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ANALYSIS OF A BILINEAR FINITE ELEMENT FOR SHALLOW SHELLS II: CONSISTENCY ERROR

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Abstract: *We consider a bilinear reduced-strain finite element of the MITC family for a shallow Reissner-Naghdi type shell. We estimate the consistency error of the element in both membrane- and bending-dominated states of deformation. We prove that in the membrane-dominated case, under severe assumptions on the domain, the finite element mesh and on the regularity of the solution, an error bound $O(h + t^{-1}h^{1+s})$ can be obtained if the contribution of transverse shear is neglected. Here t is the thickness of the shell, h the mesh spacing, and s a smoothness parameter. In the bending-dominated case, the uniformly optimal bound $O(h)$ is achievable but requires that membrane and transverse shear strains are of order $O(t^2)$ as $t \to 0$. In this case we also show that under sufficient regularity assumptions the asymptotic consistency error has the bound $O(h)$.*

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1 Introduction

Approximation of deformation states arising in thin shells by low-order finite element methods is known to be a nontrivial task. Different locking modes degrade the convergence rate of the most basic formulations when approximating bending-dominated or inextensional deformations. However, it is equally well-known by now that a suitable variational crime can be used to retain the convergence properties in such cases. This can even be done up to an optimal order and smoothness requirements for certain shell geometries as it was shown in Part I of this paper [3], see also [6]. The real challenge begins when one aims to find a formulation that has a satisfactory behavior also in the membrane-dominated states of deformation. In this case one is inevitably led to consider the questions of consistency and stability of the formulation since the approximation properties will rarely be a problem in such a case, but lack of consistency or stability can yield a very large error component.

Probably most low-order shell elements aimed to be general in nature contain the basic ideas of MITC4 by Bathe and Dvorkin [1]. In [4] it was shown that this formulation is in fact equivalent to a certain variational crime already considered in [6]. In this paper we extend our analysis of the MITC-type elements and address their consistency and stability properties. We show that at least under favorable circumstances, this kind of an element can indeed approximate well also membrane-dominated deformation. However, due to the lack of stability in the membrane-dominated case, we can bound the consistency error in this case only non-uniformly with respect to the thickness t of the shell. We need also strong assumptions on the problem setup and on the finite element mesh as in the previous part [3]. Under such hypotheses and under certain additional hypotheses on the solution we show that the consistency error is at most of order $O(h + t^{-1}h^{1+s})$ where h is the mesh spacing and $s \geq 0$ is a parameter depending on the degree of smoothness of the exact solution. As s can be arbitrarily large in principle, one can have $t^{-1}h^s = O(1)$ for reasonable sequences of (t, h) if the solution is very smooth. In such a case the consistency error is $O(h)$, which is the optimal order for bilinear elements. The conclusion from the Parts I-II of the paper is then that at least under extremely favorable circumstances, both bending- and membrane-dominated smooth deformations can be approximated with nearly optimal accuracy by the bilinear MITC4 element. To what extent this holds for more general problem setups, deformation states, and finite element meshes, is a wide open problem.

Another topic to be considered in this paper is the asymptotic behavior of the consistency error in the case of an inextensional deformation. In [3] we considered the problem of finding a best finite element approximation of a given inextensional deformation. At that point the question of consistency was deliberately left aside. However, in real computations one is inevitably faced with the fact that since the reduced inextensional space is not a subspace of the corresponding continuous space, the consistency error does not tend to zero when the thickness $t \to 0$, but to some finite value depending on h. Here we show that this error term is of the optimal order $O(h)$. However, to obtain this result we need much stronger regularity assumptions on the exact solution than in the previous Part I [3] where we bounded the approximation error. Whether our analysis here is sharp is not clear at the moment.

The plan of this paper is as follows. In section 2 we describe the problems to be considered and in section 3 we consider two slightly different FEM approximations to these. Section 4 is devoted to the consistency error in the non-asymptotic case ($t > 0$) whereas section 5 deals with the asymptotic consistency error in the inextensional deformation state.

In the following we denote by C a generic constant that may take a different value in different usage. The constants may depend on the geometry parameters of the problem but are otherwise independent of the parameters, unless indicated explicitly. The Sobolev norm and seminorm are denoted by $||\cdot||_k$ and $|\cdot|_k$ respectively on the assumed rectangular domain. Further, $||\cdot||_{L^2} = ||\cdot||_0$.

2 The shell problem

We use basically the same shell model of Reissner-Naghdi type as in [3] but with two different scalings. Denoting by $\underline{u} = (u, v, w, \theta, \psi)$ the vector of three translations and two rotations we let the (scaled) total energy of the shell be given either by

$$\mathcal{F}_M(\underline{u}) = \frac{1}{2}\big(t^2 \mathcal{A}_b(\underline{u}, \underline{u}) + \mathcal{A}_m(\underline{u}, \underline{u})\big) - Q(\underline{u})$$

or by

$$\mathcal{F}_B(\underline{u}) = \frac{1}{2}\big(\mathcal{A}_b(\underline{u}, \underline{u}) + t^{-2} \mathcal{A}_m(\underline{u}, \underline{u})\big) - Q(\underline{u})$$

where t is the thickness of the shell and Q represents the load potential. Here the subscripts M and B refer to the natural scalings of the total energy in membrane and bending-dominated deformations, respectively. We assume that in both cases $Q(\underline{u})$ defines a bounded linear functional on the corresponding energy space to be defined later. The bilinear forms $\mathcal{A}_b(\underline{u}, \underline{v})$ and $\mathcal{A}_m(\underline{u}, \underline{v})$ arising from the bending and membrane energies are given by

$$\mathcal{A}_b(\underline{u}, \underline{v}) = \int_\Omega \Big\{\nu(\overline{\kappa}_{11} + \overline{\kappa}_{22})(\underline{u})(\kappa_{11} + \kappa_{22})(\underline{v}) + (1-\nu)\sum_{i,j=1}^{2} \overline{\kappa}_{ij}(\underline{u})\kappa_{ij}(\underline{v})\Big\} dx dy$$

and

$$\mathcal{A}_m(\underline{u},\underline{v}) = 6\gamma(1-\nu)\int_\Omega \{\overline{\rho}_1(\underline{u})\rho_1(\underline{v}) + \overline{\rho}_2(\underline{u})\rho_2(\underline{v})\}dxdy$$
$$+12\int_\Omega \{\nu(\overline{\beta}_{11}+\overline{\beta}_{22})(\underline{u})(\beta_{11}+\beta_{22})(\underline{v})$$
$$+ (1-\nu)\sum_{i,j=1}^{2}\overline{\beta}_{ij}(\underline{u})\beta_{ij}(\underline{v})\}dxdy$$

where overbars denote complex conjugation. Here ν is the Poisson ratio of the material, γ is a shear correction factor and κ_{ij}, β_{ij} and ρ_i represent the bending, membrane and transverse shear strains respectively depending on $\underline{u}$ as

$$\begin{aligned}
\beta_{11} &= \frac{\partial u}{\partial x} + aw & \kappa_{11} &= \frac{\partial \theta}{\partial x} \\
\beta_{22} &= \frac{\partial v}{\partial y} + bw & \kappa_{22} &= \frac{\partial \psi}{\partial y} \\
\beta_{12} &= \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + cw = \beta_{21} & \kappa_{12} &= \frac{1}{2}(\frac{\partial \theta}{\partial y} + \frac{\partial \psi}{\partial x}) = \kappa_{21}
\end{aligned} \tag{2.1}$$

and

$$\rho_1 = \theta - \frac{\partial w}{\partial x} \qquad \rho_2 = \psi - \frac{\partial w}{\partial y}. \tag{2.2}$$

The integration is taken over the midsurface Ω of the shell which we assume to occupy the rectangular region $(0,L)\times(0,H)$ in the xy-coordinate space satisfying $d^{-1} \le L/H \le d$ for some constant $d > 0$. We are considering the shell to be shallow and assume that the parameters a, b and c defining the geometry can be taken constants. We further note that if $ab - c^2 > 0$ the shell is elliptic, if $ab - c^2 = 0$ it is parabolic and if $ab - c^2 < 0$ we have a hyperbolic shell.

