Helsinki University of Technology Institute of Mathematics Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2001 A441

DATA REDUCTION AND DOMAIN TRUNCATION IN ELECTROMAGNETIC OBSTACLE SCATTERING

Jukka Liukkonen

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

Helsinki University of Technology Institute of Mathematics Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2001 A441

DATA REDUCTION AND DOMAIN TRUNCATION IN ELECTROMAGNETIC OBSTACLE SCATTERING

Jukka Liukkonen

Dissertation for the Degree of Doctor of Philosophy to be presented with due permission for public examination and debate in Auditorium G at Helsinki University of Technology (Otakaari 1 M, Espoo, Finland) on the 19th of December, 2001, at 12 o'clock noon.

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

Jukka Liukkonen: *Data Reduction and Domain Truncation in Electromagnetic Obstacle Scattering*; Helsinki University of Technology Institute of Mathematics Research Reports A441 (2001).

Abstract: *The time harmonic electromagnetic obstacle scattering is considered. This thesis has a twofold purpose: both direct and inverse problems are studied keeping in mind their practical applications. Natural questions that arise are those of domain truncation and data reduction.*

When solving direct problems using computers it is customary to truncate the space into a bounded computational domain and then require an absorbing boundary condition at the exterior boundary. One approach is to surround the domain by a non-reflecting layer of imaginary material. The so called Perfectly Matched Layer (PML) is here regarded as complex stretching of the metric tensor. A rigorous theory in a quite general setting is developed. The existence and uniqueness of a solution to the truncated boundary value problem is proved.

Another goal is to show that an inverse problem has a unique solution provided that the measurements have been done in an arbitrarily small open neighbourhood lying on a surface outside the obstacle.

AMS subject classifications: 35Q60, 35R30, 53B30, 78A25

Keywords: electromagnetics, inversion, perfectly matched layer, scattering

Jukka.Liukkonen@hut.fi

ISBN 951-22-5744-0
ISSN 0784-3143
Espoo, 2001

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

Acknowledgments

The author is grateful to Professor Erkki Somersalo for suggesting these problems and for his guidance during the work. I would also like to thank Doctor Matti Lassas and Doctor Kirsi Peltonen for valuable discussions and Geophysicist Matti Oksama for teaching me the fundamentals of electromagnetics. Furthermore, I thank Doctor Petri Ola and Professor Eero Saksman for reviewing the manuscript and for valuable suggestions.

I would like to thank the Institute of Mathematics at Helsinki University of Technology for an inspiring atmosphere. I also wish to express my gratitude to my actual employer Geological Survey of Finland for support and understanding during the initial phase of my doctoral studies.

The thesis is based on my work at the Institute of Mathematics at Helsinki University of Technology as a participant of the Graduate School of Applied Electromagnetics coordinated by the Laboratory of Electromagnetics at Tampere University of Technology. Hence, the study was partially financially supported by the Academy of Finland.

Espoo, September 2001

Jukka Liukkonen

Publications

The thesis consists of this introductory part and of the following papers:

A. Jukka Liukkonen: Uniqueness of electromagnetic inversion by local surface measurements. Inverse Problems **15** (1999) 265–280.

B. Jukka Liukkonen: Generalized electromagnetic scattering in a complex geometry. J. Math. Anal. Appl. **254** (2001) 498–514.

C. Matti Lassas, Jukka Liukkonen and Erkki Somersalo: Complex Riemannian metric and absorbing boundary conditions. J. Math. Pures Appl. **80** (2001) 739–768.

Papers A and B are single authored. As far as paper C is concerned, in addition to being responsible for Appendix 1, the author's thankless task was to walk through those paths that lead to nowhere. Some basic ideas most important of which is the observation that the PML metric has curvature zero are the author's.

1 Introduction

The problem of finding a solution to a differential equation given certain boundary values is called a *direct problem*. On the contrary, if one knows the solution and the differential equation except the coefficient functions one has encountered an *inverse problem*.

In practice, direct problems arise when a known source like a mobile phone has been located in a known material distribution like a city and one has to compute the field caused by that source. One could, of course, measure the field. Unfortunately, measurements are usually difficult to carry out, time consuming and expensive.

Maybe the most famous inverse problems are those of geophysical exploration, mechanical engineering and medical imaging: one has to find out, *e.g.*, whether there are remarkable ore deposits hiding under the surface of the Earth, fatal cracks in the body of an aircraft or tumours inside a human head. The sounding is done by transmitters that generate fields and receivers that measure the fields. Hence, one knows a set of source terms and the corresponding solutions of an appropriate wave equation and tries to compute the coefficients representing the material.

This work treats the electromagnetic scattering caused by an obstacle. The goal is to reduce wide or unbounded measurement areas and computational domains to small or bounded ones so as to make the measurements and computations possible. It is proven that inversion yields unique material parameters even if the measurements have been carried out on a tiny piece of a surface exterior to the scatterer. On the other hand, it is shown that the solution of a whole space direct scattering problem can be approximated quite accurately by the solution of a corresponding bounded boundary value problem truncated to a neighbourhood of the scatterer. Complex stretching of the metric tensor outside the computational domain is employed to attenuate reflections due to the boundary.

1.1 Structure of the Thesis

The bunch consists of three articles and this introductory part.

Paper A concentrates on inverse problems. Two uniqueness theorems are proven. The fields are generated by tangential dipoles and tangential components are measured. It is shown that if we know every tangential source-field pair on an open piece of a surface exterior to the scatterer then we know the distributions of electric permittivity, conductivity and magnetic permeability inside the scatterer. The second result is similar except that the data consists of the admittance map restricted to an open set of a plane.

