Helsinki University of Technology Institute of Mathematics Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2001 A440

# PSEUDODIFFERENTIAL CALCULUS ON COMPACT LIE GROUPS AND HOMOGENEOUS SPACES

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# PSEUDODIFFERENTIAL CALCULUS ON COMPACT LIE GROUPS AND HOMOGENEOUS SPACES

**Ville Turunen**

**Dissertation for the degree of Doctor of Technology to be presented with due permission for public examination and debate in Council Room (Hallituksen kokoushuone) at Helsinki University of Technology (Otakaari 1 F, Espoo, Finland) on the 9th of November, 2001, at 12 o'clock noon.**

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Department of Engineering Physics and Mathematics
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**Ville Turunen**: *Pseudodifferential calculus on compact Lie groups and homogeneous spaces*; Helsinki University of Technology Institute of Mathematics Research Reports A440 (2001).

**Abstract:** *Pseudodifferential calculus is an important tool in the theory of elliptic linear partial differential equations. We introduce papers [I-IV], where pseudodifferential calculus is studied in the presence of a transitive group of symmetries for the underlying compact manifold.*

**AMS subject classifications:** 35S05, 22E30, 43A77, 43A85, 46A32, 46E40, 47G30, 58J40, 47B47

**Keywords:** pseudodifferential operators, Lie groups, homogeneous spaces, noncommutative Fourier analysis, Schwartz kernels, operator-valued symbols, symbol calculus, asymptotic expansions, parametrix, commutators, periodic pseudodifferential operators, analysis on sphere

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ISBN 951-22-5654-1
ISSN 0784-3143
Otamedia Oy, Espoo, 2001

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## Acknowledgments

First of all, I would like to thank Gennadi Vainikko for introducing me into this fascinating subject, for his advice and patience throughout the convoluted odyssey. I also thank Olavi Nevanlinna for supervising, and Petri Ola and Henrik Stetkær for reviewing the manuscript.

The influence of Michael E. Taylor has been considerable in my work, and I am greatly indebted to him in getting crucial insights in the research. The people at both Helsinki University of Technology and University of North Carolina at Chapel Hill deserve my appreciation for creating harmonious working environment.

And then there is my wife Maarit, by far the most important person in my life. Her enduring love and support have been decisive.

Finally, the *Vilho, Yrjö and Kalle Väisälä Fund* of Finnish Academy of Sciences and Letters is acknowledged for providing a three-year-in-a-row haven free from mundane worries.

The thesis consists of this overview and the following papers:

## Publications

[I] Turunen, V. and Vainikko, G.: *On symbol analysis of periodic pseudo-differential operators.* Z. Anal. Anw. 17 (1998), 9–22.

[II] Turunen, V.: *Commutator characterization of periodic pseudodifferential operators.* Z. Anal. Anw. 19 (2000), 95–108.

[III] Turunen, V.: *Pseudodifferential calculus on compact Lie groups.* Helsinki Univ. Techn. Inst. Math. Research Report A431. 2001.

[IV] Turunen, V.: *Pseudodifferential calculus on compact homogeneous spaces.* Helsinki Univ. Techn. Inst. Math. Research Report A436. 2001.

# 1 Introduction

Pseudodifferential operators are a generalization of linear partial differential operators, arising from studies of elliptic partial differential equations. Pseudodifferential calculus provides essentially constructive means for solving elliptic problems. The roots of this discipline stem from the theory of singular integral operators, notably from the work of A. P. Calderón and A. Zygmund. In 1965 J. J. Kohn and L. Nirenberg defined the first modern pseudodifferential operator classes and studied their basic properties, and soon this work was carried on especially by L. Hörmander.

The fundamental idea in pseudodifferential calculus is to replace algebras of operators by function algebras via a process resembling the freezing-coefficients technique familiar from solving variable coefficient partial differential equations. One establishes a bijective correspondence between operators and certain functions called symbols, and restricts the operator classes by setting demands on the symbol functions in order to distill some essential features of partial differential operators. Although the operator-to-symbol mapping is not an algebra homomorphism, it still approximatively preserves the elementary properties of the original operator algebra, and it is definitely easier to treat symbol functions than the corresponding operators. One starts by defining pseudodifferential operators on the Euclidean spaces, and since many of the operator classes are diffeomorphism invariant, the calculus can be transfered to smooth manifolds by using partitions of unity.

Symmetries of a manifold sometimes permit powerful tools for pseudodifferential calculus: Consider, for instance, a Dirichlet problem of an elliptic partial differential equation in a domain diffeomorphic to the unit disk of the plane $\mathbb{R}^2$, with a smooth boundary. The resulting boundary integral equation can be efficiently treated using Fourier series and fast Fourier transform (FFT). Related pseudodifferential calculus for Fourier series was studied by M. S. Agranovich ([4], [3], [2]) and others, and fast algorithms for solving equations were presented by, e.g., G. Vainikko ([34]).

Naturally, it would be desirable to construct efficient methods for solving boundary integral equations on other manifolds as well. For example, the surface $\mathbb{S}^2$ of the unit ball $\mathbb{B}^3 \subset \mathbb{R}^3$ would be an interesting case. In general, pseudodifferential calculus on a torus resembles very much the Euclidean theory, and this is due to that $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is an Abelian group; unfortunately, the sphere $\mathbb{S}^2$ lacks such an algebraic structure. However, the special orthogonal group $\mathrm{SO}(3)$ acts transitively on $\mathbb{S}^2$, and much of the analysis on the sphere can be derived from harmonic analysis on the symmetry group. E.g., spherical harmonic functions on $\mathbb{S}^2$ are a byproduct of the representation theory of $\mathrm{SO}(3)$. Actually, $\mathbb{S}^2$ is just an example of homogeneous spaces.

