Helsinki University of Technology Institute of Mathematics Research Reports

Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja

Espoo 2001 A436

PSEUDODIFFERENTIAL CALCULUS ON COMPACT HOMOGENEOUS SPACES

Ville Turunen

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

Ville Turunen: *Pseudodifferential calculus on compact homogeneous spaces*; Helsinki University of Technology Institute of Mathematics Research Reports A436 (2001).

Abstract: *Pseudodifferential operators on a compact Lie group G are projected to pseudodifferential operators on an orientable compact homogeneous space G/K. Starting with a pseudodifferential operator on a compact homogeneous space G/K with torus K, we extend the operator to act on G; a special example of such a homogeneous space is the two-sphere $\mathbb{S}^2$ as the base space for the Hopf fibration.*

AMS subject classifications: 35S05, 43A85, 22E30, 47G30

Keywords: pseudodifferential operators, homogeneous spaces, Lie groups, analysis on sphere

Ville.Turunen@hut.fi

ISBN 951-22-5524-3
ISSN 0784-3143
Libella Painopalvelu Oy, Espoo, 2001

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1 Introduction

In this article we treat pseudodifferential analysis on orientable homogeneous spaces G/K, where G is a compact Lie group with a closed subgroup K. This research continues the work in [11], where such analysis on compact Lie groups was studied. Apart from pure theoretical interests, there are applications which call for the present treatise: e.g. Dirichlet boundary value problems in a domain diffeomorphic to the unit ball of $\mathbb{R}^3$ may be considered within the framework of harmonic analysis on the two-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Taylor (see [7]) has characterized pseudodifferential operators on the spheres $\mathbb{S}^n$ by studying the smoothness of certain operator-valued functions on a large group of symmetries, but this result cannot be used for our purposes here.

We explain how a pseudodifferential operator on a compact Lie group G can be "projected" to a pseudodifferential operator on orientable compact homogeneous spaces G/K in a way respecting the algebraic structures. The other way round, given a pseudodifferential operator on G/K when K is a torus we construct an "extended" pseudodifferential operator on G; the "projection" of this "extension" in turn returns the original operator. "Extended" operators can be used to calculate asymptotic expansions for operators on G/K using operator-valued symbolic calculus on G (see [8], [11]).

Vector space notation

The space of the continuous linear operators between topological vector spaces X and Y is denoted by $\mathcal{L}(X,Y)$, and we write $\mathcal{L}(X) := \mathcal{L}(X,X)$; the dual space of X is $X' := \mathcal{L}(X,\mathbb{C})$. If X is a nuclear Fréchet space, $X \otimes X'$ stands for the complete locally convex tensor product.

2 Pseudodifferential operators on $\mathbb{R}^p \times \mathbb{T}^q$

For general treatments of pseudodifferential calculus on the Euclidean spaces or manifolds, see e.g. [3] or [9]. Periodic pseudodifferential operators, i.e. pseudodifferential operators on tori expressed utilizing Fourier series, were introduced in [1], and their complete symbolic calculus is presented in [12].

Let $\mathbb{T}^q = \mathbb{R}^q/\mathbb{Z}^q$ be the q-dimensional torus group. In the sequel we shall identify $\mathbb{R}^0$ and $\mathbb{Z}^0$ with the set $\{0\}$, and $\mathbb{R}^p \times \mathbb{T}^0$ is identified with $\mathbb{R}^p$. Let $\mathcal{S}(\mathbb{R}^p \times \mathbb{T}^q) = \{f \in C^\infty(\mathbb{R}^p \times \mathbb{T}^q) \mid \forall y \in \mathbb{T}^q : (x \mapsto f(x,y)) \in \mathcal{S}(\mathbb{R}^p)\}$ be endowed with the natural Fréchet space structure of the test functions. In this space, we define the *Fourier transform* $f \mapsto \hat{f}$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^p \times \mathbb{T}^q} f(x)\ e^{-i2\pi x\cdot\xi}\ dx_1 \cdots dx_{p+q},$$

where $\xi \in \mathbb{R}^p \times \mathbb{Z}^q$. Let $e_\xi(x) = e^{i2\pi x\cdot\xi}$, and let $A \in \mathcal{L}(\mathbb{S}'(\mathbb{R}^p \times \mathbb{T}^q))$; then $e_\xi \in \mathcal{S}'(\mathbb{R}^p \times \mathbb{T}^q)$, and we can define the *symbol* $\sigma_A : (\mathbb{R}^p \times \mathbb{T}^q) \times (\mathbb{R}^p \times \mathbb{Z}^q) \to \mathbb{C}$ of A:

$$\sigma_A(x,\xi) := e_\xi(x)^{-1}(Ae_\xi)(x), \tag{1}$$

and it is clear that σ_A is C^∞-smooth with respect to the variable $x \in \mathbb{R}^p$. Then A can be retrieved from its symbol σ_A by

$$(Af)(x) = \int_{\mathbb{R}^p} \sum_{\xi_{p+1},\ldots,\xi_{p+q} \in \mathbb{Z}} \sigma_A(x,\xi)\ \hat{f}(\xi)\ e^{i2\pi x\cdot\xi}\ d\xi_1 \cdots d\xi_p. \tag{2}$$

The symbol class $S^m(\mathbb{R}^p \times \mathbb{T}^q)$ consists of those C^∞-smooth functions $\sigma_A : (\mathbb{R}^p \times \mathbb{T}^q) \times (\mathbb{R}^p \times \mathbb{Z}^q) \to \mathbb{C}$ for which

$$\sup_{x \in \mathbb{R}^p \times \mathbb{T}^q} |\partial_\xi^{\alpha'} \triangle_\xi^{\alpha''} \partial_x^\beta \sigma_A(x,\xi)| \leq C_{A\alpha\beta m}\ \langle \xi \rangle^{m-|\alpha|} \tag{3}$$

for every multi-index $\alpha = \alpha' + \alpha'', \beta \in \mathbb{N}_0^{p+q}$; here $\alpha = \alpha' + \alpha''$, $\alpha' = (\alpha_1, \ldots, \alpha_p, 0, \ldots, 0)$, and $\langle \xi \rangle = (1 + \sum_{j=1}^{p+q} \xi_j^2)^{1/2}$. Here $\triangle_\xi^\alpha$ is the αth forward difference operator defined by

$$(\triangle_\xi^\alpha \sigma)(\xi) := \sum_{0 \leq \gamma \leq \alpha} \binom{\alpha}{\gamma} (-1)^{|\alpha-\gamma|}\ \sigma(\xi+\gamma), \tag{4}$$

$|\alpha| = 1$ implies $(\triangle_\xi^\alpha \sigma)(\xi) := \sigma(\xi+\alpha) - \sigma(\xi)$. Operator $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^p \times \mathbb{T}^q))$ is called *a pseudodifferential operator of order* $m \in \mathbb{R}$, $A \in \Psi^m(\mathbb{R}^p \times \mathbb{T}^q) = \mathrm{Op}S^m(\mathbb{R}^p \times \mathbb{T}^q)$, if $\sigma_A \in S^m(\mathbb{R}^p \times \mathbb{T}^q)$.

