We now show how the abstract theory is applied to the design and simulation of cellular networks.

3.1 Cellular network concepts

We start with the definition of the main components of a cellular network, making use of the concepts introduced above. See [5] for a more thorough and less mathematical introduction to these matters.

A *site* is the area served by one *base station* (BS). In practice this area will be of varying irregular shape and overlap other sites, but in our theoretical considerations we will assume that the sites are fixed domains having the tiling property with hexagonal symmetry.

Each BS uses a particular set of *radio frequencies* (RF's) to communicate with *mobile stations* (MS's), this being a subset of all the RF's allocated to a given operator, say. We will in particular treat the case of simple *frequency-division multiple access* (FDMA) with a frequency plan in which any two BS's either use exactly the same RF's or have no RF's in common. A macro cell type (non-urban) environment shall mainly be considered, with no landscape structure other than the BS's.

In the context of cellular networks, the significance of the notion of equivalence as defined above lies in the fact that the same MS's at equivalent points in the network are in a similar position in their relation to the surrounding network. In particular, two MS's at equivalent positions in their sites are at the same distance from their serving BS's, and other physical interpretations of equivalence will also emerge below.

From the properties of tilings above, we know that the hexagonally symmetric sites are now equivalent to sites exactly of regular hexagonal shape. This is the usual assumption in theoretical network design [5], and we now see that the a priori more general assumption of hexagonal symmetry is in effect the same. We can thus equally well use the conventional assumption.

A *cluster* is a collection of sites, such that each RF set allocated to a given operator is used exactly once by the BS's of the cluster. We further assume that the clusters, too, have the tiling property with hexagonal symmetry. Now the same RF is used at equivalent points of two clusters, so the notion of equivalence is directly related to the locations of potential sources of *co-channel interference* (CCI). It is also clear, if we denote the tiling consisting of the sites by $\mathcal{S}$ and that of clusters by $\mathcal{C}$, that we have a hexagonal inclusion $\mathcal{S} \leq \mathcal{C}$.

Note that if we take the sites to be of hexagonal shape, a cluster defined as a union of N of these hexagons will not usually be of the same shape although there is an equivalent tiling by hexagons. Now the definition of equivalence and inclusion in rather general terms above offers significant flexibility in notion. In most of the cases the exact shape of the tiling is irrelevant and all that matters is the equivalence relation induced. The idea of tiling is nevertheless useful for visualization and geometric intuition.

In the context of cellular networks the tiling ratio N between $\mathcal{S}$ and $\mathcal{C}$ is referred to as the *cluster size*, and it is well-known [5], and also shown above, that possible values are exactly all

$$N = s_6(m, n) = m^2 + mn + n^2.$$

Since the equivalence in $\mathcal{C}$ is related to *frequency reuse* (FR), i.e., the use of the same RF's in the network, the neighbour distance in $\mathcal{C}$ is called the *mean reuse distance* $D = \mathcal{D}(\mathcal{C})$. The *site radius* R (i.e., half the largest diameter of a hexagonal site) is related to $\mathcal{D}(\mathcal{S})$ by $R = \frac{1}{\sqrt{3}}\mathcal{D}(\mathcal{S})$, which follows from elementary geometry. The formula of neighbour ratio can now be written in the form more common in network design [5]:

$$\frac{D}{R} = \sqrt{3N}.$$

We can also consider the case where 120°-direction antennas are used by the BS's. The area served by one direction antenna is called a *sector* or *cell*. In this case it is common to assume that the sectors are of hexagonal shape [5], and it is clear that we could as well assume just hexagonal symmetry, which yields an equivalent situation. In this case, each site is composed of three hexagons, but $3 = s_6(1, 1)$ is a proper hexagonal tiling ratio, and thus the sites are still equivalent to hexagons, and clusters can be defined exactly as before. If the tiling formed of all the sectors is denoted by $\mathcal{S}'$, we now have successive hexagonal inclusions $\mathcal{S}' \leq \mathcal{S} \leq \mathcal{C}$.

If N is the number of sites in a cluster (i.e., the cluster size) and M is the number of FR sets in use (which equals the number of cells), then the expression N/M is called the *FR ratio*. Of course, with omnidirectional antennas we have $M = N$, and with 120°-direction antennas $M = 3N$.

MS's at equivalent positions in two sites are not only at the same distance but also in the same direction from the serving BS's, and thus receive equal power, given that the direction antennas at the BS's are identical and the same power is transmitted.