R. Strichartz treated the calculus of convolution type pseudodifferential operators on Lie groups in [26], and in 1984 M. E. Taylor (see [30]) presented a pseudodifferential calculus for Lie groups, studying operators as convolution operator -valued functions on the groups, and constructing a symbolic calculus there; however, operator classes on a group were defined using Euclidean symbol inequalities in the exponential coordinates in a neighbourhood of the neutral element; no symbol inequalities on the group were given.

In this thesis we study pseudodifferential calculus on compact homogeneous spaces, mostly on compact Lie groups, and special attention is paid to the case of the tori $\mathbb{T}^n$. We present a characterization of pseudodifferential operators on closed smooth orientable manifolds by studying Sobolev space boundedness of certain iterated commutators. We combine this commutator approach with M. E. Taylor's pseudodifferential calculus [30] to characterize symbols of pseudodifferential operators on compact Lie groups. We also define amplitudes (or generalized symbols), and we study symbol calculus. In particular, we construct asymptotic expansions for a parametrix of an elliptic pseudodifferential operator. Finally, we show how pseudodifferential calculus on compact Lie groups can be exploited to create some calculus on compact smooth orientable homogeneous spaces, especially on spaces of the form $G/K$ with $K$ a torus subgroup of $G$, like $\mathbb{S}^2 \cong \mathrm{SO}(3)/\mathbb{T}^1 \cong \mathrm{SU}(2)/\mathbb{T}^1$.

# 2 Results and connections to earlier research

Fourier transform is the main tool in pseudodifferential calculus, other important machinery including the theory of distributions (especially the Schwartz kernel theorem) and the classical Taylor expansion of smooth functions.

## 2.1 Commutative background

Already S. G. Mikhlin in the 1940s and A. P. Calderón and A. Zygmund in the 1950s represented a singular integral operator as a family of convolution operators: we start with the Schwartz kernel $K_A \in \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}'(\mathbb{R}^n)$ of a linear operator $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n))$, defined by the duality

$$\langle K_A, f \otimes \phi \rangle = \langle A\phi, f \rangle,$$

where the tensor product $\otimes$ is the completed tensor product of the nuclear locally convex spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$. Formally we can define $s_A(x) \in \mathcal{S}'(\mathbb{R}^n)$ by $s_A(x)(y) := K_A(x, x-y)$, and then

$$\begin{aligned} (A\phi)(x) &= \int_{\mathbb{R}^n} K_A(x,y)\ \phi(y)\ dy \\ &= \int_{\mathbb{R}^n} s_A(x)(x-y)\ \phi(y)\ dy. \end{aligned}$$

Hence $(A\phi)(x) = (s_A(x) * \phi)(x)$; in a sense the convolution by $s_A(x)$ is the frozen version of $A$ at point $x \in \mathbb{R}^n$. In [15] J. J. Kohn and L. Nirenberg went on to define the symbol function $\sigma_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ as the Fourier transform of $s_A$, more precisely

$$\sigma_A(x,\xi) = \widehat{s_A(x)}(\xi).$$

Let us assume that $\sigma_A \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfies

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)| \leq C_{A\alpha\beta m}\ \langle\xi\rangle^{m-|\alpha|}, \tag{1}$$

for every $x \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{N}_0^n$, where $\xi \mapsto \langle\xi\rangle = (1+|\xi|^2)^{1/2}$, and $C_{A\alpha\beta m}$ is a constant depending only on $A$, $\alpha$, $\beta$ and $m$. These *symbol inequalities* define the symbol class $S^m(\mathbb{R}^n)$ and the corresponding class $\Psi^m(\mathbb{R}^n) = \mathrm{Op}S^m(\mathbb{R}^n)$ of *pseudodifferential operators of degree* $m \in \mathbb{R}$. It should be noted that the index $m$ in the coefficients $C_{A\alpha\beta m}$ is essential: Consider, for instance, the case where $\sigma_A(x,\xi) = \log|\xi|$ for large $|\xi|$ for every $x \in \mathbb{R}$. Then $A$ is of any degree $m > 0$, but not of degree 0, and $\lim_{m\to 0+} C_{A00m} = \infty$. An operator $A \in \Psi^m(\mathbb{R}^n)$ belongs to $\mathcal{L}(\mathcal{S}(\mathbb{R}^n))$, $\mathcal{L}(\mathcal{S}'(\mathbb{R}^n))$ and $\mathcal{L}(H^s(\mathbb{R}^n), H^{s-m}(\mathbb{R}^n))$, where $H^s(\mathbb{R}^n)$ is the $L^2$-type Sobolev space of order $s \in \mathbb{R}$. We may write

$$(A\phi)(x) = \int_{\mathbb{R}^n} \sigma_A(x,\xi)\ \hat\phi(\xi)\ e^{i2\pi x\cdot\xi}\ d\xi : \tag{2}$$

the symbol $\sigma_A$ can be regarded as a weight for the inverse Fourier transform. Various other symbol inequalities, that is conditions resembling estimates (1), have been introduced to restrict symbol classes for suitable occasions.