The above two energy formulations lead naturally to two differently scaled variational formulations, the membrane (M) and bending (B) cases:

(M) Find $\underline{u} \in \mathcal{U}_M$ such that

$$\mathcal{A}_M(\underline{u},\underline{v}) = t^2\mathcal{A}_b(\underline{u},\underline{v}) + \mathcal{A}_m(\underline{u},\underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_M. \tag{2.3}$$

(B) Find $\underline{u} \in \mathcal{U}_B$ such that

$$\mathcal{A}_B(\underline{u},\underline{v}) = \mathcal{A}_b(\underline{u},\underline{v}) + t^{-2}\mathcal{A}_m(\underline{u},\underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_B. \tag{2.4}$$

Here $\mathcal{U}_M$ and $\mathcal{U}_B$ are the membrane and bending energy spaces, respectively, which we take to be subspaces of $[H^1_p(\Omega)]^5$ where $H^1_p(\Omega)$ is the usual Sobolev space with periodic boundary conditions imposed at $y = 0, H$. In $\mathcal{U}_B$ no constraints are imposed at $x = 0, L$ whereas in $\mathcal{U}_M$ we assume the constraints $u = v = w = \theta = \psi = 0$ at $x = 0, L$. In Case (B) we must also remove the rigid displacements from $\mathcal{U}_B$ so as to make (2.4) uniquely solvable. For the convenience of our error analysis, we make here somewhat stronger assumptions than are needed for the well-posedness of (2.4): We introduce the set of pseudo-rigid displacements as

$$\mathcal{Z} = \{\underline{v} \in [H_p^1(\Omega)]^5 \mid \underline{v} = \sum_{i=1}^{5} C_i \underline{e}_i\}$$

where $\underline{e}_i$ is the ith Euclidean unit vector, assume that $Q(\underline{v}) = 0$ for every $\underline{v} \in \mathcal{Z}$, and let $\mathcal{U}_B = \mathcal{Z}^{\perp}$ in $[H_p^1(\Omega)]^5$. Finally we denote the energy norms on $\mathcal{U}_M$ and on $\mathcal{U}_B$, respectively, by $|||\cdot|||_M = \sqrt{\mathcal{A}_M(\cdot,\cdot)}$ and $|||\cdot|||_B = \sqrt{\mathcal{A}_B(\cdot,\cdot)} = t^{-1}|||\cdot|||_M$.

Letting $t \to 0$ in (2.4) we obtain the inextensional formulation of the problem (B): Find $\underline{u}_0 \in \mathcal{U}_0$ such that

$$\mathcal{A}_b(\underline{u}_0, \underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_0 \tag{2.5}$$

where $\mathcal{U}_0 = \{\underline{v} \in \mathcal{U}_B \mid \mathcal{A}_m(\underline{v}, \underline{v}) = 0\} \subset \mathcal{U}_B$ is the space of inextensional deformations.

3 The reduced-strain FE scheme

We consider the bilinear MITC4 finite element formulation of the problems (2.3) – (2.5). As in [3] we make strong assumptions on the finite element mesh so as to allow the use of Fourier methods in the error analysis.

Assume that Ω is divided into rectangular elements with node points (x^k, y^n), $k = 0, \ldots, N_x$, $n = 0, \ldots, N_y$ and a constant mesh spacing h_y in the y-direction and that the aspect ratios of the elements satisfy $d^{-1} \leq h_x^k/h_y \leq d$ for some $d > 0$ where $h_x^k = x^{k+1} - x^k$. To this mesh we associate the standard space $\mathcal{V}_h \subset H_p^1(\Omega)$ of continuous piecewise bilinear functions. We then define the FE spaces $\mathcal{U}_{M,h}$ and $\mathcal{U}_{B,h}$, respectively, as subspaces of $\mathcal{V}_h^5$ where the boundary or orthogonality conditions of Problems (M) and (B) are enforced. The finite element formulation of problems (2.3) – (2.5) are then obtained by replacing $\mathcal{U}_M$, $\mathcal{U}_B$ by $\mathcal{U}_{M,h}$, $\mathcal{U}_{B,h}$ and by modifying the bilinear form $\mathcal{A}_m$ numerically as

$$\begin{aligned}\mathcal{A}_m^h(\underline{u}, \underline{v}) = 6\gamma(1-\nu) &\int_\Omega \{\overline{\tilde{\rho}}_1(\underline{u})\tilde{\rho}_1(\underline{v}) + \overline{\tilde{\rho}}_2(\underline{u})\tilde{\rho}_2(\underline{v})\} dxdy \\ &+12 \int_\Omega \{\nu(\overline{\tilde{\beta}}_{11} + \overline{\tilde{\beta}}_{22})(\underline{u})(\tilde{\beta}_{11} + \tilde{\beta}_{22})(\underline{v}) \\ &\qquad + (1-\nu)\sum_{i,j=1}^{2} \overline{\tilde{\beta}}_{ij}(\underline{u})\tilde{\beta}_{ij}(\underline{v})\} dxdy\end{aligned}$$

where $\tilde{\beta}_{ij} = R^{ij}\beta_{ij}$, $\tilde{\rho}_i = R^i\rho_i$ with suitable reduction operators R^{ij} and R^i. As in [3] we choose these operators for β_{ii} and ρ_i to be

$$\tilde{\beta}_{11} = \Pi_h^x\beta_{11}, \quad \tilde{\beta}_{22} = \Pi_h^y\beta_{22}, \quad \tilde{\rho}_1 = \Pi_h^x\rho_1 \quad \tilde{\rho}_2 = \Pi_h^y\rho_2 \tag{3.1}$$

where Π_h^x and Π_h^y are orthogonal L^2-projections onto spaces $\mathcal{W}_h^x$ and $\mathcal{W}_h^y$ consisting of functions that are constant in x an piecewise linear in y or constant in y and piecewise linear in x respectively. For the term β_{12} we consider two different alternatives

(E1) $\tilde\beta_{12} = \Pi_h^{xy}\beta_{12}$

(E2) $\tilde\beta_{12} = \beta_{12} + S_{12}$

where $\Pi_h^{xy} = \Pi_h^x\Pi_h^y$ is the orthogonal L^2-projection onto elementwise constant functions and for every element K

$$S_{12|K} = a\frac{\partial}{\partial y}(\Pi_h^x w)(x - h_x^k/2) + b\frac{\partial}{\partial x}(\Pi_h^y w)(y - h_y/2) + (\Pi_h^{xy}cw - cw)$$

is essentially the term introduced in [4].

Remark 3.1. The formulation (E1) was assumed in [3], [6]. This is a straightforward interpretation of the MITC4 finite element formulation, but as shown recently in [4], (E2) is actually a closer interpretation of MITC4. The two formulations are practically equivalent when approximating inextensional deformations but may differ in other deformation states, as noted in [4]. Our error analysis here can only detect a small difference when approximating smooth membrane-dominated deformations, see Theorem 4.4 ahead.

The above definitions give rise to two different FE-schemes for solving (2.3) and (2.4):

(M_h) Find $\underline{u}_h \in \mathcal{U}_{M,h}$ such that

$$\mathcal{A}_M^h(\underline{u}_h, \underline{v}) = t^2\mathcal{A}_b(\underline{u}_h, \underline{v}) + \mathcal{A}_m^h(\underline{u}_h, \underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_{M,h} \tag{3.2}$$

(B_h) Find $\underline{u}_h \in \mathcal{U}_{B,h}$ such that

$$\mathcal{A}_B^h(\underline{u}_h, \underline{v}) = \mathcal{A}_b(\underline{u}_h, \underline{v}) + t^{-2}\mathcal{A}_m^h(\underline{u}_h, \underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_{B,h} \tag{3.3}$$

Upon passing to the limit $t \to 0$ in (3.3) we obtain a finite element formulation of the asymptotic problem (2.5): Find $\underline{u}_h \in \mathcal{U}_{0,h}$ such that

$$\mathcal{A}_b(\underline{u}_h, \underline{v}) = Q(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_{0,h} \tag{3.4}$$

where $\mathcal{U}_{0,h} = \{\underline{v} \in \mathcal{U}_{B,h} \mid \mathcal{A}_m^h(\underline{v}, \underline{v}) = 0\}$.

To analyze the discretization errors $e_M = |||\underline{u} - \underline{u}_h|||_{M,h}$ and $e_B = |||\underline{u} - \underline{u}_h|||_{M,h}$ as originating from (3.2) and (3.3) when $t > 0$, we split e_M and e_B into two orthogonal components in both cases, namely the approximation errors

$$e_{a,M}(\underline{u}) = \min_{\underline{v} \in \mathcal{U}_{M,h}} |||\underline{u} - \underline{v}|||_{M,h}$$
$$e_{a,B}(\underline{u}) = \min_{\underline{v} \in \mathcal{U}_{B,h}} |||\underline{u} - \underline{v}|||_{B,h}$$

and the consistency errors

$$e_{c,M}(\underline{u}) = \sup_{\underline{v}\in\mathcal{U}_{M,h}} \frac{(\mathcal{A}_M - \mathcal{A}_M^h)(\underline{u},\underline{v})}{|||\underline{v}|||_{M,h}} \tag{3.5}$$

$$e_{c,B}(\underline{u}) = \sup_{\underline{v}\in\mathcal{U}_{B,h}} \frac{(\mathcal{A}_B - \mathcal{A}_B^h)(\underline{u},\underline{v})}{|||\underline{v}|||_{B,h}} \tag{3.6}$$

where $|||\cdot|||_{M,h} = \sqrt{\mathcal{A}_M^h(\cdot,\cdot)}$, $|||\cdot|||_{B,h} = \sqrt{\mathcal{A}_B^h(\cdot,\cdot)}$. These definitions imply that

$$e_M^2 = e_{a,M}^2 + e_{c,M}^2$$
$$e_B^2 = e_{a,B}^2 + e_{c,B}^2.$$

(For a detailed reasoning, see [5].) We note that standard finite element theory gives the bound $e_{a,M} \le Ch||\underline{u}||_2$ and for $e_{a,B}$ we refer to [3]. Hence, the main task of this paper is to bound $e_{c,M}$ and $e_{c,B}$. We aim to analyze these error terms with both proposed strain-reductions (E1) and (E2).