3 Analysis on closed manifolds

Let M be a C^∞-smooth, closed (i.e. compact, without a boundary) orientable manifold. The test function space $\mathcal{D}(M)$ is the space of $C^\infty(M)$ endowed with the usual Fréchet space topology. Its dual $\mathcal{D}'(M) = \mathcal{L}(\mathcal{D}(M), \mathbb{C})$ is the space of distributions, endowed with the weak-$*$-topology. The duality is expressed by the brackets $\langle \phi, f \rangle = f(\phi)$ ($\phi \in \mathcal{D}(M)$, $f \in \mathcal{D}'(M)$). Embedding $\mathcal{D}(M) \hookrightarrow \mathcal{D}'(M)$ is interpreted by

$$\langle \phi, \psi \rangle := \int_M \phi(x)\ \psi(x)\ dx.$$

The Schwartz kernel theorem states that $\mathcal{L}(\mathcal{D}(M))$ is isomorphic to $\mathcal{D}(M) \otimes \mathcal{D}'(M)$; the isomorphism is given by

$$\langle A\phi, f \rangle = \langle K_A, f \otimes \phi \rangle, \tag{5}$$

where $A \in \mathcal{L}(\mathcal{D}(M))$, $\phi \in \mathcal{D}(M)$, $f \in \mathcal{D}'(M)$, and distribution $K_A \in \mathcal{D}(M) \otimes \mathcal{D}'(M)$ is called the *Schwartz kernel* of A. Then A can uniquely be extended (by duality) to $A \in \mathcal{L}(\mathcal{D}'(M))$, and it is customary to write informally

$$(Af)(x) = \int_M K_A(x,y)\ f(y)\ dy$$

instead of $\phi \mapsto \langle \phi, Af \rangle$ ($\phi \in \mathcal{D}(M)$). Recall that $L^2(M) = H^0(M)$, $\mathcal{D}'(M) = \cup_{s \in \mathbb{R}} H^s(M)$ and $\mathcal{D}(M) = \cap_{s \in \mathbb{R}} H^s(M)$, where $H^s(M)$ is the (L^2-type) Sobolev space of order $s \in \mathbb{R}$.

An operator $A \in \mathcal{L}(\mathcal{D}(M))$ is *a pseudodifferential operator of order* $m \in \mathbb{R}$ *on* M, $A \in \Psi^m(M)$, if $(M_\phi A M_\psi)_\kappa \in \Psi^m(\mathbb{R}^{\dim(M)})$ for every chart (U,κ) of M and for every $\phi, \psi \in C_0^\infty(U)$, where M_ϕ is the multiplication operator $f \mapsto \phi f$, and

$$(M_\phi A M_\psi)_\kappa f := (M_\phi A M_\psi(f \circ \kappa)) \circ \kappa^{-1} \quad (f \in C^\infty(\kappa U)).$$

We sometimes treat write $M_\phi A M_\psi \in \Psi^m(\mathbb{R}^{\dim(M)})$, thus omitting the subscript κ and leaving the chart mapping implicit. Equivalently, pseudodifferential operators can be characterized by commutators (see [11]): $A \in \mathcal{L}(\mathcal{D}(M))$ belongs to $\Psi^m(M)$ if and only if $(A_k)_{k=0}^\infty \subset \mathcal{L}(H^m(M), H^0(M))$ for every sequence of smooth vector fields $(D_k)_{k=1}^\infty$ on M, where $A_0 = A$ and $A_{k+1} = [D_{k+1}, A_k]$.

A smooth *left transformation group* is

$$(G, M, m),$$

where G is a Lie group, M is a C^∞-manifold and $m : G \times M \to M$ is a C^∞-mapping called a left *action*, satisfying $m(e,p) = p$ and $m(x, m(y,p)) = m(xy, p)$ for every $x, y \in G$ and $p \in M$, where $e \in G$ is the neutral element of the group. The action is *free*, if $m(x,p) = p$ implies $x = e$. It is evident how one defines a *right* transformation group (G, M, m) with a *right* action $m : M \times G \to M$.

A smooth *fiber bundle* is

$$(E, B, F, p_{E\to B}),$$

where E, B, F are C^∞-manifolds and $p_{E\to B} \in C^\infty(E,B)$ is a surjective mapping such that there exists an open cover $\mathcal{U} = \{U_j \mid j \in J\}$ of B and diffeomorphisms $\phi_j : p^{-1}(U_j) \to U_j \times F$ satisfying $\phi_j(x) = (p_{E\to B}(x), \psi_j(x))$ for every $x \in p_{E\to B}^{-1}(U_j)$. The spaces E, B, F are called the *total space*, the *base space*, and the *fiber* of the bundle, respectively. The cover $\mathcal{U}$ is called a *locally trivializing cover* of the bundle. Sometimes the mapping $p_{E\to B}$ is called the fiber bundle.

A *principal fiber bundle* is

$$(E, B, F, p_{E\to B}, m),$$

where $(E, B, F, p_{E\to B})$ is a smooth fiber bundle with cover $\mathcal{U}$ and mappings ϕ_j, ψ_j as above and (F, E, m) is a smooth right transformation group with a free action satisfying $p_{E\to B}(m(x,y)) = p_{E\to B}(x)$ for every $(x,y) \in E \times F$ and $\psi_j(m(x,y)) = \psi_j(x)y$ for every $(x,y) \in p_{E\to B}^{-1}(U_j) \times F$.

4 Harmonic analysis on compact Lie groups

Let G be a compact Lie group. Let μ_G be the normalized Haar measure of G. The starting point of harmonic analysis on G is the *left regular representation* of G, which is the homomorphism $\pi_L : G \to \mathcal{L}(L^2(G))$ defined by

$$(\pi_L(y)f)(x) = f(y^{-1}x) \tag{6}$$

for almost every $x \in G$; equivalently we could begin with the *right regular representation* $\pi_R : G \to \mathcal{L}(L^2(G))$ defined by

$$(\pi_R(y)f)(x) = f(xy) \tag{7}$$

for almost every $x \in G$.