3.2 Network simulation

Computer simulation is the most common tool used in evaluating system performance in a cellular network [3], and there are numerous articles touching some side of the subject. Various concepts involved, including the question of grid size and wrap-around technique of our interest here, are treated on a general level in [4]. Important simulation aspects falling outside the scope of the present work are also discussed in [2, 3, 8]. A review of some existing (either commercially or freely available) network simulators is found in [9].

One of the basic problems in simulation models is the fact that the simulations are limited to a finite area with strict boundaries due to obvious technical reasons. The whole simulation area is referred to as the *grid* G.

If the grid has the tiling property, then a simulation field virtually without boundaries could be obtained by constructing the corresponding tiling $\mathcal{G}$ and using a mobility model in which a MS can freely move across the boundary, but is then shifted to an equivalent position on the grid G.

The main concern is the calculation of received powers (both desired and interfering), and this is where problems occur with a simple grid, since no interference is received from behind the boundaries, and thus MS and BS units close to the edges in the simulation tend to perform better than they actually would. As a solution, we could consider an equal transmitter to each one on the grid G at the equivalent positions, and so power would be received also from behind the boundaries. For these power transmissions to match with the configuration of the network inside the grid G, it is necessary and sufficient that equivalence in $\mathcal{G}$ implies equivalence in $\mathcal{C}$, i.e., $\mathcal{C} \leq \mathcal{G}$.

The method of simulation outlined above is known as *wrap-around* [4], and a grid which can be used as a tile in a tiling $\mathcal{G} \geq \mathcal{C}$ will be referred to as a *wrap-around grid*. For practical reasons, it is required that the tiling $\mathcal{G}$ be homogeneous. (Otherwise, with the mobility described above, a MS could move from point a to $a' \in [a]_{\mathcal{G}}$ to return to the same point, but the equal and parallel route from b to $b+(a'-a)$ would not take another MS to the point of its origin. Thus, two MS's moving at equal speed to the same direction would not remain at the same relative displacement! The homogeneity assumption ensures that such bizarre effects do not occur, and things work in accordance with intuition. Luckily, a simulation on a homogeneous tiling is also much easier to implement.)

With our characterization of homogeneous tilings, we hence know that the proper tilings applicable to wrap-around are exactly as in Fig. 3. Conventionally, boundary conditions with $\lambda = 0$ have been used, but we will soon see that this is not the optimal way.

3.3 Optimal grid size

In a finite grid, interference beyond the grid border will not take place. Of course, interference sufficiently far away will not be of significance. The problem of deciding the grid size is about where to draw the line; it is also a significant matter, since the simulation complexity increases rapidly as a function of the grid size (if the density of BS's and MS's is preserved). Since the simulation basically deals with interactions between objects, and the number of possible pairs is $\frac{1}{2}n(n-1)$ when the number of objects is n, a dependence of order $\theta(n^2)$ is expected. Expressed in terms of the length ℓ of the grid (e.g. of square shape), this becomes $\theta(\ell^4)$.

It is quite commonly accepted a principle [4, 5] that the so-called *first tier of interference* is necessary and sufficient to yield sufficiently accurate results. If a is the source of the signal of interest to a given unit, then the first tier of interference is $\mathcal{N}(a)$, where the neighbours are taken in the sense of $\mathcal{C}$, i.e., the closest locations, where the same RF might be in use. This definition can be extended to the tier of order n in an obvious fashion, and in the hexagonally symmetric tiling $\mathcal{C}$, the name "tier" is in consistence with intuition. It is easy to verify that the tiers of order $1, 2, 3$ contain six potential CCI's, each, whereas the fourth tier already has 12.

In order to construct a grid consistent with the above mentioned principle, we thus need at least parts of seven clusters to be in the grid so as to have the six CCI's of the first tier *and* the source of the signal of interest for a given unit. From the inclusion results it follows that $\mathcal{G}$ is equivalent to a tiling $\widetilde{\mathcal{G}}$, where each tile of $\widetilde{\mathcal{G}}$ consists of $n \geq 7$ tiles of $\mathcal{C}$. Hence the absolute minimum grid would be one consisting of seven clusters, six forming a "tier" around the centre one.

Without using wrap-around, problems would still remain with the border clusters, which do not have the full tier around them. This is, however, overcome with wrap-around, and it turns out that the above mentioned minimum grid can always be realized. This follows from the fact that $7 = s_6(2, 1)$ is a valid hexagonal tiling ratio, and thus we can form a hexagonal inclusion $\mathcal{G} \geq \mathcal{C}$.