The asymptotic formulations (2.5), (3.4) lead to a similar error decomposition. We have for $\underline{u}_0 \in \mathcal{U}_0$ the asymptotic approximation error

$$e_a^0(\underline{u}_0) = \min_{\underline{v}\in\mathcal{U}_{0,h}} |||\underline{u}_0 - \underline{v}|||_{B,h}$$

which was under consideration in [3]. On the other hand, at $t=0$ we have that

$$\mathcal{A}_b(\underline{u}_0,\underline{v}) = Q(\underline{v}) \quad \forall \underline{v}\in\mathcal{U}_0$$

for the inextensional solution $\underline{u}_0 \in \mathcal{U}_0$, and that

$$\mathcal{A}_b(\underline{u}_h,\underline{v}) = Q(\underline{v}) \quad \forall \underline{v}\in\mathcal{U}_{0,h} \tag{3.7}$$

for the corresponding finite element solution $\underline{u}_h$. Let $\underline{\tilde{u}}_h$ be the best finite element approximation to $\underline{u}_0$ in $\mathcal{U}_{0,h}$, i.e.

$$\mathcal{A}_b(\underline{\tilde{u}}_h,\underline{v}) = \mathcal{A}_b(\underline{u}_0,\underline{v}) \quad \forall \underline{v}\in\mathcal{U}_{0,h}. \tag{3.8}$$

By (3.7), (3.8) the asymptotic consistency error $\underline{u}_h - \underline{\tilde{u}}_h \in \mathcal{U}_{0,h}$ satisfies

$$\mathcal{A}_b(\underline{u}_h - \underline{\tilde{u}}_h,\underline{v}) = Q(\underline{v}) - \mathcal{A}_b(\underline{u}_0,\underline{v}) \quad \forall \underline{v}\in\mathcal{U}_{0,h} \tag{3.9}$$

and thus we can define

$$e_c^0(\underline{u}_0) \doteqdot |||\underline{u}_h - \underline{\tilde{u}}_h|||_h = \sup_{\underline{v}\in\mathcal{U}_{0,h}} \frac{Q(\underline{v}) - \mathcal{A}_b(\underline{u}^0,\underline{v})}{|||\underline{v}|||_{B,h}}. \tag{3.10}$$

As in [3], the main tool of our analysis will be the Fourier transform where we write

$$\underline{u}(x,y) = \sum_{\lambda\in\Lambda} \varphi_\lambda(y)\underline{\phi}_\lambda(x) = \sum_{\lambda\in\Lambda} \underline{\vartheta}_\lambda(x,y),$$
$$\varphi_\lambda(y) = e^{i\lambda y}, \quad \Lambda = \{\lambda = \frac{2\pi\nu}{H},\ \nu\in\mathbb{Z}\},$$

making use of the periodic boundary conditions at $y = 0, H$. For functions in the FE space we write analogously

$$\underline{v}(x,y) = \sum_{\lambda \in \Lambda_N} \tilde{\varphi}_\lambda(y) \tilde{\underline{\phi}}_\lambda(x) = \sum_{\lambda \in \Lambda_N} \tilde{\underline{\vartheta}}_\lambda(x,y)$$

where

$$\begin{aligned} \Lambda_N = \{\lambda \in \Lambda \mid -\pi \le \lambda h_y \le \pi \text{ when } N_y \text{ is odd}, \\ \text{or } -\pi < \lambda h_y \le \pi \text{ when } N_y \text{ is even}\}. \end{aligned}$$

Here $\tilde{\varphi}_\lambda(y)$ is the interpolant of $\varphi_\lambda(y)$, so that we are in fact considering a discrete Fourier transform of $\underline{v} \in \mathcal{U}_h$.

In our forthcoming analysis the following results are also needed.

Proposition 3.1 (Korn's inequality). *Let*

$$\mathcal{V} = \{\underline{v} = (v_1, v_2) \in [H^1_p(\Omega)]^2 \mid \underline{v}(0,\cdot) = \underline{v}(L,\cdot) = 0\}$$

or let

$$\mathcal{V} = \{\underline{v} = (v_1, v_2) \in [H^1_p(\Omega)]^2 \mid \int_\Omega v_1 dxdy = \int_\Omega v_2 dxdy = 0\}.$$

Then there exists a constant $c > 0$ such that

$$||\underline{v}||_1 \le c\big(||\frac{\partial v_1}{\partial x}||^2_{L^2} + ||\frac{\partial v_2}{\partial y}||^2_{L^2} + 2||\frac{1}{2}(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x})||^2_{L^2}\big)^{1/2} \quad \forall \underline{v} \in \mathcal{V}. \tag{3.11}$$

Proof. See [2]. □

Proposition 3.2. *Assume that $\underline{v} = (v_1, v_2) \in [\mathcal{V}_h]^2$. Then*

$$||\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}||_{L^2} \le C\big(||\Pi^{xy}_h(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x})||_{L^2} + ||\frac{\partial v_1}{\partial x}||_{L^2} + ||\frac{\partial v_2}{\partial y}||_{L^2}\big). \tag{3.12}$$

Proof. See Theorem 6.1 in [5]. □

4 The consistency error at $t > 0$

We start by giving a stability result for $\mathcal{U}_{M,h}$.

Lemma 4.1. *Let $\underline{v} \in \mathcal{U}_{M,h}$. Then*

$$||\underline{v}||_1 \le Ct^{-1}|||\underline{v}|||_{M,h}.$$

Proof. Assume first the modification (E1). By (3.3) we have that for $\underline{v} = (u, v, w, \theta, \psi) \in \mathcal{U}_{M,h}$

$$||\frac{\partial \theta}{\partial x}||_{L^2} + ||\frac{\partial \psi}{\partial y}||_{L^2} + ||\frac{\partial \theta}{\partial y} + \frac{\partial \psi}{\partial x}||_{L^2} \le Ct^{-1}|||\underline{v}|||_{M,h}$$

and thus by the Korn inequality (3.11)

$$||\theta||_1 + ||\psi||_1 \le Ct^{-1}|||\underline{v}|||_{M,h}. \tag{4.1}$$

Also the definitions of the membrane strains β_{ij} (2.1) imply

$$||\frac{\partial u}{\partial x}||_{L^2} + ||\frac{\partial v}{\partial y}||_{L^2} + ||\Pi_h^{xy}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})||_{L^2} \le C(|||\underline{v}|||_{M,h} + ||w||_{L^2})$$

and by (3.12) we have

$$||\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}||_{L^2} \le C(||\Pi_h^{xy}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})||_{L^2} + ||\frac{\partial u}{\partial x}||_{L^2} + ||\frac{\partial v}{\partial y}||_{L^2})$$

resulting in

$$||\frac{\partial u}{\partial x}||_{L^2} + ||\frac{\partial v}{\partial y}||_{L^2} + ||\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}||_{L^2} \le C(|||\underline{v}|||_{M,h} + ||w||_{L^2})$$

where from again by the Korn inequality (3.11)

$$||u||_1 + ||v||_1 \le C(|||\underline{v}|||_{B,h} + ||w||_{L^2}). \tag{4.2}$$

By (2.2), (3.1) we have that $\frac{\partial w}{\partial x} = \tilde{\rho}_1 - \Pi_h^x\theta$ and $\frac{\partial w}{\partial y} = \tilde{\rho}_2 - \Pi_h^y\psi$ so that

$$||\frac{\partial w}{\partial x}||_{L^2} + ||\frac{\partial w}{\partial y}||_{L^2} \le C(|||\underline{v}|||_{M,h} + ||\theta||_{L^2} + ||\psi||_{L^2}) \le Ct^{-1}|||\underline{v}|||_{M,h} \tag{4.3}$$

by (4.1). The claim follows from (4.1) – (4.3) together with the Poincaré's inequality. Similar calculations imply the result also for the modification (E2). □

Lemma 4.2. *Let* $\underline{v} \in \mathcal{U}_{B,h}$. *Then*

$$||\underline{v}||_1 \le C|||\underline{v}|||_{B,h}.$$

Proof. By the definition of $\mathcal{U}_{B,h}$ the Korn inequality (3.11) holds for the pairs (θ,ψ) and (u,v), as well as the Poincaré's inequality for w. The result follows as in Lemma 4.1. □

We derive next more specific stability results for the low-order discrete Fourier modes in the FE space.