The *Fourier transform* of a distribution $f \in \mathcal{D}'(G)$ is said to be the operator $\pi(f) \in \mathcal{L}(\mathcal{D}(G))$ defined by

$$\pi(f)g = f * g, \tag{8}$$

i.e. the left convolution by f. Let $A \in \mathcal{L}(\mathcal{D}(G))$ with the Schwartz kernel K_A. The *symbol of* A is the mapping $\sigma_A : G \to \mathcal{L}(\mathcal{D}(G))$ defined by $\sigma_A(x) = \pi(s_A(x))$, where $K_A(x, y) = (s_A(x))(xy^{-1})$ in the sense of distributions. Then we denote $A = \mathrm{Op}(\sigma_A)$, and we have

$$\begin{aligned} (Af)(x) &= (\sigma_A(x)f)(x) \\ &= \mathrm{Tr}\,(\sigma_A(x)\ \pi(f)\ \pi_L(x)^*) \quad (f \in \mathcal{D}(G),\ x \in G). \end{aligned}$$

In the sequel Δ is the bi-invariant Laplacian of G (i.e. the left and right translation invariant Laplacian, or the Laplacian corresponding to the bi-invariant Riemannian metric of G), and we define $\Xi := (I - \Delta)^{1/2}$; then Ξ^m is a Sobolev space isomorphism $H^s(G) \to H^{s-m}(G)$, and it is also bi-invariant.

In the notation of [11], let us define

$$Q^\alpha \pi(s) = \pi(y \mapsto \check{q}_\alpha(y)\ s(y)),$$

where if $s \in \mathcal{D}'(G)$, and $q_\alpha \in C^\infty(G)$ ($\alpha \in \mathbb{N}_0^{\dim(G)}$) satisfies

$$q_\alpha(\exp(x)) = \frac{1}{\alpha!} x^\alpha$$

when x belongs to a small neighbourhood of $0 \in \mathfrak{g}$, the origin of the Lie algebra $\mathfrak{g}$ of G; technical details can be found in [11], where we presented the following characterization of pseudodifferential operators:

Definition. *An operator* $A \in \mathcal{L}(\mathcal{D}(G))$ *belongs to* $\Psi^m(G)$ *if and only if* $\sigma_A \in S^m(G) = \cap_{k=0}^\infty S_k^m(G)$*; here* $\sigma_B \in S_0^m(G)$ *if and only if*

$$\|\Xi^{|\alpha|-m} Q^\alpha \partial_x^\beta \sigma_B(x)\|_{\mathcal{L}(L^2(G))} \leq C_{B\alpha\beta m} \tag{9}$$

uniformly in $x \in G$ *for every* $\alpha, \beta \in \mathbb{N}_0^{\dim(G)}$*;* $\sigma_B \in S_{k+1}^m(G)$*, if*

$$\sigma_B \in S_k^m(G), \tag{10}$$

$$[\sigma_{\partial_j}, \sigma_B] \in S_k^m(G), \tag{11}$$

$$(Q^\gamma \sigma_{\partial_j})\sigma_A \in S_k^{m+1-|\gamma|}(G) \tag{12}$$

and

$$(Q^\gamma \sigma_A)\sigma_{\partial_j} \in S_k^{m+1-|\gamma|}(G) \tag{13}$$

for every $j \in \{1, \ldots, \dim(G)\}$ *and* $\gamma \in \mathbb{N}_0^{\dim(G)}$ *with* $|\gamma| > 0$*, where* $\{\partial_j \mid 1 \leq j \leq \dim(G)\}$ *is a basis for the vector space of the right-invariant vector fields on* G.

5 Harmonic analysis on compact homogeneous spaces

Let (G, E, m) be a smooth left transformation group. The manifold M is called a *homogeneous space* if the action $m : G \times M \to M$ is *transitive*, i.e. for every $p, q \in M$ there exists $x \in G$ such that $m(x, p) = q$.

Let us give another, equivalent definition for a homogeneous space: Let G be a Lie group with a closed subgroup K. The *homogeneous space* G/K is the set of classes $xK = \{xk \mid k \in K\}$ $(x \in G)$ endowed with the topology co-induced by $x \mapsto xK$ and equipped with the unique C^∞-manifold structure such that the mapping $(x, yK) \mapsto xyK$ belongs to $C^\infty(G \times (G/K), G/K)$ and such that there is a neighbourhood $U \subset G/K$ of $eK \in G/K$ and a mapping $\psi \in C^\infty(U, G)$ satisfying $\psi(xK)K = xK$. The group G acts smoothly from the left on the manifold G/K by $(x, yK) \mapsto x^{-1}yK$. Actually a smooth homogeneous space M is diffeomorphic to G/G_p, where $G_p = \{x \in G \mid m(x, p) = p\}$.

Notice also that $(G, G/K, K, x \mapsto xK, (x, k) \mapsto xk)$ has a structure of a principal fiber bundle (see [2]).

From now on we assume the Lie group G to be compact. We can regard functions (or distributions) constant on the cosets xK $(x \in G)$ as functions (or distributions) on G/K; it is obvious how one embeds the spaces $\mathcal{D}(G/K)$ and $\mathcal{D}'(G/K)$ into the spaces $\mathcal{D}(G)$ and $\mathcal{D}'(G)$, respectively. Let us define $P_{G/K} \in \mathcal{L}(\mathcal{D}(G))$ by

$$(P_{G/K} f)(x) = \int_K f(xk)\ d\mu_K(k). \tag{14}$$

Hence $P_{G/K} f \in C^\infty(G/K)$, and $P_{G/K}$ extends uniquely to the orthogonal projection of $L^2(G)$ onto the subspace $L^2(G/K)$. Let us consider operators $A \in \mathcal{L}(\mathcal{D}(G))$ with the symbol satisfying

$$\sigma_A(xk) = \sigma_A(x) \quad (x \in G,\ k \in K); \tag{15}$$

this condition is equivalent to

$$s_A(xk)(y) = s_A(x)(y)$$

in the sense of distributions, or

$$K_A(xk, yk) = K_A(x, y).$$

Then A maps the space $\mathcal{D}(G/K)$ into itself. Of course, for a general $A \in \mathcal{L}(\mathcal{D}(G))$ this is not true, but then we can define an operator $A_{G/K} \in \mathcal{L}(\mathcal{D}(G))$ by

$$s_{A_{G/K}} = (P_{G/K} \otimes \mathrm{id}) s_A. \tag{16}$$

Recall that $\sigma_A \in C^\infty(G, \mathcal{L}(H^m(G), H^0(G)))$ when $A \in \Psi^m(G)$, so that then

$$\sigma_{A_{G/K}}(x) = \int_K \sigma_A(xk)\ d\mu_K(k) \tag{17}$$

exists as a weak integral (Pettis integral), see [4].

Suppose we are given symbols of pseudodifferential operators A_1, A_2 on G satisfying the K-invariance (15). If we look at the asymptotic expansion formulae for $\sigma_{A_1A_2}$, $\sigma_{A_1^*}$ and $\sigma_{A_1^t}$ in [11], we see that all the terms there are K-invariant in the same sense. Moreover, for an elliptic K-invariant symbol the terms in the asymptotic expansion for a parametrix are also K-invariant.