With the wrap-around technique, we now have infinitely many equivalent signal sources corresponding to each one on the grid G. We must then decide which of these equivalents to use to obtain the maximum benefit of wrap-around. (Always taking the one on G corresponds to not using wrap-around at all.) It would be natural to take the one from which the propagated signal is the strongest. When omnidirectional antennas are used, this is the same as to take the source of interference b such that the receiver $a \in \mathcal{P}(b)$. In a homogeneous tiling (such as $\mathcal{G}$), this is the same as $b \in \mathcal{P}(a)$, and thus we want to take, for each interference source b, the equivalent $\widetilde{b} \in [b]_{\mathcal{G}} \cap \mathcal{P}(a)$. Recall that all the equivalence proximities $\{\mathcal{P}(a') : a' \in [a]_{\mathcal{G}}\}$ form a tiling equivalent to $\mathcal{G}$. The algorithm to find the correct equivalent now only involves some equivalence-preserving shifts, which can be implemented easily. With such a solution, each unit experiences interference as if it was in the centre of a hexagonal grid, which exactly includes the first tier of interference.

Using the fact that a tiling by regular hexagons is equivalent to a tiling by rectangles, the simulation as described above can even be realized on a rectangular grid, which is convenient in view of the implementation.

3.4 An example

We now demonstrate with a concrete example how the ideas discussed above can be exploited in practice to construct a minimal simulation grid.

Suppose that we wish to simulate a network with FR 3/9. The building blocks of the network are now as in Fig. 4. According to the theory, the minimal simulation grid will then consist of seven of the 3-cell (9-sector) clusters, and two copies of such a grid are shown in Fig. 5. The one with stronger lines will be the base of our simulation grid, and the other one is just one of the six surrounding tiles in the tiling $\mathcal{G}$.

Let O be some point in the base grid. The choice of this point is quite arbitrary, but it is practical to take a point which is related to the geometry

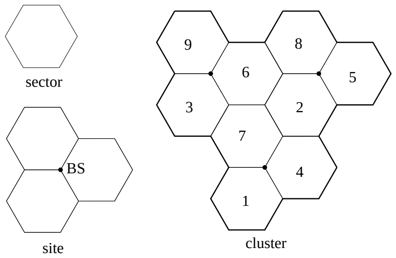

Figure 4: Sector, site and cluster in FR 3/9 with a typical FR pattern.

of the sites in a simple manner, such as the centre of the centre cluster here. The equivalent neighbours of O are labelled A through F. We may draw the segments OA through OF, and the equivalence proximity $\mathcal{P}(O)$ will be determined by the perpendicular bisectors of these segments. It is easy to see that these bisectors coincide with the lines AC, BD, CE, DF, EA and FB, and $\mathcal{P}(O)$ is then the regular hexagon with vertices G through L as in Fig. 5.

The transformation of this hexagon into a rectangle can be done following Sect. 2.4 or as in Fig. 5, where the final grid is the rectangle $JLMN$. Here M and N are the points of intersection of the line through H parallel to JL with the lines FB and EC, respectively. The empty area inside this rectangle is not, of course, left empty, but sites are placed there as they would appear in the tiling by copies of the base grid.

We observe that the locations of the BS's are typically not related to the new coordinate system (induced by the directions of the edges of the rectangle $JLMN$) in any simple manner after the two transformations. It requires some straighforward plane geometry to find the new coordinates. This needs only to be done once, when the grid is constructed, so that it does not affect the computational complexity in any way.

The minimum grid for FR 3/9 constructed here has $7 \times 3 = 21$ sites, and the similar construction for FR 4/12 yields a grid with $7 \times 4 = 28$ sites. In [4] it was claimed that 48 sites is a minimum for both FR 3/9 and FR 4/12 (with the same criterion of the first tier as here), but this depends, of course, on sticking to the conventional form of wrap-around. With our method we could, in fact, get the correct interference from *two* closest tiers with just 39 sites with FR 3/9 (see Sect. 3.5), and even this is clearly less than the 48 sites in [4].

3.5 Different cellular systems

Should we at some point want to get the interference from more tiers than just the first one, the extension of the above consideration is straightforward. For instance, for 2 or 3 tiers we require that the grid be composed of 13 or 19 clusters, respectively, and the inclusion results imply that these can be realized.