Lemma 4.3. *Let* $\underline{\tilde{\vartheta}}_\lambda = \tilde{\varphi}_\lambda \underline{\tilde{\phi}}_\lambda = \tilde{\varphi}_\lambda(\tilde{u}_\lambda, \tilde{v}_\lambda, \tilde{w}_\lambda, \tilde{\theta}_\lambda, \tilde{\psi}_\lambda) \in \mathcal{U}_{M,h}$. *Then, if* $b \neq 0$ *we have for* λ *such that* $|\lambda|h \le c < \pi$,

$$||\tilde{\varphi}_\lambda\tilde{u}_\lambda||_1 + ||\tilde{\varphi}_\lambda\tilde{v}_\lambda||_1 + ||\tilde{\varphi}_\lambda\tilde{w}_\lambda||_{L^2} \le C|\lambda|^{1-m}|||\underline{\tilde{\vartheta}}_\lambda|||_{M,h}, \tag{4.4}$$

where $m = 1$ *in the elliptic case and* $m = 0$ *in the parabolic and hyperbolic cases.*

Proof. Consider first the case (E1). The translation components $\tilde{u}_\lambda$ and $\tilde{v}_\lambda$ of $\underline{\tilde{\vartheta}}_\lambda$ satisfy the difference equation (cf. [3])

$$\begin{pmatrix}\tilde{v}_\lambda\\ \tilde{u}_\lambda\end{pmatrix}(x^{k+1})-\begin{pmatrix}\tilde{v}_\lambda\\ \tilde{u}_\lambda\end{pmatrix}(x^k)=\frac{1}{2}\tau_k M\left[\begin{pmatrix}\tilde{v}_\lambda\\ \tilde{u}_\lambda\end{pmatrix}(x^{k+1})+\begin{pmatrix}\tilde{v}_\lambda\\ \tilde{u}_\lambda\end{pmatrix}(x^k)\right]+h_x^k\tilde{F}_\lambda^k \tag{4.5}$$

where $\tau_k = 2\frac{h_x^k}{h_y}\tan\left(\frac{1}{2}\lambda h_y\right)$,

$$M = i\begin{pmatrix}\frac{2c}{b} & -1\\ \frac{a}{b} & 0\end{pmatrix} \tag{4.6}$$

and

$$\tilde{F}_\lambda^k = \frac{1}{\cos\left(\frac{1}{2}\lambda h_y\right)}\begin{pmatrix}2\tilde{f}_{12}^\lambda(x^{k+1/2})-\frac{c}{b}(\tilde{f}_{22}^\lambda(x^{k+1})+\tilde{f}_{22}^\lambda(x^k))\\ \cos\left(\frac{1}{2}\lambda h_y\right)\tilde{f}_{11}^\lambda(x^{k+1/2})-\frac{a}{2b}(\tilde{f}_{22}^\lambda(x^{k+1})+\tilde{f}_{22}^\lambda(x^k))\end{pmatrix}. \tag{4.7}$$

Here

$$\begin{aligned}\tilde{f}_{11}^\lambda(x^{k+1/2}) &= e^{-in\lambda h_y}(\tilde{\beta}_{11}(\underline{\tilde{\vartheta}}_\lambda))_{|(x^{k+1/2},y^n)}\\ \tilde{f}_{22}^\lambda(x^k) &= e^{-i(n+1/2)\lambda h_y}(\tilde{\beta}_{22}(\underline{\tilde{\vartheta}}_\lambda))_{|(x^k,y^{n+1/2})}\\ \tilde{f}_{12}^\lambda(x^{k+1/2}) &= e^{-i(n+1/2)\lambda h_y}(\tilde{\beta}_{12}(\underline{\tilde{\vartheta}}_\lambda))_{|(x^{k+1/2},y^{n+1/2})}.\end{aligned}$$

Due to the constraints at $x=0,L$ we may without loss of generality consider only the exponentially decreasing solution of (4.5) starting from $x=0$. Then if $|\lambda|h_y \le c < \pi$, the standard theory for A-stable difference schemes (see also [3]) gives us the bound

$$\begin{aligned}\left|\left|\begin{pmatrix}\tilde{v}_\lambda\\ \tilde{u}_\lambda\end{pmatrix}(x^{k+1})\right|\right| &\le e^{-\alpha|\lambda|x^{k+1}}\left|\left|\begin{pmatrix}\tilde{v}_\lambda\\ \tilde{u}_\lambda\end{pmatrix}(0)\right|\right|\\ &\quad+\int_0^{x^{k+1}} e^{-\beta|\lambda|(x^{k+1}-t)}||\tilde{F}_\lambda(t)||e^{-\alpha|\lambda|t}dt\end{aligned} \tag{4.8}$$

where $||\cdot||$ is the Euclidean norm of vectors in $\mathbb{R}^2$ and

$$\tilde{F}_\lambda = \begin{pmatrix}2\tilde{f}_{12}^\lambda-\frac{2c}{b}\tilde{f}_{22}^\lambda\\ \tilde{f}_{11}^\lambda-\frac{a}{b}\tilde{f}_{22}^\lambda\end{pmatrix}.$$

Here $\alpha > \beta > 0$ in the elliptic case and $\alpha = \beta = 0$ in the parabolic and hyperbolic cases. Since $\tilde{u}_\lambda(0) = \tilde{v}_\lambda(0) = 0$ we obtain when $\lambda \neq 0$

$$\left|\left|\begin{pmatrix}\tilde{v}_\lambda\\ \tilde{u}_\lambda\end{pmatrix}(x^{k+1})\right|\right|^2 \le \begin{cases}C|\lambda|^{-1}e^{-2\beta|\lambda|x^{k+1}}\int_0^{x^{k+1}}||\tilde{F}_\lambda(t)||^2dt & \text{in the elliptic case}\\ C\int_0^{x^{k+1}}||\tilde{F}_\lambda(t)||^2dt & \text{in the parabolic and hyperbolic case}\end{cases}$$

and consequently

$$||\tilde{v}_\lambda||^2_{L^2(0,L)} + ||\tilde{u}_\lambda||^2_{L^2(0,L)} \le C|\lambda|^{-2m}||\tilde{F}_\lambda||^2_{L^2(0,L)}. \tag{4.9}$$

Also, (4.5) gives the relation

$$\begin{pmatrix} \tilde{v}_\lambda \\ \tilde{u}_\lambda \end{pmatrix}' (x^{k+1/2}) = \frac{1}{h_y} \tan(\frac{1}{2}\lambda h_y) M \left[\begin{pmatrix} \tilde{v}_\lambda \\ \tilde{u}_\lambda \end{pmatrix} (x^{k+1}) + \begin{pmatrix} \tilde{v}_\lambda \\ \tilde{u}_\lambda \end{pmatrix} (x^k) \right] + \tilde{F}^k_\lambda$$

from which it follows that

$$\begin{aligned} ||\tilde{v}'_\lambda||^2_{L^2(0,L)} + ||\tilde{u}'_\lambda||^2_{L^2(0,L)} \leq & C|\lambda|^2(||\tilde{v}_\lambda||^2_{L^2(0,L)} + ||\tilde{u}_\lambda||^2_{L^2(0,L)}) \\ & + ||\tilde{F}_\lambda||^2_{L^2(0,L)} \\ \leq & C|\lambda|^{2(1-m)}||\tilde{F}_\lambda||^2_{L^2(0,L)}. \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10) gives

$$||\tilde{\varphi}_\lambda \tilde{u}_\lambda||_1 + ||\tilde{\varphi}_\lambda \tilde{v}_\lambda||_1 \leq C|\lambda|^{1-m}|||\underline{\tilde{\vartheta}}_\lambda|||_{M,h} \tag{4.11}$$

since $||\tilde{F}_\lambda||_{L^2} \leq C|||\underline{\tilde{\vartheta}}_\lambda|||_{M,h}$.

To consider $\tilde{w}_\lambda$ we note that (cf. [3])

$$\tilde{w}_\lambda(x^k) = \frac{-2i}{bh_y} \tan(\frac{1}{2}\lambda h_y)\tilde{v}_\lambda(x^k) + \frac{1}{b\cos(\frac{1}{2}\lambda h_y)} \tilde{f}^\lambda_{22}(x^k)$$

and thus

$$||\tilde{w}_\lambda||^2_{L(0,L)} \leq C(|\lambda|^2||\tilde{v}_\lambda||^2_{L^2(0,L)} + ||\tilde{f}^\lambda_{22}||^2_{L^2(0,L)})$$

leading to

$$||\tilde{\varphi}_\lambda \tilde{w}_\lambda||_{L^2(0,L)} \leq C|\lambda|^{1-m}|||\underline{\tilde{\vartheta}}_\lambda|||_{M,h}. \tag{4.12}$$

The claim for $\lambda \neq 0$ follows from (4.11) together with (4.12).