Theorem 1 and its corollary show how to 'project' pseudodifferential operators on G to pseudodifferential operators on G/K:

Theorem 1. *Let G be a compact Lie group with a closed Lie subgroup K. If $A \in \Psi^m(G)$, then $A_{G/K} \in \Psi^m(G)$.*

Proof. First, notice that $P_{G/K}$ is left-invariant, and hence

$$(\partial_x^\beta \otimes M_{\tilde{q}_\alpha})(P_{G/K} \otimes \mathrm{id})s_A = (P_{G/K} \otimes \mathrm{id})(\partial_x^\beta \otimes M_{\tilde{q}_\alpha})s_A$$

for a right-invariant partial differential operator ∂_x^β and a multiplication $M_{\tilde{q}_\alpha}$ for every $\alpha, \beta \in \mathbb{N}_0^{\dim(G)}$. Therefore

$$\mathrm{Op}(Q^\alpha \partial_x^\beta \sigma_{A_{G/K}}) = \left(\mathrm{Op}(Q^\alpha \partial_x^\beta \sigma_A)\right)_{G/K}.$$

Since $A \in \Psi^m(G)$, we have

$$\|Q^\alpha \partial_x^\beta \sigma_A(x)\|_{\mathcal{L}(H^{m-|\alpha|}(G), H^0(G))} \leq C_{A\alpha\beta m},$$

and so the mapping $k \mapsto Q^\alpha \partial_x^\beta \sigma_A(xk)$ belongs to $C^\infty(K, \mathcal{L}(H^{m-|\alpha|}(G), H^0(G)))$ for every $x \in G$. Then

$$\begin{aligned}
\|Q^\alpha \partial_x^\beta \sigma_{A_{G/K}}(x)\|_{\mathcal{L}(H^{m-|\alpha|}, H^0)} &= \left\| \int_K Q^\alpha \partial_x^\beta \sigma_A(xk)\ d\mu_K(k) \right\|_{\mathcal{L}(H^{m-|\alpha|}, H^0)} \\
&\leq \int_K \|Q^\alpha \partial_x^\beta \sigma_A(xk)\|_{\mathcal{L}(H^{m-|\alpha|}, H^0)}\ d\mu_K(k) \\
&\leq \sup_{k \in K} \|Q^\alpha \partial_x^\beta \sigma_A(xk)\|_{\mathcal{L}(H^{m-|\alpha|}, H^0)} \\
&\leq \sup_{y \in G} \|Q^\alpha \partial^\beta \sigma_A(y)\|_{\mathcal{L}(H^{m-|\alpha|}, H^0)} \\
&\leq C_{A\alpha\beta m}.
\end{aligned}$$

This proves that $\sigma_{A_{G/K}} \in \mathrm{Op}S_0^m(G)$. Let $B \in \mathcal{L}(\mathcal{D}(G))$ be any right-invariant (left convolution) pseudodifferential operator. Then $\sigma_B(x) = B$ for each $x \in G$ and $x \mapsto s_B(x)$ is a constant mapping $G \to \mathcal{D}'(G)$, $B = B_{G/K}$, and

$$(\mathrm{Op}(\sigma_A \sigma_B))_{G/K} = \mathrm{Op}(\sigma_{A_{G/K}} \sigma_B)$$

and

$$(\mathrm{Op}(\sigma_B \sigma_A))_{G/K} = \mathrm{Op}(\sigma_B \sigma_{A_{G/K}}).$$

Assume that we have proven $\sigma_{C_{G/K}} \in S^r_k(G)$ for every $C \in \Psi^r(G)$, for every $r \in \mathbb{R}$. Using Lemma 6, Theorem 9 and Proposition 11 in [11], we hence get

$$\mathrm{Op}([\sigma_{\partial_j}, \sigma_{A_{G/K}}]) = \mathrm{Op}([\sigma_{\partial_j}, \sigma_A])_{G/K} \in \mathrm{Op}S^m_k(G),$$

$$\mathrm{Op}((Q^\gamma \sigma_{\partial_j})\sigma_{A_{G/K}}) = \mathrm{Op}((Q^\gamma \sigma_{\partial_j})\sigma_A)_{G/K} \in \mathrm{Op}S^{m+1-|\gamma|}_k(G)$$

and

$$\mathrm{Op}((Q^\gamma \sigma_{A_{G/K}})\sigma_{\partial_j}) = \mathrm{Op}((Q^\gamma \sigma_A)\sigma_{\partial_j})_{G/K} \in \mathrm{Op}S^{m+1-|\gamma|}_k(G);$$

this means that $\sigma_{A_{G/K}} \in S^m_{k+1}(G)$, and then by induction we get $\sigma_{A_{G/K}} \in S^m(G) = \cap_{k=0}^\infty S^m_k(G)$ □

Corollary 2. *Let G/K be orientable. Then $A_{G/K}|_{\mathcal{D}(G/K)} \in \Psi^m(G/K)$ for every $A \in \Psi^m(G)$.*

Proof. Let

$$\Psi^m(G)_{G/K} = \{A_{G/K} \mid A \in \Psi^m(G)\}$$

and

$$\Psi^m(G)_{G/K}|_{\mathcal{D}(G/K)} = \{A_{G/K}|_{\mathcal{D}(G/K)} \ :\ A \in \Psi^m(G)\}.$$

By Theorem 1 we know that $\Psi^m(G)_{G/K} \subset \Psi^m(G)$. Let D be a smooth vector field on G/K. Since $(G, G/K, K, x \mapsto xK, (x,k) \mapsto xk)$ is a principal fiber bundle, there exists a smooth vector field $X = X_{G/K}$ on G such that $X|_{\mathcal{D}(G/K)} = D$ (see [5]). Then

$$[D, \Psi^m(G)_{G/K}|_{\mathcal{D}(G/K)}] = [X, \Psi^m(G)_{G/K}]|_{\mathcal{D}(G/K)} \subset \Psi^m(G)_{G/K}|_{\mathcal{D}(G/K)},$$

and this combined with $\Psi^m(G)_{G/K}|_{\mathcal{D}(G/K)} \subset \mathcal{L}(H^m(G/K), H^0(G/K))$ yields the conclusion due to the commutator characterization of pseudodifferential operators on closed manifolds □

Hence at least sometimes a pseudodifferential operator on G/K has a non-unique extension to a pseudodifferential operator on G. If $B_j \in \Psi^{m_j}(G/K)$ has an extension $C_j = (C_j)_{G/K} \in \Psi^{m_j}(G)$ (i.e. $C_j|_{\mathcal{D}(G/K)} = B_j$), then $C_j^* \in \Psi^{m_j}(G)$ is an extension of the adjoint operator $B_j^* \in \Psi^{m_j}(G/K)$, and $B_1B_2 \in \Psi^{m_1+m_2}(G/K)$ has an extension $C_1C_2 \in \Psi^{m_1+m_2}(G)$; and if C_1 is elliptic with a parametrix $D \in \Psi^{-m_1}(G)$, then $D = D_{G/K}$ and $B_1 \in \Psi^{m_1}(G/K)$ is elliptic with a parametrix $D|_{\mathcal{D}(G/K)} \in \Psi^{-m_1}(G/K)$.

