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Nonnegative Operators and the Method of Sums

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Abstract: *The method of sums, developed by P. Grisvard and G. Da Prato, is presented in a self–contained and fairly thorough fashion. Grisvard's and Da Prato's extension of the method to certain couples of operators with non–commuting resolvents is included (cf. [10]). In the resolvent commuting case explicit values for the regularity constants are given. The usefulness of the theory is demonstrated by applying it to fractional evolution equations of the form* $D_t^\alpha u + D_x^\beta u = f$ *and* $D_t^\alpha u + b(t,x)D_x u = f$ *in* $\mathcal{C}_{\partial_0 Q \mapsto 0}([0,\tau] \times [0,\xi]\,; X)$ *(see p. 114), where* X *is a complex Banach space,* $0 < \alpha, \beta \le 1$ *and* $\alpha + \beta < 2$ *in the first equation, and* $0 < \alpha < 1$ *in the second. This yields the maximal regularity of these equations with respect to the spaces* $\mathcal{C}^{\mu,\nu}_{\partial_0 Q \mapsto 0}([0,\tau] \times [0,\xi]\,; X)$ *of Hölder continuous functions and the spaces* $h^{\mu,\nu}_{\partial_0 Q \mapsto 0}([0,\tau] \times [0,\xi]\,; X)$ *of little Hölder continuous functions, where* $0 < \mu < \alpha$, *and* $0 < \nu < \beta$ *in the first case, whereas* $0 < \nu < 1$ *in the second case.*

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Introduction

In science and in mathematics one often encounters equations of the form

$$Ax + Bx = y, \tag{1}$$

where y belongs to a vector space X, A and B are linear operators in X, and $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$ is the solution to be determined. It is clear that if we have no or little knowledge of the operators A and B, then little can be said about the existence and regularity of solutions to the equation. Hence, when we develop some kind of general theory for equations of the form (1), we have to restrict our considerations to some special classes of pairs of operators. In the *method of sums*, this class consists of pairs of *nonnegative operators*.

In Chapter 2 we give an introduction to the theory of nonnegative and *positive* operators $A : \mathcal{D}(A) \longrightarrow X$ in a complex Banach space X. Among the topics included are, e.g., the interpolation spaces $\mathcal{D}_{A^n}(\theta, p)$ between $\mathcal{D}(A^n)$ and X and *fractional powers* of positive operators.

The method of sums, developed by P. Grisvard, is presented in Chapter 3. It is a powerful tool for solving the abstract operator equation (1): It provides a unique and explicit solution to the equation in the form of a complex vector–valued curve integral

$$S(A,B)y = \frac{-1}{2\pi i} \int_{\gamma} (z+A)^{-1}(z-B)^{-1} y \, dz$$

for all y in any *real interpolation space* $\mathcal{D}_A(\theta, p)$ or $\mathcal{D}_A(\theta)$ between X and $\mathcal{D}(A)$, or in any real interpolation space $\mathcal{D}_B(\theta, p)$ or $\mathcal{D}_B(\theta)$ between X and $\mathcal{D}(B)$. Here we assume that $0 < \theta < 1$ and $1 \le p \le \infty$. The restrictions that we impose on A and B in order to achieve this are, besides nonnegativity, that the *spectral angles* ω_A and ω_B of A and B satisfy the inequality $\omega_A + \omega_B < \pi$, that $0 \in \rho(A) \cup \rho(B)$, and that A and B are *resolvent commuting*.

In addition to guaranteeing the existence of a unique solution x for y in one of the above mentioned interpolation spaces, these restrictions also ensure maximal regularity of the problem with respect to the interpolation spaces in question. For example, if $y \in \mathcal{D}_A(\theta, p)$, then not only does x belong to $\mathcal{D}(A) \cap \mathcal{D}(B) \subseteq \mathcal{D}_A(\theta, p)$, but also Ax and Bx belong to this interpolation space.

The method of sums has also been generalised to pairs of operators with non–commuting resolvents. In Section 3.4 we present the method proposed by Da Prato and Grisvard (cf. [10]; a slightly different method can be found in [8]).

The method of sums may be used to investigate solutions to a number of problems, e.g. partial differential equations. In Chapter 4 we apply it to the *fractional evolution equation* $D_t^\alpha u(t,x) + D_x^\beta u(t,x) = f(t,x)$ in the space $\mathcal{C}_{\partial_0 \mapsto 0}(Q; E)$ of continuous functions $f : Q \to E$ such that $f(0,x) = f(t,0) = 0$, where $0 < \alpha, \beta \le 1$, $\alpha + \beta < 2$, $Q = [0,\tau] \times [0,\xi]$ and E is a complex Banach space, and obtain a maximal regularity result for the spaces $\mathcal{C}^{\mu,\nu}_{\partial_0 \mapsto 0}(Q;E)$, where $0 < \mu < \alpha$, $0 < \nu < \beta$. The regularity of the equation $D_t^\alpha u(t,x) + b(t,x)\, D_x u(t,x) = f(t,x)$ in the space $\mathcal{C}_{\partial_0 Q \mapsto 0}(Q;E)$ is also studied using Grisvard's and Da Prato's method for non–commuting operators.

In order to make the presentation self–contained, a chapter on graphs and linear operators as well as two appendices on integration of vector–valued functions and on interpolation spaces, respectively, are also included.

1 Graphs and linear operators

In this chapter we discuss some basic properties of linear graphs and functions, that will be needed in later chapters. Actually most of the results on graphs that we present will only be applied to functions, but we choose to prove more general versions.

1.1 Graphs, functions and linear operators

Let X and Y be two sets. A *graph*[1] G from X into Y is simply a subset of $X \times Y$. Adopting the view that a *function* (or *mapping*, or *operator*) f equals the set of ordered pairs $(x, f(x))$, we thus define a graph f from X into Y to be a function from X into Y if the implication

$$(x, y) \in f \wedge (x, z) \in f \Rightarrow y = z \tag{1.1}$$

holds for any $x \in X$ and any $y, z \in Y$. Thus, functions are many–to–one graphs. A *graph in* X is a graph from X into itself. Analogously a *function in* X is a function from X into itself.

Let us now introduce some customary generalisations to graphs of familiar concepts usually defined for functions.

The *domain* and the *range* of a graph G from X into Y are the sets

$$\mathcal{D}(G) := \{x \in X \mid \exists y \in Y : \ (x, y) \in G\}$$

and

$$\mathcal{R}(G) := \{y \in Y \mid \exists x \in X : \ (x, y) \in G\}$$

respectively.

The *inverse* of G is defined by the equality

$$G^{-1} := \{(y, x) \in Y \times X \parallel (x, y) \in G\}.$$

Then $\mathcal{D}(G^{-1}) = \mathcal{R}(G)$ and $\mathcal{R}(G^{-1}) = \mathcal{D}(G)$.

We say that f is a mapping of X into Y if f is a function from X into Y and $\mathcal{D}(L) = X$. This is denoted $f : X \longrightarrow Y$.

A function f from X into Y is said to be *one–to–one* (or *injective*) if f^{-1} is also a function. Otherwise it is *many–to–one*. It is *onto* (or *surjective*) if $\mathcal{R}(f) = Y$. If a function f from X to Y is one–to–one and onto and $\mathcal{D}(f) = X$, we say that f is *bijective* from X onto Y.

We define the *composition* $G_1 \circ G_2$ of a graph G_1 from X into Y and a graph G_2 from Y into Z by

$$G_2 \circ G_1 := \{(x, z) \in X \times Z \parallel \exists y \in Y : \ (x, y) \in G_1 \wedge (y, z) \in G_2\}.$$

Obviously

$$(G_2 \circ G_1)^{-1} = G_1^{-1} \circ G_2^{-1}.$$

[1]Set theoreticians use the term *relation*.

We define G^n $(n = 1, 2, \ldots)$ recursively by putting $G^1 = G$ and $G^{n+1} = G \circ G^n$ $(n = 1, 2, \ldots)$. Hence, we can also set $G^{-n} = (G^{-1})^n$.

We now assume that Y is a linear vector space. Then the *sum* of two graphs G_1 and G_2 from a set X into Y is the graph

$$G_1 + G_2 := \{(x, y_1 + y_2) \in X \times Y \mid (x, y_1) \in G_1 \wedge (x, y_2) \in G_2\}$$

with domain $\mathcal{D}(G_1 + G_2) = \mathcal{D}(G_1) \cap \mathcal{D}(G_2)$. For scalars λ of the space Y we set

$$\lambda G := \{(x, \lambda y) \in X \times Y \mid (x, y) \in G\}.$$

If G is a graph in a linear vector space X we can, as a special case, consider

$$\lambda - G := \lambda I - G := \{(x, \lambda x - y) \mid (x, y) \in G\},$$

where I is the identity function on X. As usual $-G := (-1)G$ and $G_1 - G_2 := G_1 + (-G_2)$.

At this point we note that $G + G$ need not equal $2G$, and that $G - G$ need not vanish (for any $x \in \mathcal{D}(G)$). But if f is a function and G_1 and G_2 are graphs with $\mathcal{D}(G_1) = \mathcal{D}(G_2) \subseteq \mathcal{D}(f)$, then

$$f + G_1 = f + G_2 \Rightarrow G_1 = G_2.$$

Let X and Y be linear vector spaces over the same scalar field. We call a graph G from X into Y *linear* if it is a linear subspace of $X \times Y$. Thus, G is linear if and only if

$$(\alpha_1 x_1 + \alpha_2 x_2, \alpha_1 y_1 + \alpha_2 y_2) \in G$$

whenever $(x_1, y_1) \in G$ and $(x_2, y_2) \in G$, and α_1 and α_2 are scalars. It is an immediate consequence of this definition that the domain and range of a linear graph from X into Y are linear subspaces of X and Y respectively. It is also easy to prove that if G_1 and G_2 are linear graphs from X into Y, and λ_1 and λ_2 are scalars, then $\lambda_1 G_1 + \lambda_2 G_2$ is linear. Also, G is linear if and only if G^{-1} is linear.

For linear graphs G_1 and G_2 we set $G_1 G_2 := G_1 \circ G_2$.

A *linear operator* from X into Y is a function from X into Y that is a linear graph. Hence, a linear graph L from X into Y is a linear operator if and only if

$$y \in Y \wedge (0, y) \in L \Rightarrow y = 0.$$

A linear operator *in* X is a linear operator L from X into itself, i.e. $L : \mathcal{D}(L) \longrightarrow X$. It is a linear operator *on* X if, in addition $\mathcal{D}(L) = X$, i.e. $L : X \longrightarrow X$.

Although the above defined concepts for graphs are straightforward generalisations of the corresponding ones for functions, some of the familiar rules of calculation with sums and composition of functions do not apply to arbitrary graphs. Thus, if H is a graph from X into Y, and F and G are graphs from Y into a linear vector space Z we can, in general, only prove that

$$(F + G)H \subseteq FH + GH.$$

The reverse inclusion holds only in special cases. For example $(F+G)H = FH + GH$ if H is a function. We also have

$$(f+G)f^{-1} = I + Gf^{-1} \tag{1.2}$$

if f (but not necessarily f^{-1}) is a function and G is a graph in X. Moreover, if G and H are graphs from X into a linear vector space Y, and F is a linear graph from Y into Z, we have

$$FG + FH \subseteq F(G+H), \tag{1.3}$$

whereas equality,

$$FG + FH = F(G+H), \tag{1.4}$$

holds if F is linear and either $\mathcal{D}(F) \supseteq \mathcal{R}(G)$ or $\mathcal{D}(F) \supseteq \mathcal{R}(H)$. One can also prove that if F is a graph from X into Y, G is a graph from Y into X and $\mathcal{R}(F) = \mathcal{D}(G)$, then

$$GF = I_{\mathcal{D}(F)}$$

implies that G is a function and $G = F^{-1}$. Here I_A is the identity function on $A \subseteq X$. All the facts stated above can be proved by simple application of the definitions of the various concepts involved.

1.2 Bounded linear operators

Let X and Y be two normed vector spaces over the same scalar field, and let L be a linear operator that maps $\mathcal{D}(L) \subseteq X$ into Y.

We say that L is *bounded* if there is a constant M such that $\| Lx \|_Y \leq M \| x \|_X$ for all $x \in \mathcal{D}(L)$. In that case the *operator norm* of L is defined by the equation

$$\| L \| := \sup_{\substack{x \in \mathcal{D}(L) \\ x \neq 0}} \frac{\| Lx \|_Y}{\| x \|_X} = \sup_{\substack{x \in \mathcal{D}(L) \\ \| x \|_X = 1}} \| Lx \|_Y .$$

It is easy to see that a bounded linear operator is continuous, since boundedness immediately implies continuity at the origin, and continuity at the origin is equivalent to continuity everywhere on account of linearity. The converse implication is also true, since if L is not bounded there is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{D}(L)$ with $\| x_n \|_X = 1$ and $\| Lx_n \|_Y \geq n \| x_n \|_X$ for $n = 1, 2, \ldots$. Hence, $x'_n = \frac{1}{n} x_n$ defines a sequence with $\| x'_n \|_X = 1/n$ and $\| Lx'_n \|_Y \geq 1$ for $n = 1, 2, \ldots$, which implies that L cannot be continuous. Thus, a linear operator is bounded if and only if it is continuous.

The set $\mathcal{L}(X, Y)$ of all bounded linear operators $L : X \longrightarrow Y$ is a normed linear vector space. We set $\mathcal{L}(X) := \mathcal{L}(X, X)$

1.3 Resolvents of graphs

Even though a graph is generally many–to–many, in special cases its inverse may, of course, be many–to–one, i.e. a function. Thus, the concept of regular value for linear operators in a Banach space X can be extended to graphs in X in the obvious way; we say that λ is a *regular* value for G if $(\lambda - G)^{-1}$ belongs to $\mathcal{L}(X)$, the space of bounded linear operators on X. In that case $(\lambda - G)^{-1}$ is said to be a *resolvent* of G. The set of all regular values of G is called the *resolvent set* of G and is denoted $\rho(G)$. Its complement with respect to $\mathbb{C}$ is the spectrum $\sigma(G)$ of G. We note that the existence of a regular value implies that the graph is linear. Thus, a function whose resolvent set is non–empty is a linear operator.

The following lemma is a straightforward generalisations of some well–known facts from the theory of linear operators.

THEOREM 1.1. *Let G be a graph in a Banach space X. Then $\rho(G) \subseteq \mathbb{C}$ is open, and the mapping $z \longrightarrow (z - G)^{-1}$ of $\rho(G)$ into $\mathcal{L}(X)$ is analytic with derivative $-(z - G)^{-2}$. Moreover, if $\lambda, \mu \in \rho(G)$, then we have the* resolvent identity

$$(\lambda - G)^{-1} - (\mu - G)^{-1} = (\mu - \lambda)(\lambda - G)^{-1}(\mu - G)^{-1}. \tag{1.5}$$

Proof. Let G be a graph in X and let z_0 be a regular value of G. We have

$$\begin{aligned} z - G &= z_0 I - G - (z_0 - z)I \\ &= (1 - (z_0 - z)(z_0 - G)^{-1})(z_0 - G), \end{aligned}$$

since if G^{-1} is a function then $G^{-1}G = I_{\mathcal{D}(G)}$. Now $1 - (z_0 - z)(z_0 - G)^{-1}$ can be inverted by means of a Neumann series if $\| (z_0 - z)(z_0 - G)^{-1} \| < 1$, and the inverse is defined on the whole of X. Thus, if $|z_0 - z| < 1/ \| (z_0 - G)^{-1} \|$, then

$$\begin{aligned} (z - G)^{-1} &= (z_0 - G)^{-1} \left(1 - (z_0 - z)(z_0 - G)^{-1}\right)^{-1} \\ &= \sum_{n=0}^{\infty} (z - z_0)^n (-1)^n (z_0 - G)^{-n-1}, \end{aligned} \tag{1.6}$$

so that

$$\rho(G) \supseteq \{z \in \mathbb{C} \mid |z_0 - z| < 1/ \left\| (z_0 - G)^{-1} \right\| \}. \tag{1.7}$$

This implies that $\rho(G)$ is open. If, in particular, z lies in the circular disc $|z - z_0| \leq (1-\delta) \| (z_0 - G)^{-1} \|$ for some $\delta \in (0, 1)$, then we have the following bound on the resolvent:

$$\left\| (z - G)^{-1} \right\| \leq \frac{1}{\delta} \left\| (z_0 - G)^{-1} \right\|. \tag{1.8}$$

The existence of a representation of $(z-G)^{-1}$ as the sum of a power series at z_0 also implies that it is analytic in z at z_0, and the derivative at z_0 is the coefficient $-(z_0 - G)^{-2}$ of $z - z_0$ (cf. Theorem A.24).

Now let $\lambda, \mu \in \rho(G)$. Then, using equation (1.2), we have

$$\left((\lambda - G)^{-1} - (\mu - G)^{-1}\right)(\mu - G) = (\lambda - G)^{-1}(\mu - G) - I_{\mathcal{D}(G)}. \tag{1.9}$$

But

$$\begin{aligned}(\lambda - G)^{-1}(\mu - G) &= (\lambda - G)^{-1}\left((\mu - \lambda)I + \lambda - G\right)\\ &= (\mu - \lambda)(\lambda - G)^{-1} + I_{\mathcal{D}(G)},\end{aligned}$$

where the second equality holds by (1.4) since $(\lambda - G)^{-1}$ is linear and defined on all of X. Inserting this into equation 1.9 we get

$$\left((\lambda - G)^{-1} - (\mu - G)^{-1}\right)(\mu - G) = (\mu - \lambda)(\lambda - G)^{-1} I_{\mathcal{D}(G)}. \tag{1.10}$$

Multiplying by $\lambda - G$ from the right, observing that

$$I_{\mathcal{D}(G)}(\lambda - G) = (\lambda - G) I_{\mathcal{D}((\mu - G)(\lambda - G))}.$$

we obtain

$$\left((\lambda - G)^{-1} - (\mu - G)^{-1}\right)(\mu - G)(\lambda - G) = (\mu - \lambda) I_{\mathcal{D}((\mu - G)(\lambda - G))}. \tag{1.11}$$

Since the domain of $(\lambda - G)^{-1} - (\mu - G)^{-1}$ and the range of $(\mu - G)(\lambda - G)$ both equal X, this implies the resolvent identity. □

1.4 Some results on the closure of a linear operator

Let G be a graph from a Banach space X into a Banach space Y. As usual, $\overline{G}$ denotes the *closure* of G, i.e., the set of all $(x, y) \in X \times X$ such that there is a sequence $\{(x_n, y_n)\}_{n=1}^{\infty}$ in G, which converges to (x, y) as $n \to \infty$. A graph is *closed* if it coincides with its closure. Thus, the closure of a graph is a closed graph.

The closure of a function is a closed graph, but it need not, however, be a function. A closed function is a function that is a closed graph. Thus, a function f is closed if and only if $f(x)$ is defined and $f(x) = y$ whenever $\{x_n\}_{n=1}^{\infty}$ converges to $x \in X$ and $\{f(x_n)\}_{n=1}^{\infty}$ converges to $y \in Y$ as $n \to \infty$. A function f is *closable* if $\overline{f}$ is also a function. We have the following simple lemma.

Lemma 1.2. *Any bounded linear operator L from X into Y is closable, and its closure $\overline{L}$ is a bounded linear operator with $\|\overline{L}\| = \|L\|$.*

Proof. It is clear that $\overline{L}$ is a linear graph, since being the closure of a linear subspace of $X \times Y$ it is a linear subspace of $X \times Y$. We have $\|Lx\| \leq \|L\|\|x\|$ for all $x \in \mathcal{D}(L)$. Hence, if $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{D}(L)$ and $x_n \to x$ as $n \to \infty$, then $Lx_n \to 0$. It follows that if $(0, y) \in \overline{L}$, then $y = 0$, which implies that $\overline{L}$ is a function. Moreover, if $x \in \mathcal{D}(\overline{L})$, then there is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq \mathcal{D}(L)$ that converges to x and such that Lx_n converges to $\overline{L}x$ as $n \to \infty$. Hence, $\|Lx\| \leq \|L\|\|x\|$ for all $x \in \mathcal{D}(\overline{L})$, so that $\|\overline{L}\| = \|L\|$. □

We observe that if L is a linear operator in X, then $\overline{(\lambda+L)^{-1}} = \overline{\lambda+L}^{\,-1} = (\lambda+\overline{L})^{-1}$. Let us prove a theorem concerning the resolvent set of the closure of a linear operator (see [7], pp. 309-314).

THEOREM 1.3. *Let X be a Banach space and let L be a linear operator in X such that*
(i) *there are constants $N \geq 1$ and $\omega_0 \geq 0$ with*

$$\| x \| \leq \frac{N}{\lambda} \| \lambda x + Lx \| \tag{1.12}$$

for all $\lambda > \omega_0$ and all $x \in \mathcal{D}(L)$, and
(ii) *there is some $\omega_1 > \omega_0$ such that $\mathcal{R}(\omega_1 + L) = (\omega_1 + L)(\mathcal{D}(L))$ is dense in X.*
Then $\big(\omega_0(1-\frac{1}{N}),\infty\big) \subseteq \rho(-\overline{L})$, and for all $\lambda > \omega_0(1-\frac{1}{N})$ the range of $\lambda + L$ is dense in X and

$$\left\| (\lambda+\overline{L})^{-1} \right\| \leq \frac{N}{\lambda}, \tag{1.13}$$

for all $\lambda > \omega_0$. If, in addition, L is densely defined in X, then it is closable. If (i) *holds with $N = 1$ and some ω_0, then it holds with $N = 1$ and $\omega_0 = 0$.*

REMARK 1.4. If an operator L satisfies condition (i) of the theorem with $N = 1$ and $\omega_0 = 0$, it is called m–accretive (cf. Section 2.1).

Proof. Let L be a linear operator in X which satisfies the hypotheses of the theorem and let $\lambda > \omega_0$. The inequality (1.12) implies that if $(0, y) \in (\lambda + L)^{-1}$, then $\| y \| \leq (N/\lambda) \| 0 \| = 0$, so that $(\lambda+L)^{-1}$ is a linear operator in X. Thus, $(\lambda+L)^{-1}$ is a bounded linear operator in X, with $\| (\lambda+L)^{-1} \| \leq N/\lambda$. Consequently, Lemma 1.2 implies that $(\lambda+\overline{L})^{-1} = \overline{(\lambda+L)^{-1}}$ is a bounded linear operator in X, and that $\left\| (\lambda+\overline{L})^{-1} \right\| \leq N/\lambda$, i.e., we have the obvious extension of the inequality (1.12) to $\overline{L}$:

$$\| x \| \leq \frac{N}{\lambda} \left\| \lambda x + \overline{L} \right\| \tag{1.14}$$

for all $x \in \mathcal{D}(\overline{L})$ and all $\lambda > \omega_0$.

Let us show that the bounded linear operator $(\omega_1 + \overline{L})^{-1}$ is defined on all of X. Thus, take an arbitrary $y \in X$. Then, since $(\omega_1 + L)(\mathcal{D}(L))$ is dense in X, there is a sequence $y_n = (\omega_1 + L)x_n$, $n = 1, 2, \ldots$, in $\mathcal{D}(L)$ with $y_n \to y$ as $n \to \infty$. Hence, by (1.12)

$$\| x_n - x_m \| \leq \frac{N}{\omega_1} \| y_n - y_m \|,$$

which means that the x_n form a Cauchy sequence, which tends to some x as $n \to \infty$. Consequently, $Lx_n = y_n - \omega_1 x_n \to y - \omega_1 x$ as $n \to \infty$, so that $(x, y - \omega_1 x) \in \overline{L}$, i.e. $(x, y) \in \omega_1 + \overline{L}$. Thus, for any $y \in X$ there is some $x \in X$ such that $(x, y) \in \omega_1 + \overline{L}$, and we have shown that $\omega_1 \in \rho(-\overline{L})$.

By (1.14) we have

$$\| (\lambda + \overline{L})^{-1} \| \le \frac{N}{\lambda} \tag{1.15}$$

for all $\lambda > \omega_0$. Hence, if $\lambda \in \rho(-\overline{L})$ and $\lambda > \omega_0$, then by (1.7) we have

$$\left(\lambda(1 - \frac{1}{N}), \lambda(1 + \frac{1}{N})\right) \subseteq \rho(-\overline{L}).$$

Now $\omega_1 \in \rho(-\overline{L})$ and $\omega_1 > \omega_0$, and thus $\big(\omega_1(1 - \frac{1}{N}), \omega_1(1 + \frac{1}{N})\big) \subseteq \rho(-\overline{L})$. Repeating this argument, we can successively enlarge the interval to obtain, finally, the inclusion $\big(\omega_0(1 - \frac{1}{N}), \infty\big) \subseteq \rho(-\overline{L})$.

Since the inclusion $\overline{\mathcal{R}(G)} \supseteq \mathcal{R}(\overline{G})$ holds for any graph G, and since $\mathcal{R}(\lambda + \overline{L}) = X$ for any $\lambda \in \rho(-\overline{L})$, it follows that $\mathcal{R}(\lambda + L)$ is dense in X for all $\lambda \in \rho(-\overline{L}) \supseteq \big(\omega_0(1 - \frac{1}{N}), \infty\big)$.

Let us now assume that L is densely defined. We shall show that this assumption in combination with the assumption (i) of the theorem implies that $\overline{L}$ is a function. For this purpose we let $x_n \in \mathcal{D}(L)$ be a sequence such that $x_n \to 0$ and $Lx_n \to y$ as $n \to \infty$. We have to show that $y = 0$. Let $z \in \mathcal{D}(L)$ be arbitrary. By (1.12) we have

$$\| \lambda x_n - z \| \le \frac{N}{\lambda} \| \lambda(\lambda x_n + Lx_n) - \lambda z - Lz \| ,$$

and letting $n \to \infty$ we get

$$\| z \| \le N \left\| y - z - \frac{1}{\lambda} Lz \right\| .$$

Since this holds for all $\lambda > \omega_0$ we can let $\lambda \to \infty$ obtaining

$$\| z \| \le N \| y - z \| . \tag{1.16}$$

Since $\mathcal{D}(L)$ is dense in X there is some sequence z_n in $\mathcal{D}(L)$ converging to y as $n \to \infty$. Applying the last inequality to these z_n and letting $n \to \infty$ we obtain $\| y \| \le N \cdot 0 = 0$, i.e. $y = 0$.

It only remains to prove that if $N = 1$, then we may assume $\omega_0 = 0$. But if $N = 1$ and $\lambda > 0$, then $\lambda + \omega_0 > \omega_0$, and by (1.12) we have

$$\begin{aligned} \| x \| &\le \frac{1}{\lambda + \omega_0} \| (\lambda + \omega_0)x + Lx \| \\ &\le \frac{1}{\lambda + \omega_0} \| \lambda x + Lx \| + \frac{\omega_0}{\lambda + \omega_0} \| x \| , \end{aligned}$$

which implies that

$$\| x \| \le \frac{1}{\lambda} \| \lambda x + Lx \| .$$

□

Definition 1.5. Let $\{L_n\}_{n=1}^{\infty}$ be a sequence of linear operators in X, its limit L as $n \to \infty$ is defined by putting

$$\mathcal{D}(L) := \{x \in \cap_{n=1}^{\infty} \mathcal{D}(L_n) \mid \lim_{n\to\infty} L_n x \text{ exists}\}$$
$$Lx := \lim_{n\to\infty} L_n x \qquad (x \in \mathcal{D}(L)).$$

Lemma 1.6. *Let L be the limit of a sequence $\{L_n\}_{n=1}^{\infty}$ of linear operators in X. Assume that $0 \in \rho(L_n)$ for $n = 1, 2, \ldots$, and that the inverses L_n^{-1} are uniformly bounded. Then $0 \in \rho(\overline{L})$, and*

$$\lim_{n\to\infty} L_n^{-1} x = \overline{L}^{-1} x$$

for all $x \in \overline{\mathcal{R}(L)}$.

Proof. Let us assume the hypotheses of the lemma to be satisfied. Then there is a constant C with $\| x \| \leq C \| L_n x \|$ for $x \in X$ and $n = 1, 2, \ldots$. Hence, if $x \in \mathcal{D}(L)$ we get $\| x \| \leq C \| Lx \|$. Consequently, L^{-1} is a bounded linear operator in X with $\| L^{-1} \| \leq C$. Hence, by Lemma 1.2, $\overline{L}^{-1} = \overline{L^{-1}}$ is a bounded linear operator in X with $\| \overline{L} \| \leq C$. For $x \in \mathcal{R}(L)$ the identity

$$L_n^{-1} x - L^{-1} x = L_n^{-1}(L - L_n) L^{-1} x.$$

holds. Since $\| L_n^{-1} \| \leq C$ for $n = 1, 2, \ldots$, the conclusion of the lemma follows for $x \in \mathcal{R}(L)$. For arbitrary $x \in \overline{\mathcal{R}(L)}$ and $y \in \mathcal{R}(L)$ we have

$$\begin{aligned}\left\| L_n^{-1} x - \overline{L}^{-1} x \right\| &\leq \left\| L_n^{-1} y - L^{-1} y \right\| + \left\| L_n^{-1}(x-y) - \overline{L}^{-1}(x-y) \right\| \\ &\leq \left\| L_n^{-1} y - L^{-1} y \right\| + 2C \| x - y \| .\end{aligned}$$

Fix $\varepsilon > 0$, and choose $y \in \mathcal{R}(L)$ such that $\| x - y \| < \varepsilon / 4C$. By the above we know that there is N such that $\| L_n^{-1} y - L^{-1} y \| < \varepsilon/2$ for all $n > N$. Hence, we have $\left\| L_n^{-1} x - \overline{L}^{-1} x \right\| < \varepsilon$ for all $n > N$, which shows that $L_n^{-1} x \to \overline{L}^{-1} x$ as $n \to \infty$ for any $x \in \overline{\mathcal{R}(L)}$. □

Let us now apply Theorem 1.3 to the limit of a sequence of operators.

Corollary 1.7. *Let L be the limit of a sequence $\{L_n\}_{n=1}^{\infty}$ of linear operators in X. Assume that the following conditions are satisfied:*

(i) *there are constants $N \geq 1$ and $\omega_0 \geq 0$ with $\rho(-L_n) \supseteq (\omega_0, \infty)$ and*

$$\| (\lambda + L_n)^{-1} \| \leq \frac{N}{\lambda}$$

for all $\lambda > \omega_0$ and $n = 1, 2, \ldots$,
and
(ii) *there is some $\omega_1 > \omega_0$ such that $\mathcal{R}(\omega_1 + L)$ is dense in X.*

Then $\big(\omega_0(1 - \frac{1}{N}), \infty\big) \subseteq \rho(-\overline{L})$, $\mathcal{R}(\lambda + L)$ is dense in X for all $\lambda > \omega_0(1 - \frac{1}{N})$ and

$$\lim_{n\to\infty} (\lambda + L_n)^{-1} x = (\lambda + \overline{L})^{-1} x \tag{1.17}$$

for all $x \in X$ and all $\lambda > \omega_0$. If, in addition, L is densely defined in X, then it is closable. Finally, if (i) *holds with $N = 1$ and some ω_0, then it holds with $N = 1$ and $\omega_0 = 0$.*

Proof. By the hypothesis (i) we have

$$\| x \| \le \frac{N}{\lambda} \| \lambda x + L_n x \|$$

for $n = 1, 2, \ldots$ and all $x \in \mathcal{D}(L_n)$. Then if $x \in \mathcal{D}(L) \subseteq \mathcal{D}(L_n)$ we get

$$\| x \| \le \frac{N}{\lambda} \| \lambda x + Lx \| + \frac{N}{\lambda} \| L_n x - Lx \| .$$

Hence, letting $n \to \infty$ we see that

$$\| x \| \le \frac{N}{\lambda} \| \lambda x + Lx \|$$

for all $x \in \mathcal{D}(L)$. Applying Theorem 1.3 we then obtain all the assertions of the corollary except (1.17). But this is obtained by Lemma 1.6, since $\mathcal{R}(\lambda + L)$ is dense in X. □

1.5 Commuting and resolvent commuting operators

Definition 1.8. Let A and B be linear operators in a Banach space X. We say that A and B *commute* if $AB = BA$, i.e. if $\mathcal{D}(AB) = \mathcal{D}(BA)$ and $ABx = BAx$ for all $x \in \mathcal{D}(AB)$. The operators A and B are called *resolvent commuting* if there are $\lambda \in \rho(A)$ and $\mu \in \rho(B)$ such that $(\lambda - A)^{-1}$ and $(\mu - B)^{-1}$ commute, i.e. if we have

$$(\lambda - A)^{-1}(\mu - B)^{-1}x = (\mu - B)^{-1}(\lambda - A)^{-1}x \tag{1.18}$$

for some $\lambda \in \rho(A)$ and $\mu \in \rho(B)$ and *all* $x \in X$.

Let us also define the *commutator* of A and B by setting

$$[\, A\,; B\,] := AB - BA \tag{1.19}$$

This operator is defined on $\mathcal{D}([\, A\,; B\,]) = \mathcal{D}(AB) \cap \mathcal{D}(BA)$.

Since $x \in \mathcal{D}((\lambda - A)B)$ if and only if $x \in \mathcal{D}(B)$ and $Bx \in \mathcal{D}(A)$, we have

$$\mathcal{D}((\lambda - A)B) = \mathcal{D}(AB)$$

for all linear operators A and B in X and all $\lambda \in \mathbb{C}$. Thus,

$$\mathcal{D}([\, \lambda - A\,; \mu - B\,]) = \mathcal{D}(A(\mu - B)) \cap \mathcal{D}(B(\lambda - A)).$$

But

$$\begin{aligned}
&x \in \mathcal{D}(A(\mu - B)) \cap \mathcal{D}(B(\lambda - A)) \\
&\qquad \Leftrightarrow x \in \mathcal{D}(A) \cap \mathcal{D}(B) \text{ and } \mu x - Bx \in \mathcal{D}(A) \text{ and } \lambda x - Ax \in \mathcal{D}(B) \\
&\qquad \Leftrightarrow x \in \mathcal{D}(A) \cap \mathcal{D}(B) \text{ and } Bx \in \mathcal{D}(A) \text{ and } Ax \in \mathcal{D}(B) \\
&\qquad \Leftrightarrow x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)
\end{aligned}$$

for all $x \in X$. Consequently,

$$\mathcal{D}([\,\lambda - A\,;\mu - B\,]) = \mathcal{D}([\,A\,;B\,]). \tag{1.20}$$

For any $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA)$ we have

$$\begin{aligned}(\lambda - A)(\mu - B)x - (\mu - B)(\lambda - A)x &= \lambda\mu x - \mu Ax - \lambda Bx + ABx \\ &\quad - (\mu\lambda x - \lambda Bx - \mu Ax + BAx) \\ &= ABx - BAx.\end{aligned}$$

Thus, we have the following lemma.

Lemma 1.9. *Let A and B be linear operators in X. Then $[\,\lambda - A\,;\mu - B\,] = [\,A\,;B\,]$ for any $\lambda, \mu \in \mathbb{C}$.*

We also have the following simple results concerning resolvent commutativity.

Proposition 1.10. *Let A and B be linear operators in a Banach space X, and assume that $\lambda \in \rho(A)$ and $\mu \in \rho(B)$. Then the following statements are equivalent.*
(i) $(\lambda - A)^{-1}$ *and* $(\mu - B)^{-1}$ *commute*
(ii) All *resolvents of A and B commute*
(iii) $\lambda - A$ *and* $\mu - B$ *commute*
(iv) *A commutes with $(\mu - B)^{-1}$ on $\mathcal{D}(A)$ and $(\mu - B)^{-1}(\mathcal{D}(A)) \subseteq \mathcal{D}(A)$.*

Proof. (ii) $\Rightarrow$ (i). This implication is trivial.

(i)$\Rightarrow$(iii). Let $A_1 = \lambda - A$ and $B_1 = \mu - B$ and assume that A_1^{-1} and B_1^{-1} commute. Firstly,

$$\begin{aligned}\mathcal{D}(A_1B_1) &= \mathcal{R}(B_1^{-1}A_1^{-1}) = \mathcal{R}(A_1^{-1}B_1^{-1}) \\ &= \mathcal{D}(B_1A_1),\end{aligned}$$

so that $\mathcal{D}([\,A_1\,;B_1\,]) = \mathcal{D}(A_1B_1) = \mathcal{D}(B_1A_1)$. Moreover

$$A_1^{-1}B_1^{-1}(A_1B_1 - B_1A_1)x = B_1^{-1}A_1^{-1}A_1B_1x - A_1^{-1}B_1^{-1}B_1A_1x = 0$$

for all $x \in \mathcal{D}([\,A_1\,;B_1\,])$. Thus, since $A_1^{-1}B_1^{-1}$ is injective, A_1 and B_1 commute.

(iii) $\Rightarrow$ (iv). Assume that $\lambda - A$ and $\mu - B$ commute. Then $\mathcal{D}(A(\mu - B)) = \mathcal{D}(B(\lambda - A))$. Let x be any element in $\mathcal{D}(A)$. Then $x = (\mu - B)y$, where $y = (\mu - B)^{-1}x \in \mathcal{D}(A(\mu - B))$. Hence, $y \in \mathcal{D}(B(\lambda - A))$ so that $(\mu - B)^{-1}x \in \mathcal{D}(A)$, and

$$\begin{aligned}(\mu - B)^{-1}(\lambda - A)x &= (\mu - B)^{-1}(\lambda - A)(\mu - B)y \\ &= (\mu - B)^{-1}(\mu - B)(\lambda - A)y = (\lambda - A)(\mu - B)^{-1}x.\end{aligned}$$

Thus, $(\mu - B)^{-1}(\mathcal{D}(A)) \subseteq \mathcal{D}(A)$, and $\lambda - A$ and $(\mu - B)^{-1}$ commute on $\mathcal{D}(A)$. Then also A and $(\mu - B)^{-1}$ commute on $\mathcal{D}(A)$.

(iv) $\Rightarrow$ (ii). Assume that $(\mu - B)^{-1}(\mathcal{D}(A)) \subseteq \mathcal{D}(A)$ and that A and $(\mu - B)^{-1}$ commute, and let $\alpha \in \rho(A)$ and $y \in X$. Then $\alpha - A$ and $(\mu - B)^{-1}$ commute, and $(\alpha - A)^{-1} y \in \mathcal{D}(A)$. Consequently,

$$(\alpha - A)(\mu - B)^{-1}(\alpha - A)^{-1} y = (\mu - B)^{-1} y.$$

Applying $(\alpha - A)^{-1}$ to this equation, we obtain

$$(\mu - B)^{-1}(\alpha - A)^{-1} y = (\alpha - A)^{-1}(\mu - B)^{-1} y,$$

for any $y \in X$. Thus, in fact, $(\mu - B)^{-1}$ commutes with any resolvent of A, and, in particular, with $(\lambda - A)^{-1}$.

(i) $\Rightarrow$ (ii). We have shown that (i) implies (iv), and that (iv) implies that $(\mu - B)^{-1}$ commutes with $(\alpha - A)^{-1}$ for any $\alpha \in \rho(A)$. Interchanging the rôles of A and B we see by (iv) that B commutes with $(\alpha - A)^{-1}$, and $(\alpha - A)^{-1}(\mathcal{D}(B)) \subseteq \mathcal{D}(B)$ and hence $(\beta - B)^{-1}$ commutes with $(\alpha - A)^{-1}$ for any $\alpha \in \rho(A)$ and any $\beta \in \rho(B)$. □

Corollary 1.11. *Let A and B be linear operators in a Banach space X such that $0 \in \rho(A) \cap \rho(B)$, $\mathcal{D}(AB) \subseteq \mathcal{D}(BA)$ and $ABx = BAx$ for all $x \in \mathcal{D}(AB)$. Then A and B are resolvent commuting.*

Proof. If the assumptions of the corollary are satisfied, then $AB \subseteq BA$ and both are bijective from their respective domains onto X. Then we must have $AB = BA$, and, in particular, $\mathcal{D}(AB) = \mathcal{D}(BA)$. Consequently, A and B commute, and, by Proposition 1.10, A and B are resolvent commuting. □

2 Nonnegative linear operators

2.1 Nonnegative and positive operators

2.1.1 Definitions

A closed linear operator A in a complex Banach space X is said to be *admissible* in the direction $\alpha \in \mathbb{R}$, if $\rho(A)$ contains the ray $\{te^{i\alpha} \mid t > 0\}$, and

$$\sup_{t>0} t \left\| (te^{i\alpha} - A)^{-1} \right\|_{\mathcal{L}(X)} < \infty.$$

If $-A$ is admissible in the direction $\alpha \in (-\pi, \pi]$, then we define $N_A(\alpha)$ by the equation

$$N_A(\alpha) := \sup\{t \left\| (te^{i\alpha} + A)^{-1} \right\|_{\mathcal{L}(X)} \mid t > 0\} \tag{2.1}$$

If $\alpha \in [0, \pi)$ and $-A$ is admissible both in the direction α and in the direction $-\alpha$, then we also set

$$M_A(\alpha) := \max(N_A(\alpha), N_A(-\alpha)) = \sup_{\substack{|\arg\lambda|=\alpha \\ \lambda\neq 0}} |\lambda| \left\| (\lambda + A)^{-1} \right\|_{\mathcal{L}(X)}. \tag{2.2}$$

Definition 2.1. A closed linear operator A in a complex Banach space X is called *nonnegative* if $-A$ is admissible in the direction $\alpha = 0$, i.e. if $(0, \infty) \subseteq \rho(-A)$, and

$$N_A := \sup_{t>0} t \left\| t(t + A)^{-1} \right\|_{\mathcal{L}(X)} < \infty. \tag{2.3}$$

If, in addition, $0 \in \rho(-A)$, i.e., if A has a bounded inverse on X, then A is said to be *positive*.

Evidently, A is admissible in the direction α if and only if $-e^{-i\alpha}A$ is nonnegative.

A simple example of a nonnegative operator in a complex Banach space X is provided by the mapping aI, where I is the identity mapping on X and $a \in \mathbb{C} \setminus (-\infty, 0)$. It is positive if $a \neq 0$.

In the sequel we shall often write $\|x\|$ and $\|A\|$ instead of $\|x\|_X$ and $\|A\|_{\mathcal{L}(X)}$, respectively, for $x \in X$ and $A \in \mathcal{L}(X)$.

Proposition 2.2. *A linear operator A in a complex Banach space X is nonnegative if and only if*

(i) *there is a constant N such that*

$$\|x\| \leq \frac{N}{t} \|tx + Ax\| \tag{2.4}$$

for all $t > 0$ and all $x \in \mathcal{D}(A)$, and
(ii) *there is some $\omega_1 > 0$ such that $\mathcal{R}(\omega_1 + A) = X$.*

Proof. The conditions of the proposition are almost identical to the hypotheses of Theorem 1.3. In fact, under these conditions Theorem 1.3 guarantees the existence of a positive constant N such that $t \in \rho(-\overline{A})$ and $t \left\| (t+\overline{A})^{-1} \right\| \leq N$ for all $t > 0$. But, by the second condition,

$$\mathcal{D}((\omega_1 + A)^{-1}) = X,$$

and, as $(\omega_1 + \overline{A})^{-1} \supseteq (\omega_1 + A)^{-1}$ is a function, we must have $(\omega_1 + \overline{A})^{-1} = (\omega_1 + A)^{-1}$. Hence, $A = \overline{A}$. Consequently, A is nonnegative.

On the other hand, if A is nonnegative, then $(0, \infty) \subseteq \rho(-A)$, and there is some constant N such that (2.3), and hence also (2.4), holds for all $t > 0$. Obviously $\mathcal{R}(t + A) = \mathcal{D}((t + A)^{-1}) = X$ for *all* $t > 0$. □

If condition (i) of Proposition 2.2 holds with $N = 1$ then A is m–accretive (cf. Remark 1.4 on p. 14).

If A is positive, then $f(t) = (1 + t) \| (t + A)^{-1} \|$, being continuous on $[0, \infty)$, is bounded on $[0, 1]$ by a constant N_1. On the other hand, since A is nonnegative, we know that there is some constant N_2, such that

$$t \left\| (t + A)^{-1} \right\| \leq N_2$$

for all $t > 0$. Consequently, $(1 + t) \| (t + A)^{-1} \| \leq 2N_2$ for all $t > 1$. It follows that with $N := \max \{N_1, 2N_2\}$ we have

$$(1 + t) \left\| (t + A)^{-1} \right\|_{\mathcal{L}(X)} \leq N \tag{2.5}$$

for all $t \geq 0$. One often *defines* a linear operator A to be positive by demanding that $[0, \infty) \subseteq \rho(-A)$, and that there is some constant $N > 0$ such that (2.5) holds for all $t \geq 0$.

2.1.2 Spectral angles

We shall now show, among other things, that a nonnegative operator A is admissible in all directions α for which $|\alpha|$ is less than some positive constant that depends on N_A.

Proposition 2.3. *Let A be a closed linear operator in a complex Banach space X. Then the set of admissible directions for A is open, and the function N_A is continuous on this set.*

Proof. If $X = \{0\}$, then any linear operator A in X is admissible in any direction $\alpha \in (-\pi, \pi]$, and $N_A(\alpha) = 0$. Thus, the statements of the proposition are trivial in this case.

We now assume that $X \neq \{0\}$. Let A be admissible in the direction α. Then $B = -e^{-i\alpha} A$ is nonnegative and $N_B = N_A(-\alpha) > 0$, so we first assume that A is nonnegative and $\alpha = 0$. Then $\mathbb{R}_+ \subseteq \rho(-A)$, and equation (1.6) shows that

$$(\mu + A)^{-1} = (t + A)^{-1} \left(1 - (t - \mu)(t + A)^{-1}\right)^{-1},$$

provided that $t > 0$ is such that $|t - \mu| \, \| (t + A)^{-1} \| < 1$. This latter condition is fulfilled as soon as $|t - \mu| \, N_A(0)/t \leq 1 - \delta$ or, equivalently,

$$|1 - \mu/t| \leq (1 - \delta)/N_A(0) \tag{2.6}$$

for some $\delta \in (0, 1)$. We then have

$$\| (\mu + A)^{-1} \| \leq \frac{1}{\delta} \| (t + A)^{-1} \| \tag{2.7}$$

by (1.8). Thus, let us choose $\delta \in (0, 1)$ and a direction $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then for every $\mu = se^{i\alpha}$ $(s > 0)$ we wish to show the existence of (at least) one positive λ such that the inequality (2.6) holds. This amounts to minimising $1 - se^{i\alpha}$ with respect to $s > 0$, or, equivalently, calculating the shortest distance between the ray $\{se^{i\alpha} \mid s > 0\}$ and the point 1, and demanding that this distance is not greater than $(1 - \delta)/N_A$. For if s_0 minimises this distance, then we may take $\lambda = s/s_0 = |\mu| / s_0$, in which case $\mu/\lambda = s_0 e^{i\alpha}$. But $|1 - se^{i\alpha}|^2 = (s - \cos \alpha)^2 + \sin^2 \alpha$, so that

$$\min\{1 - se^{i\alpha} \parallel s > 0\} = |\sin \alpha| \, ,$$

which is attained at $s = s_0 = \cos \alpha$. Hence, we deduce the following sufficient condition for the existence of a $\lambda > 0$ such that (2.6) holds for all μ on the ray $\{te^{i\alpha} \parallel t > 0\}$:

$$|\alpha| \leq \arcsin \left(\min(1, (1 - \delta)/N_A(0))\right).$$

Then $\| (\lambda + A)^{-1} \|_{\mathcal{L}(X)} \leq N_A(0)/\lambda = s_0 N_A(0)/ |\mu| = N_A(0) \cos \alpha / |\mu|$, so that

$$\| (\mu + A)^{-1} \| \leq \frac{N_A(0) \cos \alpha}{\delta \, |\mu|},$$

by (2.7). In other words, for all $\phi < \arcsin \left(\min(1, (1 - \delta)/N_A(0))\right)$ we have

$$\rho(-A) \supseteq \overline{\Sigma}_\phi \setminus \{0\} \tag{2.8}$$

and

$$\sup_{s>0} s \left\| (se^{i\phi} + A)^{-1} \right\| < \infty. \tag{2.9}$$

So if A is admissible in the direction α, then there is a $\delta > 0$ such that $-e^{i\alpha} A$ is admissible in all directions ϕ with $|\phi| < \delta$. Consequently, A is admissible in all directions β with $|\alpha - \beta| < \delta$. This means that the set of admissible directions for A is open.

Let A be admissible in the directions α and β. By the Resolvent Identity,

$$\begin{aligned}
\| t(te^{i\beta} + A)^{-1} \| - \| t(te^{i\alpha} + A)^{-1} \| \\
&\leq t \| (te^{i\beta} + A)^{-1} - (te^{i\alpha} + A)^{-1} \| \\
&= t^2 |e^{i\alpha} - e^{i\beta}| \| (te^{i\beta} + A)^{-1} (te^{i\alpha} + A)^{-1} \| \\
&\leq |e^{i\alpha} - e^{i\beta}| N_A(\alpha) N_A(\beta).
\end{aligned}$$

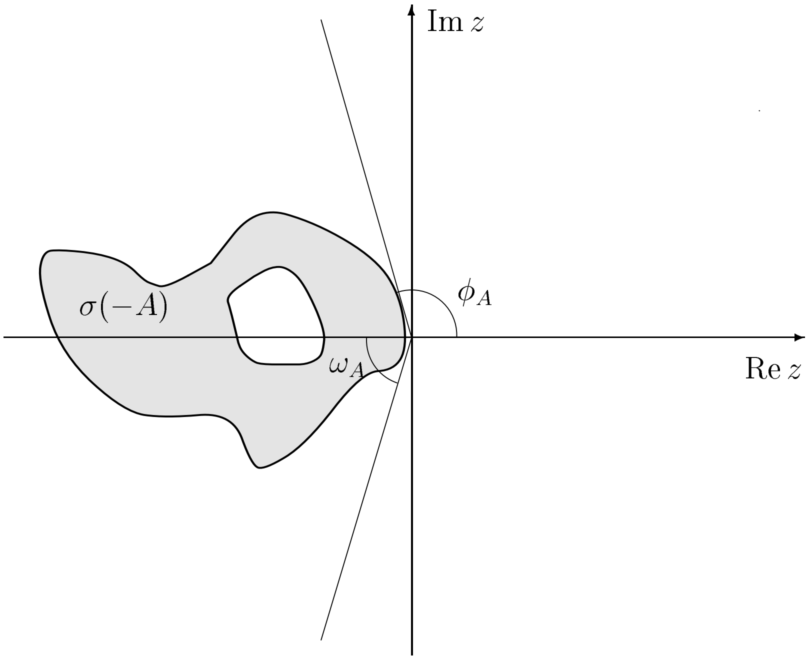

Figure 1: The spectral angle of a nonnegative linear operator.

It follows that

$$N_A(\beta) \leq N_A(\alpha) + |e^{i\alpha} - e^{i\beta}| N_A(\alpha) N_A(\beta).$$

Interchanging α and β in this formula and combining the two inequalities, we deduce the inequality

$$|N_A(\beta) - N_A(\alpha)| \leq |e^{i\alpha} - e^{i\beta}| N_A(\alpha) N_A(\beta),$$

which can also be written

$$\left| \frac{1}{N_A(\alpha)} - \frac{1}{N_A(\beta)} \right| \leq |e^{i\alpha} - e^{i\beta}|,$$

since $N_A(\alpha), N_A(\beta) > 0$. Hence, $1/N_A$, and consequently also N_A, is a continuous function on the set of admissible directions for A. □

As a consequence of the lemma, the following definition is meaningful.

Definition 2.4. Let A be a nonnegative operator in a complex Banach space X. We set

(2.10) $$\phi_A := \sup\{\phi \in (0, \pi] \parallel (2.8) \text{ and } (2.9) \text{ hold}\},$$

(2.11) $$M_A^*(\phi) := \sup_{|\alpha| \leq \phi} N_A(\alpha)$$

for any $\phi \in [0, \phi_A)$, and

$$\omega_A := \pi - \phi_A. \tag{2.12}$$

We call ω_A the *spectral angle* of A.

If A is nonnegative, then $0 < \phi_A \leq \pi$, $0 \leq \omega_A < \pi$, and $M_A^*(\alpha) \in [0, \infty)$ for any α with $|\alpha| < \phi_A$ (see Figure 1). From the above calculations it also follows that

$$\phi_A \geq \arcsin\left(\min(1, /N_A(0))\right), \tag{2.13}$$

and hence

$$\omega_A \leq \pi - \arcsin\left(\min(1, /N_A(0))\right). \tag{2.14}$$

2.2 Yosida approximations of nonnegative operators

In this section we consider Yosida approximations of nonnegative operators.

Definition 2.5. If A is a nonnegative linear operator in a complex Banach space X we put

$$I_t := t(t + A)^{-1} \tag{2.15}$$

$$A_t := AI_t = tA(t + A)^{-1}, \tag{2.16}$$

for $t > 0$, and call the A_t *Yosida approximations* of A.

We see that $I_t, A_t : X \longrightarrow X$, and that both are bounded, since

$$\| I_t \| = t \left\| (t + A)^{-1} \right\| \leq N_A(0)$$

and

$$\| A_t \| = t \, \| I - I_t \| \leq t(1 + N_A(0))$$

2.2.1 Basic convergence properties

In case A is densely defined, the families of operators $\{I_t\}_{t>0}$ and $\{A_t\}_{t>0}$ have some nice convergence properties as $t \to \infty$.

Proposition 2.6. *Let A be a nonnegative linear operator in a complex Banach space X, and define I_t and A_t as above. Then*
(i) $\lim\limits_{t\to\infty} I_t x = x$ *in X if and only if* $x \in \overline{\mathcal{D}(A)}$
(ii) $\lim\limits_{t\to\infty} A_t x$ *exists in X if and only if $x \in \mathcal{D}(A)$ and $Ax \in \overline{\mathcal{D}(A)}$, in which case* $\lim_{t\to\infty} A_t x = Ax$.

In particular, if A is densely defined, then $I_t x \to Ix$ for all $x \in X$, and $A_t x$ converges if and only if $x \in \mathcal{D}(A)$, in which case $Ax = \lim_{t\to\infty} A_t x$.

Proof. Since $I_t x \in \mathcal{D}(A)$ for all $t > 0$ and all $x \in X$, it is clear that $\lim_{t\to\infty} I_t x = x$ implies that $x \in \overline{\mathcal{D}(A)}$.

Let x be an arbitrary element in X. We have $I_t x - Ix = -A(t+A)^{-1}x$, and, for any $y \in X$,

$$
\begin{aligned}
\left\| A(t+A)^{-1}x \right\| &\le \left\| A(t+A)^{-1}(x-y) \right\| + \left\| A(t+A)^{-1}y \right\| \\
&\le (1+N_A) \| x-y \| + \left\| A(t+A)^{-1}y \right\|
\end{aligned}
\tag{2.17}
$$

Assume that $x \in \overline{\mathcal{D}(A)}$. Then we can choose $y_0 \in \mathcal{D}(A)$ such that $\| x - y_0 \| < \varepsilon/(2(1+N_A))$. As $y_0 \in \mathcal{D}(A)$ we also have $A(t+A)^{-1}y_0 = (t+A)^{-1}Ay_0$, so that

$$
\left\| A(t+A)^{-1}y_0 \right\| \le \frac{N_A \| Ay_0 \|}{t}.
$$

Thus, we can choose N so big that $\| A(t+A)^{-1}y_0 \| < \varepsilon/2$ for any $t > N$. Consequently, for such t, $\| A(t+A)^{-1}x \| < \varepsilon$, which implies that

$$
\lim_{t\to\infty} \| I_t x - Ix \| = \lim_{t\to\infty} \left\| A(t+A)^{-1}x \right\| = 0.
$$

Now take $x \in \mathcal{D}(A)$ such that $Ax \in \overline{\mathcal{D}(A)}$. Then $A_t x - Ax = A(I_t - I)x = (I_t - I)Ax$, which, by the first part of the lemma, tends to 0 as $n \to \infty$.

Conversely, if $y = \lim_{t\to\infty} A_t x$ exists, then $\lim_{t\to\infty} \| tA(t-A)^{-1}x - y \| = 0$. It follows that

$$
\lim_{t\to\infty} \| A(t-A)^{-1}x \| = \lim_{t\to\infty} \| I_t x - x \| = 0.
$$

Thus, $I_t x \to x$ and $AI_t x \to y$ as $t \to \infty$, so that $x \in \mathcal{D}(A)$ and $y = Ax$, since A is closed. It also follows that $A_t x = I_t Ax$, which belongs to $\mathcal{D}(A)$ for all $t > 0$, so that $Ax = \lim_{t\to\infty} A_t x \in \overline{\mathcal{D}(A)}$.

Finally, if A is densely defined, then $\overline{\mathcal{D}(A)} = X$ and $Ax \in \overline{\mathcal{D}(A)}$ for all $x \in \mathcal{D}(A)$, so that the last statements of the lemma follow. □

Corollary 2.7. *Let A be a nonnegative operator in a complex Banach space X. Then $\lim_{t\to\infty} I_t^n x = x$ in X for all $x \in \overline{\mathcal{D}(A)}$ and all $n \in \mathbb{N}$.*

Proof. We have

$$
I_t^n x - x = \sum_{k=0}^{n-1} I_t^k (I_t x - x),
$$

so that

$$
\begin{aligned}
\| I_t^n x - x \| &\le \sum_{k=0}^{n-1} \left\| I_t^k (I_t x - x) \right\| \\
&\le \sum_{k=0}^{n-1} N_A^k \| I_t x - x \| .
\end{aligned}
$$

But the expression on the right hand side tends to 0 as $t \to \infty$ by Proposition 2.6. □

2.2.2 Uniform nonnegativity

We have seen that a nonnegative operator A satisfies the inequality

$$t \left\| (te^{i\phi} + A)^{-1} \right\| \leq M_A^*(\phi)$$

for any ϕ with $0 \leq \phi < \phi_A$ and any $t > 0$. We shall now show that for any $t > 0$ the Yosida approximations A_t are nonnegative, that $\phi_{A_t} \geq \phi_A$, and that for all ϕ with $0 \leq \phi < \phi_A$ there is a constant $\hat{M}_A(\phi) > 0$ such that $M_{A_t}^*(\phi) \leq \hat{M}_A(\phi)$, i.e.

$$\left\| (\lambda + A_t)^{-1} \right\| \leq \frac{\hat{M}_A(\phi)}{|\lambda|} \tag{2.18}$$

for all λ with $|\arg \lambda| \leq \phi$ and $n = 1, 2, \ldots$. In fact,

$$\begin{aligned}
(\lambda + A_t)^{-1} &= \left\{ \lambda + tA(t + A)^{-1} \right\}^{-1} \\
&= \left\{ (\lambda(t + A) + tA)\,(t + A)^{-1} \right\}^{-1} \\
&= \left\{ (t\lambda + (\lambda + t)A)\,(t + A)^{-1} \right\}^{-1} \\
&= \frac{1}{\lambda + t} \left\{ \left(\frac{t\lambda}{\lambda + t} + A \right) (t + A)^{-1} \right\}^{-1} \\
&= \frac{1}{\lambda + t} (t + A) \left(\frac{t\lambda}{\lambda + t} + A \right)^{-1} \\
&= \frac{1}{\lambda + t} \left\{ 1 + \left(t - \frac{t\lambda}{\lambda + t} \right) \left(\frac{t\lambda}{\lambda + t} + A \right)^{-1} \right\} \\
&= \frac{1}{\lambda + t} \left\{ 1 + \frac{t^2}{\lambda + t} \left(\frac{t\lambda}{\lambda + t} + A \right)^{-1} \right\}
\end{aligned}$$

if $t \in \rho(-A)$ and $t\lambda/(\lambda+t) \in \rho(-A)$. This is the case provided that $\lambda \in \Sigma_{\phi_A}$, because $\Sigma_{\phi_A} \subseteq \rho(-A)$, $\arg(t\lambda/(\lambda + t)) = \arg(\lambda/(\lambda + t))$ and either

$$0 \leq \arg(t\lambda/(\lambda + t)) = \arg \lambda - \arg(\lambda + t) \leq \arg \lambda$$

or

$$0 \geq \arg(t\lambda/(\lambda + t)) = \arg \lambda - \arg(\lambda + t) \geq \arg \lambda,$$

so that $|\arg(t\lambda/(\lambda + t))| \leq |\arg \lambda|$ and consequently

$$(t\lambda/(\lambda + t)) \in \overline{\Sigma}_\phi \setminus \{0\}\,.$$

Hence, for all λ with $|\arg\lambda| \le \phi$, we have

$$\begin{aligned}
\|\,(\lambda + A_t)^{-1}\,\| &\le \frac{1}{|\lambda+t|}\left\| 1 + \frac{t^2}{\lambda+t}\left(\frac{t\lambda}{\lambda+t}+A\right)^{-1}\right\| \\
&\le \frac{1}{|\lambda+t|}\left(1 + \frac{t^2}{|\lambda+t|}\left\|\left(\frac{t\lambda}{\lambda+t}+A\right)^{-1}\right\|\right) \\
&\le \frac{1}{|\lambda+t|}\left(1 + \frac{t^2}{|\lambda+t|}\frac{M_A^*(\phi)\,|\lambda+t|}{|t\lambda|}\right) \\
&= \frac{1}{|\lambda+t|}\left(1 + \frac{tM_A^*(\phi)}{|\lambda|}\right) \\
&= \frac{1}{|\lambda|}\cdot\frac{|\lambda| + tM_A^*(\phi)}{|\lambda+t|}.
\end{aligned}$$

If $\operatorname{Re}\lambda \ge 0$, we have $|\lambda + t| > |\lambda|$ and $|\lambda+t| \ge t$. Thus, in this case $\|\,(\lambda+A_t)^{-1}\,\| \le (1 + \hat{M}_A(\phi))/\,|\lambda|$. If on the other hand $\operatorname{Re}\lambda < 0$, then $|\lambda+t| \ge |\lambda|\sin(\arg\lambda)$ and $|\lambda+t| \ge t\sin(\arg\lambda)$, so that, since now $\pi/2 < \arg\lambda < \phi_A$, we get

$$\|\,(\lambda + A_t)^{-1}\,\| \le (1 + M_A^*(\arg\lambda))/|\lambda|\sin(\pi-\phi).$$

Thus, writing

$$\hat{N}_A(\phi) := (1 + M_A^*(\phi))/\sin(\max\{\pi/2, \phi\}), \tag{2.19}$$

we have
$$\frac{|\lambda| + tM_A^*(\phi)}{|\lambda+t|} \le \hat{N}_A(\phi),$$

and (2.18) holds.

2.3 A lemma on the denseness of the domain of a sum of nonnegative operators

Let A and B be linear operators in a vector space X. Recall that the sum $L := A + B$ is defined in such a way that $\mathcal{D}(L) = \mathcal{D}(A) \cap \mathcal{D}(B)$ and $Lx = Ax + Bx$ for all $x \in \mathcal{D}(L)$.

We have the following result on the denseness of the domain of a sum of densely defined nonnegative operators in a complex Banach space.

Lemma 2.8. *Let A and B be two densely defined nonnegative linear operators in X such that $(t+B)^{-1}(\mathcal{D}(A)) \subseteq \mathcal{D}(A)$ for any $t > 0$. (We say that $\mathcal{D}(A)$ is stable under $(t+B)^{-1}$). Then $L = A + B$ is densely defined in X.*

Proof. For an arbitrary $x \in X$ and $t > 0$ we consider $x_t := t^2(t+B)^{-1}(t+A)^{-1}x \in X$. Then $x_t \in \mathcal{D}(B)$. Moreover, we have $(t+A)^{-1}x \in \mathcal{D}(A)$, and hence $x_t \in \mathcal{D}(A)$, as $\mathcal{D}(A)$ is stable under the operator $(t+B)^{-1}$ by assumption. This shows that $x_t \in \mathcal{D}(L)$, when $t > 0$. It is shown in Proposition 2.6

that $t(t+A)^{-1} \to I$ and $t(t+B)^{-1} \to I$ strongly as $t \to \infty$. Then, since $t(t+B)^{-1}$ is uniformly bounded, we have $t^2(t+A)^{-1}(t+B)^{-1} \to I$ strongly as $t \to \infty$. Hence, $x_t \to x$ as $t \to \infty$. □

Corollary 2.9. *Let A and B be closed and densely defined linear operators in X, and assume that there are positive numbers N and ω such that*
(i) $(\omega, \infty) \subseteq \rho(-A) \cap \rho(-B)$,
and the following conditions are satisfied for all $t > \omega$
(ii) $\| (t+A)^{-1} \| \leq N/t$, $\| (t+B)^{-1} \| \leq N/t$
(iii) $(t+B)^{-1}(\mathcal{D}(A)) \subseteq \mathcal{D}(A)$.
Then $L = A + B$ is densely defined in X.

Proof. If the assumptions of the corollary hold, then $\omega + A$ and $\omega + B$ are densely defined and nonnegative, $\mathcal{D}(\omega + A) = \mathcal{D}(A)$, and $(t + \omega + B)^{-1}(\mathcal{D}(A)) \subseteq \mathcal{D}(A)$ for all $t > 0$. Hence, Lemma 2.8 implies that $2\omega + A + B$, and therefore also $A + B$ is densely defined. □

Remark 2.10. Condition (iii) of Corollary 2.9 is satisfied if A and B are resolvent commuting (cf. Proposition 1.10).

2.4 Real interpolation spaces between X and $\mathcal{D}(A^n)$

2.4.1 Definitions

Let X be a complex Banach space, and let A be a closed linear operator in X. For $n \in \mathbb{N}$ we consider $\mathcal{D}(A^n)$, the domain of A^n, provided with the norm defined by

$$\| x \|_{\mathcal{D}(A^n)} := \| x \|_X + \sum_{k=1}^{n} \| A^k x \|_X. \tag{2.20}$$

The norm $\| \; \|_{\mathcal{D}(A)}$ is called the *graph norm* of $\mathcal{D}(A)$. The spaces $\mathcal{D}(A^n)$ are complex Banach spaces. Note that $A^0 = I$, so that $\mathcal{D}(A^0) = X$.

The above observations imply that $\mathcal{D}(A^n) \hookrightarrow X$, and we may consider the real interpolation spaces $(X, \mathcal{D}(A^n))_{\theta,p}$ for $0 < \theta < 1$ and $1 \leq p \leq \infty$ or $\theta \in \{0, 1\}$ 1 and $p = \infty$, as well as $(X, \mathcal{D}(A^n))_\theta$ for $0 < \theta < 1$. In the special case $n = 1$, we make the following definition.

Definition 2.11. Let X be a complex Banach space, let A be a closed linear operator in X. If $(\theta, p) \in (0,1) \times [1, \infty] \cup \{(0, \infty), (1, \infty)\}$, we set

$$\mathcal{D}_A(\theta, p) := (X, \mathcal{D}(A))_{\theta,p},$$

and if $\theta \in (0,1)$, we set

$$\mathcal{D}_A(\theta) := (X, \mathcal{D}(A))_\theta.$$

2.4.2 Closedness of A^n

If A has nonempty resolvent set, then A^n is closed and the definition (2.20) may be replaced by

$$\| x \|_{\mathcal{D}(A^n)} := \| x \|_X + \| A^n x \|_X. \tag{2.21}$$

These claims are immediate consequences of the lemma below. Thus, in this case, we may use the notation $\mathcal{D}_{A^n}(\theta, p)$ and $\mathcal{D}_{A^n(\theta)}$ for the interpolation spaces $(X, \mathcal{D}(A^n))_{\theta,p}$ and $(X, \mathcal{D}(A^n))_\theta$, respectively.