Paper B deals with the whole space direct problem on a real manifold of an arbitrary dimension. The aim is, however, at domain truncation using the *perfectly matched layer (PML)* technique. In the article a rigorous generalization of the PML concept is developed. It is also shown that electromagnetic fields in this quite general PML can be mastered by tools that resemble the conventional machinery of electromagnetics. Changing the metric outside the scatterer is proven to have no effect on the solvability and the uniqueness of a solution of a scattering problem. This result is finally applied in Euclidean spaces.

Paper C returns to a three dimensional space — although without further loss of generality — and makes a close study of the domain truncation by a generalized PML. In everyday life, field calculations are carried out in a bounded *computational domain* that surrounds the scatterer. So as to prevent spurious reflections the computational domain is surrounded by a so called *absorbing layer* or PML. One can think that it is constructed by complex stretching of the metric tensor. Requiring, *e.g.*, the perfectly conducting boundary condition at the exterior boundary of the absorbing layer one obtains a truncated boundary value problem. The main result states that the truncated problem has a unique solution and it converges exponentially to the whole space solution in the computational domain as the absorbing layer gets thicker. Throughout paper C a coordinate invariant representation is used.

In Section 2 of the introduction there are brief overviews of direct and inverse problems associated with electromagnetic obstacle scattering. Section 3 reviews the perfectly matched layer and describes the differential geometric approach. A few examples of other absorbing boundary conditions are given. Section 4 concentrates on inverse scattering laying emphasis on the uniqueness questions. The scalar Schrödinger equation is worked in as an example. Some previous results are reviewed.

2 Electromagnetic Obstacle Scattering

The real space contains billions of stars and galaxies not to speak about the interstellar dust. However, it usually is sufficient to forget the galaxies and model obstacle scattering by considering a bounded body in an empty or homogeneous space. A theoretical experiment could be organized as follows: figure a closed surface surrounding the scatterer, or alternatively, the surface of a half space containing the obstacle. Then, move a transmitter along the surface and, corresponding to each location of the transmitter make a *boundary measurement*, *i.e.*, measure the scattered field at a sufficiently dense array of points on the surface. The direct problem is: compute the scattered field at every point outside the boundary. A boundary measurement gives one boundary condition. To fix the solution uniquely we need another boundary condition at the infinity; it is called the *radiation condition.* The inverse problem is: what are the distributions of the material parameters inside the body. Before starting to make the inversion by a computer one has to know whether the model has a unique solution.

2.1 Maxwell's Equations

The *time-harmonic Maxwell's equations* represented in the classic vector formalism are

$$\begin{aligned} \nabla \times \vec{E}(x) - \mathrm{i}\omega\mu(x)\vec{H}(x) &= \vec{M}(x), & (1)\\ \nabla \times \vec{H}(x) + \mathrm{i}\omega\gamma(x)\vec{E}(x) &= \vec{J}(x). & (2) \end{aligned}$$

The *electromagnetic field* $(\vec{E}(x), \vec{H}(x))$ consists of the *electric field* $\vec{E}(x)$ and the *magnetic field* $\vec{H}(x)$. The source terms $\vec{M}(x)$ and $\vec{J}(x)$ are the *magnetic source* and the *electric source*, respectively. All of these four vector fields are functions $\mathbb{R}^3 \to \mathbb{C}^3$ or rather distributions. The scalar valued coefficients are the *magnetic permeability* $\mu(x) > 0$ and the *complex permittivity* $\gamma(x) \in \mathbb{C}$. The latter is an artificial quantity composed of the *electric permittivity* $\varepsilon(x) > 0$ and the *electric conductivity* $\sigma(x) \geq 0$ according to the formula $\gamma(x) = \varepsilon(x) + \mathrm{i}\sigma(x)/\omega$. Every quantity except the imaginary unit i and the *angular frequency* $\omega > 0$ is a function of $x \in \mathbb{R}^3$. In paper B Maxwell's equations are generalized to n-dimensional manifolds using differential geometric formalism.

2.2 Direct Problems

Assume that there are constants $\varepsilon_0 > 0$ and $\mu_0 > 0$ such that the *scatterer*

$$W_{\mu,\gamma} := \operatorname{supp}(\mu - \mu_0) \cup \operatorname{supp}(\gamma - \varepsilon_0)$$

is a compact subset of $\mathbb{R}^3$. It is also called the *obstacle* or the *inhomogeneity*. Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain with a smooth boundary. Suppose that $W_{\mu,\gamma} \subset \Omega$. The *whole space scattering problem* is to find $\vec{E}$ and $\vec{H}$ in $\mathbb{R}^3 \setminus \overline{\Omega}$ such that

$$\begin{aligned} \nabla \times \vec{E} - \mathrm{i}\omega\mu\vec{H} &= \vec{0} \text{ in } \mathbb{R}^3 \setminus \overline{\Omega}, & (3)\\ \nabla \times \vec{H} + \mathrm{i}\omega\gamma\vec{E} &= \vec{0} \text{ in } \mathbb{R}^3 \setminus \overline{\Omega}, & (4)\\ \vec{n} \times \vec{E}|_{\partial\Omega} &= \vec{f}, & (5)\\ |\hat{x} \times \vec{E}(x) - \eta\vec{H}(x)| &\leq Cr^{-2} \text{ as } r \to \infty. & (6) \end{aligned}$$

Here $\vec{n}$ is the exterior unit normal of $\partial\Omega$, $\vec{f} : \partial\Omega \to \mathbb{C}^3$ is a given function, $r := |x|$, $\hat{x} := x/r$ and C is a constant independent of $x \in \mathbb{R}^3$. The coefficient $\eta := (\mu/\varepsilon)^{1/2}$ is called the *wave impedance*. The *Silver-Müller radiation condition* (6) guarantees that (3)–(6) has a unique solution $(\vec{E}, \vec{H})$ (see [15]).