The symbol $\sigma_A$ contains all the information about an operator $A$, and while operator algebras may be difficult to handle, the symbol function algebras enable practical calculations, and the operations in these algebras are closely related to each other: For example, approximatively $\sigma_{AB}(x,\xi) \approx \sigma_A(x,\xi)\sigma_B(x,\xi)$ and $\sigma_{A^*}(x,\xi) \approx \overline{\sigma_A(x,\xi)}$, whereas actual calculations for the composition $AB$ and the adjoint $A^*$ might be tremendous tasks. For instance, if $\sigma_A \in S^{m_A}(\mathbb{R}^n)$ and $\sigma_B \in S^{m_B}(\mathbb{R}^n)$, we have $\sigma_{AB} - \sum_{|\alpha|<N} \sigma_{C_\alpha} \in S^{m_A+m_B-N}(\mathbb{R}^n)$, where

$$\sigma_{C_\alpha}(x,\xi) = (\partial_\xi^\alpha \sigma_A)(x,\xi)\ (\partial_x^\alpha \sigma_B)(x,\xi).$$

In such a case one writes $\sigma_{AB} \sim \sum_{\alpha\geq 0} \sigma_{C_\alpha}$ or $AB \sim \sum_{\alpha\geq 0} C_\alpha$, which are called asymptotic expansions for the composition of pseudodifferential operators. The constructions of such formulae mainly rely on using the traditional Taylor expansion of smooth functions, familiar from undergraduate calculus. Similarly one can construct asymptotic expansions for the adjoints and transposes of pseudodifferential operators (see [14], [31], [32]).

Since asymptotic expansions obviously reveal information only modulo

$$S^{-\infty}(\mathbb{R}^n) = \cap_{m\in\mathbb{R}}\ S^m(\mathbb{R}^n),$$

these symbols of arbitrarily low degree may be regarded as inessential. A pseudodifferential operator $A$ is called *elliptic* if it is invertible modulo class $\mathrm{Op}S^{-\infty}(\mathbb{R}^n)$, i.e. there exists a pseudodifferential operator $B$ such that $\sigma_{AB-I}$ and $\sigma_{BA-I}$ belong to $S^{-\infty}(\mathbb{R}^n)$. Such a formal inverse $B$ is called a *parametrix*, for which one may calculate an asymptotic expansion using the expansion for the composition of operators. Moreover, while differential operators are *local* in the sense that they do not increase the support of a distribution, other pseudodifferential operators are only *pseudolocal*: if a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is $C^\infty$-smooth on an open set $U \subset \mathbb{R}^n$, so is $Af$ for $A \in \Psi^m(\mathbb{R}^n)$.

We assume that the definition of a manifold contains the second countability axiom: this means that the topology has a countable base. For compact Hausdorff spaces, second countability axiom is equivalent to the metrizability of the space. On a smooth manifold $M$ one defines pseudodifferential operators by localizing with partitions of unity; the class $\Psi^m(\mathbb{R}^n)$ is diffeomorphism invariant, and hence the corresponding class $\Psi^m(M)$ is well-defined.

Let $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ be the $n$-dimensional torus. M. S. Agranovich was first to prove ([2]) that an operator $A \in \Psi^m(\mathbb{T}^n)$ can be presented in the form

$$(Af)(x) = \sum_{\xi \in \mathbb{Z}^n} \sigma_A(x,\xi)\ \hat{f}(\xi)\ e^{i2\pi x\cdot\xi} \tag{3}$$

such that $\sigma_A \in S^m(\mathbb{T}^n)$, the class $S^m(\mathbb{T}^n)$ defined by the symbol inequalities

$$|\triangle_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)| \leq C_{A\alpha\beta m}\ \langle\xi\rangle^{m-|\alpha|}, \tag{4}$$

for every $\alpha, \beta \in \mathbb{N}_0^n$ and $x \in \mathbb{T}^n$, where $C_{A\alpha\beta m}$ is a constant depending only on $A$, $\alpha$, $\beta$ and $m$. Here $\triangle_\xi^\alpha$ is the forward difference operator,

$$\triangle_\xi^\alpha \sigma(\xi) = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} (-1)^{|\alpha-\gamma|}\ \sigma(\xi+\gamma).$$

Pseudodifferential operators on the torus $\mathbb{T}^n$ are sometimes called *periodic pseudodifferential operators*. In [17] W. McLean defined the naturally generalized symbol classes $S^m_{\rho,\delta}(\mathbb{T}^n)$ (where $S^m(\mathbb{T}^n) = S^m_{1,0}(\mathbb{T}^n)$) and proved that the operator class $\mathrm{Op}S^m_{\rho,\delta}(\mathbb{T}^n)$ for $1-\rho \leq \delta < \rho \leq 1$ coincides with the traditional one obtained locally from $\mathrm{Op}S^m_{\rho,\delta}(\mathbb{R}^n)$.

In article [I] we present the elementary symbolic calculus of periodic pseudodifferential operators, providing asymptotic expansions for the compositions, adjoints and transposes of operators and for amplitude operators, and thus we generalize the results of Elschner and Amosov ([12] and [5]), who have considered the case of classical pseudodifferential operators. The form of the asymptotic expansions was proposed by G. Vainikko, and the validity of the expansions was proven by the author of this thesis. Curiously, in article [I] an asymptotic expansion for a parametrix $B$ of an elliptic periodic pseudodifferential operator $A \sim \sum_{j=0}^\infty A_j$ on $\mathbb{T}^n$ is missing, so let us fill this gap here: the construction follows the lines of the corresponding treatment in paper [III], and we get $\sigma_B \sim \sum_{k=0}^\infty \sigma_{B_k}$,

$$\sigma_{B_0}(x,\xi) = 1/\sigma_A(x,\xi) \tag{5}$$

for large $|\xi|$, and

$$\sigma_{B_N}(x,\xi) = -\sigma_{B_0}(x,\xi) \sum_{k=0}^{N-1} \sum_{j=0}^{N-k} \sum_{\gamma: j+k+|\gamma|=N} (\triangle_\xi^\gamma \sigma_{A_j}(x,\xi))\ \partial_x^{(\gamma)} \sigma_{B_k}(x,\xi). \tag{6}$$

Here

$$\partial_x^{(\gamma)} = \prod_{p=1}^{n} \prod_{q=0}^{\gamma_p - 1} (\partial_{x_p} - qI),$$

where $\partial_{x_p} = (i2\pi)^{-1}(\partial/\partial x_p)$ and $\partial_x^{(0)} := I$.