When $\lambda = 0$ we have from (4.8) and from $\tilde{w}_0(x^k) = \frac{1}{b}\tilde{f}_{22}(x^k)$ that

$$||\tilde{\varphi}_0 \tilde{u}_0||_1 + ||\tilde{\varphi}_0 \tilde{v}_0||_1 + ||\tilde{\varphi}_0 \tilde{w}_0||_{L^2} \leq C|||\underline{\tilde{\vartheta}}_0|||_{M,h}$$

in any geometry. Similar calculations show that the claim holds also for the case (E2). □

Remark 4.1. The assumption $b \neq 0$ is not superfluous. This can be seen by taking $b = 0$, $\lambda = 0$, $a = -1$, $c = 1/2$ and choosing $\underline{\tilde{\phi}}_0(x^1) = (0, 0, 2, 4/h, 0)$, then repeating the sequence

$$\begin{aligned} \underline{\tilde{\phi}}_0(x^j) &= (h, -h, 0, -8/h, 0), \\ \underline{\tilde{\phi}}_0(x^{j+1}) &= (0, 0, -2, 4/h, 0), \\ \underline{\tilde{\phi}}_0(x^{j+2}) &= (-h, h, 0, 0, 0), \\ \underline{\tilde{\phi}}_0(x^{j+3}) &= (0, 0, 2, 4/h, 0) \end{aligned}$$

for $j = 2, 6, 10, \ldots$ and finally letting

$$\begin{aligned} \underline{\tilde{\phi}}_0(x^{N_x-2}) &= (h, -h, 0, -8/h, 0), \\ \underline{\tilde{\phi}}_0(x^{N_x-1}) &= (0, 0, -2, 4/h, 0). \end{aligned}$$

For this particular choice we have that $||\frac{\partial \tilde{u}_0}{\partial x}||_{L^2} \sim \min\{\frac{1}{h}, \frac{h^2}{t}\}|||\underline{\tilde{\vartheta}}_0|||_{M,h}$ so the stability is weaker when $b = 0$.

With the help of the stability estimates given in Lemmas 4.1 – 4.3 we can now bound the consistency error.

Theorem 4.4. *Assume that $b \neq 0$, and let $m = 1$ in the elliptic case and $m = 0$ in the parabolic and hyperbolic cases.. The consistency error $e_{c,M}$ defined in (3.5) satisfies*

$$e_{c,M} \leq C_1(\underline{u})h + C_2(t,s,\underline{u})h^{1+s} + C_3(t,\underline{u})h^2, \quad s \geq 0$$

provided that

$$\begin{aligned}
C_1(\underline{u}) &= C\sum_{ij} |\beta_{ij}(\underline{u})|_{2-m} \\
C_2(t,s,\underline{u}) &= Ct^{-1}\sum_{i} |\beta_{ii}(\underline{u})|_{1+s} + \begin{cases} Ct^{-1}|\beta_{12}(\underline{u})|_{1+s} \text{ for the case(E1)} \\ Ct^{-1}(|\beta_{12}(\underline{u})|_{s} + |w|_{1+s}) \text{ for the case (E2)} \end{cases} \\
C_3(t,\underline{u}) &= Ct^{-1}(\sum_{i} |\rho_i(\underline{u})|_1)
\end{aligned}$$

are all finite. The consistency error $e_{c,B}$ defined in (3.6) satisfies

$$e_{c,B} \leq C_1(t,\underline{u})h + C_2(t,\underline{u})h^2$$

provided that

$$\begin{aligned}
C_1(t,\underline{u}) =& Ct^{-2}\sum_{ij} |\beta_{ij}(\underline{u})|_1 \\
C_2(t,\underline{u}) =& Ct^{-2}\sum_{i} |\rho_i(\underline{u})|_1
\end{aligned}$$

are both finite.

Remark 4.2. The transverse shear strains ρ_i are typically very small at small t in smooth deformation states, so the error term of $e_{c,M}$ is very likely negligible in practice. In the bending-dominated case, $e_{c,B}$ depends strongly on β_{ij} and ρ_i. For smooth deformations one could assume realistically that $|\beta_{ij}(\underline{u})|_1 \sim |\rho_i(\underline{u})|_1 \sim t^2$ as $t \to 0$, in which case $e_{c,B} = O(h)$ uniformly in t. In practice, however, boundary layer effects probably cause the growth of $e_{c,B}$, via constant $C_1(\underline{u})$ in particular.

Proof. We consider first the membrane case and write $\underline{u} = \sum_{\lambda\in\Lambda} \underline{\vartheta}_\lambda \in \mathcal{U}_M$ and $\underline{v} = \sum_{\lambda\in\Lambda_N} \underline{\tilde{\vartheta}}_\lambda \in \mathcal{U}_{M,h}$. Then by the orthogonality of the discrete and continuous modes (cf. [3])

$$
\begin{aligned}
(\mathcal{A}_M - \mathcal{A}_M^h)(\underline{u},\underline{v}) &= (\mathcal{A}_m - \mathcal{A}_m^h)(\underline{u},\underline{v}) = (\mathcal{A}_m - \mathcal{A}_m^h)(\sum_{\lambda\in\Lambda}\underline{\vartheta}_\lambda \sum_{\lambda\in\Lambda_N}\underline{\tilde{\vartheta}}_\lambda) \\
&= (\mathcal{A}_m - \mathcal{A}_m^h)(\sum_{|\lambda|\le\lambda_0}\underline{\vartheta}_\lambda, \sum_{\lambda\in\Lambda_N}\underline{\tilde{\vartheta}}_\lambda) + (\mathcal{A}_m - \mathcal{A}_m^h)(\sum_{|\lambda|>\lambda_0}\underline{\vartheta}_\lambda, \sum_{\lambda\in\Lambda_N}\underline{\tilde{\vartheta}}_\lambda) \\
&= \sum_{|\lambda|\le\lambda_0}(\mathcal{A}_m - \mathcal{A}_m^h)(\underline{\vartheta}_\lambda,\underline{\tilde{\vartheta}}_\lambda) + \sum_{|\lambda|>\lambda_0}(\mathcal{A}_m - \mathcal{A}_m^h)(\underline{\vartheta}_\lambda,\underline{v}) \\
&\le C\sum_{ij}\sum_{|\lambda|\le\lambda_0}||\beta_{ij}(\underline{\vartheta}_\lambda) - \tilde{\beta}_{ij}(\underline{\vartheta}_\lambda)||_{L^2}(||\tilde{v}_\lambda||_1 + ||\tilde{u}_\lambda||_1 + ||\tilde{w}_\lambda||_{L^2}) \qquad (4.13) \\
&\quad + C|\lambda_0|^{-s_1}\sum_i\sum_{|\lambda|>\lambda_0}|\lambda|^{s_1}\,|(\overline{\beta}_{ii}(\underline{\vartheta}_\lambda) - \overline{\tilde{\beta}}_{ii}(\underline{\vartheta}_\lambda), \beta_{ii}(\underline{v}) - \tilde{\beta}_{ii}(\underline{v}))| \\
&\quad + C|\lambda_0|^{-s_2}\sum_{i\ne j}\sum_{|\lambda|>\lambda_0}|\lambda|^{s_2}\,|(\overline{\beta}_{ii}(\underline{\vartheta}_\lambda) - \Pi_h^{xy}\overline{\beta}_{ii}(\underline{\vartheta}_\lambda), \beta_{jj}(\underline{v}))| \\
&\quad + C|\lambda_0|^{-s_3}\sum_{|\lambda|>\lambda_0}|\lambda|^{s_3}\,|\int_\Omega(\overline{\beta}_{12}(\underline{\vartheta}_\lambda)\beta_{12}(\underline{v}) - \overline{\tilde{\beta}}_{12}(\underline{\vartheta}_\lambda)\tilde{\beta}_{12}(\underline{v}))dxdy| \\
&\quad + C\sum_i\sum_{\lambda\in\Lambda}(\overline{\rho}_i(\underline{\vartheta}_\lambda) - \overline{\tilde{\rho}}_i(\underline{\vartheta}_\lambda), \rho_i(\underline{v}) - \tilde{\rho}_i(\underline{v})) \\
&= I + II + III + IV + V
\end{aligned}
$$

where we have chosen λ_0 such that $\lambda_0 h \le c < \pi$. Here we have the bounds

$$
\begin{aligned}
I &\le Ch\sum_{ij}\sum_{|\lambda|\le\lambda_0}|\lambda^{1-m}\beta_{ij}(\underline{\vartheta}_\lambda)|_1|||\underline{\tilde{\vartheta}}_\lambda|||_{M,h} \\
&\le Ch\sum_{ij}|\beta_{ij}(\underline{u})|_{2-m}|||\underline{v}|||_{M,h}
\end{aligned} \qquad (4.14)
$$

and

$$
\begin{aligned}
II &\le Ch^{2+s_1}t^{-1}\sum_i|\beta_{ii}(\underline{u})|_{1+s_1}|||\underline{v}|||_{M,h} \\
III &\le Ch^{1+s_2}t^{-1}\sum_i|\beta_{ii}(\underline{u})|_{1+s_2}|||\underline{v}|||_{M,h} \\
IV &\le \begin{cases} Ch^{1+s_3}t^{-1}|\beta_{12}(\underline{u})|_{1+s_3}|||\underline{v}|||_{M,h} \text{ for the case (E1)} \\ Ch^{1+s_3}t^{-1}(|\beta_{12}(\underline{u})|_{s_3} + |w|_{1+s_3})|||\underline{v}|||_{M,h} \text{ for the case (E2)} \end{cases} \qquad (4.15) \\
V &\le Ch^2t^{-1}\sum_i\sum_{\lambda\in\Lambda}|\rho_i(\underline{\vartheta}_\lambda)|_1|||\underline{v}|||_{M,h}
\end{aligned}
$$

by Lemmas 4.1 and 4.3. The claim for $e_{c,M}$ follows form (4.13) – (4.15) when we take $s_1 = s_2 = s_3 = s$.