The whole space problem cannot be solved using a computer since $\mathbb{R}^3 \setminus \overline{\Omega}$ is not bounded. For example, the element method would require an infinite number of elements but a computer has only a finite storage capacity. Therefore, and since the field far away from the scatterer is usually not interesting, engineers truncate the space into a bounded domain D that contains $\overline{\Omega}$ and try to solve the *truncated scattering problem* in $D \setminus \overline{\Omega}$. Unfortunately, unless we know both $\vec{n} \times \vec{E}$ and $\vec{n} \times \vec{H}$ at ∂D we cannot be sure of the uniqueness of the solution provided that $\sigma = 0$ in $D \setminus \overline{\Omega}$ (see [15]). A really intricate problem is that the boundary of D reflects the waves back towards Ω and this way spoils the solution, that is to say, the solution does not anymore resemble that of the whole space problem. To prevent reflections various *absorbing boundary conditions (ABC)* have been introduced in the literature (see [10] or [44]).

2.3 Inverse Problems

It frequently happens that an inverse problem has many solutions. Moreover, although a unique solution exists it, typically, is unstable with respect to measurement errors. This ill-posedness is often due to nonlinearity: the coefficients of even a linear partial differential equation depend nonlinearly from the solution. Before tackling an inverse problem in practice one has to choose an adequate model and, at the same time, check whether the corresponding direct problem has a unique solution. After that he should assure himself of the existence and uniqueness of the solution to the inverse problem within the model. The existence question is, in certain sense, solved by performing the measurement but the difference between the model and reality should be noted. Paper A concentrates on the uniqueness of electromagnetic inversion.

Assume that the inhomogeneity lies in Ω. An *electromagnetic inverse scattering problem* can be formulated as follows:

IP1 For each magnetic dipole $\vec{v}$ at $\partial\Omega$ tangential to $\partial\Omega$ measure the tangential component $(\vec{n} \times \vec{E}, \vec{n} \times \vec{H})$ at $\partial\Omega$ of the electromagnetic field generated by the dipole. Determine ε, μ and σ in Ω.

Another formulation is:

IP2 Determine ε, μ and σ in Ω provided that the *admittance map*

$$Y : \vec{n} \times \vec{E}|_{\partial\Omega} \mapsto \vec{n} \times \vec{H}|_{\partial\Omega}$$

is known.

If the obstacle is huge like the Earth it is not possible to arrange a measurement that surrounds the whole scatterer. For this reason, one is obliged to restrict the measurements to a local area and mathematician's job is to find out whether the material parameters can, even in principle, be derived from this reduced data.

3 Battle against Reflections

When computing waves in truncated domains artificial boundaries that have no physical counterparts give rise to extraneous reflections. In order to moderate these false corrugations it is necessary to impose absorbing boundary conditions at the nonphysical boundary. The requirement of a perfectly absorbing boundary leads to nonlocal pseudodifferential operators (see [18]) that need extravagant use of resources in practical computations. Hence they have to be approximated by local operators. Unfortunately, the latter ones only yield more or less highly — not perfectly — absorbing boundaries.

It is useful to have a closer look at one of the local approximations. The presentation follows that of [10]. Fourier analysis shows that an arbitrary solution to the scalar wave equation

$$\Delta u - \frac{1}{c^2}\partial_t u = 0$$

in $\mathbb{R}^3$ can be composed of plane waves

$$u(x,t) = Ae^{-i(\omega t - k\cdot x)}.$$

Hence, a plane wave is the basic object to construct an ABC for. The above u propagates to the direction $k/|k|$ at the velocity $c := \omega/|k|$. Suppose $k_1 < 0$ and consider the scattering of u from a wall at $x_1 = 0$. From $|k| = \omega/c$ it follows that

$$k_1 = -\frac{\omega}{c}\sqrt{1 - \frac{c^2}{\omega^2}(k_2^2 + k_3^2)}.$$

The partial derivatives of u are

$$\partial_j u = \mathrm{i}k_j u, \quad j = 1,2,3, \tag{7}$$

$$\partial_t u = -\mathrm{i}\omega u, \tag{8}$$

and particularly

$$\mathrm{i}\omega\partial_1 u + \omega k_1 u = 0. \tag{9}$$

When the wave is close to normal incidence

$$\frac{c^2}{\omega^2}(k_2^2 + k_3^2) = \frac{k_2^2 + k_3^2}{|k|^2} \ll 1.$$

From the Taylor approximation $\sqrt{1-\zeta} \approx 1 - \frac{1}{2}\zeta$ one obtains the so called *paraxial approximation*

$$k_1 \approx -\frac{\omega}{c}\left(1 - \frac{1}{2}\frac{c^2}{\omega^2}(k_2^2 + k_3^2)\right).$$

In accordance with (9) then

$$\mathrm{i}\omega\partial_1 u - \frac{\omega^2}{c}u + \frac{1}{2}c(k_2^2 + k_3^2)u \approx 0. \tag{10}$$

On substituting (7)–(8) into (10), one has

$$\partial_1\partial_t u + \frac{1}{c}\partial_t^2 u - \frac{1}{2}c(\partial_2^2 + \partial_3^2)u \approx 0.$$

Thus one of the famous *Engquist-Majda ABC's* (see [18]) has been derived:

$$\left(\partial_1\partial_t + \frac{1}{c}\partial_t^2 - \frac{1}{2}c(\partial_2^2 + \partial_3^2)\right) u \Bigg|_{x_1=0} = 0.$$

It should be noted that for normally incident plane waves the wall is perfectly absorbing.