Lemma 2.12. *Let A be a closed linear operator in a Banach space X. Assume that A has nonempty resolvent set. Then for $n = 2, 3, 4, \dots$ and $k = 1, 2, 3, \dots, n-1$ there exist constants $c_{n,k}$ such that the inequality*

$$\| A^k x \|_X \leq c_{n,k}(\| x \|_X + \| A^n x \|_X)$$

holds for all $x \in \mathcal{D}(A^n)$.

Proof. Assume that $-\lambda \in \rho(A)$. Let us first prove the statement of the lemma for $n = 2, 3, \dots$ and $k = 1$. Assuming that $x \in \mathcal{D}(A^n)$ and writing

$$Ax = (\lambda + A)^{-(n-1)}(\lambda + A)^n x - \lambda x,$$

we get

$$\begin{aligned} Ax &= -\lambda x + (\lambda + A)^{-(n-1)} \left((\lambda + A)^n x - (-\lambda + (\lambda + A))^n x + A^n x\right) \\ &= -\lambda x + (\lambda + A)^{-(n-1)} \left((\lambda + A)^n x - \sum_{i=0}^{n} \binom{n}{i} (-\lambda)^{n-i} (\lambda + A)^i x \right) \\ &\quad + (\lambda + A)^{-(n-1)} A^n x \\ &= -\lambda x - \sum_{i=0}^{n-1} \binom{n}{i} (-\lambda)^{n-i} (\lambda + A)^{i-(n-1)} x + (\lambda + A)^{-(n-1)} A^n x. \end{aligned}$$

Hence $\| Ax \|_X \leq c_{n,1}(\| x \|_X + \| A^n x \|_X)$ for all $x \in \mathcal{D}(A^n)$, where we can choose

$$c_{n,1} \leq |\lambda| + \| (\lambda + A)^{-1} \|^{-1} \left((1 + |\lambda| \| (\lambda + A)^{-1} \|)^n - 1\right).$$

In particular, the statement of the lemma holds with $n = 2$ and $k = 1$.

Let us now assume that the statement of the lemma has been proved for some $n \geq 2$ and $k = 1, 2, \dots, n-1$. If $x \in \mathcal{D}(A^{n+1})$, we then have

$$\| Ax \|_X \leq c_{n+1,1}(\| x \|_X + \| A^{n+1} \|_X)$$

and hence

$$\begin{aligned} \| A^k x \|_X &\leq c_{n,k-1}(\| Ax \|_X + \| A^{n+1} x \|_X) \\ &\leq c_{n,k-1}(1 + c_{n+1,1})(\| x \|_X + \| A^{n+1} \|_X) \end{aligned}$$

for $k = 2, 3, \dots, n$. Thus the statement of the lemma follows by induction. □

In case $0 \in \rho(A)$ we can even use the definition

$$\| x \|_{\mathcal{D}(A^n)} := \| A^n x \|_X \,, \tag{2.22}$$

since

$$\| A^k x \|_X = \| A^{-(n-k)} A^n x \| \le \| A^{-1} \|_{\mathcal{L}(X)}^{n-k} \| A^n x \| .$$

2.4.3 Characterisation of $\mathcal{D}_{A^n}(\theta, p)$ and $\mathcal{D}_{A^n}(\theta)$

In case $-A$ is admissible in some direction α, we may characterise the interpolation spaces $\mathcal{D}_{A^n}(\theta, p)$ and $\mathcal{D}_{A^n}(\theta)$ without using the methods of the general theory of real interpolation spaces. In particular, the functionals K and J are dispensable (cf. Section B.2). We start with the following lemma.

Lemma 2.13. *Let A be an operator in a complex Banach space X such that is $-A$ is admissible in the direction α. Let n and k be positive integers, and let $t > 0$. Then*

$$\begin{aligned} x = {} & \sum_{j=0}^{k-1} \binom{n+j-1}{j} t^n e^{in\alpha} A^j (te^{i\alpha} + A)^{-n-j} x \\ & + n \binom{n+k-1}{k-1} \int_t^\infty \tau^{n-1} e^{in\alpha} A^k (\tau e^{i\alpha} + A)^{-n-k} x \, d\tau, \end{aligned} \tag{2.23}$$

for any $x \in \overline{\mathcal{D}(A)}$. If, in addition, $0 \in \rho(A)$, then

$$x = n \binom{n+k-1}{n} \int_0^\infty \tau^{n-1} e^{in\alpha} A^k (\tau e^{i\alpha} + A)^{-n-k} x \, d\tau \tag{2.24}$$

for any $x \in \overline{\mathcal{D}(A)}$. The integrals $\int_t^\infty \ldots$ and $\int_0^\infty \ldots$ in these formulas are improper Bochner integrals, and should be interpreted as $\lim_{s\to\infty} \int_t^s \ldots$ and $\lim_{s\to\infty} \int_0^s \ldots$, respectively.[2]

Proof. Since $te^{i\alpha}(te^{i\alpha} + A)^{-1} = t(t + e^{-i\alpha}A)^{-1}$, we see that $e^{-i\alpha}A$ is nonnegative, and (2.23) can be written as follows

$$\begin{aligned} & x - \sum_{j=0}^{k-1} \binom{n+j-1}{j} t^n e^{ij\alpha} A^j (t + e^{-i\alpha} A)^{-n-j} x \\ & \quad = n \binom{n+k-1}{k-1} \lim_{s\to\infty} \int_t^s \tau^{n-1} e^{-ik\alpha} A^k (\tau + e^{-i\alpha} A)^{-n-k} \, d\tau \end{aligned}$$

[2] These integrals become proper Bochner integrals if $x \in \mathcal{D}_{A^k}(\theta, \infty)$ for some $\theta \in (0,1)$, for then $\| \tau^{n-1} e^{in\alpha} A^k (\tau e^{i\alpha} + A)^{-n-k} \| \le N_A(\alpha)^N \| x \|_{\mathcal{D}_{A^k}(\theta,\infty)} \tau^{-1-\theta}$ a.e. at infinity (more precisely: for a.e. $\tau \ge 1$) (cf. Proposition 2.14).

Hence, it suffices to prove (2.23) for nonnegative A and $\alpha = 0$, i.e. it suffices to prove

$$n\binom{n+k-1}{k-1} \lim_{s\to\infty} \int_t^s \tau^{n-1} A^k (\tau + A)^{-n-k}\, d\tau = x - \sum_{j=0}^{k-1} \binom{n+j-1}{j} t^n A^j (t+A)^{-n-j} x. \tag{2.25}$$

for nonnegative A.

First we observe that

$$\frac{d}{d\tau} \tau^{n+k} A^k (\tau + A)^{-n-k} = (n+k)\tau^{n+k-1} A^{k+1} (\tau + A)^{-n-k-1},$$

For $k = 1$ we have

$$\begin{aligned} n \lim_{s\to\infty} \int_t^s \tau^{n-1} A(t+A)^{-n-1}\, d\tau &= \lim_{s\to\infty} \int_t^s \frac{d}{d\tau} \left(\tau^n (\tau + A)^{-n}\right) x\, d\tau \\ &= \lim_{s\to\infty} s^n (s+A)^{-n} x - t^n (t+A)^{-n} x \\ &= x - t^n (t+A)^{-n} x, \end{aligned}$$

where the last equality follows by Corollary 2.7. Thus, (2.25) holds when $k = 1$.

Next, we assume that (2.25) has been proved for some $k \geq 1$. Then, using integration by parts, we get

$$\begin{aligned} & n\binom{n+k}{n} \lim_{s\to\infty} \int_t^s \tau^{n-1} A^{k+1} (\tau + A)^{-n-k-1} x\, d\tau \\ &= n\frac{(n+k-1)!}{n!k!} \lim_{s\to\infty} \int_t^s \tau^{-k} \frac{d}{d\tau} \tau^{n+k} A^k (\tau + A)^{-n-k} x\, d\tau \\ &= \binom{n+k-1}{k} \lim_{s\to\infty} \left[s^n A^k (s+A)^{-n-k} x - t^n A^k (t+A)^{-n-k}\right] \\ &\quad + n\frac{(n+k-1)!k}{n!k!} \lim_{s\to\infty} \int_t^s \tau^{n-1} A^{n+k} (\tau + A)^{-n-k} A^{-n} x\, d\tau \\ &= x - \sum_{j=0}^{k} \binom{n+j-1}{j} t^n A^j (t+A)^{-n-j} x, \end{aligned}$$

and (2.23) is proved. Here the last equality was obtained using the induction hypothesis, and the fact that

$$\lim_{s\to\infty} \left\| s^n A^k (s+A)^{-n-k} x \right\| \leq N_A^n (1 + N_A)^{k-1} \lim_{s\to\infty} \left\| A(s+A)^{-1} x \right\| = 0$$

for any $x \in \overline{\mathcal{D}(A)}$ by Proposition 2.6, since $A(t+A)^{-1} = I - I_t$.

To prove the second formula of the lemma, we note that if $0 \in \rho(A)$, then

$$\lim_{t \downarrow 0} \left\| t^n A^j (t+A)^{-n-j} \right\| = 0$$

for all positive integers n and all nonnegative integers j. Letting $t \downarrow 0$ in the first part of the lemma, we thus obtain

$$x = n \binom{n+k-1}{n} \lim_{s \to \infty} \int_0^s \tau^{n-1} A^k (\tau + A)^{-n-k} x \, d\tau$$

for any $x \in \overline{\mathcal{D}(A)}$. $\square$

Proposition 2.14. *Assume that $-A$ is admissible in the direction α, and that $(\theta, p) \in (0,1) \times [1,\infty] \cup \{(0,\infty),(1,\infty)\}$. Let $n \in \mathbb{N}$, and define*

$$\| x \|_{\mathcal{D}_{A^n}(\theta,p)} := \| x \|_X + [\, x\,]_{\mathcal{D}_{A^n}(\theta,p)}, \tag{2.26}$$

where

$$[\, x\,]_{\mathcal{D}_{A^n}(\theta,p)} := \left\| \underline{t}^{n\theta} A^n (\underline{t} e^{i\alpha} + A)^{-n} x \right\|_{L^p_*((1,\infty);X)} . \tag{2.27}$$

Then $\| \ \|_{\mathcal{D}_{A^n}(\theta,p)}$ and $\| \ \|_{(X,\mathcal{D}(A^n))_{\theta,p}}$ are equivalent norms on $\mathcal{D}_{A^n}(\theta,p)$. If $\theta \in (0,1)$, we also have

$$\mathcal{D}_{A^n}(\theta) = \{ x \in X \mid \lim_{t \to \infty} t^{n\theta} \left\| A^n (t e^{i\alpha} + A)^{-n} x \right\|_X = 0 \}.$$

In case $0 \in \rho(A)$ we may use the definition

$$\| x \|_{\mathcal{D}_{A^n}(\theta,p)} := \left\| t^{n\theta} A^n (t e^{i\alpha} + A)^{-n} x \right\|_{L^p_*((1,\infty);X)} .$$

Proof. Again, since $\| A^n (t e^{i\alpha} + A)^{-n} x \|_X = \| A^n (t + e^{-i\alpha} A)^{-n} x \|_X$ and $-A$ is admissible in the direction α if and only if $e^{-i\alpha} A$ is nonnegative, it suffices to consider the case $\alpha = 0$, which means that we can assume that A is nonnegative.

We observe that substituting of t^{-n} for t in the definition of $\| x \|_{(X,\mathcal{D}(A^n))_{\theta,p}}$ (cf. Section B.3), we get

$$\| x \|_{(X,\mathcal{D}(A^n))_{\theta,p}} = n \left\| \underline{t}^{n\theta} K(\underline{t}^{-n}, x, X, \mathcal{D}(A^n)) \right\|_{L^p_*(1,\infty)}$$

if $0 < \theta < 1$ and $1 \le p < \infty$,

$$\| x \|_{(X,\mathcal{D}(A^n))_{\theta,\infty}} = \sup_{t > 1} t^{n\theta} K(t^{-n}, x, X, \mathcal{D}(A^n)),$$

if $0 \le \theta \le 1$, and

$$\mathcal{D}_{A^n}(\theta) = \{ x \in X \mid \lim_{t \to \infty} t^{n\theta} K(t^{-n}, x, X, \mathcal{D}(A^n)) = 0 \}$$

if $0 < \theta < 1$.

One easily checks that $[\ \ ]_{\mathcal{D}_{A^n}(\theta,p)}$ is a seminorm in $\{x \in X \mid [\,x\,]_{\mathcal{D}_{A^n}(\theta,p)} < \infty\}$.

Let z be an arbitrary element of X, and assume that $z = x + y$, where $x \in X$ and $y \in \mathcal{D}(A^n)$. Then

$$A^n(t+A)^{-n}z = A^n(t+A)^{-n}x + (t+A)^{-n}A^n y$$

so that

$$\left\| A^n(t+A)^{-n}z \right\|_X \le (1+N_A)^n \|\,x\,\|_X + N_A^n t^{-n} \|\,A^n y\,\|_X ,$$

and, consequently,

$$\left\| A^n(t+A)^{-n}z \right\|_X \le (1+N_A)^n \left(\|\,x\,\|_X + t^{-n} \|\,A^n y\,\|_X\right), \tag{2.28}$$

for $k = 0, 1, \dots, n$. Since this holds for any partition $z = x + y$ with $x \in X$ and $y \in \mathcal{D}(A^n)$, we get

$$\left\| A^n(t+A)^{-n}z \right\|_X \le (1+N_A)^n K(t^{-n}, z, X, \mathcal{D}(A^n)) \tag{2.29}$$

for all $z \in X$ and all $t \ge 1$, whence

$$[\,z\,]_{\mathcal{D}_{A^n}(\theta,p)} \le \frac{(1+N_A)^n}{c(n,p)} \|\,z\,\|_{(X,\mathcal{D}(A^n))_{\theta,p}} \tag{2.30}$$

where $c(n,p) = n$ if $1 \le p < \infty$, and $c(n,\infty) = 1$. It also follows that if $x \in \mathcal{D}_{A^n}(\theta)$, then $\lim_{t\to\infty} t^{n\theta} \left\| A^k(t+A)^{-k}z \right\|_X = 0$, and hence

$$\mathcal{D}_{A^n}(\theta) \subseteq \{x \in X \parallel \lim_{t\to\infty} t^{n\theta} \sum_{k=0}^{n} t^{k-n} \left\| A^k(t+A)^{-k}x \right\|_X = 0\}.$$

To prove that $\|\ \ \|_{\mathcal{D}_{A^n}(\theta,p)}$ is finer than the norm $\|\ \ \|_{(X,\mathcal{D}(A^n))_{\theta,p}}$, let us define $U(t) : X \longrightarrow X$ and $V(t) : X \longrightarrow \mathcal{D}(A^n)$ by

$$V(t) := t^n \sum_{j=0}^{n-1} \binom{n+j-1}{j} A^j(t+A)^{-n-j}$$

and

$$U(t) := I - V(t)$$

for $t > 0$.

By the definition of K, we have

$$K(t^{-n}, x, X, \mathcal{D}(A^n)) \le \|\,U(t)x\,\|_X + t^{-n} \|\,V(t)x\,\|_X , \tag{2.31}$$

for any $x \in X$ and any $t > 0$, and, consequently,

$$\begin{aligned} \|\,x\,\|_{(X,\mathcal{D}(A^n))_{\theta,p}} \le \frac{1}{c(n,p)} \Big(& \left\| t^{n\theta}U(t)x \right\|_{L^p_*((1,\infty);X)} \\ &+ \left\| t^{-n(1-\theta)}V(t)x \right\|_{L^p_*((1,\infty);X)}\Big) . \end{aligned} \tag{2.32}$$

In case $n=1$, we have $V(t)=t(t+A)^{-1}$ and $U(t)=A(t+A)^{-1}$, so that

$$\begin{aligned}
&\| U(t)x \|_X = \left\| A(t+A)^{-1}x \right\|_X \\
&t^{-1} \| V(t)x \|_X \le N_A \| x \|_X \\
&t^{-1} \| AV(t)x \|_X = \left\| A(t+A)^{-1}x \right\|_X
\end{aligned}$$

Hence, if $\| x \|_{\mathcal{D}(A)} := \| x \|_X + \| Ax \|_X$, we deduce that

$$K(t^{-1}, x, X, \mathcal{D}(A)) \le N_A \| x \|_X + 2 \left\| A(t+A)^{-1} \right\|_X .$$

If $0 \in \rho(A)$, we may employ the definition $\| x \|_{\mathcal{D}(A)} := \| Ax \|_X$. Then

$$K(t^{-1}, x, X, \mathcal{D}(A)) \le 2 \left\| A(t+A)^{-1} \right\|_X .$$

In both cases there is a constant C such that

$$\| x \|_{(X,\mathcal{D}(A))_{\theta,p}} \le C \| x \|_{\mathcal{D}_A(\theta,p)}$$

for all $x \in X$.

We now assume that $n > 1$. Let us estimate $\left\| t^{-n(1-\theta)} V(t)x \right\|_{L^p_*((1,\infty);X)}$. First we consider the case $\| x \|_{\mathcal{D}(A^n)} := \| x \|_X + \| A^n x \|_X$. Then

$$\begin{aligned}
&t^{-n(1-\theta)} \| V(t)x \|_{\mathcal{D}(A^n)} \\
&\quad \le t^{n\theta} \sum_{j=0}^{n-1} \binom{n+j-1}{j} \left(\left\| A^j (t+A)^{-n-j} x \right\|_X + \left\| A^{n+j} (t+A)^{-n-j} x \right\|_X \right) \\
&\quad \le t^{n\theta} \sum_{j=0}^{n-1} \binom{n+j-1}{j} (1+N_A)^j \left(N_A^n t^{-n} \| x \|_X + \left\| A^n (t+A)^{-n} x \right\|_X \right),
\end{aligned}$$

so that

$$t^{-n(1-\theta)} \| V(t)x \|_{\mathcal{D}(A^n)} \le C(t^{-n(1-\theta)} \| x \|_X + t^{n\theta} \left\| A^k (t+A)^{-k} x \right\|_X \tag{2.33}$$

for any $x \in X$ and any $t > 0$ and some constant C.

If $\| x \|_{\mathcal{D}(A^n)} := \| A^n x \|_X$, then

$$\begin{aligned}
t^{-n(1-\theta)} \| V(t)x \|_{\mathcal{D}(A^n)} &= t^{n\theta} \sum_{j=0}^{n-1} \binom{n+j-1}{j} \left\| A^{n+j} (t+A)^{-n-j} x \right\|_X \\
&\le t^{n\theta} \sum_{j=0}^{n-1} \binom{n+j-1}{j} (1+N_A)^j \left\| A^n (t+A)^{-n} x \right\|_X ,
\end{aligned}$$

so that

$$t^{-n(1-\theta)} \| V(t)x \|_{\mathcal{D}(A^n)} \le C t^{n\theta} \left\| A^n (t+A)^{-n} x \right\|_X \tag{2.34}$$

for any $x \in X$, $t \ge 1$, where $C > 0$ is a constant.

In both cases we have

$$\| t^{-n(1-\theta)} V(t)x \|_{L^p_*((1,\infty);X)} \leq C \| x \|_{\mathcal{D}_{A^n}(\theta,p)} \tag{2.35}$$

for some constant $C > 0$.

Let us estimate $\| t^{n\theta} U(t)x \|_{L^p_*((0,\infty);X)}$. By Lemma 2.13 we have

$$\begin{aligned} U(t)x &= n\binom{2n-1}{n-1} \int_t^\infty \tau^n A^n (\tau + A)^{2n} x \frac{d\tau}{\tau} \\ &= n\binom{2n-1}{n-1} \int_1^\infty (t\sigma)^n A^n (t\sigma + A)^{2n} x \frac{d\sigma}{\sigma}, \end{aligned}$$

and, consequently,

$$t^{n\theta} \| U(t)x \|_X \leq n\binom{2n-1}{n-1} N_A^n \int_1^\infty (t\sigma)^{n\theta} \| A^n (t\sigma + A)^n x \|_X \frac{d\sigma}{\sigma^{n\theta}} \tag{2.36}$$

for $n = 1, 2, \dots$ and $x \in \mathcal{D}_{A^n}(\theta,p) \hookrightarrow \mathcal{D}_{A^n}(\theta,\infty)$. Moreover, using Theorem A.17, we get

$$\| t^{n\theta} U(t)x \|_{L^p_*((1,\infty);X)} \leq \frac{1}{\theta}\binom{2n-1}{n} N_A^n \int_1^\infty t^{n\theta} \| A^n (t+A)^{-n} x \|_X \, dt. \tag{2.37}$$

Hence, combining (2.32), (2.37) and (2.35), we deduce that there is a constant $C = C(n, \theta, p, A)$, such that

$$\| x \|_{(X,\mathcal{D}(A^n))_{\theta,p}} \leq C \| x \|_{\mathcal{D}_{A^n}(\theta,p)}$$

for any $x \in X$.

Finally, assume that

$$\lim_{t\to\infty} t^{n\theta} \| A^n (t+A)^{-n} x \|_X = 0,$$

or that A is positive, $\| x \|_{\mathcal{D}(A^n)} := \| A^n x \|_X$ and

$$\lim_{t\to\infty} t^{n\theta} \| A(t+A)^{-1} x \|_X .$$

Then (2.33) or (2.34) shows that

$$\lim_{t\to\infty} t^{-n(1-\theta)} \| V(t)x \|_{\mathcal{D}(A^n)} = 0. \tag{2.38}$$

Moreover, there is a constant C such that

$$t^{n\theta} \| A^n (t+A)^n x \|_X \leq C$$

for all $t \geq 1$. By (2.37), we have

$$\| t^{n\theta} U(t)x \|_X \leq n \binom{2n-1}{n-1} N_A(\alpha)^n \int_1^\infty (t\sigma)^{n\theta} \| A^n (t\sigma e^{i\alpha} + A)^n x \|_X \frac{d\sigma}{\sigma^{1+n\theta}}$$

Hence,

$$\lim_{t\to\infty} \| t^{n\theta} U(t)x \|_X = 0 \tag{2.39}$$

by the Dominated Convergence Theorem. Summarising, we get

$$\lim_{t\to\infty} t^{n\theta} K(t^{-n}, x, X, \mathcal{D}(A^n)) = 0,$$

i.e. $x \in (X, \mathcal{D}(A^n))_\theta$. It follows that

$$\{x \in X \parallel \lim_{t\to\infty} t^{n\theta} \| A^n (t+A)^{-n} x \|_X = 0\} \subseteq \mathcal{D}_{A^n}(\theta).$$

Since the reverse inclusion has already been obtained, there is nothing more to prove. □

2.4.4 More equivalent norms

As a special case of the definition in Proposition 2.14 we have the following definition.

Definition 2.15. Let A be a nonnegative operator in a complex Banach space X, and assume that $(\theta, p) \in (0,1) \times [1,\infty] \cup \{(0,\infty), (1,\infty)\}$. We then set

$$\| x \|_{\mathcal{D}_A(\theta,p)} := \| x \|_X + [\, x \,]_{\mathcal{D}_A(\theta,p)}, \tag{2.40}$$

where

$$[\, x \,]_{\mathcal{D}_A(\theta,p)} := \| t^\theta A(t+A)^{-1} x \|_{L^p_*((0,\infty);X)}. \tag{2.41}$$

This norm will often be used in the sequel.

Proposition 2.14 implies that if A is admissible in two different directions α and β, then the two norms given by (2.26) are, in fact, equivalent for all $(\theta, p) \in (0,1) \times [1,\infty] \cup \{(0,\infty), (1,\infty)\}$. However, this result can be proved more directly. Indeed, we have

$$\begin{aligned} A(te^{i\alpha} + A)^{-1} &= A(te^{i\alpha} + A)^{-1}\{A(te^{i\beta} + A)^{-1} + te^{i\beta}(te^{i\beta} + A)^{-1}\} \\ &= \{A(te^{i\alpha} + A)^{-1} + te^{i\beta}(te^{i\alpha} + A)^{-1}\}A(te^{i\beta} + A)^{-1} \\ &= \{t(e^{i\beta} - e^{i\alpha})(te^{i\alpha} + A)^{-1} + 1\}(te^{i\beta} + A)^{-1} \end{aligned}$$

and, since

$$\| t(e^{i\beta} - e^{i\alpha})(te^{i\alpha} + A)^{-1} \|_{\mathcal{L}(X)} \leq 2 \left| \sin \frac{\beta - \alpha}{2} \right| N_A(\alpha),$$

we obtain the following lemma.

Lemma 2.16. *Let α and β be two admissible directions for $-A$. Then*
(2.42)
$$\left\| A^n(te^{i\alpha}+A)^{-n}x \right\|_X \le \left(1+2\left|\sin\frac{\beta-\alpha}{2}\right| N_A(\alpha)\right)^n \left\| A^n(te^{i\beta}+A)^{-n}x \right\|_X$$

for all $x \in X$ and $n \in \mathbb{N}$.

Assuming that A is nonnegative with $\phi_A > \phi \ge |\alpha|$ we have $N_A(\alpha) \le M_A^*(\phi)$. It follows that

$$\left\| A^n(te^{i\alpha}+A)^{-n}x \right\|_X \le (1+2M_A^*(\phi))^n \left\| A^n(te^{i\phi}+A)^{-n}x \right\|_X$$

for all $x \in X$. Hence, we could take

$$\| x \|_{\mathcal{D}_{A^n}(\theta,p)} := \| x \|_X + \sup_{|\alpha|\le\phi} \left\| t^\theta A^n(te^{i\alpha}+A)^{-n}x \right\|_{L^p_*((0,\infty);X)}$$

as a definition of the norm in $\mathcal{D}_{A^n}(\theta,p)$. In particular, if $p=\infty$, we could use the definition

$$\| x \|_{\mathcal{D}_{A^n}(\theta,\infty)} := \| x \|_X + \sup_{|\arg\lambda|\le\phi} \left\| \lambda^\theta A^n(\lambda+A)^{-n}x \right\|_X . \tag{2.43}$$

2.4.5 The spaces $(\mathcal{D}(A^n),\mathcal{D}(A^{n+1}))_{\theta,p}$

Let A be a linear operator in a complex Banach space X, and let the linear operator $A|_n : \mathcal{D}(A^{n+1}) \longrightarrow \mathcal{D}(A^n)$ be the restriction of A to $\mathcal{D}(A^{n+1})$, i.e. $A|_n := A \cap (\mathcal{D}(A^{n+1}) \times X)$, for $n=1,2,\ldots$ We also set $A|_0 := A$.

Proposition 2.17. *Let A be a nonnegative operator in a complex Banach space X. Then $A|_n : \mathcal{D}(A^{n+1}) \longrightarrow \mathcal{D}(A^n)$ is nonnegative in $\mathcal{D}(A^n)$, $\rho(-A|_n) \supseteq \rho(-A)$ for $n=1,2,\ldots$ and*

$$\mathcal{D}_{A|_n}(\theta,p) = (\mathcal{D}(A^n),\mathcal{D}(A^{n+1}))_{\theta,p} = \{x \in \mathcal{D}(A^n) \parallel A^n x \in \mathcal{D}_A(\theta,p)\}$$
$$\mathcal{D}_{A|_n}(\theta) = (\mathcal{D}(A^n),\mathcal{D}(A^{n+1}))_{\theta} = \{x \in \mathcal{D}(A^n) \parallel A^n x \in \mathcal{D}_A(\theta)\}$$

if $0<\theta<1$, $1\le p\le\infty$ and $n \in \mathbb{N}$. Moreover, there is a constant C, depending only on A, n, θ and p, such that

$$[\, A^n x \,]_{\mathcal{D}_A(\theta,p)} \le [\, x \,]_{\mathcal{D}_{A|_n}(\theta,p)} \le C\| x \|_{\mathcal{D}(A^n)} + [\, A^n x \,]_{\mathcal{D}_A(\theta,p)} \tag{2.44}$$

In particular, if A is positive, then $A|_n$ is positive in $\mathcal{D}(A^n)$, and we define $\| x \|_{\mathcal{D}(A^n)} := \| A^n x \|_X$, then

$$[\, x \,]_{\mathcal{D}_{A|_n}(\theta,p)} = [\, A^n x \,]_{\mathcal{D}_A(\theta,p)} \tag{2.45}$$

for any $x \in \mathcal{D}(A^n)$ and any $n \in \mathbb{N}$. Hence, if A is positive, then A^n is an isometric isomorphism of $\mathcal{D}_{A|_n}(\theta,p)$ onto $\mathcal{D}_A(\theta,p)$ and of $\mathcal{D}_{A|_n}(\theta)$ onto $\mathcal{D}_A(\theta)$.

Proof. Assume that $\lambda \in \rho(-A)$. Clearly $\lambda + A|_n : \mathcal{D}_{A^{n+1}} \longrightarrow \mathcal{D}_{A^n}$ is bijective with inverse $(\lambda + A|_n)^{-1}$ such that

$$(\lambda + A|_n)^{-1}x = (\lambda + A)^{-1}x$$

for any $x \in \mathcal{D}(A^n)$. We have

$$\begin{aligned} \| (\lambda + A|_n)^{-1}x \|_{\mathcal{D}(A^n)} &= \| (\lambda + A)^{-1}x \|_X + \| (\lambda + A)^{-1}A^n x \|_X \\ &\leq \| (\lambda + A)^{-1} \|_{\mathcal{L}(X)} \| x \|_{\mathcal{D}(A^n)} \end{aligned}$$

for any $\lambda \in \rho(-A)$, any $x \in \mathcal{D}(A^n)$ and $n = 1, 2, \ldots$. Consequently, $\rho(-A|_n) \supseteq \rho(-A)$ and $\| (\lambda + A|_n)^{-1} \|_{\mathcal{L}(\mathcal{D}(A^n))} \leq \| (\lambda + A)^{-1} \|_{\mathcal{L}(\mathcal{D}(A))}$ for any $\lambda \in \rho{-}A$ $(\lambda + A|_n)^{-1}$ is bounded and $\lambda \in \rho(-A|_n)$.

For $x \in \mathcal{D}(A^n)$, $k = 1, 2, \ldots, n$ and $\lambda \in \rho(-A)$ we also get

$$\left\| \lambda A^k (\lambda + A)^{-1}x \right\| = \left\| \lambda(\lambda + A)^{-1}A^k x \right\| \leq N_A(\arg\lambda) \left\| A^k x \right\|,$$

which implies that

$$\left\| \lambda(\lambda + A)^{-1} \right\|_{\mathcal{L}(\mathcal{D}(A^n))} \leq N_A(\arg\lambda).$$

Hence, the nonnegativity of $A|_n$ in $\mathcal{D}(A^n)$ is proved. We also see that, for $x \in \mathcal{D}(A^n)$ and $t > 0$,

$$t^\theta \left\| A^n A(t + A)^{-1}x \right\| = t^\theta \left\| A(\lambda + A)^{-1}A^n x \right\|$$

and

$$\begin{aligned} t^\theta \left\| A(t + A)^{-1}x \right\| &= t^\theta \left\| (\lambda + A)^{-1}Ax \right\| \\ &\leq t^{\theta-1}(1 + M_A) \| Ax \| \\ &\leq C t^{\theta-1} \| x \|_{\mathcal{D}(A^n)}, \end{aligned}$$

where C is a constant that only depends on A and n. These relations easily yield the remaining statements of the proposition. $\square$

2.4.6 Further results on $\mathcal{D}_{A^n}(\theta, p)$ and $\mathcal{D}_{A^n}(\theta)$

We close this section with the following result together with an interesting corollary.

Theorem 2.18. *Let X be a Banach, space and let A be an operator in X such that $-A$ is admissible in the direction α. Let $n \geq 1$ and m be integers with $0 \leq m \leq n$. Then*

$$\mathcal{D}(A^m) \in J_{\frac{m}{n}}(X, \mathcal{D}(A^n)) \cap K_{\frac{m}{n}}(X, \mathcal{D}(A^n)).$$

Proof. The statement is obvious if $m = 0$ or $m = n$. Thus, it remains to prove the embeddings

$$\mathcal{D}_{A^n}(m/n, 1) \hookrightarrow \mathcal{D}(A^m) \hookrightarrow \mathcal{D}_{A^n}(m/n, \infty)$$

for $1 \le m < n$. To prove the second one, we assume that $x \in \mathcal{D}(A^m)$. We have

$$\begin{aligned}
&t^m \left\| A^n(te^{i\alpha}+A)^{-n}x \right\|_X \\
&\quad = \left\| t^m(te^{i\alpha}+A)^{-m}A^{n-m}(te^{i\alpha}+A)^{-(n-m)}A^m x \right\| \\
&\quad \le N_A(\alpha)^m(1+N_A(\alpha))^{n-m} \| A^m x \|
\end{aligned}$$

for all $t > 1$. Consequently,

$$\| x \|_{\mathcal{D}_{A^n}(m/n,\infty)} \le N_A(\alpha)^m(1+N_A(\alpha))^{n-m} \| x \|_{\mathcal{D}(A^m)} .$$

To prove the embedding $\mathcal{D}(A^n)(m/n,1) \hookrightarrow \mathcal{D}(A^m)$, we first observe that if $x \in \mathcal{D}_{A^n}(m/n,1)$, then

$$\begin{aligned}
x = &\sum_{j=0}^{n-m-1} \binom{n+j-1}{j} e^{in\alpha}A^j(e^{i\alpha}+A)^{-n-j}x \\
&+ n\binom{2n-m-1}{n} \int_1^\infty t^{n-1}e^{in\alpha}A^{n-m}(te^{i\alpha}+A)^{-2n+m}x\,dt,
\end{aligned}$$

which is obtained from (2.23) by taking $t = 1$ and $k = n-m$, noting that $\mathcal{D}_{A^n}(m/n,1) \subseteq \overline{\mathcal{D}(A)}$. Now,

$$\left\| t^{n-1}A^n(te^{i\alpha}+A)^{-2n+m}x \right\|_X \le t^{m-1}N_A(\alpha)^{n-m} \left\| A^n(te^{i\alpha}+A)^{-n}x \right\|_X ,$$

for $t \ge 1$, so that the closed operator A^m can be applied to the above expression for x by moving it under the integral sign, and all the integrals are proper ones. This way we get $x \in \mathcal{D}(A^m)$, and

$$\begin{aligned}
A^m x = &\sum_{j=0}^{n-m-1} \binom{n+j-1}{j} e^{in\alpha}A^{m+j}(e^{i\alpha}+A)^{-n-j}x \\
&+ n\binom{2n-m-1}{n} \int_1^\infty e^{in\alpha}t^{n-m}(te^{i\alpha}+A)^{-(n-m)} \\
&\qquad \cdot t^{m-1}A^n(te^{i\alpha}+A)^{-n}x\,dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
\| A^m x \|_X \le &\sum_{j=0}^{n-m-1} \binom{n+j-1}{j} N_A(\alpha)^{n-m}(1+N_A(\alpha))^{m+j} \| x \|_X \\
&n\binom{2n-m-1}{n} N_A(\alpha))^{n-m}[\, x \,]_{\mathcal{D}_{A^n}(\frac{m}{n},1)}
\end{aligned}$$

which completes the proof. $\square$

Corollary 2.19. *Let X be a Banach, space and let A be an operator in X such that $-A$ is admissible in the direction α. Let $n \geq 1$ and m be integers with $0 \leq m \leq n$ Then*

$$\mathcal{D}_{A^m}(\theta, p) = \mathcal{D}_{A^n}(m\theta/n, p)$$

and

$$\mathcal{D}_{A^m}(\theta) = \mathcal{D}_{A^n}(m\theta/n)$$

for all $(\theta, p) \in (0,1) \times [1, \infty]$.

Proof. Since

$$X \in J_1(X, \mathcal{D}(A^n)) \cap K_1(X, \mathcal{D}(A^n))$$

and

$$\mathcal{D}(A^m) \in J_{m/n}(X, \mathcal{D}(A^n)) \cap K_{m/n}(X, \mathcal{D}(A^n)),$$

the statement of the corollary is a consequence of the Reiteration Theorem. □

The corollary implies that $\mathcal{D}_{A^m}(\theta, p) = \mathcal{D}_{A^n}(\eta, p)$ and $\mathcal{D}_{A^m}(\theta) = \mathcal{D}_{A^n}(\eta)$ whenever $m\theta = n\eta$, $0 < \theta, \eta < 1$, and $1 \leq p \leq \infty$.

In the next subsection we define fractional powers A^z for $\operatorname{Re} z \neq 0$, and show that Theorem 2.18, and Corollary 2.19, can be generalised to fractional powers A^z with $\operatorname{Re} z > 0$ (cf. Theorem 2.21 (f)).

Remark 2.20. Theorem 2.18 is used in Subsection 4.1.3 to prove that

$$\mathcal{C}^k_{0\mapsto 0}([0,T]; X) \in J_{k/n}(\mathcal{C}_{0\mapsto 0}([0,T]; X) \cap K_{k/n}(\mathcal{C}_{0\mapsto 0}([0,T]; X).$$

2.5 Fractional powers of positive operators

In this section we consider fractional powers A^z of a positive operator A for complex exponents z that are not purely imaginary.

2.5.1 Definition of A^z when $\operatorname{Re} z < 0$. Special cases

Let A be a positive operator in a complex Banach space X. In this section we define the operator A^z for $\operatorname{Re} z \neq 0$, and examine some of its basic properties.

If $\operatorname{Re} z < 0$ we define the operator A^z by the equality

$$A^z := \frac{1}{2\pi i} \int_\gamma (-\lambda)^z (\lambda + A)^{-1}\, d\lambda, \tag{2.46}$$

where γ goes from $\infty e^{-i\sigma_1}$ to $\infty e^{i\sigma_2}$, for some σ_1, σ_2 with $0 < \sigma_1, \sigma_2 < \phi_A$, and γ lies in $\rho(-A)$ and crosses the real axis to the left of the origin. A practical choice would be a curve of the form $\gamma^-_{\sigma,r}$ for a sufficiently small r (see

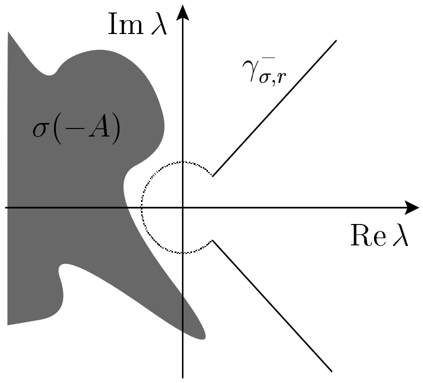

Figure 2: The curve $\gamma^-_{\sigma,r}$ in the definition of A^z, $\operatorname{Re} z < 0$

Figure 2; this curve is defined in Section 3.1.5). The expression $(-\lambda)^z$ is calculated according to the principal value of $\arg(-\lambda)$, i.e. $(-\lambda)^z = e^{z \log(-\lambda)} = e^{z[\ln|\lambda| + i \arg(-\lambda)]}$, where we take the principal branch of the logarithm.

Let us show that the above definition is meaningful, and, in fact, defines a bounded linear operator with domain X. Firstly, the integrand is analytic in $\mathbb{C} \setminus \mathbb{R}_+$. Secondly, we note that

$$\| A^z \| \leq \frac{e^{|\operatorname{Im} z|(\pi-\sigma)}}{2\pi} \int\limits_{\gamma^-_{\sigma,r}} |\lambda|^{\operatorname{Re} z} \left\| (\lambda + A)^{-1} \right\| \, d\,|\lambda|\,.$$

The integrand is of order $|\lambda|^{\operatorname{Re} z}$ at the origin, since $0 \in \rho(-A)$, whereas it is of order $|\lambda|^{1+\operatorname{Re} z}$ at infinity, since A is nonnegative. The boundedness of A^z follows from these observations.

Choosing, as above, $\gamma = \gamma^-_{\sigma,r}$, we can integrate (2.46) by parts m times. In this manner we obtain

$$(2.47) \qquad A^z = \frac{(-1)^m}{2\pi i (z+1)(z+2)\cdots(z+m)} \int\limits_{\gamma} (-\lambda)^{z+m} (\lambda + A)^{-m-1} \, d\lambda,$$

for $m = 0, 1, 2, 3, \ldots$, provided that $z \notin \{0, -1, -2, \ldots, -m\}$. Cauchy's integral theorem shows that in this formula we may take any γ of the kind accepted in the definition of A^z.

In (2.47) the path γ' along γ from $R_1 e^{-i\sigma_1}$ to $R_2 e^{i\sigma_2}$ can be continuously deformed into a path consisting of the curve $\{R_1 e^{i\phi} \mid -\sigma_1 \leq \phi \leq 0\}$, the line segment from R_1 to r, the circle $C_r : |\lambda| = r$ run through in the negative sense, the line segment from r to R_2, and the curve $\{R_2 e^{i\phi} \mid 0 \leq \phi \leq \sigma_2\}$, where r is so small that $B_r(0) \subseteq \rho(-A)$. We get

$$\int\limits_{\gamma'} (-\lambda)^{z+m}(\lambda+A)^{-m-1}\,d\lambda$$

$$= e^{-i\pi(z+m)} \int\limits_r^{R_1} t^{z+m}(t+A)^{-m-1}\,dt - e^{i\pi(z+m)z} \int\limits_r^{R_2} t^{z+m}(t+A)^{-m-1}\,dt$$

$$+ \oint\limits_{C_r} (-\lambda)^{z+m}(z+A)^{-m-1}\,d\lambda$$

$$+ \int\limits_{-\sigma_1}^{0} R_1^{z+m} e^{i(z+m)(\pi+\phi)}(R_1 e^{i\phi}+A)^{-m-1} i R_1 e^{i\phi}\,d\phi$$

$$+ \int\limits_{0}^{\sigma_2} R_2^{z+m} e^{i(z+m)(\pi+\phi)}(R_2 e^{i\phi}+A)^{-m-1} i R_2 e^{i\phi}\,d\phi.$$

Letting $R_1, R_2 \to \infty$, we see that

$$\begin{aligned} A^z = &-\frac{\sin \pi z}{\pi}\frac{m!}{(z+1)(z+2)\cdots(z+m)} \int\limits_r^{\infty} t^{z+m}(t+A)^{-m-1}\,dt \\ &+ \frac{m!}{2\pi i(z+1)(z+2)\cdots(z+m)} \oint\limits_{C_r} (-\lambda)^{z+m}(\lambda+A)^{-m-1}\,d\lambda, \end{aligned} \tag{2.48}$$

since

$$\left\| R_k^{z+m} e^{i(z+m)(\pi+\phi)}(R_k e^{i\phi}+A)^{-m-1} i R_k e^{i\phi} \right\| \le e^{2\pi|\operatorname{Im} z|} R_k^{\operatorname{Re} z}$$

for $k = 1, 2$ by the nonnegativity of A.

The positivity of A also implies that $\| t^{z+m}(t+A)^{-m-1} \| \le c t^{\operatorname{Re} z+m}$ and

$$\left\| (-re^{i\phi})^{z+m}(re^{i\phi}+A)^{-m-1} \right\| \le c e^{\pi|\operatorname{Im} z|} r^{\operatorname{Re} z+m}$$

in the integrals of (2.48), where c is a constant. Consequently, if $-m-1 < \operatorname{Re} z < 0$, we can let $r \downarrow 0$ in that formula, and get

$$A^z = -\frac{\sin \pi z}{\pi}\frac{m!}{(z+1)(z+2)\cdots(z+m)} \int\limits_0^{\infty} t^{z+m}(t+A)^{-m-1}\,dt \tag{2.49}$$

whenever $-m-1 < \operatorname{Re} z < 0$ and $z \neq -1, -2, \dots, -m$. Since $\sin \pi z = -\pi/\Gamma(z+1)\Gamma(-z)$, the last formula may also be written

$$A^z = \frac{\Gamma(m)}{\Gamma(z+m)\Gamma(-z)} \int\limits_0^{\infty} t^{z+m-1}(t+A)^{-m}\,dt, \tag{2.50}$$

where $-m < \operatorname{Re} z < 0$ and $z \neq -1, -2, \dots, -m+1$.

If $-z = n \in \mathbb{N}$, then $\sin \pi z = 0$ in (2.48), and, taking $m = 0$, we obtain

$$A^{-n} = \frac{1}{2\pi i} \int\limits_{C_r} (-\lambda)^{-n} (\lambda + A)^{-1} \, d\lambda.$$

Moreover, $(-\lambda)^{-n}(\lambda + A)^{-1}$ is analytic in $\rho(-A) \setminus \{0\}$. Since

$$\begin{aligned}
(-\lambda)^{-n}(\lambda + A)^{-1} &= (-1)^n \lambda^{-n} [(1 + \lambda A^{-1})A]^{-1} \\
&= (-1)^n \lambda^{-n} A^{-1} \sum_{k=0}^{\infty} (-1)^k \lambda^k (A^{-1})^k \\
&= \sum_{k=-n}^{\infty} (-1)^k \lambda^k (A^{-1})^{k+n+1},
\end{aligned}$$

we have $\operatorname*{Res}_{\lambda=0} (-\lambda)^{-n} (\lambda + A)^{-1} = -(A^{-1})^n$, so that

$$A^{-n} = -\frac{1}{2\pi i} \cdot 2\pi i \operatorname*{Res}_{z=0} (-\lambda)^{-n} (\lambda + A)^{-1} = (A^{-1})^n.$$

Thus, our new definition of A^{-n} agrees with the familiar one.

Since the right hand member of (2.46) can be differentiated under the integral sign, $z \longrightarrow A^z$ defines an analytic mapping of $(-\infty, 0)$ into $\mathcal{L}(X)$.

2.5.2 The product formula

Let us now show that if $\operatorname{Re} z < 0$ and $\operatorname{Re} w < 0$, then $A^z A^w = A^{z+w}$. For this purpose we choose paths of integration $\gamma_1 := \gamma_{r_1, \sigma_1}$ and $\gamma_2 := \gamma_{r_2, \sigma_2}$, with $r_1 > r_2$ and $\sigma_1 > \sigma_2$, so that γ_1 lies to the left of γ_2. Then

$$\begin{aligned}
A^z A^w &= \frac{1}{2\pi i} \int\limits_{\gamma_1} (-\lambda)^z (\lambda + A)^{-1} \left(\frac{1}{2\pi i} \int\limits_{\gamma_2} (-\mu)^w (\mu + A)^{-1} \, d\mu \right) d\lambda \\
&= \frac{1}{2\pi i} \int\limits_{\gamma_1} (-\lambda)^z \left(\frac{1}{2\pi i} \int\limits_{\gamma_2} (-\mu)^w (\lambda + A)^{-1} (\mu + A)^{-1} \, d\mu \right) d\lambda \\
&= \frac{1}{2\pi i} \int\limits_{\gamma_1} (-\lambda)^z \Big(\frac{1}{2\pi i} \int\limits_{\gamma_2} (-\mu)^w (\mu - \lambda)^{-1} \big((\lambda + A)^{-1} \\
&\qquad\qquad - (\mu + A)^{-1} \big) \, d\mu \Big) \, d\lambda \\
&= \frac{1}{2\pi i} \int\limits_{\gamma_1} (-\lambda)^z (\lambda + A)^{-1} \left(\frac{1}{2\pi i} \int\limits_{\gamma_2} \frac{(-\mu)^w}{\mu - \lambda} \, d\mu \right) d\lambda \\
&\quad + \frac{1}{2\pi i} \int\limits_{\gamma_2} (-\lambda)^z (\mu + A)^{-1} \left(\frac{1}{2\pi i} \int\limits_{\gamma_1} \frac{(-\lambda)^z}{\lambda - \mu} \, d\lambda \right) d\mu.
\end{aligned}$$

The integrand $(\lambda-\mu)^{-1}(-\lambda)^z$ in the last integral is analytic on and to the left of γ_1, and it is of order $|\lambda|^{\operatorname{Re} z-1}$ at infinity. Consequently, the path of integration can be replaced by a closed path, and, by Cauchy's integral theorem (Theorem A.25), the integral evaluates to zero. The integrand $(\mu-\lambda)^{-1}(-\mu)^w$ is analytic on and to the left of γ_2, except at $\lambda=\mu$, and it is of order $|\mu|^{\operatorname{Re} w-1}$ at infinity in this region. Hence, the integration can be performed along a closed path with λ inside. According to Cauchy's integral formula A.13, the value of this integral is $2\pi i(-\lambda)^w$. Therefore,

$$A^z A^w = \frac{1}{2\pi i}\int_{\gamma_1}(-\lambda)^z(-\lambda)^w(\lambda+A)^{-1}\,d\lambda = A^{z+w}$$

whenever $\operatorname{Re} z$, $\operatorname{Re} w<0$.

2.5.3 Definition of A^z when $\operatorname{Re} z>0$

Using the above formula, one easily shows that A^z is injective if $\operatorname{Re} z<0$. In fact, if $A^z x=A^z y$, then $A^w x=A^{w-z}A^z x=A^{w-z}A^z y=A^w y$, if $\operatorname{Re} w<\operatorname{Re} z<0$, so that, in particular, $A^{-n}x=A^{-n}y$ for all $n\in\mathbb{N}$ such that $n>-\operatorname{Re} z$. It follows that $x=y$. Consequently,

$$A^z := (A^{-z})^{-1} \qquad (\operatorname{Re} z>0)$$

defines a linear operator if $\operatorname{Re} z>0$, and the definition agrees with the usual one for $n\in\mathbb{N}$. The operator A^z is closed since A^{-z} is. It also follows from the definition that

$$\mathcal{D}(A^z)=\mathcal{R}(A^{-z}) \qquad (\operatorname{Re} z\neq 0).$$

Observe that if $\operatorname{Re} z>0$, then $0\in\rho(A^z)$. Consequently, $\|x\|_{\mathcal{D}(A^z)}=\|A^z x\|_X$ defines a norm on $\mathcal{D}(A^z)$ that is equivalent with the graph norm of associated with A^z.

2.5.4 Some properties of fractional powers

In the following theorem we have gathered some useful properties of fractional powers of positive operators.

Theorem 2.21. *Let A be a positive linear operator with spectral angle $\omega_A\in[0,\pi)$. Assume that $z,w\in\mathbb{C}$ and $\operatorname{Re} z,\operatorname{Re} w\neq 0$. Then the following assertions hold.*

(a) *A^z is an injective, closed linear operator, and $A^{-z}=(A^z)^{-1}$.*

(b) *$A^z\in\mathcal{L}(X)$, if $\operatorname{Re} z<0$.*

(c) *If $\operatorname{Re} z<\operatorname{Re} w$, then $\mathcal{D}(A^z)\supseteq\mathcal{D}(A^w)$.*

(d) *If A is densely defined, then so is A^z.*

(e) *If $\operatorname{Re} z,\operatorname{Re} w,\operatorname{Re}(z+w)\neq 0$, then $A^zA^w=I_{\mathcal{D}(A^{-z})}A^{z+w}I_{\mathcal{D}(A^w)}$. If, in* *addition,* $\operatorname{Re} w < \min\{0, -\operatorname{Re} z\}$ *or* $\operatorname{Re} z > \max\{0, -\operatorname{Re} w\}$*, then* $A^z A^w = A^{z+w}$.

(f) *If* B *is an operator in* X *such that* A *and* B *are resolvent commuting, and if* $0 < \operatorname{Re} z$*, then* A^z *and* B *are resolvent commuting. In particular, if* B *is positive,* $\operatorname{Re} w > 0$ *and* A *and* B *are resolvent commuting, then so are* A^z *and* B^w.

(g) *If* $0 < \alpha := \operatorname{Re} z < n$*, where* $n \in \mathbb{N}$*, then*

$$\mathcal{D}(A^z) \in J_{\frac{\alpha}{n}}(X, \mathcal{D}(A^n)) \cap K_{\frac{\alpha}{n}}(X, \mathcal{D}(A^n)).$$

Hence, if $0 < \alpha := \operatorname{Re} z \leq \beta := \operatorname{Re} w$*, then*

$$\begin{aligned}\mathcal{D}_{A^z}(\theta, p) &= \mathcal{D}_{A^w}(\alpha\theta/\beta, p)\\ \mathcal{D}_{A^z}(\alpha) &= \mathcal{D}_{A^w}(\alpha\theta/\beta),\end{aligned}$$

with equivalence of norms, for any θ *and* p *with* $0 < \theta < 1$ *and* $1 \leq p \leq \infty$.

(h) *If* $0 < \alpha < 1$*, then* A^α *is positive with spectral angle* $\omega_{A^\alpha} \leq \alpha\omega_A$*, and we have*

$$(z + A^\alpha)^{-1} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{r^\alpha}{r^2\alpha + 2r^\alpha z \cos \pi\alpha + z^2} (r + A)^{-1}\, dr \tag{2.51}$$

Moreover, $\mathcal{D}_{A^\alpha}(\theta, p) = \mathcal{D}_A(\alpha\theta, p)$ *and* $\mathcal{D}_{A^\alpha}(\theta) = \mathcal{D}_A(\alpha\theta)$ *for any* θ *and* p *with* $0 < \theta < 1$ *and* $1 \leq p \leq \infty$.

Proof. Statements (a)–(b) have already been proved.

(c) Obviously $\mathcal{D}(A) \supseteq \mathcal{D}(A^2) \supseteq \ldots$. We generalise this property to non-integer exponents as follows: Assume that $x \in \mathcal{D}(A^w)$, and that $\operatorname{Re} z < \operatorname{Re} w$. We may also assume that $\operatorname{Re} z > 0$ and $\operatorname{Re} w > 0$, since otherwise $\mathcal{D}(A^z) = X$. Then $x \in \mathcal{R}(A^{-w})$, i.e. $x = A^{-w}y = A^{-z}A^{-(w-z)}y$ for some $y \in X$. Consequently, $x \in \mathcal{R}(A^{-z}) = \mathcal{D}(A)^z$, and (c) is proved.

To prove (d), assume that A is densely defined. Thanks to (c) it suffices to show that $\overline{\mathcal{D}(A^n)} = X$ for any $n \in \mathbb{N}$. We start by showing an auxiliary result:

Let us assume that A is densely defined in X, that $0 \in \rho(A)$ and that $0 \in \rho(B)$. Take an arbitrary $x \in \mathcal{D}(B)$ and $\varepsilon > 0$. By the denseness of $\mathcal{D}(A)$ there is some $u \in \mathcal{D}(A)$ with $\| u - y \| < \varepsilon / \| B^{-1} \|$. We put $v = Au$. Then

$$\| B^{-1}A^{-1}v - x \| = \| B^{-1}u - B^{-1}y \| \leq \varepsilon.$$

Hence, $\mathcal{D}(B) \subseteq \overline{\mathcal{D}(AB)}$, so that $\overline{\mathcal{D}(B)} \subseteq \overline{\mathcal{D}(AB)}$. Therefore, if B is also densely defined, then so is AB.

Applying this result to A, we get $\overline{\mathcal{D}(A^2)} = X$. Using induction, we conclude that $\overline{\mathcal{D}(A^n)} = X$ for any $n \in \mathbb{N}$.

(e) It has been shown above that if $\operatorname{Re} z < 0$ and $\operatorname{Re} w < 0$, then $A^{z+w} = A^z A^w$. If $\operatorname{Re} z > 0$ and $\operatorname{Re}(z + w) < 0$, then $\operatorname{Re}(-z) < 0$, so that $A^z A^w =$

$A^z(A^{-z}A^{z+w}) = A^{z+w}$. If $\operatorname{Re} w > 0$ and $\operatorname{Re}(z+w) < 0$, then $\operatorname{Re} z < -\operatorname{Re} w < 0$, so that $A^zA^w = A^{z+w}A^{-w}A^w) = A^{z+w}I_{\mathcal{D}(A^w)}$. This formula also holds in the previous two cases, since then $\operatorname{Re} w < 0$, and hence $I_{\mathcal{D}(A^w)} = I_X$. Thus, $A^zA^w = A^{z+w}I_{\mathcal{D}(A^w)}$ whenever $\operatorname{Re}(z+w) < 0$ and $\operatorname{Re} z, \operatorname{Re} w \neq 0$. Finally, if $\operatorname{Re}(z+w) > 0$ and $\operatorname{Re} z, \operatorname{Re} w \neq 0$, then $\operatorname{Re}(-z-w) < 0$, so that

$$\begin{aligned} A^zA^w &= (A^{-z})^{-1}(A^{-w})^{-1} = (A^{-w}A^{-z})^{-1} = (A^{-z-w}I_{\mathcal{D}(A^{-z})})^{-1} \\ &= I_{\mathcal{D}(A^{-z})}A^{z+w}. \end{aligned}$$

It follows that $A^zA^w = I_{\mathcal{D}(A^{-z})}A^zA^wI_{\mathcal{D}(A^w)} = I_{\mathcal{D}(A^{-z})}A^{z+w}I_{\mathcal{D}(A^w)}$, provided that $\operatorname{Re} z, \operatorname{Re} w, \operatorname{Re}(z+w) \neq 0$.

(f) By claim (b) and the definition of A^z, we have $0 \in \rho(-A^z)$, and

$$(A^z)^{-1} = A^{-z} := \frac{1}{2\pi i}\int_\gamma (-\lambda)^{-z}(\lambda + A)^{-1}\,d\lambda.$$

Thus, to prove the first statement it suffices to show that $(\lambda + B)^{-1}$ and A^{-z} commute. But this is immediate from the above formula for A^{-z}. If B is positive and $\operatorname{Re} w > 0$, this result may be applied once again to the pair B and A^z, and we immediately obtain the second statement of (f).

(g) By Theorem 2.18 the first statement of part (g) holds when z is a nonnegative integer. Therefore we assume that $0 < \alpha := \operatorname{Re} z < n \in \mathbb{N}$, and $\alpha \notin \mathbb{N}$. We should prove that

$$\mathcal{D}_{A^n}(\tfrac{\alpha}{n}, 1) \hookrightarrow \mathcal{D}(A^z) \hookrightarrow \mathcal{D}_{A^n}(\tfrac{\alpha}{n}, \infty)$$

To prove the first of these embeddings, note that, by formula (2.50),

$$A^{z-n}x = \frac{\Gamma(n)}{\Gamma(z)\Gamma(n-z)}\int_0^\infty t^{z-1}(t+A)^{-n}x\,dt.$$

Proposition 2.14 guarantees that the closed operator A^n can be applied to the last integral by moving it under the integral sign, since the resulting integral is absolutely convergent, and we get

$$A^zx = A^nxA^{z-n}z = \frac{\Gamma(n)}{\Gamma(z)\Gamma(n-z)}\int_0^\infty t^{z-1}A^n(t+A)^{-n}x\,dt,$$

so that

$$\|\, A^zx \,\|_X \leq \frac{\Gamma(n)}{\Gamma(z)\Gamma(n-z)}[\,x\,]_{\mathcal{D}_{A^n}(\frac{\alpha}{n},1)}$$

for any $x \in \mathcal{D}_{A^n}(\frac{\alpha}{n}, 1)$. Thus $\mathcal{D}_{A^n}(\alpha/n, 1) \hookrightarrow \mathcal{D}(A^z)$.

To prove the embedding $\mathcal{D}(A^z) \hookrightarrow \mathcal{D}_{A^n}(\alpha/n, \infty)$, we take an arbitrary $x \in \mathcal{D}(A^z)$. Then $x = A^{-z}y$, where $y = A^zx$. By (2.50),

$$t^\alpha A^n(t+A)^{-n}A^{-z}y = \frac{\Gamma(n)}{\Gamma(n-z)\Gamma(z)}t^\alpha A^n(t+A)^{-n}\int_0^\infty s^{-z+n-1}(s+A)^{-n}y\,ds$$

Now recall that $\| (\lambda + A)^{-1} \| \leq M_A \lambda^{-1}$ and $\| A(\lambda+A)^{-1} \| \leq 1 + M_A$ for any $\lambda > 0$. Using this, and dividing the interval of integration into two parts, we obtain

$$\begin{aligned}
& t^\alpha \left\| A^n (t+A)^{-n} \int_0^t s^{-z+n-1} (s+A)^{-n} y \, ds \right\| \\
& = t^\alpha \left\| (t+A)^{-n} \int_0^t s^{-z+n-1} A^n (s+A)^{-n} y \, ds \right\| \\
& \qquad \leq \left\| t^n (t+A)^{-n} \right\| \| y \| \, t^{\alpha-n} \int_0^t s^{-\alpha+n-1} \left\| A^n (s+A)^{-n} \right\| \, ds \\
& \qquad \leq M_A^n (1+M_A)^n \| y \| \, t^{\alpha-n} \int_0^t s^{n-1-\alpha} \, ds = \frac{M_A^n (M_A+1)^n}{n-\alpha} \| A^z x \| ,
\end{aligned}$$

and

$$\begin{aligned}
& t^\alpha \left\| A^n (t+A)^{-n} \int_t^\infty s^{-z+n-1} (s+A)^{-n} y \, ds \right\| \\
& \qquad \leq \| A^n (t+A)^{-1} \| \, \| y \| \, t^\alpha \int_t^\infty s^{-\alpha+n-1} \left\| (s+A)^{-n} \right\| \, ds \\
& \qquad M_A^n (1+M_A)^n \| y \| \, t^\alpha \int_t^\infty s^{-\alpha-1} \, ds = \frac{M_A^n (1+M_A)^n}{\alpha} \| A^z x \| ,
\end{aligned}$$

so that

$$t^\alpha \left\| A(t+A)^{-n} \right\| \leq \frac{\Gamma(n+1)}{\Gamma(n-z)\Gamma(z)} \frac{M_A^n (M_A+1)^n}{\alpha(n-\alpha)} \| A^z x \| .$$

It follows that

$$[\, x \,]_{\mathcal{D}_{A^n}(\frac{\alpha}{n},\infty)} \leq \frac{\Gamma(n+1)}{\Gamma(n-z)\Gamma(z)} \frac{M_A^n (M_A+1)^n}{\alpha(n-\alpha)} \| A^z x \| .$$

This completes the proof of the first statement of (g).

Since

$$\mathcal{D}(A^z) \in J_{\alpha/n}(X, \mathcal{D}(A^n)) \cap K_{\alpha/n}(X, \mathcal{D}(A))$$

and

$$\mathcal{D}(A^w) \in J_{\beta/n}(X, \mathcal{D}(A^n)) \cap K_{\beta/n}(X, \mathcal{D}(A))$$

if $0 < \alpha := \operatorname{Re} z \le \beta := \operatorname{Re} w < n \in \mathbb{N}$, the Reiteration Theorem shows that

$$\mathcal{D}_{A^z}(\theta, p) = \mathcal{D}_{A^n}(\alpha\theta/n, p) = \mathcal{D}_{A^n}(\beta(\alpha\theta/\beta)/n, p) = \mathcal{D}_{A^w}(\alpha\theta/\beta, p),$$

and

$$\mathcal{D}_{A^z}(\theta) = \mathcal{D}_{A^n}(\alpha\theta/n) = \mathcal{D}_{A^n}(\beta(\alpha\theta/\beta)/n) = \mathcal{D}_{A^w}(\alpha\theta/\beta).$$

The proof that A^α is positive with spectral angle $\omega_{A^\alpha} \le \alpha\omega_A$ as well as the formula for $(z + A^\alpha)^{-1}$ can be found in [1], pp. 159–160. The last statement of (h) is a direct consequence of (g). □

Remark 2.22. One can also define A^z for purely imaginary z as the closure of the operator $A_z : \mathcal{D}(A) \to X$ defined by

$$A_z x := \frac{\sin \pi z}{\pi z} \int_0^\infty t^z (t + A)^{-2} A x \, dt \qquad (x \in \mathcal{D}(A))$$

(cf. [1] or [13]).

2.6 The operator $A_\lambda := \lambda + A$

2.6.1 Positivity

Let A be a nonnegative operator with spectral angle ω_A. Then for any $\lambda > 0$ the operator $A_\lambda := \lambda + A$ is positive with spectral angle $\omega_{A_\lambda} \le \omega_A$. In fact we have the following result.

Proposition 2.23. *Let A be a nonnegative operator, let $\lambda \in \mathbb{C}$ be such that $\phi := |\arg \lambda| < \phi_A$. Then A_λ is positive and $\phi_{A_\lambda} \ge \min(\phi_A, \pi - \phi)$.*

Proof. Let $0 \le \alpha < \min(\phi_A, \pi - \phi)$ and let $|\arg z| \le \alpha$. Then $|\arg(z + \lambda)| \le \max(\alpha, \phi) < \phi_A$. We also have $z + \lambda \ne 0$, since $z = -\lambda$ would imply that $|\arg z| = \pi - |\arg \lambda| = \pi - \phi$, which contradicts the assumptions. Hence, $z + \lambda \in \rho(-A)$ so that $z \in \rho(-A_\lambda)$ and $|z + \lambda| \, \| (z + A_\lambda)^{-1} \| \le M_A^*(\alpha + \phi)$ (see Figure 3).

In order to find an upper bound for $|z| \, \| (z + A_\lambda)^{-1} \|$, we write $|z + \lambda| = |z||1 + \lambda\overline{z}/|z|^2|$. We have

$$\begin{aligned} |\arg(\lambda/z)| &= |\arg \lambda + \arg \overline{z}| \\ &\le |\arg \lambda| + |\arg \overline{z}| \\ &\le \phi + \alpha < \pi, \end{aligned}$$

whence $(\lambda/z) \in \overline{\Sigma}_{\alpha+\phi}$. But

$$\min\{|1 + \zeta| \parallel \zeta \in \overline{\Sigma}_\omega\} = \begin{cases} \sin \omega & \text{if } \frac{\pi}{2} < \omega \le \pi \\ 1 & \text{if } 0 < \omega \le \frac{\pi}{2} \end{cases}.$$

Hence, we deduce that

$$M_{A_\lambda}^*(\alpha) \le \frac{M_A^*(\alpha + \phi)}{\sin(\max\{\pi/2, \alpha + \phi\})}. \tag{2.52}$$

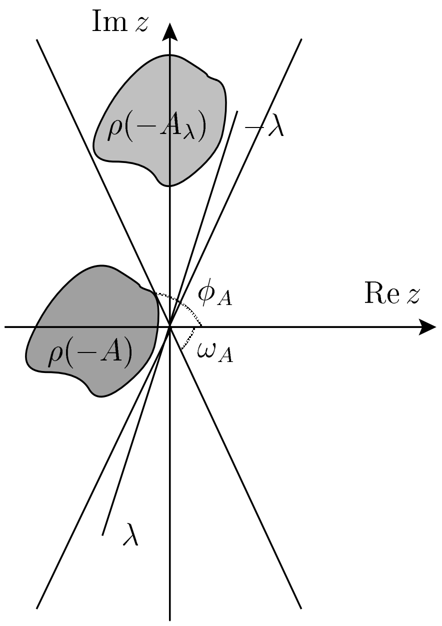

Figure 3: The spectrum of $A_\lambda = \lambda + A$

In particular,

$$M^*_{A_\lambda}(\alpha) \leq \frac{M^*_A(\alpha)}{\sin(\max\{\alpha, \pi/2\})} \tag{2.53}$$

if $\lambda > 0$.

We have shown that A_λ is nonnegative and $\phi_{A_\lambda} \geq \alpha$ for any α belonging to $[0, \min(\phi_A, \pi - \phi))$. But it was assumed that $|\arg \lambda| = \phi < \phi_A$, whence $\lambda \in \rho(-A)$, i.e. $0 \in \rho(A_\lambda)$. Consequently, A_λ is positive. $\square$

Remark 2.24. Observe that we do not have $\lim_{\epsilon\downarrow 0} M_{A_\epsilon}(\alpha) = M_A(\alpha)$. In fact, if A is the zero operator, then $M_A(\alpha) = 1$, whereas

$$M_{A_\epsilon}(\alpha) = 1/\sin(\max\{\alpha, \pi/2\})$$

for any $\epsilon > 0$.