For the case $e_{c,B}$ the claim follows by the same arguments when we note that

$$(\mathcal{A}_B - \mathcal{A}_B^h)(\underline{u},\underline{v}) = t^{-2}(\mathcal{A}_m - \mathcal{A}_m^h)(\underline{u},\underline{v})$$

and use the stability result given in Lemma 4.2. □

5 The asymptotic consistency error

In this section we bound the asymptotic consistency error in an inextensional deformation state, as defined by (3.10). In [3] we showed that the approximation error in the inextensional state is of order $O(h)$ under nearly optimal regularity assumptions on $\underline{u}_0$. Here we find that the consistency error is likewise of order $O(h)$ at $t = 0$, but we need a very strong regularity assumption on $\underline{u}_0$.

We also make the additional assumption that the load is given by

$$Q(\underline{v}) = \int_\Omega (q_1 u + q_2 v + q_3 w) dxdy$$

for some suitable $q_i \in L^2_p(\Omega)$, $i = 1, 2, 3$ where $L^2_p(\Omega)$ denotes the usual L^2-space with periodic boundary conditions imposed at $y = 0, H$ and define the Fourier components of the load by

$$Q^\lambda(\underline{v}) = \int_\Omega (q_1^\lambda u + q_2^\lambda v + q_3^\lambda w) dxdy$$

where for each q_i we write $q_i(x,y) = \sum_{\lambda\in\Lambda} q_i^\lambda(x,y) = \sum_{\lambda\in\Lambda} \hat{q}_i^\lambda(x)\varphi_\lambda(y)$. We define the (semi-) norms

$$|Q|_s = \left(\sum_{\lambda\in\Lambda} |Q^\lambda|_s^2\right)^{1/2}$$

where

$$|Q^\lambda|_s = |\lambda|^s(||q_1^\lambda||^2_{L^2} + ||q_2^\lambda||^2_{L^2} + ||q_3^\lambda||^2_{L^2})^{1/2}$$

and write frequently $|Q|_0 = ||Q||_{L^2}$, $|Q^\lambda|_0 = ||Q^\lambda||_{L^2}$ and $||Q||_k = (\sum_{j=0}^k |Q|_j^2)^{1/2}$.

Theorem 5.1. *Assume that $b \neq 0$, $\underline{u}^0 \in [H^5_p(\Omega)]^5$. Then the asymptotic consistency error $e^0_{c,B}(\underline{u}_0)$, as defined by (3.10), satisfies*

$$e^0_{c,B}(\underline{u}_0) \leq C(||\underline{u}_0||_5 + ||Q||_1)h.$$

Proof. Let $\underline{v} \in \mathcal{U}_{0,h}$ and write $\underline{v} = \sum_{\lambda\in\Lambda_N} \underline{\tilde{\vartheta}}_\lambda(x,y) = \sum_{\lambda\in\Lambda_N} A_\lambda\tilde{\varphi}_\lambda(y)\underline{\tilde{\zeta}}_\lambda(x) = \sum_{\lambda\in\Lambda_N} A_\lambda\tilde{\varphi}_\lambda(y)(\tilde{u}_\lambda(x), \tilde{v}_\lambda(x), \tilde{w}_\lambda(x), \tilde{\theta}_\lambda(x), \tilde{\psi}_\lambda(x))$,
$\underline{u}_0 = \sum_{\lambda\in\Lambda} \underline{u}_0^\lambda = \sum_{\lambda\in\Lambda}(u_0^\lambda, v_0^\lambda, w_0^\lambda, \theta_0^\lambda, \psi_0^\lambda)$ and $Q(\underline{v}) = \sum_{\lambda\in\Lambda} Q^\lambda(\underline{v})$. Then by the orthogonality of the Fourier modes [3] we have that

$$\begin{aligned}
\mathcal{A}_b(\underline{u}_0, \underline{v}) - Q(\underline{v}) =& \mathcal{A}_b(\sum_{\lambda\in\Lambda} \underline{u}_0^\lambda, \sum_{\lambda\in\Lambda_N} \underline{\tilde{\vartheta}}_\lambda) - \sum_{\lambda\in\Lambda} Q^\lambda(\sum_{\lambda\in\Lambda_N} \underline{\tilde{\vartheta}}_\lambda) \\
=& \mathcal{A}_b(\sum_{|\lambda|\leq\lambda_0} \underline{u}_0^\lambda, \sum_{\lambda\in\Lambda_N} \underline{\tilde{\vartheta}}_\lambda) - \sum_{|\lambda|\leq\lambda_0} Q^\lambda(\sum_{\lambda\in\Lambda_N} \underline{\tilde{\vartheta}}_\lambda) \\
&+ \mathcal{A}_b(\sum_{|\lambda|>\lambda_0} \underline{u}_0^\lambda, \sum_{\lambda\in\Lambda_N} \underline{\tilde{\vartheta}}_\lambda) - \sum_{|\lambda|>\lambda_0} Q^\lambda(\sum_{\lambda\in\Lambda_N} \underline{\tilde{\vartheta}}_\lambda) \\
=& \sum_{|\lambda|\leq\lambda_0} (\mathcal{A}_b(\underline{u}_0^\lambda, \underline{\tilde{\vartheta}}_\lambda) - Q^\lambda(\underline{\tilde{\vartheta}}_\lambda)) + \sum_{|\lambda|>\lambda_0} (\mathcal{A}_b(\underline{u}_0^\lambda, \underline{v}) - Q^\lambda(\underline{v})) \\
=& I + II
\end{aligned}$$

for any λ_0 such that $\lambda_0 h_y \le c < \pi$.

Let us first bound the term II. Here we have that

$$\sum_{|\lambda|>\lambda_0} \mathcal{A}_b(\underline{u}_0^\lambda, \underline{v}) \le \lambda_0^{-1} \sum_{|\lambda|>\lambda_0} \mathcal{A}_b(|\lambda|\underline{u}_0^\lambda, \underline{v}) \le Ch||\underline{u}_0||_2|||\underline{v}|||_{B,h} \tag{5.1}$$

for $\lambda_0 = \frac{c}{h}$, c sufficiently small and similarly

$$\sum_{|\lambda|>\lambda_0} Q^\lambda(\underline{v}) \le \lambda_0^{-1} \sum_{|\lambda|>\lambda_0} |\lambda| Q^\lambda(\underline{v}) \le Ch|Q|_1|||\underline{v}|||_{B,h}. \tag{5.2}$$

To bound the term I when $b \neq 0$ we note that for any $\underline{\vartheta}_\lambda = A_\lambda \varphi_\lambda \underline{\zeta}_\lambda \in \mathcal{U}_0$ we can write

$$\begin{aligned} \mathcal{A}_b(\underline{u}_0^\lambda, \underline{\tilde{\vartheta}}_\lambda) - Q^\lambda(\underline{\tilde{\vartheta}}_\lambda) =& \mathcal{A}_b(\underline{u}_0^\lambda, \underline{\tilde{\vartheta}}_\lambda - \underline{\vartheta}_\lambda) - Q^\lambda(\underline{\tilde{\vartheta}}_\lambda - \underline{\vartheta}_\lambda) \\ =& A_\lambda(\mathcal{A}_b(\underline{u}_0^\lambda, \tilde{\varphi}_\lambda \underline{\tilde{\zeta}}_\lambda - \varphi_\lambda \underline{\zeta}_\lambda) - Q^\lambda(\tilde{\varphi}_\lambda \underline{\tilde{\zeta}}_\lambda - \varphi_\lambda \underline{\zeta}_\lambda)). \end{aligned} \tag{5.3}$$