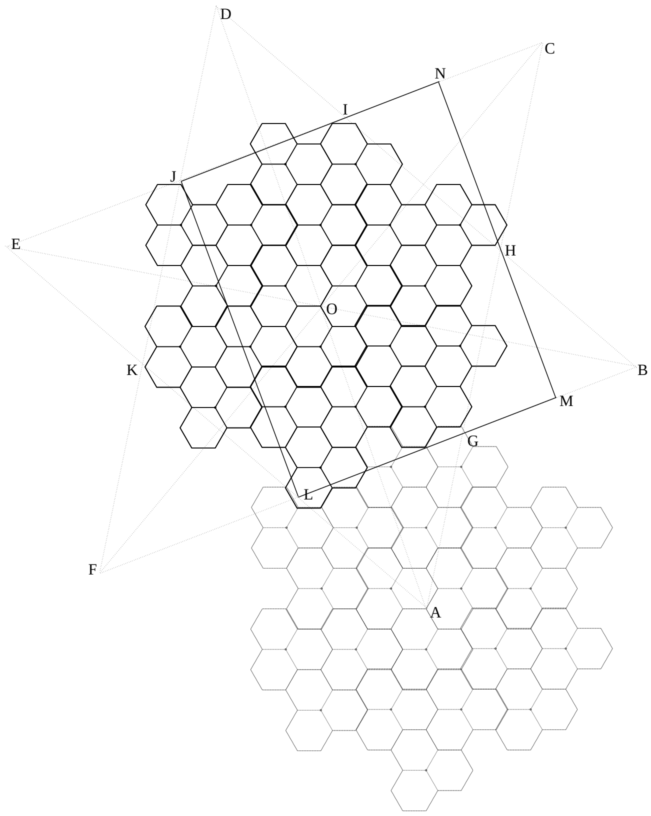

Figure 5: Grid construction for FR 3/9.

It is also of interest to simulate other than pure FDMA systems. The use of time-division multiple access (TDMA) causes no special problem as such, since the above considerations apply during each fixed time slot, and the equivalence relations are preserved, although different connections are active during different time slots. Code-division multiple access (CDMA), in which all connections can operate on the same RF, can also be handled, once we realize that a cluster in the sense of the above considerations is now the same as a site, and thus we just have a special case with cluster size $N = 1$. There could also appear some other ways, possibly related to the code used in separating connections, of determining the most significant sources of interference, but we will not treat this question further.

It should be noted that when no RF separation is made between uplink (MS to BS) and downlink (BS to MS) transmission, it is also possible to have interference between two MS's or two BS's, and significant CCI may occasionally appear closer than the first tier of interference. (Interference modelling for simulation purposes in such a case is discussed in [8].) However, this definitely does not reduce the CCI from the first tier, and it is still assumed that this tier is necessary.

Extra considerations are needed when dealing with T/FDMA systems using frequency hopping (FH). When FH is used, the allocation of RF's changes from a time slot to another, so that the CCI's will not stay at fixed locations, and the definition of equivalence as used above is not related to the

RF's as strictly as before. Neither will there be a fixed first tier of interference, which was the crucial part in the above considerations. However, we can still define a cluster as an area of sufficiently many sites, so that the total number of RF sets used in one cluster is equal to the overall total of RF sets allocated to the operator. In such an area we typically expect one CCI, although its location can be other than the equivalent position. Thus we could still do with seven clusters, which ensure the presence of the six potential CCI's to a MS at a in the clusters containing the points $\mathcal{N}(a)$.

If directional antennas are used in BS's, the definition of the first tier of interference from the point of view of a MS also becomes somewhat fuzzy, since the variation in antenna gain to different directions is typically large enough to make the BS-CCI's in the second tier with the antenna directed towards the MS more significant than the BS-CCI's in the first tier with opposite orientation. Thus the optimal way of exploiting wrap-around would now require taking the equivalents of the CCI's from a domain other than the hexagonal equivalence proximity, which is evaluated quite easily. This would cause quite a lot of complications, but the above approach still yields a reasonably good solution, though not absolutely optimal as before. But if we accept the first tier to be sufficient (which is the case in [4]), we still have a solution. It is nevertheless worth pointing out that the exploitation of the gain pattern of the directional antennas could yield some enhancement, unless it gets too complicated.

If we wish to add some structure to the landscape, extra considerations are also needed. No sudden changes in the environment should be caused by the boundary conditions, but the landscape on two sides of an edge of the wrap-around grid should match smoothly. This is analogous to the requirement $\mathcal{C} \leq \mathcal{G}$, which in effect states the same thing for the FR plan and BS locations. This point becomes crucial in particular in micro cell type (urban) simulations, where MS's typically move in street canyons, which should match on the border with the streets on the other side of the grid.