## 2.2 Commutators

A striking feature calls for our attention: pseudodifferential operators $A \in \Psi^{m_A}(\mathbb{R}^n)$ and $B \in \Psi^{m_B}(\mathbb{R}^n)$ almost commute in the sense that $[A, B] = AB - BA \in \Psi^{m_A+m_B-1}(\mathbb{R}^n)$ although both $AB$ and $BA$ belong usually only to $\Psi^{m_A+m_B}(\mathbb{R}^n)$. This property is a not-so-distant-relative to the degree drop in the commutator of two partial differential operators, due to the Leibniz formula for derivations,

$$D(f\ g) = (Df)\ g + f\ (Dg).$$

Special cases of the almost-commutation property are

$$\sigma_{[\partial_{x_j}, A]}(x, \xi) = \partial_{x_j} \sigma_A(x, \xi) \tag{7}$$

and

$$\sigma_{[A, M_{x \mapsto x_k}]}(x, \xi) = \partial_{\xi_k} \sigma_A(x, \xi), \tag{8}$$

where $M_f$ is the multiplication operator $(M_f g)(x) = f(x)\ g(x)$. Beals ([6]) used this discovery to characterize pseudodifferential operators in terms of Sobolev space boundedness of iterated commutators of the type (7) and (8). This gave a spark for several other investigations, e.g. [8] and [9]. In article [II] one form of this commutator characterization is given on the Euclidean spaces.

Commutator characterizations for pseudodifferential operators on compact manifolds have been given by R. R. Coifman and Y. Meyer in [8] and J. Dunau in [11], and we present other commutator characterizations in papers [II] and [III]. In article [II] this is applied to prove that the definition of the periodic pseudodifferential operator class $\mathrm{Op}S^m(\mathbb{T}^n)$ coincides with the class $\Psi^m(\mathbb{T}^n)$. Let $M$ be a compact smooth manifold without a boundary. R. R. Coifman and Y. Meyer have proven that $A \in \Psi^0(M)$ if and only if $[D_{k+1}, [D_k, \cdots [D_1, A] \cdots]] \in \mathcal{L}(L^2(M))$ for every sequence of smooth vector fields $D_k$ on $M$. We generalize this to get the following:

*$A \in \Psi^m(M)$ if and only if $[D_{k+1}, [D_k, \cdots [D_1, A] \cdots]] \in \mathcal{L}(H^m(M), H^0(M))$ for every sequence of smooth vector fields $D_k$ on $M$,*

where $H^s(M)$ is the Sobolev space of order $s \in \mathbb{R}$ on $M$. This result is used in paper [III] to characterize operator-valued symbols of pseudodifferential operators on compact Lie groups.

## 2.3 Harmonic analysis on Lie groups

On our way to pseudodifferential calculus on homogeneous spaces, the next step is to establish analysis on Lie groups.

Let $G$ be a compact Lie group; the group structure is now encoded in of the test function algebra $\mathcal{D}(G)$ in a subtle way. Let $(x,y) \mapsto xy$ denote the group operation $G \times G \to G$, $e \in G$ the neutral element, and $x^{-1} \in G$ the inverse element of $x \in G$. Let $\mathcal{D}(G) \otimes \mathcal{D}(G)$ denote the complete tensor product of the Schwartz spaces, isomorphic to $\mathcal{D}(G \times G)$ (see [33] or [19]). Let us define the *co-product*

$$\Delta : \mathcal{D}(G) \to \mathcal{D}(G) \otimes \mathcal{D}(G), \quad (\Delta f)(x,y) := f(xy),$$

the *co-unit*

$$\varepsilon : \mathcal{D}(G) \to \mathbb{C}, \quad \varepsilon(f) := f(e),$$

and the *antipode*

$$S : \mathcal{D}(G) \to \mathcal{D}(G), \quad (Sf)(x) := f(x^{-1}).$$

The multiplication of the function algebra can be extended linearly to

$$m : \mathcal{D}(G) \otimes \mathcal{D}(G) \to \mathcal{D}(G), \quad m(f \otimes g)(x) := f(x)\ g(x),$$

and there is the natural embedding

$$\eta : \mathbb{C} \to \mathcal{D}(G), \quad \eta(z)(x) := z.$$

The structure $(\mathcal{D}(G), m, \eta, \Delta, \varepsilon, S)$ is called a Hopf algebra (see [1], [28]), and from it one can retrieve the group structure of $G$. The *convolution product* is the transpose $\Delta^t : \mathcal{D}'(G) \otimes \mathcal{D}'(G) \to \mathcal{D}'(G)$ of $\Delta$ with respect to the dual pairing

$$((\phi, f) \mapsto \langle \phi, f \rangle) : \mathcal{D}(G) \times \mathcal{D}'(G) \to \mathbb{C};$$