Since $k_1 = \omega c^{-1}\cos\theta$ one can write (9) in the form

$$\frac{1}{\cos\theta}\,\partial_1 u - \mathrm{i}\omega c^{-1} u = 0.$$

Determination of α_m and β_m, $m = 1, \ldots, M$, such that

$$\frac{1}{\cos\theta} \approx 1 + \sum_{m=1}^{M} \frac{\alpha_m \sin^2\theta}{1 - \beta_m \sin^2\theta}$$

leads to *Lindman ABC's*. They can be adjusted to absorb waves at oblique incidence quite efficiently (see [10] for details).

Bayliss-Turkel ABC assumes that the wave has the form

$$u(r,\theta,\phi,t) = \sum_{j=1}^{\infty} \frac{f_j(ct - r, \theta, \phi)}{r^j}$$

in spherical coordinates. This is a general expansion of radiating solutions u to the wave equation. Bayliss and Turkel (see [3]) define recursively a sequence (B_m) of differential operators such that $B_m u = \mathcal{O}(1/r^{2m+1})$. A family of boundary conditions $B_m u = 0$ results.

A curiosity among domain truncation methods that deserves to be mentioned is the *measured equation of invariance (MEI)* by Mei *et al.* in 1994 (see [34]). This method permits the finite difference mesh to be terminated very close to the physical boundary that may be concave. In [26] Lassas *et al.* introduce a similar *double surface radiation condition (DSRC)* method which "can be viewed as a justification and theoretical background for the MEI method". Such a double surface near-field radiation condition is also encountered in [29] and in paper C.

As the above discussion may reveal absorbing boundary conditions are rather complicated to implement particularly in general geometries and for an arbitrary incidence. Fortunately, this is not the case with PML.

3.1 An Introductory Example

Consider the scattering of a one dimensional scalar wave

$$V(x,t) = v(x)e^{-\mathrm{i}\omega t}$$

from a wall at $x = 0$:

$$\begin{aligned} v''(x) + k^2 v(x) &= 0, \quad x > 0, && (11) \\ v(0) &= v_0. && (12) \end{aligned}$$

The general solution to (11) is $v(x) = v_-(x) + v_+(x)$ where $v_-(x) := B_- e^{-\mathrm{i}kx}$ and $v_+(x) := B_+ e^{\mathrm{i}kx}$ propagate to the left and to the right, respectively. Assume that the transmitted signal v_- is known and, based on the measurement (12), one has to compute the scattered wave v_+ in the vicinity of the wall. The analytical solution is obviously $v_+(x) = u_0 e^{\mathrm{i}kx}$ with $u_0 := v_0 - v_-(0)$. Nevertheless, to illustrate how PML works it is quite elucidating to have a detailed look at the one dimensional case.

The scattered wave $u := v_+$ satisfies

$$\begin{aligned} u''(x) + k^2 u(x) &= 0, \quad x > 0, && (13) \\ u(0) &= u_0. && (14) \end{aligned}$$

The aim is to determine u from (13)–(14) in a computational domain $]0, x_0[$, $x_0 > 0$. So as to make sure of the uniqueness of the solution another boundary value $u(R) = u_R$ has to be fixed, say, at $R \in \,]x_0, \infty[$. For the sake of simplicity assume that

$$u(R) = 0. \tag{15}$$

A general solution to (13) is

$$u(x) = A_- e^{-\mathrm{i}kx} + A_+ e^{\mathrm{i}kx}. \tag{16}$$

The conditions (14) and (15) yield $u_0 = A_-(1 - e^{-2\mathrm{i}kR})$. If $kR \notin \pi\mathbb{Z}$ it follows

$$A_- = \frac{u_0}{1 - e^{-2\mathrm{i}kR}}, \quad A_+ = \frac{u_0}{1 - e^{2\mathrm{i}kR}}. \tag{17}$$

If $kR \in \pi\mathbb{Z}$ one has $e^{\mathrm{i}kR} = e^{-\mathrm{i}kR} \neq 0$. The boundary conditions imply that $u_0 = 0$. Hence there are three possible cases:

1° If $kR \notin \pi\mathbb{Z}$ there exists a unique solution to (13)–(15) given by (16)–(17);

2° If $kR \in \pi\mathbb{Z}$ and $u_0 \neq 0$ then (13)–(15) has no solution;

3° If $kR \in \pi\mathbb{Z}$ and $u_0 = 0$ then (13)–(15) has infinitely many solutions (16) given by $A_- = -A_+$.

Since $k = 2\pi\lambda^{-1}$ and $\omega = 2\pi T^{-1}$ where λ is the wavelength and T is the period of the oscillation the phase velocity c has the expression $c = \lambda T^{-1} = \omega k^{-1}$ and thus $k = \omega c^{-1}$. The relation $kR = \pi n \in \pi\mathbb{Z}$ is equivalent to $\omega = cn\pi R^{-1}$, $n \in \mathbb{Z}$. These are the resonance (angular) frequencies.