2.6.2 Convergence of the seminorms $[\,x\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)}$

Since $\|\,x\,\| + \|\,\epsilon x + Ax\,\| \leq (1 + |\epsilon|)(\|\,x\,\| + \|\,Ax\,\|)$ and $\|\,x\,\| + \|\,Ax\,\| \leq (1 + |\epsilon|)(\|\,x\,\| + \|\,Ax\,\|)$, the graph norms of the operators A and $\epsilon + A$ are equivalent. Hence, the interpolation spaces $\mathcal{D}_A(\theta, p)$ and $\mathcal{D}_{A_\epsilon}(\theta, p)$ must be equal with equivalent norms, and the same is true of $\mathcal{D}_A(\theta)$ and $\mathcal{D}_{A_\epsilon}(\theta)$. We even have convergence of the seminorms $[\,x\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)}$ to $[\,x\,]_{\mathcal{D}_A(\theta,p)}$ as $\epsilon \to 0$. (Recall that $[\,x\,]_{\mathcal{D}_A(\theta,p)} = \left\|\, t^\theta A(t+A)^{-1}x \,\right\|_{L^p_*((0,\infty);X)}$).

Proposition 2.25. *Let A be a nonnegative operator and let $\epsilon \in \Sigma_{\phi_A}$. Then $\mathcal{D}_A(\theta,p) = \mathcal{D}_{A_\epsilon}(\theta,p)$ and $\mathcal{D}_A(\theta) = \mathcal{D}_{A_\epsilon}(\theta)$ with equivalence of norms if $0 < \theta < 1$ and $1 \leq p \leq \infty$. Moreover $[\,x\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)} \to [\,x\,]_{\mathcal{D}_A(\theta,p)}$ as $\epsilon \to 0$ in Σ_α for some $\alpha < \pi - \omega_A$.*

Proof. Using the resolvent identity (formula (1.5)) we get

$$\begin{aligned}
&A_\epsilon(t+A_\epsilon)^{-1} - A(t+A)^{-1} \\
&\quad = (\epsilon + A)(t+\epsilon+A)^{-1} - A(t+A)^{-1} \\
&\quad = \epsilon(t+\epsilon+A)^{-1} + A\left((t+\epsilon+A)^{-1} - (t+A)^{-1}\right) \\
&\quad = \epsilon(t+\epsilon+A)^{-1} - \epsilon A(t+\epsilon+A)^{-1}(t+A)^{-1} \\
&\quad = \epsilon(t+\epsilon+A)^{-1} - \epsilon\left(1 - t(t+A)^{-1}\right)(t+\epsilon+A)^{-1}(t+A)^{-1} \\
&\quad = \epsilon t(t+A_\epsilon)^{-1}(t+A)^{-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left| t^\theta \left\| A_\epsilon(t+A_\epsilon)^{-1}x \right\|_X - t^\theta \left\| A(t+A)^{-1}x \right\|_X \right| \\
&\quad \leq t^\theta \left\| \epsilon t(t+A_\epsilon)^{-1}(t+A)^{-1}x \right\|_X \\
&\quad \leq N_A(0)N_{A_\epsilon}(0)\frac{\epsilon t^\theta}{t+\epsilon} \left\| x \right\|_X
\end{aligned}$$

and

$$\begin{aligned}
&\left| [\,x\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)} - [\,x\,]_{\mathcal{D}_A(\theta,p)} \right| \\
&\quad = \left| \left\| t^\theta A_\epsilon(t+A_\epsilon)^{-1}x \right\|_{L^p_*((0,\infty);X)} - \left\| t^\theta (t+A)^{-1}x \right\|_{L^p_*((0,\infty);X)} \right| \\
&\quad \leq \left\| t^\theta A_\epsilon(t+A_\epsilon)^{-1}x - t^\theta(t+A)^{-1}x \right\|_{L^p_*((0,\infty);X)} \\
&\quad \leq N_A(0)N_{A_\epsilon}(0) \left\| x \right\|_X \left\| \frac{\epsilon}{\underline{t}+\epsilon}\underline{t}^\theta \right\|_{L^p_*(0,\infty)} \\
&\quad = N_A(0)N_{A_\epsilon}(0)\epsilon^\theta \left\| t^\theta(t+1)^{-1} \right\|_{L^p_*(0,\infty)} \left\| x \right\|_X .
\end{aligned}$$

Clearly this implies that if one of the seminorms $[\,x\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)}$ and $[\,x\,]_{\mathcal{D}(A)(\theta,p)}$ is finite, then so is the other. Moreover,

$$\frac{1}{c}\left\| x \right\|_{\mathcal{D}_A(\theta,p)} \leq \left\| x \right\|_{\mathcal{D}(A_\epsilon)(\theta,p)} \leq c \left\| x \right\|_{\mathcal{D}_A(\theta,p)},$$

where

$$c = c(\theta,p,\epsilon,A) = 1 + N_A(0)N_{A_\epsilon}(0)\epsilon^\theta \left\| \frac{t^\theta}{t+1} \right\|_{L^p_*}.$$

Consequently, $\mathcal{D}_{A_\epsilon}(\theta,p) = \mathcal{D}_A(\theta,p)$ with equivalence of norms. It is also obvious from (2.52) that $c \to 1$ as $\epsilon \to 0$ in $\overline{\Sigma}_\alpha$, where $\alpha < \pi - \omega_A$, which implies that $\left\| x \right\|_{\mathcal{D}_{A_\epsilon}(\theta,p)}$ converges to $\left\| x \right\|_{\mathcal{D}_A(\theta,p)}$ for any x as $\epsilon \to 0$ in $\overline{\Sigma}_\alpha$.

One also sees that if $t^\theta \left\| A(t+A)^{-1}x \right\|_X$ tends to 0 as $t \to \infty$, then the same is true for $t^\theta \left\| A_\epsilon(t+A_\epsilon)^{-1}x \right\|_X$. Hence, $\mathcal{D}_A(\theta) = \mathcal{D}_{A_\epsilon}(\theta)$. ☐

2.7 A perturbation lemma

We end this chapter with a sufficient condition for an operator CB to be nonnegative (positive) if B is assumed to be nonnegative (positive).

Lemma 2.26. *Let C and B be linear operators in a complex Banach space $\tilde{X}$. Assume that C is closed with $0 \in \rho(C)$ and that B is nonnegative (positive). Also assume that $\phi < \phi_B$ and $\| 1 - C^{-1} \|_{\mathcal{L}(\tilde{X})} M_B(\phi) < 1$. Then CB is nonnegative (positive), and $\phi_{CB} \geq \phi$.*

Proof. We have

$$\begin{aligned} \lambda + CB &= C(\lambda C^{-1} + B) \\ &= C(\lambda + B + \lambda(C^{-1} - 1)) \\ &= C(1 - \lambda(1 - C^{-1})(\lambda + B)^{-1})(\lambda + B). \end{aligned}$$

Now, by assumption, $\beta M_B(\phi) < 1$, where $\beta = \| 1 - C^{-1} \|_{\mathcal{L}(\tilde{X})}$, so that $1 - \lambda(1 - C^{-1})(\lambda + B)^{-1}$ can be inverted by means of a Neumann series. Thus,

$$(\lambda + CB)^{-1} = (\lambda + B)^{-1}(1 - \lambda(1 - C^{-1})(\lambda + B)^{-1})^{-1}C^{-1},$$

where

$$(1 - \lambda(1 - C^{-1})(\lambda + B)^{-1})^{-1} = \sum_{n=0}^{\infty} \lambda^n [(1 - C^{-1})(\lambda + B)^{-1}]^n$$

and

$$\| (1 - \lambda(1 - C^{-1})(\lambda + B)^{-1})^{-1} \|_{\mathcal{L}(\tilde{X})} \leq \frac{1}{1 - \beta M_B(\phi)}.$$

Hence,

$$\| (\lambda + CB)^{-1} \| \leq \frac{\| (\lambda + B^{-1} \| \| C^{-1} \|}{1 - \beta M_B(\phi)} \leq \frac{(\beta + 1) M_B(\phi)}{1 - \beta M_B(\phi)} \frac{1}{|\lambda|} \tag{2.54}$$

for all λ with $|\arg \lambda| < \phi$. ∎

3 The method of sums

In this chapter we study equations of the form

$$Ax + Bx = y, \tag{3.1}$$

where A and B are nonnegative operators in a complex Banach space X, $y \in X$ and $x \in X$ is the solution whose existence and regularity we are investigating. We shall impose some additional conditions arriving at a so called *parabolic* problem. This parabolic problem will be treated as two separate cases depending on whether A and B are *(resolvent) commuting* (cf. Proposition 1.10) or not. In both cases the operator $S(A, B)$, to be defined below, will play a fundamental rôle in expressing and analysing solutions to these equations. In the resolvent commuting case $S(A, B)y$ actually proves to be the unique solution to the equation (3.1), provided that A and B and y satisfy certain additional conditions.

3.1 The operator $S(A, B)$

In this section we first carry out a short and somewhat formal calculation the outcome of which is a candidate for a solution operator $S(A, B)$ of the equation $Ax + Bx = y$. We then discuss some conditions under which this linear operator is well–defined, as well as some of its basic properties.

3.1.1 A formal derivation

Let us rewrite our equation $Ax+Bx = y$ as $(z+A)x-(z-B)x = y$, where z is a complex number. Assuming that $z \in \rho(-A)\cap\rho(B)$, and that the resolvents of A and B commute, we can multiply this equation by $(z + A)^{-1}(z - B)^{-1}$ to get $(z-B)^{-1}x-(z+A)^{-1}x = (z+A)^{-1}(z-B)^{-1}y$. We then observe that $(z - B)^{-1} = (1/z)(1 + B(z - B)^{-1})$ and $(z + A)^{-1} = (1/z)(1 - A(z + A)^{-1}$, so that we have

$$\frac{1}{z}(z + A)^{-1}Ax + \frac{1}{z}(z - B)^{-1}Bx = (z + A)^{-1}(z - B)^{-1}y. \tag{3.2}$$

Here we have used the fact that $A(z + A)^{-1}x = (z + A)^{-1}Ax$ and $B(z - B)^{-1}x = (z - B)^{-1}Bx$ for $x \in \mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$. We now integrate the two terms of the left hand member along a curve γ lying completely in $\rho(-A) \cap \rho(B)$ and passing from $\infty e^{-i\sigma}$ to $\infty e^{i\sigma}$, where σ is some suitable angle. We also assume that it crosses the real axis to the left of the origin, that the region to the left of γ is contained in $\rho(B)$, and that the region to the right of γ is contained in $\rho(-A)$. (Below we discuss conditions under which such curves exists.) In order to be able to carry out the integration, we also assume that $(z+A)^{-1}Ax$ and $(z-B)^{-1}Bx$ are of order $z^{-\theta}$ at infinity for some $\theta > 0$ ($\theta = 1$ if A and B are nonnegative). Hence, $\frac{1}{z}(z - B)^{-1}Bx$ is of order $z^{-1-\theta}$ at infinity. consequently, using standard methods of complex analysis, we easily see that

$$\int_\gamma (z-B)^{-1}Bx\frac{dz}{z} = 0.$$

As for the expression $\frac{1}{z}(z+A)^{-1}x$, it is also of order $z^{-1-\theta}$ at infinity. It is not analytic at the origin, but if we assume that $\rho(-A)$ contains γ and the region to the right of γ, then, using the calculus of residues and the fact that the residue of $\frac{1}{z}(z+A)^{-1}Ax$ at the origin is

$$\lim_{z\to 0} z\cdot\frac{1}{z}(z+A)^{-1}Ax = x - \lim_{z\to 0} z(z+A)^{-1}Ax = x$$

we get

$$\int_\gamma (z+A)^{-1}Ax\frac{dz}{z} = -2\pi i x,$$

where the negative sign stems from the fact that we first integrate along the part of γ where $|z| \le R$ and some curve to the right of γ far from the origin, so that we get a closed path encircling the origin in a *clockwise* fashion. Letting $R \to \infty$, the integral along the circular path vanishes. Thus, we can also integrate the right hand member of (3.2) along γ, and we obtain

$$x = \frac{-1}{2\pi i}\int_\gamma (z+A)^{-1}(z-B)^{-1}y\,dz.$$

The right hand member of this equation is what we shall shortly define as $S(A,B)y$.

3.1.2 Definition of S. Boundedness

Let us now assume that A and B are nonnegative, and that their spectral angles ω_A and ω_B satisfy the inequality

$$\omega_A + \omega_B < \pi, \tag{3.3}$$

or, equivalently, $\omega_B < \phi_A$. Let us also assume that $0 \in \rho(A) \cup \rho(B)$. These assumptions guarantee that $\rho(-A) \cap \rho(B)$ covers a certain portion of the complex plane including simple curves γ from $\infty e^{i\sigma}$ to $\infty e^{i\sigma}$, where $\sigma \in (0,\pi)$ is some suitable angle (see Figure 4). More precisely, if $0 \in \rho(A)$ we let $r_1 = \sup\{r > 0 \,\|\, B(0,r) \subseteq \rho(A)\}$, and if $0 \in \rho(B) \setminus \rho(A)$ we let $r_1 = \sup\{r > 0 \,|\, B(0,r) \subseteq \rho(B)\}$. In the first case we let $\Gamma = (\Sigma_{\phi_A} \cup B(0,r_1)) \setminus \overline{\Sigma}_{\omega_B}$, and in the second $\Gamma = (\complement\overline{\Sigma}_{\omega_B} \cup B(0,r_1)) \setminus \complement\Sigma_{\phi_A}$, where $B(0,r_1)$ is the open disc with centre at the origin and radius r_1. Then Γ is a simply connected domain that includes $\Sigma_{\phi_A} \setminus \overline{\Sigma}_{\omega_B}$ and is contained in $\rho(-A) \cap \rho(B)$.

Definition 3.1. For any $r > 0$ and $0 < \sigma < \pi$ we define simple curves $\gamma^-_{\sigma,r}$ and $\gamma^+_{\sigma,r}$ by

(3.4)

$$\gamma^-_{\sigma,r} := \{te^{-i\sigma} \,|\, \infty > t > r\} \cup \{re^{i\tau} \,\|\, 2\pi - \sigma \ge \tau \ge \sigma\} \cup \{te^{-i\sigma} \,\|\, r < t < \infty\}$$

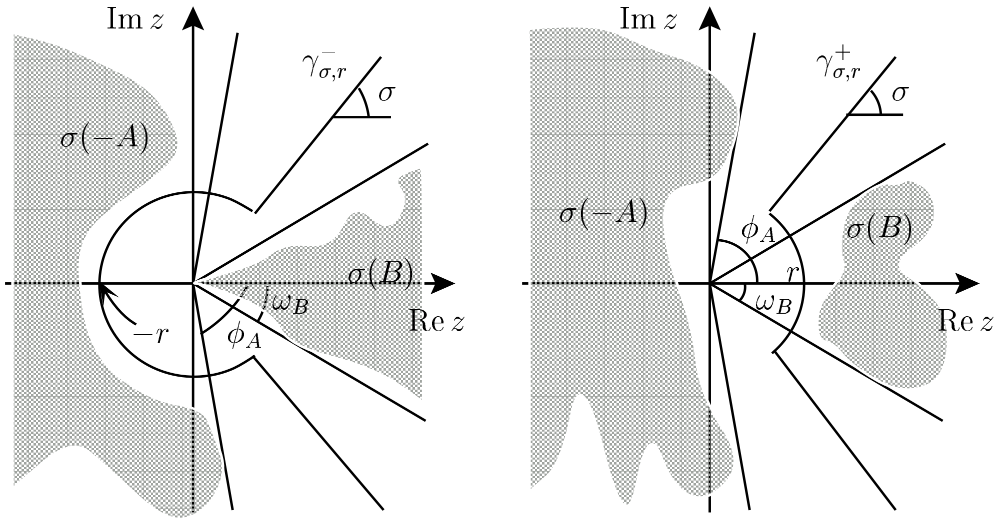

Figure 4: The paths $\gamma^-_{\sigma,r}$ and $\gamma^+_{\sigma,r}$.

and
(3.5)
$$\gamma^+_{\sigma,r} := \{te^{-i\sigma} \mid \infty > t > r\} \cup \{re^{i\tau} \mid -\sigma \le \tau \le \sigma\} \cup \{te^{-i\sigma} \mid r < t < \infty\}.$$

The curves $\gamma^-_{\sigma,r}$ and $\gamma^+_{\sigma,r}$ pass from $\infty e^{-i\sigma}$ to $\infty e^{i\sigma}$ along the ray from $\infty e^{-i\sigma}$ to $re^{-i\sigma}$, along a circular path from $re^{-i\sigma}$ to $re^{i\sigma}$ with radius r and centre at the origin, and finally along the ray from $re^{i\sigma}$ to $\infty e^{i\sigma}$ (see Figure 4). Let γ_σ be the limit of these curves as $r \downarrow 0$, i.e.

$$\gamma_\sigma := \{te^{-i\sigma} \mid \infty > t > 0\} \cup \{te^{-i\sigma} \mid 0 \le t < \infty\}. \tag{3.6}$$

If $0 \in \rho(A)$ let $r_1 = \sup\{r > 0 \mid B(0,r) \subseteq \rho(A)\}$, and if $0 \in \rho(B) \setminus \rho(A)$ let $r_1 = \sup\{r > 0 \mid B(0,r) \subseteq \rho(B)\}$, as above. We then take $r_0 = \min(1, r/2)$. Also let $\sigma_0 = (\phi_A + \omega_B)/2$, so that $\omega_B < \sigma_0 < \phi_0$, since we have assumed that $\omega_B < \phi_A$.

For any $y \in X$ we get an analytic function $f : \rho(-A) \cap \rho(B) \longrightarrow \mathcal{L}(X)$ by putting $f(z) = (z+A)^{-1}(z-B)^{-1}$. For $z \in \mathbb{C}$ with $\omega_B < \sigma = |\arg z| < \phi_A$ we have

$$\left\| (z+A)^{-1}(z-B)^{-1} \right\|_{\mathcal{L}(X)} \le \frac{M_A(\sigma)M_B(\pi-\sigma)}{|z|^2}. \tag{3.7}$$

Hence, this function is absolutely integrable over $\gamma^-_{\sigma_0,r_0}$ or $\gamma^+_{\sigma_0,r_0}$.

Definition 3.2. We set

$$S(A,B)y := \frac{-1}{2\pi i}\int_\gamma (z+A)^{-1}(z-B)^{-1}y\,dz, \tag{3.8}$$

where $\gamma := \gamma^-_{\sigma_0,r_0}$ if $0 \in \rho(A)$, and $\gamma := \gamma^+_{\sigma_0,r_0}$ if $0 \in \rho(B) \setminus \rho(A)$. Here σ_0 and r_0 are defined as above.

When there is no risk of confusion, we simply write S instead of $S(A,B)$. Since

$$\| Sy \| \leq \frac{1}{2\pi} \| y \| \int\limits_{\gamma} \| (z+A)^{-1}(z-B)^{-1} \| \, d|z| \tag{3.9}$$

the estimate (3.7) shows that $S(A,B)$ is a bounded linear operator on X.

Proposition 3.3. *If A and B are nonnegative operators in a complex Banach space X, if $\phi_A + \phi_B < \pi$, and if $0 \in \rho(A) \cup \rho(B)$, then $S(A,B) \in \mathcal{L}(X)$.*

3.1.3 Independence of the choice of path

We shall now prove that the curve γ in the above definition of $S(A,B)$ can be replaced by a whole range of simple curves.

Lemma 3.4. *Let A and B be two nonnegative operators in a complex Banach space X. Assume that $0 \in \rho(A) \cup \rho(B)$ and that the spectral angles ω_A and ω_B satisfy the inequality $\omega_A + \omega_B < \pi$. Let $\omega_B < \sigma_1 \leq \sigma_2 < \phi_A$ and let $\gamma = \gamma(\tau)$ $(-\infty < \tau < \infty)$ be a simple path in $\rho(-A) \cap \rho(B)$, which can be continuously deformed into γ_σ without passing any point in the spectra of $-A$ or B and such that*

(i) $\lim\limits_{|\tau| \to \infty} |\gamma(\tau)| = \infty$,

(ii) *and there is a $\tau_0 \geq 0$ such that*
$\tau \geq \tau_0$ implies that $\sigma_1 \leq \arg(\gamma(\tau)) \leq \sigma_2$
$\tau \leq -\tau_0$ implies that $-\sigma_2 \leq \arg(\gamma(\tau)) \leq -\sigma_1$.

Then

$$S(A,B) = \frac{-1}{2\pi i} \int\limits_{\gamma} (z+A)^{-1}(z-B)^{-1} \, dz.$$

Proof. Assuming that $\sigma < \pi - \omega_A$ we have

$$\| (z+A)^{-1} \| \leq \frac{M^*_A(\sigma)}{|z|} \tag{3.10}$$

for all z with $|\arg z| \leq \sigma$. When $\omega_B < \sigma$ we get the estimate

$$\| (z-B)^{-1} \| \leq \frac{M^*_B(\pi - \sigma)}{|z|} \tag{3.11}$$

for all $z \neq 0$ with $|\arg z| \geq \sigma$, since for such z we have $|\arg(-z)| \leq \pi - \sigma < \pi - \omega_B$. Consequently, we have

$$\| (z+A)^{-1}(z-B)^{-1} \|_{\mathcal{L}(X)} \leq \frac{M^*_A(\sigma_2) M^*_B(\pi - \sigma_1)}{|z|^2}.$$

As a consequence, the integral $\int_\gamma (z+A)^{-1}(z-B)^{-1}\,dz$ converges absolutely.

By the assumptions on γ there is a $\tau_1 > \tau_0$ such that if $\tau \geq \tau_1$ then $|\gamma(\pm\tau)| \geq r_0$, where r_0 has been defined above. Thus, let $\tau \geq \tau_1$. Then the integral along γ_{σ_0,r_0} from $|\gamma(-\tau)|\,e^{-i\sigma_0}$ to $|\gamma(\tau)|\,e^{i\sigma_0}$ equals the sum of the integral along γ from $\gamma(-\tau)$ to $\gamma(\tau)$ and the integrals from $|\gamma(-\tau)|\,e^{-i\sigma_0}$ to $\gamma(-\tau)$ and from $|\gamma(\tau)|\,e^{i\sigma_0}$ to $\gamma(\tau)$ along circles with radii $|\gamma(-\tau)|$ and $|\gamma(\tau)|$ respectively. Letting $\tau \to \infty$ these latter integrals tend to 0, and thus the statement of the theorem follows. □

A simple consequence of the above theorem is that the curve $\gamma^-_{\sigma_0,r_0}$ or $\gamma^+_{\sigma_0,r_0}$ in the definition of $S(A,B)$ can be replaced by any $\gamma^-_{\sigma,r}$ or $\gamma^+_{\sigma,r}$, respectively, where $\omega_B < \sigma < \phi_B$ and $B(0,r') \subseteq \rho(A)$ or $B(0,r') \subseteq \rho(B)$, respectively, for some $r' > r$.

3.1.4 Commutativity

Among the basic properties of S we note the following, which is a direct consequence of Theorem A.6.

Lemma 3.5. *Let X, A, B and γ be as in Lemma 3.4, let $x \in X$, let P be a closed linear operator in X, and assume that*

$$(z+A)^{-1}(z-B)^{-1}Px = P(z+A)^{-1}(z-B)^{-1}x$$

for all z on γ. Then

$$SPx = PSx. \tag{3.12}$$

In particular, if A and B are resolvent commuting,

$$SAx = ASx \qquad (x \in \mathcal{D}(A)), \tag{3.13}$$

and

$$SBx = BSx \tag{3.14}$$

for all $x \in \mathcal{D}(B)$.

When A and B are resolvent commuting, we also have $S(B,A) = S(A,B)$.

Lemma 3.6. *Let A and B satisfy the hypotheses of Proposition 3.3, and assume, in addition, that they are resolvent commuting. Then*

$$S(B,A) = S(A,B). \tag{3.15}$$

Proof. We have

$$S(B,A)x = \frac{-1}{2\pi i}\int_\gamma (z+B)^{-1}(z-A)^{-1}x\,dz = \frac{1}{2\pi i}\int_{\gamma'} (z+A)^{-1}(z-B)^{-1}x\,dz,$$

where $\gamma' = -\gamma$, $\gamma = \gamma_{\sigma,r}$, $\omega_A < \sigma < \phi_B$, and the sign in $\gamma_{\sigma,r}$ is "+" or "−" depending on whether A is invertible or B is invertible. If we set $\sigma' = \pi - \sigma$, we get $\omega_B < \sigma' < \phi_A$. Now, changing the direction of $\gamma' = -\gamma_{\sigma,r}$, we get $\gamma'' = \gamma^{\mp}_{\sigma',r}$. This amounts to changing the sign in the last integral, and so

$$\begin{aligned} S(B,A)x &= \frac{-1}{2\pi i}\int\limits_{\gamma''}(z+A)^{-1}(z-B)^{-1}x\,dz \\ &= S(A,B)x. \end{aligned}$$

□

3.1.5 The operator $S_\lambda(A,B)$

We still assume that A and B are nonnegative operators, whose spectral angles satisfy the inequality $\omega_A + \omega_B < \pi$. In Section 2.1 we saw that if A is a nonnegative linear operator with spectral angle ω_A, then the operator $A_\lambda := \lambda + A$ is positive for any complex number λ with $|\arg\lambda| < \phi_A$. In this case the spectral angle ω_{A_λ} of A_λ satisfies the inequality $\omega_{A_\lambda} \le \max(\omega_A, |\arg\lambda|)$. Hence, if $|\arg\lambda| < \min(\phi_A, \phi_B)$, then the operator A_λ is positive, and $\omega_{A_\lambda} < \pi - \omega_B$. Consequently, the operator

$$S_\lambda(A,B) := S(A_\lambda, B) \tag{3.16}$$

is well defined. (The same applies to $S(A, B_\lambda)$). When there is no risk of confusion, we simply denote this operator by S_λ. The case $\lambda \in (0,\infty)$ is illustrated in Figure 6. In this case, by Lemma 3.4, the path of integration γ may be, *e.g.*, any simple curve, which coincides with $\gamma_\sigma - \lambda'$ at infinity and lies between the broken lines $\gamma_{\phi_A} - \lambda$ and γ_{ω_B} for some σ and λ' with $\omega_B < \sigma < \phi_A$ and $0 < \lambda' < \lambda$. Examples of such paths are provided by the curves $\gamma^{+}_{\sigma,r} - \lambda$, $\gamma_\sigma - \lambda'$ and $\gamma^{-}_{\sigma,r}$, where $\omega_B < \sigma < \phi_A$, $0 < \lambda' < \lambda$ and $r > 0$ is sufficiently small, so that the curve under consideration lies in $\rho(-A_\lambda) \cap \rho(B)$.

We have the following estimate on the norm of S_λ.

Proposition 3.7. *Let A and B be nonnegative operators with $\omega_A + \omega_B < \pi$, and let $\lambda \in \mathbb{C}$ be such that $|\arg\lambda| < \min(\phi_A, \phi_B)$. Then there is a constant m depending on M_A, M_B, ϕ_A, ϕ_B and $\arg\lambda$, but not on $|\lambda|$, such that*

$$\| S_\lambda \|_{\mathcal{L}(X)} \le \frac{m}{|\lambda|}.$$

If $\lambda \in (0,\infty)$, we may take $m = m_0$, where

$$m_0 := \frac{2}{\pi} M_A^*(\sigma) M_B^*(\pi - \sigma) \int\limits_0^\infty \frac{dr}{|r^2 e^{2i\sigma} - 1|}$$

and $\omega_B < \sigma < \pi - \omega_A$. If, in addition, either A or B is positive, then

$$\lim_{\substack{\lambda \to 0 \\ \lambda \in \overline{\Sigma}_\phi}} \| S_\lambda(A,B) - S(A,B) \| = 0,$$

for any ϕ with $0 < \phi < \min(\phi_A, \phi_B)$

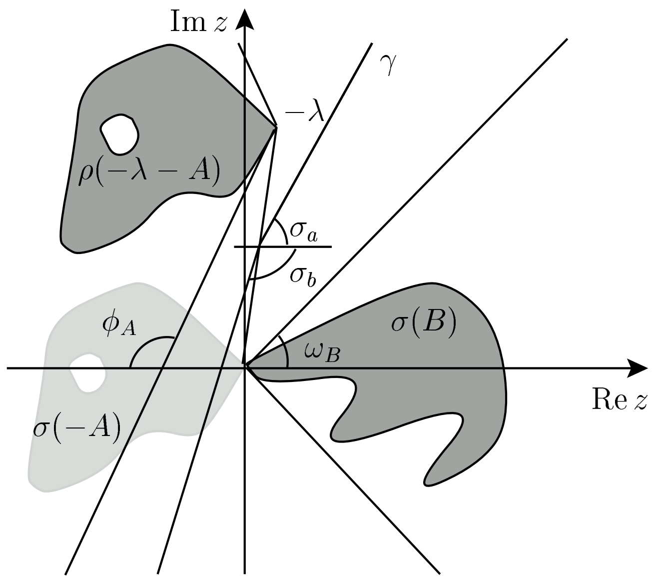

Figure 5: Paths of integration used in the proof of Proposition 3.7

Proof. We may let γ consist of two rays of the form $\gamma_1(\tau) = -\frac{\lambda}{2} + \tau e^{-i\sigma_b}$ $(\infty > \tau > 0)$ and $\gamma_2(\tau) = -\frac{\lambda}{2} + \tau e^{i\sigma_a}$ $(0 \le \tau < \infty)$, where

$$\begin{aligned} \omega_B < \sigma_a < \min(\phi_A, \arg(-\lambda)) \\ \max(\arg\lambda, \omega_B) < \sigma_b < \phi_A \end{aligned}$$

if $\arg(-\lambda) \ge 0$, and

$$\begin{aligned} \max(\arg\lambda, \omega_B) < \sigma_a < \phi_A \\ \omega_B < \sigma_b < \min(\phi_A, \arg(-\lambda)) \end{aligned}$$

if $\arg(-\lambda) < 0$ (see Figure 5). Let

$$S^{(i)} := \frac{-1}{2\pi i} \int\limits_{\gamma_i} (z + \lambda + A)^{-1} (z - B)^{-1} \, dz,$$

so that $S_\lambda x = S^{(i)} + S^{(2)}$ for all $x \in X$. There are angles σ_a^* and σ_b^* depending only on σ_a, σ_b and $\arg\lambda$, such that $|\arg(z+\lambda)| \le \sigma_a^*$ and $|\arg z| \ge \sigma_b^*$ on γ. Hence, on the first ray we have $\|\,(z + \lambda + A)^{-1}\,\| \le M(\sigma_a^*)\,|z + \lambda|$ and

$\| (z-B)^{-1} \| \le M_B(\pi - \sigma_b *)/ |z|$, and thus

$$\| S^{(1)} \| \le \frac{M_A^*(\sigma_a^*) M_B^*(\pi - \sigma_b^*)}{2\pi} \int\limits_{\gamma_1} \frac{1}{|z|\,|z+\lambda|}\, d\,|\lambda|$$

$$= \frac{M_A^*(\sigma_a^*) M_B^*(\pi - \sigma_b^*)}{2\pi} \int\limits_0^\infty \frac{dr}{\left|re^{i\sigma_a} + \frac{\lambda}{2}\right| \left|re^{-i\sigma_a} - \frac{\lambda}{2}\right|}$$

$$= \frac{M_A^*(\sigma_a^*) M_B^*(\pi - \sigma_b^*)}{\pi\,|\lambda|} \int\limits_0^\infty \frac{dr}{|r^2 e^{2i(\sigma_a - \arg\lambda)} - 1|}$$

Analogously we get

$$\| S^{(2)} \| \le \frac{M_A^*(\sigma_a^*) M_B^*(\pi - \sigma_b^*)}{2\pi} \int\limits_{\gamma_1} \frac{1}{|z|\,|z+\lambda|}\, d\,|\lambda|$$

$$= \frac{M_A^*(\sigma_a^*) M_B^*(\pi - \sigma_b^*)}{2\pi} \int\limits_0^\infty \frac{dr}{\left|re^{i\sigma_b} + \frac{\lambda}{2}\right| \left|re^{-i\sigma_b^*} - \frac{\lambda}{2}\right|}$$

$$= \frac{M_A^*(\sigma_a^*) M_B^*(\pi - \sigma_b^*)}{\pi\,|\lambda|} \int\limits_0^\infty \frac{d(\lambda + A)^{-(n-1)} r}{|r^2 e^{2i(\sigma_b - \arg\lambda)} - 1|}.$$

Thus, if

$$m := \frac{M_A^*(\sigma_a^*) M_B^*(\pi - \sigma_b^*)}{\pi} \int\limits_0^\infty \frac{dr}{|r^2 e^{2i(\sigma_a - \arg\lambda)} - 1|} + \frac{M_A^*(\sigma_a^*) M_B^*(\pi - \sigma_b^*)}{\pi} \int\limits_0^\infty \frac{dr}{|r^2 e^{2i(\sigma_b - \arg\lambda)} - 1|},$$

we get $\| S_\lambda \| \le \frac{m}{\|\lambda\|}$.

If $\lambda \in (0, \infty)$ we can take $\gamma = -\frac{\lambda}{2} + \gamma_\sigma$, and $\sigma_a = \sigma_a^* = \sigma_b = \sigma_b^* = \sigma$, so that $m = m_0$.

If we assume that either A or B is positive, and that the argument of λ satisfies $|\arg\lambda| < \min\{\phi_A, \phi_B\}$, then, using the Resolvent Identity, we obtain

$$S_\lambda(A,B) - S(A,B) = \lambda \frac{-1}{2\pi i} \int\limits_\gamma (z + \lambda + A)^{-1} (z + A)^{-1} (z - B)^{-1}\, dz,$$

where $\gamma = \gamma^-_{\sigma,r}$ or $\gamma = \gamma^+_{\sigma,r}$ is a path for $S(A,B)$ with $\sigma \in (\omega_B, \phi_A)$ and $r > 0$ sufficiently small. It follows that there is a constant $C(A,B,\phi) > 0$ such that

$$\| S_\lambda(A,B) - S(A,B) \| \le C|\lambda| \| S \|$$

for any $\lambda \in \overline{\Sigma}_\phi$, where $\phi < \min(\phi_A, \phi_B)$. This implies that $\| S_\lambda - S \| \to 0$ as $\lambda \to 0$, $\lambda \in \overline{\sigma}_\phi$. Hence, using the first part, we infer that the mapping $\lambda \mapsto S_\lambda$ is bounded on $\overline{\sigma}_\phi$. $\square$

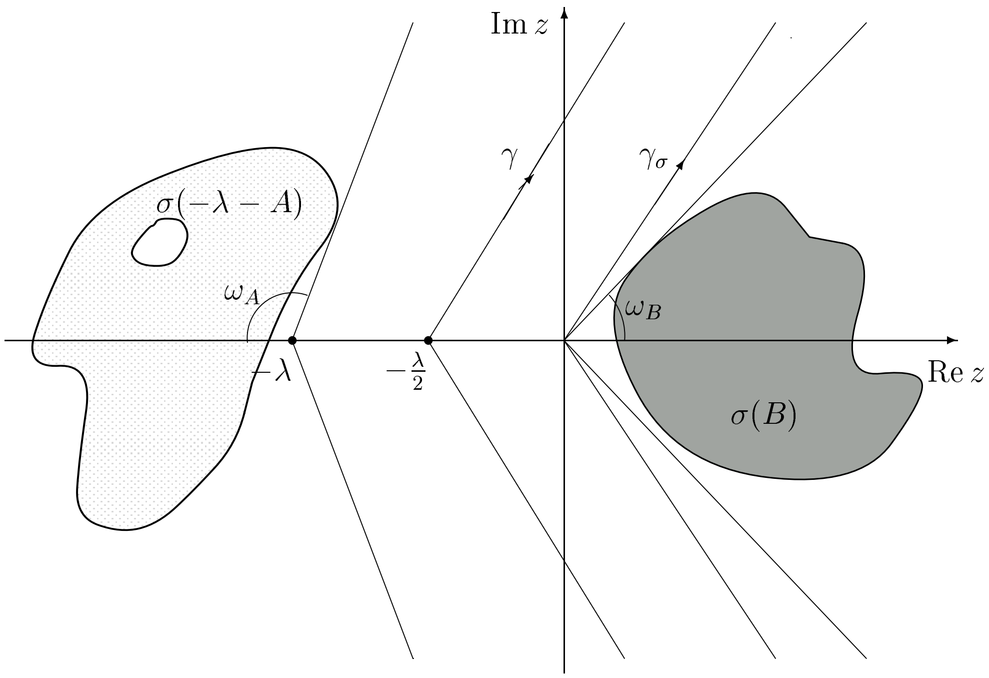

Figure 6: Spectral angles and paths of integration involved in the calculation of S_λ ($\lambda > 0$).

The following lemma shows that if A and B are resolvent commuting they may change places in the definition of $S_\lambda(A, B)$ without changing the operator.

Lemma 3.8. *Let A and B be resolvent commuting nonnegative linear operators with $\omega_A + \omega_B < \pi$. Assume that $\lambda, \mu \geq 0$, and $\mu + \lambda > 0$. Then $S_{\lambda+\mu}(A, B) = S(A_\lambda, B_\mu)$.*

Proof. The statement of the lemma follows from the equalities

$$
\begin{aligned}
S_{\lambda+\mu}(A, B) &= \frac{-1}{2\pi i} \int\limits_{\gamma_\sigma - \frac{\lambda+\mu}{2}} (z + \lambda + \mu + A)^{-1}(z - B)^{-1}\, dz \\
&= \frac{-1}{2\pi i} \int\limits_{\gamma_\sigma + \frac{\mu-\lambda}{2}} (z + \lambda + A)^{-1}(z - \mu - B)^{-1}\, dz = S(A_\lambda, B_\mu).
\end{aligned}
$$

□

The result of the lemma may, of course, be generalised to non–real λ and μ.

3.2 The resolvent commuting case

In this section we study the existence, uniqueness and maximal regularity of solutions to the equation $Ax + Bx = y$ in the case where the resolvents of A and B commute. According to the previous section we can then expect $x = Sy = S(A, B)y$ to be a good candidate for a solution to the equation.

Throughout the rest of this chapter we shall assume that A and B are (at least) nonnegative linear operators in X, and that their spectral angles satisfy the inequality $\omega_A + \omega_B < \pi$. In this section we also assume that $0 \in \rho(A) \cup \rho(B)$. Thus, the linear operator $S = S(A, B)$ will belong to $\mathcal{L}(X)$. We shall also assume that γ is a simple curve from $\infty e^{-i\sigma}$ to $\infty e^{i\sigma}$ for some $\sigma \in (\omega_B, \phi_A)$ as described in Lemma 3.4. Moreover, A and B are usually assumed to be resolvent commuting. However, resolvent commutativity will not be needed in all the lemmas and theorems of the section and will therefore be stated explicitly where needed.

3.2.1 Some useful integrals

Let us now state and prove a simple lemma that will turn out to be useful in the sequel.

Lemma 3.9. *Let A be a nonnegative operator in a complex Banach space X, let $0 < \sigma < \phi_A$ and let $r > 0$. Then*

$$\int_{\gamma^+_{\sigma,r}} \frac{1}{z}(z + A)^{-1}\, dz = 0 \tag{3.17}$$

and

$$\int_{\gamma^+_{\sigma,r}} \frac{1}{z}A(z + A)^{-1}x\, dz = 0 \tag{3.18}$$

for any $x \in \mathcal{D}_A(\theta, \infty)$ $(0 < \theta < 1)$. If, in addition, A is positive and r is so small that $\gamma^+_{\sigma,r}$ lies in $\rho(-A)$, then

$$\int_{\gamma^-_{\sigma,r}} \frac{1}{z}(z + A)^{-1}\, dz = -2\pi i A^{-1}, \tag{3.19}$$

and

$$\int_{\gamma^-_{\sigma,r}} \frac{1}{z}A(z + A)^{-1}x\, dz = -2\pi i x, \tag{3.20}$$

for any $x \in \mathcal{D}_A(\theta, \infty)$ $(0 < \theta < 1)$.

Proof. Under the assumptions of the lemma, the function $f : \rho(-A)\backslash\{0\} \longrightarrow \mathcal{L}(X)$ defined by $f(z) := \frac{1}{z}(z+A)^{-1}$ is analytic. It is absolutely integrable over $\gamma^+_{\sigma,r}$, since it is bounded of order $|z|^{-2}$ at infinity in any sector Σ_ϕ, where $0 \leq \phi < \phi_A$. Choosing $R > r$, we can continuously deform the part $\gamma^+_{\sigma,r,R}$ of $\gamma^+_{\sigma,r}$ that lies between $Re^{-i\sigma}$ and $Re^{i\sigma}$ into the circular path $C_R = \{Re^{-i\tau} \parallel -\sigma \leq \tau \leq \sigma\}$ without changing the value of the integral $\int_{\gamma^+_{\sigma,r}} f(z)\,dz$. Letting $R \to \infty$, we make the integral along C_R tend to 0, so that (3.17) holds.

For $x \in \mathcal{D}_A(\theta,\infty)$, we define $g : \rho(-A) \setminus \{0\} \longrightarrow X$ by $g(z) := \frac{1}{z}A(z+A)^{-1}$. This function is bounded of order $|z|^{-1-\theta}$ at infinity in any sector Σ_ϕ, where $0 \leq \phi < \phi_A$, and imitating the above argument, we arrive at (3.18).

Assume now that all the hypotheses of the second part of the lemma hold. Then the function f introduced in the first part of the proof is defined and analytic in some region containing $\gamma^-_{\sigma,r}$ and all points "to the right of" this curve, except the origin. Letting $R > r$, deforming the part of $\gamma^-_{\sigma,r}$ that lies between $Re^{-i\sigma}$ and $Re^{i\sigma}$ into the circular path $C_R = \{Re^{-i\tau} \mid -\sigma \leq \tau \leq \sigma\}$, adding $2\pi i$ times the residue of f at the origin, and finally, letting $R \to \infty$, we see that

$$\int_{\gamma^+_{\sigma,r}} f(z)\,dz = -2\pi i A^{-1},$$

since the residue is $\lim_{z\to 0} zf(z) = \lim_{z\to 0}(z+A)^{-1} = A^{-1}$ by Theorem A.31.

Formula 3.20 is obtained by an analogous argument, using the function g. □

3.2.2 Uniqueness

We shall now prove that for any $y \in X$, $x = S_\lambda y$ is the only possible solution to the equation $Ax + Bx = y$, provided that A and B are resolvent commuting. We recall that

$$S = \frac{-1}{2\pi i}\int_\gamma (z+A)^{-1}(z-B)^{-1}\,dz.$$

Theorem 3.10 (Uniqueness). *Assume that $y \in X$ and that $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$ is a solution to the equation $Ax + Bx = y$, where A and B are non-negative linear operators such that $0 \in \rho(A) \cup \rho(B)$ and $\omega_A + \omega_B < \pi$. Then we have*

$$S(A,B)y = x + J(A,B)x,$$

where

(3.21)
$$J(A,B)x := \frac{-1}{2\pi i}\int_\gamma \left[(z+A)^{-1}(z-B)^{-1}(z+A)x - (z-B)^{-1}x\right]\,dz.$$

In particular, if A and B are resolvent commuting, then $J(A,B)$ vanishes, and the equation has at most *one solution x given by $x = S(A,B)y$.*

Proof. Assume that $y = Ax + Bx$. Let us put $D := (z+A)^{-1}(z-B)^{-1}$. We then have

$$\begin{aligned}
Dy &= (z+A)^{-1}(z-B)^{-1}\big((z+A)x + (B-z)x\big) \\
&= (z+A)^{-1}(z-B)^{-1}(z+A)x + (z+A)^{-1}x \\
&= -(z+A)^{-1}x + (z-B)^{-1}x \\
&\quad + \big\{(z+A)^{-1}(z-B)^{-1}(z+A)x - (z-B)^{-1}x\big\}
\end{aligned}$$

But $-(z+A)^{-1}x = \frac{1}{z}\big((z+A)^{-1}Ax - x\big)$ and $(z-B)^{-1}x = \frac{1}{z}\big(x + (z-B)^{-1}Bx\big)$, and we get

$$\begin{aligned}
(z+A)^{-1}(z-B)^{-1}y &= \frac{1}{z}(z+A)^{-1}Ax + \frac{1}{z}(z-B)^{-1}Bx \\
&\quad + \big\{(z+A)^{-1}(z-B)^{-1}(z+A)x - (z-B)^{-1}x\big\}
\end{aligned} \tag{3.22}$$

Since $0 \in \rho(A) \cup \rho(B)$ either A or B has a bounded inverse defined on all of X. If $0 \in \rho(A)$, then we take a curve γ of the form $\gamma = \gamma^-_{\sigma,r}$ and get

$$\frac{-1}{2\pi i}\int_\gamma (z+A)^{-1}Ax\frac{dz}{z} = x.$$

by Lemma 3.9. The same lemma also yields

$$\frac{-1}{2\pi i}\int_\gamma (z-B)^{-1}Bx\frac{dz}{z} = 0.$$

If, on the other hand, $0 \notin \rho(A)$, then $0 \in \rho(B)$, and by an argument analogous to the one used above we get

$$\frac{-1}{2\pi i}\int_\gamma (z+A)^{-1}Ax\frac{dz}{z} = 0$$

and

$$\frac{-1}{2\pi i}\int_\gamma (z-B)^{-1}Bx\frac{dz}{z} = x,$$

since in this case we may take $\gamma = \gamma^+_{\sigma,r}$. Thus, we have shown that the first three expressions in (3.22) are absolutely integrable over γ. Hence, also the term inside the curly brackets in the right hand member must be absolutely integrable over γ, i.e. the integral defining $Jx := J(A,B)x$ converges absolutely, and the equation $Sy = x + Jx$ holds.

In the resolvent commuting case the integrand in the definition of Jx vanishes, so that $x = Sy$. $\square$

Remark 3.11. The operator $J(\lambda + A, B)$ was introduced by Labbas and Terreni in [8]. They used it to obtain a *left inverse* to $\lambda + A + B$ in the non-commutative case, much in the spirit of Section 3.4, where we consider Grisvard's and Da Prato's method of obtaining a *right inverse*.

3.2.3 Existence

We shall proceed to prove that $x = Sy$ really is a solution to (3.1) when A and B are resolvent commuting. More precisely, we shall prove that $(A+B)Sy = y$ for $y \in \mathcal{D}(A) + \mathcal{D}(B)$ and, more generally, for $y \in \mathcal{D}_A(\theta, p) + \mathcal{D}_B(\theta, p)$, where $0 < \theta < 1$ and $1 \le p \le \infty$. Note that, as $\mathcal{D}(A) \subset \mathcal{D}_A(\theta, p) \subset \mathcal{D}_A(\theta) \subset \mathcal{D}_A(\theta, \infty)$ and $\mathcal{D}(B) \subset \mathcal{D}_B(\theta, p) \subset \mathcal{D}_B(\theta) \subset \mathcal{D}_B(\theta, \infty)$ for $1 \le p < \infty$, it suffices to consider the case $y \in \mathcal{D}_A(\theta, \infty) + \mathcal{D}_B(\theta, \infty)$. We first present the ideas of the proof through a sequence of formal calculations. Then we state and prove a number of lemmas that together yield a rigorous proof.

Assume that $y \in \mathcal{D}_A(\theta, \infty)$ or $y \in \mathcal{D}_B(\theta, \infty)$ and that $0 \in \rho(A)$. Using the identity $(z-B)^{-1}y = \frac{1}{z}\big((z-B)+B\big)(z-B)^{-1}y = \frac{1}{z}y + \frac{1}{z}B(z-B)^{-1}y$ we rewrite the integral defining S as follows.

$$\begin{aligned} Sy &= \frac{-1}{2\pi i}\int_\gamma \big((z+A)^{-1}y - (z+A)^{-1}B(z-B)^{-1}y\big)\frac{dz}{z} \\ &= A^{-1}y + \frac{1}{2\pi i}\int_\gamma (z+A)^{-1}B(z-B)^{-1}y\frac{dz}{z}, \end{aligned}$$

where we have used Proposition 1.10 to compute the integral of $\frac{1}{z}(z+A)^{-1}y$ over γ. Applying A to this new expression for S we get

$$\begin{aligned} ASy &= y + \frac{1}{2\pi i}\int_\gamma A(z+A)^{-1}B(z-B)^{-1}y\frac{dz}{z} \\ &= y + \frac{1}{2\pi i}B\int_\gamma \big((z+A)-z\big)(z+A)^{-1}(z-B)^{-1}y\frac{dz}{z} \\ &= y + \frac{1}{2\pi i}B\int_\gamma (z+A)^{-1}(z-B)^{-1}y\,dz - \frac{1}{2\pi i}B\int_\gamma (z-B)^{-1}y\frac{dz}{z} \\ &= y - BSy, \end{aligned}$$

where the last equality follows from (3.17) (applied to B) and the definition of S. Thus, $AS(A,B)y + BS(A,B)y = y$.

In case $0 \notin \rho(A)$, one has $0 \in \rho(B)$. Hence, the above argument shows that $AS(B,A)y + BS(B,A)y = y$. But since A and B are resolvent commuting, we have $S(B,A) = S(A,B)$. (The last equality also follows from the uniqueness result of Theorem 3.10).

These calculations will now be made precise. Actually, resolvent commutativity can be used to obtain an algebraic relation involving only A, B and S (which holds on X and not just on interpolation spaces between $\mathcal{D}(A)$ or $\mathcal{D}(B)$ and X).

Proposition 3.12. *Let A and B be nonnegative linear operators in a complex Banach space X such that $\omega_A + \omega_B < \pi$. Then, if $0 \in \rho(A)$,*

$$S = A^{-1} + \frac{-1}{2\pi i}\int_\gamma (z+A)^{-1}B(z-B)^{-1}\frac{dz}{z}. \tag{3.23}$$

If, in addition, A and B are resolvent commuting, then

$$S + BA^{-1}S = A^{-1}. \tag{3.24}$$

If $0 \in \rho(B)$ and A and B are resolvent commuting, then

$$S + AB^{-1}S = B^{-1}. \tag{3.25}$$

Proof. Assume first that $0 \in \rho(A)$. Then we may assume that the curve γ in the definition of S lies to the left of the broken line γ_{ω_B} (so that it crosses the real axis to the left of the origin). Using the identity $(z-B)^{-1}y = \frac{1}{z}\{(z-B)+B\}(z-B)^{-1}y = \frac{1}{z}y + \frac{1}{z}B(z-B)^{-1}y$ we obtain

$$\begin{aligned}(z+A)^{-1}(z-B)^{-1}y = {} & \frac{1}{z}(z+A)^{-1}y \\ & - \frac{1}{z}(z+A)^{-1}B(z-B)^{-1}y.\end{aligned} \tag{3.26}$$

By Proposition 1.10 we have

$$\frac{-1}{2\pi i}\int\limits_{\gamma}(z+A)^{-1}y\frac{dz}{z} = A^{-1}y.$$

Hence, we have proved (3.23).

As for the second part, still assuming $0 \in \rho(A)$, we note that if A and B are resolvent commuting, then

$$\begin{aligned}(z+A)^{-1}B(z-B)^{-1} &= B(z-B)^{-1}(z+A)^{-1} \\ &= B(z+A)^{-1}(z-B)^{-1}.\end{aligned}$$

Since B is closed, it can be moved in front of the integral in (3.23), and we obtain

$$S = A^{-1} + \frac{-1}{2\pi i}B\int\limits_{\gamma}(z+A)^{-1}(z-B)^{-1}\frac{dz}{z}.$$

Next, observe that, by the resolvent identity, $(z+A)^{-1} = -zA^{-1}(z+A)^{-1} + A^{-1}$. Inserting this into the last integral, and moving A^{-1} in front of the integral, we get

$$S = A^{-1} + \frac{1}{2\pi i}BA^{-1}\int\limits_{\gamma}\left[(z+A)^{-1}(z-B)^{-1} - \frac{1}{z}(z-B)^{-1}\right]dz.$$

But, by Proposition 1.10, $\int_\gamma (z-B)^{-1}y\frac{dz}{z} = 0$, so that (3.24) holds.

In case $0 \notin \rho(A)$, one has $0 \in \rho(B)$. Hence, $S(B,A) + AB^{-1}S(B,A) = B^{-1}$. But $S(B,A) = S(A,B)$, and formula (3.25) follows. ☐

Corollary 3.13. *Let A and B be resolvent commuting nonnegative operators in X such that $0 \in \rho(A) \cup \rho(B)$ and $\omega_A + \omega_B < \pi$. Then, if $y \in X$ and $Sy \in \mathcal{D}(A) \cup \mathcal{D}(B)$, we have $Sy \in \mathcal{D}(A) \cap \mathcal{D}(B)$ and $ASy + BSy = y$.*

Proof. Assume that $0 \in \rho(A)$ so that

$$S + BA^{-1}S = A^{-1}.$$

If $Sy \in \mathcal{D}(B)$, then $Sy \in \mathcal{D}(BA^{-1})$ and $BA^{-1}Sy = A^{-1}BSy$ by Proposition 1.10. Consequently, $Sy + A^{-1}BSy = A^{-1}y$. Therefore, Sy belongs to $\mathcal{D}(A)$, and $ASy + BSy = y$.

If $Sy \in \mathcal{D}(A)$, then , since $A^{-1}y \in \mathcal{D}(A)$, also $BA^{-1}Sy$ belongs to $\mathcal{D}(A)$. Hence, $ASy + ABA^{-1}Sy = y$. Now note that A commutes with $\mu + B$ for any $\mu \in \rho(-B)$. Take such a μ. Thus, $\mathcal{D}(BA) = \mathcal{D}((\mu + B)A) = \mathcal{D}(A(\mu + B))$, which implies that $\mathcal{D}(BA) = \mathcal{D}(AB) \cap \mathcal{D}(A)$, and $ABx = BAx$ for all $x \in \mathcal{D}(BA)$. Since $A^{-1}Sy \in \mathcal{D}(AB) \cap \mathcal{D}(A)$, we have $ABA^{-1}Sy = BSy$, and thus $ASy + BSy = y$.

The case $0 \in \rho(B)$ follows from the previous case, since $S(A, B) = S(B, A)$ so that we can simply let A and B change places in the above formulas. □

Using the preceding results we are now ready to prove that, in the resolvent commuting case, $(A + B)Sy = y$ for all y in the real interpolation spaces $\mathcal{D}_A(\theta, p)$, $\mathcal{D}_A(\theta)$, $\mathcal{D}_B(\theta, p)$ and $\mathcal{D}_B(\theta)$. Since $\mathcal{D}_L(\theta, p) \hookrightarrow \mathcal{D}_L(\theta) \hookrightarrow \mathcal{D}_L(\theta, \infty)$ for any closed linear operator L, any $\theta \in (0, 1)$, and any $p \in [1, \infty)$, it suffices to consider the spaces $\mathcal{D}_A(\theta, \infty)$ and $\mathcal{D}_B(\theta, \infty)$.

Theorem 3.14 (Existence). *Assume that A and B are resolvent commuting. If $y \in \mathcal{D}_A(\theta, \infty) + \mathcal{D}_B(\theta, \infty)$, where $0 < \theta < 1$, and if $x = Sy$, then $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$ and x is the unique solution to the equation $Ax + Bx = y$.*

Proof. The uniqueness of solutions x to the equation $Ax + Bx = y$ has been proved in Lemma 3.10. Thus, we only have to prove the existence part.

Let $y \in \mathcal{D}_A(\theta, \infty) \cup \mathcal{D}_B(\theta, \infty)$. We first prove that if $0 \in \rho(A)$ and $x = Sy$, then $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$ and $Ax + Bx = y$. By Proposition 3.12 we have

$$Sy = A^{-1}y + \frac{-1}{2\pi i} \int_{\gamma} (z + A)^{-1} B (z - B)^{-1} y \frac{dz}{z}. \tag{3.27}$$

Let us show that the assumption $y \in \mathcal{D}_A(\theta, \infty) \cup \mathcal{D}_B(\theta, \infty)$ implies that the integral in the above formula belongs to $\mathcal{D}(A)$, and that A can be applied under the integral sign. We have $A(z + A)^{-1} = I - z(z + A)^{-1}$ and $B(z - B)^{-1} = z(z - B)^{-1} - I$ so that

$$\| A(z + A)^{-1} \| \leq (1 + M_A(\sigma))$$

and

$$\| B(z - B)^{-1} \| \leq (1 + M_B(\pi - \sigma))$$

for z on γ_σ. Now, if $y \in \mathcal{D}_B(\theta, \infty)$, then Lemma 2.16 shows that

$$\| B(z - B)^{-1} y \| \leq (1 + 2M_B(\pi - \sigma)) [\, y \,]_{\mathcal{D}_B(\theta,\infty)} \cdot \frac{1}{|z|^\theta}.$$

on γ_σ, except at $z = 0$. Thus, $f(z) = (1/z)A(z+A)^{-1}B(z-B)^{-1}y$ is analytic on $\gamma_{\sigma,r}^-$, and is bounded of order $|z|^{-1-\theta}$ at infinity. Hence, it is absolutely integrable over $\gamma_{\sigma,r}^-$, and since A is closed we infer that

$$(3.28) \qquad A\int_\gamma (z+A)^{-1}B(z-B)^{-1}y\frac{dz}{z} = \int_\gamma A(z+A)^{-1}B(z-B)^{-1}y\frac{dz}{z}.$$

If $y \in \mathcal{D}_A(\theta,\infty)$, we first note that $A(z+A)^{-1}B(z-B)^{-1}y = B(z-B)^{-1}A(z+A)^{-1}y$ by resolvent commutativity, so that a perfectly analogous argument again shows that formula (3.28) holds. Thus, $Sy \in \mathcal{D}(A)$, and

$$(3.29) \qquad ASy = y + \frac{-1}{2\pi i}\int_\gamma A(z+A)^{-1}B(z-B)^{-1}y\frac{dz}{z}$$

for any $y \in \mathcal{D}_A(\theta,p) \cup \mathcal{D}_B(\theta,p)$. By Corollary 3.13, $ASy + BSy = y$.

If $0 \notin \rho(A)$, then $0 \in \rho(B)$, so that $AS(B,A)y + BS(B,A)y = y$ by the above argument. But $S(B,A) = S(A,B)$, and there is nothing more to prove in this case.

Finally, if $y = y_1 + y_2$, where $y_1 \in \mathcal{D}_A(\theta,\infty)$ and $y_2 \in \mathcal{D}_B(\theta,\infty)$, we put $x_i := Sy_i$ and get $x_i \in \mathcal{D}(A) \cap \mathcal{D}(B)$ and $Ax_i + Bx_i = y_i$ for $i = 1, 2$. Hence, $x := x_1 + x_2 = Sy$ is in $\mathcal{D}(A) \cap \mathcal{D}(B)$ and $Ax + Bx = (A+B)x_1 + (A+B)x_2 = y_1 + y_2 = y$. ∎

The following corollary states that if $\mathcal{D}(A) + \mathcal{D}(B)$ is dense in X, then $\overline{A+B}$ is an invertible linear operator with inverse $S(A,B) \in \mathcal{L}(X)$.

Corollary 3.15. *Let A and B be resolvent commuting nonnegative operators in a complex Banach space X, and assume that $\omega_A + \omega_B < \pi$ and $0 \in \rho(A) \cup \rho(B)$. Then $A + B$ is closable. If, in addition, $\mathcal{D}(A) + \mathcal{D}(D)$ is dense in X, then $0 \in \rho(\overline{A+B})$ and $\overline{A+B}^{-1} = S(A,B)$.*

Proof. Let $\{x_n\}_{n=1}^\infty \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$ be such that $\lim_{n\to\infty} x_n = x$ and $Ax_n + Bx_n$ converges to some y in X. Then, by Theorem 3.14 and the continuity of S,

$$Sy = \lim_{n\to\infty} S(Ax_n + Bx_n) = \lim_{n\to\infty} x_n = x.$$

Hence, $A + B$ is closable and $S\overline{A+B}x = x$ whenever $x \in \mathcal{D}(\overline{A+B})$. It also follows that $\overline{A+B}$ is injective.

If $\mathcal{D}(A) + \mathcal{D}(B)$ is dense in X, then for any $y \in X$, we may find sequences $\{a_n\}_{n=1}^\infty \subseteq \mathcal{D}(A)$ and $\{b_n\}_{n=1}^\infty \subseteq \mathcal{D}(B)$ with $\lim_{n\to\infty}(a_n + b_n) = y$. Then the continuity of S implies that $\lim_{n\to\infty} S(a_n + b_n) = Sy$. We also have

$$\lim_{n\to\infty} (A+B)S(a_n + b_n) = \lim_{n\to\infty} a_n + b_n = y.$$

Consequently, $\overline{A+B}\,Sy = y$, showing that $\overline{A+B}$ is surjective. Since $\overline{A+B}$ is also injective, we conclude that $\overline{A+B}^{-1} = S(A,B)$. ∎

3.2.4 Maximal regularity

We continue to consider the equation $Ax + Bx = y$. In the previous section we proved that if $y \in \mathcal{D}_A(\theta, p)$, then there is a (unique) solution $x = Sy$, which then, of course, belongs to $\mathcal{D}(A) \cap \mathcal{D}(B) \subseteq \mathcal{D}_A(\theta, p)$. The question is now: Do we also have $Ax, Bx \in \mathcal{D}_A(\theta, p)$? The answer to this question is affirmative, i.e., we have *maximal regularity.*

By the maximal regularity of the equation $Ax+Bx = y$ in X with respect to the space $Y \hookrightarrow X$ we mean that, whenever $y \in Y$ and $x \in X$ is a solution to the equation, we have $x, Ax, Bx \in Y$. We start by proving a useful lemma.

Lemma 3.16. *Let $D_{k,n}(z) := (z+A)^{-1}B^k(t+B)^{-n}(z-B)^{-1}$, where $n = 0, 1$; $k = 0, 1, \ldots, n+1$; and $z \in \rho(-A) \cap \rho(B)$. Let γ be a curve as in Lemma 3.4, and assume that γ crosses the real axis to the right of $z = -t$. Then if $k \leq n$, the following equation holds*

$$\int_\gamma D_{k,l}(z)\, y\, dz = \int_\gamma \frac{z^k}{z+t}(z+A)^{-1}(z-B)^{-1}y\, dz \quad (y \in X).$$

If $y \in \mathcal{D}_B(\theta, \infty)$, then we also have

$$\int_\gamma D_{n+1,n}(z)\, y\, dz = \int_\gamma \frac{z^n}{(z+t)^n}(z+A)^{-1}B(z-B)^{-1}y\, dz.$$

In case A and B are resolvent commuting, the above formulas may be written

$$B^k(t+B)^{-n}Sy = \frac{-1}{2\pi i}\int_\gamma \frac{z^k}{(z+t)^n}(z+A)^{-1}(z-B)^{-1}y\, dz,$$

if $k \leq n$, and

$$B^{n+1}(t+B)^{-n}Sy = \frac{-1}{2\pi i}\int_\gamma \frac{z^n}{(z+t)^n}(z+A)^{-1}B(z-B)^{-1}y\, dz,$$

respectively. In the right–hand members of these formulas the path of integration γ may be deformed into γ_σ, provided that $k \geq 1$ (whenever k is involved).

Proof. We have

$$\| D_{k,n}(z) \|_{\mathcal{L}(X)} \leq M_A(\sigma)(1 + M_B(\sigma))^k M_B(\sigma)^{n-k} M_B(\pi - \sigma) \cdot \| y \| \, t^{-(n-k)} |z|^{-2}$$

on γ for large $|z|$. Hence, $D_{k,n}(z)$ is absolutely integrable over γ.

In case $n = 0$ there is nothing new to prove. Let us now calculate the integral $\int_\gamma (z+A)^{-1}(t+B)^{-1}(z-B)^{-1}y\, dz$. By the resolvent identity,

$$(t+B)^{-1}(z-B)^{-1} = \frac{1}{z+t}(t+B)^{-1} + \frac{1}{z+t}(z-B)^{-1}. \tag{3.30}$$

Since $\frac{1}{z+t}(z+A)^{-1}(t+B)^{-1}y$ is analytic, as a function of z, on the whole of $\rho(-A)$ except for the point $-t$, and is bounded of order $|z|^{-2}$ on and to the right of γ as $|z| \to \infty$, we have

$$\int_\gamma \frac{1}{z+t}(z+A)^{-1}(t+B)^{-1}y\,dz = 0,$$

and the integral converges absolutely. Thus, also $\frac{1}{z+t}(z+A)^{-1}(z-B)^{-1}y$ is absolutely integrable over γ, and we have

$$\int_\gamma D_{0,1}(z)\,dz = \int_\gamma \frac{1}{z+t}(z+A)^{-1}(z-B)^{-1}y\,dz,$$

provided that γ passes to the right of $z=-t$. We also observe that $B(t+B)^{-1} = 1 - t(t+B)^{-1}$, so that we also obtain

$$\begin{aligned}\int_\gamma D_{1,1}(z)\,dz &= \int_\gamma \left(1 - \frac{t}{z+t}\right)(z+A)^{-1}(z-B)^{-1}y\,dz \\ &= \int_\gamma \frac{z}{z+t}(z+A)^{-1}(z-B)^{-1}y\,dz.\end{aligned}$$

It is obvious from the above argument that the particular choice of γ is unimportant, provided that γ passes to the right of $z=-t$. In fact,

$$\begin{aligned}&\left\| \frac{z}{z+t}(z+A)^{-1}(z-B)^{-1}y \right\| \\ &\qquad \le M_B(\pi-\sigma)\frac{1}{|z+t|}\,\|\,(z+A)^{-1}\,\|_{\mathcal{L}(X)}\,\|\,y\,\|\end{aligned}$$

which remains bounded as $z \to 0$ to the left of γ_σ if $0 \in \rho(A)$, and analogously if $0 \in \rho(B)$. Thus, the path of integration can even be deformed into γ_σ.

Assume that $y \in \mathcal{D}_B(\theta,\infty)$. Using formula (3.30), we get

$$\begin{aligned}B(t+B)^{-1}B(z-B)^{-1} &= B(1-t(t+B)^{-1})(z-B)^{-1} \\ &= B(z-B)^{-1} - \frac{t}{z+t}B(t+B)^{-1} \\ &\qquad - \frac{t}{z+t}B(z-B)^{-1} \\ &= -\frac{t}{z+t}B(t+B)^{-1} + \frac{z}{z+t}B(z-B)^{-1}.\end{aligned}$$

Now $\frac{t}{z+t}(z+A)^{-1}B(t+B)^{-1}y$ is analytic in z to the right of γ_{ω_A}, and is bounded of order $|z|^{-2}$ on and to the right of γ as $|z| \to \infty$, since

$$\left\| \frac{t}{z+t}(z+A)^{-1}B(t+B)^{-1}y \right\| \le \frac{M_A(\sigma)(1+N_B)\,|t|\,\|\,y\,\|}{|z+t|\;|z|},$$

where we have used the estimate $\| B(t+B)^{-1} \| = \| 1 - t(t+B)^{-1} \| \leq 1 + N_B$. Consequently,

$$\int_\gamma \frac{t}{z+t}(z+A)^{-1}B^{n+1}(t+B)^{-n-1}y\,dz = 0.$$

We now turn to the expression $\frac{z}{z+t}(z+A)^{-1}B(z-B)^{-1}y$. This expression defines an analytic function of z on $(\Sigma_{\phi_A} - \lambda) \setminus \overline{\Sigma}_{\omega_B}$, except for the point $z = -t$ if this point lies in that region. On $\gamma_\sigma \setminus \{0\}$, and thus on γ when $|z|$ is large, we get, using the assumption $y \in \mathcal{D}_B(\theta, \infty)$,

$$\left\| \frac{z}{z+t}(z+A)^{-1}B(z-B)^{-1}y \right\| \leq \frac{C \left\| \, |z|^\theta B(|z|-B)^{-1} \right\| \| y \|}{|z|^\theta \; |z+t|},$$

where C is a constant, which can be chosen to be $M_A(\sigma)(2M_B(\pi-\sigma)+1)$ (cf. Section 2.4 and Lemma 2.16). Thus, the expression is bounded of order $|z|^{-1-\theta}$ as $|z| \to \infty$. Hence, it is absolutely integrable over γ and the value of the integral is independent of the particular choice of γ.