notice that $\Delta^t(f \otimes g) = f * g$, the usual convolution product of distributions, formally

$$(f * g)(x) = \int_G f(xy^{-1})\ g(y)\ d\mu_G(y). \tag{9}$$

Here $\mu_G$ is the Haar measure of the compact Lie group $G$, i.e. the unique translation-invariant regular Borel probability measure on $G$. The "global" *Fourier transform* (in the words of Stinespring, [25], see also [22]) of a distribution $f \in \mathcal{D}'(G)$ is defined to be the convolution operator $\pi(f)$,

$$\pi(f)g = f * g, \tag{10}$$

and $\pi(f) \in \mathcal{L}(\mathcal{D}(G))$. In the weak sense,

$$\pi(f) = \int_G f(x)\ \pi_L(x)\ d\mu_G(x),$$

where and $\pi_L : G \to \mathcal{L}(L^2(G))$ is the *left regular representation*,

$$(\pi_L(x)g)(y) = g(x^{-1}y) \tag{11}$$

for almost every $y \in G$.

Due to the Schwartz kernel theorem, operator $A \in \mathcal{L}(\mathcal{D}(G))$ can be represented by its *Schwartz kernel* $K_A \in \mathcal{D}(G) \otimes \mathcal{D}'(G)$ by

$$\langle K_A, f \otimes \phi \rangle = \langle A\phi, f \rangle,$$

where the duality is evaluated by help of the Haar measure. On the other hand,

$$k_A(x)f := (Af)(x) \tag{12}$$

allows us to think $K_A$ as a function $k_A : G \to \mathcal{D}'(G)$, $k_A(x)(y) = K_A(x, y)$, and in a sense $k_A \in C^\infty(G, \mathcal{D}'(G))$. Now let us define

$$s_A(x)(y) := k_A(x)(y^{-1}x) \tag{13}$$

in the sense of distributions; then $s_A \in \mathcal{D}(G) \otimes \mathcal{D}'(G)$, or $s_A \in C^\infty(G, \mathcal{D}'(G))$. The *symbol* of an operator $A \in \mathcal{L}(\mathcal{D}(G))$ is the mapping $\sigma_A : G \to \mathcal{L}(\mathcal{D}(G))$ defined by the Fourier transform,

$$\sigma_A(x) := \pi(s_A(x)). \tag{14}$$

In [30] M. E. Taylor examined a more restricted class of symbols. To our knowledge, the definition of the symbol $\sigma_A$ of $A \in \mathcal{L}(\mathcal{D}(G))$ is a new concept, presented first time in paper [III].

Let $f \in \mathcal{D}(G)$. Then

$$\begin{aligned}
\mathrm{Tr}(\pi(f)) &= \int_G K_{\pi(f)}(x, y)|_{y=x} \, d\mu_G(x) \\
&= \int_G f(xy^{-1})|_{y=x} \, d\mu_G(x) \\
&= f(e);
\end{aligned}$$

Let $\delta_x \in \mathcal{D}'(G)$ be the Dirac delta distribution at $x \in G$. Thereby

$$\begin{aligned}
(Af)(x) &= (\sigma_A(x)f)(x) \\
&= (s_A(x) * f)(x) \\
&= (s_A(x) * f * \delta_{x^{-1}})(e) \\
&= \mathrm{Tr}\,(\pi(s_A(x) * f * \delta_{x^{-1}})) \\
&= \mathrm{Tr}\left(\sigma_A(x)\ \pi(f)\ \pi_L(x^{-1})\right).
\end{aligned}$$

When $A = I$, this yields the inverse Fourier transform

$$f(x) = \mathrm{Tr}\left(\pi(f)\ \pi_L(x^{-1})\right).$$

These formulae are important in numerical calculations, but we do not consider those questions in the thesis. One should yet notice the resemblance of

$$(Af)(x) = \mathrm{Tr}\left(\sigma_A(x)\ \pi(f)\ \pi_L(x^{-1})\right) \tag{15}$$

and formulas (2) and (3).

In paper [III], we characterize $\Psi^m(G)$ by operator norm inequalities for symbols; we define classes $S^m(G)$ of symbols such that for the corresponding operator classes

$$\mathrm{Op}S^m(G) = \Psi^m(G). \tag{16}$$

Let us be more precise: Let $\Xi = I - \Delta$, where $\Delta$ is the bi-invariant Laplacian of $G$, and let $Q^\alpha \sigma_A(x) = \pi(y \mapsto \check{q}_\alpha(y)\ s_A(x)(y))$ with $\check{q}_\alpha \in C^\infty(G)$ satisfying $\check{q}_\alpha(\exp(z)) = z^\alpha$ for $z$ in a small neighbourhood of the origin on the Lie algebra $\mathfrak{g}$ of $G$. The underlying idea for operators $Q^\alpha$ comes from the observation how the Euclidean Fourier transform converts multiplications by polynomials to differential operators: $\widehat{Mf}(\xi) = (\partial_\xi^\alpha \hat{f})(\xi)$ for $(Mf)(x) = x^\alpha f(x)$. The symbol $\sigma_A$ of an operator $A \in \mathcal{L}(\mathcal{D}(G))$ is said to belong to $S_0^m(G)$, if

$$\|\Xi^{|\alpha|-m} Q^\alpha \partial_x^\beta \sigma_A(x)\|_{\mathcal{L}(L^2(G))} \le C_{A\alpha\beta m} \tag{17}$$

uniformly in $x \in G$, for every multi-index $\alpha, \beta \in \mathbb{N}_0^{\dim(G)}$, where the constants $C_{A\alpha\beta m}$ depend only on $A$, $\alpha$, $\beta$ and $m$; notice similarities to (1) and (4). Then we recursively define $S_{k+1}^m(G) \subset S_k^m(G)$ with the idea that $\sigma_{[D,A]} \in S_k^m(G)$ for every smooth vector field $D$ if $\sigma_A \in S_{k+1}^m(G)$ (for a more accurate definition, see paper [III]). Then $S^m(G) = \cap_{k=0}^\infty S_k^m(G)$.