The problem is that 2° and 3° are possible due to the truncation at R and if $u_0 \neq 0$ in 1° the nonvanishing amplitude A_- of the wave component reflected by the artificial boundary at R spoils the solution.

The real analytic function $u(x)$ in (16)* can be uniquely continued to a complex analytic function

$$u(z) = A_- e^{-\mathrm{i}kz} + A_+ e^{\mathrm{i}kz}$$

onto the closed half plane $\overline{\mathbb{C}}^+ = \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0\}$. As such $u(z)$ is the general solution to the Helmholtz equation

$$u''(z) + k^2 u(z) = 0$$

in the open half plane $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$. Here $u''(z)$ is the second complex derivative of the analytic function $u(z)$. Let $\tilde{R} = a + \mathrm{i}b \in \mathbb{C}^+$, $b > 0$. Consider the boundary value problem

$$\begin{aligned} u''(z) + k^2 u(z) &= 0, \quad z \in \mathbb{C}^+, && (18)\\ u(0) &= u_0, && (19)\\ u(\tilde{R}) &= 0. && (20) \end{aligned}$$

Since $k\tilde{R} \notin \pi\mathbb{Z}$ there exists a unique solution

$$\begin{aligned} u(z) &= A_- e^{-\mathrm{i}kz} + A_+ e^{\mathrm{i}kz}, \\ A_- &= \frac{u_0}{1 - e^{-2\mathrm{i}k\tilde{R}}} = \frac{u_0}{1 - e^{-2\mathrm{i}ka} e^{2kb}}, \\ A_+ &= \frac{u_0}{1 - e^{2\mathrm{i}k\tilde{R}}} = \frac{u_0}{1 - e^{2\mathrm{i}ka} e^{-2kb}}. \end{aligned}$$

Note that

$$\lim_{b\to\infty} A_+ = u_0, \quad \lim_{b\to\infty} A_- = 0.$$

It is obvious that instead of solving (13)–(15) one should rather search for the solution to (18)–(20) for sufficiently large $\operatorname{Im} \tilde{R} > 0$. The latter problem always is uniquely solvable and the solution restricted to $]0, x_0[$ tends to the "physical" scattering solution exponentially along with $b = \operatorname{Im} \tilde{R}$ since

$$\frac{|A_-|}{|A_+|} = e^{-2kb} \to 0 \quad \text{as} \quad b \to \infty.$$

This is the great idea of PML.

*The formula is actually analogous to the Stratton-Chu formula (see [15]).

In practice, PML is constructed using a smooth stretching function

$$[0,\infty[\to\mathbb{C},\quad x\mapsto\tilde{x}=x+sa(x).$$

Here $s\in\mathbb{C}^+$, $\operatorname{Im}s>0$, and $a:[0,\infty[\to[0,\infty[$ is an increasing smooth function such that

$$a|_{[0,x_0]}=0,\quad \lim_{x\to\infty}a(x)=\infty.$$

The stretching induces a complex valued metric $g:=(\mathrm{d}\tilde{x}/\mathrm{d}x)^2$ to the real manifold $]0,\infty[$. The one dimensional Laplacian $\mathrm{d}^2/\mathrm{d}\tilde{x}^2$ is, in accordance with the chain rule, at least formally equivalent to the covariant Laplace operator

$$\tilde{\Delta}=\frac{1}{\sqrt{g}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{g}\,\frac{1}{g}\frac{\mathrm{d}}{\mathrm{d}x}\right)=\frac{\mathrm{d}x}{\mathrm{d}\tilde{x}}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}x}{\mathrm{d}\tilde{x}}\frac{\mathrm{d}}{\mathrm{d}x}\right)=\frac{\mathrm{d}^2}{\mathrm{d}\tilde{x}^2}.$$

Instead of the original boundary value problem one solves the following PML boundary value problem:

$$\begin{aligned}\tilde{\Delta}u(x)+k^2u(x)&=0,\quad x\in]0,R[,\\ u(0)&=u_0,\\ u(R)&=0.\end{aligned}$$

Restricted to $]0,x_0[$ the covariant Laplacian equals the ordinary Laplacian which in this one dimensional case is the second derivative. As the thickness $R-x_0$ of the perfectly matched layer increases the solution tends to the desired scattering solution exponentially in the computational domain $]0,x_0[$ provided that $a'(x)\geq C$ for some positive constant C as x tends to ∞.

3.2 Perfectly Matched Layer as an Absorbing Boundary Condition

Let Ω, B and D be sufficiently regular bounded open domains in $\mathbb{R}^3$ such that the scatterer is included in Ω and $\overline{\Omega}\subset D\subset\overline{D}\subset B$. In what follows $D\setminus\overline{\Omega}$ stands for the computational domain and $B\setminus\overline{D}$ is the absorbing layer. If the radiation condition (6) is replaced by

$$\vec{n}\times\vec{E}|_{\partial B}=\vec{0}$$

the solution of the boundary value problem in Section 2.2 will be contaminated by spurious reflections due to the artificial boundary ∂B. In order to prevent the contamination one could define some mathematical conditions that correspond to a kind of sponge absorbing material filling up the zone $B\setminus\overline{D}$. The perfectly matched layer technique offers means for that purpose. It should be emphasized that PML can be used in connection with various wave phenomena — not just with electromagnetic waves.