Now $\| B(z-B)^{-1}y \| = \| y - z(z-B)^{-1}y \| \leq (1 + M_B(\pi - \sigma)) \| y \|$. If $0 \in \rho(A)$, then $\| z(z+A)^{-1} \|_{\mathcal{L}(X)}$ tends to 0 as $z \to 0$. If $0 \notin \rho(A)$, then γ lies in $\overline{\Sigma}_\sigma \setminus \{0\}$. But $\| z(z+A)^{-1} \|_{\mathcal{L}(X)} \leq M_A(\sigma)$ in this region. Hence, in both cases there is a constant C such that

$$\left\| \frac{z}{(z+t)}(z+A)^{-1}B(z-B)^{-1}y \right\| \leq C \| y \|$$

at $z = 0$ when γ is deformed into γ_σ. Consequently,

$$\int_\gamma D_{2,1}(z)\,y\,dz = \int_\gamma \frac{z}{z+t}(z+A)^{-1}B(z-B)^{-1}y\,dz.$$

If A and B are resolvent commuting $B(t+B)^{-1}$ commutes with $(z+A)^{-1}$. On $\mathcal{D}(B)$ the operators B and $(z+A)^{-1}$ commute. Since $B(t+B)^{-1}$ is bounded, and B is closed, both can also be moved out in front of the integral sign in the formulas we have just proved, and we obtain the last two formulas of the lemma. These observations complete the proof. $\square$

Remark 3.17. It is easy to show that the lemma just proved is true not just for $n = 0, 1$, but for any nonnegative integer n.

By Lemma 3.16 we have $BSy = \frac{-1}{2\pi i} \int_\gamma (z+A)^{-1}B(z-B)^{-1}y\,dz$ for any $y \in \mathcal{D}_B(\theta, \infty)$. From this it is not difficult to show the existence of a constant M such that $\| BSy \| \leq M \| y \|_{\mathcal{D}_B(\theta,\infty)}$ for any $y \in \mathcal{D}_B(\theta, \infty)$. Since $ASy = y - BSy$, we also have $\| ASy \| \leq (1+M) \| y \|_{\mathcal{D}_B(\theta,\infty)}$. However, it is not easy to find a nice expression for M, since the integration cannot be performed along a curve γ_σ passing through the origin. With the seminorms $[\ \ ]_{\mathcal{D}_B(\theta,\infty)}$ the situation is different.

Theorem 3.18. *Let A and B be resolvent commuting nonnegative operators with $0 \in \rho(A) \cup \rho(B)$ and $\omega_A + \omega_B < \pi$, and let $x = Sy$ be the unique solution to the equation $Ax + Bx = y$. Then if $y \in \mathcal{D}_B(\theta, p)$, where $(\theta, p) \in (0,1) \times [1, \infty]$, we have $Ax \in \mathcal{D}_B(\theta, p)$ and $Bx \in \mathcal{D}_A(\theta, p) \cap \mathcal{D}_B(\theta, p)$. If $y \in \mathcal{D}_B(\theta)$, then $Ax \in \mathcal{D}_B(\theta)$ and $Bx \in \mathcal{D}_A(\theta) \cap \mathcal{D}_B(\theta)$. Moreover, we have the following estimates*

$$[\, Ax \,]_{\mathcal{D}_B(\theta,p)} \leq (1 + c_1)[\, y \,]_{\mathcal{D}_B(\theta,p)} \tag{3.31a}$$

$$[\, Bx \,]_{\mathcal{D}_B(\theta,p)} \leq c_1[\, y \,]_{\mathcal{D}_B(\theta,p)} \tag{3.31b}$$

$$[\, Bx \,]_{\mathcal{D}_A(\theta,p)} \leq c_2[\, y \,]_{\mathcal{D}_B(\theta,p)}, \tag{3.31c}$$

where

(3.32a)

$$c_1 := c_1(\sigma, \theta; A, B) := \frac{1}{\pi} M_A(\sigma)(1 + 2\sin\left(\tfrac{\sigma}{2}\right) M_B(\pi - \sigma)) \int_0^\infty \frac{dt}{t^\theta \left|te^{i\sigma} + 1\right|},$$

(3.32b)

$$c_2 := c_2(\sigma, \theta; A, B) := \frac{1}{\pi} M_A(\sigma)(1 + 2\sin\left(\tfrac{\sigma}{2}\right) M_B(\pi - \sigma)) \int_0^\infty \frac{dt}{t^\theta \left|te^{i\sigma} - 1\right|}$$

for any $\sigma \in (\omega_B, \phi_A)$.

The fact that we also have $Bx \in \mathcal{D}_A(\theta, p)$, when $y \in \mathcal{D}_B(\theta, p)$, is a somewhat unexpected result referred to as *cross regularity*. We also note that if $y \in \mathcal{D}_A(\theta, p)$ or $y \in \mathcal{D}_A(\theta)$ we get perfectly analogous results, as $S(B, A) = S(A, B)$.

Proof. We shall estimate $[\, BSy \,]_{\mathcal{D}_B(\theta,p)}$, $[\, ASy \,]_{\mathcal{D}_B(\theta,p)}$ and $[\, BSy \,]_{\mathcal{D}_A(\theta,p)}$.

Let us start with $[\, BSy \,]_{\mathcal{D}_B(\theta,p)}$. According to Lemma 3.16, we have

$$B(t + B)^{-1} BSy = \frac{-1}{2\pi i} \int_{\gamma_\sigma} \frac{z}{z + t}(z + A)^{-1} B(z - B)^{-1} y \, dz, \tag{3.33}$$

i.e.

$$B(t + B)^{-1} BSy = \frac{-1}{2\pi i} \int_0^\infty \left[\frac{re^{2i\sigma}}{re^{i\sigma} + t} \psi_\sigma(r) + \frac{re^{-2i\sigma}}{re^{-i\sigma} + t} \psi_{-\sigma}(r) \right] \frac{dr}{r}, \tag{3.34}$$

for any $t > 0$, where

$$\psi_\sigma(r) := \psi_\sigma(r; A, B) := r(re^{i\sigma} + A)^{-1} B(re^{i\sigma} - B)^{-1} y. \tag{3.35}$$

Substituting ts for r in (3.34), we get

$$B(t + B)^{-1} BSy = \frac{-1}{2\pi i} \int_0^\infty \left[\frac{se^{2i\sigma}}{se^{i\sigma} + 1} \psi_\sigma(st) + \frac{se^{-2i\sigma}}{se^{-i\sigma} + 1} \psi_{-\sigma}(st) \right] \frac{ds}{s}. \tag{3.36}$$

Both terms of the integrand are of the form required in the hypotheses of Corollary A.18. It is also clear that

$$\| s^{-\theta} \frac{se^{\pm 2i\sigma}}{se^{\pm i\sigma}+1} \|_{L^1_*(0,\infty)} = \int_0^\infty \frac{ds}{s^\theta |s^{i\sigma}+1|} < \infty.$$

Regarding $\psi_{\pm\sigma}(t)$, we have, by Lemma 2.16, the estimate

$$\| \psi_{\pm\sigma}(t) \| \leq c(\sigma) \| B(t+B)^{-1} y \|, \tag{3.37}$$

where $c(\sigma := c(\sigma; A, B) := M_A(\sigma)((1 + 2\sin\left(\frac{\sigma}{2}\right) M_B(\pi - \sigma))$. It follows from the assumption $y \in \mathcal{D}_B(\theta, p)$ that

$$[\, y \,]_{\mathcal{D}_B(\theta,p)} = \| t^\theta B(t+B)^{-1} y \|_{L^p_*((0,\infty);X)} < \infty,$$

so that $\underline{t}^\theta \psi_{\pm\sigma}(\underline{t}; A; B) \in L^p_*((0,\infty); X)$ and

$$\| \underline{t}^\theta \psi_{\pm\sigma}(\underline{t}; A; B) \|_{L^p_*((0,\infty);X)} \leq c(\sigma; A, B) [\, y \,]_{\mathcal{D}_B(\theta,p)}. \tag{3.38}$$

Consequently, we can apply Corollary A.18, and get

$$[\, B(t+B)^{-1} BSy \,]_{\mathcal{D}_B(\theta,p)} \leq c_1(\sigma, \theta) [\, y \,]_{\mathcal{D}_B(\theta,p)}, \tag{3.39}$$

where

$$c_1(\sigma, \theta) := c_1(\sigma, \theta; A, B) := \frac{1}{\pi} c(\sigma; A, B) \int_0^\infty \frac{dt}{t^\theta |t^{i\sigma}+1|}.$$

In particular, if $y \in \mathcal{D}_B(\theta)$, then $y \in \mathcal{D}_B(\theta, \infty)$ and

$$\lim_{r\to\infty} r^\theta \left\| B(r+B)^{-1} y \right\| = 0.$$

Using (3.36) and (3.37), we get

$$t^\theta \left\| B(t+B)^{-1} Bx \right\| \leq \frac{1}{\pi} c(\sigma) \int_0^\infty \frac{s^{-\theta}}{|se^{i\sigma}+1|} (ts)^\theta \left\| B(ts+B)^{-1} y \right\| \, ds. \tag{3.40}$$

Since $(ts)^\theta \| B(ts+B)^{-1} y \| \leq [\, y \,]_{\mathcal{D}_B(\theta,p)}$, $(ts)^\theta \| B(ts+B)^{-1} y \|$ tends to 0 as $t \to \infty$ $(s > 0)$ and $\int_0^\infty s^{-\theta}(s+1)^{-1} \, ds < \infty$, we can apply the Dominated Convergence Theorem to formula (3.40) and obtain

$$\lim_{t\to\infty} t^\theta \left\| B(t+B)^{-1} Bx \right\| = 0,$$

i.e. $Bx \in \mathcal{D}_B(\theta)$.

Using the equality $Ax = y - Bx$, where $y, Bx \in \mathcal{D}_B(\theta, p)$, we immediately infer that $Ax \in \mathcal{D}_B(\theta, p)$, and that $Ax \in \mathcal{D}_B(\theta)$ if $y \in \mathcal{D}_B(\theta)$. Moreover, we get

$$\| Ax \|_{\mathcal{D}_B(\theta,p)} \leq (1 + c_1(\sigma, \theta; A, B)) \| y \|_{\mathcal{D}_B(\theta,p)} .$$

We also have $(t+A)^{-1}(z+A)^{-1} = \frac{1}{t-z}((z+A)^{-1} - (t+A)^{-1})$, so that

$$(t+A)^{-1}Bx = \frac{-1}{2\pi i} \int\limits_{\gamma_{\sigma,r}} \frac{1}{t-z}(z+A)^{-1}B(z-B)^{-1}y\,dz,$$

since

$$\int\limits_{\gamma_{\sigma,r}} \frac{1}{t-z}(t+A)^{-1}B(z-B)^{-1}y\,dz = (t+A)^{-1}B \int\limits_{\gamma_{\sigma,r}} \frac{1}{t-z}(z-B)^{-1}y\,dz = 0,$$

provided that we choose $r < t$. As $A(t+A)^{-1} = 1 - t(t+A)^{-1}$, we then obtain

$$A(t+A)^{-1}Bx = \frac{-1}{2\pi i} \int\limits_{\gamma_{\sigma,r}} \frac{z}{z-t}(z+A)^{-1}B(z-B)^{-1}y\,dz.$$

Now the path of integration can be deformed into γ_σ without changing the value of the integral, so that

$$A(t+A)^{-1}BSy = \frac{-1}{2\pi i} \int\limits_0^\infty \left[\frac{re^{2i\sigma}}{re^{i\sigma}-t}\psi_\sigma(r) + \frac{re^{-2i\sigma}}{re^{-i\sigma}-t}\psi_{-\sigma}(r)\right] \frac{dr}{r}, \tag{3.41}$$

where $\psi_\sigma(r; A, B)$ has been defined above. Again we can substitute ts for r, and get

$$A(t+A)^{-1}BSy = \frac{-1}{2\pi i} \int\limits_0^\infty \left[\frac{se^{2i\sigma}}{se^{i\sigma}-1}\psi_\sigma(st) + \frac{se^{-2i\sigma}}{se^{-i\sigma}-1}\psi_{-\sigma}(st)\right] \frac{ds}{s}. \tag{3.42}$$

Consequently, we can argue exactly as above to deduce

$$[\,Bx\,]_{\mathcal{D}_A(\theta,p)} \le c_2(\sigma,\theta)[\,y\,]_{\mathcal{D}_B(\theta,p)},$$

where

$$c_2(\sigma,\theta) := c_2(\sigma,\theta; A, B) := \frac{1}{\pi}c(\sigma; A, B) \int\limits_0^\infty \frac{dt}{t^\theta|t^{i\sigma}-1|}.$$

If $y \in \mathcal{D}_B(\theta)$, then $y \in \mathcal{D}_B(\theta,\infty)$ and

$$\lim_{r\to\infty} r^\theta \,\|\, B(r+B)^{-1}y \,\| = 0.$$

On account of (3.42) and (3.37), we have

$$t^\theta \,\|\, A(t+A)^{-1}BSy \,\| \le \frac{1}{\pi}c(\sigma) \int\limits_0^\infty \frac{1}{s^\theta\,|se^{i\sigma}-1|}(ts)^\theta \,\|\, B(ts+B)^{-1}y \,\|\, ds. \tag{3.43}$$

Since $(ts)^\theta \, \| \, B(ts+B)^{-1}y \, \| \le [\, y \,]_{\mathcal{D}_B(\theta,\infty)}$, $(ts)^\theta \, \| \, B(ts+B)^{-1}y \, \|$ tends to 0 as $t \to \infty$ $(s > 0)$, and

$$\int_0^\infty \frac{ds}{s^\theta \, |se^{i\sigma} - 1|} < \infty,$$

we can apply the Dominated Convergence Theorem to formula (3.43) and get

$$\lim_{t\to\infty} t^\theta \, \big\| \, A(t+A)^{-1}Bx \, \big\| = 0,$$

i.e. $Bx \in \mathcal{D}_A(\theta)$. $\square$

Let us assume that A and B are resolvent commuting and nonnegative with $\omega_A + \omega_B < \pi$. However, we now drop the assumption that either of the operators is positive. Let x be a solution to the equation $Ax + Bx = y$, where $y \in \mathcal{D}_B(\theta, p)$. Then, for any $\epsilon > 0$, we have $Ax + Bx + \epsilon x = y + \epsilon x$ and $y + \epsilon x \in \mathcal{D}_B(\theta, p)$ (since $x \in \mathcal{D}(B)$). Hence, $x = S_\epsilon(A,B)(y + \epsilon x)$, $Ax \in \mathcal{D}_B(\theta,p)$ and $Bx \in \mathcal{D}_A(\theta,p) \cap \mathcal{D}_B(\theta,p)$. In particular, if $y \in \mathcal{D}_B(\theta)$, then $Ax \in \mathcal{D}_B(\theta)$ and $Bx \in \mathcal{D}_A(\theta) \cap \mathcal{D}_B(\theta)$. We have

$$Bx = BS_\epsilon y + \epsilon BS_\epsilon x. \tag{3.44}$$

By (3.31b),

$$[\, BS_\epsilon x \,]_{\mathcal{D}_B(\theta,p)} \le c_1(\sigma, \theta; A_\epsilon, B) [\, x \,]_{\mathcal{D}_B(\theta,p)},$$

where $\omega_B < \sigma < \phi_A$ and

$$\begin{aligned} c_1(\sigma,\theta;A_\epsilon,B) &= \frac{1}{\pi} M_{A_\epsilon}(\sigma) \left(1 + 2\sin(\tfrac{\sigma}{2}) M_B(\pi - \sigma)\right) \int_0^\infty \frac{dt}{t^\theta \, |te^{i\sigma} + 1|} \\ &\le \frac{M_A(\sigma) \left(1 + 2\sin(\tfrac{\sigma}{2}) M_B(\pi - \sigma)\right)}{\pi \sin[\max\{\pi/2, \sigma\}]} \int_0^\infty \frac{dt}{t^\theta \, |te^{i\sigma} + 1|} \end{aligned}$$

Hence,

$$\lim_{\epsilon \downarrow 0} \epsilon [\, BS_\epsilon x \,]_{\mathcal{D}_B(\theta,p)} = 0. \tag{3.45}$$

To handle $[\, BS_\epsilon y \,]_{\mathcal{D}_B(\theta,p)}$, we first note that (3.36) implies

$$\begin{aligned} B(t+B)^{-1}BS_\epsilon y = \frac{-1}{2\pi i} \int_0^\infty \bigg[&\frac{se^{2i\sigma}}{se^{i\sigma}+1} \psi_\sigma(st; A_\epsilon, B) \\ &+ \frac{se^{-2i\sigma}}{se^{-i\sigma}+1} \psi_{-\sigma}(st; A_\epsilon, B) \bigg] \frac{ds}{s}, \end{aligned} \tag{3.46}$$

and $[\, BS_\epsilon y\,]_{\mathcal{D}_B(\theta,p)} \leq c(\sigma,\theta;A_\epsilon,B)[\,y\,]_{\mathcal{D}_B(\theta,p)}$. Clearly, by the above proof, the integral in the definition

$$\begin{aligned}\Psi(t;A,B) &:= \frac{-1}{2\pi i}\int\limits_{\gamma_\sigma} \frac{z}{z+t}(z+A)^{-1}B(z-B)^{-1}y\,dz \\ &= \frac{-1}{2\pi i}\int\limits_0^\infty \left[\frac{se^{2i\sigma}}{se^{i\sigma}+1}\psi_\sigma(st;A,B) + \frac{se^{-2i\sigma}}{se^{-i\sigma}+1}\psi_{-\sigma}(st;A,B)\right]\frac{ds}{s}\end{aligned}$$

is also convergent for all $t>0$, and

$$\|\,\underline{t}^\theta\Psi(\underline{t})\,\|_{L^p_*((0,\infty);X)} \leq c_1(\sigma,\theta;A,B)[\,y\,]_{\mathcal{D}_B(\theta,p)}$$

where we have simplified notation by writing $\Psi(t)$ instead of $\Psi(t;A,B)$. Now, by the resolvent identity,

$$\begin{aligned}&\psi_{\pm\sigma}(t;A_\epsilon,B) - \psi_{\pm\sigma}(t;A,B) \\ &\qquad = -\epsilon t(te^{\pm i\sigma}+\epsilon+A)^{-1}(te^{\pm i\sigma}+A)^{-1}B(te^{\pm i\sigma}-B)^{-1}y,\end{aligned}$$

so that

$$\begin{aligned}&\|\,\psi_{\pm\sigma}(t;A_\epsilon,B) - \psi_{\pm\sigma}(t;A,B)\,\| \\ &\qquad \leq M_A(\sigma)M_A^*(\sigma)\,(1+M_B(\pi-\sigma))\,\|\,y\,\|\frac{\epsilon}{|te^{i\sigma}+\epsilon|}.\end{aligned}$$

But

$$\left\|\,\frac{\epsilon\underline{t}^\theta}{\underline{t}e^{i\sigma}+\epsilon}\,\right\|_{L^p_*(0,\infty)} = \epsilon^\theta\left\|\,\frac{\underline{t}^\theta}{\underline{t}e^{i\sigma}+1}\,\right\|_{L^p_*(0,\infty)}.$$

Hence, formula A.7 of Corollary A.18 yields

$$\|\,\underline{t}^\theta B(\underline{t}+B)^{-1}BS_\epsilon y - \underline{t}^\theta\Psi(\underline{t})\,\|_{L^p_*((0,\infty);X)} \leq C\epsilon^\theta,$$

where

$$C := M_A(\sigma)M_A^*(\sigma)\,(1+M_B(\pi-\sigma))\,\|\,y\,\|\int\limits_0^\infty \frac{dt}{t^\theta\,|te^{i\sigma}+1|}\left\|\,\frac{\underline{t}^\theta}{\underline{t}e^{i\sigma}+1}\,\right\|_{L^p_*(0,\infty)},$$

so that we finally get

$$\lim_{\epsilon\to 0}\|\,\underline{t}^\theta B(\underline{t}+B)^{-1}BS_\epsilon y - \underline{t}^\theta\Psi(\underline{t})\,\|_{L^p_*((0,\infty);X)} = 0$$

which tends to 0 as $\epsilon\downarrow 0$. Consequently,

$$\lim_{\epsilon\downarrow 0}[\,BS_\epsilon y\,]_{\mathcal{D}_B(\theta,p)} = \|\,\underline{t}^\theta\Psi(\underline{t})\,\|_{L^p_*((0,\infty);X)} \leq c_1(\sigma,\theta;A,B)[\,y\,]_{\mathcal{D}_B(\theta,p)}. \tag{3.47}$$

Equations (3.44), (3.45), and (3.47) together imply that

$$\begin{aligned}[\,Bx\,]_{\mathcal{D}_B(\theta,p)} &= \lim_{\epsilon\downarrow 0}[\,BS_\epsilon y\,]_{\mathcal{D}_B(\theta,p)} + \lim_{\epsilon\downarrow 0}\epsilon[\,BS_\epsilon x\,]_{\mathcal{D}_B(\theta,p)} \\ &= \|\,\underline{t}^\theta\Psi(\underline{t})\,\|_{L^p_*((0,\infty);X)} \\ &\leq c_1(\sigma,\theta;A,B)[\,y\,]_{\mathcal{D}_B(\theta,p)}.\end{aligned}$$

Since $Ax = y - Bx$, we infer

$$[\,Ax\,]_{\mathcal{D}_B(\theta,p)} \le (1 + c_1(\sigma,\theta;A,B))[\,y\,]_{\mathcal{D}_B(\theta,p)}.$$

As for the cross regularity estimate, we have

$$[\,Bx\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)} \le [\,BS_\epsilon y\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)} + \epsilon[\,BS_\epsilon x\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)}. \tag{3.48}$$

By (3.31c),

$$[\,BS_\epsilon x\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)} \le c_2(\sigma,\theta;A_\epsilon,B),$$

where

$$\begin{aligned} c_2(\sigma,\theta;A_\epsilon,B) &= \frac{1}{\pi} M_{A\_eps}(\sigma)\left(1 + 2\sin(\tfrac{\sigma}{2})M_B(\pi-\sigma)\right)\int_0^\infty \frac{dt}{t^\theta\,|te^{i\sigma}-1|} \\ &\le \frac{M_A(\sigma)\left(1 + 2\sin(\tfrac{\sigma}{2})M_B(\pi-\sigma)\right)}{\pi\sin[\max\{\pi/2,\sigma\}]}\int_0^\infty \frac{dt}{t^\theta\,|te^{i\sigma}-1|}. \end{aligned}$$

Hence,

$$\lim_{\epsilon\downarrow 0}\epsilon[\,BS_\epsilon x\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)} = 0. \tag{3.49}$$

By (3.41),

$$\begin{aligned} A_\epsilon(t + A_\epsilon)^{-1}BSy = \frac{-1}{2\pi i}\int_0^\infty \Bigg[&\frac{re^{2i\sigma}}{re^{i\sigma}-t}\psi_\sigma(r;A_\epsilon,B) \\ &+ \frac{re^{-2i\sigma}}{re^{-i\sigma}-t}\psi_{-\sigma}(r;A_\epsilon,B)\Bigg]\frac{dr}{r}. \end{aligned}$$

Again the proof of Theorem 3.18 shows that

$$\begin{aligned} \Psi(-t) &= \frac{-1}{2\pi i}\int_{\gamma_\sigma} \frac{z}{z-t}(z+A)^{-1}B(z-B)^{-1}y\,dz \\ &= \frac{-1}{2\pi i}\int_0^\infty \left[\frac{se^{2i\sigma}}{se^{i\sigma}-1}\psi_\sigma(st;A,B) + \frac{se^{-2i\sigma}}{se^{-i\sigma}-1}\psi_{-\sigma}(st;A,B)\right]\frac{ds}{s} \end{aligned}$$

is convergent for all $t > 0$, and

$$\|\,\underline{t}^\theta\Psi(-\underline{t})\,\|_{L^p_*((0,\infty);X)} \le c_2(\sigma,\theta;A,B)[\,y\,]_{\mathcal{D}_B(\theta,p)},$$

where $c_2(\sigma,\theta;A,B)$ is given by (3.32b). Arguing as above, we then infer that

$$\|\,\underline{t}^\theta B(\underline{t}+B)^{-1}BS_\epsilon y - \underline{t}^\theta\Psi(-\underline{t})\,\|_{L^p_*((0,\infty);X)} \le C'\epsilon^\theta,$$

where

$$C' := M_A(\sigma)M_A^*(\sigma)\,(1 + M_B(\pi - \sigma))\,\|\,y\,\| \int_0^\infty \frac{dt}{t^\theta\,|te^{i\sigma} - 1|} \left\| \frac{t^\theta}{te^{i\sigma} + 1} \right\|_{L_*^p(0,\infty)}.$$

We conclude that

$$(3.50)\quad \lim_{\epsilon\downarrow 0} [\, BS_\epsilon y\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)} = \|\, \underline{t}^\theta \Psi(-\underline{t})\,\|_{L_*^p((0,\infty);X)} \leq c_2(\sigma,\theta;A,B)[\,y\,]_{\mathcal{D}_B(\theta,p)}.$$

Proposition 2.25 and equations (3.48), (3.49), and (3.50) together yield

$$\begin{aligned}
[\,Bx\,]_{\mathcal{D}_A(\theta,p)} &= \lim_{\epsilon\downarrow 0} [\,Bx\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)} \\
&= \lim_{\epsilon\downarrow 0} [\,BS_\epsilon y\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)} + \lim_{\epsilon\downarrow 0} \epsilon[\,BS_\epsilon x\,]_{\mathcal{D}_{A_\epsilon}(\theta,p)} \\
&= \|\,\underline{t}^\theta \Psi(-\underline{t})\,\|_{L_*^p((0,\infty);X)} \\
&\quad \leq c_2(\sigma,\theta;A,B)[\,y\,]_{\mathcal{D}_B(\theta,p)}.
\end{aligned}$$

Summing up these observations, we can state the following corollary to Theorem 3.18:

Corollary 3.19. *Let A and B be resolvent commuting nonnegative operators with spectral angles ω_A and ω_B, respectively, satisfying the inequality $\omega_A + \omega_B < \pi$. Assume that $Ax + Bx = y$. Then if $y \in \mathcal{D}_B(\theta,p)$ we have $Ax \in \mathcal{D}_B(\theta,p)$ and $Bx \in \mathcal{D}_A(\theta,p) \cap \mathcal{D}_B(\theta,p)$ for all $(\theta,p) \in (0,1) \times [1,\infty]$. Analogously, if $y \in \mathcal{D}_B(\theta)$ then $Ax \in \mathcal{D}_B(\theta)$ and $Bx \in \mathcal{D}_A(\theta) \cap \mathcal{D}_B(\theta)$. In addition, we have the following estimates*

$$\begin{aligned}
[\,Ax\,]_{\mathcal{D}_B(\theta,p)} &\leq (1 + c_1)[\,y\,]_{\mathcal{D}_B(\theta,p)} \\
[\,Bx\,]_{\mathcal{D}_B(\theta,p)} &\leq c_1[\,y\,]_{\mathcal{D}_B(\theta,p)} \\
[\,Bx\,]_{\mathcal{D}_A(\theta,p)} &\leq c_2[\,y\,]_{\mathcal{D}_B(\theta,p)},
\end{aligned}$$

where c_1 and c_2 are as in Theorem 3.18.

Remark 3.20. The above proof of the corollary avoids a mistake in [6], where it was wrongly assumed that $\lim_{\epsilon\downarrow 0} M_{B_\epsilon}(\phi) = M_B(\phi)$.

3.3 The operator S_λ revisited

We have seen above that if A and B are nonnegative, if $\omega_A + \omega_B < \pi$ and if $|\arg\lambda| < \min(\phi_A, \phi_B)$, then $A_\lambda = \lambda + A$ and $B_\lambda = \lambda + B$ are positive, and we have $\omega_{A_\lambda} + \omega_B < \pi$ and $\omega_A + \omega_{B_\lambda} < \pi$. Hence, the above results on $S(A,B)$ and the equation $Ax + Bx = y$ apply to the pairs (A_λ, B) and (A, B_λ). Consequently, $S_\lambda(A,B) = S(A_\lambda, B) = S(A, B_\lambda)$ is the unique solution to the equation $\lambda x + Ax + Bx = y$ for any $y \in \mathcal{D}_A(\theta,\infty) + \mathcal{D}_B(\theta,\infty)$ $(\theta \in (0,1))$.

3.3.1 Additional regularity of S_λ

We shall prove a result on the range of the bounded operator S_λ, which strengthens Proposition 3.7.

Theorem 3.21. *Let A and B be resolvent commuting and let $\lambda > 0$. Then S_λ is a bounded linear operator mapping X into $\mathcal{D}_A(\theta,p) \cap \mathcal{D}_B(\theta,p)$ for any $\theta \in (0,1)$ and any $p \in [1,\infty]$, and we have*

$$\| S_\lambda x \|_X \le m_0 \lambda^{-1} \| x \|_X \,, \tag{3.51}$$

$$[\, S_\lambda x \,]_{\mathcal{D}_A(\theta,p)} \le m_1 \lambda^{-(1-\theta)} \| x \|_X \,, \tag{3.52}$$

$$[\, S_\lambda x \,]_{\mathcal{D}_B(\theta,p)} \le m_2 \lambda^{-(1-\theta)} \| x \|_X \,, \tag{3.53}$$

where

$$m_0 := \frac{2}{\pi} M_A^*(\sigma) M_B(\pi - \sigma) \int_0^\infty \frac{dt}{|t^2 e^{2i\sigma} - 1|}, \tag{3.54}$$

$$m_1 := \frac{2}{\pi} M_A(\sigma) M_B^*(\pi - \sigma) \left\| \frac{t^{1-\theta}}{te^{i\sigma} - 1} \right\|_{L^1_*} \left\| \frac{t^\theta}{te^{i\sigma} - 1} \right\|_{L^p_*}, \tag{3.55}$$

$$m_2 := \frac{2}{\pi} M_A^*(\sigma) M_B(\pi - \sigma) \left\| \frac{t^{1-\theta}}{te^{i\sigma} + 1} \right\|_{L^1_*} \left\| \frac{t^\theta}{te^{i\sigma} + 1} \right\|_{L^p_*}. \tag{3.56}$$

Proof. The estimate (3.51) has already been stated and proved in Proposition 3.7. Let us now show that (3.53) holds for all $x \in X$.

By Lemma 3.16 we have

$$B(t+B)^{-1} S_\lambda x = \frac{-1}{2\pi i} \int_{\gamma_\sigma} \frac{z}{z+t} (z + \lambda + A)^{-1} (z - B)^{-1} x \, dz.$$

On $\gamma_\sigma \setminus \{0\}$ we have $z = re^{\pm i\sigma}$, and the following estimate holds

$$\left\| \frac{z}{z+t} (z + \lambda + A)^{-1} (z - B)^{-1} x \right\| \le \frac{M_A^*(\sigma) M_B(\pi - \sigma)}{|re^{i\sigma} + t| \, |re^{i\sigma} + \lambda|} \| x \| ,$$

where we have used the fact that $|e^{-ia} + b| = |e^{ia} + b|$ for all $a, b \in \mathbb{R}$. Hence, writing $c := \frac{1}{\pi} M_A^*(\sigma) M_B(\pi - \sigma)$, we get

$$\begin{aligned} \| B(t+B)^{-1} S_\lambda x \| &\le c \| x \| \int_0^\infty \frac{dr}{|re^{i\sigma} + t| \, |re^{i\sigma} + \lambda|} \\ &= c \| x \| \int_0^\infty \frac{s}{|se^{i\sigma} + 1|} \frac{1}{|tse^{i\sigma} + \lambda|} \frac{ds}{s}. \end{aligned}$$

We now take $f(t) := t \, |te^{i\sigma} + 1|^{-1}$ and $g(t) := |te^{i\sigma} + \lambda|^{-1}$ in Corollary A.18 and deduce the following inequality

$$[\, S_\lambda x \,]_{\mathcal{D}_B(\theta,p)} \le c \left\| \frac{t^{1-\theta}}{(te^{i\sigma} + 1)} \right\|_{L^1_*} \left\| \frac{t^\theta}{te^{i\sigma} + \lambda} \right\|_{L^p_*} \| x \| .$$

But

$$\left\| \frac{t^\theta}{te^{i\sigma}+\lambda} \right\|_{L^p_*} = \lambda^{-(1-\theta)} \left\| \frac{t^\theta}{te^{i\sigma}+1} \right\|_{L^p_*},$$

and we thus have $[\, S_\lambda x \,]_{\mathcal{D}_B(\theta,p)} \le m_2 \lambda^{-(1-\theta)} \|\, x \,\|$, where

$$m_2 = \frac{1}{\pi} M_A^*(\sigma) M_B(\pi-\sigma) \left\| \frac{t^{1-\theta}}{te^{i\sigma}+1} \right\|_{L^1_*} \left\| \frac{t^\theta}{te^{i\sigma}+1} \right\|_{L^p_*}.$$

We have shown that $S_\lambda(A,B)$ maps X continuously into $\mathcal{D}_B(\theta,p)$ for any $\theta \in (0,1)$ and any $p \in [1,\infty]$. Since $S_\lambda(A,B) = S_\lambda(B,A)$, it follows that $S_\lambda(A,B)$ maps X continuously into $\mathcal{D}_A(\theta,p)$. Moreover, we have $\omega_A < \pi - \sigma < \phi_B$, so that, interchanging the rôles of A and B and replacing σ by $\pi-\sigma$ in formula (3.56), we see that we can choose

$$m_1 := m_2(B,A) = \frac{2}{\pi} M_A(\sigma) M_B^*(\pi-\sigma) \left\| \frac{t^{1-\theta}}{te^{i\sigma}-1} \right\|_{L^1_*} \left\| \frac{t^\theta}{te^{i\sigma}-1} \right\|_{L^p_*}$$

(where the meaning of the arguments A and B of m_2 should be obvious). This ends the proof of the theorem. □

Remark 3.22. One might wonder why the estimates allow $\|\, S_\lambda x \,\|_X$ to grow faster than $[\, S_\lambda x \,]_{\mathcal{D}_B(\theta,p)}$ and $[\, S_\lambda x \,]_{\mathcal{D}_A(\theta,p)}$ as λ decreases. Note that this is, however, supported by the fact that if $A = B = 0$, then $S_\lambda = (1/\lambda)I$, so that $\|\, S_\lambda \,\| = 1/|\lambda|$, whereas $[\, S_\lambda x \,]_{\mathcal{D}_A(\theta,p)} = [\, S_\lambda x \,]_{\mathcal{D}_B(\theta,p)} = 0$ for all $x \in X$.

3.3.2 Regularity constants for the equation $\lambda x + Ax + Bx = y$

When applied to A_λ and B, Theorem 3.18 yields some of the statements of the following theorem on the maximal regularity of the equation $\lambda x + Ax + Bx = y$.

Theorem 3.23. *Let A and B be resolvent commuting nonnegative operators with $\omega_A + \omega_B < \pi$, let $\lambda > 0$, let $\omega_B < \sigma < \phi_A$ and let $x = S_\lambda y$. If $y \in \mathcal{D}_B(\theta,p)$, where $(\theta,p) \in (0,1) \times [1,\infty]$, then $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$, x is the unique solution to the equation $\lambda x + Ax + Bx = y$, and we have $Ax \in \mathcal{D}_B(\theta,p)$ and $Bx \in \mathcal{D}_A(\theta,p) \cap \mathcal{D}_B(\theta,p)$. If $y \in \mathcal{D}_B(\theta)$, where $\theta \in (0,1)$, then $Ax \in \mathcal{D}_B(\theta)$ and $Bx \in \mathcal{D}_A(\theta) \cap \mathcal{D}_B(\theta)$. Moreover,*

$$\begin{array}{ll}
\|\, x \,\| \le m_0 \lambda^{-1} \|\, y \,\| & [\, x \,]_{\mathcal{D}_B(\theta,p)} \le c_0 \lambda^{-1} [\, y \,]_{\mathcal{D}_B(\theta,p)} \\
\|\, Ax \,\| \le (1+m_0) \|\, y \,\| + \tilde{c}_1 \lambda^{-\theta} [\, y \,]_{\mathcal{D}_B(\theta,p)} & [\, Ax \,]_{\mathcal{D}_B(\theta,p)} \le (1 + c_0 + c_1') \\
\|\, Bx \,\| \le \tilde{c}_1 \lambda^{-\theta} [\, y \,]_{\mathcal{D}_B(\theta,p)} & \qquad \cdot [\, y \,]_{\mathcal{D}_B(\theta,p)} \\
[\, Bx \,]_{\mathcal{D}_{A_\lambda}(\theta,p)} \le c_2' [\, y \,]_{\mathcal{D}_B(\theta,p)} & [\, Bx \,]_{\mathcal{D}_B(\theta,p)} \le c_1' [\, y \,]_{\mathcal{D}_B(\theta,p)}
\end{array}$$

Here m_0 is given by Proposition 3.7; $c_1' := c_1/\sin\sigma$; $c_2' := c_2/\sin\sigma$, with c_1 and c_2 given by Theorem 3.18;

$$c_0 := \frac{M_A^*(\sigma)(1 + 2\sin(\frac{\sigma}{2}) M_B(\pi-\sigma)}{\pi \sin\sigma} \left\| \frac{t^{1-\theta}}{te^{i\sigma}+1} \right\|_{L^1_*}$$

$$\tilde{c}_1 := \frac{M_A^*(\sigma)(1 + 2\sin(\frac{\sigma}{2}) M_B(\pi-\sigma))}{\pi \sin\sigma} \left\| \frac{t^{1-\theta}}{te^{i\sigma}+1} \right\|_{L^q_*},$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Since $S_\lambda(A,B)=S_\lambda(B,A)$, interchanging the rôles of A and B in the above theorem yields a set of analogous estimates.

Proof. By Lemma 2.25, $\mathcal{D}_{A_\lambda}(\theta,p)=\mathcal{D}_A(\theta,p)$ and $\mathcal{D}_{A_\lambda}(\theta)=\mathcal{D}_A(\theta)$. Therefore, all the statements of the theorem, except the estimates, follow directly from Theorem 3.18.

By Proposition 3.7 we have $\|\lambda x\|_X\le m_0\|y\|_X$.

By Lemma 3.16

$$B(t+B)^{-1}x=\frac{1}{2\pi i}\int\limits_\gamma \frac{1}{z+t}(z+\lambda+A)^{-1}B(z-B)^{-1}y\,dz. \tag{3.57}$$

On $\gamma_\sigma\setminus\{0\}$ we have $z=re^{\pm i\sigma}$, and by Lemma 2.16 we get

$$\left\|\frac{1}{z+t}(z+\lambda+A)^{-1}B(z+B)^{-1}y\right\|\le\frac{c}{|re^{i\sigma}+t|\,|re^{i\sigma}+\lambda|}\|B(r+B)^{-1}y\|,$$

where $c:=M_A^*(\sigma)(1+2\sin(\frac{\sigma}{2})M_B(\pi-\sigma))$. This shows that the integrand remains bounded in $\rho(-A)\cap\rho(B)$ as $z\to 0$. The path of integration may consequently be deformed into γ_σ. Combining these results, and using the fact that $\|re^{i\sigma}+\lambda\|\ge\lambda\sin\sigma$, we get

$$\begin{aligned}\|B(t+B)^{-1}x\|&=\left\|\frac{1}{2\pi i}\int\limits_{\gamma_\sigma}\frac{1}{z+t}(z+\lambda+A)^{-1}B(z-B)^{-1}y\,dz\right\|\\ &\le\frac{c}{2\pi}\int\limits_{\gamma_\sigma}\frac{\|B(t+B)^{-1}y\|}{|z+\lambda|\,|z+t|}\,d|z|\\ &=\frac{c}{\pi\lambda\sin\sigma}\int\limits_0^\infty\frac{r}{|re^{i\sigma}+t|}\|B(t+B)^{-1}y\|\frac{dr}{r}\\ &=\frac{c}{\pi\lambda\sin\sigma}\int\limits_0^\infty\frac{s}{se^{i\sigma}+1}\|B(ts+B)^{-1}y\|\frac{ds}{s},\end{aligned}$$

where the last equality was obtained by substituting ts for r in the integral. Taking $f(t):=\frac{t}{te^{i\sigma}+1}$ and $g(t):=\|B(t+B)^{-1}y\|$ in Corollary A.18,we obtain

$$\left\|t^\theta\|B(t+B)^{-1}x\|_X\right\|_{L_*^p}\le c_0\lambda^{-1}\left\|t^\theta\|B(t+B)^{-1}y\|_X\right\|_{L_*^p}, \tag{3.58}$$

i.e. $[x]_{\mathcal{D}_B(\theta,p)}\le c_0\lambda^{-1}[y]_{\mathcal{D}_B(\theta,p)}$, where c_0 is given in the theorem.

Let us now estimate $\|Bx\|$. By Lemma 3.16 we have

$$Bx=\frac{-1}{2\pi i}\int_{\gamma_\sigma}(z+\lambda+A)^{-1}B(z-B)^{-1}y\,dz.$$

Hence, using Hölder's inequality and the formula

$$\left\| \frac{r^{1-\theta}}{te^{i\sigma}+\lambda} \right\|_{L^q_*} = \lambda^{-\theta} \left\| \frac{t^{1-\theta}}{te^{i\sigma}+1} \right\|_{L^q_*},$$

we get

$$\begin{aligned} \| Bx \| &\le \frac{1}{\pi} M^*_A(\sigma)(1+2\sin\left(\tfrac{\sigma}{2}\right) M_B(\pi-\sigma)) \int_0^\infty \frac{t^{1-\theta}}{te^{i\sigma}+\lambda} \left\| t^\theta B(t+B)^{-1} y \right\| \frac{dt}{t} \\ &\le \tilde{c}_1 \lambda^{-\theta} [\, y\,]_{\mathcal{D}_B(\theta,p)}. \end{aligned}$$

By Theorem 3.18 we have

$$[\, BS_\lambda y\,]_{\mathcal{D}_B(\theta,p)} \le c' [\, y\,]_{\mathcal{D}_B(\theta,p)},$$

where $c' := \frac{1}{\pi} M_{\lambda+A}(\sigma)(1+2\sin(\frac{\sigma}{2}) M_B(\pi-\sigma)) \int_0^\infty \frac{dt}{t^\theta |te^{i\sigma}+1|}$. But $M_{\lambda+A}(\sigma) \le M^*_A(\sigma)/\sin\sigma$ by (2.53), whence

$$[\, BS_\lambda y\,]_{\mathcal{D}_B(\theta,p)} \le c'_1 [\, y\,]_{\mathcal{D}_B(\theta,p)}.$$

Since $Ax = y - \lambda x - Bx$, we also have

$$\| AS_\lambda y \| \le (1+m_0) \| y \| + \tilde{c}_1 \lambda^{-\theta} [\, y\,]_{\mathcal{D}_B(\theta,p)}$$

and

$$[\, AS_\lambda y\,]_{\mathcal{D}_B(\theta,p)} \le (1+c_0+c'_1) [\, y\,]_{\mathcal{D}_B(\theta,p)}.$$

Finally, by (3.52) we have $[\, Bx\,]_{\mathcal{D}_{A_\lambda}(\theta,p)} \le c [\, y\,]_{\mathcal{D}_B(\theta,p)}$, where

$$c := \frac{1}{\pi} M_{A_\lambda}(\sigma)(1+2\sin\left(\tfrac{\sigma}{2}\right) M_B(\pi-\sigma)) \int_0^\infty \frac{dt}{t^\theta \, |te^{i\sigma}-1|}$$

and $M_{A_\lambda}(\sigma) \le M^*_A(\sigma)/\sin\sigma$. □

Remark 3.24. The proof of Lemma 2.25 provides formulas that could be used to obtain cross regularity estimates that do not include $[\, Bx\,]_{\mathcal{D}_{A_\lambda}(\theta,p)}$ or $\| Bx \|_{\mathcal{D}_{A_\lambda}(\theta,p)}$.

Remark 3.25. The theorem could be generalised to apply to all $\lambda \in \mathbb{C}$ with $|\arg\lambda| < \min(\phi_A, \phi_B)$. In particular, there is a function c_0 such that $[\, S_\lambda y\,]_{\mathcal{D}_B(\theta,p)} \le c_0(|\arg\lambda|)\, |\lambda|^{-1} [\, y\,]_{\mathcal{D}_B(\theta,p)}$ for any $y \in \mathcal{D}_B(\theta,p)$. Hence, there is also a function m'_0 such that $\| S_\lambda \|_{\mathcal{L}(\mathcal{D}_B(\theta,p))} \le m'_0(|\arg\lambda|)\, |\lambda|^{-1}$.

3.3.3 The nonnegativity of $\overline{A+B}$

As a consequence of Proposition 2.23, we note that if A and B are nonnegative with $\omega_A + \omega_B < \pi$, and if $|\arg\lambda| < \min(\phi_A, \phi_B)$, then A_λ and B are resolvent commuting and nonnegative, A_λ is positive and $\omega_{A_\lambda} + \omega_B < \pi$. Hence, using Corollary 3.15 and Proposition 3.7, we arrive at the following theorem.

Theorem 3.26. *Let A and B be nonnegative resolvent commuting operators with $\omega_A + \omega_B < \pi$, and assume that $\mathcal{D}(A) + \mathcal{D}(B)$ is dense in X. Then $\overline{A+B}$ is nonnegative, $\omega_{\overline{A+B}} \le \max(\omega_A, \omega_B)$ (or, equivalently, $\phi_{\overline{A+B}} \ge \min(\phi_A, \phi_B)$) and $(\lambda + \overline{A+B})^{-1} = S_\lambda$ for all λ with $|\arg\lambda| < \min(\phi_A, \phi_B)$.*

Remark 3.27. The above theorem can be used to generalise the method of sums to sums of more than two nonnegative operators.

One may also consider the restrictions of the operators A and B that map into $\mathcal{D}_A(\theta, p)$ or into $\mathcal{D}_B(\theta, p)$, where $(\theta, p) \in (0,1) \times [1, \infty]$. Thus, let $\tilde{X} = \mathcal{D}_B(\theta, p)$, let $\tilde{A} = A|_{\tilde{X}}$ and let $\tilde{B} = B|_{\tilde{X}}$. Here $L|_Y$ is defined by putting

$$\begin{aligned} &\mathcal{D}(L|_Y) = \{x \in \mathcal{D}(L) \cap Y \mid Lx \in Y\} \\ &L|_Y x = Lx \qquad (x \in \mathcal{D}(L|_Y)), \end{aligned}$$

i.e. $L|_Y = L \cap (Y \times Y)$, for any linear operator L in X and any subspace Y of X. We have the following simple result that shows that $\tilde{A}$ and $\tilde{B}$ are nonnegative in $\tilde{X}$.

Lemma 3.28. *Let A and B be two resolvent commuting nonnegative linear operators in a complex Banach space X, let $\tilde{X} = \mathcal{D}_B(\theta, p)$, where $(\theta, p) \in (0,1) \times [1, \infty] \cup \{(0, \infty), (1, \infty)\}$, and put $\tilde{A} = A|_{\tilde{X}}$ and $\tilde{B} = B|_{\tilde{X}}$. Then $\tilde{A}$ is nonnegative, $\rho(-\tilde{A}) \supseteq \rho(-A)$, $\phi_{\tilde{A}} \ge \phi_A$, and $N_{\tilde{A}}(\phi) \le N_A(\phi)$ for all $\phi \in \Sigma_{\phi_B}$. If A is positive, then so is $\tilde{A}$. In particular, $\tilde{B}$ is nonnegative, $\phi_{\tilde{B}} \ge \phi_B$ and $N_{\tilde{B}}(\phi) \le N_B(\phi)$ for all $\phi \in \Sigma_{\phi_B}$. Moreover, $\tilde{A}$ and $\tilde{B}$ are resolvent commuting.*

Proof. Take $\lambda \in \rho(-A)$. It is clear that $\lambda + \tilde{A}$ is one–to–one. Let us show that

$$(\lambda + A)^{-1}(\mathcal{D}_B(\theta, p)) \subseteq \mathcal{D}_B(\theta, p).$$

In fact, for any $x \in \mathcal{D}_B(\theta, p)$ we have

$$\begin{aligned} t^\theta \left\| B(t+B)^{-1}(\lambda + A)^{-1}x \right\| &= t^\theta \left\| (\lambda + A)^{-1} B(t+B)^{-1} x \right\| \\ &\le \left\| (\lambda + A)^{-1} \right\|_{\mathcal{L}(X)} t^\theta \left\| B(t+B)^{-1}x \right\|. \end{aligned}$$

Hence, $[\,(\lambda + A)^{-1}x\,]_{\mathcal{D}_B(\theta,p)} \le \|\,(\lambda + A)^{-1}\,\|_{\mathcal{L}(X)}\,[\,x\,]_{\mathcal{D}_B(\theta,p)} < \infty$, so that $(\lambda + A)^{-1}x \in \mathcal{D}_B(\theta, p)$. As a consequence, we have $A(\lambda + A)^{-1}x = x - \lambda(\lambda + A)^{-1} \in \mathcal{D}_B(\theta, p)$. It follows that $x \in \mathcal{D}((\lambda + \tilde{A})^{-1})$, and thus $\mathcal{R}(\lambda + \tilde{A}) = \tilde{X}$ and $(\lambda + \tilde{A})^{-1}x = (\lambda + A)^{-1}x$ for all $x \in \tilde{X}$.

The above calculation also shows that $\rho(-\tilde{A}) \supseteq \rho(-A)$, and that

$$\left\| (\lambda + \tilde{A})^{-1} \right\|_{\mathcal{L}(\mathcal{D}_B(\theta,p))} \le \left\| (\lambda + \tilde{A})^{-1} \right\|_{\mathcal{L}(X)}$$

for all $\lambda \in \rho(-A)$. In particular, if $\lambda \in \Sigma_{\phi_A}$ we have

$$\left\| (\lambda + \tilde{A})^{-1} \right\|_{\mathcal{L}(\mathcal{D}_B(\theta,p))} \leq \frac{N_A(\arg\lambda)}{|\lambda|}$$

for all $\lambda \in \Sigma_{\phi_A}$.

Having a nonempty resolvent set, $\tilde{A}$ is closed.

Summing up what has been proved so far, we see that $\tilde{A}$ is nonnegative in $\tilde{X}$, $\rho(-\tilde{A}) \supseteq \rho(-A)$, $\phi_{\tilde{A}} \geq \phi_A$, and $N_{\tilde{A}}(\phi) \leq N_A(\phi)$ for any $\phi \in \Sigma_{\phi_A}$. As a special case we see that if A is positive, then so is $\tilde{A}$.

Since all resolvents of B commute with each other, the statements regarding $\tilde{B}$ follow.

Finally, it is clear that $\tilde{A}$ and $\tilde{B}$ are resolvent commuting, since

$$\begin{aligned}(\lambda + \tilde{A})^{-1}(\mu + \tilde{B})^{-1}x &= (\lambda + A)^{-1}(\mu + B)^{-1}x \\ &= (\mu + B)^{-1}(\lambda + A)^{-1}x \\ &= (\mu + \tilde{B})^{-1}(\lambda + \tilde{A})^{-1}x\end{aligned}$$

for all $x \in \tilde{X}$. □

We have seen that $x = S_\lambda(\tilde{A}, \tilde{B})y = S_\lambda(A, B)$ is the unique solution to the equation $\lambda x + \tilde{A}x + \tilde{B}x = y$ for any $y \in \tilde{X} = \mathcal{D}_B(\theta, p)$ and $\lambda \in \Sigma_{\min\{\phi_A,\phi_B\}}$. Thus, we have $S_\lambda(\tilde{A}, \tilde{B})y = (\lambda + \tilde{A} + \tilde{B})^{-1}y$ for such λ and y. If A or B is positive, then $S(\tilde{A}, \tilde{B})y = (\tilde{A} + \tilde{B})^{-1}y$ for all $y \in \tilde{X}$. For any ϕ with $|\phi| < \min\{\phi_A, \phi_B\}$ there is a number $m > 0$ such that $\arg\lambda = \phi$ and $\lambda \neq 0$ implies that

$$\left\| S_\lambda(\tilde{A}, \tilde{B}) \right\|_{\mathcal{L}(\tilde{X})} \leq \frac{m_0(\phi)}{|\lambda|},$$

by Proposition 3.7. From these observations we obtain the following Theorem.

Theorem 3.29. *Let A and B be resolvent commuting nonnegative operators with $\omega_A + \omega_B < \pi$, let $(\theta, p) \in (0,1) \times [1, \infty]$, and put $\tilde{X} = \mathcal{D}_B(\theta, p)$, $\tilde{A} = A|_{\tilde{X}}$ and $\tilde{B} = B|_{\tilde{X}}$. Then $\tilde{A} + \tilde{B}$ is nonnegative in $\tilde{X}$, $\phi_{\tilde{A}+\tilde{B}} \geq \min(\phi_A, \phi_B)$ and*

$$M_{\tilde{A}+\tilde{B}}(\phi) \leq m_0(\phi)$$

for all $\phi \in \mathbb{R}$ with $|\arg\phi| < \min(\phi_A, \phi_B)$, where $m_0(\phi)$ is given in Proposition 3.7. If, in addition, $0 \in \rho(A) \cup \rho(B)$, then $\tilde{A} + \tilde{B}$ is positive.

Remark 3.30. The theorem remains true if we take $\tilde{X} = \mathcal{D}_B(\theta)$. Since $S_\lambda(\tilde{A}, \tilde{B}) = S_\lambda(\tilde{B}, \tilde{A})$, it is also clear that we can take $\tilde{X} = \mathcal{D}_A(\theta, p)$ or $\tilde{X} = \mathcal{D}_A(\theta)$.

3.4 The non–commutative case

In this section we consider the equation $\lambda x + Ax + Bx = y$, where $\lambda > 0$, and the linear operators A and B are nonnegative with spectral angles ω_A and ω_B that satisfy the inequality $\omega_A + \omega_B < \pi$. However we do not assume the operators A and B to be resolvent commuting. The material is based on Section 6 of [10].

3.4.1 Introductory discussion

In the case where A and B are resolvent commuting we were able to show, among other things, that $(A+B)Sy = y$ for any y in $\mathcal{D}_B(\theta,\infty)$ (or in $\mathcal{D}_A(\theta,\infty)$), whenever $0<\theta<1$. This means that S_λ is a right inverse to $A+B$ on these interpolation spaces. Let us now look at the expression $(A+B)Sy$ without assuming A and B to be resolvent commuting. First we observe that if $0 \in \rho(A)$, then formula (3.29) is still valid for $y \in \mathcal{D}_B(\theta,\infty)$, where $0<\theta<1$, i.e.

$$ASy = y + \frac{-1}{2\pi i}\int_\gamma A(z+A)^{-1}B(z-B)^{-1}y\frac{dz}{z}.$$

For $z \in \rho(-A)$ we have the identity $A(z+A)^{-1} = I - z(z+A)^{-1}$, and, using Lemma 3.9 to calculate $\int_\gamma B(z-B)^{-1}y\frac{dz}{z} = 0$, we get

$$ASy = y - Uy, \tag{3.59}$$

where

$$Uy := U(A,B)y := \frac{-1}{2\pi i}\int_\gamma (z+A)^{-1}B(z-B)^{-1}y\,dz. \tag{3.60}$$

The last integral converges absolutely by Lemma 3.16.

In the resolvent commuting case considered above B commuted with all resolvents of A. Since B is closed we were allowed to move it in front of the integral in the right hand member of (3.59). Thus, in that case, we were able to infer that $ASy = y - BSy$. In the present case, these steps are not feasible. Instead we examine the difference

$$\begin{aligned} B(z+A)^{-1}(z-B)^{-1}y - &(z+A)^{-1}B(z-B)^{-1}y \\ &= [\,B\,;(z+A)^{-1}\,]\,(z-B)^{-1}y, \end{aligned} \tag{3.61}$$

which is well–defined provided that we have $(z+A)^{-1}(\mathcal{D}(B)) \subseteq \mathcal{D}(B)$ for every $z \in \rho(-A)$.[3] By Lemma 3.16 the second expression of the left hand member of (3.61) is absolutely integrable over γ whenever $y \in \mathcal{D}_B(\theta,p)$ for some $(\theta,p) \in (0,1)\times[1,\infty]$. Thus, the first expression is (absolutely) integrable , provided that the right hand member has this property. Since B is closed, it can be moved in front of the integral sign, and we get

$$BSy = Uy + Ry,$$

where

$$Ry := R(A,B)y := \frac{-1}{2\pi i}\int_\gamma [\,B\,;(z+A)^{-1}\,]\,(z-B)^{-1}y\,dz. \tag{3.62}$$

[3] Also note that $[\,B\,;(z+A)^{-1}\,]\,(z-B)^{-1} = (z+A)^{-1} - (z-B)(z+A)^{-1}(z-B)^{-1}$.

Therefore, using (3.59), we obtain

$$ASy + BSy = y + Ry.$$

However, for the above calculations to be valid, we must impose some conditions on $[\,B\,;(z+A)^{-1}\,]\,(z-B)^{-1}$ to make it integrable over some suitable curve γ.

Going one step further we realise that it would be useful to have R bounded with $\|\,R\,\| < 1$, rendering possible, under certain circumstances, the construction of the inverse $(1+R)^{-1}$ as a Neumann series. This would give $(A+B)S(1+R)^{-1}y = y$, i.e., we would have a right inverse to $(A+B)$. But of course we have no reason to expect the condition $\|\,R\,\| < 1$ to be satisfied very often. However, replacing A by $A_\lambda := \lambda + A$, where A is nonnegative, in the above argument, it seems plausible that the norm of $[\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}$ becomes small for large λ. Thus, it appears more likely that

$$R_\lambda := R(A_\lambda, B) = \frac{-1}{2\pi i}\int_\gamma [\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}\,dz. \tag{3.63}$$

exists with norm less than 1 for large λ.

3.4.2 An augmented set of hypotheses $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$

The above observations were first used by Da Prato and Grisvard in [10], p. 346, where they introduced a set of sufficient conditions for the line of reasoning sketched above to work out. We now present a somewhat simplified version of that set of conditions:

Definition 3.31. We say that A and B satisfy $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ if
(i) A and B are nonnegative operators with $\omega_A + \omega_B < \pi$,
(ii) σ_A and σ_B are positive numbers such that

$$\omega_B < \sigma_B \le \sigma_A < \phi_A,$$

(iii) $\mathcal{D}(B)$ is stable under resolvents of A, i.e.,

$$(z+A)^{-1}(\mathcal{D}(B)) \subseteq \mathcal{D}(B), \tag{3.64}$$

for all $z \in \overline{\Sigma}_{\sigma_A} \setminus \{0\}$,
and
(iv) The mapping $(z, w) \mapsto [\,B\,;(z+A)^{-1}\,]\,(w-B)^{-1}$ is continuous into $\mathcal{L}(X)$, and ψ is a measurable function

$$\psi :\, (0,\infty)\times(0,\infty) \longrightarrow [0,\infty)$$

such that the integral $\int_\gamma \psi(|z+\lambda|\,, |z|)\,d\,|z|$ is convergent for all curves $\gamma = -\lambda' + \gamma_\sigma$, where $0 < \lambda' < \lambda$ and $\sigma_B \le \sigma \le \sigma_A$, and we have

$$\lim_{\lambda\to\infty}\int_\gamma \psi(|z+\lambda|\,, |z|)\,d\,|z| = 0 \tag{3.65}$$

and

$$\| \, [\, B\, ; (z+A)^{-1} \,] \, (w-B)^{-1} \, \| \leq \psi(|z| \, , |w|) \tag{3.66}$$

for all $z, w \in \mathbb{C} \setminus \{0\}$ with $|\arg z| \leq \sigma_A$ and $|\arg w| \geq \sigma_B$.[4]

Note that if we fix a curve γ for some $\lambda = \lambda_0$, then this γ will do for all $\lambda \geq \lambda_0$, so that (3.65) makes sense.

In the rest of this chapter we usually assume that $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ holds, that $\lambda > 0$ and that γ is of the form $\gamma = \gamma_\sigma - \lambda'$, where $0 < \lambda' < \lambda$.

3.4.3 Some simple consequences of $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$

Assuming that $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ holds, we have

$$\| \, [\, B\, ; (z+\lambda+A)^{-1} \,] \, (z-B)^{-1} \, \| \leq \psi(|z+\lambda| \, , |z|),$$

for all $y \in X$ and all $z \in \big((\overline{\Sigma}_{\sigma_A} - \lambda) \setminus \Sigma_{\sigma_B}\big) \setminus \{0, -\lambda\}$, and, consequently, for all z on any curve $\gamma = \gamma_\sigma - \lambda'$ as described above. Let $y \in \mathcal{D}_B(\theta, \infty)$, where $0 < \theta < 1$. It follows by Definition 3.31 (iv) that the integral defining $R_\lambda y$ converges absolutely for such γ, and (3.4.1) holds with A and R replaced by A_λ and R_λ, respectively, i.e.

$$(\lambda + A + B) S_\lambda y = y + R_\lambda y. \tag{3.67}$$

We also have the estimate

$$\| \, R_\lambda \, \| \leq \frac{1}{2\pi} \int_\gamma \psi(|z+\lambda| \, , |z|) \, d\, |z| \, . \tag{3.68}$$

Since S_λ is bounded by Proposition 3.7, the above argument results in the following lemma.

Lemma 3.32. *Assume that A and B satisfy $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ for some $\sigma_A, \sigma_B \in (0, \pi)$ and $\psi : (0, \infty) \times (0, \infty) \longrightarrow [0, \infty)$, and let $0 < \theta < 1$. Then S_λ maps $\mathcal{D}_B(\theta, \infty)$ continuously into $\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$, and*

$$(\lambda + A + B) S_\lambda x = (1 + R_\lambda) x \tag{3.69}$$

for all $x \in \mathcal{D}_B(\theta, \infty)$, where the linear operator $R_\lambda : X \longrightarrow X$ is defined by (3.62)*. For $x \in \mathcal{D}_B(\theta, \infty)$ the particular choice of γ does not influence the value of $R_\lambda x$.*

As we have proved that $(\lambda + A + B) S_\lambda u = (1 + R_\lambda) u$ for $u \in \mathcal{D}_B(\theta, \infty)$, we could try to invert $1 + R_\lambda$ to get $(\lambda + A + B) S_\lambda (1 + R_\lambda)^{-1} y = y$, provided that $(1 + R_\lambda)^{-1}$ exists and $(1 + R_\lambda)^{-1} y \in \mathcal{D}_B(\theta, \infty)$ for some $\theta \in (0, 1)$. Now

[4] It will become clear below that (3.66) does not necessarily have to be satisfied for small $|z|$.

the inverse of $(1 + R_\lambda)$ can be constructed as a Neumann series when λ is large enough, because, as has been seen above,

$$\| R_\lambda \| \le \frac{1}{2\pi} \int_\gamma \psi(|z+\lambda|, |z|)\, d|z|,$$

and according to the assumption (3.65) there is some constant λ_0 such that $\| R_\lambda \| < \frac{1}{2}$ for $\lambda > \lambda_0$. In order to make this work out in all detail we would still need to establish that $(1 + R_\lambda)^{-1} y \in \mathcal{D}_B(\theta, \infty)$, for some $\theta \in (0, \infty)$, hopefully for all $y \in \mathcal{D}(A + B) = \mathcal{D}(A) \cap \mathcal{D}(B)$; only then can we apply formula (3.69). This would be a major restriction to the usefulness of the method.

At this point we may, nevertheless, draw the following conclusion from the previous lemma.

Corollary 3.33. *Assume that A and B satisfy $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ for some ψ, and that B is densely defined in X. Then there is some $\lambda_0 \ge 0$ such that the range of $\lambda + A + B$ is dense in X for all $\lambda > \lambda_0$.*

Proof. By Lemma 3.32

$$\mathcal{R}(\lambda + A + B) \supseteq (1 + R_\lambda)(\mathcal{D}_B(\theta, \infty)) \supseteq (1 + R_\lambda)(\mathcal{D}(B)), \tag{3.70}$$

and thus, since $1 + R_\lambda$ is bounded and bijective for λ large enough, the range $\mathcal{R}(\lambda + A + B)$ is dense in X for λ large enough, provided that $\mathcal{D}(B)$ is dense in X.[5] □

3.4.4 Inverting $\lambda + A + B_n$

We shall avoid the above described difficulty in inverting $\lambda + L = \lambda + A + B$ directly by means of a procedure where the operator B is replaced by its Yosida approximations $B_n := nB(n + B)^{-1}$ $(n = 1, 2, \ldots)$. We thus consider the operators $L_n := A + B_n$ defined on $\mathcal{D}(L_n) = \mathcal{D}(A) \cap \mathcal{D}(B_n) = \mathcal{D}(A)$. We intend to apply Corollary 1.7 to the sequence $\{L_n\}_{n=1}^\infty$. By Lemma 3.33 the hypothesis (ii) of that corollary is satisfied, i.e. $\mathcal{R}(\lambda + L)$ is dense in X, as soon as B is densely defined. Furthermore, we know that if B is densely defined, then $\{B_n x\}_{n=1}^\infty$ is convergent exactly when $x \in \mathcal{D}(B)$, and then $\lim_{n\to\infty} B_n x = Bx$. Consequently, $Ax + B_n x$ converges precisely when $x \in \mathcal{D}(L) = \mathcal{D}(A) \cap \mathcal{D}(B)$, and the limit is $Ax + Bx$ (see Proposition 2.6). Thus, we have

$$L = \lim_{n\to\infty} L_n \tag{3.71}$$

in the sense of Definition 1.5.

[5] When $L : X \longrightarrow X$ is bounded and bijective, $Y \subseteq X$ is dense in X, $x \in X$ and $\varepsilon > 0$ we can take $y \in Y$ such that $\| L^{-1}x - y \| \le \varepsilon / \| L \|$. Then $\| x - Ly \| \le \| L \| \| L^{-1}x - y \| < \varepsilon$. As such a $y \in Y$ can be found for any $x \in X$ and $\varepsilon > 0$ we conclude that $L(Y)$ is dense in X.

It remains to show that (i) of the corollary in question is satisfied, i.e. that there are constants $N \geq 1$ and $\lambda_0 \geq 0$ satisfying $\rho(-L_n) \supseteq (\lambda_0, \infty)$ and

$$\| (\lambda + L_n)^{-1} \| \leq \frac{N}{\lambda}$$

for all $\lambda > \lambda_0$ and $n = 1, 2, \ldots$. This is the object of this subsection. The idea is to apply the method suggested in Subsection 3.4.3 to $\lambda + A + B_n$. This will be possible since $\mathcal{D}(B_n) = X$.

First we show that A and B_n satisfy $\mathcal{H}(A, B_n, \sigma_A, \sigma_B, \hat{\psi})$ for some $\hat{\psi}$ and all $n \in \mathbb{N}$.