In [30] M. E. Taylor characterized pseudodifferential operators on $G$ by studying the distributions $(x,y) \mapsto s_A(x)(y)$ in the exponential coordinates when $\mathrm{supp}(s_A)$ is contained in a small neighbourhood of $(e,e) \in G \times G$; his results are essential in the characterization of $\Psi^m(G)$ in paper [III]. Taylor also presented asymptotic expansions for compositions, adjoints and parametrices of pseudodifferential operators on $G$. We review these results in paper [III], but we also construct an asymptotic expansion for the transpose, and we give a more readily applicable expansion for a parametrix: Those pseudodifferential operators that are invertible modulo infinitely smoothing operators are called *elliptic*, and an inverse of such an operator modulo infinitely smoothing operators is called a parametrix. This definition for ellipticity is more general than the standard one, and this condition may be often difficult to check, but our definition has some aesthetic appeal. Let $A$ be a pseudodifferential operator with expansion $A \sim \sum_{j=0}^\infty A_j$, $A_j \in \Psi^{m-j}(G)$, and assume that $x \mapsto \sigma_{A_0}(x)^{-1}$ is a symbol of a pseudodifferential operator $B_0 \in \Psi^{-m+1-\varepsilon}(G)$ (for some $\varepsilon > 0$). Then $A$ is elliptic with a parametrix $B$, $\sigma_B \sim \sum_{k=0}^\infty \sigma_{B_k}$, where

$$\sigma_{B_0}(x) = \sigma_A(x)^{-1}, \tag{18}$$

$$\sigma_{B_N}(x) = -\sigma_{B_0}(x) \sum_{k=0}^{N-1} \sum_{j=0}^{N-k} \sum_{\gamma: j+k+|\gamma|=N} (Q^\gamma \sigma_{A_j}(x))\ \partial_x^\gamma \sigma_{B_k}(x); \tag{19}$$

compare this to (5) and (6). Moreover, we introduce the concept of amplitudes on $G$, in analogy to the Euclidean pseudodifferential calculus.

## 2.4 Harmonic analysis on homogeneous spaces

A homogeneous space is a quotient of a group by a subgroup, and thus harmonic analysis on Lie groups provides the crucial step-stone for analysis on homogeneous spaces.

A left *transformation group* is a triple

$$(G, M, m),$$

where $G$ is a group, $M$ is a set and $m : G \times M \to M$ a mapping called a left *action of $G$ on $M$*, satisfying $m(e, p) = p$ and $m(x, m(y, p)) = m(xy, p)$ for every $x, y \in G$ and $p \in M$. The action is *free* if $m(x, p) = p$ for every $p \in M$ implies $x = e$. The action is *transitive* if for every $p, q \in M$ there exists $x \in G$ such that $m(x, p) = q$, and in such case $M$ is called a *homogeneous space*. One defines right transformation groups with right actions in the obvious manner.

Let $K$ be a subgroup of a group $G$. Let $G/K$ be the set $\{xK \mid x \in G\}$ of the classes $xK = \{xk \mid k \in K\}$, and let us define an action $m : G \times G/K \to G/K$ by $(x, yK) \mapsto x^{-1}yK$. Evidently, $G/K$ is a homogeneous space.

We assume everything to be *smooth*: hence above $G$ is a Lie group, $M$ is a $C^\infty$-manifold, and $m$ a $C^\infty$-mapping. Let $K$ be a closed subgroup of $G$. The set $G/K$ is endowed with the co-induced topology via mapping $(x \mapsto xK) : G \to G/K$, and there is a unique $C^\infty$-structure such that the action $(x, yK) \mapsto x^{-1}yK$ is $C^\infty$-smooth and such that there exists a mapping $\psi \in C^\infty(U, G)$ in a neighbourhood $U \subset G/K$ of $eK$ satisfying $\psi(xK)K = xK$ for every $xK \in U$. Homogeneous space $G/K$ is in fact typical: if $(G, M, m)$ is a smooth transformation group with a transitive left action and $p \in M$, then $M$ is diffeomorphic to $G/G_p$, where $G_p = \{x \in G \mid m(x, p) = p\}$ is the isotropy group of $p \in M$, i.e. that subgroup of $G$ which fixes the point $p$. For more information on homogeneous spaces we refer to [36] and [7].

A smooth *fiber bundle* is

$$(E, B, F, p_{E\to B}),$$

where $E, B, F$ are $C^\infty$-manifolds and $p_{E\to B} \in C^\infty(E, B)$ is a surjective mapping, and $B$ has an open cover $\mathcal{U} = \{U_j \mid j \in J\}$ and diffeomorphisms $\phi_j : p_{E\to B}^{-1}(U_j) \to U_j \times F$ such that $\phi_j(x) = (p_{E\to B}(x), \psi_j(x))$. Spaces $E$, $B$ and $F$ are called the *total space*, the *base space* and the *fiber*, respectively; $p_{E\to B}$ is the *projection* of the bundle. A *principal fiber bundle* is

$$(E, B, F, p_{E\to B}, m),$$

where $(E, B, F, p_{E\to B})$ is a smooth fiber bundle and $(F, E, m)$ is a smooth transformation group with a free right action satisfying $p_{E\to B}(m(x, y)) =$

$p_{E\to B}(x)$ and such that the mappings $\psi_j : E \to F$ above satisfy $\psi_j(m(x,y)) = \psi_j(x)y$ for every $(x,y) \in E \times F$.