The whole story began in 1994 when Bérenger introduced PML for Cartesian coordinates and planar interfaces in the article [4]. He was computing electromagnetic waves using the finite-difference time-domain (FDTD) method. Later in the same year Katz *et al.* (see [22]) constructed a three dimensional PML for FDTD. Still in 1994, Chew and Weedon (see [11]) observed that in the Fourier domain PML can be derived by complex coordinate stretching. During 1996–1997 PML was generalized to cylindrical and spherical coordinates by several authors and in 1998 to a general orthogonal curvilinear coordinate system by Teixeira and Chew (see [50]). Finally, in 1999 Teixeira and Chew gave a unified differential geometric description of PML by means of differential forms (see [51]). The absorbing layer is constructed by changing the metric tensor. In the so called *Maxwellian formulation* Maxwell's equations are preserved everywhere: only the constitutive relations change along with the Hodge-$*$ operators.

The above referenced articles concentrate on the construction of absorbing layers in different geometries. There still remained the questions of existence and uniqueness of the solutions to the adequate boundary value problems in the presence of the absorbing material. Moreover, the convergence and stability of solutions had to be considered. Inspired by the work of Collino and Monk (see [14]) Lassas and Somersalo gave in 1998 an answer for the scalar Helmholtz equation in a two dimensional cylindrical geometry (see [27]). Soon they continued with a pioneering work [29] handling the case of a general convex geometry in an arbitrary dimension. Although dealing with Maxwell's equations in three dimensions paper C is the next milestone on this road to a *complexified scattering theory* as the scenario is named in the introduction of [29].

In Section 3.1 the absorbing layer is constructed using coordinate stretching in one dimension. The method is readily generalized to three dimensions when D is a strictly convex domain with a smooth boundary. For $x \in \mathbb{R}^3 \setminus \overline{D}$ let $p(x) \in \partial D$ be the unique point such that the distance between x and ∂D equals

$$h(x) := |x - p(x)| > 0.$$

Then x has a unique representation

$$x = p(x) + h(x)n(x)$$

where $n(x) := (x - p(x))/|x - p(x)|$ is the exterior unit normal vector of ∂D at $p(x)$. For $x \in \partial D$ define $p(x) = x$ and $h(x) = 0$. Let $\tau : \mathbb{R} \to \mathbb{R}$ be an increasing smooth function such that $\tau|_{]-\infty,0]} = 0$. For each

$$s \in \overline{\mathbb{C}}^{++} := \{z \in \mathbb{C} \mid \operatorname{Re} z \geq 0,\ \operatorname{Im} z \geq 0\}$$

define

$$F_s : \mathbb{R}^3 \to \mathbb{C}^3, \quad F_s(x) = x + sa(x),$$

where

$$a : \mathbb{R}^3 \to \mathbb{R}^3, \quad a(x) = \begin{cases} 0, & x \in \overline{D}, \\ \tau(h(x))n(x) & x \in \mathbb{R}^3 \setminus \overline{D}. \end{cases}$$

The mapping F_s is called the *stretching function* and it is analytic with respect to s. Note that $F_s(x) = x$ if $x \in \overline{D}$ or $s = 0$. As in the one dimensional case the stretching $\tilde{x} := F_s(x)$ induces a complex valued pseudo-Riemannian metric

$$g_{jl} := \sum_{m=1}^{3} \frac{\partial \tilde{x}^m}{\partial x^j} \frac{\partial \tilde{x}^m}{\partial x^l}, \quad j, l = 1, 2, 3,$$

to $\mathbb{R}^3$. This process can be regarded as complex stretching of the metric tensor. At first sight it may seem unbelievable that the curvature of the stretched manifold vanishes. Indeed, the components of the Riemannian curvature tensor are analytic functions with respect to the stretching parameter s. Since they vanish for $s \in [0, \infty[$ they have to vanish for all $s \in \overline{\mathbb{C}}^{++}$.

In accordance with the Stratton-Chu representation (see [15]) the electromagnetic field $(\vec{E}(x), \vec{H}(x))$ is a real analytic function of $x \in \mathbb{R}^3$. Following the argument in [29] one is able to show that the field can be analytically continued to a unique field $(\vec{E}(z), \vec{H}(z))$ that is defined in an open neighbourhood of the manifold $F_s(\mathbb{R}^3) \setminus \overline{\Omega}$ and in which the complex Maxwell's equations

$$\begin{aligned} \nabla_z \times \vec{E}(z) - \mathrm{i}\omega\mu_0 \vec{H}(z) &= \vec{0}, \\ \nabla_z \times \vec{H}(z) + \mathrm{i}\omega\varepsilon_0 \vec{E}(z) &= \vec{0}, \end{aligned}$$

are satisfied. The "pullback" of $(\vec{E}(z), \vec{H}(z))$ with respect to the stretching function obeys certain covariant Maxwell's equations in the pseudo-Riemannian manifold $(\mathbb{R}^3, (g_{jl}))$. Written in the language of differential forms the covariant equations have the appearance of the ordinary Maxwell's equations: the change is hiding in the Hodge-$*$ operator.

Papers B and C together with [29] reveal that by properly adjusting the stretching function one can efficiently eliminate the reflections caused by the truncation at ∂B. In other words, the solution of the covariant boundary value problem tends to the physical scattering solution exponentially along with the increasing thickness of the absorbing layer.