In Section 2.2 it was shown that the B_n are uniformly nonnegative: for all ϕ with $0 \leq \phi < \pi - \omega_B$ there is a $\hat{M}_B(\phi) > 0$ such that

$$\| (z + B_n)^{-1} \| \leq \frac{\hat{M}_B(\phi)}{|z|}$$

for all z with $|\arg z| \leq \phi$ and $n = 1, 2, \ldots$. From this it follows that $\omega_A + \omega_{B_n} < \pi$, and $\omega_{B_n} < \sigma_B \leq \sigma_A < \phi_A$.

The inclusion $(z+A)^{-1}(\mathcal{D}(B_n)) \subseteq \mathcal{D}(B_n)$ is trivially satisfied, as $\mathcal{D}(B_n) = X$.

In order to show that (iv) of Definition 3.31 holds with B replaced by B_n and ψ replaced by some $\hat{\psi}$, we assume that $|\arg z| \leq \sigma_A$ and $|\arg u| \geq \sigma_B$. We observe that $B_n = (-n^2 + n(n+B))(n+B)^{-1} = n - n^2(n+B)^{-1}$. We also have the identities $[\, z + P\,; Q\,] = [\, P\,; Q\,]$ and, if P is invertible (i.e. injective), $[\, P\,; Q\,]\, x = P\, [\, Q\,; P^{-1}\,]\, Px$, for all $x \in \mathcal{D}(PQ) \cap \mathcal{D}(P^{-1}QP)$. Hence,

$$\begin{aligned} [\, B_n\,; (z+A)^{-1}\,] &= -n^2\, [\, (n+B)^{-1}\,; (z+A)^{-1}\,] \\ &= -n^2 (n+B)^{-1}\, [\, (z+A)^{-1}\,; B\,]\, (n+B)^{-1}, \end{aligned}$$

where we have used the fact that $\mathcal{D}((n+B)(z+A)^{-1}(n+B)^{-1}) = X$ by (3.64). By the definition of B_n we also have $(u - B_n)(n+B) = nu + (u-n)B$, so that

$$(n+B)^{-1}(u - B_n)^{-1} = \frac{1}{u-n} \left(\frac{nu}{u-n} + B \right)^{-1}.$$

Moreover,

$$\begin{aligned} \left\{ \frac{nu}{u-n} + B \right\}^{-1} &= (u-B)^{-1}(u-B) \left(\frac{nu}{u-n} + B \right)^{-1} \\ &= -(u-B)^{-1} \left\{ 1 - \frac{u^2}{u-n} \left(\frac{nu}{u-n} + B \right)^{-1} \right\}. \end{aligned}$$

Combining these equalities, we obtain

(3.72)

$$\begin{aligned} [\, B_n\,; (z+A)^{-1}\,]\, (u - B_n)^{-1} = \frac{n^2}{u-n} &(n+B)^{-1}\, [\, B\,; (z+A)^{-1}\,]\, (u-B)^{-1} \\ &\cdot \left\{ 1 - \frac{u^2}{u-n} \left(\frac{nu}{u-n} + B \right)^{-1} \right\}. \end{aligned}$$

At this point we note that either $\arg(u-n) \geq \arg nu = \arg u \geq 0$ or $\arg(u-n) \leq \arg nu = \arg u \leq 0$, so that

$$\begin{aligned}
|\arg(nu/(u-n))| &= |\arg(u-n)| - |\arg u| \\
&\leq \pi - |\arg u| \\
&\leq \pi - \sigma_B .
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|\, [\, B_n \,;(z+A)^{-1}\,]\,(u-B_n)^{-1}\,\| \\
&\quad \leq \frac{n^2}{|u-n|}\frac{N_B}{n}\psi(|z|\,,|u|)\left\{1+\frac{|u|^2}{|u-n|}M_B^*(\pi-|\arg u|)\frac{|u-n|}{n\,|u|}\right\} \\
&\quad \leq \frac{n+|u|\,M_B^*(\pi-|\arg u|)}{|n-u|}N_B\psi(|z|\,,|u|) \\
&\quad \leq N_B\hat{M}_B(\pi-\sigma_B)\psi(|z|\,,|u|).
\end{aligned}$$

Hence, we can take $\hat{\psi} = N_B\hat{M}_B(\pi-\sigma_B)\psi$, and conclude that

$$\|\, [\, B_n \,;(z+A)^{-1}\,]\,(u-B_n)^{-1}\,\| \leq \hat{\psi}(|z|\,,|u|) \tag{3.73}$$

for all $z, u \in \mathbb{C}\setminus\{0\}$ with $|\arg z| \leq \sigma_A$ and $|\arg u| \geq \sigma_B$ and $n = 1, 2, \ldots$ (For the definition of $\hat{M}_A$, see formula (2.19)). Thus, we have the following lemma.

Lemma 3.34. *If $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ holds, then $\mathcal{H}(A, B_n, \sigma_A, \sigma_B, \hat{\psi})$ holds for $n = 1, 2, \ldots$, where $\hat{\psi} := N_B\hat{M}_B(\pi-\sigma_B)\psi$.*

We can now prove that $\lambda \in \rho(-(A+B_n))$ for large $n \in \mathbb{N}$, and, in fact, that the sequence $L_n := A + B_n$ satisfies condition (i) of Corollary 1.7.

Lemma 3.35. *Let A and B satisfy $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$, and let $L_n := A + B_n = A + nB(n+B)^{-1}$. Then there is a constant $\lambda_0 \geq 0$ and a constant $N > 0$ depending only on λ_0, A and B such that $(\lambda_0, \infty) \subseteq \rho(-L_n)$, and*

$$\|\,(\lambda + L_n)^{-1}\,\| \leq \frac{N}{\lambda} \tag{3.74}$$

for all $\lambda > \lambda_0$ and $n = 1, 2, \ldots$

Proof. Assume that $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ holds. By Lemma 3.34 there is some $\hat{\psi}$ such that $\mathcal{H}(A, B, \sigma_A, \sigma_B, \hat{\psi})$ holds for all $n \in \mathbb{N}$. Let us define

$$S_{n,\lambda} := S_\lambda(A, B_n) = \frac{-1}{2\pi i}\int_\gamma (z+\lambda+A)^{-1}(z-B_n)^{-1}\,dz,$$

and

$$R_{n,\lambda} := R_\lambda(A, B_n) = \frac{-1}{2\pi i}\int_\gamma\, [\, B_n \,;(z+\lambda+A)^{-1}\,]\,(z-B_n)^{-1}\,dz,$$

for $\lambda > 0$ We then have

$$(\lambda + L_n)S_{n,\lambda} = (1 + R_{n,\lambda}) \tag{3.75}$$

for all $\lambda > 0$ and $n \in \mathbb{N}$, by Lemma 3.32. Here we have used the fact that $\mathcal{D}(B_n) = X$.

By (2.18) and Proposition 3.7 there is a constant $m > 0$, with

$$\| S_{n,\lambda} \| \leq \frac{m}{\lambda} \tag{3.76}$$

for all $\lambda > 0$ and all $n \in \mathbb{N}$. Since A and B_n satisfy $\mathcal{H}(A, B, \sigma_A, \sigma_B, \hat{\psi})$ for all n, Lemma 3.34 shows that there is some λ_0 such that $\| R_{n,\lambda} \| \leq \frac{1}{2}$ for all $\lambda > \lambda_0$, $n = 1, 2, \ldots$ This implies that $(1 + R_{n,\lambda})^{-1}$ is defined on the whole of X, and

$$\left\| (1 + R_{n,\lambda})^{-1} \right\| \leq 2$$

for $\lambda > \lambda_0$, $n \in \mathbb{N}$. It follows that

$$(\lambda + L_n)^{-1} = S_{n,\lambda}(1 + R_{n,\lambda})^{-1}, \tag{3.77}$$

and

$$\left\| (\lambda + L_n)^{-1} \right\| \leq \frac{2m}{\lambda}$$

for all $\lambda > \lambda_0$ and all $n \in \mathbb{N}$. □

3.4.5 Inverting $\lambda + \overline{A + B}$

We have already verified the conditions of Corollary 1.7 for the operators $L_n = A + B_n$ and $L = A + B$, where $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ holds for some ψ and B is densely defined. Hence, there is some constant $\lambda_0 \geq 0$ such that $(\lambda_0, \infty) \subseteq \rho(-\overline{L})$, $\mathcal{R}(\lambda + L)$ is dense in X and

$$\lim_{n \to \infty} (\lambda + L_n)^{-1} x = (\lambda + \overline{L})^{-1} x$$

for any $\lambda > \lambda_0$ and any $x \in X$. Moreover, if A is also densely defined in X, then Definition 3.31 (iii) and Lemma 2.8 show that L is densely defined so that $L = A + B$ becomes closable, i.e. $\overline{L} = \overline{A + B}$ is a (closed) linear operator. We shall now show that, for large λ,

$$(\lambda + \overline{L})^{-1} x = S_\lambda (1 + R_\lambda)^{-1} x$$

for any $x \in X$. We begin by proving a simple result on the convergence of the operator sequences $S_{n,\lambda}$ and $R_{n,\lambda}$.

Lemma 3.36. *Let A and B be nonnegative operators with $\omega_A + \omega_B < \pi$, then $S_{n,\lambda} \to S_\lambda$ in operator norm as $n \to \infty$. If A and B satisfy $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$, then $R_{n,\lambda} \to R_\lambda$ strongly on X as $n \to \infty$.*

Proof. We have

$$S_{n,\lambda}x - S_\lambda x := \frac{-1}{2\pi i}\int_\gamma (z+\lambda+A)^{-1}\left((z-B_n)^{-1}x - (z-B)^{-1}x\right)\, dz,$$

and thus

$$\|\, S_{n,\lambda} - S_\lambda \,\| := \frac{1}{2\pi}\int\limits_\gamma \|\,(z+\lambda+A)^{-1}\,\|\,\|\,(z-B_n)^{-1} - (z-B)^{-1}\,\|\; d\,|z|\,,$$

The integrand is bounded by an integrable function for all n since

$$\begin{aligned}
&\|\,(z+\lambda+A)^{-1}\,\| \le M_A^*(\sigma)/\,|z+\lambda|\,,\\
&\|\,(z-B)^{-1}\,\| \le M_B^*(\pi-\sigma)/\,|z|
\end{aligned}$$

and

$$\|\,(z-B_n)^{-1}\,\| \le \hat{M}_B(\pi-\sigma)/\,|z|$$

on γ, where $\hat{M}_B$ does not depend on n. Moreover the integrand tends to 0 as $n\to\infty$, since $\|\,(z-B_n)^{-1} - (z-B)^{-1}\,\| \to 0$ as $n\to\infty$ by Proposition 2.6 and Lemma 1.6. Hence, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \|\, S_{n,\lambda} - S_\lambda \,\|_{\mathcal{L}(X)} = 0. \tag{3.78}$$

Turning now to $R_{n,\lambda} - R_\lambda$, we have

$$\begin{aligned}
R_{n,\lambda}x - R_\lambda x = \frac{-1}{2\pi i}\int\limits_\gamma &\big(\,[\,B_n\,;(z+\lambda+A)^{-1}\,]\,(z-B_n)^{-1}x\\
&-\,[\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}x\big)\,dz.
\end{aligned}$$

Here

$$\|\,[\,B_n\,;(z+\lambda+A)^{-1}\,]\,(z-B_n)^{-1}\,\| \le \hat{\psi}(|z+\lambda|\,,|z|),$$

for $n = 1, 2, \ldots$, and

$$\|\,[\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}\,\| \le \psi(|z+\lambda|\,,|z|),$$

so that the integrand is bounded above by $\psi(|z+\lambda|\,,|z|)x + \hat{\psi}(|z+\lambda|\,,|z|)x$, which is absolutely integrable over γ (chosen to be of the form $\gamma = \gamma_\sigma - \lambda'$, where $0 < \lambda' < \lambda$, as before).

The integrand in the above expression for $R_{n,\lambda}x - R_\lambda x$ also satisfies

$$\begin{aligned}
&[\,B_n\,;(z+\lambda+A)^{-1}\,]\,(z-B_n)^{-1}y - [\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}y\\
&\quad= [\,B_n - B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}y\\
&\qquad+ [\,B_n\,;(z+\lambda+A)^{-1}\,]\,\left((z-B_n)^{-1}y - (z-B)^{-1}y\right).
\end{aligned}$$

Let us show that both terms of the right hand member tend to 0 as $n \to \infty$. First note that

$$\begin{aligned}\|\, [\, B_n - B\,; (u+A)^{-1}\,]\, x \,\| \le \|\, (B_n - B)(u+A)^{-1} x \,\| \\ + \|\, (u+A)^{-1} \,\|\, \|\, (B_n - B) x \,\|\end{aligned}$$

for all $u \in \Sigma_{\phi_A}$ and all $x \in \mathcal{D}(B)$, so that $[\, B_n - B\,; (u+A)^{-1}\,]\, x \to 0$ for all $x \in \mathcal{D}(B)$ as $n \to \infty$. Hence,

$$[\, B_n - B\,; (z+\lambda+A)^{-1}\,]\, (z-B)^{-1} x \to 0$$

for all $x \in X$ as $n \to \infty$ whenever z is on γ.

Since $(z-B_n)^{-1} y - (z-B)^{-1} = (z-B_n)^{-1}(B_n - B)(z-B)^{-1}$, we also have

$$\begin{aligned}&\|\, [\, B_n\,; (z+\lambda+A)^{-1}\,]\, \big((z-B_n)^{-1} y - (z-B)^{-1} y\big) \,\| \\ &\quad \le \|\, [\, B_n\,; (z+\lambda+A)^{-1}\,]\, (z-B_n)^{-1} \,\|\, \|\, (B_n - B)(z-B)^{-1} y \,\| \\ &\quad \le \hat{\psi}(|z+\lambda|\,, |z|)\, \|\, (B_n - B)(z-B)^{-1} y \,\|\end{aligned}$$

for $n = 1, 2, \ldots$, where we have applied Lemma 3.34 to obtain the last inequality. As $B_n x \to Bx$ for all $x \in \mathcal{D}(B)$ and $(z-B)^{-1} y \in \mathcal{D}(B)$, it follows that

$$[\, B_n\,; (z+\lambda+A)^{-1}\,]\, \big((z-B_n)^{-1} y - (z-B)^{-1} y\big) \to 0$$

for all $y \in X$ as $n \to \infty$. Summing this up, we have

$$[\, B_n\,; (z+\lambda+A)^{-1}\,]\, (z-B_n)^{-1} y - [\, B\,; (z+\lambda+A)^{-1}\,]\, (z-B)^{-1} \to 0$$

as $n \to \infty$. Consequently, we can again apply the Dominated Convergence Theorem to conclude that

$$\lim_{n\to\infty} R_{n,\lambda} x = R_\lambda x \tag{3.79}$$

for all $x \in X$. □

We are now ready to prove one of the main results of this section.

Theorem 3.37. *Assume that A and B satisfy $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ and that B is densely defined in X. Then there exists some $\lambda_0 \ge 0$ such that we have $\rho(\overline{A+B}) \supseteq (\lambda_0, \infty)$ and*

$$(\lambda + \overline{A+B})^{-1} = S_\lambda (1 + R_\lambda)^{-1} \qquad (\lambda > \lambda_0).$$

If A, too, is densely defined, then $A+B$ is closable. Finally if $\|\, (\lambda+A)^{-1} \,\| \le \frac{1}{\lambda}$ and $\|\, (\lambda+B)^{-1} \,\| \le \frac{1}{\lambda}$ for all $\lambda > 0$, then the above statements hold with $\lambda_0 = 0$.

Proof. We assume that A and B satisfy $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ and that B is densely defined. We put $L := A + B$, $B_n := nB(n+B)^{-1}$, $L_n := A + B_n$, $S_{n,\lambda} := S_\lambda(A, B_n)$ and $R_{n,\lambda} := R_\lambda(A, B_n)$, as before, and choose λ_0 so large that $\| R_{n,\lambda} \| \le \frac{1}{2}$ $(n = 1, 2, \ldots)$ and $\| R_\lambda \| \le \frac{1}{2}$ for all $\lambda > \lambda_0$. We have seen above that

$$(\lambda + \overline{L})^{-1} x = \lim_{n\to\infty} (\lambda + L_n)^{-1} x$$

for all $x \in X$ and all $\lambda > \lambda_0$. By (3.77), we also have

$$(\lambda + L_n)^{-1} = S_{n,\lambda}(1 + R_{n,\lambda})^{-1}$$

for all $n \in \mathbb{N}$ and all $\lambda > \lambda_0$.

We must show that $(\lambda + L_n)^{-1}x = S_{n,\lambda}(1 + R_{n,\lambda})^{-1}x$ converges to $S_\lambda(1 + R_\lambda)^{-1}x$ for any $x \in X$ as $n \to \infty$. Let us write

$$\begin{aligned} S_{n,\lambda}(1 + R_{n,\lambda})^{-1}x - S_\lambda(1 + R_\lambda)^{-1}x =& S_{n,\lambda}\left((1 + R_{n,\lambda})^{-1}x - (1 + R_\lambda)^{-1}x\right) \\ &+ (S_{n,\lambda} - S_\lambda)(1 + R_\lambda)^{-1}x. \end{aligned}$$

The estimate (3.76) shows that $S_{n,\lambda}$ is uniformly bounded. We have also proved that $S_{n,\lambda} \to S_\lambda$ (even in operator norm) as $n \to \infty$ (see Lemma 3.36). Hence, it suffices to prove that $(1+R_{n,\lambda})^{-1} \to (1+R_\lambda)^{-1}$ strongly as $n \to \infty$. But, by Lemma 3.36, $R_{n,\lambda} \to R_\lambda$ strongly as $n \to \infty$, so that the result follows by Lemma 1.6. Summing this up, we have proved that $S_{n,\lambda}(1 + R_{n,\lambda})^{-1}y \to S_\lambda(1 + R_\lambda)^{-1}y = (\lambda + \overline{L})^{-1}y$ as $n \to \infty$, for all $y \in X$ and all $\lambda > \lambda_0$.

If A is densely defined, then, by Lemma 2.8, $\mathcal{D}(L) = \mathcal{D}(A) \cap \mathcal{D}(B)$ is dense in X and L is closable as noted above.

Let us now assume that $\| (\lambda + A)^{-1} \| \le \frac{1}{\lambda}$ and $\| (\lambda + B)^{-1} \| \le \frac{1}{\lambda}$ for all $\lambda > 0$. Then

$$\begin{aligned} \lambda + L_n &= (\lambda + n + A) - (n - B_n) \\ &= (\lambda + n + A) - n^2(n - B)^{-1} \\ &= \{1 - n^2(n + B)^{-1}(\lambda + n + A)^{-1}\}(\lambda + n + A). \end{aligned}$$

Since

$$\begin{aligned} \left\| n^2(n + B)^{-1}(\lambda + n + A)^{-1} \right\| &\le n^2 \left\| (n + B)^{-1} \right\| \left\| (\lambda + n + A)^{-1} \right\| \\ &\le n^2 \frac{1}{n} \frac{1}{\lambda + n} = \frac{n}{\lambda + n} < 1, \end{aligned}$$

we get the inverse

$$(\lambda + L_n)^{-1} = (\lambda + n + A)^{-1}\{1 - n^2(n + B)^{-1}(\lambda + n + A)^{-1}\}^{-1}$$

Its norm can be estimated as follows

$$\begin{aligned} \left\| (\lambda + L_n)^{-1} \right\| &\le \left\| (\lambda + n + A)^{-1} \right\| \left\| \{1 - n^2(n + B)^{-1}(\lambda + n + A)^{-1}\}^{-1} \right\| \\ &\le \frac{1}{\lambda + n} \cdot \frac{1}{1 - \frac{n}{\lambda + n}} = \frac{1}{\lambda}. \end{aligned}$$

We can thus take $N = 1$ in Corollary 1.7 and consequently $\lambda_0 = 0$. $\square$

REMARK 3.38. It should be stressed that $\overline{A+B}$, and hence also $\lambda+\overline{A+B}$, are graphs, but need not be functions; they are functions precisely when $A+B$ is closable. Thus, Theorem 3.37 implies that for $y \in X$ we have $(x, y) \in \lambda + \overline{A+B}$ if an only if $x = S_\lambda(1+R_\lambda)^{-1}y$, provided that λ is sufficiently large. In particular, if $y \in \mathcal{R}(\lambda + A + B)$, which is dense in X, then $x = S_\lambda(1+R_\lambda)^{-1}y$ is the unique solution to the equation $\lambda x + Ax + Bx = y$ for sufficiently large λ.

3.4.6 The continuity of S_λ

Let us prove the analogue of Theorem 3.21 for the non–commutative case.

THEOREM 3.39. *Assume that $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ holds. Then the linear operator S_λ maps X continuously into $\mathcal{D}_A(\theta, p) \cap \mathcal{D}_B(\theta, p)$ for any $\theta \in (0, 1)$ and $1 \le p \le \infty$. Moreover, we have the estimates*

$$\| S_\lambda x \|_X \le m_0 \lambda^{-1} \| x \|_X \,, \tag{3.80}$$

$$[\, S_\lambda x \,]_{\mathcal{D}_A(\theta,p)} \le m_1 \lambda^{-(1-\theta)} \| x \|_X \,, \tag{3.81}$$

and

$$[\, S_\lambda x \,]_{\mathcal{D}_B(\theta,p)} \le m_2'(\lambda) \| x \|_X \,, \tag{3.82}$$

where m_0 and m_1 are as in Theorem 3.21, and $m_2'(\lambda) \to 0$ as $\lambda \to \infty$.

Proof. Formula(3.80) has been stated and proved before, and has only been included here for completeness. Let us estimate $[\, S_\lambda x \,]_{\mathcal{D}_A(\theta,p)}$ and $[\, S_\lambda x \,]_{\mathcal{D}_B(\theta,p)}$.

In the integral defining S_λ we carry out the substitution $\zeta := z + \lambda$ and obtain

$$S_\lambda x = \frac{-1}{2\pi i} \int_\gamma (\zeta + A)^{-1} (\zeta - \lambda - B)^{-1} x \, d\zeta$$

As path of integration we may take $\gamma = \gamma^+_{\sigma,r}$, where $\omega_B < \sigma < \phi_A$ and r is chosen so small that $0 < r < \lambda$ and $\arg z < \sigma$ on γ (see Section 3.1.5). Then the integral converges absolutely, and, since the linear operator $(t + A)^{-1} : X \to X$ is bounded, we have

$$(t + A)^{-1} x S_\lambda x = \frac{-1}{2\pi i} \int_\gamma (t + A)^{-1} (\zeta + A)^{-1} (\zeta - \lambda - B)^{-1} x \, d\zeta.$$

By the resolvent identity

$$(t + A)^{-1} (\zeta + A)^{-1} = \frac{1}{t - \zeta} \left((\zeta + A)^{-1} - (t + A)^{-1} \right).$$

Let $0 < r < \min(\lambda, t)$. It follows that

$$\int_\gamma \frac{1}{\zeta - t} (t + A)^{-1} (\zeta - \lambda - B)^{-1} x \, d\zeta = 0,$$

since the integrand is bounded of order at most $|z|^{-2}$ at infinity and analytic in a domain containing γ and the region to the left of γ, so that the integration can be performed on along a circular path with centre at the origin. Consequently,

$$(t+A)^{-1}S_\lambda x = \frac{-1}{2\pi i}\int_\gamma \frac{1}{t-\zeta}(\zeta+A)^{-1}(\zeta-\lambda-B)^{-1}x\,d\zeta.$$

Using the identity $A(t+A)^{-1} = 1 - t(t+A)^{-1}$, we hence get

$$A(t+A)^{-1}S_\lambda x = \frac{-1}{2\pi i}\int_\gamma \frac{\zeta}{t-\zeta}(\zeta+A)^{-1}(\zeta-\lambda-B)^{-1}x\,d\zeta.$$

On the circular part of $\gamma^+_{\sigma,r}$ we have $\zeta = re^{i\tau}$ $(-\sigma \le \tau \le \sigma)$, so that

$$\left\| \frac{\zeta}{t-\zeta}(\zeta+A)^{-1}(\zeta-\lambda-B)^{-1}x\,d\zeta \right\| \le \frac{M_A^*(\sigma)M_B^*(\pi-\sigma)}{|re^{i\tau}-t|\,|re^{i\tau}-\lambda|}\,\|\,x\,\|\,,$$

which clearly remains bounded as $r \to 0$. Hence, the path of integration $\gamma^+_{\sigma,r}$ may be deformed into γ_σ without changing the value of the integral, and

$$A(t+A)^{-1}S_\lambda x = \frac{-1}{2\pi i}\int_{\gamma_\sigma} \frac{\zeta}{t-\zeta}(\zeta+A)^{-1}(\zeta-\lambda-B)^{-1}x\,d\zeta.$$

On γ_σ we have $\zeta = e^{\pm i\sigma}$, so that

$$\begin{aligned}\|\,A(t+A)^{-1}S_\lambda x\,\| &\le c\,\|\,x\,\|\int_0^\infty \frac{dr}{|re^{i\sigma}-t|\,|re^{i\sigma}-\lambda|}\\ &= c\,\|\,x\,\|\int_0^\infty \frac{s}{|tse^{i\sigma}-1|}\,\frac{1}{|se^{i\sigma}-\lambda|}\,\frac{ds}{s},\end{aligned}$$

where $c := M_A^*(\sigma)M_B^*(\pi-\sigma)/\pi$. Taking $f(t) := t\,|te^{i\sigma}-1|^{-1}$ and $g(t) := |te^{i\sigma}-\lambda|^{-1}$ in Corollary A.18, we deduce that

$$[\,S_\lambda x\,]_{\mathcal{D}_A(\theta,p)} \le c\left\|\frac{t^{1-\theta}}{te^{i\sigma}-1}\right\|_{L^1_*}\left\|\frac{t^\theta}{te^{i\sigma}-\lambda}\right\|_{L^p_*}\|\,x\,\|\,.$$

But

$$\left\|\frac{t^\theta}{te^{i\sigma}-\lambda}\right\|_{L^p_*} = \lambda^{-(1-\theta)}\left\|\frac{t^\theta}{te^{i\sigma}-1}\right\|_{L^p_*},$$

and thus we have $[\,S_\lambda x\,]_{\mathcal{D}_A(\theta,p)} \le m_1\lambda^{-(1-\theta)}\,\|\,x\,\|$, where

$$m_1 := \frac{1}{\pi}M_A^*(\sigma)M_B^*(\pi-\sigma)\left\|\frac{t^{1-\theta}}{te^{i\sigma}-1}\right\|_{L^1_*}\left\|\frac{t^\theta}{te^{i\sigma}-1}\right\|_{L^p_*}.$$

Since $B(t+B)^{-1} :\, X \longrightarrow X$ is bounded for $t > 0$ we also have

$$
\begin{aligned}
B(t+B)^{-1} S_\lambda x &= \frac{-1}{2\pi i} \int\limits_\gamma B(t+B)^{-1}(z+\lambda+A)^{-1}(z-B)^{-1}x \, dz \\
&= \frac{-1}{2\pi i} \int\limits_\gamma (z+\lambda+A)^{-1}B(t+B)^{-1}(z-B)^{-1}x \, dz \\
&\quad + \frac{-1}{2\pi i} \int\limits_\gamma \left[\, B(t+B)^{-1} \,;\, (z+\lambda+A)^{-1} \,\right] (z-B)^{-1}x \, dz.
\end{aligned}
$$

In the proof of Theorem 3.21 it was shown that

$$
\begin{aligned}
&\left\| \int\limits_\gamma (z+\lambda+A)^{-1}B(\underline{t}+B)^{-1}(z-B)^{-1}x \, dz \right\|_X \\
&\qquad \leq c \, \| \, x \, \| \int\limits_0^\infty \frac{s}{|se^{i\sigma}+1|} \frac{1}{|tse^{i\sigma}+\lambda|} \frac{ds}{s},
\end{aligned}
$$

with c as above, and, from this, that

$$
\begin{aligned}
&\left\| \underline{t}^\theta \left\| \frac{-1}{2\pi i} \int\limits_\gamma (z+\lambda+A)^{-1}B(\underline{t}+B)^{-1}(z-B)^{-1}x \, dz \right\|_X \right\|_{L^p_*} \\
&\qquad \leq m_2 \lambda^{\theta-1} \, \| \, x \, \| ,
\end{aligned}
$$

where

$$
m_2 := \frac{1}{\pi} M_A^*(\sigma) M_B^*(\pi-\sigma) \left\| \frac{t^{1-\theta}}{te^{i\sigma}+1} \right\|_{L^1_*} \left\| \frac{t^\theta}{te^{i\sigma}+1} \right\|_{L^p_*} .
$$

Using the identity $B(t+B)^{-1} = 1 - t(t+B)^{-1}$ we deduce

$$
\begin{aligned}
&\left[\, B(t+B)^{-1} \,;\, (z+\lambda+A)^{-1} \,\right] \\
&\qquad = -t \left[\, (t+B)^{-1} \,;\, (z+\lambda+A)^{-1} \,\right] \\
&\qquad = t(t+B)^{-1} \left[\, t+B \,;\, (z+\lambda+A)^{-1} \,\right] (t+B)^{-1} \\
&\qquad = t(t+B)^{-1} \left[\, B \,;\, (z+\lambda+A)^{-1} \,\right] (t+B)^{-1},
\end{aligned}
$$

and, consequently,

$$
\begin{aligned}
&\left[\, B(t+B)^{-1} \,;\, (z+\lambda+A)^{-1} \,\right] (z-B)^{-1} \\
&\qquad = t(t+B)^{-1} \left[\, B \,;\, (z+\lambda+A)^{-1} \,\right] (z-B)^{-1}(t+B)^{-1}.
\end{aligned}
\tag{3.83}
$$

Therefore,

$$
\begin{aligned}
&\left\| \int\limits_\gamma \left[\, B(t+B)^{-1} \,;\, (z+\lambda+A)^{-1} \,\right] (z-B)^{-1}x \, dz \right\| \\
&\qquad \leq \frac{N_B^2}{t} \, \| \, x \, \| \int\limits_\gamma \psi(|z+\lambda| \, , |z|) \, d\,|z| \, .
\end{aligned}
$$

On the other hand, since $\| B(t+B)^{-1} \| \leq 1 + N_B$, we have

$$\| B(t+B)^{-1} S_\lambda x \| \leq (1+N_B) m_0 \lambda^{-1} \| x \|,$$

where m_0 is given by formula (3.54). It follows that

$$\begin{aligned}\| B(t+B)^{-1} S_\lambda x \| \leq \frac{M_A^*(\sigma) M_B^*(\pi-\sigma)}{\pi} \| x \| \int_0^\infty \frac{s}{|se^{i\sigma}+1|} \frac{1}{|tse^{i\sigma}+\lambda|} \frac{ds}{s} \\ + \min\Big\{ N_B m_0 \lambda^{-1}, N_B^2 t^{-1} \int_\gamma \psi(|z+\lambda|, |z|)\, d|z| \Big\} \| x \|.\end{aligned}$$

We then multiply by t^θ, replace the min by the first term for $0 < t < 1$ and by the second for $1 < t$, and take the L_*^p–norm of both members. This results in the estimate

$$[\, S_\lambda x \,]_{\mathcal{D}_B(\theta,p)} \leq m_2'(\lambda) \| x \|, \tag{3.84}$$

where

$$\begin{aligned} m_2'(\lambda) &:= m_2 \lambda^{\theta-1} \\ &\quad + (p\theta)^{-\frac{1}{p}} m_0 N_B \lambda^{-1} (p(1-\theta))^{-\frac{1}{p}} N_B^2 \int_\gamma \psi(|z+\lambda|, |z|)\, d|z| \end{aligned} \tag{3.85}$$

for $0 < \theta < 1$ and $1 \leq p < \infty$, and

$$m_2'(\lambda) = m_2 \lambda^{\theta-1} + m_0 N_B \lambda^{-1} + N_B^2 \int_\gamma \psi(|z+\lambda|, |z|)\, d|z| \tag{3.86}$$

if $0 < \theta < 1$ and $p = \infty$. Here m_0 and m_2 are as in Theorem 3.21. Finally, we let $\lambda \to \infty$ to obtain $m_2'(\lambda) \to 0$. $\square$

3.4.7 A maximal regularity result

We close this section with a result on the maximal regularity of the equation $\lambda x + Ax + Bx = y$, where A and B satisfy $\mathcal{H}(\mathcal{A}, \mathcal{B}, \sigma_{\mathcal{A}}, \sigma_{\mathcal{B}}, \psi)$. But let us first prove the following lemma.

Lemma 3.40. *Assume that $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ holds, and let $(\theta, p) \in (0,1) \times [1, \infty]$. Then AS_λ maps $\mathcal{D}_B(\theta,p)$ continuously into itself. If we also have $R_\lambda(\mathcal{D}_B(\theta,p)) \subseteq \mathcal{D}_B(\theta,p)$ then $BS_\lambda(\mathcal{D}_B(\theta,p)) \subseteq \mathcal{D}_B(\theta,p)$. If, in addition, R_λ maps $\mathcal{D}_B(\theta,p)$ continuously into itself, then so does BS_λ.*

Proof. Let $x \in \mathcal{D}_B(\theta,p)$. According to formula (3.59) we have

$$AS_\lambda x = x - \lambda S_\lambda x + U_\lambda x, \tag{3.87}$$

where

$$U_\lambda x = \frac{-1}{2\pi i}\int_\gamma (z+\lambda+A)^{-1}B(z-B)^{-1}x\,dz. \tag{3.88}$$

By the previous lemma, S_λ maps $\mathcal{D}_B(\theta,p)$ continuously into itself. Now $B(t+B)^{-1} : X \longrightarrow X$ is bounded and the integral that defines U_λ is absolutely convergent. Hence, we get

$$B(t+B)^{-1}U_\lambda x = \frac{-1}{2\pi i}\int_\gamma B(t+B)^{-1}(z+\lambda+A)^{-1}B(z-B)^{-1}x\,dz.$$

In the proof of Theorem 3.39 it was shown that

$$[\,B(t+B)^{-1}\,;(z+\lambda+A)^{-1}\,] = t(t+B)^{-1}\,[\,B\,;(z+\lambda+A)^{-1}\,]\,(t+B)^{-1}.$$

Consequently,

$$\begin{aligned}
&B(t+B)^{-1}(z+\lambda+A)^{-1}B(z-B)^{-1}x\\
&\qquad = [\,B(t+B)^{-1}\,;(z+\lambda+A)^{-1}\,]\,B(z-B)^{-1}x\\
&\qquad\quad + (z+\lambda+A)^{-1}B(t+B)^{-1}B(z-B)^{-1}x\\
&\qquad = t(t+B)^{-1}\,[\,B\,;(z+\lambda+A)^{-1}\,]\,(t+B)^{-1}B(z-B)^{-1}x\\
&\qquad\quad + (z+\lambda+A)^{-1}B(t+B)^{-1}B(z-B)^{-1}x.
\end{aligned}$$

By Lemma 3.16,

$$\begin{aligned}
&\frac{-1}{2\pi i}\int_\gamma (z+\lambda+A)^{-1}B(t+B)^{-1}B(z-B)^{-1}x\,dz\\
&\qquad\qquad = \frac{-1}{2\pi i}\int_\gamma \frac{z}{z+t}(z+\lambda+A)^{-1}B(z-B)^{-1}x\,dz.
\end{aligned}$$

It can be seen from the proof of Theorem 3.23 that

$$\underline{t}^\theta \Big\| \int_\gamma (z+\lambda+A)^{-1}B(\underline{t}+B)^{-1}B(z-B)^{-1}x\,dz\,\Big\| \in L^p_*$$

and

$$\begin{aligned}
&\Big\|\,\underline{t}^\theta \Big\|\,\frac{-1}{2\pi i}\int_\gamma (z+\lambda+A)^{-1}B(\underline{t}+B)^{-1}B(z-B)^{-1}x\,dz\,\Big\|_X\,\Big\|_{L^p_*}\\
&\qquad\qquad \le c_2(A_\lambda,B)[\,x\,]_{\mathcal{D}_B(\theta,p)} \quad .
\end{aligned}$$

Concerning the expression $t(t+B)^{-1}\,[\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}B(t+B)^{-1}x$, observe that

$$\begin{aligned}
&\big\|\,t(t+B)^{-1}\,[\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}B(t+B)^{-1}x\,\big\|\\
&\qquad \le N_B\psi(|z+\lambda|\,,|z|)\,\big\|\,B(t+B)^{-1}x\,\big\| \quad .
\end{aligned}$$

Hence,

$$\Big\| \int_\gamma t(t+B)^{-1}\,[\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}B(t+B)^{-1}x\,dz\,\Big\|$$
$$\leq N_B \|\,B(t+B)^{-1}x\,\| \int_\gamma \psi(|z+\lambda|\,,|z|)\,dz$$

and

$$\Big\|\, t^\theta \Big\| \int_\gamma t(t+B)^{-1}\,[\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}B(t+B)^{-1}x\,dz\,\Big\|_X \Big\|_{L^p_*}$$
$$\leq N_B\,[\,x\,]_{\mathcal{D}_B(\theta,p)} \int_{\gamma_\sigma} \psi(|z+\lambda|\,,|z|)\,dz \quad .$$

Summarising, we have shown that

$$\begin{aligned} &B(t+B)^{-1}U_\lambda x \\ &\quad = \frac{-1}{2\pi i}\int_\gamma t(t+B)^{-1}\,[\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}B(t+B)^{-1}x\,dz \\ &\quad\quad + \frac{-1}{2\pi i}\int_\gamma \frac{z}{z+t}(z+\lambda+A)^{-1}B(z-B)^{-1}x\,dz, \end{aligned}$$

and there is a constant

$$c := c_1(A_\lambda, B) + \frac{N_B}{2\pi}\int_\gamma \psi(|z+\lambda|\,,|z|)\,dz$$

such that

$$[\,U_\lambda x\,]_{\mathcal{D}_B(\theta,p)} \leq c\,[\,x\,]_{\mathcal{D}_B(\theta,p)} \tag{3.89}$$

for any $x \in \mathcal{D}_B(\theta,p)$.

By the proof of Theorem 3.23, we have $\|\,U_\lambda x\,\| \leq c_1'\lambda^{-\theta}\,\|\,x\,\|_{\mathcal{D}_B(\theta,p)}$, where c_1' does not depend on y. Thus, we have, in fact, shown that U_λ maps $\mathcal{D}_B(\theta,p)$ continuously into itself. By (3.59), $AS_\lambda = y - \lambda S_\lambda y - U_\lambda y$. It follows that AS_λ maps $\mathcal{D}_B(\theta,p)$ continuously into itself.

Let us now assume that $R_\lambda(\mathcal{D}_B(\theta,p)) \subseteq \mathcal{D}_B(\theta,p)$. By (3.4.1)

$$BS_\lambda x = U_\lambda x + R_\lambda x,$$

so that $BS_\lambda((\mathcal{D}_B(\theta,p)) \subseteq \mathcal{D}_B(\theta,p)$. Moreover, if R_λ maps $\mathcal{D}_B(\theta,p)$ continuously into itself, then so does $BS_\lambda = U_\lambda + R_\lambda$. $\square$

In the following theorem we strengthen the inequality (3.66) of the hypothesis $\mathcal{H}$ in order to satisfy all the assumptions of the previous lemma for sufficiently large λ.

Theorem 3.41. *Let A and B be nonnegative operators in a complex Banach space X. Assume that $\mathcal{H}(A, B, \sigma_A, \sigma_B, \psi)$ holds, B is densely defined in X, $[\,B\,;(z+A)^{-1}\,]\,(w-B)^{-1} \in \mathcal{L}(\mathcal{D}_B(\theta, p))$ and that*

$$\|\,[\,B\,;(z+A)^{-1}\,]\,(w-B)^{-1}\,\|_{\mathcal{L}(\mathcal{D}_B(\theta,p))} \le \psi(|z|\,,|w|) \tag{3.90}$$

for all $z, w \in \mathbb{C}\setminus\{0\}$ such that $|\arg z| \le \sigma_A$ and $\sigma_B \le |\arg w|$, where $(\theta, p) \in (0,1)\times[1,\infty]$. Then there is a $\lambda_0 \ge 0$ such that the problem $\lambda x + Ax + Bx = y$ has a unique solution $x = S_\lambda(1+R_\lambda)^{-1}y \in \mathcal{D}(A)\cap\mathcal{D}(B)$ for any $y \in \mathcal{D}_B(\theta, p)$ and any $\lambda > \lambda_0$. The mappings $S_\lambda(1+R_\lambda)^{-1}$, $AS_\lambda(1+R_\lambda)^{-1}$ and $BS_\lambda(1+R_\lambda)^{-1}$, are continuous from $\mathcal{D}_B(\theta, p)$ into itself.

Proof. By Lemma 3.32, we have $(\lambda + A + B)S_\lambda z = (1+R_\lambda)z$ for all $z \in \mathcal{D}_B(\theta, p)$. Assume now that $y \in \mathcal{D}_B(\theta, p)$. We then have

$$\begin{aligned}
&\|\,t^\theta B(t+B)^{-1}R_\lambda y\,\| \\
&\quad = \frac{1}{2\pi}\left\|\int_\gamma t^\theta B(t+B)^{-1}\,[\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}y\,dz\right\| \\
&\quad \le \frac{1}{2\pi}\int_\gamma \|\,t^\theta B(t+B)^{-1}\,[\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}y\,\|\;d\,|z|\,.
\end{aligned}$$

If $p = \infty$, we immediately infer that

$$[\,R_\lambda y\,]_{\mathcal{D}_B(\theta,\infty)} \le \frac{1}{2\pi}\,\|\,y\,\|_{\mathcal{D}_B(\theta,\infty)}\int_\gamma \psi(|z+\lambda|\,,|z|)\,d\,|z|\,,$$

since in this case

$$\begin{aligned}
\|\,t^\theta B(t+B)^{-1}\,&[\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}y\,\| \\
&\le [\,[\,B\,;(z+\lambda+A)^{-1}\,]\,(z-B)^{-1}y\,]_{\mathcal{D}_B(\theta,\infty)} \\
&\le \psi(|z+\lambda|\,,|z|)\,\|\,y\,\|_{\mathcal{D}_B(\theta,\infty)}\,.
\end{aligned}$$

If $1 \le p < \infty$, using Fubini's Theorem, we deduce the same inequality, i.e.

$$[\,R_\lambda y\,]_{\mathcal{D}_B(\theta,p)} \le \frac{1}{2\pi}\,\|\,y\,\|_{\mathcal{D}_B(\theta,p)}\int_\gamma \psi(|z+\lambda|\,,|z|)\,d\,|z|\,.$$

Hence, there is some constant K such that

$$\|\,R_\lambda\,\|_{\mathcal{L}(\mathcal{D}_B(\theta,p))} \le K\int_\gamma \psi(|z+\lambda|\,,|z|)\,d\,|z|\,. \tag{3.91}$$

By assumption, the expression to the right tends to zero as $\lambda \to \infty$, so that $\|\,R_\lambda\,\|_{\mathcal{L}(\mathcal{D}_B(\theta,\infty))} < 1$ for all λ larger than some $\lambda_0 \ge 0$.

Consequently, R_λ maps $\mathcal{D}_B(\theta, p)$ continuously into itself, and there is some $\lambda_0 \ge 0$ such that $1 + R_\lambda : \mathcal{D}_B(\theta, p) \longrightarrow \mathcal{D}_B(\theta, p)$ is invertible, by means of a Neumann series, as soon as $\lambda > \lambda_0$. Hence, we have $(\lambda + A + B)S_\lambda(1 + R_\lambda)^{-1}y = y$, by Lemma 3.32, i.e., $S_\lambda(1+R_\lambda)^{-1}y$ is a solution to the equation $\lambda x + Ax + Bx = y$. Choosing λ_0 sufficiently large, we can apply Theorem 3.37 to get uniqueness.

We also see that $(1+R_\lambda)^{-1}$ maps $\mathcal{D}_B(\theta, p)$ continuously onto itself for $\lambda > \lambda_0$, so that, by Lemma 3.40, $S_\lambda(1+R_\lambda)^{-1}$, $AS_\lambda(1+R_\lambda)^{-1}$, and $BS_\lambda(1+R_\lambda)^{-1}$ are continuous from $\mathcal{D}_B(\theta, p)$ into itself. □

3.4.8 Resolvent commuting operators with perturbation

Let us apply the theory developed above to equations of the form

$$\lambda x + Ax + CBx = y,$$

where A and CB are nonnegative, $\omega_A + \omega_{CB} < \pi$, and B commutes with resolvents of A. The operator C can be thought of as a perturbation. This idea will be pursued further in Section 4.3, where the following theorem will be applied in combination with Lemma 2.26.

Theorem 3.42. *Let A, B and C be linear operators in a complex Banach space X, and let $(\theta, p) \in (0,1) \times [1, \infty]$. Assume that the following conditions are satisfied:*

(i) *A and CB are nonnegative and $\omega_A + \omega_{CB} < \pi$.*
(ii) *C bounded and injective with bounded inverse, and $\mathcal{D}(C) \supseteq \mathcal{R}(B)$.*
(iii) *B is densely defined in X.*
(iv) *B commutes with resolvents of A on $\mathcal{D}(B)$.*
(v) *There is a number σ_A with $\omega_{CB} < \sigma_A < \phi_A$, as well as constants $K, R, \delta > 0$ such that $[\,C\,; (z + A)^{-1}\,]\,C^{-1} \in \mathcal{L}(\mathcal{D}_{CB}(\theta, p))$ and*

$$\|\,[\,C\,; (z + A)^{-1}\,]\,C^{-1}\,\|_{\mathcal{L}(\mathcal{D}_{CB}(\theta,p))} \le K|z|^{-1-\delta}$$

for all z with $|\arg z| \le \sigma_A$ and $|z| > R$.

Then there is a $\lambda_0 \ge 0$ such that the problem $\lambda x + Ax + CBx = y$ has a unique solution $x = S_\lambda(A, CB)(1 + R_\lambda(A, CB))^{-1}y \in \mathcal{D}(A) \cap \mathcal{D}(B)$ for any $y \in \mathcal{D}_B(\theta, p)$ and any $\lambda > \lambda_0$. Moreover, we have

$$\|\,x\,\|_{\mathcal{D}_B(\theta,p)} + \|\,Ax\,\|_{\mathcal{D}_B(\theta,p)} + \|\,Bx\,\|_{\mathcal{D}_B(\theta,p)} \le M\,\|\,y\,\|_{\mathcal{D}_B(\theta,p)}$$

for some positive constant M that only depends on λ, A and B.

Proof. Assumption (ii) implies that

$$\mathcal{D}(CB) = \mathcal{D}(B)$$
$$\|\,Bx\,\| \le \|\,C^{-1}\,\|\,\|\,CBx\,\|$$

and

$$\|\,CBx\,\| \le \|\,C\,\|\,\|\,Bx\,\|,$$

so that $\mathcal{D}(CB)$ and $\mathcal{D}(B)$ are equal with equivalent graph norms. Hence, $\mathcal{D}_{CB}(\theta,p) = \mathcal{D}_B(\theta,p)$ with equivalent norms.

From (iv) it follows that $\mathcal{D}(CB) = \mathcal{D}(B)$ is stable under resolvents of A.

Let us fix some σ_{CB} with

$$\omega_{CB} < \sigma_{CB} < \sigma_A < \phi_A.$$

Thanks to Theorem 3.41 it suffices to show that there is some measurable ψ such that (3.65) holds, the mapping $(z,w) \mapsto [\, CB\,; (z+A)^{-1}\,]\,(w-CB)^{-1}$ is continuous into $\mathcal{L}(X)$ and

$$\big\|\, [\, CB\,; (z+A)^{-1}\,]\,(w-CB)^{-1}\,\big\|_{\mathcal{L}(\mathcal{D}_B(\theta,p))} \le \psi(|z|\,,|w|)$$

for all $z,w \in \mathbb{C}\setminus\{0\}$ such that $|\arg z| \le \sigma_A$ and $|\arg w| \ge \sigma_{CB}$ with $|z|$ large enough.

First we note that, since B commutes with resolvents of A,

$$\begin{aligned}[\, CB\,; (z+A)^{-1}\,]\,(w-CB)^{-1} &= [\, C\,; (z+A)^{-1}\,]\,B(w-CB)^{-1} \\ &= [\, C\,; (z+A)^{-1}\,]\,C^{-1}CB(w-CB)^{-1}.\end{aligned}$$

This defines a continuous function from $\{(z,w) \in \mathbb{C}\times\mathbb{C} \;\|\; |\arg z| < \phi_A \wedge |\arg w| \ge \omega_{CB}\}$ into $\mathcal{L}(X)$.

Let us show that for any w with $\omega_{CB} < |\arg w| \le \pi$, we have $CB(w-CB)^{-1} \in \mathcal{L}(\mathcal{D}_{CB}(\theta,p))$ and

$$\|\, CB(w-CB)^{-1}\,\|_{\mathcal{L}(\mathcal{D}_{CB}(\theta,p))} \le 1 + M_{CB}(\pi - \arg w).$$

In fact, if $x \in \mathcal{D}_{CB}(\theta,p)$, then

$$\begin{aligned}&\|\, CB(t+CB)^{-1}CB(w-CB)^{-1}x\,\|_X \\ &\quad = \|\, CB(w-CB)^{-1}CB(t+CB)^{-1}x\,\|_X \\ &\quad\quad \le (1 + M_{CB}(\pi - \arg w))\|\, CB(t+CB)^{-1}x\,\|_X,\end{aligned}$$

and hence

$$[\, CB(w-CB)^{-1}x\,]_{\mathcal{D}_{CB}(\theta,p)} \le (1 + M_{CB}(\pi - \arg w))[\, x\,]_{\mathcal{D}_{CB}(\theta,p)}.$$

Using the assumption of the theorem, we deduce that there is a constant $c = K(1 + M_{CB}(\pi - \sigma_{CB}))$ such that

$$\big\|\, [\, CB\,; (z+A)^{-1}\,]\,(w-CB)^{-1}\,\big\|_{\mathcal{L}(\mathcal{D}_B(\theta,p))} \le c|z|^{-1-\delta}$$

whenever $|\arg z| < \sigma_A$, $\pi \ge |\arg w| \ge \sigma_{CB}$ and $|z| > R$. Clearly $\psi(x,y) := cx^{-1-\delta}$ satisfies (3.65). In fact, if $\sigma_{CB} \le \sigma \le \sigma_A$ and $R/\sin(\min\{\sigma,\pi/2\}) < \lambda_0 < \lambda$, then

$$\begin{aligned}\int_{-\lambda_0+\gamma_\sigma} |z+\lambda|^{-1-\delta}\,d|z| &= 2\int_0^\infty \frac{dt}{|te^{i\sigma}+\lambda-\lambda_0|^{1+\delta}} \\ &\le \frac{2}{(\lambda-\lambda_0)^\delta}\int_0^\infty \frac{dt}{|te^{i\sigma}+1|^{1+\delta}},\end{aligned}$$

which tends to 0 as $\lambda \to \infty$. $\square$

4 Fractional evolution equations in Hölder spaces

In this chapter we apply the theory on abstract operator equations developed in the previous chapter to fractional evolution equations of the form

$$D_t^\alpha u(t,x) + b(t,x)\, D_x^\beta u(t,x) = f(t,x)$$

in the space $\mathcal{C}_{\partial_0 Q \mapsto 0}(Q;E)$ of all continuous functions $f : [0,\tau] \times [0,\xi] \longrightarrow \mathbb{C}$ such that $f(0,x) = f(t,0) = 0$ for all $t \in [0,\tau]$ and all $x \in [0,\xi]$. The partial differential operators D_t^α and D_x^β will be of *fractional orders* α and β, respectively, where $0 < \alpha, \beta \leq 1$, and $\alpha + \beta < 2$.

In the first section we study differential and *fractional differential operators* D^α in spaces of the form $\mathcal{C}_{0\mapsto 0}([0,T];E) := \{f \in \mathcal{C}([0,T];X) \mid f(0) = 0\}$, as well as the relationship between these operators and the subspaces $\mathcal{C}^\alpha_{0\mapsto 0}([0,T];E)$ of the *Hölder spaces* $\mathcal{C}^\alpha([0,T];E)$.

In the second section we prove some results on the existence and maximal regularity of solutions to a fractional evolution equation of the form $D_t^\alpha u(t,x) + D_x^\beta u(t,x) = f(t,x)$ in $\mathcal{C}_{\partial_0 Q \mapsto 0}(Q;E)$. The technique that we use is to consider the equation as an equation in $\mathcal{C}_{0\mapsto 0}([0,\xi];\mathcal{C}_{0\mapsto 0}([0,\tau];E))$ and apply the method of sums to the new equation.

In the last section we consider the equation $D_t^\alpha u(t,x) + b(t,x) D_x u(t,x) = f(t,x)$, which involves two operators that do not commute. By imposing some rather severe restrictions on the function b, we obtain existence, uniqueness and maximal regularity of solutions to this equation.

4.1 Hölder spaces, interpolation and fractional derivatives

This section is devoted to the differential operator D (see Definition 4.2) and its fractional powers D^α in $\mathcal{C}_{0\mapsto 0}([0,T];X)$. in particular, we investigate the relationship between the interpolation spaces $\mathcal{D}_{\mathsf{D}^\alpha}(\theta,\infty)$ and Hölder spaces.

We start by defining the Hölder spaces $\mathcal{C}^\alpha([0,T];E)$, the little Hölder spaces $h^\alpha([0,T];E)$, and their subspaces $\mathcal{C}^\alpha_{0\mapsto 0}([0,T];E)$ and $h^\alpha_{0\mapsto 0}([0,T];E)$. We then show that the differential operator D in $\mathcal{C}_{0\mapsto 0}([0,T];E)$ is positive and densely defined, and that if $\alpha \notin \mathbb{N}$ and $\alpha < n \in \mathbb{N}$, then $\mathcal{C}^\alpha_{0\mapsto 0}([0,T];E) = \mathcal{D}_{\mathsf{D}^n}(\alpha/n,\infty)$ and $h^\alpha_{0\mapsto 0}([0,T];E) = \mathcal{D}_{\mathsf{D}^n}(\alpha/n)$, where $\mathcal{D}_{\mathsf{D}^n}(\theta,\infty)$ and $\mathcal{D}_{\mathsf{D}}(\theta)$ are real interpolation spaces between $\mathcal{C}_{0\mapsto 0}([0,T];E)$ and $\mathcal{C}^n_{0\mapsto 0}([0,T];E)$.

In the last subsection we define fractional derivatives, and show that the fractional differential operator of order $\alpha \in (0,1)$ in $\mathcal{C}_{0\mapsto 0}([0,T];E)$ is, in fact, D^α. Hence, we may apply general results on fractional powers to fractional derivatives.

4.1.1 Spaces of Hölder and little Hölder continuous functions

Let Ω be a set and let E be a normed vector space. By $\mathcal{B}(\Omega; E)$ we denote the vector space of all bounded functions $f : \Omega \longrightarrow E$. We can define a norm on $\mathcal{B}(\Omega; E)$ by

$$\| f \|_{\mathcal{B}(\Omega;E)} := \sup_{t \in \Omega} \| f(t) \|_E .$$

We shall also use $\| f \|_\infty$ to denote this norm when it is clear from the context what Ω and E are. If E is a complex Banach space, then so is $\mathcal{B}(\Omega; E)$.

From now on we let $I = [a, b] \subset \mathbb{R}$ be a bounded and closed interval and E a complex Banach space. Then $\mathcal{C}(I; E)$ is the space consisting of all continuous functions $f : I \longrightarrow E$, where continuity at the end points a and b means right and left continuity respectively. For $t \in I$, we define the derivative of f at t by

$$f'(t) = \lim_{\substack{h \to 0 \\ t+h \in I}} \frac{1}{h}(f(t+h) - f(t)),$$

whenever the limit exists in E, in which case f is said to be differentiable at t.

If $f'(t)$ exists for any $t \in I$, then the derivative $f'(\underline{t})$ of f defines a function $f' : I \longrightarrow E$. We define the spaces $\mathcal{C}^n(I; E)$ of n times continuously differentiable functions by

$$\begin{aligned}
&\mathcal{C}^0(I; E) := \mathcal{C}(I; E) \\
&\mathcal{C}^1(I; E) := \{ f \in \mathcal{C}(I; E) \mid f'(t) \text{ exists for all } t \in I \text{ and } f' \in \mathcal{C}(I; E) \} \\
&\mathcal{C}^n(I; E) := \{ f \in \mathcal{C}^1(I; E) \mid f' \in \mathcal{C}^{n-1}(I; E) \} \qquad (n = 2, 3, \ldots).
\end{aligned}$$

Then $\mathcal{C}^n(I; E)$, provided with the norm $\| f \|_{\mathcal{C}^n(I;E)} := \sum_{i=0}^n \left\| f^{(i)} \right\|_\infty$, is a complex Banach space for any integer n. We also define $\mathcal{C}^\infty(I; E) := \bigcap_{n=0}^\infty \mathcal{C}^n(I; E)$. One can show that $\mathcal{C}^\infty(I; E)$ is dense in $\mathcal{C}^n(I; E)$ for $n = 0, 1, \ldots$

Let $0 < \alpha < 1$ and let I be a closed and bounded interval as above. The *Hölder space* $\mathcal{C}^\alpha(I; E)$ is defined by

$$\mathcal{C}^\alpha(I; E) := \{ f \in \mathcal{C}(I; E) \,\|\, [\, f \,]_\alpha < \infty \},$$

where $[\ \ ]_\alpha$ is given by

$$[\, f \,]_\alpha := [\, f \,]_{\mathcal{C}^\alpha(I;E)} := \sup_{\substack{s,t \in I \\ s \neq t}} \frac{\| f(s) - f(t) \|_E}{|s - t|^\alpha}.$$

We also put

$$\| f \|_\alpha := \| f \|_{\mathcal{C}^\alpha(I;E)} := \| f \|_\infty + [\, f \,]_\alpha .$$

Then $[\ \ ]_\alpha$ is a seminorm and $\|\ \ \|_\alpha$ is a norm on $\mathcal{C}^\alpha(I;E)$.

We can also define $\mathcal{C}^\alpha(I;E)$ for arbitrary $\alpha \geq 0$ in the following way: If $0 < \alpha < 1$ and $n \in \mathbb{N}$, we put

$$\mathcal{C}^{n+\alpha}(I;E) := \{f \in \mathcal{C}^n(I;E) \mid [\, f^{(n)}\,]_{\mathcal{C}^\alpha(I;E)} < \infty\}$$

and

$$\|\, f\,\|_{\mathcal{C}^{n+\alpha}(I;E)} := \|\, f\,\|_{\mathcal{C}^n(I;E)} + \left\|\, f^{(n)}\,\right\|_{\mathcal{C}^\alpha(I;E)}.$$

The *little Hölder spaces* $h^\alpha(I;E)$ are defined by

$$h^\alpha(I;E) := \{f \in \mathcal{C}^\alpha(I;E) \mid \lim_{\delta\downarrow 0} \sup_{\substack{0<|t-s|<\delta \\ t,s\in I}} \frac{\|\, f(t) - f(s)\,\|}{|t-s|^\alpha}\}$$

$$h^{n+\alpha}(I;E) := \{f \in \mathcal{C}^n(I;E) \mid f^{(n)} \in h^\alpha(I;E)\}.$$

for $0 < \alpha < 1$ and $n = 1, 2, \ldots$ As a norm in $h^\alpha(I;E)$ we take the norm of $\mathcal{C}^\alpha(I;E)$.

Let us now introduce spaces of Hölder continuous and spaces of little Hölder continuous functions $f : I \longrightarrow E$ with initial value zero.

Definition 4.1. Assume that $n \in \mathbb{N}$, $\alpha \in [0,1)$ and $I = [0,T]$, where $T > 0$. We then set

$$\mathcal{C}^{n+\alpha}_{0\mapsto 0}(I;E) := \{f \in \mathcal{C}^{n+\alpha}(I;E) \mid f(0) = f'(0) = \ldots = f^{(n)}(0) = 0\},$$
$$\mathcal{C}_{0\mapsto 0}(I;E) := \mathcal{C}^0_{0\mapsto 0}(I;E),$$

and, if $\alpha \in (0,1)$,

$$h^{n+\alpha}_{0\mapsto 0}(I;E) := \{f \in h^{n+\alpha}(I;E) \,\|\, f(0) = f'(0) = \ldots = f^{(n)}(0) = 0\}.$$

Obviously, the spaces $\mathcal{C}^{n+\alpha}_{0\mapsto 0}(I;E)$ and $h^{n+\alpha}_{0\mapsto 0}(I;E)$ are closed subspaces of the spaces $\mathcal{C}^{n+\alpha}(I;E)$ and $h^{n+\alpha}(I;E)$, respectively.

We also set $\mathcal{C}^\infty_{0\mapsto 0}(I;E) := \bigcap_{n=0}^\infty \mathcal{C}^n_{0\mapsto 0}(I;E)$. One can show that $\mathcal{C}^\infty_{0\mapsto 0}(I;E)$ is dense in $\mathcal{C}^n_{0\mapsto 0}(I;E)$ for $n = 0, 1, \ldots$.

When $E = \mathbb{R}$ or $E = \mathbb{C}$, we omit '$;\mathbb{R}$' and '$;\mathbb{C}$' from the notation, and simply write $\mathcal{C}(I)$, $\mathcal{C}_{0\mapsto 0}(I)$, $\mathcal{C}^\alpha_{0\mapsto 0}(I)$, etc.

Corollary 4.5 below states that if $0 < \alpha < \theta < 1$ and n is a nonnegative integer, then $h^{n+\alpha}_{0\mapsto 0}(I;E)$ is the closure of $\mathcal{C}^{n+\theta}_{0\mapsto 0}(I;E)$ in $\mathcal{C}^{n+\alpha}_{0\mapsto 0}(I;E)$. With the same assumptions on α, θ and n it can also be shown that $h^{n+\alpha}(I;E)$ is the closure of $\mathcal{C}^{n+\theta}(I;E)$ in $\mathcal{C}^\alpha(I;E)$ (see [9]).

It is a well-known fact that the Hölder spaces and the little Hölder spaces are complex Banach spaces. Since the spaces $\mathcal{C}^{n+\alpha}_{0\mapsto 0}([0,T];E)$ and $h^{n+\alpha}_{0\mapsto 0}([0,T];E)$ are closed subspaces of the corresponding Hölder spaces and little Hölder spaces, respectively, they are complex Banach spaces, a result that we also obtain in Corollary 4.6.

4.1.2 The differential operator D in $\mathcal{C}_{0\mapsto 0}(I;E)$

We have already defined the concept of derivative for functions in $\mathcal{C}([0,T]\,;E)$. Let us now define the linear operator that produces the derivative of a function in $\mathcal{C}_{0\mapsto 0}([0,T]\,;E)$, provided that this derivative also belongs to $\mathcal{C}_{0\mapsto 0}([0,T]\,;E)$.

Definition 4.2. Let $I=[0,T]$, where $T>0$. We define the differential operator D in $\mathcal{C}_{0\mapsto 0}(I;E)$ as follows:

$$\begin{aligned}\mathcal{D}(\mathsf{D}) &:= \mathcal{C}^1_{0\mapsto 0}(I;E)\\ \mathsf{D}\,f &:= f' \qquad (f\in\mathcal{D}(\mathsf{D})).\end{aligned}$$

We immediately see that the norm of $\mathcal{C}^1_{0\mapsto 0}(I;E)$ is the graph norm associated with D.

Let us state and prove some simple, but fundamental, properties of D.

Proposition 4.3. *Let D be the differential operator in $\mathcal{C}_{0\mapsto 0}([0,T]\,;E)$ defined above. Then D is densely defined and positive with spectral angle $\pi/2$. Moreover, $\rho(\mathsf{D})=\mathbb{C}$, and*

$$((\lambda+\mathsf{D})^{-1}f)(t)=\int_0^t e^{-\lambda(t-\tau)}f(\tau)\,d\tau \tag{4.1}$$

for any $f\in\mathcal{C}_{0\mapsto 0}([0,T]\,;E)$, any $t\in[0,T]$, and any $\lambda\in\mathbb{C}$.

Proof. We have noted above that $\mathcal{C}^\infty_{0\mapsto 0}(I;E)$ is dense in $\mathcal{C}_{0\mapsto 0}(I;E)$, so that D is densely defined.

Next, we show that $\rho(\mathsf{D})=\mathbb{C}$. In fact, if $f\in\mathcal{C}_{0\mapsto 0}(I;E)$ and $g\in\mathcal{C}^1_{0\mapsto 0}(I;E)$, then $(\lambda+\mathsf{D})g=f$ if and only if $\lambda e^{\lambda t}g(t)+e^{\lambda t}g'(t)=e^{\lambda t}f(t)$ for all $t\in[0,T]$. But, since the left hand member is the derivative of $e^{\lambda t}g(t)$, and g(0)=0 by assumption, the Fundamental Theorem of Calculus (see Subsection A.1.6) shows that the last equality holds precisely when $e^{\lambda t}g(t)=\int_0^t e^{\lambda\tau}f(\tau)\,d\tau$, i.e., if and only if

$$g(t)=\int_0^t e^{-\lambda(t-\tau)}f(\tau)\,d\tau \qquad (0\le t\le T).$$

Moreover, if g is defined in this manner, then it is continuously differentiable, and $g(0)=g'(0)=0$, i.e. $g\in\mathcal{C}^1_{0\mapsto 0}(I;E)$, so that $\lambda\in\rho(-\mathsf{D})$ and (4.1) holds for any $f\in\mathcal{C}_{0\mapsto 0}(I;E)$. From the last equality we obtain

$$\left\|\,(\lambda+\mathsf{D})^{-1}f\,\right\|_\infty\le\|\,f\,\|_\infty\int_0^t e^{-\operatorname{Re}\lambda(t-\tau)}\,d\tau\le\frac{\|\,f\,\|_\infty}{\operatorname{Re}\lambda}(1-e^{-t\operatorname{Re}\lambda}),$$

and hence

$$\left\|\,(\lambda+\mathsf{D})^{-1}f\,\right\|_\infty\le\frac{\|\,f\,\|_\infty}{|\operatorname{Re}\lambda|}(e^{T|\operatorname{Re}\lambda|}+1),$$

if $\operatorname{Re}\lambda \neq 0$. If $\operatorname{Re}\lambda = 0$, we get $\|\,(\lambda+\mathsf{D})^{-1}f\,\|_\infty \le T\,\|\,f\,\|_\infty$. Thus, $(\lambda+\mathsf{D})^{-1}$ is bounded for any $\lambda \in \mathbb{C}$.

We also deduce that

$$\|\,(\lambda+\mathsf{D})^{-1}f\,\|_\infty \le \frac{\|\,f\,\|_\infty}{|\lambda|\cos\phi}$$

for all $\lambda \in \Sigma_\phi$, where $0 < \phi < \pi/2$. It follows that the spectral angle ω_{D} of D is at most $\pi/2$.