Homogeneous spaces can be thought as base spaces for principal fiber bundles: $(G, G/K, K, x \mapsto xK, (k,x) \mapsto xk)$ is a principal fiber bundle for a Lie group $G$ with a closed subgroup $K$ (see [7]). In the sequel, we assume that $G$ is compact and $G/K$ is orientable.

Those functions (or distributions) on $G$ that are constant on each class $xK$ can be thought of being functions (or distributions) on $G/K$. The Haar measure on $K$ provides us with a procedure to "project" functions (or distributions) via mapping $P_{G/K} : C^\infty(G) \to C^\infty(G/K)$,

$$(P_{G/K}f)(xK) = \int_K f(xk)\ d\mu_K(k). \tag{20}$$

In fact, $P_{G/K}$ is surjective, and we consider $C^\infty(G/K)$ to be embedded into $C^\infty(G)$. Since $(G, G/K, K, x \mapsto xK, (k,x) \mapsto xk)$ is a principal fiber bundle, every smooth vector field on on $G/K$ is a "projection" of a smooth vector field on $G$ (see [23]). Let $A \in \mathcal{L}(\mathcal{D}(G))$ with $(Af)(x) = (\sigma_A(x)f)(x) = (s_A(x) * f)(x)$ as above. In paper [IV] we introduce the "averaged" operator $A_{G/K} \in \mathcal{L}(\mathcal{D}(G))$ defined by

$$s_{A_{G/K}} = (P_{G/K} \otimes \mathrm{id}) s_A. \tag{21}$$

We prove that $A_{G/K}$ maps $\mathcal{D}(G/K)$ into $\mathcal{D}(G/K)$, and if $A$ belongs to $\Psi^m(G)$, so does $A_{G/K}$, and even $A_{G/K}|_{\mathcal{D}(G/K)} \in \Psi^m(G/K)$. The characterization of pseudodifferential operators on $G$ in paper [III] is exploited in obtaining these results for $G/K$. The history of averaging processes for pseudodifferential operators can be traced at least back to the work of M. F. Atiyah and I. M. Singer in the 1960s, and H. Stetkær studied related topics for classical pseudodifferential operators in [24].

Should an elliptic pseudodifferential operator $B$ on $G/K$ be extendable to an elliptic pseudodifferential operator $A = A_{G/K}$ on $G$, we could use the symbol-operator calculus on $G$ (see paper [III] and [30]) to calculate a parametrix $C = C_{G/K}$ for $A$, and then $C|_{\mathcal{D}(G/K)}$ would be a parametrix for $B = A|_{\mathcal{D}(G/K)}$.

Paper [IV] continues with assumption that $K$ is a *torus subgroup* of $G$, i.e. $K$ is isomorphic to the torus group $\mathbb{T}^q$ for some $q \in \mathbb{N}$. Although this follows from the previous results, we prove in another way that $A_{G/K}|_{\mathcal{D}(G/K)} \in \Psi^m(G/K)$ whenever $A \in \Psi^m(G)$. Then, given $B \in \Psi^m(G/K)$, we present a method to construct an operator $A = A_{G/K} \in \Psi^m(G)$ such that $A|_{\mathcal{D}(G/K)} = B$. The method involves extending a symbol $\sigma_{B'} \in S^m(\mathbb{R}^p)$ to a symbol $\sigma_{A'} \in S^m(\mathbb{R}^{p+q})$ so that

$$\sigma_{A'}(x; \xi_1, \ldots, \xi_p, 0, \ldots, 0) = \sigma_{B'}(x_1, \ldots, x_p; \xi_1, \ldots, \xi_p). \tag{22}$$

It may also happen that $B$ above is elliptic on $G/K$, but its extension $A$ is not elliptic on $G$; this is discussed especially in the case $G/K \cong \mathbb{S}^2$, and we make other remarks on the subject.

Hence paper [IV] provides some methods for solving pseudodifferential equations on compact homogeneous spaces $G/K$, especially for a torus subgroup $K$.

In [27] Svensson dealt with pseudodifferential operators on the sphere $\mathbb{S}^2$: Let $f \in \mathcal{D}(\mathbb{S}^2)$ and $B \in \mathcal{L}(\mathcal{D}(G))$ such that

$$(Bf)(\phi, \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b(l)\ \hat{f}(l)_m\ Y_l^m(\phi, \theta) \tag{23}$$

where $\{Y_l^m \mid l, m\}$ is the $L^2(\mathbb{S}^2)$-orthogonal basis of the spherical harmonic functions and $\hat{f}(l)_m$ is a spherical Fourier coefficient (see paper [IV]). Assume that $l \mapsto b(l)$ is a rational function. In this case Svensson proved that $B \in \Psi^r(\mathbb{S}^2)$ if and only if $b(l) = \mathcal{O}(l^r)$; the class of such operators contains many interesting operators originating from geophysics ([27]), but the class is quite restricted as it is not diffeomorphism-invariant. Our results in paper [IV] imply that actually every $B \in \Psi^m(\mathbb{S}^2)$ can be presented in the form

$$(Bf)(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \sum_{n=-l}^{l} b(x, l)_{mn}\ \hat{f}(l)_n \right) Y_l^m(x) \qquad (x \in \mathbb{S}^2), \tag{24}$$

where $x \mapsto b(x, l)_{mn}$ belongs to $C^\infty(\mathbb{S}^2)$ such that there exists $A \in \Psi^m(\mathrm{SU}(2))$ with the symbol $\sigma_A : \mathrm{SU}(2) \to \mathcal{L}(\mathcal{D}(\mathrm{SU}(2)))$ having a matrix presentation

$$\sigma_A(x) \sim \bigoplus_{l=0}^{\infty} c_l\ \left(b(x, l)_{mn}\right)_{m,n=-l}^{l} \qquad (x \in \mathrm{SU}(2)) \tag{25}$$

for some constants $c_l$ depending only on $l \in \mathbb{N}_0$.