3.3 Differential Geometric Approach

Material or computational boundaries usually consist of curved surfaces also called two dimensional manifolds. If one has to compute, *e.g.*, the magnetic flux across such a surface he should be able to integrate over a manifold. What kind of entities can be integrated over manifolds? The answer is: differential forms. There is a plethora of reasons why use differential forms instead of vectors in electromagnetic theory (see [25], [51], [53], [54]). One important aspect is the purely topological nature of Maxwell's equations.

In the differential topological context Maxwell's equations are written as

$$\mathrm{d}E = \mathrm{i}\omega B,$$

$$\mathrm{d}H = -\mathrm{i}\omega D.$$

Since the *exterior derivative* d is independent of any coordinate system or a metric so are these equations. The field intensities E and H are 1-forms whereas the *magnetic flux density* B and the *electric flux density* D are 2-forms† (see [37] for definitions of differential forms). The metric appears in the constitutive relations

$$\begin{aligned} B &= \mu_0 * H, \\ D &= \varepsilon_0 * E, \end{aligned}$$

as Hodge-$*$ operator that maps 1-forms to 2-forms or, more generally, p-forms to $(n-p)$-forms n being the dimension of the manifold. Hence the coordinate invariant Maxwell's equations are

$$\begin{aligned} \mathrm{d}E &= \mathrm{i}\omega\mu_0 * H, \\ \mathrm{d}H &= -\mathrm{i}\omega\varepsilon_0 * E, \end{aligned}$$

both in the free space and in PML.

In the late 1980s Bossavit gave a detailed description of how the so called Whitney elements can be used in electromagnetic field computations by the finite element method (FEM). These elements of degrees 0–3 can be regarded as discretized differential forms. They conform to the electromagnetic boundary conditions: field or flux continuities and discontinuities across material boundaries are easy to model. In a finite element mesh consisting of tetrahedra elements of degree p are associated with p-simplices. In addition to nodal values, degrees of freedom can be circulations along edges, fluxes across facets or volume integrals over tetrahedra. In [6] Bossavit considers electromagnetic obstacle scattering. As he says FEM works well with complicated geometries near the scatterer contrary to boundary-integral methods that are excellent tools for far-field computations but require smooth and simple surfaces. Recently Järvenpää in his thesis [21] has implemented Whitney elements in a generally shaped PML. He solves numerically a two dimensional electromagnetic scattering problem but the work also contains a discussion how the solver can be extended to three dimensional domains.

As the metric within the absorbing layer is complex valued it is necessary to study real manifolds with complexified tangent and cotangent bundles. This complexification gives more symmetry. As it was seen in Section 3.2 the curvature tensor will vanish also for several manifolds that originally have a nonvanishing curvature. One consequence of the flatness is that it is possible

†The reader should be careful about the notations: it is obvious from the context whether B and D mean forms or domains.

to employ global orthonormal frames and express the fundamental solutions of Maxwell's equations, *i.e.*, dipoles in such frames. Forced by necessity, one drawback has to be accepted: in general the manifold does not contain geodesics. An alternative way to express the state of affairs is that geodesics usually go outside the manifold. This is better understood if the manifold is embedded in some $\mathbb{C}^n$ as in connection with the stretching function. The lack of geodesics makes large chunks of the standard Riemannian geometry useless.

4 Inversion and Data Reduction

A standard reference in papers dealing with electromagnetic inversion is Calderón's work [8] published in 1980. Instead of time varying electromagnetic fields he considered stationary electric currents that are governed by the *conductivity equation*

$$\nabla \cdot (\sigma \nabla u) = 0.$$

It describes the potential u in the absence of sinks and sources of current in a bounded domain Ω with a smooth boundary. According to Ohm's law $\sigma \nabla u$ represents the current flux. If one knows the currents and voltages at the boundary $\partial\Omega$, Calderón asked, is it possible to determine the conductivity $\sigma(x)$ at every point $x \in \Omega$? To be precise, define the so called *voltage-to-current map*, or *Dirichlet-to-Neumann map*, Λ_σ by

$$\Lambda_\sigma f = \left(\sigma \frac{\partial u}{\partial \vec{n}} \right) \Big|_{\partial\Omega}$$

where $u \in H^1(\Omega)$ solves the Dirichlet problem

$$\begin{aligned} \nabla \cdot (\sigma \nabla u) &= 0 \text{ in } \Omega, \\ u|_{\partial\Omega} &= f \in H^{1/2}(\partial\Omega), \end{aligned}$$

and $\vec{n}$ is the exterior unit normal of $\partial\Omega$. Calderón's problem was whether σ is uniquely determined by Λ_σ and, if the answer is affirmative, whether one can calculate σ in terms of Λ_σ. The uniqueness was shown in 1980's by several authors. In 1988 Nachman published a reconstruction method for dimensions $n \geq 3$ (see [35]) and in 1996 for the remaining dimension $n = 2$ (see [36]). Siltanen tested Nachman's method in two dimensions both theoretically and numerically in his thesis [45] in 1999.