6 Harmonic analysis on G/K, K a torus

In the sequel we always assume that the subgroup K of G is a torus, $K \cong \mathbb{T}^q$.

Example of special interest: Let $\mathbb{B}^n$ be the unit ball of the Euclidean space $\mathbb{R}^n$, and $\mathbb{S}^{n-1}$ its boundary, the $(n-1)$-sphere. The two-sphere $\mathbb{S}^2$ can be considered as the base space of the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$, where the fibers are diffeomorphic to the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. In the context of harmonic analysis, $\mathbb{S}^3$ is diffeomorphic to the compact non-commutative Lie group $G = \mathrm{SU}(2)$, having a maximal torus $K \cong \mathbb{S}^1 \cong \mathbb{T}^1$. Then the homogeneous space G/K is diffeomorphic to $\mathbb{S}^2$, so that the canonical projection $p_{G\to G/K} : x \mapsto xK$ is interpreted as the Hopf fiber bundle $G \to G/K$; in the sequel we treat the two-sphere $\mathbb{S}^2$ always as the homogeneous space G/K. Notice that also $\mathbb{S}^2 \cong \mathrm{SO}(3)/\mathbb{T}^1$.

In [6] a subalgebra of $\Psi^m(\mathbb{S}^2)$ was described in terms of so called spherical symbols. Functions $f \in \mathcal{D}(\mathbb{S}^2)$ can be expanded in series

$$f(\phi,\theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{f}(l)_m \; Y_l^m(\phi,\theta), \tag{18}$$

where $(\phi,\theta) \in [0,2\pi] \times [0,\pi]$ are the spherical coordinates, the functions Y_l^m the spherical harmonics with Fourier coefficients

$$\hat{f}(l)_m := \int_0^{\pi} \int_0^{2\pi} f(\phi,\theta) \; \overline{Y_l^m(\phi,\theta)} \sin(\theta) \; d\phi \; d\theta. \tag{19}$$

Let us define

$$(Af)(\phi,\theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a(l) \; \hat{f}(l)_m \; Y_l^m(\phi,\theta), \tag{20}$$

where $a : \mathbb{N}_0 \to \mathbb{C}$ is a rational function; in [6], Svensson states that $A \in \Psi^m(\mathbb{S}^2)$ if and only if

$$|a(l)| \leq C_{A,m}(l+1)^m. \tag{21}$$

Let us present another proof for a special case of Theorem 1 and Corollary 2.

Theorem 3. *Let G be a compact Lie group with a torus subgroup K. If $A \in \Psi^m(G)$, then $A_{G/K} \in \Psi^m(G)$ and the restriction $A_{G/K}|_{\mathcal{D}(G/K)} \in \Psi^m(G/K)$.*

Proof. Let $\dim(G) = p + q$, $K \cong \mathbb{T}^q$. Let $\mathcal{V} = \{V_i \mid i \in \mathcal{I}\}$ be a locally trivializing open cover of G/K for the principal fiber bundle $(G, G/K, K, x \mapsto xK, (x,k) \mapsto xk)$; Let $\mathcal{U} = \{U_j \mid 1 \leq j \leq N\}$ be an open cover of G/K such that for every $j_1, j_2 \in \{1, \ldots, N\}$ there exists $V_i \in \mathcal{V}$ containing $U_{j_1} \cup U_{j_2}$ whenever $U_{j_1} \cap U_{j_2} \neq \emptyset$. Notice that we can always refine any open cover on a finite-dimensional manifold to get a new cover satisfying this additional requirement (proving this is easy, see an analogous treatment for partitions of unity in [10]). Then each $U_i \cup U_j$ $(1 \leq i,j \leq N)$ is a chart neighbourhood on G/K, and furthermore there exist diffeomorphisms $\phi_{ij} : (U_i \cup U_j) \times K \to p_{G\to G/K}^{-1}(U_i \cup U_j)$ such that $p_{G\to G/K}(\phi_{ij}(x,k)) = x$ for every $x \in U_i \cup U_j$ and

$k \in K$. To simplify notation, we treat the neighbourhood $U_i \cup U_j \subset G/K$ as a set $U_i \cup U_j \subset \mathbb{R}^p$, and $p^{-1}_{G \to G/K}(U_i \cup U_j) \subset G$ as a set $(U_i \cup U_j) \times \mathbb{T}^q \subset \mathbb{R}^p \times \mathbb{T}^q$.

Let $\{(U_j, \psi_j) \mid 1 \leq j \leq N\}$ be a partition of unity subordinate to $\mathcal{U}$, and let $A_{ij} = M_{\psi_i} A M_{\psi_j} \in \Psi^m(G)$. With the localized notation we consider $A_{ij} \in \Psi^m(\mathbb{R}^p \times \mathbb{T}^q)$, so that it has the symbol $\sigma_{A_{ij}} \in S^m(\mathbb{R}^p \times \mathbb{T}^q)$. Then

$$\begin{aligned}
\sigma_{(A_{G/K})_{ij}}(x,\xi) &= \sigma_{(A_{ij})_{G/K}}(x,\xi) \\
&= \int_{\mathbb{T}^q} \sigma_{A_{ij}}(x_1, \ldots, x_p, x_{p+1} + z_1, \ldots, x_{p+q} + z_q; \xi)\ dz_1\ \cdots\ dz_q,
\end{aligned}$$

and it is now easy to check that $\sigma_{(A_{G/K})_{ij}} \in S^m(\mathbb{R}^p \times \mathbb{T}^q)$. This yields $(A_{G/K})_{ij} \in \Psi^m(G)$, thus

$$A_{G/K} = \sum_{i,j} (A_{G/K})_{ij} \in \Psi^m(G)$$

$\square$

Theorem 4. *Let G be a compact Lie group with a torus subgroup K. Let $B \in \Psi^m(G/K)$. Then there exists an operator $A = A_{G/K} \in \Psi^m(G)$ such that $A|_{\mathcal{D}(G/K)} = B$.*