To see that $\omega_{\mathsf{D}} \ge \pi/2$, we consider $f \in \mathcal{C}_{0\mapsto 0}(I;E)$ defined by $f(t) = tx_0$, where $x_0 \in E$ and $\|\,x_0\,\| = 1$. Assume that $\lambda \neq 0$. Then $\|\,f\,\|_\infty = T$ and

$$((\lambda+\mathsf{D})^{-1}f)(t) = \left(\int_0^t \tau e^{\lambda(\tau-t)}\,d\tau\right) x_0 = \left(\frac{1}{\lambda^2}(e^{-t\operatorname{Re}\lambda}-1) - \frac{t}{\lambda}\right) x_0,$$

so that, for $\operatorname{Re}\lambda < 0$,

$$\|\,\lambda(\lambda+\mathsf{D})^{-1}\,\| \ge \frac{1}{|\lambda|}(e^{-T\operatorname{Re}\lambda}-1) - T.$$

Taking $\lambda = Re^{i\phi}$, where $R > 0$ and $\phi \in (\pi/2,\pi)$ is fixed, we see that $\|\,\lambda(\lambda+\mathsf{D})^{-1}\,\|$ tends to ∞ as $R \to \infty$. Consequently, $\phi_{\mathsf{D}} < \phi$ for any $\phi \in (\pi/2,\pi)$, so that $\omega_{\mathsf{D}} \ge \pi/2$. The reverse inequality has already been proved. $\square$

4.1.3 Interpolation of the spaces $\mathcal{C}^{n+\alpha}_{0\mapsto 0}([0,T]\,;E)$

Let $I = [0,T]$, where $T > 0$. It is an immediate consequence of the definition of a Hölder space that if $0 \le \beta \le \alpha$, then $\mathcal{C}^\alpha(I;E) \hookrightarrow \mathcal{C}^\beta(I;E)$ and $\mathcal{C}^\alpha_{0\mapsto 0}(I;E) \hookrightarrow \mathcal{C}^\beta_{0\mapsto 0}(I;E)$. We shall now show that the spaces $\mathcal{C}^{n+\alpha}_{0\mapsto 0}(I;E)$ and $h^{n+\alpha}_{0\mapsto 0}(I;E)$ are, in fact, interpolation spaces between $\mathcal{C}^n_{0\mapsto 0}(I;E)$ and $\mathcal{C}^{n+1}_{0\mapsto 0}(I;E)$ for $n = 0,1,\ldots$ and $0 < \alpha < 1$.

Proposition 4.4. *Let $T > 0$, let n be a nonnegative integer, and let $0 < \alpha < 1$. Then*

$$(\mathcal{C}^n_{0\mapsto 0}(I;E), \mathcal{C}^{n+1}_{0\mapsto 0}(I;E))_{\alpha,\infty} = \mathcal{C}^{n+\alpha}_{0\mapsto 0}([0,T]\,;E)$$

and

$$(\mathcal{C}^n_{0\mapsto 0}(I;E), \mathcal{C}^{n+1}_{0\mapsto 0}(I;E))_{\alpha} = h^{n+\alpha}_{0\mapsto 0}([0,T]\,;E)$$

with equivalence of norms. In particular, $\mathcal{D}_{\mathsf{D}}(\alpha,\infty) = \mathcal{C}^\alpha_{0\mapsto 0}(I;E)$ and $\mathcal{D}_{\mathsf{D}}(\alpha) = h^\alpha_{0\mapsto 0}(I;E)$.

Proof. The second claim of the theorem is obtained from the first one by choosing $n = 0$. By Proposition 2.17 it suffices to prove the first claim in this special case. Thus we only have to establish the embeddings

$$\mathcal{D}_{\mathsf{D}}(\alpha,\infty) \hookrightarrow \mathcal{C}^\alpha_{0\mapsto 0}(I;E)$$

and

$$\mathcal{C}^{\alpha}_{0\mapsto 0}(I;E) \hookrightarrow \mathcal{D}_{\mathsf{D}}(\alpha,\infty)$$

together with the equality of the sets $\mathcal{D}_{\mathsf{D}}(\alpha)$ and $h^{\alpha}_{0\mapsto 0}(I;E)$.

We have

$$\left((\lambda+\mathsf{D})^{-1}f\right)(t) = \int_0^t e^{-\lambda(t-s)} f(s)\,ds,$$

and, since $\int_0^t \lambda e^{\lambda(s-t)} f(t)\,ds = (1-e^{-\lambda t})f(t)$, we infer that

$$\begin{aligned}\left(\mathsf{D}(\lambda+\mathsf{D})^{-1}f\right)(t) &= f(t) - \lambda\int_0^t e^{\lambda(s-t)} f(s)\,ds \\ &= e^{-\lambda t} f(t) + \int_0^t \lambda e^{\lambda(s-t)}(f(t)-f(s))\,ds,\end{aligned}$$

and hence

(4.2)

$$\lambda^{\alpha}\left\|\left(\mathsf{D}(\lambda+\mathsf{D})^{-1}f\right)(t)\right\| \leq \lambda^{\alpha} e^{-\lambda t}\,\|f(t)\| + \int_0^t \lambda^{1+\alpha} e^{\lambda(s-t)}\,\|f(t)-f(s)\|\,ds.$$

Let us first assume that $f \in \mathcal{C}^{\alpha}_{0\mapsto 0}(I;E)$. Then

$$\|f(t)-f(s)\| \leq [\,f\,]_{\alpha}\,|t-s|^{\alpha}$$

for all $s,t \in I$. In particular, $\|f(t)\|_E = \|f(t)-f(0)\|_E \leq [\,f\,]_{\alpha} t^{\alpha}$, so that

$$\lambda^{\alpha}\left\|e^{-\lambda t} f(t)\right\|_E \leq (\lambda t)^{\alpha} e^{-\lambda t}[\,f\,]_{\alpha} \leq [\,f\,]_{\alpha}\sup_{t>0} t^{\alpha} e^{-t}.$$

For all $\lambda > 0$ we also have

$$\begin{aligned}\lambda^{\alpha}\int_0^t \lambda e^{-\lambda(t-s)}\,\|f(t)-f(s)\|\,ds &\leq [\,f\,]_{\alpha}\int_0^t e^{-\lambda(t-s)}\lambda^{\alpha}(t-s)^{\alpha}\lambda\,ds \\ &= [\,f\,]_{\alpha}\int_0^{\lambda t} \tau^{\alpha} e^{-\tau}\,d\tau \\ &\leq [\,f\,]_{\alpha}\int_0^{\infty} \tau^{\alpha} e^{-\tau}\,d\tau.\end{aligned}$$

Since the last integral is finite, we conclude that

$$[\,f\,]_{\mathcal{D}_{\mathsf{D}}(\alpha,\infty)} \leq c[\,f\,]_{\alpha},$$

where $c = \sup_{t>0} t^\alpha e^{-t} + \int_0^\infty \tau^\alpha e^{-\tau}\, d\tau$. Therefore, $\mathcal{C}^\alpha_{0\mapsto 0}(I;E) \hookrightarrow \mathcal{D}_{\mathsf{D}}(\alpha,\infty)$.

To prove the converse embedding, we assume that $f \in \mathcal{D}_{\mathsf{D}}(\alpha,\infty)$, and that $s,t \in I$. Then, by the definition of the functional K (see p. 160), there exist functions $g \in \mathcal{C}_{0\mapsto 0}(I;E)$ and $h \in \mathcal{C}_{0\mapsto 0}(I;E)$ such that $f = g + h$ and

$$\| g \|_{\mathcal{C}(I;E)} + |t-s| \, \| h \|_{\mathcal{C}^1(I;E)} \le 2K(|t-s|\,, f, \mathcal{C}_{0\mapsto 0}(I;E), \mathcal{C}_{0\mapsto 0}(I;E)).$$

We also have

$$\begin{aligned} \| f(t) - f(s) \| &\le \| g(t) \| + \| g(s) \| + \| h(t) - h(s) \| \\ &\le 2 \| g \|_{\mathcal{C}(I;E)} + |t-s| \, [\, h \,]_{\mathcal{C}^1(I;E)} \\ &\le 2 \| g \|_{\mathcal{C}(I;E)} + 2\, |t-s| \, \| h \|_{\mathcal{C}^1(I;E)} \,. \end{aligned}$$

Combining these inequalities, we get

$$\frac{\| f(t) - f(s) \|}{|t-s|^\alpha} \le 4\, |t-s|^{-\alpha} \, K(|t-s|\,, f, \mathcal{C}_{0\mapsto 0}(I;E), \mathcal{C}_{0\mapsto 0}(I;E)), \tag{4.3}$$

and hence

$$[\, f \,]_{\mathcal{C}^\alpha(I;E)} \le 4 \, \| f \|_{(\mathcal{C}_{0\mapsto 0}(I;E), \mathcal{C}^1_{0\mapsto 0}(I;E))_{\alpha,\infty}} \,.$$

it follows that $\mathcal{D}_{\mathsf{D}}(\alpha,\infty) = \mathcal{C}^\alpha_{0\mapsto 0}(I;E)$ with equivalence of norms.

It remains to show that $\mathcal{D}_{\mathsf{D}}(\alpha) = h^\alpha_{0\mapsto 0}(I;E)$. Assume, first, that $f \in h^\alpha_{0\mapsto 0}(I;E)$. We take an arbitrary $\varepsilon > 0$ and some $\delta > 0$ such that

$$\sup_{\substack{|t-s|<\delta \\ t,s\in I}} \frac{\| f(t) - f(s) \|}{|t-s|^\alpha} \le \varepsilon.$$

The inequality (4.2) is still valid. For $0 < t < \delta$, taking $s = 0$, we get $\lambda^\alpha e^{-\lambda t} \, \| f(t) \| \le \varepsilon \, (\lambda t)^\alpha e^{-\lambda t}$, which implies that

$$\lambda^\alpha e^{-\lambda t} \, \| f(t) \| \le \varepsilon \sup_{\tau>0} \tau^\alpha e^{-\tau} \qquad (0 < t < \delta, \lambda > 0).$$

If $t \ge \delta$, we have

$$\lambda^\alpha e^{-\lambda t} \, \| f(t) \| \le [\, f \,]_\alpha \sup_{\tau > \lambda\delta} \tau^\alpha e^{-\tau} \qquad (\lambda > 0).$$

Consequently,

$$\lambda^\alpha \left\| \, e^{-\lambda \underline{t}} \, \| f(\underline{t}) \| \, \right\|_\infty \le \varepsilon \sup_{\tau>0} \tau^\alpha e^{-\tau} + [\, f \,]_\alpha \sup_{\tau>\lambda\delta} \tau^\alpha e^{-\tau}$$

for any $\lambda > 0$. Choosing λ big enough we can force the second term of the right hand side to be less than ε. Since this can be done for any $\varepsilon > 0$, it follows that $\lambda^\alpha \left\| \, e^{-\lambda \underline{t}} \, \| f(\underline{t}) \| \, \right\|_\infty \to 0$ as $\lambda \to \infty$.

Let us now look at the second term in the right hand member of (4.2). We have

$$\begin{aligned}
&\int_0^t \lambda^{1+\alpha} e^{-\lambda(t-s)} \, \| f(t) - f(s) \| \, ds \\
&\qquad = \int_0^{\lambda t} \lambda^{\alpha} e^{-\tau} \, \| f(t) - f(t-\tau/\lambda) \| \, d\tau \\
&\qquad = \varepsilon \int_0^{\lambda\delta} \lambda^{\alpha} (\tau/\lambda)^{\alpha} e^{-\tau} \, d\tau + [\, f \,]_{\alpha} \int_{\min\{\lambda\delta, \lambda t\}}^{\lambda t} \lambda^{\alpha} (\tau/\lambda)^{\alpha} e^{-\tau} \, d\tau \\
&\qquad \le \varepsilon \int_0^{\infty} \tau^{\alpha} e^{-\tau} \, d\tau + [\, f \,]_{\alpha} \int_{\lambda\delta}^{\infty} \tau^{\alpha} e^{-\tau} \, d\tau.
\end{aligned}$$

The last inequality yields

$$\left\| \int_0^{\underline{t}} \lambda^{1+\alpha} e^{-\lambda(\underline{t}-s)} \, \| f(\underline{t}) - f(s) \| \, ds \right\|_{\infty} \le \varepsilon \int_0^{\infty} \tau^{\alpha} e^{-\tau} \, d\tau + [\, f \,]_{\alpha} \int_{\delta/\lambda}^{\infty} \tau^{\alpha} e^{-\tau} \, d\tau.$$

Choosing $\lambda > 0$ that is big enough, we can force the last term of the right hand side of this inequality to be less than ε. It follows that

$$\lim \lambda \to \infty \left\| \int_0^{\underline{t}} \lambda^{1+\alpha} e^{-\lambda(\underline{t}-s)} \, \| f(\underline{t}) - f(s) \| \, ds \right\|_{\infty} = 0,$$

which means that $f \in \mathcal{D}_{\mathsf{D}}(\alpha)$. Thus, we have shown that $h^{\alpha}_{0 \mapsto 0}(I; E) \subseteq \mathcal{D}_{\mathsf{D}}(\alpha)$.

To prove the reverse inclusion, we assume that $f \in \mathcal{D}_{\mathsf{D}}(\alpha)$. Then for any $\varepsilon > 0$ there is some $\delta > 0$ such that $0 < \tau < \delta$ implies that

$$\tau^{-\alpha} K(\tau, f, \mathcal{C}_{0 \mapsto 0}(I; E), \mathcal{C}_{0 \mapsto 0}(I; E)) < \varepsilon/4.$$

Hence, by (4.3),

$$\sup_{\substack{|t-s|<\delta \\ t,s \in I}} \frac{\| f(t) - f(s) \|}{|t-s|^{\alpha}} \le \varepsilon.$$

Consequently,

$$\lim_{\delta \downarrow 0} \sup_{\substack{|t-s|<\delta \\ t,s \in I}} \frac{\| f(t) - f(s) \|}{|t-s|^{\alpha}} = 0,$$

so that $f \in h^{\alpha}_{0 \mapsto 0}(I; E)$. This shows that $\mathcal{D}_{\mathsf{D}}(\alpha) \subseteq h^{\alpha}_{0 \mapsto 0}(I; E)$, and the proof is complete. $\square$

We recall that $\mathcal{D}_{\mathsf{D}|_n}(\theta)$ is the closure in $\mathcal{D}_{\mathsf{D}|_n}(\alpha,\infty)$ of $\mathcal{D}(\mathsf{D}|_n)$, and thus of any of the spaces $\mathcal{D}_{\mathsf{D}|_n}(\theta,p)$, where $n = 0,1,\dots$ and $1 > \theta > \alpha$. Hence, the above theorem yields the following characterisation of the spaces $h^{n+\alpha}_{0\mapsto 0}([0,T]\,;E)$.

Corollary 4.5. *Let $0 < \alpha < \theta < 1$ and let n be a nonnegative integer. Then $h^{n+\alpha}_{0\mapsto 0}(I;E)$ is the closure of $\mathcal{C}^{n+\theta}_{0\mapsto 0}(I;E)$ in $\mathcal{C}^{n+\alpha}_{0\mapsto 0}(I;E)$.*

Since the spaces $\mathcal{C}^{n+\alpha}_{0\mapsto 0}([0,T]\,;E)$ and $h^{n+\alpha}_{0\mapsto 0}([0,T]\,;E)$ are interpolation spaces between complex Banach spaces, Proposition 4.4 also provides a proof of the following well-known result.

Corollary 4.6. *Let $T > 0$, let $0 < \alpha < 1$, and let n be a nonnegative integer. Then the spaces $\mathcal{C}^{n+\alpha}_{0\mapsto 0}([0,T]\,;E)$ and $h^{n+\alpha}_{0\mapsto 0}([0,T]\,;E)$ are complex Banach spaces.*

For the sake of completeness, we also include the following generalisation of Proposition 4.4.

Theorem 4.7. *If $T > 0$, $0 \le \alpha \le \beta$, $0 < \theta < 1$ and $(1-\theta)\alpha + \theta\beta \notin \mathbb{N}$, then*

$$(\mathcal{C}^{\alpha}_{0\mapsto 0}([0,T]\,;E), \mathcal{C}^{\beta}_{0\mapsto 0}([0,T]\,;E))_{\theta,\infty} = \mathcal{C}^{(1-\theta)\alpha+\theta\beta}_{0\mapsto 0}([0,T]\,;E). \tag{4.4}$$

Proof. Let us omit $([0,T]\,;E)$ from the notation. Theorem 2.18 shows that $\mathcal{C}^{k}_{0\mapsto 0} \in J_{k/n}(\mathcal{C}_{0\mapsto 0}, \mathcal{C}^{n}_{0\mapsto 0}) \cap K_{k/n}(\mathcal{C}_{0\mapsto 0}, \mathcal{C}^{n}_{0\mapsto 0})$ for integers k and n, with $1 \le k \le n$.

If m and n are integers with $m < \alpha < m+1 \le n$, then $\mathcal{C}^{\alpha}_{0\mapsto 0} = (\mathcal{C}^{m}_{0\mapsto 0}, \mathcal{C}^{m+1}_{0\mapsto 0})_{\alpha-m,\infty}$ and $h^{\alpha}_{0\mapsto 0} = (\mathcal{C}^{m}_{0\mapsto 0}, \mathcal{C}^{m+1}_{0\mapsto 0})_{\alpha-m}$. Using the Reiteration Theorem, we infer that

$$\mathcal{C}^{\alpha}_{0\mapsto 0} = (\mathcal{C}_{0\mapsto 0}, \mathcal{C}^{n}_{0\mapsto 0})_{\alpha/n,\infty},$$

and

$$h^{\alpha}_{0\mapsto 0} = (\mathcal{C}_{0\mapsto 0}, \mathcal{C}^{n}_{0\mapsto 0})_{\alpha/n},$$

for any non-integer α and any integer n with $0 < \alpha < n$. Thus, if n is an integer and $0 \le \alpha \le \beta < n$, then $\mathcal{C}^{\alpha}_{0\mapsto 0} \in J_{\alpha/n}(\mathcal{C}_{0\mapsto 0}, \mathcal{C}^{n}_{0\mapsto 0}) \cap K_{\alpha/n}(\mathcal{C}_{0\mapsto 0}, \mathcal{C}^{n}_{0\mapsto 0})$, and $\mathcal{C}^{\beta}_{0\mapsto 0} \in J_{\beta/n}(\mathcal{C}_{0\mapsto 0}, \mathcal{C}^{n}_{0\mapsto 0}) \cap K_{\beta/n}(\mathcal{C}_{0\mapsto 0}, \mathcal{C}^{n}_{0\mapsto 0})$. It follows that if $0 < \theta < 1$, then

$$(\mathcal{C}^{\alpha}_{0\mapsto 0}, \mathcal{C}^{\beta}_{0\mapsto 0})_{\theta,\infty} = (\mathcal{C}_{0\mapsto 0}, \mathcal{C}^{n}_{0\mapsto 0})_{\frac{(1-\theta)\alpha+\theta\beta}{n},\infty}.$$

which equals $\mathcal{C}^{(1-\theta)\alpha+\theta\beta}_{0\mapsto 0}$ if $(1-\theta)\alpha+\theta\beta \notin \mathbb{N}$. Analogously

$$(\mathcal{C}^{\alpha}_{0\mapsto 0}, \mathcal{C}^{\beta}_{0\mapsto 0})_{\theta} = (\mathcal{C}_{0\mapsto 0}, \mathcal{C}^{n}_{0\mapsto 0})_{\frac{(1-\theta)\alpha+\theta\beta}{n}}.$$

which equals $h^{(1-\theta)\alpha+\theta\beta}_{0\mapsto 0}$ if $(1-\theta)\alpha+\theta\beta \notin \mathbb{N}$. ∎

Remark 4.8. A similar result holds for interpolation between Hölder spaces. In fact, if $0 \leq \alpha \leq \beta$, $0 < \theta < 1$ and $\alpha + \theta(\beta - \alpha) \notin \mathbb{N}$, then

$$(\mathcal{C}^{\alpha}(I;E), \mathcal{C}^{\beta}(I;E))_{\theta,\infty} = \mathcal{C}^{\alpha+\theta(\beta-\alpha)}(I;E) \tag{4.5}$$

and

$$(\mathcal{C}^{\alpha}(I;E), \mathcal{C}^{\beta}(I;E))_{\theta} = h^{\alpha+\theta(\beta-\alpha)}(I;E). \tag{4.6}$$

Also note that if $\alpha + \theta(\beta - \alpha) \in \mathbb{N}$, then, for example,

$$(\mathcal{C}^{\alpha}(I;E), \mathcal{C}^{\beta}(I;E))_{\theta,\infty} \subsetneq \mathcal{C}^{\alpha+\theta(\beta-\alpha)}(I;E),$$

the interpolation space to the left of the inclusion being a so called *Zygmund class*.

Since we will not make use of these results, the proofs are omitted. A proof of formula (4.5) that makes frequent use of the Reiteration Theorem can be found in [9], pp. 5–8, 28–32.

4.1.4 Fractional derivatives

For $\alpha > 0$ we define $g_\alpha \in L^1(\mathbb{R}_+)$ by

$$g_\alpha(t) := \frac{1}{\Gamma(\alpha)} t^{\alpha-1}. \tag{4.7}$$

If $f \in L^1([0,T])$ and $0 < \alpha < 1$, then the convolution

$$(g_{1-\alpha} * f)(t) = \int_0^t g_{1-\alpha}(t-s) f(s)\, ds. \tag{4.8}$$

exists for a.e. $t \in [0,T]$. If $f \in \mathcal{C}([0,T];E)$ is continuous, then $g_{1-\alpha} * f$ exists for all $t \in [0,T]$. We define the *fractional derivative* $(D^\alpha f)(t)$ of $f \in \mathcal{C}_{0\mapsto 0}([0,T];E)$ by

$$(D^\alpha f)(t) := \frac{\mathrm{d}}{dt}(g_{1-\alpha} * f)(t), \quad t > 0, \tag{4.9}$$

if $0 < t < T$, as the right hand derivative

$$(D^\alpha f)(0) := \lim_{h \downarrow 0} \frac{(g_{1-\alpha} * f)(h)}{h} \tag{4.10}$$

at $t = 0$, and as the left hand derivative

$$(D^\alpha f)(T) := \lim_{h \uparrow 0} \frac{(g_{1-\alpha} * f)(T+h) - (g_{1-\alpha} * f)(T)}{h} \tag{4.11}$$

at $t = T$, whenever the respective limit exists.

One motivation for this definition of $(D^\alpha f)(t)$ is that if we formally calculate the Laplace transform of $\frac{\mathrm{d}}{dt}(g_{1-\alpha} * f)(t)$, we obtain

$$s \mapsto s \cdot \frac{1}{s^{1+\alpha}} F(s) = \frac{1}{s^\alpha} F(s),$$

where F is the Laplace transform of f, whereas the Laplace transform of $Df = f'$ is $s \mapsto \frac{1}{s} F(s)$.

If u is defined on a rectangle $Q = [0, \tau] \times [0, \xi]$ and takes values in a complex Banach space E, then the *fractional partial derivatives* of u can be defined by

$$(D_t^\alpha u)(t, x) := (D^\alpha u(\underline{t}, x))(t) \tag{4.12}$$

$$(D_x^\alpha u)(t, x) := (D^\alpha u(t, \underline{x}))(x) \tag{4.13}$$

whenever these fractional derivatives exist.

Let us now show that the fractional differential operator in $\mathcal{C}_{0\mapsto 0}([0, T]\,; E)$, whose value at f is the function $D^\alpha f(\underline{t})$, is actually the operator D^α.

THEOREM 4.9. *Let $I = [0, T]$, where $T > 0$, let D be the differential operator in $\mathcal{C}_{0\mapsto 0}(I; E)$ defined above, let n be a nonnegative integer, and let $0 < \alpha < 1$. Then D^α is densely defined and positive with spectral angle $\alpha\pi/2$, and we have*

$$\mathcal{D}(\mathsf{D}^{n+\alpha}) = \{ f \in \mathcal{C}^n_{0\mapsto 0}(I; E) \parallel g_{1-\alpha} * f^{(n)} \in \mathcal{C}^1_{0\mapsto 0}(I; E) \} \tag{4.14}$$

and

$$\mathsf{D}^\alpha f = D^\alpha D^n f \qquad (f \in \mathcal{D}(\mathsf{D}^{n+\alpha})). \tag{4.15}$$

Moreover, for any $\alpha > 0$ and any $\theta \in (0, 1)$ such that $\alpha\theta \notin \mathbb{N}$ we have

$$\mathcal{D}_{\mathsf{D}^\alpha}(\theta, \infty) = \mathcal{C}^{\theta\alpha}_{0\mapsto 0}(I; E)$$

and

$$\mathcal{D}_{\mathsf{D}^\alpha}(\theta) = h^{\theta\alpha}_{0\mapsto 0}(I; E).$$

Proof. By Proposition 4.3 and Theorem 2.21, D^α is densely defined and positive, and $\omega_{\mathsf{D}^\alpha} \le \alpha\pi/2$. That $\omega_{\mathsf{D}^\alpha}$ is exactly $\alpha\pi/2$ is not difficult to show (see [5]), but we do not need this result, and therefore omit the proof.

We have also shown that

$$((\lambda + \mathsf{D})^{-1} f)(t) = \int_0^t e^{-\lambda(t-\tau)} f(\tau)\, d\tau \qquad (0 \le t \le T)$$

for any $\lambda \in \mathbb{C}$ and any $f \in \mathcal{C}_{0\mapsto 0}(I;E)$. Hence, by (2.49) and Fubini's Theorem (Theorem A.16),

$$\begin{aligned}(\mathsf{D}^{\alpha-1} f)(t) &= \frac{\sin\pi\alpha}{\pi}\int_0^\infty s^{\alpha-1}\int_0^t e^{-s(t-\tau)}f(\tau)\,d\tau\,ds\\ &= \frac{\sin\pi\alpha}{\pi}\int_0^t\left(\int_0^\infty s^{\alpha-1}e^{-s(t-\tau)}\,ds\right)f(\tau)\,d\tau\\ &= \int_0^t g_{1-\alpha}(t-\tau)f(\tau)\,d\tau = (g_{1-\alpha}*f)(t),\end{aligned}$$

where we have used the fact that

$$\frac{\sin\pi\alpha}{\pi} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}.$$

It follows that

$$\mathcal{D}(\mathsf{D}^\alpha) = \mathcal{D}(\mathsf{D}\,\mathsf{D}^{\alpha-1}) = \{f \in \mathcal{C}_{0\mapsto 0}(I;E) \mid g_{1-\alpha}*f \in \mathcal{C}^1_{0\mapsto 0}(I;E)\}, \tag{4.16}$$

and that

$$\mathsf{D}^\alpha f = \mathsf{D}\,\mathsf{D}^{\alpha-1} f = D^\alpha f$$

for $f \in \mathcal{D}(\mathsf{D}^\alpha)$. This proves (4.14) when $n=0$.

If $n>0$ we get $\mathsf{D}^{n+\alpha} = \mathsf{D}^\alpha\,\mathsf{D}^n$, and (4.14) follows from the case $n=0$.

The formulas for $\mathcal{D}_{\mathsf{D}^\alpha}(\theta,\infty)$ and $\mathcal{D}_{\mathsf{D}^\alpha}(\theta)$ follow from Theorem 2.21 (g) and Theorem 4.7. □

4.1.5 Hölder and little Hölder continuous functions on a rectangle

We now introduce Hölder spaces and little Hölder spaces on a rectangle, and prove some simple results for these spaces.

Definition 4.10. Let $0<\mu,\nu<1$, let E be a Banach space, let $Q := [0,\tau]\times[0,\xi]$, and let $\partial_0 Q := \{(t,x)\in Q \parallel t=0 \text{ or } x=0\}$, where $\tau,\xi>0$. We introduce the spaces $\mathcal{C}^{\mu,\nu}(Q;E)$ and $\mathcal{C}^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E)$ of Hölder continuous functions $f : Q \longrightarrow E$ by setting

$$\begin{aligned}\mathcal{C}^{\mu,\nu}(Q;E) &:= \{f\in\mathcal{C}(Q;E) \mid [\,f\,]_{\mathcal{C}^{\mu,\nu}(Q;E)}<\infty\}\\ \mathcal{C}^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E) &:= \{f\in\mathcal{C}^{\mu,\nu}(Q;E) \parallel f(t,x)=0 \text{ for all } (t,x)\in\partial_0 Q\},\end{aligned}$$

where the seminorm $[\ ]_{\mathcal{C}^{\mu,\nu}(Q;E)}$ and the norm $\|\ \|_{\mathcal{C}^{\mu,\nu}(Q;E)}$ on these spaces are defined by are defined by

$$\begin{aligned}[\,f\,]_{\mathcal{C}^{\mu,\nu}(Q;E)} &:= \sup_{\substack{(t,x)(s,y)\in Q\\(t,x)\neq(s,y)}}\frac{\|\,f(t,x)-f(s,y)\,\|_E}{|t-s|^\mu+|x-y|^\nu},\\ \|\,f\,\|_{\mathcal{C}^{\mu,\nu}(Q;E)} &:= \|\,f\,\|_{\mathcal{C}(Q;E)}+[\,f\,]_{\mathcal{C}^{\mu,\nu}(Q;E)}.\end{aligned}$$

We also define the spaces $h^{\mu,\nu}(Q;E)$ and $h^{\mu,\nu}_{\partial_0 Q}$ of little Hölder continuous functions $f : Q \longrightarrow E$ by

$$h^{\mu,\nu}(Q;E) := \{f \in \mathcal{C}^{\mu,\nu}(Q;E) \parallel \lim_{\delta\downarrow 0} \sup_{\substack{0<|(t,x)-(s,y)|<\delta \\ (t,x),(s,y)\in Q}} \frac{\| f(t,x) - f(s,y) \|_E}{|t-s|^\mu + |x-y|^\nu} = 0\}$$

$$h^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E) := \{f \in h^{\mu,\nu}(Q;E) \mid f(t,x) = 0 \text{ for all } (t,x) \in \partial_0 Q\}.$$

The last two spaces are also provided with the seminorm $[\ ]_{\mathcal{C}^{\mu,\nu}(Q;E)}$ and the norm $\|\ \|_{\mathcal{C}^{\mu,\nu}(Q;E)}$.

When $E = \mathbb{R}$ or $E = \mathbb{C}$, we usually omit the '$;\mathbb{R}$' or '$;\mathbb{C}$' from the notation and simply write $\mathcal{C}(Q)$, $\mathcal{C}^{\mu,\nu}(Q)$ etc.

Lemma 4.11. *Let E be a complex Banach space and let $Q := [0,\tau] \times [0,\xi]$, where $\tau, \xi > 0$. To any function $f : Q \to \mathbb{C}$ we can define a function $\tilde{f} = \tilde{\imath}(f) : [0,\xi] \to ([0,\tau] \to \mathbb{C})$ by $\tilde{f}(x) = f(\underline{t}, x)$ for $x \in [0,\xi]$. This defines a bijection $\tilde{\imath}$ from the set of functions $f : Q \longrightarrow E$ to the set of functions $\tilde{f}$ that map $t \in [0,\tau]$ to $\tilde{f}(t) : [0,\xi] \longrightarrow E$. When $\tilde{f} = \tilde{\imath}(f)$, the following assertions are hold:*

(a) *The mapping $\tilde{\imath}$ defines an isometric isomorphism between*

$$\mathcal{C}(Q;E) \text{ and } \mathcal{C}([0,\tau]\,;\mathcal{C}([0,\xi]\,;E))$$

and between

$$\mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E) \text{ and } \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))$$

(b) *For $0 < \mu, \nu < 1$, the mapping $\tilde{\imath}$ defines an isomorphism between*

$$\mathcal{C}^{\mu,\nu}(Q;E) \text{ and } \mathcal{C}^{\mu}([0,\tau]\,;\mathcal{B}([0,\xi]\,;E)) \cap \mathcal{B}([0,\tau]\,;\mathcal{C}^{\nu}([0,\xi]\,;E))$$

and between

$$\mathcal{C}^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E) \text{ and } \mathcal{C}^{\mu}_{0\mapsto 0}([0,\tau]\,;\mathcal{B}([0,\xi]\,;E)) \cap \mathcal{B}([0,\tau]\,;\mathcal{C}^{\nu}_{0\mapsto 0}([0,\xi]\,;E)).$$

Moreover,

$$\| f \|_{\mathcal{C}^{\mu,\nu}(Q;E)} \leq \| \tilde{f} \|_{\mathcal{C}^{\nu}([0,\tau];\mathcal{B}([0,\xi];E))} + \| \tilde{f} \|_{\mathcal{B}([0,\tau];\mathcal{C}^{\mu}([0,\xi];E))} \leq 2\| f \|_{\mathcal{C}^{\mu,\nu}(Q;E)}.$$

(c)*For $0 < \mu, \nu < 1$, the mapping $\tilde{\imath}$ defines an isomorphism between*

$$h^{\mu,\nu}(Q;E). \text{ and } h^{\nu}([0,\tau]\,;\mathcal{B}([0,\xi]\,;E)) \cap \mathcal{B}([0,\tau]\,;h^{\mu}([0,\xi]\,;E))$$

and between

$$h^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E) \text{ and } h^{\nu}_{0\mapsto 0}([0,\tau]\,;\mathcal{B}([0,\xi]\,;E)) \cap \mathcal{B}([0,\tau]\,;h^{\mu}_{0\mapsto 0}([0,\xi]\,;E)).$$

(d) *The spaces $\mathcal{C}^{\mu,\nu}(Q;E), \mathcal{C}^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E), h^{\mu,\nu}(Q;E),$ and $h^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E)$ are Banach spaces.*

Proof. The bijectivity of $\tilde{\imath}$ is trivial.

Now let $\tilde{f} = \tilde{\imath}(f)$, where $f : Q \longrightarrow E$.

(a) Assume first that $f \in \mathcal{C}(Q; E)$. Then

$$|\tilde{f}(t)(x) - \tilde{f}(s)(y)| = |f(t,x) - f(s,y)|.$$

Since f is uniformly continuous on the bounded square in question, for any $\varepsilon > 0$ there is some $\delta > 0$, such that $|\tilde{f}(t)(x) - \tilde{f}(y)(s)| < \varepsilon$ for any two pairs $(t,x),(s,y) \in [0,\xi] \times [0,\tau]$ with $|x-y| < \delta$ and $|t-s| < \delta$. Thus, taking $t = s$, we see that $\tilde{f}(t)$ is (uniformly) continuous from $[0,\xi]$ to E. If we take $x = y$, we also see that $\| \tilde{f}(t) - \tilde{f}(s) \|_{\mathcal{C}([0,\xi];E)} \leq \varepsilon$ as soon as $|t-s| < \delta$, which implies that $\tilde{f}$ is continuous.

We also have

$$|f(t,x) - f(s,y)| \leq |\tilde{f}(t)(x) - \tilde{f}(t)(y)| + |\tilde{f}(t)(y) - \tilde{f}(s)(y)|.$$

If $\tilde{f} \in \mathcal{C}([0,\xi]\,;\mathcal{C}([0,\tau]\,;E))$ we deduce the continuity of f from this. It follows that the restriction of $\tilde{\imath}$ to $\mathcal{C}(Q;E)$ is bijective. Obviously

$$\| f \|_{\mathcal{C}(Q;E)} = \sup_{t\in[0,\tau]} \sup_{x\in[0,\xi]} \| \tilde{f}(t)(x) \|_E = \| \tilde{f} \|_{\mathcal{B}([0,\tau];\mathcal{B}([0,\xi];E))},$$

so that it is an isometric isomorphism.

Clearly f maps $\partial_0 Q$ to $\{0\}$ iff and only if $\tilde{f}(0)(x) = \tilde{f}(t)(0)$ for all $(t,x) \in Q$.

(b) We bear in mind the previous equation. We also have

$$\begin{aligned}
[\, f \,]_{\mathcal{C}^{\mu,\nu}(Q;E)} &= \sup_{\substack{(t,x),(s,y)\in Q\\(t,x)\neq(s,y)}} \frac{\| f(t,x) - f(s,y) \|}{|t-s|^{\mu} + |x-y|^{\nu}} \\
&\leq \sup_{\substack{(t,x),(s,y)\in Q\\(t,x)\neq(s,y)}} \frac{\| f(t,x) - f(t,y) \|_E}{|t-s|^{\mu} + |x-y|^{\nu}} + \sup_{\substack{(t,x),(s,y)\in Q\\(t,x)\neq(s,y)}} \frac{\| f(t,y) - f(s,y) \|_E}{|t-s|^{\mu} + |x-y|^{\nu}} \\
&\leq \sup_{\substack{t,s\in[0,\tau]\\t\neq s}} \frac{\| \tilde{f}(t) - \tilde{f}(s) \|_{\mathcal{B}([0,\xi];E)}}{|t-s|^{\mu}} + \sup_{t\in[0,\tau]} [\, \tilde{f}(t) \,]_{\mathcal{C}^{\nu}([0,\xi];E)} \\
&= [\, \tilde{f} \,]_{\mathcal{C}^{\mu}([0,\tau];\mathcal{B}([0,\xi];E))} + \| \, [\, \tilde{f}(\underline{t}) \,]_{\mathcal{C}^{\nu}([0,\xi];E)} \, \|_{\mathcal{B}([0,\tau];\mathbb{R})}.
\end{aligned}$$

It follows that

$$\| f \|_{\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\xi];E)} \leq \| \tilde{f} \|_{\mathcal{C}^{\mu}([0,\tau];\mathcal{B}([0,\xi];E))} + \| \tilde{f} \|_{\mathcal{B}([0,\tau];\mathcal{C}^{\nu}([0,\xi];E))}.$$

On the other hand

$$\begin{aligned}
\| \tilde{f} \|_{\mathcal{C}^{\mu}([0,\tau];\mathcal{B}([0,\xi];E))} &= \sup_{\substack{t,s\in[0,\tau]\\t\neq s}} \frac{\| \tilde{f}(t) - \tilde{f}(s) \|_{\mathcal{B}([0,\xi];E)}}{|t-s|^{\mu}} + \sup_{t\in[0,\tau]} \| \tilde{f}(t) \|_{\mathcal{B}([0,\xi];E)} \\
&= \sup_{\substack{t,s\in[0,\tau]\\t\neq s}} \sup_{x\in[0,\xi]} \frac{|f(t,x) - f(t,y)|}{|t-s|^{\mu}} + \sup_{t\in[0,\tau]} \sup_{x\in[0,\xi]} \| f(t,x) \|_E \\
&\leq \sup_{\substack{(t,x),(s,y)\in Q\\(t,x)\neq(s,y)}} \frac{|f(t,x) - f(s,y)|}{|t-s|^{\mu} + |x-y|^{\nu}} + \sup_{\substack{t\in[0,\tau]\\x\in[0,\xi]}} \| f(t,x) \|_E \\
&= \| f \|_{\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\tau];E)},
\end{aligned}$$

and

$$\begin{aligned}\| \tilde{f} \|_{\mathcal{B}([0,\tau];\mathcal{C}^{\mu}([0,\xi];E))} &= \sup_{t\in[0,\tau]} \left\{ \sup_{\substack{x,y\in[0,\xi]\\ x\neq y}} \frac{\| \tilde{f}(t)(x) - \tilde{f}(t)(y) \|_E}{|x-y|^{\mu}} + \| \tilde{f}(t) \|_{\mathcal{C}([0,\xi];E)} \right\} \\
&= \sup_{t\in[0,\tau]} \sup_{\substack{x,y\in[0,\xi]\\ x\neq y}} \frac{\| f(t,x) - f(t,y) \|_E}{|x-y|^{\nu}} + \sup_{\substack{t\in[0,\tau]\\ x\in[0,\xi]}} \| f(t,x) \|_E \\
&\leq \sup_{\substack{(t,x),(s,y)\in Q\\ (t,x)\neq(s,y)}} \frac{\| f(t,x) - f(s,y) \|_E}{|t-s|^{\mu} + |x-y|^{\nu}} + \sup_{\substack{t\in[0,\tau]\\ x\in[0,\xi]}} \| f(t,x) \|_E \\
&= \| f \|_{\mathcal{C}^{\mu,\nu}([0,\tau]\times[0,\tau];E)},\end{aligned}$$

so that the equivalence of the norms asserted in (b) is proved. From this the other statements of part (b) follow.

(c) Assume that $f \in h^{\mu,\nu}(Q;E)$. Then

$$\tilde{f} \in \mathcal{C}^{\mu}([0,\tau]\,;\mathcal{B}([0,\xi]\,;E)) \cap \mathcal{B}([0,\tau]\,;\mathcal{C}^{\mu}([0,\xi]\,;E)),$$

and

$$\lim_{\delta\downarrow 0} \sup_{\substack{0<|(t,x)-(s,y)|<\delta\\ (t,x),(s,y)\in Q}} \frac{\| f(t,x) - f(s,y) \|_E}{|t-s|^{\mu} + |x-y|^{\nu}} = 0.$$

From this we see that

$$\lim_{\delta\downarrow 0} \sup_{\substack{0<|x-y|<\delta\\ x,y\in[0,\tau]}} \frac{\| \tilde{f}(t)(x) - \tilde{f}(t)(y) \|_E}{|x-y|^{\nu}} = 0,$$

i.e. $\tilde{f}(x) \in h^{\mu}([0,\xi]\,;E)$ for any $t \in [0,\tau]$. Hence, $f \in \mathcal{B}([0,\tau]\,;h^{\mu}([0,\xi]\,;E))$. We also deduce that

$$\lim_{\delta\downarrow 0} \sup_{\substack{0<|t-s|<\delta\\ t,s\in[0,\tau],x\in[0,\xi]}} \frac{\| \tilde{f}(t)(x) - \tilde{f}(s)(x) \|_E}{|t-s|^{\mu}} = 0,$$

i.e.,

$$\lim_{\delta\downarrow 0} \sup_{\substack{0<|t-s|<\delta\\ t,s\in[0,\tau],t\in[0,\xi]}} \frac{\| \tilde{f}(t) - \tilde{f}(s) \|_{\mathcal{C}([0,\xi];E)}}{|t-s|^{\mu}} = 0,$$

so that $\tilde{f} \in h^{\mu}([0,\tau]\,;\mathcal{C}([0,\xi]\,;E))$. Thus, we have shown that if $f \in h^{\mu,\nu}(Q;E)$ then $\tilde{f} \in h^{\mu}([0,\tau]\,;\mathcal{B}([0,\xi]\,;E)) \cap \mathcal{B}([0,\tau]\,;h^{\nu}([0,\xi]\,;E))$.

To prove the reverse implication, we first note that if $\mu < \mu' < 1$ and $\nu < \nu' < 1$, then $h^{\mu}([0,\tau]\,;\mathcal{B}([0,\xi]\,;E)) \cap \mathcal{B}([0,\tau]\,;h^{\nu}([0,\xi]\,;E))$ is included in the closure of

$$\mathcal{C}^{\mu'}([0,\tau]\,;\mathcal{B}([0,\xi]\,;E)) \cap \mathcal{B}([0,\tau]\,;\mathcal{C}^{\nu'}([0,\xi]\,;E))$$

in

$$\mathcal{C}^{\mu}([0,\tau]\,;\mathcal{B}([0,\xi]\,;E) \cap \mathcal{B}([0,\tau]\,;\mathcal{C}^{\nu}([0,\xi]\,;E)).$$

This is seen by considering the sequence $\left\{\widetilde{f}_n\right\}_{n=1}^{\infty}$, where

$$f_n(x) = \iint_{\mathbb{R}^2} \rho_n(t)\rho_n(x)\hat{f}(t,x)\,dt\,dx;$$

$$\hat{f}(t,x) = \begin{cases} f(t,x), & \text{if } (t,x) \in Q, \\ f(0,x), & \text{if } (t,x) \in (-\infty,0) \times [0,\xi], \\ f(\tau,x), & \text{if } (t,x) \in (\tau,\infty) \times [0,\xi], \\ f(t,0), & \text{if } (t,x) \in [0,\tau] \times (-\infty,0), \\ f(t,\xi), & \text{if } (t,x) \in [0,\tau] \times (\xi,\infty), \\ f(0,0), & \text{if } (t,x) \in (-\infty,0) \times (-\infty,0), \\ f(\tau,\xi), & \text{if } (t,x) \in (\tau,\infty) \times (\xi,\infty) \end{cases};$$

$\rho_n(t) = \frac{1}{n}\rho(\frac{t}{n})$; $\rho \in \mathcal{C}^\infty(\mathbb{R})$; $\operatorname{supp}\rho \subseteq (-1,1)$ and $\|\,\rho\,\|_{L^1(\mathbb{R})} = 1$. It can also be seen that $h^{\mu,\nu}(Q;E)$ contains the closure of $\mathcal{C}^{\mu',\nu'}(Q;E)$ in $\mathcal{C}^{\mu,\nu}(Q;E)$. In fact, if $\{f_n\}_{n=1}^{\infty} \subseteq \mathcal{C}^{\mu',\nu'}(Q;E)$ and $\lim_{n\to\infty}\|\,f_n - f\,\|_{\mathcal{C}^{\mu,\nu}(Q;E)}$, where $f \in \mathcal{C}^{\mu',\nu'}(Q;E)$, then for any $\varepsilon > 0$, we can first choose n so big that $\|\,f_n - f\,\|_{\mathcal{C}^{\mu,\nu}(Q;E)} < \varepsilon/3$. Then we choose $\delta' > 0$ so small that

$$\sup_{\substack{0<|(t,r)-(s,y)|<\delta \\ (t,x),(s,y)\in Q}} \frac{\|\,\widetilde{f}_n(t,x) - \widetilde{f}_n(s,y)\,\|_E}{|t-s|^\mu + |x-y|^\nu} < \varepsilon/3$$

for any δ with $0 < \delta < \delta'$. For such δ we thus have

$$\sup_{\substack{0<|(t,r)-(s,y)|<\delta \\ (t,x),(s,y)\in Q}} \frac{\|\,\tilde{f}(t)(x) - \tilde{f}(s)(y)\,\|_E}{|t-s|^\mu + |x-y|^\nu} < \varepsilon.$$

We conclude that

$$\lim_{\delta\downarrow 0} \sup_{\substack{0<|(t,x)-(s,y)|<\delta \\ (t,x),(s,y)\in Q}} \frac{\|\,f(t,x) - f(s,y)\,\|_E}{|t-s|^\mu + |x-y|^\nu} = 0,$$

and the first claim of part (c) has been established. The second claim is an immediate consequence of the first one.

(d) All the spaces are isomorphic to spaces of the form $X_1 \cap X_2$ with norms of the form $\|\ \|_{X_1\cap X_2} := \|\ \|_{X_1} + \|\ \|_{X_2}$, where the Banach spaces X_1 and X_2 are continuously embedded in $\mathcal{C}([0,\tau]\,;\mathcal{C}([0,\xi]\,;E))$. Hence, Cauchy sequences in $X_1 \cap X_2$ converge to the same function in X_1 and in X_2, and thus also in $X_1 \cap X_2$. □

4.2 An equation with resolvent commuting operators

Let α and β be positive numbers with $\alpha, \beta \leq 1$, $\alpha + \beta < 2$ and let E be a complex Banach space. In this section we prove some maximal regularity results for the equation

$$D_t^\alpha u(t,x) + D_x^\beta u(t,x) = f(t,x) \tag{4.17}$$

in the space $X := \mathcal{C}_{\partial_0 Q \mapsto 0}(Q; E)$, consisting of those continuous functions $u : Q \longrightarrow E$ defined on the rectangle $Q := [0, \tau] \times [0, \xi]$ for which $u(t, x) = 0$ for all $(t, x) \in \partial_0 Q := \{(t, x) \in Q \mid t = 0 \vee x = 0\}$, where $\tau, \xi > 0$. In order to be able to apply the method of sums to this problem, we transform equation (4.17) into an equation

$$\mathsf{D}^{\alpha}_{\tilde{X}} \tilde{u} + \mathsf{D}^{\beta}_{Y} \circ \tilde{u} = \tilde{f} \tag{4.18}$$

in $\tilde{X} := \mathcal{C}_{0 \mapsto 0}([0, \tau]\,; Y)$, where $Y := \mathcal{C}_{0 \mapsto 0}([0, \xi]\,; E)$. Here the lower indices of D denote the spaces of functions in which the differential operators have there domains. Thus, $\mathsf{D}^{\alpha}_{\tilde{X}}$ is fractional differentiation of functions in $\mathcal{C}_{0 \mapsto 0}([0, \tau]\,; Y)$, whereas $\mathsf{D}^{\beta}_{Y} \circ$ means fractional differentiation in $\mathcal{C}_{0 \mapsto 0}(Y; E)$.

Regarding $\mathsf{D}^{\beta}_{Y} \circ u$ we note that

$$(\mathsf{D}^{\alpha}_{Y} \circ \tilde{u})(x) = \mathsf{D}^{\beta}_{Y}(\tilde{u}(x)) \tag{4.19}$$

if $\tilde{u}(x) \in \mathcal{D}(\mathsf{D}^{\beta}_{Y})$. We will presently show that the mapping $u \mapsto \mathsf{D}^{\beta}_{Y} \circ u$ defines a positive operator $\mathsf{D}^{\beta}_{Y} \circ$ in $\tilde{X}$ with dense domain and spectral angle $\omega_{\mathsf{D}^{\beta}_{Y} \circ} = \beta\pi/2$. Our aim is therefore to apply Lemma 3.10 and Theorems 3.14 and 3.18 to this equation (with $A = \mathsf{D}^{\alpha}_{\tilde{X}}$ and $B = \mathsf{D}^{\beta}_{Y} \circ$) in order to show that it has at most one solution $\tilde{u}$, and that, for $\tilde{f}$ in interpolation spaces of the form $\mathcal{D}_{\mathsf{D}^{\alpha}_{\tilde{X}}}(\theta, \infty)$ and $\mathcal{D}_{\mathsf{D}^{\beta}_{Y}}(\theta, \infty)$, it has, indeed, a unique solution. Moreover, we obtain maximal regularity for these spaces.

4.2.1 Properties of the operator $A\circ$

Using the following theorem, we can show that $\mathsf{D}^{\beta}_{Y} \circ$ inherits properties such as "closedness" and "positiveness" from D^{β}_{Y}.

Theorem 4.12. *Let A be a closed linear operator in a complex Banach space Y. Define the operator $A\circ$ by*

$$\begin{aligned} &\mathcal{D}(A\circ) := \mathcal{C}([0, \tau]\,; \mathcal{D}(A)) \\ &(A \circ u)(t) := A(u(t)) \qquad u \in \mathcal{C}([0, \tau]\,; \mathcal{D}(A)), t \in [0, \tau] \end{aligned}$$

where $\mathcal{D}(A)$ is provided with the graph norm. Then $A\circ$ is a well–defined closed linear operator in $\mathcal{C}([0, \tau]\,; Y)$, and the following statements hold:

(a) *If A is densely defined, then so is $A\circ$.*

(b) *$\rho(-A\circ) = \rho(-A)$, and for all $\lambda \in \rho(-A)$ and all $u \in \mathcal{D}(A\circ)$ we have*

$$(\lambda + A\circ)^{-1} u = (\lambda + A)^{-1} \circ u$$

and

$$\| (\lambda + A\circ)^{-1} \|_{\mathcal{L}(\mathcal{C}([0,\tau];Y))} = \| (\lambda + A)^{-1} \|_{\mathcal{L}(Y)}$$

(c) *$A\circ$ is nonnegative iff A is nonnegative, and in that case*

$$\phi_{A\circ} = \phi_A; \quad N_{A\circ} = N_A; \quad M_{A\circ} = M_A; \quad M^*_{A\circ} = M^*_A.$$

(d) $A\circ$ *is positive iff* A *is positive.*

(e) *If* A *is positive and* $\operatorname{Re} z \neq 0$, *then*

$$(A\circ)^z = A^z \circ .$$

(f) *If* A *is nonnegative, then*

$$\mathcal{D}_{A\circ}(\theta, \infty) = \mathcal{C}([0,\tau]\,; Y) \cap \mathcal{B}([0,\tau]\,; \mathcal{D}_A(\theta, \infty)) \tag{4.20}$$

and

$$\mathcal{D}_{A\circ}(\theta) = \mathcal{C}([0,\tau]\,; \mathcal{D}_A(\theta)) \tag{4.21}$$

for all $\theta \in (0,1)$, *with equivalence of the respective norms.*

The above statements remain true if all spaces of the form $\mathcal{C}([0,\tau]\,; E)$ *involved are replaced by* $\mathcal{C}_{0\mapsto 0}([0,\tau]\,; E)$.

Proof. It is quite obvious that $A\circ$ is a linear operator in the set of functions from $[0,\tau]$ to Y. The definitions of $A\circ$ and the graph norm $\| \tilde{u} \|_{\mathcal{D}(A)} = \| u \|_Y + \| A \circ u \|_Y$ guarantee that if $u \in \mathcal{D}(A\circ)$, then both u and $A \circ u$ are continuous functions from $[0,\tau]$ to Y. In particular, this implies that $A\circ : \mathcal{C}([0,\tau]\,; \mathcal{D}(A)) \longrightarrow \mathcal{C}([0,\tau]\,; Y)$

Let A be closed, and let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{C}([0,\tau]\,; \mathcal{D}(A))$ such that $u_n \to u$ in that space and $A \circ u_n \to v$ in $\mathcal{C}([0,\tau]\,; Y)$ as $n \to \infty$. Then $u_n(t) \to u(t)$ and $A[u_n(t)] \to v(t)$ for all $t \in [0,\tau]$. Since A is closed, we must have $u(t) \in \mathcal{D}(A)$ and $v(t) = A[u(t)]$ for all $t \in [0,\tau]$, or, equivalently, $v = A \circ u$. Since $u, v \in \mathcal{C}([0,\tau]\,; Y)$, it follows that $u \in \mathcal{D}(A\circ) = \mathcal{C}([0,\tau]\,; \mathcal{D}(A))$ and $v = A \circ u$, i.e, $A\circ$ is closed.

(a) The set subspace of polynomial functions from $[0,\tau]$ into Y is dense in $\mathcal{C}([0,\tau]\,; Y)$. For any such polynomial function, there are constants $\{c_i\}_{i=0}^{n}$ in Y such that

$$p(t) = \sum_{k=0}^{n} t^k c_k.$$

Since $\mathcal{D}(A)$ is dense in X, for any $\varepsilon > 0$ one can choose $a_0, a_1, \ldots, a_n \in \mathcal{D}(A)$ so that $\| c_k - a_k \|_Y \leq \varepsilon/(n+1)\tau^k$. Hence, $\| p_n - \sum_{k=1}^{n} a_k t^k \|_{\mathcal{C}([0,\tau];Y)} \leq \varepsilon$. It follows that the set of polynomial functions from $[0,\tau]$ into $\mathcal{D}(A)$ is dense in $\mathcal{C}([0,\tau]\,; Y)$. These polynomials belong to $\mathcal{D}(A\circ) = \mathcal{C}([0,\tau]\,; \mathcal{D}(A))$.

(b) Let $\lambda \in \mathbb{C}$, and assume that the linear operator $\lambda + A$ in Y is one–to–one and onto, so that $(\lambda + A)^{-1} : Y \longrightarrow Y$ exists. Define a linear operator B by putting $\mathcal{D}(B) = \mathcal{C}([0,\tau]\,; E)$ and $Bu = (\lambda + A)^{-1} \circ u : [0,\tau] \longrightarrow \mathcal{D}(A)$ for all $u \in \mathcal{D}(P)$. Then

$$[B(\lambda + A\circ)u](t) = (\lambda + A)^{-1}(\lambda + A)[u(t)] = u(t)$$

for all $u \in \mathcal{D}(A\circ)$ and all $t \in [0,\tau]$. Hence, $B(\lambda + A\circ)u = u$ for all $u \in \mathcal{D}(A\circ)$. This implies that $\lambda + A\circ$ is one–to–one and B is a left inverse. Assume now that $\lambda \in \rho(-A)$. Then $(\lambda + A)^{-1}$ and $A(\lambda + A)^{-1} = I - \lambda(\lambda + A)^{-1}$ belong to $\mathcal{L}(Y)$. Hence, $Bu = (\lambda + A)^{-1} \circ u \in \mathcal{C}([0,\tau]\,;Y)$ and $A \circ (Bu) = A(\lambda + A)^{-1} \circ u \in \mathcal{C}([0,\tau]\,;Y)$. Consequently, $Bu \in \mathcal{C}([0,\tau]\,;\mathcal{D}(A))$ and

$$[(\lambda + A\circ)Bu](t) = (\lambda + A)(\lambda + A)^{-1}[u(t)] = u(t)$$

for all $u \in \mathcal{C}([0,\tau]\,;Y)$, showing that $\lambda + A\circ$ is onto. Thus,

$$(\lambda + A\circ)^{-1}u = (\lambda + A)^{-1} \circ u \tag{4.22}$$

for all $u \in \mathcal{C}([0,\tau]\,;Y)$. We also have

$$\begin{aligned}\left\| (\lambda + A\circ)^{-1}u \right\|_{\mathcal{C}([0,\tau];Y)} &= \sup_{t\in[0,\tau]} \left\| (\lambda + A)^{-1}[u(t)] \right\|_Y \\ &\le \sup_{t\in[0,\tau]} \left\| (\lambda + A)^{-1} \right\|_{\mathcal{L}(Y)} \| u(t) \|_Y \\ &= \left\| (\lambda + A)^{-1} \right\|_{\mathcal{L}(Y)} \| u \|_{\mathcal{C}([0,\tau];Y)}\end{aligned}$$

for each $u \in \mathcal{D}(A\circ)$, so that $(\lambda + A\circ)^{-1}$ is bounded and

$$\left\| (\lambda + A\circ)^{-1} \right\|_{\mathcal{L}(\mathcal{C}([0,\tau];Y))} \le \left\| (\lambda + A)^{-1} \right\|_{\mathcal{L}(Y)}.$$

Consequently, $\lambda \in \rho(-A\circ)$.

Assume now that $\lambda + A\circ$ is injective from $\mathcal{D}(A\circ) = \mathcal{C}([0,\tau]\,;Y)$ into $\mathcal{C}([0,\tau]\,;Y)$. If $(\lambda+A)u_0 = (\lambda+A)v_0$, where $u_0, v_0 \in Y$, we put $u(t) = tu_0$ and $v(t) = tv_0$ for any $t \in [0,\tau]$. Then both u and v belong to $\mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{D}(A))$, and we get

$$\begin{aligned}[(\lambda + A\circ)u](t) &= (\lambda + A)[u(t)] = t(\lambda + A)u_0 \\ &= (\lambda + A)[v(t)] = [(\lambda + A\circ)v](t)\end{aligned}$$

for all $t \in [0,\tau]$ Hence, $u = v$, so that $u_0 = v_0$ and $\lambda + A$ is one–to–one.

If $\lambda + A\circ$ is also onto and $v_0 \in Y$, then there is a $u \in \mathcal{D}(A\circ)$ such that $(\lambda + A\circ)u = v$, where $v(t) = (t/\tau)v_0$ for all $t \in [0,\tau]$. Hence, $(\lambda + A)[u(\tau)] = v_0$, so that $\lambda + A$ is onto.

By the above, B is a left inverse to $\lambda + A\circ$. Since the latter operator maps $\mathcal{D}(A\circ)$ onto $\mathcal{C}([0,\tau]\,;Y)$, it follows that $(\lambda + A\circ)^{-1}v = (\lambda + A)^{-1} \circ v$ for all $v \in \mathcal{C}([0,\tau]\,;Y)$. Moreover, if $\lambda \in \rho(-A\circ)$, then for any $u_0 \in Y$ we consider $u \in \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{D}(A))$ defined by $u(t) = (t/\tau)u_0$. It follows that

$$\begin{aligned}\left\| (\lambda + A)^{-1}u_0 \right\| &= \left\| (\lambda + A\circ)^{-1}[u(\tau)] \right\| \\ &\le \left\| (\lambda + A\circ)^{-1}u \right\|_{\mathcal{C}([0,\tau];Y)} \\ &\le \left\| (\lambda + A\circ)^{-1} \right\|_{\mathcal{L}(\mathcal{C}([0,\tau];Y))} \| u \|_{\mathcal{C}([0,\tau];Y)} \\ &= \left\| (\lambda + A\circ)^{-1} \right\|_{\mathcal{L}(\mathcal{C}([0,\tau];Y))} \| u_0 \|,\end{aligned}$$

and hence $(\lambda + A)^{-1}$ is bounded with

$$\left\| (\lambda + A)^{-1} \right\|_{\mathcal{L}(Y)} \le \left\| (\lambda + A\circ)^{-1} \right\|_{\mathcal{L}(\mathcal{C}([0,\tau];Y))}.$$

Consequently, $\lambda \in \rho(-A)$. Summarising what has been proved so far yields (b).

(c) and (d) are immediate consequences of (b).

(e) Assume first that $\operatorname{Re} z < 0$. Then $\mathcal{D}(A^z) = Y$. Let $u \in \mathcal{D}(A^z) = \mathcal{C}([0,\tau]\,;Y)$, and let $t \in [0,\tau]$. Then

$$\begin{aligned}
(A^z \circ u)(t) &= \frac{1}{2\pi i}\left\{\int\limits_\gamma (-\lambda)^z(\lambda + A)^{-1}\,d\lambda\right\}[u(t)] \\
&= \frac{1}{2\pi i}\left\{\int\limits_\gamma (-\lambda)^z(\lambda + A)^{-1}[u(t)]\,d\lambda\right\} \\
&= \frac{1}{2\pi i}\int\limits_\gamma (-\lambda)^z[(\lambda + A\circ)^{-1}u](t)\,d\lambda \\
&= \frac{1}{2\pi i}\left\{\int\limits_\gamma (-\lambda)^z[(\lambda + A\circ)^{-1}u]\,d\lambda\right\}(t) \\
&= \left\{(A\circ)^\alpha u\right\}(t).
\end{aligned}$$

If $\operatorname{Re} z > 0$, we choose $n \in \mathbb{N}$ with $n > \operatorname{Re} z$. Then

$$\begin{aligned}
(A^z \circ u)(t) &= A^n A^{z-n}[u(t)] = A^n(A^{z-n}[u(t)]) \\
&= A^n([(A\circ)^{z-n}u](t)) = [(A\circ)^n(A\circ)^{z-n}u](t) \\
&= [(A\circ)^z u](t)
\end{aligned}$$

for any $t \in [0,\tau]$, and any $u \in \mathcal{D}(A^z)$, so that $A^z\circ = (A\circ)^z$.

(f) We have

$$\begin{aligned}
\sup_{r>0}\left\| r^\theta A\circ(r + A\circ)^{-1}u\right\|_{\mathcal{C}([0,\tau];Y)} &= \sup_{r>0}\sup_{t\in[0,\tau]}\left\| r^\theta A(r+A)^{-1}u(t)\right\|_Y \\
&= \sup_{t\in[0,\tau]}\sup_{r>0}\left\| r^\theta A(r+A)^{-1}u(t)\right\|_Y,
\end{aligned}$$

so that

$$[\,u\,]_{\mathcal{D}_{A\circ}(\theta,\infty)} = \sup_{t\in[0,\tau]}[\,u(t)\,]_{\mathcal{D}_A(\theta,\infty)} \tag{4.23}$$

for $u \in \mathcal{C}([0,\tau]\,;Y)$. For such u we also have $\sup\limits_{t\in[0,\tau]}\|\,u(t)\,\|_Y = \|\,u\,\|_{\mathcal{C}([0,\tau];Y)} < \infty$. Hence, (4.20) holds. In addition, we see that

$$\sup_{t\in[0,\tau]}\|\,u(t)\,\|_{\mathcal{D}_A(\theta,\infty)} \le \|\,u\,\|_{\mathcal{D}_{A\circ}(\theta,\infty)} \le 2\sup_{t\in[0,\tau]}\|\,u(t)\,\|_{\mathcal{D}_A(\theta,\infty)},$$

which establishes the equivalence of the norms.

Next, let $u \in \mathcal{C}([0,\tau]\,;\mathcal{D}_A(\theta))$. We intend to show that

$$\lim_{r\to\infty}\sup_{t\in[0,\tau]}\left\| r^\theta A\circ(r + A\circ)u(t)\right\|_Y = 0.$$

By formula (4.20) and the assumption on u, it is clear that $u \in \mathcal{D}_{A\circ}(\theta, \infty)$. Hence, putting $f_r(t) = r^\theta \, \| A(r+A)^{-1}(u(t)) \|_Y$, we see that

$$f_r(t) \le [\, u \,]_{\mathcal{D}_{A\circ}(\theta,\infty)}$$

for all $r > 0$ and $t \in [0, \tau]$. By assumption, we also have $\lim_{r\to\infty} f_r(t) = 0$ for any $t \in [0, \tau]$. Moreover,

$$\begin{aligned} &\left| \left\| r^\theta A \circ (r + A\circ)^{-1} u(t) \right\|_Y - \left\| r^\theta A \circ (r + A\circ)^{-1} u(s) \right\|_Y \right| \\ &\quad \le \left\| r^\theta A(r+A)^{-1}\{u(t) - u(s)\} \right\|_Y \\ &\quad \le \sup_{r>0} \left\| r^\theta A(r+A)^{-1}\{u(t) - u(s)\} \right\|_Y = [\, u(t) - u(s) \,]_{\mathcal{D}_A(\theta,\infty)}, \end{aligned}$$

i.e.,

$$|f_r(t) - f_r(s)| \le [\, u(t) - u(s) \,]_{\mathcal{D}_A(\theta,\infty)},$$

for all $r > 0$ and all $t, s \in [0, \tau]$. Hence, the f_r are uniformly continuous. Let us put $\alpha_r = \| f_r \|_{\mathcal{C}([0,\tau];\mathbb{R})}$, and assume that $\{r_k\}_{k=1}^\infty \subset (0, \infty)$ is such that $\alpha_{r_k} \to \alpha$ as $k \to \infty$. Then for any k there is some $t_k \in [0, \tau]$ with $\alpha_{r_k} = f_{r_k}(t_k)$. There is some convergent subsequence $\{s_j\}_{j=1}^\infty := \left\{t_{k_j}\right\}_{j=0}^\infty$ of $\{t_k\}_{k=1}^\infty$, with limit $s \in [0, \tau]$. Put $r_{k_j} = q_j$, $j = 1, 2, \ldots$. We have $f_{q_j}(s)) = (f_{q_j}(s) - f_{q_j}(s_j)) + f_{r_j}(s_j)$, which tends to α as $j \to \infty$, since $\left|f_{r_j}(s) - f_{r_j}(s_j)\right| \le [\, u(s) - u(s_j) \,]_{\mathcal{D}_A(\theta,\infty)}$ tends to 0 by the uniform continuity of the f_r, and $f_{r_j}(s_j) = \alpha_{r_j}$ tends to α as $j \to \infty$. Consequently, $\alpha = 0$. It follows from this argument that $\lim_{r\to\infty} \alpha_r = 0$, so that $u \in \mathcal{D}_{A\circ}(\theta)$. Thus, $\mathcal{C}([0,\tau]\,; \mathcal{D}_A(\theta)) \subseteq \mathcal{D}_{A\circ}(\theta)$.