Integration by parts in the first term in (5.3) gives

$$\begin{aligned} \mathcal{A}_b(\underline{u}_0^\lambda, \tilde{\varphi}_\lambda \underline{\tilde{\zeta}} - \varphi_\lambda \underline{\zeta}_\lambda) = &\int_0^H \Big|_0^L \overline{\alpha}_1^\lambda (\tilde{\varphi}_\lambda \tilde{\theta}_\lambda - \varphi_\lambda \theta_\lambda) + \overline{\alpha}_2^\lambda (\tilde{\varphi}_\lambda \tilde{\psi}_\lambda - \varphi_\lambda \psi_\lambda) dy \\ &+ \int_\Omega \overline{\delta}_1^\lambda (\tilde{\varphi}_\lambda \tilde{\theta}_\lambda - \varphi_\lambda \theta_\lambda) + \overline{\delta}_2^\lambda (\tilde{\varphi}_\lambda \tilde{\psi}_\lambda - \varphi_\lambda \psi_\lambda) dx dy \end{aligned}$$

where

$$\begin{cases} \alpha_1^\lambda = \frac{\partial^2 w_0^\lambda}{\partial x^2} + \nu \frac{\partial^2 w_0^\lambda}{\partial y^2} \\ \alpha_2^\lambda = (1-\nu) \frac{\partial^2 w_0^\lambda}{\partial x \partial y} \\ \delta_1^\lambda = -\frac{\partial}{\partial x} \Delta w_0^\lambda \\ \delta_2^\lambda = -\frac{\partial}{\partial y} \Delta w_0^\lambda \end{cases}$$

so that

$$\begin{aligned} \mathcal{A}_b(\underline{u}_0^\lambda, \tilde{\varphi}_\lambda \underline{\tilde{\zeta}} - \varphi_\lambda \underline{\zeta}_\lambda) \le& ||\alpha_1^\lambda(L,\cdot)||_{L^2(0,H)} ||\tilde{\varphi}_\lambda \tilde{\theta}_\lambda(L,\cdot) - \varphi_\lambda \theta_\lambda(L,\cdot)||_{L^2(0,H)} \\ &+ ||\delta_1^\lambda(L,\cdot)||_{L^2(0,H)} ||\tilde{\varphi}_\lambda \tilde{\psi}_\lambda(L,\cdot) - \varphi_\lambda \psi_\lambda(L,\cdot)||_{L^2(0,H)} \\ &+ ||\alpha_1^\lambda(0,\cdot)||_{L^2(0,H)} ||\tilde{\varphi}_\lambda \tilde{\theta}_\lambda(0,\cdot) - \varphi_\lambda \theta_\lambda(0,\cdot)||_{L^2(0,H)} \\ &+ ||\delta_1^\lambda(0,\cdot)||_{L^2(0,H)} ||\tilde{\varphi}_\lambda \tilde{\psi}_\lambda(0,\cdot) - \varphi_\lambda \psi_\lambda(0,\cdot)||_{L^2(0,H)} \\ &+ ||\alpha_2^\lambda||_{L^2} ||\tilde{\varphi}_\lambda \tilde{\theta}_\lambda - \varphi_\lambda \theta_\lambda||_{L^2} + ||\delta_2^\lambda||_{L^2} ||\tilde{\varphi}_\lambda \tilde{\psi}_\lambda - \varphi_\lambda \psi_\lambda||_{L^2} \\ \le& C||\underline{u}_0^\lambda||_3 \big(||\tilde{\varphi}_\lambda \tilde{\theta}_\lambda(L,\cdot) - \varphi_\lambda \theta_\lambda(L,\cdot)||_{L^2(0,H)} \\ &+ ||\tilde{\varphi}_\lambda \tilde{\psi}_\lambda(L,\cdot) - \varphi_\lambda \psi_\lambda(L,\cdot)||_{L^2(0,H)} \\ &+ ||\tilde{\varphi}_\lambda \tilde{\theta}_\lambda(0,\cdot) - \varphi_\lambda \theta_\lambda(0,\cdot)||_{L^2(0,H)} \\ &+ ||\tilde{\varphi}_\lambda \tilde{\psi}_\lambda(0,\cdot) - \varphi_\lambda \psi_\lambda(0,\cdot)||_{L^2(0,H)} \\ &+ ||\tilde{\varphi}_\lambda \tilde{\theta}_\lambda - \varphi_\lambda \theta_\lambda||_{L^2} + ||\tilde{\varphi}_\lambda \tilde{\psi}_\lambda - \varphi_\lambda \psi_\lambda||_{L^2} \big). \end{aligned} \tag{5.4}$$

Also for Q^λ in (5.3) we have the bound

$$\begin{aligned} Q^\lambda(\tilde{\varphi}_\lambda \underline{\tilde{\zeta}}_\lambda - \varphi_\lambda \underline{\zeta}_\lambda) \le& C||Q^\lambda||_{L^2} \big(||\tilde{\varphi}_\lambda \tilde{u}_\lambda - \varphi_\lambda u_\lambda||_{L^2} \\ &+ ||\tilde{\varphi}_\lambda \tilde{v}_\lambda - \varphi_\lambda v_\lambda||_{L^2} + ||\tilde{\varphi}_\lambda \tilde{w}_\lambda - \varphi_\lambda w_\lambda||_{L}{}^2 \big). \end{aligned} \tag{5.5}$$

To continue we need the following approximation results. The proof will be postponed to the end of this section.

Lemma 5.2. *For every λ such that $|\lambda|h_y \leq c < \pi$ there exists a $\varphi_\lambda\underline{\zeta}_\lambda \in \mathcal{U}_0$ such that*

$$\begin{aligned} &||\tilde{\varphi}_\lambda\tilde{\theta}_\lambda(L,\cdot) - \varphi_\lambda\theta_\lambda(L,\cdot)||_{L^2(0,H)} + ||\tilde{\varphi}_\lambda\tilde{\psi}_\lambda(L,\cdot) - \varphi_\lambda\psi_\lambda(L,\cdot)||_{L^2(0,H)} \\ &+||\tilde{\varphi}_\lambda\tilde{\theta}_\lambda(0,\cdot) - \varphi_\lambda\theta_\lambda(0,\cdot)||_{L^2(0,H)} + ||\tilde{\varphi}_\lambda\tilde{\psi}_\lambda(0,\cdot) - \varphi_\lambda\psi_\lambda(0,\cdot)||_{L^2(0,H)} \\ &\qquad +||\tilde{\varphi}_\lambda\tilde{\theta}_\lambda - \varphi_\lambda\theta_\lambda||_{L^2} + ||\tilde{\varphi}_\lambda\tilde{\psi}_\lambda - \varphi_\lambda\psi_\lambda||_{L^2} \\ &\qquad \leq C(h^2|\lambda|^{5-m} + h^2\lambda^4) + Ch\lambda^2|||\tilde{\varphi}_\lambda\tilde{\underline{\zeta}}_\lambda|||_{B,h} \end{aligned} \tag{5.6}$$

and

$$||\tilde{\varphi}_\lambda\tilde{u}_\lambda - \varphi_\lambda u_\lambda||_{L^2} + ||\tilde{\varphi}_\lambda\tilde{v}_\lambda - \varphi_\lambda v_\lambda||_{L^2} + ||\tilde{\varphi}_\lambda\tilde{w}_\lambda - \varphi_\lambda w_\lambda||_{L^2} \leq Ch^2|\lambda|^{4-3m/2} \tag{5.7}$$

where $m = 1$ in the elliptic case and $m = 0$ in the hyperbolic and parabolic cases.

To complete the proof of Theorem 5.1 we note that since $|\lambda|^{3-m/2} \leq C|||\tilde{\varphi}_\lambda\tilde{\underline{\zeta}}_\lambda|||_{B,h}$ and $|\lambda|h \leq \lambda_0 h \leq c < \pi$ we obtain from (5.4) with the help of Lemma 5.2

$$\mathcal{A}_b(\underline{u}_0^\lambda, \tilde{\varphi}_\lambda\tilde{\underline{\zeta}} - \varphi_\lambda\underline{\zeta}_\lambda) \leq Ch||\lambda^2\underline{u}_0^\lambda||_3|||\tilde{\varphi}_\lambda\tilde{\underline{\zeta}}_\lambda|||_{B,h} \tag{5.8}$$

and from (5.5)

$$Q^\lambda(\tilde{\varphi}_\lambda\tilde{\underline{\zeta}}_\lambda - \varphi_\lambda\underline{\zeta}_\lambda) \leq Ch||Q^\lambda||_{L^2}|||\tilde{\varphi}_\lambda\tilde{\underline{\zeta}}_\lambda|||_{B,h} \tag{5.9}$$

so that by (5.3), (5.8) and (5.9)