3.6 Further remarks

The importance of different boundary condition geometries in the context of molecular dynamics (MD) simulation was already recognised in [6]. These simulations, of course, involve three dimensions, but it is readily observed from Sect.'s 2.1 and 2.2 that all the basic definitions of tilings and homogeneity immeadiately extend to higher dimensions. Indeed, the two-dimensionality above only occurs in the terminology, such as discs rather than balls. This is even the case with inclusions (Sect. 2.5), although this part of the theory appears to be more connected to the cellular networks, where it plays a prominent role. The notion of symmetry does require some reconsideration in three dimensions, but this is also achievable when the regular polygons are replaced by regular or semiregular polyhedra [7].

The particle interactions of interest in MD are typically due to central forces with an essentially finite range. This is analogous to the problem with cellular networks, where the first tier of interference is to be included in the simulations, but the rest may be ignored. In MD, the corresponding part of space could be referred to as the ball of interaction, and in order to minimise the grid size we need to find the smallest solid with the tiling property, in which the ball can be inscribed.

Thus there is a direct connection to the classical problem of mathematics called *sphere packing* [7]. This has been studied, and it is known that the best possible packing ratio with a regular lattice is achieved by *hexagonal close packing* (hcp), where the tiles have the shape of a *rhombic dodecahedron*. Three dimensional boundary conditions derived from this and other possible geometries are discussed in more detail in [6].

The packing ratio is the ratio of volumes of the balls and all space, or equivalently the ratio of the ball inscribed in a polyhedron tile to the whole tile. In view of the applications to simulation theory, it is the portion of "effective grid content". For hcp this ratio is $\frac{\pi}{3\sqrt{2}} = 74.05\%$, whereas a simple cubic structure would only give $\frac{\pi}{6} = 52.36\%$ [7]. Similar ratios can also be computed for plane tilings, now assuming the range of interaction to be a disc inscribed in the grid. For the square the ratio is $\frac{\pi}{4} = 78.54\%$; for the regular hexagon we have $\frac{\pi}{2\sqrt{3}} = 90.69\%$, which is, again, the best possible packing ratio of discs on the plane [7]. Thus it is clear that any objects with a fixed radius of interaction can be simulated in both two and three dimensions more efficiently, when boundary conditions are derived from the special tile shapes discussed here.

In cellular network simulation, the method performs even better, as due to the technical nature of the network, the range of interaction can be regarded as the hexagon, as discussed above, so that the approximation of the effective range is in fact exact. We should note, on the other hand, that the hexagons are used in network design exactly for the reason that they approximate a disc so well [5], and our method works so nicely, since there is no further need for approximation.

A fascinating observation from applications to both cellular networks and MD is the fact that the optimal periodic boundary conditions in some sense reflect the geometry of the objects simulated: in the former, the BS locations and the channel allocation plan, in the latter, the crystal structure of matter. In a sense, the solution is very natural.

There is another interesting observation related to the topology of wrap-around: The conventionally used periodic boundary conditions (in 2 dimensions) can be visualised by folding the (rectangular) simulation grid in 3 dimensions as sketched in Fig. 6. The optimal wrap-around in the plane can be similarly visualised, except that then the points A and N in Fig. 6 must be joined with M and B, respectively, not with B and M as in the figure. The surface so obtained will be the famous Möbius strip [7].

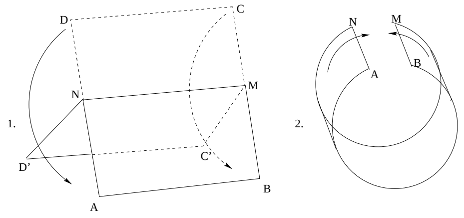

Figure 6: Illustration of wrap-around.

4 Summary

New kinds of simulation methods were considered, based on the mathematical theory of tilings, which gives a convenient description of some of the relevant aspects of cellular networks. These extend the ideas involved in the well-known simulation principle of wrap-around, which is used to overcome the effects of a finite grid. We found that certain "natural" periodic boundary conditions can be used to reduce the size of the simulation grid in network simulations. The results show that with the criterion of having the first tier of interference correctly simulated, the minimal grid consists of seven clusters, and such a grid can always be realized with the techniques discussed herein. When T/FDMA with omnidirectional antennas is simulated, our solution cannot be improved, but in some other cases, taking into account special properties of the system could yield further enhancement.

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