## 2.5 Applications to boundary integral equations

Aspects of numerical analysis and computation are not treated in papers [I-IV], but let us have a word on potential applications. Pseudodifferential calculus plays an important role in solving boundary integral equations suggested by many problems in physics, arising from, e.g., electro-magnetics and mechanics (as in the case of geophysics in [27]).

Integral equations on closed curves (diffeomorphic to $\mathbb{T}^1$) and on open arcs have been studied profoundly (see [21] and [20]). Sometimes equations on $\mathbb{R}^n$ can be treated as periodic, i.e. equations on $\mathbb{T}^n$, as for instance in [34]. Usually the integral equations from applications are of classical type ([35]), but also general pseudodifferential equations can be studied easily. Efficiency in numerical computations on $\mathbb{T}^n$ is essentially due to the *fast Fourier transform* (FFT).

So far, little is known of computationally efficient numerical solving of pseudodifferential equations on compact spaces other than the tori. In recent years, FFT methods for compact Lie groups and homogeneous spaces have been introduced ([16]). Efficient Fourier transforms have been studied especially in the case of the sphere $\mathbb{S}^2$ (see [10], [13]). From the application point of view, it will be an important task to provide computational algorithms for solving pseudodifferential equations particularly on $\mathbb{S}^2$, but this is outside the scope of the thesis.

## 2.6 Related research on operator-valued mappings

In Beals–Cordes -type characterizations of pseudodifferential operators on homogeneous spaces one examines an operator-valued mapping obtained by conjugating an operator with a unitary representation of a symmetry group ([6], [9]).

Let $M = G/K$ be a compact smooth homogeneous space and let $\pi : G \to \mathcal{L}(L^2(M))$ be the regular representation of $G$ on $L^2(M)$,

$$(\pi(g)f)(x) = f(g^{-1}x)$$

for almost every $x \in M$. Given $A \in \mathcal{L}(L^2(M))$, one studies the operator-valued mapping $g \mapsto A_g = \pi(g)A\pi(g)^{-1}$, $G \to \mathcal{L}(L^2(G))$. The smoothness of this mapping is not enough to guarantee that $A \in \Psi^0(M)$ ([18]), and $A$ with smooth $g \mapsto A_g$ might even be non-pseudolocal. Enlarging the group of symmetries may help: In that way pseudodifferential operators on $\mathbb{R}$ were characterized by H. O. Cordes in [9]. In [29] M. E. Taylor described $\Psi^0(\mathbb{S}^n)$ by the smoothness of the mapping $g \mapsto A(g) = U(g)AU(g)^{-1}$, where $U : \mathrm{SO}_e(n+1,1) \to \mathcal{L}(L^2(\mathbb{S}^n))$ is the representation $(U(g)f)(x) = f(g^{-1}x)$ of the conformal group.

Nevertheless, we have not treated operator-valued functions of Beals–Cordes -type in the thesis, as they do not help us in constructing a symbolic calculus for pseudodifferential operators. It should also be noted that the structure of the operators $A_g$ and $A(g)$ is essentially as complicated as the structure of $A \in \mathcal{L}(\mathcal{D}(G))$, whereas for the symbol $\sigma_A : G \to \mathcal{L}(\mathcal{D}(G))$ the evaluation $\sigma_A(x)$ is a left convolution operator for every $x \in G$.

# 3 Summary

In this thesis we study Hörmander $(1,0)$-type pseudodifferential calculus on an orientable homogeneous space $G/K$, where $G$ is a compact Lie group and $K$ its closed subgroup. In a joint paper [I] with G. Vainikko we start with the case $G = G/K = \mathbb{T}^n$, and develop symbolic calculus there. The rest of the papers are written solely by the author of the thesis. In article [II] we present a commutator characterization of pseudodifferential operators on compact smooth orientable manifolds without boundary; a derivation of the symbol inequalities on a torus from commutators is a special example. In paper [III] this commutator characterization is modified to construct symbol inequalities for pseudodifferential operators on compact Lie groups, and we present symbolic calculus there building on earlier results [30] by M. E. Taylor. Finally, in paper [IV] the pseudodifferential operators are studied

on a compact smooth orientable homogeneous space $G/K$. We calculate the average of a pseudodifferential operator $A$ on $G$ along classes $xK$ ($x \in G$) to obtain a pseudodifferential operator $A_{G/K}$ on $G$ such that the restriction $A_{G/K}|_{\mathcal{D}(G/K)}$ is a pseudodifferential operator on $G/K$. If $K$ is a torus subgroup, we show that every pseudodifferential operator on $G/K$ is obtained by this averaging process, as we extend or lift operators from $G/K$ to $G$.

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ISBN 951-22-5654-1
ISSN 0784-3143
Otamedia Oy, Espoo, 2001