$$\begin{aligned} \sum_{|\lambda|\leq\lambda_0} (\mathcal{A}_b(\underline{u}_0^\lambda, \underline{\vartheta}_\lambda) - Q^\lambda(\underline{\vartheta}_\lambda)) &= \sum_{|\lambda|\leq\lambda_0} A_\lambda(\mathcal{A}_b(\underline{u}_0^\lambda, \tilde{\varphi}_\lambda\tilde{\underline{\zeta}}_\lambda) - Q^\lambda(\tilde{\varphi}_\lambda\tilde{\underline{\zeta}}_\lambda)) \\ &\leq Ch\sum_{|\lambda|\leq\lambda_0} (||\lambda^2\underline{u}_0^\lambda||_3 + ||Q^\lambda||_{L^2})|||A_\lambda\tilde{\varphi}_\lambda\tilde{\underline{\zeta}}_\lambda|||_{B,h} \\ &\leq Ch(||\underline{u}_0||_5 + ||Q||_{L^2})|||\underline{v}|||_{B,h} \end{aligned} \tag{5.10}$$

and Theorem 5.1 follows from the estimates (5.1), (5.2) and (5.10). $\square$

Proof of Lemma 5.2. In [3] it was shown that for every discrete mode $\tilde{\varphi}_\lambda\tilde{\underline{\zeta}}_\lambda \in \mathcal{U}_{0,h}$ with $|\lambda|h_y \leq c < \pi$ there corresponds a continuous mode $\varphi_\lambda\underline{\zeta}_\lambda = \varphi_\lambda(y)(u_\lambda(x), v_\lambda(x), w_\lambda(x), \theta_\lambda(x), \psi_\lambda(x)) \in \mathcal{U}_0$ satisfying $u_\lambda(0) = \tilde{u}_\lambda(0)$ and $v_\lambda(0) = \tilde{v}_\lambda(0)$ and such that

$$\begin{aligned} |u_\lambda(x^k) - \tilde{u}_\lambda(x^k)| &\leq Ch^2|\lambda|^{3-m}e^{-\beta|\lambda|x^k} \\ |v_\lambda(x^k) - \tilde{v}_\lambda(x^k)| &\leq Ch^2|\lambda|^{3-m}e^{-\beta|\lambda|x^k} \\ |w_\lambda(x^k) - \tilde{w}_\lambda(x^k)| &\leq Ch^2|\lambda|^{4-m}e^{-\beta|\lambda|x^k} \\ |\psi_\lambda(x^k) - \tilde{\psi}_\lambda(x^k)| &\leq Ch^2|\lambda|^{5-m}e^{-\beta|\lambda|x^k} \end{aligned}$$

with $\beta > 0$ in the elliptic case and $\beta = 0$ in the hyperbolic and parabolic cases, so that

$$\begin{aligned} ||\tilde{\varphi}_\lambda \tilde{\psi}_\lambda(0,\cdot) - \varphi_\lambda \psi_\lambda(0,\cdot)||_{L^2(0,H)} \leq & Ch^2|\lambda|^{5-m} \\ ||\tilde{\varphi}_\lambda \tilde{\psi}_\lambda(L,\cdot) - \varphi_\lambda \psi_\lambda(L,\cdot)||_{L^2(0,H)} \leq & Ch^2|\lambda|^{5-m} \\ ||\tilde{\varphi}_\lambda \tilde{\psi}_\lambda - \varphi_\lambda \psi_\lambda||_{L^2} \leq & Ch^2|\lambda|^{5-3m/2}. \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} ||\tilde{\varphi}_\lambda \tilde{u}_\lambda - \varphi_\lambda u_\lambda||_{L^2} \leq & Ch^2|\lambda|^{3-3m/2} \\ ||\tilde{\varphi}_\lambda \tilde{v}_\lambda - \varphi_\lambda v_\lambda||_{L^2} \leq & Ch^2|\lambda|^{3-3m/2} \\ ||\tilde{\varphi}_\lambda \tilde{w}_\lambda - \varphi_\lambda w_\lambda||_{L^2} \leq & Ch^2|\lambda|^{4-3m/2} \end{aligned} \tag{5.12}$$

Also, by [3] we have that

$$\begin{aligned} \frac{1}{2}(\tilde{\theta}_\lambda(x^{k+1}) + \tilde{\theta}_\lambda(x^k)) = & \frac{2}{bh_y^2}\tan^2(\frac{1}{2}\lambda h_y)\Big(\frac{2c}{b}(\tilde{v}_\lambda(x^{k+1}) + \tilde{v}_\lambda(x^k)) \\ & - (\tilde{u}_\lambda(x^{k+1}) + \tilde{u}_\lambda(x^k))\Big) \\ = & \frac{1}{2}(\tilde{g}(x^{k+1}) + \tilde{g}(x^k)) \end{aligned}$$

so that

$$\tilde{\theta}_\lambda(x^{k+1}) = \tilde{g}(x^{k+1}) + (-1)^k(\tilde{\theta}_\lambda(x^0) - \tilde{g}(x^0)) \tag{5.13}$$

and

$$\begin{aligned} \frac{\partial\tilde{\theta}_\lambda}{\partial x}(x^{k+1/2}) = & \frac{\tilde{\theta}_\lambda(x^{k+1}) - \tilde{\theta}_\lambda(x^k)}{h_x^k} \\ = & \frac{\tilde{g}(x^{k+1}) - \tilde{g}(x^k)}{h_x^k} + \frac{2}{h_x^k}(-1)^k(\tilde{\theta}_\lambda(x^0) - \tilde{g}(x^0)). \end{aligned} \tag{5.14}$$

Since

$$\theta_\lambda(x^k) = \frac{\lambda^2}{b}(\frac{2c}{b}v_\lambda(x^k) - u_\lambda(x^k)) = g(x^k) \tag{5.15}$$

it follows from (5.13) – (5.15) that

$$\begin{aligned} \tilde{\theta}_\lambda(x^{k+1}) - \theta_\lambda(x^{k+1}) = & \tilde{g}(x^{k+1}) - g(x^{k+1}) \\ & + \frac{h_x^k}{2}\frac{\partial\theta_\lambda}{\partial x}(x^{k+1/2}) - \frac{h_x^k}{2}\frac{\tilde{g}(x^{k+1}) - \tilde{g}(x^k)}{h_x^k} \end{aligned}$$

and finally that

$$\begin{aligned} |\tilde{\theta}_\lambda(x^{k+1}) - \theta_\lambda(x^{k+1})| \leq & C\Big(h^2|\lambda|^{5-m}e^{-\beta|\lambda|x^{k+1}} + h|\frac{\partial\tilde{\theta}_\lambda}{\partial x}(x^{k+1/2})| \\ & + h\lambda^2(|\frac{\partial\tilde{v}_\lambda}{\partial x}(x^{k+1/2})| + |\frac{\partial\tilde{u}_\lambda}{\partial x}(x^{k+1/2})|\Big). \end{aligned}$$

For the values at the endpoints we get similarly

$$\begin{aligned}\tilde\theta_\lambda(x^0)-\theta_\lambda(x^0) =&(-1)^k(\tilde\theta_\lambda(x^{k+1})-\theta_\lambda(x^{k+1})+g(x^{k+1})-\tilde g(x^{k+1}))\\&+\tilde g(x^0)-g(x^0)\end{aligned}$$

and

$$\begin{aligned}\tilde\theta_\lambda(x^{N_x})-\theta_\lambda(x^{N_x}) =&(-1)^{N_x}(\tilde\theta_\lambda(x^0)-\theta_\lambda(x^0)+g(x^0)-\tilde g(x^0))\\&+\tilde g(x^{N_x})-g(x^{N_x}).\end{aligned}$$

Thus, we have the following bounds

$$\begin{aligned}||\tilde\varphi_\lambda\tilde\theta_\lambda(0,\cdot)-\varphi_\lambda\theta_\lambda(0,\cdot)||_{L^2(0,H)} \le& C(h^2\lambda^4+h^2|\lambda|^{5-3m/2})\\&+C(h+h\lambda^2)|||\tilde\varphi_\lambda\tilde{\underline{\zeta}}_\lambda|||_{B,h}\\
||\tilde\varphi_\lambda\tilde\theta_\lambda(L,\cdot)-\varphi_\lambda\theta_\lambda(L,\cdot)||_{L^2(0,H)} \le& C(h^2\lambda^4+h^2|\lambda|^{5-3m/2})\\&+C(h+h\lambda^2)|||\tilde\varphi_\lambda\tilde{\underline{\zeta}}_\lambda|||_{B,h} \qquad (5.16)\\&+C(h^2|\lambda|^{5-m}+h^2\lambda^4)\\
||\tilde\varphi_\lambda\tilde\theta_\lambda-\varphi_\lambda\theta_\lambda||_{L^2} \le& Ch^2|\lambda|^{5-3m/2}\\&+C(h+h\lambda^2)|||\tilde\varphi_\lambda\tilde{\underline{\zeta}}_\lambda|||_{B,h}.\end{aligned}$$

and Lemma 5.2 follows from (5.11), (5.12) and (5.16) since $|\lambda|h_y \le c < \pi$. $\square$

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