Proof. Let $K \cong \mathbb{T}^q$, $\dim(G) = p + q$, and let $\{(U_j, \psi_j) \mid 1 \leq j \leq N\}$ be the same partition of unity as in the proof of Theorem 3. Let $B_{ij} = M_{\psi_i} B M_{\psi_j} \in \Psi^m(G/K)$. With the localized notation we consider $B_{ij} \in \Psi^m(\mathbb{R}^p)$, so that it has the symbol $\sigma_{B_{ij}} : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{C}$, and the mapping $(x,\xi) \mapsto \sigma_{B_{ij}}(x,\xi)$ is zero when $x \in \mathbb{R}^p \setminus (U_i \cup U_j)$. We use Lemma 5 in Appendix to construct a pseudodifferential operator $A_{ij} \in \Psi^m(\mathbb{R}^p \times \mathbb{T}^q)$ such that $\sigma_{A_{ij}} : (\mathbb{R}^p \times \mathbb{T}^q) \times (\mathbb{R}^p \times \mathbb{Z}^q) \to \mathbb{C}$,

$$\sigma_{A_{ij}}(x; P\xi, 0, \ldots, 0) = \sigma_{B_{ij}}(Px; P\xi),$$

where $Py = (y_1, \ldots, y_p)$ $(y \in \mathbb{R}^{p+q})$. Hence $A = A_{G/K} = \sum_{i,j} A_{ij} \in \Psi^m(G)$ and $A|_{\mathcal{D}(G/K)} \in \Psi^m(G/K)$. Let $f = \sum_k f_k \in C^\infty(G/K) \subset C^\infty(G)$, $f_k = f \psi_k$; then

$$\begin{aligned}
(Af)(x) &= \sum_{i,j,k} (A_{ij} f_k)(x) \\
&= \sum_{i,j,k} \int_{\mathbb{R}^p} \sum_{\xi_{p+1}, \ldots, \xi_{p+q} \in \mathbb{Z}} \sigma_{A_{ij}}(x,\xi)\ \hat{f}_k(\xi)\ e^{i2\pi x \cdot \xi}\ d\xi_1 \cdots d\xi_p \\
&= \sum_{i,j,k} \int_{\mathbb{R}^p} \sigma_{A_{ij}}(x; P\xi, 0, \ldots, 0)\ \hat{f}_k(P\xi, 0, \ldots, 0)\ e^{i2\pi (Px) \cdot (P\xi)}\ d\xi_1 \cdots d\xi_p \\
&= \sum_{i,j,k} \int_{\mathbb{R}^p} \sigma_{B_{ij}}(Px; P\xi)\ \hat{f}_k(P\xi, 0, \ldots, 0)\ e^{i2\pi (Px) \cdot (P\xi)}\ d\xi_1 \cdots d\xi_p \\
&= \sum_{i,j,k} (B_{ij} f_k)(Px) \\
&= (Bf)(xK)
\end{aligned}$$

$\square$

7 Discussion

Theorem 4 combined with Lemma 5 provides just one way of extending operators, unfortunately destroying ellipticity: this is due to the apparent non-ellipticity of the symbol χ in Lemma 5. Let us discuss this problem and provide other extensions.

Let us extend the identity operator $I \in \Psi^0(\mathbb{R}^p)$ using the process suggested by Lemma 5. Of course, it would be desirable if $I \in \Psi^0(\mathbb{R}^p)$ could be extended to the identity in $\Psi^0(\mathbb{R}^{p+q})$, but now $\sigma_I(x,\xi) \equiv 1$, and thereby its extension $A \in \Psi^0(\mathbb{R}^{p+q})$ has the non-elliptic homogeneous symbol $\sigma_A = \chi \in S^0(\mathbb{R}^{p+q})$.

Given an elliptic symbol $\sigma_B \in S^m(\mathbb{R}^p)$ we can occasionally modify the construction in Lemma 5 to get an extended elliptic symbol in $S^m(\mathbb{R}^{p+q})$. Sometimes the following trick helps: Let $\sigma_{A_1} \in S^m(\mathbb{R}^{p+q})$ be an extension of σ_{B_1} as in Lemma 5,

$$\sigma_{A_1}(x,\xi) = \chi_1(\xi)\ \sigma_{B_1}(x_1,\ldots,x_p;\xi_1,\ldots,\xi_p),$$

where $\chi_1 \in S^0(\mathbb{R}^{p+q})$ is a homogeneous symbol satisfying $\chi_1|_{(U\times\mathbb{R}^q)\setminus\mathbb{B}(0,1)} \equiv 0$, $\chi_1|_{\mathbb{R}^p\times V} \equiv 1$, where $U \subset \mathbb{R}^p$ and $V \subset \mathbb{R}^q$ are neighborhoods of zeros. Take any elliptic symbol $\sigma_{B_2} \in S^m(\mathbb{R}^q)$, and modify Lemma 5 to construct an extension $\sigma_{A_2} \in S^m(\mathbb{R}^{p+q})$ such that

$$\sigma_{A_2}(x,\xi) = \chi_2(\xi)\ \sigma_{B_2}(x_p,\ldots,x_{p+q};\xi_p,\ldots,\xi_{p+q})$$

for a homogeneous symbol $\chi_2 \in S^0(\mathbb{R}^{p+q})$ satisfying $\chi_2|_{(U\times\mathbb{R}^q)\setminus\mathbb{B}(0,1)} \equiv 1$, $\chi_2|_{(\mathbb{R}^p\times V)\setminus\mathbb{B}(0,1)} \equiv 0$. Then $\sigma_{A_1} + \sigma_{A_2} \in S^m(\mathbb{R}^{p+q})$ is an extension for σ_{B_1} (modulo infinitely smoothing operators). For instance, if $B_1 = I \in \Psi^0(\mathbb{R}^p)$, let $B_2 = I \in \Psi^0(\mathbb{R}^q)$ and $\chi_2(\xi) = 1 - \chi_1(\xi)$ (for $|\xi| > 1$), then $A_1 + A_2 = I \in \Psi^0(\mathbb{R}^{p+q})$ (modulo infinitely smoothing operators).

It may happen that any extension process for an elliptic symbol $\sigma_B \in S^m(\mathbb{R}^p)$ constructs a non-elliptic symbol in $S^m(\mathbb{R}^{p+q})$. Consider, for instance, a case where $B \in \Psi^m(\mathbb{R}^2)$ is an elliptic convolution operator and $\xi \mapsto f(\xi) \equiv \sigma_B(x,\xi)$ is homogeneous outside the unit ball $\mathbb{B}(0,1) \subset \mathbb{R}^2$. If the mapping $f|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{C} \setminus \{0\}$ is not homotopic to a constant mapping (i.e. $f|_{\mathbb{S}^1}$ has a non-zero winding number) then no extension $\sigma_A \in S^m(\mathbb{R}^3)$ of σ_B can be elliptic.