To prove the reverse inclusion, let $u \in \mathcal{D}_{A\circ}(\theta)$. Then

$$u \in \mathcal{D}_{A\circ}(\theta, \infty) = \mathcal{C}([0,\tau]\,; Y) \cap \mathcal{B}([0,\tau]\,; \mathcal{D}_A(\theta, \infty))$$

and

$$\lim_{r\to\infty} \left\| r^\theta A \circ (r + A\circ)^{-1} u \right\|_{\mathcal{C}([0,\tau];Y)} = 0.$$

Thus, for all $t \in [0, \tau]$ we have $u(t) \in \mathcal{D}_A(\theta, \infty)$ and

$$\lim_{r\to\infty} \left\| r^\theta A(r+A)^{-1}(u(t)) \right\|_Y = 0,$$

so that $u \in \mathcal{B}([0,\tau]\,; \mathcal{D}_A(\theta))$. We know that $\lim_{s\to t} \| u(t) - u(s) \| = 0$ for all $t \in [0, \tau]$, since $u \in \mathcal{C}([0,\tau]\,; Y)$. In order to show that $u \in \mathcal{C}([0,\tau]\,; \mathcal{D}_A(\theta))$, we should therefore show that $\lim_{s\to t} [\, u(t) - u(s) \,]_{\mathcal{D}_A(\theta,\infty)} = 0$. For this purpose, take $t, s \in [0, \tau]$. We have

$$[\, u(t) - u(s) \,]_{\mathcal{D}_A(\theta,\infty)} = \sup_{r>0} r^\theta \left\| A(r+A)^{-1}(u(t) - u(s)) \right\|,$$

and

$$r^\theta \left\| A(r+A)^{-1}(u(t) - u(s)) \right\| \le 2r^\theta \left\| A \circ (r + A\circ)^{-1} u \right\|_{\mathcal{C}([0,\tau];Y)}.$$

By assumption, the expression on the right hand side tends to 0 as $r \to \infty$. Hence, given an arbitrary $\varepsilon > 0$, we can find $R > 0$ such that

$$r^\theta \left\| A(r+A)^{-1}(u(t)-u(s)) \right\| < \varepsilon$$

for all $r > R$ and all $t, s \in [0,\tau]$. We also have $\| A(r+A)^{-1} \|_{\mathcal{L}(Y)} \leq 1 + N_A$ for all $r > 0$, since A is nonnegative. Consequently,

$$r^\theta \left\| A(r+A)^{-1}(u(t)-u(s)) \right\| \leq R^\theta (1+N_A) \| u(t) - u(s) \|$$

for $0 < r \leq R$ and $t, s \in [0,\tau]$. Hence, there is some $\delta > 0$ such that

$$r^\theta \left\| A(r+A)^{-1}\{u(t)-u(s)\} \right\| < \varepsilon,$$

provided that $0 < r \leq R$, $|t-s| < \delta$ and $t, s \in [0,\tau]$. Consequently, for all such t and s we have

$$[\, u(t) - u(s) \,]_{\mathcal{D}_A(\theta,\infty)} = \sup_{r>0} r^\theta \left\| A(r+A)^{-1}(u(t)-u(s)) \right\| < \varepsilon.$$

Thus, $\mathcal{D}_{A\circ}(\theta,\infty) \subseteq \mathcal{C}([0,\tau]\,;\mathcal{D}_A(\theta,\infty)$. Since the converse inclusion was already proved, (4.21) holds.

Finally, consider the restriction of $A\circ$ to $\mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{D}(A))$. But in case $u \in \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{D}(A))$, we see that $(A \circ u)(0) = A(u(0)) = A0 = 0$, so that the range of this restriction is a subset of $\mathcal{C}_{0\mapsto 0}(I;Y)$. It is easy to check that the above proof remains true if $\mathcal{C}([0,\tau]\,;E)$ and $\mathcal{C}(I;\mathcal{D}(A))$ are replaced by $\mathcal{C}_{0\mapsto 0}([0,\tau]\,;E)$ and $\mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{D}(A))$, respectively. ☐

4.2.2 Solving the equation in $\mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))$

As a step on the way to a solution of

$$D_t^\alpha u(t,x) + D_x^\beta u(t,x) = f(t,x)$$

in some subspace of $\mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E)$, we now consider solutions to

$$\mathsf{D}^\alpha_{\tilde{X}}\, \tilde{u} + \mathsf{D}^\beta_{\tilde{X}} \circ \tilde{u} = \tilde{f} \tag{4.24}$$

in $\tilde{X} = \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))$. We assume that $0 < \alpha, \beta \leq 1$, and that we do not have $\alpha = \beta = 1$.

By Theorems 4.9 and 4.12, the operators $\mathsf{D}^\alpha_{\tilde{X}}$ and $\mathsf{D}^\beta_Y \circ$ are densely defined and positive with spectral angles $\omega_{\mathsf{D}^\beta_Y \circ} = \alpha\pi/2$ and $\omega_{\mathsf{D}^\beta_{\tilde{X}} \circ} = \omega_{\mathsf{D}^\beta_{\tilde{X}}} = \beta\pi/2$, respectively. It follows that

$$\omega_{\mathsf{D}^\alpha_{\tilde{X}}} + \omega_{\mathsf{D}^\beta_Y \circ} = \tfrac{\alpha+\beta}{2}\pi < \pi$$

when $\alpha + \beta < 2$. By Lemma 4.15 (c) below, the operators $\mathsf{D}^\alpha_{\tilde{X}}$ and $\mathsf{D}^\beta_Y \circ$ are resolvent commuting. We conclude that the operators $A = \mathsf{D}^\alpha_{\tilde{X}}$ and $B = \mathsf{D}^\beta_Y \circ$ in $\tilde{X} = \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))$ satisfy the hypotheses of Theorem 3.10.

In combination with Proposition 4.4, the above mentioned theorems also show that

$$\mathcal{D}_{\mathsf{D}^{\alpha}_{\tilde{X}}}(\theta,\infty) = \mathcal{C}^{\alpha\theta}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E)) \tag{4.25}$$

and

$$\mathcal{D}_{\mathsf{D}^{\beta}_{Y}\circ}(\theta,\infty) = \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{B}([0,\xi]\,;E)) \cap \mathcal{B}([0,\tau]\,;\mathcal{C}^{\beta\theta}_{0\mapsto 0}([0,\xi]\,;E)) \tag{4.26}$$

for $\theta \in (0,1)$. Let us summarise these results as a lemma.

Lemma 4.13. *Let E be a complex Banach space and let $0 \le \alpha, \beta \le 1$. Define $\tilde{X}$ and Y as above. Then $\mathsf{D}^{\alpha}_{\tilde{X}}$ and $\mathsf{D}^{\beta}_{Y}\circ$ are positive and resolvent commuting operators with $\omega_{\mathsf{D}^{\alpha}_{\tilde{X}}} = \alpha\pi/2$ and $\omega_{\mathsf{D}^{\beta}_{Y}\circ} = \beta\pi/2$. Moreover, if $\theta \in (0,1)$, then* (4.25) *and* (4.26) *hold.*

Thus, Theorem 3.10 readily implies that if $\tilde{f} \in \tilde{X}$, then (4.24) has at most one solution.

Let us now turn to the existence part. If $\tilde{f} \in \mathcal{D}_{\mathsf{D}^{\alpha}_{\tilde{X}}}(\mu/\alpha,\infty)$, where we assume $0 < \mu < \alpha$, then, by Theorems 3.14 and 3.18, there is a unique solution $\tilde{u}$ to the equation $\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{u} + \mathsf{D}^{\beta}_{Y}\circ\tilde{u} = \tilde{f}$, such that

$$\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{u}, \mathsf{D}^{\beta}_{Y}\circ\tilde{u} \in \mathcal{D}_{\mathsf{D}^{\alpha}_{\tilde{X}}}(\mu/\alpha,\infty). \tag{4.27}$$

Analogously, if $\tilde{u} \in \mathcal{D}_{\mathsf{D}^{\beta}_{Y}\circ}(\nu/\beta,\infty)$, where $0 < \nu < \beta$, then there is a unique solution $\tilde{u}$ to (4.24), such that

$$\mathsf{D}^{\alpha}_{\tilde{X}}\tilde{u}, \mathsf{D}^{\beta}_{Y}\circ\tilde{u} \in \mathcal{D}_{\mathsf{D}^{\beta}_{Y}\circ}(\nu/\beta,\infty). \tag{4.28}$$

By (4.25) and (4.26),

$$\begin{aligned}
&\mathcal{D}_{\mathsf{D}^{\alpha}_{\tilde{x}}}(\mu/\alpha,\infty) \cap \mathcal{D}_{\mathsf{D}^{\beta}_{Y}\circ}(\nu/\beta,\infty)\\
&\quad = \mathcal{C}^{\mu}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E)) \cap \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))\\
&\qquad\qquad \cap \mathcal{B}([0,\tau]\,;\mathcal{C}^{\nu}_{0\mapsto 0}([0,\xi]\,;E))\\
&\quad = \mathcal{C}^{\mu}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E)) \cap \mathcal{B}([0,\tau]\,;\mathcal{C}^{\nu}_{0\mapsto 0}([0,\xi]\,;E))
\end{aligned}$$

Hence, if

$$\tilde{f} \in \tilde{X}_{\mu,\nu} := \mathcal{C}^{\mu}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E)) \cap \mathcal{B}([0,\tau]\,;\mathcal{C}^{\nu}_{0\mapsto 0}([0,\xi]\,;E)),$$

then (4.24) has a unique solution $\tilde{X}_{\mu,\nu}$, such that

$$\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{u}, \mathsf{D}^{\beta}_{Y}\circ\tilde{u} \in \tilde{X}_{\mu,\nu}. \tag{4.29}$$

Since $\mathsf{D}_{\tilde{X}}\,\alpha$ and $\mathsf{D}^{\beta}_{Y}\circ$ are positive, Theorem 3.18 also implies that there is a constant M that does not depend on f such that

$$\|\,\tilde{u}\,\|_{\tilde{X}_{\mu,\nu}} + \|\,\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{u}\,\|_{\tilde{X}_{\mu,\nu}} + \|\,\mathsf{D}^{\beta}_{Y}\circ\tilde{u}\,\|_{\tilde{X}_{\mu,\nu}} \le M\|\,\tilde{f}\,\|_{\tilde{X}_{\mu,\nu}} \tag{4.30}$$

In particular, if $\tilde f \in \mathcal{D}_{\mathsf{D}^\alpha_{\tilde X}}(\mu/\alpha)\cap\mathcal{D}_{\mathsf{D}^\beta_Y\circ}(\nu/\beta)$, then, again by Theorems 3.14 and 3.18, there is a unique solution $\tilde u$ to the equation $\mathsf{D}^\alpha_{\tilde X}\,\tilde u + \mathsf{D}^\beta_Y\circ\tilde u = \tilde f$, such that $\mathsf{D}^\alpha_{\tilde X}\,\tilde u$ and $\mathsf{D}^\beta_Y\circ\tilde u$ belong to $\mathcal{D}_{\mathsf{D}^\alpha_{\tilde X}}(\mu/\alpha)\cap\mathcal{D}_{\mathsf{D}^\beta_Y\circ}(\nu/\beta)$. But

$$\begin{aligned}&\mathcal{D}_{\mathsf{D}^\alpha_{\tilde X}}(\mu/\alpha)\cap\mathcal{D}_{\mathsf{D}^\beta_Y\circ}(\nu/\beta)\\ &\qquad = h^\mu_{0\mapsto 0}([0,\tau]\,;\mathcal{B}_{0\mapsto 0}([0,\xi]\,;E))\cap\mathcal{B}([0,\tau]\,;h^\nu_{0\mapsto 0}([0,\xi]\,;E)),\end{aligned}$$

so that we have

$$\mathsf{D}^\alpha_{\tilde X}\tilde u, \mathsf{D}^\beta_Y\circ\tilde u \in h^\mu_{0\mapsto 0}([0,\tau]\,;\mathcal{B}_{0\mapsto 0}([0,\xi]\,;E))\cap\mathcal{B}([0,\tau]\,;h^\nu_{0\mapsto 0}([0,\xi]\,;E)). \tag{4.31}$$

Summarising these observations, we obtain the following lemma.

Lemma 4.14. *Let E be a complex Banach space, let $\tau,\xi>0$, and assume that $0<\mu<\alpha\le 1$, $0<\nu<\beta\le 1$, and $\alpha+\beta<2$. Then for any $\tilde f\in\tilde X = \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))$ the equation (4.24) in $\tilde X$ has at most one strict solution. If $\tilde f\in\mathcal{C}^\mu_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))\cap\mathcal{B}([0,\tau]\,;\mathcal{C}^\nu_{0\mapsto 0}([0,\xi]\,;E))$ then the equation has a unique solution $\tilde u$, and (4.29) as well as the estimate (4.30) holds for this solution. In particular, if $\tilde f\in h^\mu_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))\cap\mathcal{B}([0,\tau]\,;h^\nu_{0\mapsto 0}([0,\xi]\,;E))$, then (4.31) holds.*

4.2.3 Carrying over the results to $\mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E)$

In order to carry over these results on the equation $\mathsf{D}^\alpha_{\tilde X}\tilde u + \mathsf{D}^\beta_Y\circ\tilde u = \tilde f$ to the original evolution equation $D^\alpha_t u(t,x) + D^\beta_x u(t,x) = f(t,x)$ we use Lemma 4.11 together with the following lemma, the proof of which is postponed.

Lemma 4.15. *Let $Y=\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E)$ and $\tilde X = \mathcal{C}_{0\mapsto 0}([0,\tau]\,;Y)$, where $\tau,\xi>0$. For $u : Q\longrightarrow\mathbb{C}$, where $Q:=[0,\tau]\times[0,\xi]$, let $\tilde u$ denote the function $\tilde u : [0,\xi]\longrightarrow([0,\tau]\longrightarrow\mathbb{C})$ such that $\tilde u(t)(x) = u(t,x)$. Assume that $0<\alpha,\beta\le 1$. Then the following statements hold.*

(a) *We have*

$$\mathcal{D}(\mathsf{D}_{\tilde X}) = \{\tilde u \parallel u, D_t u\in\mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E)\}$$

and

$$\mathcal{D}(\mathsf{D}^\alpha_{\tilde X}) = \{\tilde u \parallel u,\, g_{1-\alpha}*u,\, D_t(g_{1-\alpha}*u)\,\in\mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E)\}$$

for $0<\alpha<1$. Moreover, if $\tilde u\in\mathcal{D}(\mathsf{D}^\alpha_{\tilde X})$, then

$$(\mathsf{D}^\alpha_{\tilde X}\,\tilde u) = \widetilde{D^\alpha_t u}$$

for $0<\alpha\le 1$.

(b) *$\mathcal{D}(\mathsf{D}^\beta_Y\circ) = \{\tilde u\parallel u, D^\beta_x u\in\mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E)\}$, and if $\tilde u\in\mathcal{D}(\mathsf{D}^\beta_Y\circ)$, then*

$$\mathsf{D}^\beta_Y\circ\tilde u = \widetilde{D^\beta_x u}.$$

(c) *$\mathsf{D}^\alpha_{\tilde X}$ and $\mathsf{D}^\beta_Y\circ$ are (resolvent) commuting operators in $\tilde X$.*

We assume that $u, f : Q \longrightarrow \mathbb{C}$ and $\tilde{u}, \tilde{f} : [0,\xi] \longrightarrow ([0,\tau] \longrightarrow \mathbb{C})$ are functions such that $\tilde{u}(t)(x) = u(t,x)$ and $\tilde{f}(t)(x) = f(t,x)$. Then by Lemma 4.11 (a) $f \in \mathcal{C}(Q;E)$ if and only if $\tilde{f} \in \mathcal{C}([0,\tau]\,;\mathcal{C}([0,\xi]\,;E))$ and analogously for u and $\tilde{u}$. By Lemma 4.15, u is therefore a solution to the equation $D_t^\alpha u + D_x^\beta u = f$ in $\mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E)$ if and only if $\tilde{u}$ is a solution to the equation $\mathsf{D}_{\tilde{X}}^\alpha \tilde{u} + \mathsf{D}_Y^\beta \circ \tilde{u} = \tilde{f}$ in $\mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))$. Consequently, by Lemma 4.14, also the former equation has at most one solution.

If $f \in \mathcal{C}^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E)$, then by Lemmas 4.14 and 4.15 we get a unique solution $u \in \mathcal{C}^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E)$ to the equation $D_t^\alpha u + D_x^\beta u = f$ in $\mathcal{C}_{\partial_0\mapsto 0}(Q;E)$, and for this solution we have

$$D_t^\alpha u, D_x^\beta u \in \mathcal{C}^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E) \tag{4.32}$$

and

$$\| u \|_{\mathcal{C}(Q;E)} + \| D_t^\alpha u \|_{\mathcal{C}(Q;E)} + \| D_x^\beta u \|_{\mathcal{C}(Q;E)} \leq M \| f \|_{\mathcal{C}(Q;E)}, \tag{4.33}$$

where M is a constant that does not depend on f.

In particular, if $f \in h^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E)$, then

$$D_t^\alpha u, D_x^\beta u \in h^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E).$$

We have now proved the final theorem of this section.

Theorem 4.16. *Let E be a complex Banach space, let $Q = [0,\tau] \times [0,\xi]$, where $\tau, \xi > 0$, and assume that $0 < \mu < \alpha \leq 1$, $0 < \nu < \beta \leq 1$, and $\alpha + \beta < 2$. If $f \in \mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E)$, then the fractional partial differential equation*

$$D_t^\alpha u + D_x^\beta u = f$$

in $\mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E) = \{u \in \mathcal{C}(Q;E) \mid u(t,x) = 0 \text{ for all } (t,x) \in \partial_0 Q\}$ has at most one solution, where $\partial_0 Q := \{(t,x) \in Q \parallel t = 0 \vee x = 0\}$. The equation possesses maximal regularity with respect to the spaces $\mathcal{C}^{\mu,\nu}_{\partial_0 Q\mapsto 0}$ and $h^{\mu,\nu}_{\partial_0 Q\mapsto 0}$. More precisely, if $f \in \mathcal{C}^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E)$, then it has a unique solution u. This solution satisfies $D_t^\alpha u, D_x^\beta u \in \mathcal{C}^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E)$, as well as the estimate (4.33). Moreover, if $f \in h^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E)$, then also $D_t^\alpha u, D_x^\beta u \in h^{\mu,\nu}_{\partial_0 Q\mapsto 0}(Q;E)$.

We close this section with a proof of Lemma 4.15.

Proof of Lemma 4.15. (a) First we consider the case $\alpha = 1$. Assume that $u, u_t \in \mathcal{C}(Q;E)$. We have

$$\begin{aligned}\left\| \frac{\tilde{u}(t+h)(x) - \tilde{u}(t)(x)}{h} - \widetilde{u_t}(t)(x) \right\|_E &= \left\| \frac{1}{h} \int_t^{t+h} (u_t(s,x) - u_t(t,x))\, ds \right\|_E \\ &\leq \frac{1}{h} \int_t^{t+h} \| u_t(s,x) - u_t(t,x) \|_E \, ds.\end{aligned}$$

Since $u_t : Q \longrightarrow Y$ is uniformly continuous, for any $\varepsilon > 0$ it is possible to find $\delta > 0$ such that $\| u_t(s,y) - u_t(t,x) \|_Y < \varepsilon$ whenever $|s-t| < \delta$ and $|y-x| < \delta$. In particular, if $|h| < \delta$, we have $\| u_t(s,x) - u_t(t,x) \|_Y < \varepsilon$ provided that $(t,x), (t+h,x) \in Q$ and s lies between t and $t+h$. It follows that

$$\begin{aligned}
&\left\| \frac{\tilde{u}(t+h) - \tilde{u}(t)}{h} - \widetilde{u_t}(t) \right\|_{\mathcal{C}([0,\xi];E)} \\
&\qquad = \sup_{x \in [0,\xi]} \left\| \frac{\tilde{u}(t+h)(x) - \tilde{u}(t)(x)}{h} - \widetilde{u_t}(t)(x) \right\|_E \\
&\qquad \leq \varepsilon.
\end{aligned}$$

This implies that $\tilde{u}' = \widetilde{u_t}$. Hence, we can apply part (b) of Lemma 4.11, with u_t and $\tilde{u}'$ instead of u and $\tilde{u}$, to conclude that $\tilde{u}' \in \mathcal{C}([0,\tau]\,; \mathcal{C}([0,\xi]\,;E))$. If u and u_t vanish on $\partial_0 Q$, then $\tilde{u}(t)(0) = \tilde{u}(0)(x) = 0$ and $\widetilde{u_t}(t)(0) = \widetilde{u_t}(0)(x) = 0$ for all $(t,x) \in Q$.

For the reverse implication, assume that $\tilde{u}, \tilde{u}' \in \mathcal{C}([0,\tau]\,; \mathcal{C}([0,\xi]\,;E))$. We have

$$\left\| \frac{u(t+h,x) - u(t,x)}{h} - \tilde{u}'(t)(x) \right\| = \left\| \frac{\tilde{u}(t+h)(x) - \tilde{u}(t)(x)}{h} - \tilde{u}'(t)(x) \right\| \to 0$$

as $h \to 0$ for any $(t,x) \in Q$. This shows that $u_x(t,x) = \tilde{u}'(t)(x)$. Using part (b) of Lemma 4.11, this time in the right–to–left direction, we obtain $u' \in \mathcal{C}([0,\xi] \times [0,\xi]\,;E)$. If $\tilde{u}(t)(0) = \tilde{u}(0)(x) = 0$ and $\widetilde{u_t}(t)(0) = \widetilde{u_t}(0)(x) = 0$ for all $(t,x) \in Q$, then u and u_t vanish on $\partial_0 Q$.

Now assume that $0 < \alpha < 1$. If $u, D_t^\alpha \in \mathcal{C}_{\partial_0 Q}(Q;E)$, then $\tilde{u}, g_{1-\alpha} * u(t,x) \in \mathcal{C}^1(Q;E)$. Hence,

$$\widetilde{g_{1-\alpha} * u} \in \mathcal{C}_{\partial_0 Q \mapsto 0}([0,\tau]\,;[0,\xi]\,;E))$$

and

$$D\widetilde{g_{1-\alpha} * u} = \widetilde{\overbrace{D_t g_{1-\alpha} * u}}$$

by the above. But $\widetilde{g_{1-\alpha} * u} = g_{1-\alpha} * \tilde{u}$, so that we get

$$\begin{aligned}
\mathsf{D}_{\tilde{X}}^\alpha &= \mathsf{D}_{\tilde{X}}\, g_{1-\alpha} * \tilde{u} = \widetilde{\overbrace{D_t g_{1-\alpha} * u}} \\
&= \widetilde{D_t^\alpha u}
\end{aligned}$$

by the above.

Conversely, if $\tilde{u}$, $g_{1-\alpha} * \tilde{u}$, $D(g_{1-\alpha} * \tilde{u}) \in \mathcal{C}_{0 \mapsto 0}([0,\tau]\,; \mathcal{C}_{0 \mapsto 0}([0,\xi]\,;E))$, then, again by the previous case u, $g_{1-\alpha} * u$, $D_t(g_{1-\alpha}) \in \mathcal{C}_{\partial_0 Q \mapsto 0}([0,\tau]\,;[0,\xi]\,;E))$, and $\widetilde{D_t g_{1-\alpha}} * u = \mathsf{D}_{\tilde{X}}^\alpha\, \tilde{u}$.

(b) Assume, first, that $\beta = 1$. Let $u, u_x \in \mathcal{C}([0,\tau]\times[0,\xi]\,;E)$. Then $\tilde{u}, \widetilde{u_x} \in \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E)))$ by Lemma 4.11 (a). By definition $\widetilde{u_x}(t)(x) = u_x(t,x) = \tilde{u}(t)'(x)$, and we infer that

$$\tilde{u} \in \mathcal{D}(\mathsf{D}_Y \circ) = \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}^1_{0\mapsto 0}([0,\xi]\,;E))),$$

and that $\mathsf{D}_Y \circ \tilde{u} = \widetilde{u_x}$.

For the reverse inclusion, assume that

$$\tilde{u} \in \mathcal{D}(\mathsf{D}_Y \circ) = \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}^1_{0\mapsto 0}([0,\xi]\,;E))).$$

We have $\widetilde{u_x}(t)(x) = u_x(t,x) = \tilde{u}(t)'(x)$ by definition. Consequently, we deduce $\tilde{u}, \widetilde{u_x} \in \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E)))$, which yields $u, u_x \in \mathcal{C}([0,\xi] \times [0,\xi]\,;E)$ by Lemma 4.11 (a).

We proceed to the case $0 < \beta < 1$. Let $\tilde{u} \in \mathcal{D}(\mathsf{D}^\beta_Y \circ)$. This means that $\tilde{u}$ is continuous from $[0,\tau]$ to

$$\mathcal{D}(\mathsf{D}^\beta_Y) = \{u \in \mathcal{C}_{0\mapsto 0}([0,\tau]\,;E) \parallel g_{1-\beta} * u \in \mathcal{C}^1_{0\mapsto 0}([0,\tau]\,;E)\}$$

(provided with the graph norm). Thus, $\tilde{u}(t) \in \mathcal{D}(\mathsf{D}^\alpha_Y)$ for all $t \in [0,\tau]$, and the functions $\tilde{u}$ and $\mathsf{D}^\beta_Y \circ \tilde{u}$ from $[0,\tau]$ to $Y = \mathcal{C}([0,\xi]\,;E)$ are continuous. Consequently, $u \in \mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E)$ by Lemma 4.11 (a). By the definition of $\mathsf{D}^\beta_Y \circ$, we have

$$\begin{aligned}\left(\mathsf{D}^\beta_Y \circ \tilde{u}\right)(t)(x) &= \mathsf{D}^\beta_Y(\tilde{u}(t))(x)\\ &= D(g_{1-\beta} * \tilde{u}(t))(x) = D(g_{1-\beta} * u(t,\underline{x}))(x)\\ &= D^\beta_t u(t,x).\end{aligned}$$

for all $(t,x) \in Q$. Hence, Lemma 4.11 (a) can be applied with $f = D^\beta_x u$ and $\tilde{f} = \mathsf{D}^\beta_Y \circ \tilde{u}$, so that, since $\mathsf{D}^\beta_Y \circ \tilde{u} \in \mathcal{C}([0,\tau]\,;Y)$ and $\mathsf{D}^\beta_Y \circ \tilde{u}(t)(0) = \mathsf{D}^\beta_Y \circ \tilde{u}(0)(x) = 0$, we have $D^\beta_x u \in \mathcal{C}_{\partial_0 Q\mapsto 0}([0,\tau] \times [0,\tau]\,;E)$.

We now assume that $u, D^\beta_x u \in \mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E)$ and prove that $\tilde{u} \in \mathcal{D}(\mathsf{D}^\beta_Y \circ)$. By Lemma 4.11 (a), we have $\tilde{u} \in \mathcal{C}_{0\mapsto 0}([0,\tau]\,;Y)$, where $Y = \mathcal{C}_{0\mapsto 0}([0,\xi]\,;E)$. Therefore, it suffices to show that $\tilde{u}(t) \in \mathcal{D}(\mathsf{D}^\beta_Y)$ for all $t \in [0,\tau]$, and that the mapping $\mathsf{D}^\beta_Y \circ \tilde{u}$ is continuous from $[0,\xi]$ to $\mathcal{C}([0,\xi]\,;E)$. We already noted that $\tilde{u}(t) \in \mathcal{C}_{0\mapsto 0}([0,\xi]\,;E)$. We have

$$D^\beta_x u(t,x) = D(g_{1-\beta} * u(t,\underline{x}))(x) = D(g_{1-\beta} * \tilde{u}(t))(x) = D^\beta(\tilde{u}(t))(x).$$

Thus, Lemma 4.11 (a) can be applied to $D^\beta_x u$ and $\tilde{v} = D^\beta(\tilde{u}(\underline{t}))$, and we have $\tilde{v} \in \mathcal{C}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))$. This implies that $\tilde{v}(t) \in \mathcal{C}_{0\mapsto 0}([0,\xi]\,;E)$, i.e. $g_{1-\beta} * \tilde{u}(t) \in \mathcal{C}^1([0,\xi]\,;E)$ for all $t \in [0,\tau]$, so that, since $D^\beta(\tilde{u}(t))(0) = D^\beta_x u(t,0) = 0$, we get $\tilde{u}(t) \in \mathcal{D}(\mathsf{D}^\beta_Y)$ and $\mathsf{D}^\beta_Y \circ \tilde{u} = D^\beta(\tilde{u}(t)) = \tilde{v}$, which is continuous. Thus, $\mathsf{D}^\beta_Y \circ \tilde{u} \in \mathcal{C}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))$.

(c) In view of Theorem 2.21 (e), Proposition 1.10 and the fact that $\mathsf{D}_{\tilde{X}}$ and $\mathsf{D}_Y \circ$ are positive, it suffices to show that $\mathsf{D}^{-1}_{\tilde{X}}$ and $(\mathsf{D}_Y \circ)^{-1} = \mathsf{D}^{-1}_Y \circ$ commute. Thus, let $u \in \mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E)$. By (4.1), Theorem 4.12 (a) and the fact that the mapping $f \mapsto f(x)$ from $\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E)$ into E is bounded for any $x \in [0,\xi]$, we get

$$(\mathsf{D}_{\tilde{X}}\,\tilde{u})(t)(x) = \left\{\int_0^t \tilde{u}(s)\,ds\right\}(x) = \int_0^t u(s,x)\,ds$$

and

$$(\mathsf{D}_Y \circ \tilde{u})(t)(x) = \int_0^x u(t,y)\,dy$$

for all $(t,x) \in Q$. It follows that

$$\begin{aligned}(\mathsf{D}_{\tilde{X}}\,\mathsf{D}_Y \circ \tilde{u})(x)(t) &= \int_0^t \left[\int_0^x u(s,y)\,dy\right] ds \\ &= \int_0^x \left[\int_0^t u(s,y)\,ds\right] dy \\ &= (\mathsf{D}_Y \circ \mathsf{D}_{\tilde{X}}\,\tilde{u})(t)(x)\end{aligned}$$

for all $(t,x) \in Q$, which completes the proof. □

Remark 4.17. Lemma 4.15 (c) and Proposition 1.10 imply that $\mathsf{D}_{\tilde{X}}$ and $\mathsf{D}_Y\circ$ commute. Therefore, if $u, u_t, u_{xt} \in \mathcal{C}_{\partial_0 Q\mapsto 0}(Q;\tilde{X})$, then, by Lemma 4.15, u_x and u_{tx} exist, $u_x, u_{tx} \in \mathcal{C}_{\partial_0 Q\mapsto 0}(Q;\tilde{X})$, and $u_{tx} = u_{xt}$.

4.3 An equation with non–commuting operators

In this section we prove some results regarding the existence and maximal regularity of solutions u to the equation

$$D_t^\alpha u + b\,D_x u = f \quad (0 < \alpha < 1), \tag{4.34}$$

again in the space $\mathcal{C}_{\partial_0 Q\mapsto 0}(Q;E)$, where $Q = [0,\tau]\times[0,\xi]$. Here $f : Q \longrightarrow E$ and the coefficient $b : Q \longrightarrow \mathbb{C}$ are given functions. The restrictions that we impose on b will be specified below. At all events b will be assumed to be continuous and nonzero everywhere on Q.

As in the previous section, we transform the original equation (4.34) into an equation in $\tilde{X} := \mathcal{C}_{0\mapsto 0}([0,\xi]\,;\mathcal{C}_{0\mapsto 0}([0,\tau]\,;E))$. We have defined $\mathsf{D}^\alpha_{\tilde{X}}$ in $\tilde{X}$ so that

$$\mathsf{D}^\alpha_{\tilde{X}}\,u(t)(x) = D_t^\alpha u(t,x)$$

whenever $u \in \mathcal{D}(\mathsf{D}^\alpha_{\tilde{X}}) = \{u \in \mathcal{C}_{0\mapsto 0}([0,\tau]\,;Y) \parallel g_{1-\alpha} * u \in \mathcal{C}^1_{0\mapsto 0}([0,\tau]\,;Y)\}$, where $Y = \mathcal{C}_{0\mapsto 0}([0,\xi]\,;E)$. We have also defined the operator $\mathsf{D}_Y\circ$ in $\tilde{X}$ so that

$$\mathsf{D}_Y \circ \tilde{u}(t)(x) = D[\tilde{u}(t)](x) = u_x(t,x)$$

whenever $\tilde{u} \in \mathcal{D}(\mathsf{D}_Y \circ) = \mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}^1_{0\mapsto 0}([0,\xi]\,;E))$.

Since we assume that b belongs to $\mathcal{C}(Q)$, we know that $\tilde{b}$ belongs to $\mathcal{C}([0,\xi]\,;\mathcal{C}([0,\tau]))$. To any such function $\tilde{b} \in \tilde{X}$ we may associate a linear operator in $\tilde{X}$, also written $\tilde{b}$ by abuse of notation, by putting

$$(\tilde{b}\,\tilde{u})(t)(x) := \tilde{b}(t)(x)\,\tilde{u}(t)(x)$$

for all $\tilde{u} \in \tilde{X}$ and all $(t,x) \in Q$. Obviously $\mathcal{D}(\tilde{b}) = \tilde{X}$. Moreover, $\tilde{b}$ is bounded, and

$$\|\,\tilde{b}\,\|_{\mathcal{L}(\tilde{X})} = \|\,b\,\|_{\mathcal{C}(Q)}. \tag{4.35}$$

Therefore, we replace $D_t^\alpha u$ and $b\,D_x u$ by $\mathsf{D}^\alpha_{\tilde{X}}\,\tilde{u}$ and $\tilde{b}\ \mathsf{D}^\alpha_Y \circ\tilde{u}$, respectively, and consider the equation

$$\mathsf{D}^\alpha_{\tilde{X}}\,\tilde{u} + \tilde{b}\ \mathsf{D}_Y \circ\tilde{u} = \tilde{f} \tag{4.36}$$

in $\tilde{X}$. Since we assume $b(t,x)$ to be nonzero for all $(t,x) \in Q$, $\tilde{b}$ has a bounded inverse $\tilde{a}$, where $a = 1/b$. It follows that the equation can also be written

$$\tilde{a}\ \mathsf{D}^\alpha_{\tilde{X}}\,\tilde{u} + \mathsf{D}_Y \circ\tilde{u} = \tilde{a}\tilde{f}. \tag{4.37}$$

In the previous section we showed that the operators $\mathsf{D}^\alpha_{\tilde{X}}$ and $\mathsf{D}_Y \circ$ are resolvent commuting, positive and densely defined with spectral angles $\alpha\pi/2$ and $\pi/2$, respectively. It is quite evident, however, that the operators $\mathsf{D}^\alpha_{\tilde{X}}$ and $\tilde{b}\,\mathsf{D}_Y \circ$ are not, in general, resolvent commuting. Hence, we cannot apply the results on sums of resolvent commuting operators as in the previous section. Instead, our objective will be to apply the theory by Da Prato and Grisvard on the sum of two operators with non–commuting resolvents that was presented in Section 3.4.

4.3.1 Solving the equation in $\mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))$

The operator $\tilde{b}\,\mathsf{D}_Y \circ$ is of the form CB, where $B = \mathsf{D}_Y \circ$ is positive with spectral angle $\omega_{\mathsf{D}_Y^\circ} = \pi/2$. Assuming that

$$\|\,1 - a\,\|_{\mathcal{C}(Q)} \le 1/M_{\mathsf{D}_Y}(\phi_1)$$

for some ϕ_1 with $\omega_B < \omega_1 := \pi - \phi_1$, Lemma 2.26 shows that $\tilde{b}\,\mathsf{D}^\beta_Y \circ$ is positive with spectral angle $\omega_{\tilde{b}\,\mathsf{D}_Y \circ} < \omega_1$. If we also have $\omega_{\mathsf{D}^\alpha_{\tilde{X}}} + \omega_1 < \pi$, then we shall see that Theorem 3.42 may be applied to the operators $A = \mathsf{D}^\alpha_{\tilde{X}}$, $B = \mathsf{D}_Y \circ$ and $C = \tilde{b}$. Under similar conditions the same theorem may also be applied with $A = \mathsf{D}_Y \circ$, $B = \mathsf{D}^\alpha_{\tilde{X}}$ and $C = \tilde{a}$. The following lemma ensures that we can satisfy the hypotheses of that theorem with a suitable choice of the function $b \in \mathcal{C}(Q)$.

Lemma 4.18. *Let $0 < \mu < \alpha \le 1$, $0 < \nu < \beta \le 1$, and let $b \in \mathcal{C}^{\mu',\nu'}(Q)$, where $\mu < \mu' < 1$, and $\nu < \nu' < 1$. Assume that $b(t,x) \ne 0$ on Q, and set*

$a = 1/b$. Then for any $\phi_1 < (1-\alpha/2)\pi$ and any $\phi_2 < (1-\beta/2)\pi$ there are constants $K_1, K_2 > 0$ such that

$$(4.38)\quad \| \, [\, \tilde{b}\,;(z+\mathsf{D}_{\tilde{X}}^{\alpha})^{-1}\,]\,\tilde{a}\,\|_{\mathcal{L}(\mathcal{D}_{\mathsf{D}_Y^{\beta}\circ}(\nu/\beta))} \leq K_1(|z|^{-1-\mu'/\alpha} + |z|^{-1-(1-\nu/\nu')\mu'/\alpha})$$

for all z with $|\arg z| \leq \phi_1$, and

$$(4.39)\quad \| \, [\, \tilde{a}\,;(z+\mathsf{D}_{Y}^{\beta}\circ)^{-1}\,]\,\tilde{b}\,\|_{\mathcal{L}(\mathcal{D}_{\mathsf{D}_{\tilde{X}}^{\alpha}}(\mu/\alpha))} \leq K_2(|z|^{-1-\nu'/\beta} + |z|^{-1-(1-\mu/\mu')\nu'/\beta})$$

for all z with $|\arg z| \leq \phi_2$.

Proof. Recall that

$$\mathcal{D}_{\mathsf{D}_Y^{\beta}\circ}(\nu/\beta) = \tilde{X} \cap \mathcal{B}([0,\tau]\,;\mathcal{C}^{\nu}([0,\xi]\,;E))$$

and

$$\mathcal{D}_{\mathsf{D}_{\tilde{X}}^{\alpha}}(\mu/\alpha) = \mathcal{C}^{\mu}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))$$

with equivalence of the respective norms (cf. Lemma 4.11). We have

$$\{\,[\,\tilde{b}\,;(z+\mathsf{D}_{\tilde{X}})^{-1}\,]\,\tilde{a}\,\tilde{u}\}(t)(x) = \int_0^t e^{-z(t-s)}\left(\frac{b(t,x)}{b(s,x)} - 1\right) u(s,x)\,ds.$$

Let us set

$$\begin{aligned}
&\gamma := \operatorname{Re} z\\
&\|\, b\,\|_\infty := \|\, b\,\|_{\mathcal{C}(Q)}\\
&\|\, \tilde{u}\,\|_\infty := \|\, \tilde{u}\,\|_{\mathcal{C}([0,\tau];\mathcal{C}([0,\xi];E))}\\
&[\, b\,]_{\mu',\nu'} := [\, b\,]_{\mathcal{C}^{\mu',\nu'}(Q)}\\
&[\, \tilde{u}\,]_{\mu} := [\, \tilde{u}\,]_{\mathcal{C}^{\mu}([0,\tau];\mathcal{C}([0,\xi];E))}
\end{aligned}$$

and

$$[\, \tilde{u}\,]_{\nu} := \|\,[\, \tilde{u}(t)\,]_{\mathcal{C}^{\nu}([0,\xi];E)}\,\|_{\mathcal{C}([0,\tau];\mathbb{R})}$$

Then

$$\left|\frac{b(t,x)}{b(s,x)} - 1\right| = |a(t,x)|\,|b(t,x) - b(s,x)| \leq \|\, a\,\|_\infty [\, b\,]_{\mu',\nu'}|t-s|^{\mu'},$$

so that

$$\begin{aligned}
&\| \, [\, \tilde{b}\,;(z+\mathsf{D}_{\tilde{X}})^{-1}\,]\,\tilde{b}\,\tilde{u}\,\|_{\mathcal{C}([0,\tau];\mathcal{C}([0,\xi];E))}\\
&\qquad \leq \|\, a\,\|_\infty [\, b\,]_{\mu'\nu'}\|\, u\,\|_\infty \int_0^t e^{-\gamma(t-s)}(t-s)^{\mu'}\,dt\\
&\qquad = \|\, a\,\|_\infty [\, b\,]_{\mu'\nu'}\|\, u\,\|_\infty \gamma^{-1-\mu'}.
\end{aligned}$$

We also have

(4.40)
$$\left(\frac{b(t,x)}{b(s,x)}-1\right)u(s,x)-\left(\frac{b(t,y)}{b(s,y)}-1\right)u(s,y)$$
$$=\frac{b(t,x)-b(s,x)}{b(s,x)}(u(s,x)-u(s,y))+\left(\frac{b(t,x)}{b(s,x)}-\frac{b(t,y)}{b(s,y)}\right)u(s,y),$$

where

$$\left\|\frac{b(t,x)-b(s,x)}{b(s,x)}(u(s,x)-u(s,y))\right\|_E \leq (\|a\|_\infty[b]_{\mu',\nu'}[u]_\nu|t-s|^{\mu'}|x-y|^\nu$$

and

$$\begin{aligned}\frac{b(t,x)}{b(s,x)}-\frac{b(t,y)}{b(s,y)} &= \frac{b(t,x)b(s,y)-b(s,x)b(t,y)}{b(s,x)b(s,y)}\\ &= \frac{(b(t,x)-b(s,x))b(s,y)+b(s,x)(b(s,y)-b(t,y))}{b(s,x)b(s,y)}\\ &= \frac{(b(t,x)-b(t,y))b(s,y)+b(t,y)(b(s,y)-b(s,x))}{b(s,x)b(s,y)}\end{aligned}$$

The last two equalities give

$$\left|\frac{b(t,x)}{b(s,x)}-\frac{b(t,y)}{b(s,y)}\right| \leq 2\|b\|_\infty\|a\|_\infty^2[b]_{\mu',\nu'}|t-s|^{\mu'}$$

and

$$\left|\frac{b(t,x)}{b(s,x)}-\frac{b(t,y)}{b(s,y)}\right| \leq 2\|b\|_\infty\|a\|_\infty^2[b]_{\mu',\nu'}|x-y|^{\nu'},$$

whence

$$\left|\frac{b(t,x)}{b(s,x)}-\frac{b(t,y)}{b(s,y)}\right| \leq 2\|b\|_\infty\|a\|_\infty^2[b]_{\mu',\nu'}|t-s|^{\mu'(1-\nu/nu')}|x-y|^{\nu}.$$

Using (4.40), we therefore deduce

$$\begin{aligned}&\|\{[\tilde b;(z+\mathsf{D}_{\tilde X})^{-1}]\,\tilde a\,\tilde u\}(t)(x)-\{[\tilde b;(z+\mathsf{D}_{\tilde X})^{-1}]\,\tilde b\,\tilde u\}(t)(y)\|_E\\ &\quad\leq |x-y|^\nu\int_0^t e^{-\gamma(t-s)}\Big(\|a\|_\infty[b]_{\mu',\nu'}[u]_\nu|t-s|^{\mu'}\\ &\qquad\qquad + 2\|b\|_\infty\|a\|_\infty^2[b]_{\mu',\nu'}\|u\|_\infty|t-s|^{\mu'(1-\nu/\nu')}\Big)\,ds.\end{aligned}$$

It follows that there is a constant k_1 such that

$$\begin{aligned}&\|[\tilde b;(z+\mathsf{D}_{\tilde X})^{-1}]\,\tilde a\,\tilde u\|_{\mathcal{D}_{\mathsf{D}^\beta_Y\circ}(\nu/\beta)}\\ &\quad\leq k_1(|\gamma|^{-1-\mu'}+|\gamma|^{-1-\mu'(1-\nu/\nu')})(\|u\|_\infty+[\tilde u]_\nu),\end{aligned}$$

and consequently

$$
\begin{aligned}
&\| \, [\, \tilde{b}\,; (z + \mathsf{D}_{\tilde{X}})^{-1}\,]\, \tilde{a}\, \|_{\mathcal{L}(\mathcal{D}_{\mathsf{D}_Y^\beta \circ}(\nu/\beta))} \\
&\qquad \le k_1(|\gamma|^{-1-\mu'} + |\gamma|^{-1-\mu'(1-\nu/\nu')}).
\end{aligned}
$$

Since $\gamma = \operatorname{Re} z > |z| \sin\phi_1$, this takes care of the case $\alpha = 1$.

If $0 < \alpha < 1$, then, using formula (2.51), we get

$$
\begin{aligned}
&\{\, [\, \tilde{b}\,; (z + \mathsf{D}_{\tilde{X}}^\alpha)^{-1}\,]\, \tilde{a}\tilde{u}\}(t)(x) \\
&\quad = \frac{\sin\pi\alpha}{\pi} \int_0^\infty r^\alpha (r^\alpha + z e^{i\pi\alpha})^{-1}(r^\alpha + z e^{-i\pi\alpha})^{-1}\, [\, \tilde{b}\,; (r + A)^{-1}\,]\, \tilde{a}(t)(x)\, dr
\end{aligned}
$$

for all z with $|\arg z| < \phi_{\mathsf{D}_{\tilde{X}}^\alpha} = (1 - \alpha/2)\pi$. Using the above estimates, we hence obtain

$$
\begin{aligned}
&\| \{\, [\, \tilde{b}\,; (z + \mathsf{D}_{\tilde{X}}^\alpha)^{-1}\,]\, \tilde{a}\, \tilde{u}\}(t)\, \|_{\mathcal{D}_{\mathsf{D}_Y^\beta \circ}(\nu/\beta)} \\
&\qquad \le k_\alpha(|\gamma|^{-1-\mu'/\alpha} + |\gamma|^{-1-(1-\nu/\nu')\mu'/\alpha})(\| u \|_\infty + [\, u\,]_\nu)
\end{aligned}
$$

for some constant k_α This proves (4.38).

The proof of (4.39) is perfectly analogous, since

$$
\{\, [\, \tilde{a}\,; (z + \mathsf{D}_{\tilde{X}})^{-1}\,]\, \tilde{b}\, \tilde{u}\}(t)(x) = \int_0^x e^{-z(x-y)} \left(\frac{b(t,x)}{b(t,y)} - 1 \right) u(t,y)\, dy,
$$

for all z with $|\arg z| < \pi/2$, and

$$
\begin{aligned}
\{\, [\, \tilde{a}\,; (z + \mathsf{D}_Y^\beta \circ)^{-1}\,]\, \tilde{b}\, \tilde{u}\}(t)(x) = \frac{\sin\pi\beta}{\pi} \int_0^\infty & r^\alpha (r^\beta + z e^{i\pi\beta})^{-1}(r^\beta + z e^{-i\beta})^{-1} \\
& [\, \tilde{a}\,; (r + \mathsf{D}_Y^\beta)^{-1} \circ\,]\, \tilde{b}(t)(x)\, dr.
\end{aligned}
$$

for all z with $|\arg z| < \phi_{\mathsf{D}_Y^\beta \circ} = (1 - \beta/2)\pi$. □

With the aid of Lemmas 4.18 and 2.26 we may now apply Theorem 3.42 with $A = \mathsf{D}_{\tilde{X}}^\alpha$, $B = \mathsf{D}_Y \circ$ and $C = \tilde{b}$ to the equation

$$
\lambda\tilde{u} + \mathsf{D}_{\tilde{X}}^\alpha\, \tilde{u} + \tilde{b}\, \mathsf{D}_Y \circ \tilde{u} = \tilde{f}
$$

provided that $\| 1 - a \|_{\mathcal{C}(Q)} < 1/M_{\mathsf{D}_Y}(\phi_1)$ for some ϕ_1 with $\omega_{\mathsf{D}_Y \circ} = \pi/2 < \omega_1 := \pi - \phi_1 < (1 - \alpha/2)\pi = \phi_{\mathsf{D}_{\tilde{X}}^\alpha}$. In that case there is some $\lambda_0 \ge 0$ such that the equation under consideration has a unique solution $\tilde{u}$ for any $\tilde{f} \in \mathcal{D}_{\mathsf{D}_Y \circ}(\nu, \infty) = \tilde{X} \cap \mathcal{C}([0,\tau]\,; \mathcal{C}^\nu([0,\xi]\,; E))$ and any $\lambda > \lambda_0$. Moreover, we have

$$
\mathsf{D}_{\tilde{X}}^\alpha\, \tilde{u},\ \mathsf{D}_Y \circ \tilde{u} \in \mathcal{D}_{\mathsf{D}_Y \circ}(\nu, \infty). \tag{4.41}
$$

Let us also consider the equation

$$\lambda\tilde{v} + \tilde{a}\,\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{v} + \mathsf{D}_Y\circ\tilde{v} = \tilde{g}.$$

If we assume that there is a ϕ_2 with $\omega_{\mathsf{D}^{\alpha}_{\tilde{X}}} = \alpha\pi/2 < \omega_2 := \pi - \phi_2 < \pi/2 = \phi_{\mathsf{D}_Y\circ}$ such that $\|\,1-b\,\|_{\mathcal{C}(Q)} < 1/M_{\mathsf{D}^{\alpha}_{\tilde{X}}}(\phi_2)$, then this equation has a unique solution $\tilde{v}$ for any $\tilde{g} \in \mathcal{D}_{\mathsf{D}^{\alpha}_{\tilde{X}}}(\mu/\alpha) = \mathcal{C}^{\mu}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))$ and any $\lambda > \lambda_0'$, where $0<\mu<\alpha$, and λ_0' is a nonnegative constant.

If we take $g(t,x) = e^{-\lambda x}a(t,x)f(t,x)$, we obtain a solution $\tilde{v}$. Then $\tilde{u}$, where $u(t,x) = e^{\lambda x}v(t,x)$ solves the equation $\tilde{a}\,\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{u} + \mathsf{D}_Y\circ\tilde{u} = \tilde{a}\tilde{f}$, i.e.

$$\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{u} + \tilde{b}\,\mathsf{D}_Y\circ\tilde{u} = \tilde{f}.$$

We also have

$$\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{u},\ \mathsf{D}_Y\circ\tilde{u} \in \mathcal{D}_{\mathsf{D}^{\alpha}_{\tilde{X}}}(\mu/\alpha), \tag{4.42}$$

For this solution $\tilde{u}\in\mathcal{D}(\mathsf{D}_Y\circ)$ to the equation

$$\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{u} + \tilde{b}\,\mathsf{D}_Y\circ\tilde{u} = \tilde{f}.$$

we let w be the solution to

$$\lambda\tilde{w} + \mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{w} + \tilde{b}\,\mathsf{D}_Y\circ\tilde{w} = \lambda\tilde{u},$$

where $\lambda > \lambda_0$. Then, by what has been said above,

$$\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{w}, \mathsf{D}_Y\circ\tilde{w} \in \mathcal{D}_{\mathsf{D}_Y\circ}(\nu,\infty).$$

Subtracting the last two equations, get

$$\lambda(\tilde{u}-\tilde{w}) + \mathsf{D}^{\alpha}_{\tilde{X}}(\tilde{u}-\tilde{w}) + \tilde{b}\,\mathsf{D}_Y\circ(\tilde{u}-\tilde{w}) = \tilde{f}.$$

If $\tilde{f}\in\mathcal{D}_{\mathsf{D}^{\alpha}_{\tilde{X}}}(\mu/\alpha,\infty)\cap\mathcal{D}_{\mathsf{D}_Y\circ}(\nu,\infty)$, it follows that both $\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{w}$ and $\mathsf{D}^{\alpha}_{\tilde{X}}(\tilde{u}-\tilde{w})$ belong to $\mathcal{D}_{\mathsf{D}_Y\circ}(\nu,\infty)$, so that $\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{u}$ also belongs to this interpolation space. In the same fashion we see that $\mathsf{D}_Y\circ\tilde{u}$ belongs to $\mathcal{D}_{\mathsf{D}_Y\circ}(\nu,\infty)$. Recalling (4.42), we hence get

$$\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{u}, \mathsf{D}_Y\circ\tilde{u} \in \mathcal{D}_{\mathsf{D}^{\alpha}_{\tilde{X}}}(\mu/\alpha)\cap\mathcal{D}_{\mathsf{D}_Y\circ}(\nu,\infty). \tag{4.43}$$

Summing up what has been proved so far, we arrive at the following lemma.

Lemma 4.19. *Let $0<\mu<\alpha<1$, $0<\nu<1$, and let $b\in\mathcal{C}^{\mu',\nu'}(Q)$ and $b(t,x)\neq 0$ for all $(t,x)\in Q=[0,\tau]\times[0,\xi]$, where $\mu<\mu'<1$, $\nu<\nu'<1$ and $\tau,\xi>0$. Assume, in addition, that $\|\,1-a\,\|_{\mathcal{C}(Q)} < 1/M_{\mathsf{D}_Y}(\phi_1)$ and $\|\,1-b\,\|_{\mathcal{C}(Q)} < 1/M_{\mathsf{D}^{\alpha}_{\tilde{X}}}(\phi_2)$ where $\alpha\pi/2<\phi_1$, $\pi-\phi_2<\pi/2$. Then the equation*

$$\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{u} + \tilde{b}\,\mathsf{D}_Y\circ\tilde{u} = \tilde{f}$$

has a unique solution $\tilde{u}$ for any $\tilde{f}$ in

$$\tilde{X}_{\mu,\nu} := \mathcal{C}^{\mu}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}_{0\mapsto 0}([0,\xi]\,;E))\cap\mathcal{C}_{0\mapsto 0}([0,\tau]\,;\mathcal{C}^{\nu}_{0\mapsto 0}([0,\xi]\,;E)).$$

Moreover,

$$\mathsf{D}^{\alpha}_{\tilde{X}}\,\tilde{u}, \mathsf{D}_Y\circ\tilde{u} \in \tilde{X}_{\mu,\nu}.$$

4.3.2 Carrying over the results to $X = \mathcal{C}_{\partial_0 Q \mapsto 0}(Q; E)$

Imitating the procedure used in the unperturbed case, we can show that if b, α, μ and ν are as in the previous lemma, then the equation

$$D_t^\alpha u + b\, D_x u = f$$

has a unique solution $u \in \mathcal{C}_{\partial_0 Q \mapsto 0}(Q; E)$ for any $f \in \mathcal{C}^{\mu,\nu}_{\partial_0 Q \mapsto 0}(Q; E)$, and the solution as well as $D_t^\alpha u$ and $D_x u$ belong to $\mathcal{C}^{\mu,\nu}_{\partial_0 Q \mapsto 0}(Q; E)$. We thus obtain the final result of this section.

Theorem 4.20. *Let $0 < \mu < \alpha < 1$, $0 < \nu < 1$. Assume that $b \in \mathcal{C}^{\mu',\nu'}(Q)$ and $b(t,x) \neq 0$ for all $(t,x) \in Q = [0,\tau] \times [0,\xi]$, where $\mu < \mu' < 1$, $\nu < \nu' < 1$ and $\tau, \xi > 0$. Assume, in addition, that $\| 1 - a \|_{\mathcal{C}(Q)} < 1/M_{\mathsf{D}_Y}(\phi_1)$and $\| 1 - b \|_{\mathcal{C}(Q)} < 1/M_{\mathsf{D}^\alpha_X}(\phi_2)$ where $\alpha\pi/2 < \phi_1$, $\pi - \phi_2 < \pi/2$. Then for any $f \in \mathcal{C}^{\mu,\nu}_{\partial_0 \mapsto 0}(Q; E)$ the equation*

$$D_t^\alpha u + b\, D_x u = f$$

has a unique solution u, and we have

$$D_t^\alpha u,\ D_x u \in \mathcal{C}^{\mu,\nu}_{\partial_0 \mapsto 0}(Q; E).$$

Remark 4.21. If $b \in \mathcal{C}^{\mu',\nu'}(Q)$, we can define $\hat{b} \in \mathcal{C}^{\mu',\nu'}(\hat{Q})$ by putting $\hat{b}(t,x) = b(t, b_0 x)/b_0$, where $b_0 = |b(0,0)|$ and $\hat{Q} = [0,\tau] \times [0, \xi/b_0]$. Clearly, if u solves the equation

$$D_t^\alpha u + \hat{b} D_x u = f \tag{4.44}$$

in $\mathcal{C}_{\partial_0 \hat{Q} \mapsto 0}(\hat{Q}; E)$, then $v(t,x) = u(t, b_0 x)$ solves the equation $D_t^\alpha u + b D_t^\beta u = 0$ in $\mathcal{C}_{\partial_0 Q \mapsto 0}(Q; E)$ and vice versa.

If we developed a technique to accommodate nonzero boundary values at $t = 0$ and $x = 0$ we would thus be able to state the theorem for any rectangle $Q = [0,\tau] \times [0,\xi]$ and any $b \in \mathcal{C}^{\mu',\nu'}(Q)$ with $b(t,x) \neq 0$ for all $(t,x) \in Q$. A result of this kind has been presented in [4].

A Vector–valued calculus

In the first section of this appendix we give short introduction to Bochner integrals of vector–valued functions defined on Lebesgue measurable subsets of $\mathbb{R}$. In the second we prove some results from vector–valued complex analysis, the final goal being the Residue Theorem together with some observations on residues at poles. The material is mainly based on [2].

A.1 Bochner integrals

A.1.1 Measurable functions

Let X be a complex Banach space, and let $\Omega \subseteq \mathbb{R}$ be Lebesgue measurable. Analogously to the scalar–valued case, a function $: \Omega \longrightarrow X$ is a *simple function* if

$$f = \sum_{i=1}^{n} \chi_{\Omega_i} x_i,$$

where $n \in \mathbb{N}$, $\{x_i\}_{i=1}^n \subset X$ is a sequence of vectors, and is $\{\Omega_i\}_{i=1}^n$ a sequence of pairwise disjoint Lebesgue measurable sets $\Omega_i \subseteq \Omega$ with finite Lebesgue measure $m(\Omega_i) > 0$ for $i = 1, 2, \ldots, n$.

If the sets Ω_i can be chosen to be intervals, then f is a *step function*.

A function $f : \Omega \longrightarrow X$ is *strongly measurable* if there is a sequence $\{g_n\}_{n=1}^\infty$ of simple functions $g_n : \Omega \longrightarrow X$ such that $f(t) = \lim_{n\to\infty} g_n(t)$ for almost all $t \in \Omega$. From now on we omit the word t̕stronglyt̕ and simply say that f is *measurable* when f is strongly measurable.

We note that if $X = \mathbb{C}$, the above definition of a measurable function agrees with the usual one. It is also obvious that if $f : \Omega \longrightarrow X$ is the pointwise limit (a.e.) of a sequence of measurable functions $f_n : \Omega \longrightarrow X$, then f is measurable.

If X and X_i $(i = 1, 2, \ldots, m)$ are Banach spaces, the functions $f_i : \Omega \longrightarrow X_i$ are measurable, and $k : X_1 \times X_2 \times \ldots \times X_m \longrightarrow X$ is continuous, then $k(f_1, f_2, \ldots, f_m)$ is measurable. In fact, if $g_i^n : \Omega \longrightarrow X_i$ are sequences of simple functions with $\lim_{n\to\infty} g_i^n(t) = f_i(t)$ for a.e. t and $i = 1, 2, \ldots, m$, then $k(g_1^n, g_2^n, \ldots, g_m^n) : \Omega \longrightarrow X$ is a sequence of simple functions that converges pointwise to $k(f_1, f_2, \ldots, f_m)$ a.e. in Ω. In particular, if $f, g : \Omega \longrightarrow X$ and $h : \Omega \longrightarrow \mathbb{C}$ are measurable, then so are $\| f \|$, $f + g$ and hf.

We say that a function $f : \Omega \longrightarrow X$ is *almost separably valued* if there is a nullset $\Omega_0 \subseteq \Omega$ (i.e. Ω_0 has zero measure) such that $f(\Omega \setminus \Omega_0)$ is separable. A function $f : \Omega \longrightarrow X$ is *weakly measurable* if $\langle f(\underline{z}), \phi \rangle : \Omega \longrightarrow \mathbb{C}$ is measurable for all $\phi \in X^*$. Thus, a measurable function $f : \Omega \longrightarrow X$ is weakly measurable. In fact, using these concepts, we can state an equivalent characterisation of measurability for vector valued functions.

Lemma A.1 (Pettis). *A function $f : \Omega \longrightarrow X$ is measurable if and only if it is weakly measurable and almost separably valued.*

As a consequence of this lemma we note that if X is separable, then $f : \Omega \longrightarrow X$ is measurable if and only if it is weakly measurable.

A.1.2 Bochner integrability

If $f : \Omega \longrightarrow X$ is a simple function with $f = \sum_{i=1}^{n} x_i \chi_{\Omega_i}$, we define the integral of f over Ω by

$$\int\limits_{\Omega} f(t)\, dt = \sum_{i=1}^{n} m(\Omega_i) x_i, \tag{A.1}$$

where μ is the Lebesgue measure on Ω. It is a simple exercise to show that this definition is independent of the particular representation of f as a simple function.[6]

If $f : \Omega \longrightarrow X$ is a simple function, then $\| f \| : \Omega \longrightarrow \mathbb{R}^+$ is a simple function, and it is an immediate consequence of the above definition that

$$\left\| \int\limits_{\Omega} f(t)\, dt \right\| \leq \int\limits_{\Omega} \| f(t) \|\, dt. \tag{A.2}$$

A function $f : \Omega \longrightarrow X$ is called *Bochner integrable* if there exist simple functions $g_n : \Omega \longrightarrow X$ such that

$$\lim_{n\to\infty} g_n(t) = f(t) \tag{A.3}$$

for a.e. $t \in \Omega$, and

$$\lim_{n\to\infty} \int\limits_{\Omega} \| f(t) - g_n(t) \|\, dt = 0. \tag{A.4}$$

In that case one easily checks that the sequence of integrals $\int_\Omega g_n(t)\, dt$ is a Cauchy sequence in X, so that one can define

$$\int\limits_{\Omega} f(t)\, dt = \lim_{n\to\infty} \int\limits_{\Omega} g_n(t)\, dt. \tag{A.5}$$

This definition is unambiguous, since if $\{h_n\}_{n=1}^{\infty}$ is another sequence of simple functions with properties (A.3) and (A.4), then

$$\left\| \int\limits_{\Omega} g_n(t)\, dt - \int\limits_{\Omega} h_n(t)\, dt \right\| \leq \int\limits_{\Omega} \| g_n(t) - f(t) \|\, dt + \int\limits_{\Omega} \| f(t) - h_n(t) \|\, dt,$$

[6] Starting from any representation $f = \sum_{i=1}^{n} \chi_{\Omega_i} x_i$ of f, as in the definition of a simple function, we can take the union of those Ω_i for which the x_i are equal ($i = 1, 2, \ldots, n$). In this manner we obtain a representation $f = \sum_{i=1}^{n} \chi_{\Omega_i} x_i$, where (i) $m(\Omega_i) \neq 0$; (ii) $x_i \neq 0$; (iii) $\Omega_i \cap \Omega_j = \emptyset$; and (iv) $x_i \neq x_j$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$. For such a representation one has $\mathcal{R}(f) \setminus \{0\} = \{x_1, x_2, \ldots, x_n\}$ and $\Omega_i = f^{-1}(x_i)$ for $i = 1, 2, \ldots, n$, which implies that this representation of f is unique except for the order of the terms $\chi_{\Omega_i} x_i$. It is clear that the construction of this new representation of f does not change the value of the left hand member of (A.1).

and the integrals on the right hand side vanish as $n \to \infty$.

Let us also define $\int_a^b f(t)\,dt$ by setting

$$\int\limits_a^b f(t)\,dt = \begin{cases} \int\limits_{(a,b)} f(t)\,dt & \text{if} \quad a \le b, \\ -\int\limits_{(b,a)} f(t)\,dt & \text{if} \quad b < a, \end{cases}$$

where $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ and the combinations $a = b = -\infty$ and $a = b = \infty$ are excluded.

One easily shows that $\int_\Omega f(t)\,dt \in \overline{\operatorname{span}\{f(t) \mid t \in \Omega\}}$, the closure of the linear span of the set $\{f(t) \mid t \in \Omega\}$.

As in the scalar valued case, the mapping $f \mapsto \int_\Omega f(t)\,dt$ is linear from the set $L^1(\Omega; X)$ of Bochner integrable functions into X.[7]

It is obvious from the definition of Bochner integrability, that if f is Bochner integrable, then f, and hence also $\| f \|$, is measurable. Moreover, the inequality

$$\int\limits_\Omega \| f(t) \| \, dt \le \int\limits_\Omega \| g_n(t) \| \, dt + \int\limits_\Omega \| f(t) - g_n(t) \| \, dt$$

shows that $\| f \|$ is integrable, or, in other words, f is *absolutely integrable*. In fact, we have the following theorem.

Theorem A.2 (Bochner). *A function $f : \Omega \longrightarrow X$ is Bochner integrable if and only if f is measurable and $\| f \|$ is integrable. Moreover, if f is Bochner integrable, then*

$$\left\| \int_\Omega f(t)\,dt \right\| \le \int_\Omega \| f(t) \| \, dt.$$

Proof. We have proved that if f is Bochner integrable, then it is measurable and $\| f \|$ is integrable. Therefore, let us assume that f is measurable, and that $\| f \|$ is integrable. Let $\{g_n\}_{n=1}^\infty$ be a sequence such that $g_n \to f$ pointwise on $\Omega \setminus \Omega_0$ as $n \to \infty$, where $m(\Omega_0) = 0$. Let us define

$$h_n(t) = \begin{cases} g_n(t) & \text{if } \| g_n \| \le 2 \, \| f \|, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously h_n is a simple function, $\| h_n(t) \| \le 2 \, \| f(t) \|$ for $n = 1, 2, \ldots$ and $t \in \Omega$, and $h_n \to f$ on $\Omega \setminus \Omega_0$. Hence, $\| h_n(t) - f(t) \| \le 3 \, \| f \|$ and $\| h_n(t) - f(t) \| \to 0$ as $n \to \infty$ for $t \in \Omega \setminus \Omega_0$. Consequently, the scalar form of the Dominated Convergence Theorem shows that

$$\lim_{n \to \infty} \int_\Omega \| h_n(t) - f(t) \| \, dt = 0.$$

[7]For the notation $L^1(\Omega; X)$, see Subsection A.1.9.

It follows that f is Bochner integrable, and

$$\int_\Omega f(t)\,dt = \lim_{n\to\infty} \int_\Omega h_n(t)\,dt.$$

Moreover, we have

$$\begin{aligned}\left\| \int_\Omega f(t)\,dt \right\| &= \lim_{n\to\infty} \left\| \int_\Omega h_n(t)\,dt \right\| \\ &\le \lim_{n\to\infty} \int_\Omega \| h_n(t) \|\,dt = \int_\Omega \| f(t) \|\,dt,\end{aligned}$$

where we have applied, again, the Dominated Convergence Theorem for scalar–valued integrals and the fact that $\left\| \int_\Omega h_n(t)\,dt \right\| \le \int_\Omega \| h_n(t) \|\,dt$ (since each h_n is a simple functions). □

A.1.3 The Dominated Convergence Theorem

One of the most important results of integration theory is the Dominated Convergence Theorem. In the vector–valued case it can be stated as follows.

Theorem A.3. *Let $f_n : \Omega \longrightarrow X$ be Bochner integrable functions ($n = 1, 2, \ldots$). If $\lim_{n\to\infty} f_n(t) = f(t)$ for a.e. $t \in \Omega$, and if there is an integrable function $g : \Omega \longrightarrow \mathbb{R}$ with $\| f_n(t) \| \le g(t)$ a.e. for all n, then f is Bochner integrable, and*

$$\int_\Omega f(t)\,dt = \lim_{n\to\infty} \int_\Omega f_n(t)\,dt.$$

Moreover,

$$\lim_{n\to\infty} \int_\Omega \| f_n(t) - f(t) \|\,dt = 0.$$

Proof. Since, under the assumptions of the theorem, f is the pointwise limit (a.e.) of a sequence of measurable functions, it is measurable. Then so are the functions $f_n - f$ ($n = 1, 2, \ldots$), and $\| f_n - f \|$. Obviously $\| f_n - f \| \le 2g$ a.e., so that we can apply the familiar scalar–valued version of the Dominated Convergence Theorem to $\| f_n - f \|$. Hence,

$$\lim_{n\to\infty} \int_\Omega \| f_n(t) - f(t) \|\,dt = 0.$$

But $\left\| \int_\Omega (f_n(t) - f(t))\,dt \right\| \le \int_\Omega \| f_n(t) - f(t) \|\,dt$, and we deduce that

$$\int_\Omega f(t)\,dt = \lim_{n\to\infty} \int_\Omega f_n(t)\,dt.$$

□

Let us also mention the following easy corollary to the Dominated Convergence Theorem.