Multiplications on G/K have already been extended to multiplications G via $x \mapsto xK$, and $A = A_{G/K}$ for any left convolution operator (multiplier) $A \in \mathcal{L}(\mathcal{D}(G))$ (in fact, then $\sigma_A(x) = A$ for every $x \in G$). Sometimes on G/K we have operators that resemble convolution operators. Suppose we are given a left convolution operator $A \in \Psi^m(\mathrm{SU}(2))$. Then the restriction $B = A|_{\mathcal{D}(\mathbb{S}^2)} \in \Psi^m(\mathbb{S}^2)$ is of the form

$$(Bf)(\phi,\theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\sum_{n=-l}^{l} a(l)_{mn}\ \hat{f}(l)_n \right)\ Y_l^m(\phi,\theta), \tag{22}$$

where the coefficients $a(l)_{mn} \in \mathbb{C}$ can be calculated from the data

$$\{BY_l^m \mid l \in \mathbb{N}_0,\ m \in \{-l, -l+1, \ldots, l-1, l\}\}.$$

It is even true that the original operator A can be retrieved from the coefficients $a(l)_{mn}$. In fact, any operator $B \in \mathcal{L}(\mathcal{D}(\mathbb{S}^2))$ of the form (22) can be extended to a unique left convolution operator belonging to $\mathcal{L}(\mathcal{D}(\mathrm{SU}(2)))$. Now a natural question arises: given a pseudodifferential operator $B \in \Psi^m(\mathbb{S}^2)$ of the form (22), does its extension to the left convolution operator belong to $\Psi^m(\mathrm{SU}(2))$? This is an open problem. An interesting special case is

$$(Bf)(x) = \int_{\mathbb{S}^2} \kappa(x \cdot y)\ f(y)\ dy, \tag{23}$$

where $\kappa \in \mathcal{D}'(\mathbb{S}^2)$, $(x, y) \mapsto x \cdot y$ is the scalar product of $\mathbb{R}^3$, and the integration is with respect to the angular part of the Lebesgue measure of $\mathbb{R}^3$. Then

$$(Bf)(\phi, \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_l\ \hat{\kappa}(l)_0\ \hat{f}(l)_m\ Y_l^m(\phi, \theta)$$

for some normalizing constants c_l depending only on $l \in \mathbb{N}_0$.

8 Appendix

Lemma 5. *Let $\chi \in C^\infty(\mathbb{R}^{p+q})$ be homogeneous of order 0 in $\mathbb{R}^{p+q} \setminus \mathbb{B}(0,1)$, i.e. $\chi(\xi) = \chi(\xi/\|\xi\|)$ when $\|\xi\| \geq 1$. Furthermore, assume that χ satisfies $\chi|_{(U \times \mathbb{R}^q) \setminus \mathbb{B}(0,1)} \equiv 0$, $\chi|_{\mathbb{R}^p \times V} \equiv 1$, where $U \subset \mathbb{R}^p$ and $V \subset \mathbb{R}^q$ are neighborhoods of zeros. Let $\sigma_B \in S^m(\mathbb{R}^p)$ and*

$$\sigma_A(x, \xi) := \chi(\xi)\ \sigma_B(Px, P\xi),$$

where $P(x_1, \ldots, x_{p+q}) = (x_1, \ldots, x_p)$. Then $\sigma_A \in S^m(\mathbb{R}^{p+q})$. Moreover, $\sigma_A|_{(\mathbb{R}^p \times \mathbb{R}^q) \times (\mathbb{R}^p \times \mathbb{Z}^q)} \in S^m(\mathbb{R}^p \times \mathbb{T}^q)$.

Proof. We shall first prove that

$$|(\partial_\xi^\gamma \chi)(\xi)| \leq C_{\gamma r}\ \langle P\xi \rangle^{-r}\ \langle \xi \rangle^{r-|\gamma|} \tag{24}$$

for every $r \in \mathbb{R}$ and for every $\gamma \in \mathbb{N}_0^{p+q}$. It is trivial that $(x, \xi) \mapsto \chi(\xi)$ belongs to $S^0(\mathbb{R}^{p+q})$. If $r \geq 0$ then obviously (24) is true. Since we are not interested in the behaviour of the symbols when $\|\xi\|$ is small, we assume that $\|\xi\| > 1$ from here on. There exists $r_0 \in (0,1)$ such that $\chi(\xi) = 0$ when $\|P\xi\| < r_0$. Let $r < 0$ and $\xi \in \mathrm{supp}(\chi)$. Then $\|P\xi\| \geq r_0 \|\xi\|$, and thus

$$\begin{aligned}
|(\partial_\xi^\gamma \chi)(\xi)| &\leq C_\gamma\ \langle \xi \rangle^{-|\gamma|} \\
&= C_\gamma\ \langle P\xi \rangle^{-r}\ \langle P\xi \rangle^{r}\ \langle \xi \rangle^{-|\gamma|} \\
&\leq C_\gamma\ \langle P\xi \rangle^{-r}\ \langle r_0 \xi \rangle^{r}\ \langle \xi \rangle^{-|\gamma|} \\
&\leq C_\gamma\ r_0^r\ \langle P\xi \rangle^{-r}\ \langle \xi \rangle^{r-|\gamma|}.
\end{aligned}$$

Hence the inequality (24) is proven. Now

$$\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)| &\leq \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \ |(\partial_\xi^\gamma \chi)(\xi)| \ |(\partial_\xi^{\alpha-\gamma}\partial_x^\beta \sigma_B)(Px,P\xi)| \\
&\leq \sum_{\gamma\leq\alpha} \binom{\alpha}{\gamma} \ C_{\gamma r_\gamma} \ \langle P\xi\rangle^{-r_\gamma} \ \langle \xi\rangle^{r_\gamma-|\gamma|} \ C_{B(\alpha-\gamma)\beta m} \ \langle P\xi\rangle^{m-|\alpha-\gamma|} \\
&\leq C_{B\alpha\beta m\chi} \ \langle\xi\rangle^{m-|\alpha|},
\end{aligned}$$

if we choose $r_\gamma = m-|\alpha-\gamma|$. Thereby $\sigma_A \in S^m(\mathbb{R}^{p+q})$. Clearly we can consider this symbol as a function $\sigma_A : (\mathbb{R}^p \times \mathbb{T}^q) \times (\mathbb{R}^p \times \mathbb{R}^q) \to \mathbb{C}$ and study its restriction $\sigma_A|_{(\mathbb{R}^p\times\mathbb{T}^q)\times(\mathbb{R}^p\times\mathbb{Z}^q)}$ we claim that this restriction belongs to $S^m(\mathbb{R}^p \times \mathbb{T}^q)$. Indeed, Taylor expansion of a function $\sigma \in C^\infty(\mathbb{R}^q)$ yields

$$\begin{aligned}
\triangle_\xi^\gamma \sigma(\xi) &= \sum_{\delta\leq\gamma} \binom{\gamma}{\delta} (-1)^{|\gamma-\delta|} \ \sigma(\xi+\delta) \\
&= \sum_{\delta\leq\gamma} \binom{\gamma}{\delta} (-1)^{|\gamma-\delta|} \\
&\quad \times \left(\sum_{|\rho|<|\gamma|} \frac{1}{\rho!}\delta^\rho \ (\partial_\xi^\rho\sigma)(\xi) + \sum_{|\rho|=|\gamma|} \frac{1}{\rho!}\delta^\rho \ (\partial_\xi^\rho\sigma)(\xi+\theta_\delta\delta) \right) \\
&= \sum_{|\rho|<|\gamma|} \frac{1}{\rho!} \ (\partial_\xi^\rho\sigma)(\xi) \sum_{\delta\leq\gamma} \binom{\gamma}{\delta} (-1)^{|\gamma-\delta|}\delta^\rho \\
&\quad + \sum_{\delta\leq\gamma} \sum_{|\rho|=|\gamma|} \frac{1}{\rho!}\delta^\rho \ (\partial_\xi^\rho\sigma)(\xi+\theta_\delta\delta) \\
&= \sum_{\delta\leq\gamma} \sum_{|\rho|=|\gamma|} \frac{1}{\rho!}\delta^\rho \ (\partial_\xi^\rho\sigma)(\xi+\theta_\delta\delta),
\end{aligned}$$

because

$$\sum_{\delta\leq\gamma} \binom{\gamma}{\delta} (-1)^{|\gamma-\delta|}\delta^\rho = \triangle_\xi^\gamma \xi^\rho|_{\xi=0} = 0$$

whenever $|\rho| < |\gamma|$. Therefore

$$\begin{aligned}
|\triangle_\xi^\gamma \sigma(\xi)| &\leq \sum_{\delta\leq\gamma} \sum_{|\rho|=|\gamma|} \frac{1}{\rho!}\delta^\rho \ |(\partial_\xi^\rho\sigma)(\xi+\theta_\delta\delta)| \\
&\leq c_\gamma \sup_{\eta\in S_\gamma, |\rho|=|\gamma|} |(\partial_\xi^\rho\sigma)(\xi+\eta)|,
\end{aligned}$$

where S_γ is the hyper-rectangle $\prod_{j=1}^q [0,\gamma_j]$. Let $\alpha' = (P\alpha, 0, \ldots, 0)$, $\alpha'' = \alpha - \alpha'$; then

$$\begin{aligned}
|\partial_\xi^{\alpha'} \triangle_\xi^{\alpha''} \partial_x^\beta \sigma_A(x,\xi)| &\leq C_\alpha \sup_{\eta\in S_{\alpha''}, |\rho|=|\alpha''|} |\partial_\xi^{\alpha'+\rho}\partial_x^\beta \sigma_A(x,\xi+\eta)| \\
&\leq C_\alpha \ C_{A\alpha\beta m} \sup_{\eta\in S_\alpha} \langle \xi+\eta\rangle^{m-|\alpha|}
\end{aligned}$$

$$
\begin{aligned}
&\leq C_\alpha\, C_{A\alpha\beta m}\, 2^{|m-|\alpha||} \sup_{\eta\in S_\alpha} \langle\eta\rangle^{|m-|\alpha||}\, \langle\xi\rangle^{m-|\alpha|} \\
&\leq C_\alpha\, C_{A\alpha\beta m}\, 2^{|m-|\alpha||}\, \langle\alpha\rangle^{|m-|\alpha||}\, \langle\xi\rangle^{m-|\alpha|} \\
&= C'_{A\alpha\beta m}\, \langle\xi\rangle^{m-|\alpha|};
\end{aligned}
$$

notice the application of the Peetre inequality

$$\langle\xi+\eta\rangle^s \leq 2^{|s|}\, \langle\xi\rangle^s\, \langle\eta\rangle^{|s|}.$$

Hence $\sigma_A|_{(\mathbb{R}^p\times\mathbb{T}^q)\times(\mathbb{R}^p\times\mathbb{Z}^q)} \in S^m(\mathbb{R}^p\times\mathbb{T}^q)$ $\square$

References

[1] Agranovich, M. S.: *Spectral properties of elliptic pseudodifferential operators on a closed curve.* Funct. Anal. Appl. 13 (1979), 279–281.

[2] Bröcker, Th. and tom Dieck, T.: *Representations of Compact Lie Groups.* New York: Springer-Verlag 1985.

[3] Hörmander, L.: *The Analysis of Linear Partial Differential Operators III.* Berlin: Springer-Verlag 1985.

[4] Rudin, W.: *Functional Analysis.* New Delhi: Tata McGraw-Hill 1974.

[5] Spivak, M.: *A Comprehensive Introduction to Differential Geometry, Vol. II.* Houston: Publish or Perish 1979.

[6] Svensson, S. L.: *Pseudodifferential operators — a new approach to the boundary problems of physical geodesy.* Manuscripta Geodaetica 8 (1983), 1–40.

[7] Taylor, M. E.: *Beals–Cordes -type characterizations of pseudodifferential operators.* Proc. Amer. Math. Soc. 125 (1997), 6, 1711–1716.

[8] Taylor, M. E.: *Noncommutative microlocal analysis.* Mem. Amer. Math. Soc. 52 (1984), No. 313.

[9] Taylor, M. E.: *Pseudodifferential Operators.* Princeton Univ. Press 1981.

[10] Turunen, V.: *Commutator characterization of periodic pseudodifferential operators.* Z. Anal. Anw. 19 (2000), 95–108.

[11] Turunen, V.: *Pseudodifferential calculus on compact Lie groups.* Helsinki Univ. Techn. Inst. Math. Research Report A431. 2001.

[12] Turunen, V. and Vainikko, G.: *Symbolic calculus of periodic pseudodifferential operators.* Z. Anal. Anw. 17 (1998), 9–22.

[13] Wallach, N. R.: *Harmonic Analysis on Homogeneous Spaces.* New York: Marcel Dekker, Inc. 1973.

(continued from the back cover)

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The list of reports is continued inside. Electronical versions of the reports are available at *http://www.math.hut.fi/reports/* .

A435 Timo Salin
On Quenching with Logarithmic Singularity, May 2001

A434 Ville Havu
An Analysis of Asymptotic Consistency Error in a Parameter Dependent Model Problem, Mar 2001

A433 Ville Havu and Juhani Pitkäranta
Analysis of a bilinear finite element for shallow shells I: Consistency error, Jan 2001

A432 Tuomas Hytönen
Optimal Wrap-Araoud Network Simulation, Dec 2000

ISBN 951-22-5524-3
ISSN 0784-3143
Libella Painopalvelu Oy, Espoo, 2001