Corollary A.4. *Let $f : (a,\infty) \longrightarrow X$ be measurable, where $a \in \mathbb{R}$, and assume that $\lim_{T\to\infty} \int_a^T \| f(t) \|_X \, dt$ is finite. Then f is Bochner integrable, and*

$$\int_a^\infty f(t)\,dt = \lim_{T\to\infty} \int_a^T f(t)\,dt.$$

Proof. By assumption the functions $\chi_{(a,T)} f$ are Bochner integrable over the interval (a,∞). Moreover, the monotone convergence theorem for scalar–valued functions shows that

$$\int_0^\infty \| f(t) \|_X \, dt = \lim_{T\to\infty} \int_a^\infty \| \chi_{(0,T)} f(t) \|_X \, dt = \lim_{T\to\infty} \int_a^T \| f(t) \|_X \, dt < \infty,$$

so that f is also Bochner integrable. Since $\left\| \chi_{(a,T)}(t) f(t) \right\|_X \leq \| f(t) \|_X$ for all $t \in (a,\infty)$, the corollary follows by an easy application of the Dominated Convergence Theorem. □

A.1.4 The action of closed linear operators on Bochner integrals

If $f = \sum_{i=1}^n x_i \chi_{\Omega_i}$ is a simple function mapping Ω into a Banach space X, and if $L : X \longrightarrow Y$ is a linear mapping of X into another Banach space Y, then Lf is obviously a simple function mapping Ω into Y, and we have $L \int_\Omega f(t)\,dt = \int_\Omega Lf(t)\,dt$. The following two theorems give sufficient conditions on L for this formula to hold for functions f that are not simple.

Theorem A.5. *Let $L : X \longrightarrow Y$ be a bounded linear operator that maps a Banach space X into another Banach space Y. Assume that $f : \Omega \longrightarrow X$ is Bochner integrable. Then $L \circ f : \Omega \longrightarrow Y$ is Bochner integrable and*

$$L \int_\Omega f(t)\,dt = \int_\Omega L(f(t))\,dt.$$

Proof. Let $\{g_n\}_{n=1}^\infty$ be a sequence of simple functions from Ω into X, with

$$\lim_{n\to\infty} g_n(t) = f(t)$$

X for a.e. $t \in \Omega$ and $\lim_{n\to\infty} \int_\Omega \| g_n(t) - f(t) \| \, dt = 0$. Then $L \circ g_n$ is a simple function for $n = 1, 2, \ldots$, and $\lim_{n\to\infty} Lg_n(t) = Lf(t)$ in X for a.e. $t \in \Omega$. We also have

$$\int_\Omega \| Lf(t) - Lg_n(t) \|_Y \, dt \leq \| L \| \int_\Omega \| f(t) - g_n(t) \|_X \, dt,$$

and the integral on the right hand side vanishes as $n \to \infty$. Hence, $L \circ f$ is Bochner integrable and

$$\int_\Omega Lf(t)\,dt = \lim_{n\to\infty} \int_\Omega Lg_n(t)\,dt = L\left(\lim_{n\to\infty} \int_\Omega g_n(t)\,dt\right) = L\int_\Omega f(t)\,dt.$$

Here we have used the fact that $\int_\Omega Lg(t)\,dt = L\int_\Omega g(t)\,dt$ for simple functions g, which is an immediate consequence of the definition of the Bochner integral of a simple function. □

Theorem A.6. *Let A be a closed linear operator from a Banach space X into a Banach space Y, and let $f : \Omega \longrightarrow X$ be Bochner integrable. Assume that $f(t) \in \mathcal{D}(A)$ for every $t \in \Omega$, and that $A \circ f : \Omega \longrightarrow Y$ is Bochner integrable. Then $\int_\Omega f(t)\,dt \in \mathcal{D}(A)$, and*

$$A\int_\Omega f(t)\,dt = \int_\Omega A(f(t))\,dt.$$

Proof. Let us define the function $g : \Omega \longrightarrow A \subseteq X \times Y$ by the equality $g(t) := (f(t), A(f(t)))$ for any $t \in \Omega$. If we provide $X \times Y$ with a norm defined by $\|(x,y)\| = \|x\|_X + \|y\|_Y$, then $X \times Y$ becomes a Banach space. As both $f : \Omega \longrightarrow X$ and $A \circ f : \Omega \longrightarrow Y$ are measurable, it is clear that g is measurable. Moreover, $\int_\Omega \|g(t)\|\,dt = \int_\Omega (\|f(t)\|_X + \|A(f(t))\|_Y)\,dt$ is finite, so that g is Bochner integrable.

Since the operator A is closed, it is a closed linear subspace of $X \times Y$. It follows that $\int_\Omega g(t)\,dt \in \overline{\operatorname{span}\{(f(t), A(f(t))) \,\|\, t \in \Omega\}} \subseteq A$. Let $\pi_1 : X \times Y \longrightarrow X$ and $\pi_2 : X \times Y \longrightarrow Y$ be the projections of $X \times Y$ on X and Y, respectively. These projections are bounded linear operators, and, by the previous theorem, we have

$$\pi_1 \int_\Omega g(t)\,dt = \int_\Omega f(t)\,dt$$
$$\pi_2 \int_\Omega g(t)\,dt = \int_\Omega A(f(t))\,dt.$$

Hence,

$$\left(\int_\Omega f(t)\,dt, \int_\Omega A(f(t))\,dt\right) = \int_\Omega g(t)\,dt \in A.$$

It follows that $\int_\Omega f(t)\,dt \in \mathcal{D}(A)$ and $A\int_\Omega f(t)\,dt = \int_\Omega A(f(t))\,dt$. □

The proof of the following useful lemma illustrates how many results for vector–valued integrals can be obtained by a simple application of Theorem A.5 and the Hahn-Banach theorem (Theorem A.8).

Lemma A.7. *If $h : \Omega \longrightarrow \mathbb{C}$ is absolutely integrable and $x \in X$, then $f(t) = h(t)x$ defines a Bochner integrable mapping of Ω into X, and $\int_\Omega h(t)x\,dt = \left(\int_\Omega h(t)\,dt\right) x$.*

Proof. It is clear that f, defined as in the lemma, is measurable. By assumption, $\| f \| = \| x \| \, |h|$ is integrable. Hence, f is Bochner integrable.

Now take $\phi \in X^*$. Then

$$\begin{aligned}\langle \int_\Omega f(t)\, dt, \phi \rangle &= \int_\Omega \langle f(t), \phi \rangle \, dt \\ &= \int_\Omega \langle x, \phi \rangle h(t)\, dt \\ &= \langle \left(\int_\Omega h(t)\, dt \right) x, \phi \rangle.\end{aligned}$$

Since these equalities hold for any $\phi \in X^*$, the Hahn-Banach theorem, stated below, implies that $\int_\Omega f(t)\, dt = \left(\int_\Omega h(t)\, dt \right) x$. □

Theorem A.8 (Hahn-Banach). *Let X be a normed vector space. If $x, y \in X$ and $\langle x, \phi \rangle = \langle y, \phi \rangle$ for any $\phi \in X^*$, then $x = y$.*

Another way of putting the statement of the Hahn-Banach theorem is to say that the dual of a normed space X separates points on X. A proof of this result can be found in e.g. [12]. (There X is only assumed to be a *locally convex* vector space). The Hahn-Banach theorem will prove very useful in the sequel.

A.1.5 Change of variables

In this subsection we investigate how the familiar change–of–variables formula for Lebesgue integrals can be carried over to our present vector–valued setting.

Theorem A.9. *Let $T : \Omega' \longrightarrow \Omega$ be a differentiable bijection with continuous inverse, where $\Omega' \subseteq \mathbb{R}$ is open and $\Omega \subset \mathbb{R}$ is open and bounded. Assume that $f : \Omega \longrightarrow X$ is Bochner integrable. Then $T'(f \circ T)$ is Bochner integrable, and*

$$\int_\Omega f(t)\, dt = \int_{\Omega'} |T'(s)|\, f(T(s))\, ds.$$

Proof. Let $T : \Omega' \longrightarrow \Omega$ and $f : \Omega \longrightarrow X$ be as in the theorem. Then the function $\| f \| : \Omega \longrightarrow \mathbb{R}^+$ is integrable over Ω, and we have[8]

$$\int_\Omega \| f(t) \| \, dt = \int_{\Omega'} |T'(s)| \, \| f(T(s)) \| \, ds.$$

Hence, $|T| \, (f \circ T)$ is Bochner integrable over Ω'.

[8] See e.g. [11], pp. 185–187.

Let us choose an arbitrary $\phi \in X^*$. Then[9]

$$\begin{aligned}\langle \int\limits_{\Omega'} |T'(s)|\, f(T(s))\, ds, \phi\rangle &= \int\limits_{\Omega'} |T'(s)|\, \langle f(T(s)), \phi\rangle\, ds \\ &= \int\limits_{\Omega} \langle f(t), \phi\rangle\, dt = \langle \int\limits_{\Omega} f(t)\, dt, \phi\rangle.\end{aligned}$$

Hence, by the Hahn-Banach theorem, the statement of the theorem is true. □

Remark A.10. By the method used in the proof of Corollary A.4, one can easily show that the above theorem remains true when Ω' is an interval, and Ω is an interval that is possibly unbounded.

A.1.6 The Fundamental Theorem of Calculus

Let $f : \Omega \longrightarrow X$. An integral function of f is a function $F : \Omega \longrightarrow X$ such that $F' = f$. The following lemma shows that integral functions on intervals differ by a constant.

Lemma A.11. *Let $I \subseteq \mathbb{R}$ be an interval. If $f : I \longrightarrow X$ is differentiable and $f' = 0$, then f is constant.*

Proof. Take $\phi \in X^*$. Then $\langle f(\underline{t}), \phi\rangle' = \langle f'(\underline{t}), \phi\rangle = 0$, so that $\langle f(\underline{t}), \phi\rangle$ is constant. Consequently, f is constant by the Hahn-Banach theorem (Theorem A.8). □

We now prove that a function that a continuous function on a closed interval always has an integral function.

Theorem A.12. *Let $f : (a, b) \longrightarrow X$ be Bochner integrable and continuous at $t \in (a, b)$ $(a, b \in \mathbb{R}, a < b)$. Then the function $F : (a, b) \longrightarrow X$ defined by $F(s) = \int_a^s f(\tau)\, d\tau$ is differentiable at t, and $S'(t) = f(t)$.*

Proof. Take an arbitrary $\varepsilon > 0$. If $h \in \mathbb{R} \setminus \{0\}$ is so small that $t + h \in (a, b)$ and $\| f(s) - f(t) \| < \varepsilon$ for $|s - t| < h$, then

$$\begin{aligned}\left\| \frac{1}{h}(F(t+h) - F(t)) - f(t) \right\| &= \left\| \frac{1}{h} \int\limits_{t}^{t+h} (f(s) - f(t))\, ds \right\| \\ &\le \left| \frac{1}{h} \int\limits_{t} t + h \, \| f(s) - f(t) \| \, ds \right| \\ &\le \varepsilon.\end{aligned}$$

□

[9]For the second equality, see the previous reference.

Remark A.13. If $f : (a, b] \longrightarrow X$ is Bochner integrable and left continuous at b, we can put $f(t) = f(b)$ for $t > b$. Then $f : (a, \infty) \longrightarrow X$ is Bochner integrable and continuous at b, so that F is left differentiable with left derivative $f(b)$ at b. In the same manner one shows that if $f : [a, b) \longrightarrow X$ is Bochner integrable and right continuous at a, then F is right differentiable at with right derivative $f(a)$ at a.

Let us finally state and prove the vector–valued counterpart to the classical result on the correspondence between integration and differentiation.

Theorem A.14 (The Fundamental Theorem of Calculus). *Let f : $[a, b]$ be continuous, and let F be an integral function of f, i.e. $F' = f$. Then*

$$\int_a^b f(t)\,dt = F(b) - F(a)$$

Proof. Take $\phi \in X^*$. Then $\langle F(\underline{t}), \phi\rangle' = \langle F'(\underline{t}), \phi\rangle = \langle f(\underline{t}), \phi\rangle$, so that

$$\begin{aligned}\langle \int_a^b f(t)\,dt, \phi\rangle &= \int_a^b \langle f(t), \phi\rangle\,dt \\ &= \langle F(b), \phi\rangle - \langle F(a), \phi\rangle = \langle F(b) - F(a), \phi\rangle,\end{aligned}$$

by the familiar scalar version of the Fundamental Theorem of Calculus. Applying the Hahn-Banach theorem, we arrive at the statement of the theorem. □

A.1.7 Integration by parts

Let $\lambda : [a, b] \longrightarrow \mathbb{C}$ and $f : [a, b] \longrightarrow X$ be continuous functions, and assume that $\lambda = \Lambda'$ and $f = F'$. Then $\{\Lambda F\}' = \lambda F + \Lambda f$ is continuous, so that, by the Fundamental Theorem of Calculus, $\Lambda(b)F(b) - \Lambda(a)F(a) = \int_a^b \lambda(t)F(t)\,dt + \int_a^b \Lambda(t)f(t)\,dt$. Consequently, we have the following theorem.

Theorem A.15. *Let $a < b$, and assume that $\lambda : [a, b] \longrightarrow \mathbb{C}$ and $f : [a, b] \longrightarrow X$ are continuous functions with integral functions Λ and F, respectively. Then*

$$\int_a^b \lambda(t)F(t)\,dt = \Lambda(b)F(b) - \Lambda(a)F(a) - \int_a^b \Lambda(t)f(t)\,dt$$

and

$$\int_a^b \Lambda(t)f(t)\,dt = \Lambda(b)F(b) - \Lambda(a)F(a) - \int_a^b \lambda(t)F(t)\,dt.$$

A.1.8 Fubini's Theorem

It should be perfectly clear how the concepts of measurability and Bochner integrability can be generalised to functions $f : \Omega \longrightarrow X$, where (Ω, μ) is some general measure space. In particular, we can consider measure spaces $(\Omega_1 \times \Omega_2, m)$, where $\Omega_1, \Omega_2 \subseteq \mathbb{R}$ are Lebesgue measurable and m is the Lebesgue measure. We have the following vector–valued version of Fubini's Theorem.

THEOREM A.16. *Let $\Omega_1, \Omega_2 \subseteq \mathbb{R}$ be Lebesgue measurable, set $\Omega := \Omega_1 \times \Omega_2$, and assume that $f : \Omega \longrightarrow X$ is Bochner integrable (with respect to the Lebesgue measure). Then the integrals*

$$\int\limits_{\Omega_1}\int\limits_{\Omega_2} f(s,t)\,dt\,ds, \int\limits_{\Omega_2}\int\limits_{\Omega_1} f(s,t)\,ds\,dt$$

exist and are equal to the double integral $\int_\Omega f(s,t)\,d(s,t)$.

The proof of this theorem is omitted. Note that f is Bochner integrable if and only if it is measurable and one of the integrals $\int_\Omega \| f(s,t) \| \, d(s,t)$, $\int\limits_{\Omega_1}\int\limits_{\Omega_2} \| f(s,t) \| \, dt\,ds$ and $\int\limits_{\Omega_2}\int\limits_{\Omega_1} \| f(s,t) \| \, ds\,dt$ is finite, in which case they are all finite and equal.

A.1.9 L^p-spaces of vector–valued functions

Analogously to the scalar–valued case one defines the spaces $L^p(\Omega; X)$ for $1 \leq p \leq \infty$ by setting $L^p(\Omega; X) = \{f : \Omega \longrightarrow X \parallel f \text{ is measurable and } \| f \| \in L^p(\Omega)\}$, and $\| f \|_{L^p(\Omega;X)} = \| \, \| f \|_X \, \|_{L^p(\Omega)}$, i.e.,

$$\begin{aligned} \| f \|_{L^p(\Omega;X)} &= \left(\int_\Omega \| f(t) \|_X^p \, dt \right)^{1/p} \qquad (p \in [1, \infty)) \\ \| f \|_{L^\infty(\Omega;X)} &= \operatorname{ess\,sup}\{ \| f(t) \|_X \;\mid t \in \Omega\}. \end{aligned}$$

In particular, we note that $L^1(\Omega; X)$ is the precisely the collection of Bochner integrable functions $f : \Omega \longrightarrow X$.

Hölder's inequality for scalar–valued functions shows that if $f \in L^p(I)$ and $g \in L^q(I; X)$, where $1 \leq p, q \leq \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, then $fg \in L^1(I; X)$ and

$$\| fg \|_{L^1(I;X)} \leq \| f \|_{L^p(I)} \| g \|_{L^q(I;X)} . \tag{A.6}$$

The spaces $L^p(\Omega; X)$ are, in fact, Banach spaces for $1 \leq p \leq \infty$.

For $0 \leq a < b \leq \infty$, $1 \leq p \leq \infty$, and a Banach space X, we also define the spaces $L^p_*((a,b)\,; X)$ by

$$L^p_*((a,b)\,; X) := \{ f : (a,b) \longrightarrow X \parallel f \text{ is measurable and } \\ \| f \|_{L^p_*((a,b);X)} < \infty\},$$

where

$$\| f \|_{L^p_*((a,b);X)} := \begin{cases} \left(\int\limits_a^b \| f(t) \|_X^p \frac{dt}{t} \right)^{\frac{1}{p}} & \text{if } \ 1 \le p < \infty \\ \operatorname{ess\,sup}_{a<t<b} \| f(t) \|_X & \text{if } \ p = \infty \end{cases} .$$

In particular, we set $L^p_*(a,b) := L^p_*((a,b)\,;\mathbb{C})$.

A.1.10 Convolutions

If $h \in L^1(\mathbb{R})$ and $f \in L^1(\mathbb{R};X)$, then $|h|\,, \| f \| \in L^1(\mathbb{R})$, so that the convolution

$$(|h| * \| f \|)(t) = \int\limits_{\mathbb{R}} |h(t-s)| \, \| f(s) \| \, ds$$

of $|h|$ and $\| f \|$ exists for a.e. $t \in \mathbb{R}$. Hence, $h(t-s)f(s)$ is integrable over $\mathbb{R}$ with respect to s for a.e. $t \in \mathbb{R}$, and we define the convolution $h * f$ by

$$(h * f)(t) = \int\limits_{\mathbb{R}} h(t-s) f(s) \, ds \qquad (\text{a.e. } \ t \in \mathbb{R}).$$

For intervals $I \subseteq \mathbb{R}$, $h \in L^1(I)$, and $f \in L^1(I;X)$ we define $h * f = (\chi_I h) * (\chi_I f)$ on I. In particular, if $h \in L^1((0,\tau))$ and $f \in L^1((0,\tau)\,;X)$ for some $\tau \in \mathbb{R}_+$, then

$$(h * f)(t) = \int\limits_0^t h(t-s) f(s) \, ds \qquad (\text{a.e. } \ t \in (0,\tau)\,).$$

A.1.11 Young's theorem

Theorem A.17 (Young's theorem). *Let let $f \in L^p(\mathbb{R})$ and $g \in L^p(\mathbb{R};X)$, where X is a Banach space and $1 \le p,q \le \infty$. Let $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then $f * g$ belongs to $L^r(I;X)$, and*

$$\| f * g \|_{L^r(\mathbb{R};X)} \le \| f \|_{L^p(\mathbb{R})} \, \| g \|_{L^q(\mathbb{R};X)} \, .$$

Proof. (Outline) The measurability of $f * g : \mathbb{R} \longrightarrow X$ follows by Pettis's theorem and the measurability of $f * h$ for all $h \in L^q(\mathbb{R})$ obtained in the scalar–valued version of the theorem. Young's inequality for scalar–valued functions then gives

$$\begin{aligned} \| f * g \|_{L^r(I;X)} &\le \| \, |f| * \| g \|_X \, \|_{L^r(\mathbb{R})} \\ &\le \| f \|_{L^p(\mathbb{R})} \| g \|_{L^q(\mathbb{R};X)}. \end{aligned}$$

Note that, in case $p,q,r \neq \infty$, the second (scalar–valued) estimate is obtained by first applying the generalised Hölder inequality to

$$(f * h)(t) = \int\limits_{\mathbb{R}} f(s) h(t-s) \, ds = \int\limits_{R} f(s)^{\frac{r-p}{r}} h(t-s)^{\frac{r-q}{r}} \left(f(t)^{\frac{p}{r}} h(t-s)^{\frac{q}{r}} \right) \, ds,$$

with $p_1 = \frac{rp}{r-p}$, $p_2 = \frac{rq}{r-q}$, and $p_3 = r$, which gives

$$|(f*h)(t)| \le \| f \|_{L^p(\mathbb{R})}^{\frac{r}{r-p}} \| h \|_{L^q(\mathbb{R})}^{\frac{r}{q-p}} \int_{\mathbb{R}} |f(s)|^p |h(t-s)|^q \, ds,$$

and then integrating over $\mathbb{R}$ with respect to t, using Fubini's theorem. $\square$

The last estimate of the following corollary to Young's theorem is extensively used in Chapter 3.

Corollary A.18. *Let X be a Banach space, let $I = (0,\infty)$ or $I = (1,\infty)$, and assume that $\underline{t}^{-\theta} f(\underline{t}) \in L_*^p(I)$ and $t^\theta g(\underline{t}) \in L_*^q(I;X)$, where $\theta \in \mathbb{R}$ and $1 \le p,q \le \infty$. Let $h(s) := \int_I f(t)g(st)\,dt/t$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Then $\underline{t}^\theta h(\underline{t}) \in L_*^r(I;X)$, and*

$$\| \underline{t}^\theta h(\underline{t}) \|_{L_*^r(I;X)} \le \| \underline{t}^{-\theta} f(\underline{t}) \|_{L_*^p(I)} \| \underline{t}^\theta g(\underline{t}) \|_{L_*^q(I;X}.$$

In case $p = 1$ and $r = q$, we have $\underline{t}^\theta h(\underline{t}) \in L_*^q(I;X)$, and

$$\| \underline{t}^\theta h(\underline{t}) \|_{L_*^q} \le \| \underline{t}^{-\theta} f(\underline{t}) \|_{L_*^1} \| \underline{t}^\theta g(\underline{t}) \|_{L_*^q}. \tag{A.7}$$

Proof. First note that

$$h(s) = \int_I f(t/s) g(t)\, dt/t.$$

Assume that $I = (0,\infty)$. Let us define $f_1(t) := f(e^{-t})$ and $g_1(t) := g(e^t)$. Then $\underline{t}^{-\theta} f(\underline{t}) \in L_*^p(I)$ iff $e^{\theta \underline{t}} f_1(e^{\underline{t}}) \in L^p(\mathbb{R})$, $\underline{t}^\theta g(\underline{t}) \in L_*^q(I;X)$ iff $e^{\theta \underline{t}} g_1(e^{\underline{t}}) \in L^q(I;X)$, $\| \underline{t}^{-\theta} f(\underline{t}) \|_{L_*^p(I)} = \| e^{\theta \underline{t}} f_1(e^{\underline{t}}) \|_{L^p(\mathbb{R})}$, $\| \underline{t}^\theta g(\underline{t}) \|_{L_*^q(I;X)} = \| e^{\theta \underline{t}} g_1(e^{\underline{t}}) \|_{L^p(I;X)}$, and

$$\begin{aligned} e^{\theta s} h(e^s) &= \int_{\mathbb{R}} e^{\theta(s-t)} f_1(s-t) e^{\theta t} g_1(t)\, dt \\ &= e^{\theta \underline{t}} f_1(\underline{t}) * e^{\theta t} g_1(\underline{t}). \end{aligned}$$

Hence, we only have to apply Young's theorem to the functions $e^{\theta \underline{t}} f_1(e^{\underline{t}})$ and $e^{\theta \underline{t}} g_1(e^{\underline{t}})$ to deduce that $e^{\theta \underline{s}} h(e^{\underline{s}}) \in L^r(\mathbb{R};X)$, and

$$\| e^{\theta \underline{s}} h(e^{\underline{s}}) \|_{L^r(\mathbb{R};X)} \le \| e^{\theta \underline{t}} f_1(\underline{t}) \|_{L^p(\mathbb{R};X)} \| e^{\theta t} g_1(\underline{t}) \|_{L^q(\mathbb{R};X)}$$

from which the first statement of the corollary follows.

To handle the case $I = (1,\infty)$, we define $f_0 \in L_*^p(0,\infty)$ and $g_0 \in L_*^q((0,\infty);X)$ by setting $f_0(t) = f(t)$, $g_0(t) = g(t)$ if $t > 1$, and $f_0(t) = 0$, $g_0(t) = 0$ if $0 < t \le 1$. We also define $h_0 \in L_*^r((0,\infty);X)$ by

$$h_0(s) := \int_0^\infty f_0(t) g_0(st) \frac{dt}{t} = \int_1^\infty f(t) g_0(st) \frac{dt}{t}.$$

If $s > 1$, then $g_0(st) = g(st)$ for all $t > 1$, so that $h \subseteq h_1$. Hence, $\underline{t}^{-\theta} h(\underline{t}) \in L^r_*((1,\infty)\,;X)$ and

$$\begin{aligned}\| \underline{t}^{-\theta} h(\underline{t}) \|_{L^r_*((1,\infty);X)} &\leq \| \underline{t}^{\theta} h_0(\underline{t}) \|_{L^r_*((0,\infty);X)} \\ &\leq \| \underline{t}^{-\theta} f_0(\underline{t}) \|_{L^p_*(0,\infty)} \| \underline{t}^{\theta} g_0(\underline{t}) \|_{L^q_*((0,\infty);X)} \\ &= \| \underline{t}^{-\theta} f(\underline{t}) \|_{L^p_*(1,\infty)} \| \underline{t}^{\theta} g_0(\underline{t}) \|_{L^q_*((1,\infty);X)} .\end{aligned}$$

□

A.1.12 Operator–valued integrals

As a special case of vector–valued integrals we can consider integrals of the form $B = \int_\Omega A(t)\,dt$, where $A : \Omega \longrightarrow \mathcal{L}(X,Y)$ is Bochner integrable, Y is a Banach space and X is a normed vector space. Then $B \in \mathcal{L}(X,Y)$.

Theorem A.19. *Let $A : \Omega \longrightarrow \mathcal{L}(X,Y)$ be Bochner integrable, where $\Omega \subseteq \mathbb{R}$ is Lebesgue measurable, X is a normed vector space, and Y. Assume that $x \in X$. Then $A(\cdot)x : \Omega \longrightarrow Y$ Bochner integrable, and*

$$\int_\Omega A(t)x\,dt = \left\{ \int_\Omega A(t)\,dt \right\} x.$$

Proof. As the mapping $B \longrightarrow Bx$ of $B \in \mathcal{L}(X,Y)$ to $Bx \in Y$ is continuous and $A : \Omega \longrightarrow \mathcal{L}(X,Y)$ is measurable, $A(\cdot)x : \Omega \longrightarrow Y$ is measurable. Also $\| A(t)x \|_Y \leq \| A(t) \|_{\mathcal{L}(X,Y)} \| x \|_X$, so that $A(\cdot)x$ is Bochner integrable since A is.

Let $\{A_n\}_{n=1}^\infty$ be a sequence of simple functions $A_n : \Omega \longrightarrow \mathcal{L}(X,Y)$, such that $A_n(t) \to A(t)$ for a.e. $t \in \Omega$ as $n \to \infty$, and

$$\int_\Omega A(t)\,dt = \lim_{n\to\infty} \int_\Omega A_n(t)\,dt$$

in $\mathcal{L}(X,Y)$.

Since the A_n are simple functions, so are the $A_n(\cdot)x$, and

$$\left\{ \int_\Omega A_n(t)\,dt \right\} x = \int_\Omega A_n(t)x\,dt.$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega A_n(t)x\,dt &= \lim_{n\to\infty} \left\{ \int_\Omega A_n(t)\,dt \right\} x \\ &= \left\{ \int_\Omega A(t)\,dt \right\} x,\end{aligned}$$

so that $\int_\Omega A(t)x\,dt = \left\{ \int_\Omega A(t)\,dt \right\} x$. □

A.2 Complex integration

A.2.1 Curve integrals

Let γ be a path in $\mathbb{C}$, i.e. a piecewise continuously differentiable function from a closed bounded interval $I = [a, b] \subseteq \mathbb{R}$ into $\mathbb{C}$. If f is a function that maps a subset of $\mathbb{C}$ into a Banach space X, and if f is continuous on the range of γ, then the function $\gamma' f \circ \gamma : I \longrightarrow X$ is Bochner integrable, and we can define the integral of f over γ by

$$\int_\gamma f(z)\, dz := \int_a^b f(\gamma(t))\gamma'(t)\, dt. \tag{A.8}$$

The value of this integral does not depend on the particular parameter representation of the path of integration γ. In fact, if γ is a path as above, and if T is a continuously differentiable bijection of an interval $J = [\alpha, \beta]$ onto $I = [a, b]$ such that $T(\alpha) = a$ and $T(\beta) = b$, then $\gamma \circ T$ is a path with the same range as γ, and, by Theorem A.9,

$$\int_{\gamma\circ T} f(z)\, dz = \int_\alpha^\beta \gamma'(T(t))T'(t)f(T(t))\, dt = \int_a^b \gamma(t)f(t)\, dt = \int_\gamma f(z)\, dz,$$

since T is strictly increasing, so that $|T'| = T'$ on J.

It is also clear that

$$\left\| \int_\gamma f(z)\, dz \right\| \le \int_a^b \| \gamma'(t)f(\gamma(t)) \|\, dt = \int_\gamma \| f(z) \|\, d\,|z|\,. \tag{A.9}$$

A.2.2 Integration over unbounded paths

Let us consider a function $\gamma : I \longrightarrow \mathbb{C}$, where $I \subseteq \mathbb{R}$ is an interval. Let us further assume that the restriction of γ to any bounded closed interval $[a, b] \subseteq I$ is a path. We then define

$$\int_\gamma f(z)\, dz := \int_I \gamma'(t)f(\gamma(t))\, dt \tag{A.10}$$

provided that $\gamma'(f \circ \gamma)$ is absolutely integrable over I. The reason for this definition is that in the main text we consider integrals

$$\int_\gamma f(z)\, dz = \int_{-\infty}^{\infty} \gamma'(t)f(\gamma(t))\, dt,$$

where $|\gamma(t)| \to \infty$ as $t \to \pm\infty$.

Note that in this context it would suffice to extend the original definition of a path to curves γ defined on intervals of one of the forms $[a, \infty)$,

$(-\infty, b]$ and $(-\infty, \infty)$ (whose restrictions to closed bounded subintervals are paths in the original sense), where $a, b \in \mathbb{R}$ since all other types of intervals can be bijectively mapped onto one of these by means of some continuously differentiable "change of variables function".

A.2.3 The action of closed linear operators on curve integrals

Theorems A.5 and A.6 have the following obvious corollaries that are frequently used in the main text.

Corollary A.20. *Let $L : X \longrightarrow Y$ be a bounded linear operator that maps a Banach space X into another Banach space Y. Let γ be a path in $\mathbb{C}$ (bounded or unbounded), and assume that $f : \mathcal{R}(\gamma) \longrightarrow X$ is continuous. Then if γ is a (bounded) path, we have*

$$L \int\limits_{\gamma} f(z)\, dz = \int\limits_{\gamma} L(f(z))\, dz.$$

If γ is unbounded, the same statement holds, provided that $\int_\gamma f(z)\, dz$ exists.

Corollary A.21. *Let A be a closed linear operator from a Banach space X into a Banach space Y, let f be a function from $\mathbb{C}$ into X, and let γ be a path in $\mathbb{C}$ (bounded or unbounded). Assume that $\mathcal{R}(\gamma) \subseteq \mathcal{D}(f)$, $f(\mathcal{R}(\gamma)) \subseteq \mathcal{D}(A)$, and that f and $A \circ f$ map $\mathcal{R}(\gamma)$ continuously into X and Y respectively. Then if γ is a (bounded) path, we have $\int_\Omega f(z)\, dz \in \mathcal{D}(A)$, and*

$$A \int\limits_{\gamma} f(z)\, dz = \int\limits_{\gamma} A(f(z))\, dz.$$

If γ is an unbounded path, then the same conclusion holds, provided that $\int_\gamma f(z)\, dz$ and $\int_\gamma A(f(z))\, dz$ exist (at least as improper curve integrals).

A.2.4 Holomorphic functions

Let Ω be an open subset of $\mathbb{C}$ and let X be a Banach space as above. A function $f : \Omega \longrightarrow X$ is called *analytic*, if it has a derivative

$$f'(z) = \lim_{\substack{h \to 0 \\ h \in \mathbb{C} \setminus \{0\}}} \frac{f(z+h) - f(z)}{h}$$

at every point $z \in \Omega$. It is clear that any analytic function $f : \Omega \longrightarrow X$ is continuous. It is also *weakly analytic*, i.e. $\langle f(\underline{z}), \phi \rangle$ is analytic for any $\phi \in X^*$. Moreover (with abuse of notation) $\langle f(z), \phi \rangle' = \langle f'(z), \phi \rangle$.

A.2.5 Power series

Let X be a Banach space and let $\{x_n\}_{n=0}^\infty$ be a sequence in X. We can then consider the sequence $\{s_N\}_{N=0}^\infty$ of partial sums $s_N = \sum_{n=0}^N x_n$. If these partial sums form a convergent sequence with limit s in X, we say that $\sum_{n=0}^\infty x_n$ is a convergent series and denote s by $\sum_{n=0}^\infty x_n$. Otherwise the series $\sum_{n=0}^\infty x_n$ is called divergent. The series is absolutely convergent if $\sum_{n=0}^\infty \| x_n \|$ is convergent in $\mathbb{R}$. In that case $\sum_{n=0}^\infty x_n$ is convergent in X, since

$$\| s_N - s_M \| = \left\| \sum_{n=M+1}^N x_n \right\| \le \sum_{n=M+1}^N \| x_n \| = \sum_{n=0}^N \| x_n \| - \sum_{n=0}^M \| x_n \|$$

for $0 \le M < N$, and X is complete.

One observes that if $\sum_{n=0}^\infty x_n$ is convergent, then $\| x_n \| = \| s_N - s_{N-1} \| \to 0$ as $N \to \infty$. In particular, the sequence $\{x_n\}_{n=0}^\infty$ is bounded in X.

If $\{a_n\}_{n=1}^\infty$ is a sequence in a Banach space X and $z_0 \in \mathbb{C}$, we can consider the *power series*

$$\sum_{n=0}^\infty (z - z_0)^n a_n,$$

for any $z \in \mathbb{C}$. It can be regarded as a function defined on the set of all $z \in \mathbb{C}$ such that the series is convergent, and we can define $s(z) = \sum_{n=0}^\infty (z - z_0)^n a_n$ for such z. By definition, this series converges absolutely if $\sum_{n=0}^\infty |z - z_0|^n \| a_n \|$ converges.

Let $\sum_{n=0}^\infty (z - z_0)^n a_n$ be a power series in X, and define

$$R := \liminf_{n\to\infty} 1/\sqrt[n]{\| a_n \|}, \tag{A.11}$$

the radius of convergence of $\sum_{n=0}^\infty (z - z_0)^n \| a_n \|$. Then we know that the series $\sum_{n=0}^\infty (z - z_0)^n a_n$ converges absolutely for all z with $|z - z_0| < R$.

Assume, now, that $z_1 \in \mathbb{C}$ and $s(z_1)$ is convergent. Then there is a number $M > 0$ such that $\| (z_1 - z_0)^n a_n \| \le M$ for $n = 0, 1, \ldots$ Hence, $\| (z - z_0)^n a_n \| \le M \left(|z - z_0| \,/\, |z - z_0|\right)^n$. It follows that $\sum_{n=0}^\infty (z - z_0)^n a_n$ converges absolutely for all z with $|z - z_0| < |z_1 - z_0|$, and we have proved the following lemma.

Lemma A.22. *Let $z_1 \in \mathbb{C}$ be such that the power series $\sum_{n=0}^\infty (z_1 - z_0)^n a_n$ converges. Then $\sum_{n=0}^\infty (z - z_0)^n a_n$ converges absolutely for all z with $|z - z_0| < |z_1 - z_0|$.*

Using this lemma, we see that if $|z_1 - z_0| > R$, then $\sum_{n=0}^n (z_1 - z_0)^n a_n$ must diverge, for otherwise there would exist z with $|z - z_0| > R$, such that $\sum_{n=0}^n (z - z_0)^n a_n$ converges absolutely, which is impossible. We have thus proved the following theorem.

Theorem A.23. *Let X be a Banach space, let $\{a_n\}_{n=0}^\infty \subseteq X$ and define R by* (A.11)*. Then the power series $\sum_{n=0}^\infty (z - z_0)^n a_n$ converges absolutely if $|z - z_0| < R$ and diverges if $|z - z_0| > R$, where $R = \liminf_{n\to\infty} 1/\sqrt[n]{\| a_n \|}$ is the radius of convergence of the series.*

We shall now prove that (the sum of) a power series is an analytic function within its radius of convergence. To simplify notation, we restrict ourselves to power series about the origin, i.e. power series of the form $\sum_{n=0}^{\infty} z^n a_n$. Substituting $w - w_0$ for z in such a series we obtain the general case.

First we note that if the series $s(z) = \sum_{n=0}^{\infty} z^n a_n$ is differentiated termwise, we obtain a new power series $s_1(z) = \sum_{n=1}^{\infty} n z^{n-1} a_n = \sum_{n=0}^{\infty} (n+1) z^n a_{n+1}$ with radius of convergence

$$\liminf_{n\to\infty} 1/\sqrt[n]{(n+1)\,\|\, a_{n+1}\,\|} = \lim_{n\to\infty} 1/\sqrt[n]{n+1}\,\liminf_{n\to\infty} 1/\sqrt[n]{\|\, a_{n+1}\,\|} = R,$$

where R is the radius of convergence of s. Hence, the termwise differentiated series s_1 converges within R. Repeating this procedure, we see that s can be k times termwise differentiated for any k within its radius of convergence R.

We also need to show that s_1 is the derivative of s. Thus, let $|z| < R$ and let $\Delta z \neq 0$ be so small that $|z + \Delta z| < R_1$ for some $R_1 < R$. We have

$$\frac{s(z+\Delta z) - s(z)}{\Delta z} - s_1(z) = \Delta z \sum_{n=2}^{\infty} \sum_{k=2}^{n} \binom{n}{k} z^{n-k} (\Delta z)^{k-2} a_n. \tag{A.12}$$

The inequality $\binom{n}{k} \leq \binom{n}{2}\binom{n-2}{k-2}$, which holds for $k = 2, 3, \ldots, n$, yields the following estimate for the terms of the series in (A.12)

$$\begin{aligned}\left\| \binom{n}{k} z^{n-k} (\Delta z)^{k-2} a_n \right\| &\leq \binom{n}{2} \left(\sum_{k=2}^{n} \binom{n-2}{k-2} |z|^{n-k} |\Delta z|^{k-2} \right) \|\, a_n \,\| \\ &= \frac{n(n-1)}{2} |z + \Delta z|^{n-2} \,\|\, a_n \,\|.\end{aligned}$$

Hence, the series is absolutely convergent, and

$$\begin{aligned}\left\| \frac{s(z+\Delta z) - s(z)}{\Delta z} - s_1(z) \right\| &\leq \frac{|\Delta z|}{2} \sum_{n=2}^{\infty} n(n-1) \, |z + \Delta z|^{n-2} \,\|\, a_n \,\| \\ &\leq \frac{|\Delta z|}{2} \sum_{n=2}^{\infty} n(n-1) R_1^{n-2} \,\|\, a_n \,\|.\end{aligned}$$

This implies that the difference quotient $(s(z+\Delta z) - s(z))/\Delta z$ tends to $s_1(z)$ as $\Delta z \to 0$. Thus, we have proved that a power series can be differentiated termwise within its radius of convergence.

Theorem A.24. *Let X be a complex Banach space, and let*

$$s(z) = \sum_{n=0}^{\infty} (z - z_0)^n a_n$$

be a power series in X with radius of convergence R. Then s is analytic in the disc $B(z_0, R) := \{z \in \mathbb{C} \mid |z - z_0| < R\}$ with derivative given by

$$s'(z) = \sum_{n=1}^{\infty} n(z - z_0)^{n-1} a_n.$$

A.2.6 Cauchy's integral formula and Cauchy's integral theorem

If Γ is a closed simple curve that is positively oriented about $z \in \Omega \subseteq \mathbb{C}$, where Γ is contained in Ω, and if $f : \Omega \longrightarrow X$ is analytic, then, Cauchy's integral formula in combination with Theorem A.5 yields

$$\langle f(z_0), \phi \rangle = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\langle f(z), \phi \rangle}{z - z_0} \, dz = \langle \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - z_0} \, dt, \phi \rangle.$$

Consequently, by the Hahn-Banach theorem (A.8), we have the vector–valued version of Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - z_0} \, dt. \tag{A.13}$$

In particular, if $g : \Omega \longrightarrow X$ defined by $g(z) = (z - z_0)f(z)$ if $z \in \Omega \setminus \{z_0\}$, and $g(z_0) = \lim_{z \to z_0} (z - z_0)f(z)$ is analytic, then we have

$$2\pi i \operatorname*{Res}_{z=z_0} f(z) = \oint_{\Gamma} f(t) \, dt \tag{A.14}$$

where

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z) \tag{A.15}$$

is the *residue* of f at z_0. This special case of the general Residue Theorem[10] covers the case where f has at most a simple pole at z_0 and is analytic elsewhere. In particular, if f is also analytic at z_0, it follows that $g(z_0) = (z - z_0)f(z_0) = 0$, and hence $\oint_\Gamma f(z)\, dz = 0$. Changing the orientation of the curve does not change this formula. Thus, we have proved the vector–valued version of Cauchy's integral theorem.

Theorem A.25. *Let $f : \Omega \longrightarrow X$ be analytic, and let Γ be a closed simple curve in Ω. Then*

$$\oint_{\Gamma} f(z) \, dz = 0.$$

A.2.7 Taylor series and Laurent series

Let $D \subseteq \mathbb{C}$ be the annulus $\{z \in \mathbb{C} \mid r < |z - z_0| < R\}$, where $0 \le r < R \le \infty$. Assume that f is an analytic mapping of D into X and let Γ be a simple curve in D that encircles $\{z \in \mathbb{C} \,\|\, |z - z_0| < r\}$ in the positive sense. Let us imitate the Laurent series representation of scalar valued analytic functions on D, putting

$$c_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \qquad (n \in \mathbb{Z}) \tag{A.16}$$

[10]See Subsection A.2.10.

and considering the series

$$s(z) = \sum_{n=-\infty}^{\infty} (z - z_0)^n c_n. \tag{A.17}$$

First we note that if Γ' is another simple curve that is positively oriented around the disc $\{z \in \mathbb{C} \parallel |z - z_0| \leq r\}$ and lies within the annulus, then

$$c_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz) = \frac{1}{2\pi i} \oint_{\Gamma'} \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (n \in \mathbb{Z}).$$

To see this, we take an arbitrary $\phi \in X^*$ and obtain

$$\begin{aligned}
\langle \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \phi\rangle &= \frac{1}{2\pi i} \oint_\Gamma \frac{\langle f(z), \phi\rangle}{(z - z_0)^{n+1}}\, dz \\
&= \frac{1}{2\pi i} \oint_{\Gamma'} \frac{\langle f(z), \phi\rangle}{(z - z_0)^{n+1}}\, dz \\
&= \langle \frac{1}{2\pi i} \oint_{\Gamma'} \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \phi\rangle.
\end{aligned}$$

The Hahn-Banach theorem then yields the result that we wished to prove.

Now let Γ be a circle with centre at z_0 and radius R_1, where $r < R_1 < R$. Hence, for $n \in \mathbb{Z}$, we have

$$\| c_n \| \leq \alpha R_1^{-n-1},$$

where $\alpha = \frac{1}{2\pi} \oint_\Gamma \| f(z) \|\, d|z|$ Consequently,

$$\liminf_{n\to\infty} 1/\sqrt[n]{\| c_{-n} \|} \geq \lim_{n\to\infty} 1/R_1 \sqrt[n]{\alpha R_1} = 1/R_1.$$

It follows that $\sum_{n=1}^{\infty} (z - z_0)^{-n} c_{-n}$ converges absolutely if $1/|z - z_0| < 1/R_1$, i.e. if $|z - z_0| > R_1$. Since this is true for any $R_1 \in (r, R)$, we conclude that the series converges absolutely for all z with $|z - z_0| > r$.

We also have

$$\liminf_{n\to\infty} 1/\sqrt[n]{\| c_n \|} \geq \lim_{n\to\infty} R_1 \sqrt[n]{R_1/\alpha} = R_1.$$

As this holds for any $R_1 \in (r, R)$, we see that the radius of convergence of the series $\sum_{n=0}^{\infty} (z - z_0)^n c_n$ is at least R. Therefore, we infer that the Laurent series $s(z) = \sum_{n=-\infty}^{\infty} (z - z_0)^n c_n$ converges absolutely in the annulus $\{z \in \mathbb{C} \mid r < |z - z_0| < R\}$.

Now, take an arbitrary $\phi \in X^*$ and z with $r < |z - z_0| < R$. Then

$$\begin{aligned}
\langle f(z), \phi\rangle &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i} (z - z_0)^n \oint_\Gamma \frac{\langle f(z), \phi\rangle}{(z - z_0)^{n+1}}\, dz \\
&= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i} (z - z_0)^n \langle \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz, \phi\rangle \\
&= \langle \sum_{n=-\infty}^{\infty} (z - z_0)^n c_n, \phi\rangle.
\end{aligned}$$

We use the Hahn-Banach theorem, again, to conclude that

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i}(z - z_0)^n c_n. \tag{A.18}$$

Moreover the representation of f as a Laurent series is unique. This is seen as follows: Let us assume that

$$\sum_{n=-\infty}^{\infty} \frac{1}{2\pi i}(z - z_0)^n c_n = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i}(z - z_0)^n c'_n$$

for all z with $r < |z - z_0| < R$, so that

$$\sum_{n=-\infty}^{\infty} \frac{1}{2\pi i}(z - z_0)^n d_n = 0,$$

where $d_n = c_n - c'_n$. Take $\phi \in X^*$. Then

$$\sum_{n=-\infty}^{\infty} \frac{1}{2\pi i}(z - z_0)^n \langle d_n, \phi\rangle = \langle \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i}(z - z_0)^n d_n, \phi\rangle = 0,$$

and the corresponding result for complex–valued Laurent series shows that $\langle d_n, \phi\rangle = 0$ for all $n \in \mathbb{Z}$. Since this holds for all $\phi \in X^*$, we conclude that $c_n - c'_n = d_n = 0$ for all $n \in \mathbb{Z}$.

In case f is analytic in the whole of D, Cauchy's integral theorem shows that c_n vanishes for all negative n, and the Laurent series reduces to a power series at z_0. Differentiating this series n times at z_0, shows that $c_n = f^{(n)}(z_0)/n!$, so that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (n = 0, 1, \ldots)$$

which is the generalised Cauchy integral formula. Thus, a function f that is analytic in the open disc $\{z \in \mathbb{C} \mid |z - z_0| < R\}$ can be represented as a Taylor series within that disk.

We now sum up the results proved in this subsection.

Theorem A.26. *Let f be an analytic function on the annulus $\{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. Then f has a unique representation as a* Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi i}(z - z_0)^n c_n,$$

in $\{z \in \mathbb{C} \mid r < |z - z_0| < R\}$, where

$$c_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

for any closed simple curve that encircles the disc $\{z \in \mathbb{C} \parallel |z - z_0| \leq r\}$ *in the positive sense. In particular, if* f *is analytic in* $\{z \in \mathbb{C} \mid |z - z_0| < R\}$*, then* $c_n = 0$ *for* $n = -1, -2, \ldots$*, and the Laurent series reduces to a* Taylor series*, and we have* $c_n = f^{(n)}(z_0)/n!$ *and the generalised Cauchy integral formula*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz \qquad (n = 0, 1, \ldots).$$

A.2.8 The identity theorem

Theorem A.27. *Let* f *and* g *be analytic mappings of* $\Omega \subseteq \mathbb{C}$ *into a Banach space* X*. Assume that* $f(z) = g(z)$ *for all* $z \in \Omega_0 \subseteq \Omega$*, and that* Ω_0 *has a point of accumulation in* Ω_0*. Then* $f(x) = g(x)$ *for all* $x \in \Omega$*.*

Proof. Take an arbitrary $phi \in X^*$. Then $\phi \circ f$ and $\phi \circ g$ are analytic mappings of Ω into $\mathbb{C}$, that coincide on Ω_0. Consequently, $\phi(f(z)) = \phi(g(z))$ for all $z \in \Omega$ and all $\phi \in X^*$. The Hahn–Banach theorem therefore implies that $f(z) = g(z)$ for all $z \in \Omega$. □

A.2.9 Liouville's theorem

Assume that $f : \Omega \longrightarrow X$ is analytic in a domain that contains the disc $\{z \in \mathbb{C} \parallel |z - z_0| \leq r\}$ for some $r > 0$. Then, by the generalised Cauchy integral formula applied to f with Γ being the circle $|z - z_0|$ run through in the positive sense,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{r^n},$$

where $M = \max\{\| f(z) \| \parallel |z - z_0| = r\}$. Hence, if $f : \mathbb{C} \longrightarrow X$ is an entire function, i.e. if f is analytic everywhere, and if f is also bounded, then

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{r^n},$$

holds for any $r > 0$, where $M = \sup\{\| f(z) \| \parallel z \in \mathbb{C}\}$. Consequently, $f^{(n)}(z_0) = 0$, so that the Taylor series of f reduces to $f(z_0)$.[11]

Theorem A.28 (Liouville's theorem). *Assume that* $f : \mathbb{C} \longrightarrow X$ *is a bounded entire function. Then* f *is constant.*

Corollary A.29. *Let* f *be analytic in a punctured neighbourhood of* $z_0 \in \mathbb{C}$*, and bounded at* z_0*. Then* f *is analytic in an neighbourhood of* z_0*.*

Proof. By Theorem A.26, $f(z) = \sum_{n=-\infty}^{\infty}(z - z_0)^n c_n$ in some annulus $\{z \in \mathbb{C} \parallel 0 < |z - z_0| < R\}$. The series $\sum_{n=0}^{\infty} w^n c_{-n}$ then converges for any

[11] Of course we could equally well have applied Liouville's theorem for complex–valued functions to obtain this result: If $f : \mathbb{C} \longrightarrow X$ is bounded and entire, then so is $\langle f(\underline{z}), \phi\rangle : \mathbb{C} \longrightarrow \mathbb{C}$ for any $\phi \in X^*$, so that $\langle f(\underline{z}), \phi\rangle$ is constant. Since this holds for any $\phi \in X^*$, it follows by the Hahn-Banach theorem that f is constant.

w (since it must have infinite radius of convergence as $\sum_{n=0}^{\infty}(z-z_0)^{-n}c_{-n}$ converges for arbitrarily small $z-z_0$). It follows that $g(w)=\sum_{n=0}^{\infty}w^n c_{-n}$ defines an entire function $g:\mathbb{C}\longrightarrow X$. By assumption $\sum_{n=-\infty}^{\infty}(z-z_0)^n c_n$ is bounded at z_0. Hence, g is bounded at infinity. Being an entire function, it is therefore bounded everywhere. By Liouville's theorem, g must be constant, i.e. $g(w)=c_0$ for any $w\in\mathbb{C}$. Consequently, f can be represented by the Taylor series $f(z)=\sum_{n=0}^{\infty}(z-z_0)^n c_n$ in $\{z\in\mathbb{C}\parallel |z-z_0|<R\}$, and the power series is analytic. $\square$

A.2.10 The Residue Theorem

One of the important consequences of the Laurent series representation of an analytic function f in an annulus $\{z\in\mathbb{C}\mid r<|z-z_0|<R\}$ is that we have

$$\oint_{\Gamma} f(z)\,dz = 2\pi i c_{-1} \tag{A.19}$$

In particular, if z_0 is an isolated singularity of f, we define

$$\operatorname*{Res}_{z=z_0} f(z) = c_{-1},$$

where c_{-1} is as above. Imitating the procedure used in the scalar case, we get the following version of the residue theorem.

Theorem A.30. *Let X be a Banach space, let $\Omega\subseteq\mathbb{C}$ be a simply connected domain, let γ be a closed simple curve in γ, and assume that $f:\Omega\longrightarrow X$ has N singular points $z_1,z_2,\dots,z_N$ that all lie within γ, which is positively oriented around these points. Then*

$$\oint_{\Gamma} f(z)\,dz = 2\pi i\sum_{i=1}^{N}\operatorname*{Res}_{z=z_i} f(z)$$

Proof. (Outline) The proof is based on the fact that there are N closed simple curves $\gamma_1,\gamma_2,\dots,\gamma_N$

$$\oint_{\Gamma} f(z)\,dz = 2\pi i\sum_{i=1}^{N}\oint_{\gamma_i} f(z)\,dz,$$

z_i is inside γ_i, and z_j is outside γ_j for $i,j=1,2,\dots,N$, $i\neq j$ (see Figure 7). $\square$

A.2.11 Calculating residues at poles

Let $f:\Omega\longrightarrow X$, where $\Omega\subseteq\mathbb{C}$ is an open set. Then f is said to have a pole of order $n\in\mathbb{N}$ at z_0 if there is a function h mapping dome disc $\{z\in\mathbb{C}\parallel |z-z_0|<R\}$ analytically into X, and such that $f(z)=h(z)/(z-z_0)^n$

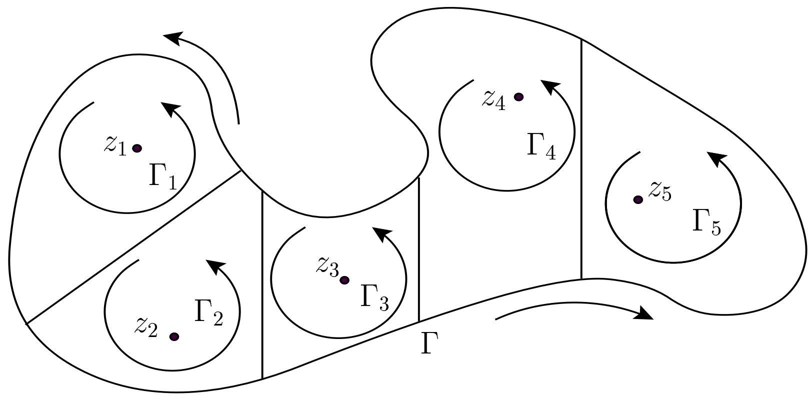

Figure 7: Curves used in the proof of the Residue Theorem

in $\{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$ and $h(z_0) \neq 0$. Then f is analytic in $\{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$, and

$$f(z) = \sum_{k=-n}^{\infty} (z - z_0)^k c_n,$$

where $f(z) = \sum_{k=0}^{\infty} (z - z_0)^k c_{k-n}$ is the Taylor series of h, and $c_{-n} = h(0) \neq 0$. Thus, if f has a removable singularity, or has a pole of order at most n at $z = z_0$, then $c_k = 0$ for $k < -n$, where the c_k are the coefficients of the Laurent series of f. Conversely, it is clear that if the Laurent series of f has the above form, then either f has a removable singularity or has a pole of order $k \leq n$ at z_0. Hence, if we put $h(z) = \sum_{k=0}^{\infty} (z - z_0)^k c_{k-n}$, we get

$$h^{(n-1)}(z) = \sum_{k=n-1}^{\infty} \frac{(n-1)!}{(k-n+1)!} (z - z_0)^{k-n+1} c_{k-1}$$

so that $h^{(n-1)}(z_0) = (n-1)! c_{-1}$, and consequently

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^n}{dz^n} \left[(z - z_0)^n f(z) \right]$$

By Corollary A.29, the condition that $f(z) = h(z) = h(z)/(z - z_0)^n$, where h is analytic in some disc $\{z \in \mathbb{C} \parallel |z - z_0| < R\}$ and can be replaced by the condition that f is analytic and $h(z) = (z - z_0)^n f(z)$ is bounded in $\{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$. Thus, we have the following theorem.

Theorem A.31. *Let us assume that $f : \Omega \longrightarrow X$ is analytic in some annulus $\{z \in \mathbb{C} \parallel 0 < |z - z_0| < R\} \subseteq \Omega$. Then f has a removable singularity or a pole of order $k \leq n$ at z_0 if and only if $(z - z_0)^n f(z)$ is bounded at z_0, in which case*

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^n}{dz^n} \left[(z - z_0)^n f(z) \right].$$

B Interpolation spaces

In this appendix we introduce some basic notions and state some basic results from the theory of interpolation spaces. The material is included for reference only, and no proofs are given; More systematic introductions to interpolation spaces can be found in [9] and in [3].

B.1 Intermediate spaces and interpolation spaces

Let X and Y be normed spaces such that $Y \hookrightarrow X$, i.e. Y is continuously embedded in X by the identity mapping. We then say that a normed space Z is an *intermediate space* between Y and X if $Y \hookrightarrow Z \hookrightarrow X$.

Definition B.1. Let X and $Y \hookrightarrow X$ be complex Banach spaces and Let Z be an intermediate space between Y and X. Then Z is said to be an *interpolation space* between Y and X if any bounded linear operator T on X whose restriction $T|_Y$ to Y is a bounded mapping on Y is also a bounded linear mapping on Z, where each space is considered with its own norm.

Remark B.2. The concept of interpolation space can be introduced in a more general setting. Two normed vector spaces X and Y are said to be compatible if there is a Hausdorff space Z with both X and Y as subspaces. An interpolation space between X and Y is then a subspace of Z that is intermediate between $X \cap Y$ and $X + Y = \{x + y \mid x \in X \wedge y \in Y\}$ (see [3], pp. 24–28)

B.2 Real interpolation spaces

Let X and Y be complex Banach spaces such that $Y \hookrightarrow X$. We define the function K by

$$K(t, z, X, Y) := \inf_{\substack{z=x+y \\ (x,y)\in X\times Y}} (\| x \|_X + t \| y \|_Y) \tag{B.1}$$

for any $t > 0$ and any $z \in X$. For later use we also define J by

$$J(t, z, X, Y) := \max(\| z \|_X , t \| z \|_Y). \tag{B.2}$$

for all $t > 0$ and all $z \in Y$. We note that these definitions make K and J into norms on X and Y respectively for any fixed $t \geq 0$. In the sequel we shall use the shorter notation $K(t, z)$ and $J(t, z)$ whenever it is clear from the context what the spaces X and Y are.

Recall that for $0 \leq a < b \leq \infty$, and $1 \leq p \leq \infty$, we have defined (see p. 146) the spaces $L^p_*((a, b))$ by

$$L^p_*((a, b)) := \{f : (a, b) \longrightarrow \mathbb{C} \parallel f \text{ is measurable and } \| f \|_{L^p_*((a,b))} < \infty\},$$

where

$$\| f \|_{L^p_*((a,b))} := \begin{cases} \left(\int_a^b |f(t)|^p_X \frac{dt}{t} \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \operatorname{ess\,sup}_{a<t<b} |f(t)|_X & \text{if } p = \infty \end{cases}.$$

DEFINITION B.3. Assume that X and Y are complex Banach spaces with $Y \hookrightarrow X$. If $0 < \theta < 1$ and $1 \leq p \leq \infty$ or $\theta \in \{0, 1\}$ and $p = \infty$, we set

$$\| x \|_{(X,Y)_{\theta,p}} := \left\| t^{-\theta} K(t, x) \right\|_{L_*^p(0,1} , \tag{B.3}$$

and define

$$(X, Y)_{\theta,p} := \{x \in X \parallel \| x \|_{(X,Y)_{\theta,p}} < \infty\} \tag{B.4}$$

If $0 < \theta < 1$, we also define

$$(X, Y)_\theta := \{x \in X \parallel \lim_{t \to 0} t^{-\theta} K(t, x) = 0\}. \tag{B.5}$$

One can show that the sets $(X, Y)_{\theta,p}$ and $(X, Y)_\theta$ are interpolation spaces, called *real interpolation spaces*[12], between X and Y. This method of introducing the real interpolation spaces is called the *K-method*. (There are also other methods, eg. the *trace method* and the *J-method*). All real interpolation spaces between X and Y are complex Banach spaces (see [9], p. 18).

It is obvious from the definition of $K(t, z, X, Y)$ that if the norms $\| \ \|_X$ and $\| \ \|_Y$ a re replaced by equivalent ones $\| \ \|'_X$ and $\| \ \|'_Y$, respectively, then there are constants c_1 and c_2 such that if $K_1(t, z, X, Y)$ is the K that results, then

$$c_1 K(t, z, X, Y) \leq K_1(t, z, X, Y) \leq K(t, z, X, Y)$$

for all $z \in X$ and all $t \geq 0$. This, in turn, makes the interpolation spaces obtained with the different pairs of norms in X and Y equal with equivalent norms.

The following lemma compares the real interpolation spaces between two fixed complex Banach spaces X and $Y \hookrightarrow X$ to each other.

LEMMA B.4. *Let X be a complex Banach space and let $Y \hookrightarrow X$. If $0 \leq \theta_1 < \theta_2 \leq 1$ and $1 \leq p, q \leq \infty$, where $q = \infty$ if $\theta_1 = 0$ and $p = \infty$ if $\theta_2 = 1$, then*

$$Y \hookrightarrow (X, Y)_{\theta_2,p} \hookrightarrow (X, Y)_{\theta_1,q} \hookrightarrow X \tag{B.6}$$

If $1 \leq p_1 \leq p_2 < \infty$ and $0 < \theta < 1$, then

$$Y \hookrightarrow (X, Y)_{\theta,p_1} \hookrightarrow (X, Y)_{\theta,p_2} \hookrightarrow (X, Y)_\theta \hookrightarrow (X, Y)_{\theta,\infty} \hookrightarrow X. \tag{B.7}$$

Moreover, if $0\eta < \theta \leq 1$, then $(X, Y)_\eta$ is the closure of $(X, Y)_{\eta,p}$ in $(X, Y)_{\eta,\infty}$, and $(X, Y)_\theta$ is included in the closure of Y in X.

The proof is not difficult, and can be found in e.g. [9], pp. 16–18. As a special case we note that $(X, Y)_{0,\infty} = X$ with equivalence of norms.

REMARK B.5. If X and Y are complex Banach spaces, then $\| x \|_{(X,Y)_{\theta,p}}$ is often defined by $\| x \|_{(X,Y)_{\theta,p}} := \left\| \underline{t}^{-\theta} K(\underline{t}, x) \right\|_{L_*^p(0,\infty)}$. It is easy to show that this defines a norm on $(X, Y)_{\theta,p}$ that is equivalent to the one introduced above.

[12] The word 'real' refers to the fact that real interpolation spaces are obtained by means of techniques of real analysis.

B.3 The Reiteration Theorem

One of the most useful results in the theory of interpolation spaces is the so called *Reiteration Theorem.* This theorem implies that, if we take two real interpolation spaces Z_1 and Z_2 between X and Y, then, under certain conditions, forming a real interpolation space between Z_1 and Z_2 results in a new real interpolation space between X and Y. In order to be able to state the theorem in a somewhat more general form, we first make some definitions.

Let Z be an intermediate space between Y and X and fix $\theta \in [0,1]$. Then Z is said to belong to the *class* $J_\theta(X,Y)$ if there is a constant $c > 0$ such that $\| x \|_Z \leq ct^{-\theta} J(t,x)$ for any $t > 0$ and any $x \in Y$. Analogously we define Z to belong to the *class* $K_\theta(X,Y)$ if there is a constant $c > 0$ $ct^{-\theta}K(t,x) \leq \| x \|_Z$ for any $t > 0$ and any $x \in Z$. When the spaces X and Y are clear from the context, we shall use the abbreviated notation J_θ and K_θ, respectively. We immediately conclude that Z belongs to K_θ if and only if $Y \hookrightarrow Z \hookrightarrow (X,Y)_{\theta,\infty}$. One can also show, that if $0 < \theta < 1$, then Z belongs to J_θ if and only if $(X,Y)_{\theta,1} \hookrightarrow Z \hookrightarrow X$. Hence, if $0 < \theta < 1$, we have $Z \in J_\theta \cap K_\theta$ precisely when $(X,Y)_{\theta,1} \hookrightarrow Z \hookrightarrow (X,Y)_{\theta,\infty}$. In view of Lemma B.4, both $(X,Y)_{\theta,p}$ and $(X,Y)_\theta$ belong to $J_\theta \cap K_\theta$ if $(\theta,p) \in (0,1) \times [1,\infty]$ and $\theta \in (0,1)$ respectively. In addition $X \in K_0 \cap \mathcal{J}_\prime$ and $Y \in J_1 \cap K_1$, which is an immediate consequence of the definition of $\mathcal{C}_{\mathcal{J}}(\theta)$ and K_θ for $\theta = 0, 1$.

We can now state the Reiteration Theorem

Theorem B.6. *Let $0 \leq \theta_1 < \theta_2 \leq 1$ and $0 < \theta < 1$. Put $\eta = (1-\theta)\theta_1 + \theta\theta_2$. Then the following statements hold for all $p \in [0,\infty]$.*

(i) If $Z_i \in K_{\theta_i}$ (i=1,2), then

$$(Z_1, Z_2)_{\theta,p} \hookrightarrow (X,Y)_{\eta,p} \text{ and } (Z_1,Z_2)_\theta \hookrightarrow (X,Y)_\eta$$

(ii) If $Z_i \in J_{\theta_i}$ (i=1,2), then

$$(X,Y)_{\eta,p} \hookrightarrow (Z_1,Z_2)_{\theta,p} \text{ and } (X,Y)_\eta \hookrightarrow (Z_1,Z_2)_\theta$$

(iii) If $Z_i \in K_{\theta_i} \cap J_{\theta_i}$ (i=1,2), then

$$(X,Y)_{\eta,p} = (Z_1,Z_2)_{\theta,p} \text{ and } (X,Y)_\eta = (Z_1,Z_2)_\theta$$

with equivalence of norms.

We see that the third assertion of the theorem is an immediate consequence of the first two. For a proof of the theorem, see [3], pp. 50–51.

Corollary B.7. *Let $0 < \theta, \theta_1, \theta_2 < 1$ and $1 \le p, q_1, q_2 \le \infty$. Then*

$$
\begin{aligned}
&((X,Y)_{\theta_1,q_1}, (X,Y)_{\theta_2,q_2})_{\theta,p} = (X,Y)_{(1-\theta)\theta_1+\theta\theta_2,p}\\
&((X,Y)_{\theta_1}, (X,Y)_{\theta_2,q_2})_{\theta,p} = (X,Y)_{(1-\theta)\theta_1+\theta\theta_2,p}\\
&((X,Y)_{\theta_1,q_1}, (X,Y)_{\theta_2})_{\theta,p} = (X,Y)_{(1-\theta)\theta_1+\theta\theta_2,p}\\
&((X,Y)_{\theta_1}, (X,Y)_{\theta_2})_{\theta,p} = (X,Y)_{(1-\theta)\theta_1+\theta\theta_2,p}\\
&((X,Y)_{\theta_1,q_1}, Y)_{\theta,p} = (X,Y)_{(1-\theta)\theta_1+\theta,p}\\
&((X,Y)_{\theta_1}, Y)_{\theta,p} = (X,Y)_{(1-\theta)\theta_1+\theta}\\
&(X, (X,Y)_{\theta_2,q_2})_{\theta,p} = (X,Y)_{\theta\theta_2,p}\\
&(X, (X,Y)_{\theta_2})_{\theta,p} = (X,Y)_{\theta\theta_2,p}.
\end{aligned}
$$

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