4.1 Inversion of the Scalar Schrödinger Equation

To understand the philosophy of inverse problems consider, for simplicity, the scalar Schrödinger equation

$$\Delta\psi(x) + q(x)\psi(x) = 0$$

with zero energy in $\mathbb{R}^n$. It is very closely related to the conductivity equation. This presentation follows the lectures on inversion theory given by Prof. Erkki Somersalo at Helsinki University of Technology in the spring 1999. Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain with a smooth boundary $\partial\Omega$ such that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected. Assume that $q \in C_0(\Omega)$, that is to say, q is a continuous compactly supported function in Ω. For every $f \in H^{1/2}(\partial\Omega)$ the boundary value problem

$$\begin{aligned} \Delta\psi + q\psi &= 0 \text{ in } \Omega, \\ \tau\psi &= f, \end{aligned}$$

is known to have a unique weak solution $\psi \in H^1(\Omega)$:

$$\forall\varphi \in H_0^1(\Omega): \quad \int_\Omega (\nabla\psi \cdot \nabla\varphi + q\psi\varphi)\,\mathrm{d}x = 0.$$

The *trace map* $\tau : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is surjective and it has at least one right inverse $R : H^{1/2}(\partial\Omega) \to H^1(\Omega)$, $\tau R = \mathrm{id}_{H^{1/2}(\partial\Omega)}$. The Dirichlet-to-Neumann map

$$\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

is the unique map with the property that

$$\forall g \in H^{1/2}(\partial\Omega): \quad <\Lambda_q f, g> = \int_\Omega (\nabla\psi \cdot \nabla Rg + q\psi Rg)\,\mathrm{d}x.$$

On the left hand side $H^{-1/2}(\partial\Omega)$ is regarded as the dual of $H^{1/2}(\partial\Omega)$.

Suppose there are two continuous potentials q_1 and q_2 with compact supports in Ω. If $\Lambda_{q_1} = \Lambda_{q_2}$ is it guaranteed that $q_1 = q_2$? The answer turns out to be affirmative in dimensions $n \geq 3$. Up to our knowledge the case $n = 2$ is open.

Assume that $n \geq 3$ and $\Lambda_{q_1} = \Lambda_{q_2}$. From the symmetry of the Dirichlet-to-Neumann map with respect to the duality $< \cdot, \cdot >$ it follows that

$$\int_\Omega (q_1 - q_2) u_1 u_2\,\mathrm{d}x = 0$$

for all weak solutions $u_j \in H^1(\Omega)$ to $(\Delta - q_j)u_j = 0$, $j = 1, 2$. Let $\xi \in \mathbb{R}^n$. Choose complex vectors $\zeta_j \in \mathbb{C}^n$, $j = 1, 2$, such that

$$\begin{aligned} \zeta_j \cdot \zeta_j &= 0, \\ \zeta_1 + \zeta_2 &= \xi, \\ |\zeta_j| &\geq \max(||q_1||_\infty, ||q_2||_\infty) + 1. \end{aligned}$$

Using the Banach fixed point theorem it can be proved that there exists a solution $u_j \in H^2(\Omega)$ to $(\Delta - q_j)u_j = 0$ of the form

$$u_j = e^{\mathrm{i}\zeta_j \cdot x}(1 + w_{\zeta_j}), \quad ||w_{\zeta_j}||_{L^2} \leq \frac{C}{|\mathrm{Im}\,\zeta_j|}.$$

Particularly,

$$\begin{aligned} 0 &= \int_\Omega (q_1(x)-q_2(x))e^{\mathrm{i}\xi\cdot x}(1+w_{\zeta_1}(x))(1+w_{\zeta_2}(x))\,\mathrm{d}x \\ &= \int_\Omega (q_1(x)-q_2(x))e^{\mathrm{i}\xi\cdot x}\,\mathrm{d}x + \mathcal{R}_{\zeta_1,\zeta_2} \end{aligned}$$

where

$$\mathcal{R}_{\zeta_1,\zeta_2} := \int_\Omega (q_1(x)-q_2(x))e^{\mathrm{i}\xi\cdot x}(w_{\zeta_1}(x)+w_{\zeta_2}(x)+w_{\zeta_1}(x)w_{\zeta_2}(x))\,\mathrm{d}x.$$

The Cauchy-Schwarz inequality implies, if the Lebesgue measure of Ω is denoted by $|\Omega|$,

$$\begin{aligned} |\mathcal{R}_{\zeta_1,\zeta_2}| &\le \int_\Omega |q_1(x)-q_2(x)|\Big(|w_{\zeta_1}(x)|+|w_{\zeta_2}(x)|+|w_{\zeta_1}(x)w_{\zeta_2}(x)|\Big)\,\mathrm{d}x \\ &\le ||q_1-q_2||_\infty\Big(|\Omega|^{1/2}\left(||w_{\zeta_1}||_{L^2}+||w_{\zeta_2}||_{L^2}\right)+||w_{\zeta_1}||_{L^2}||w_{\zeta_2}||_{L^2}\Big). \end{aligned}$$

It follows from the equation $\zeta_j\cdot\zeta_j=0$ that $|\mathrm{Im}\,\zeta_j|=|\mathrm{Re}\,\zeta_j|$. Hence

$$|\mathcal{R}_{\zeta_1,\zeta_2}| \le ||q_1-q_2||_\infty\left[C|\Omega|^{1/2}\left(\frac{1}{|\mathrm{Im}\,\zeta_1|}+\frac{1}{|\mathrm{Im}\,\zeta_2|}\right)+\frac{C}{|\mathrm{Im}\,\zeta_1||\mathrm{Im}\,\zeta_2|}\right]\to 0$$

as $|\zeta_1|,|\zeta_2|\to 0$. Thus the Fourier transform

$$\mathcal{F}(q_1-q_2)(\xi) = \int_\Omega (q_1(x)-q_2(x))e^{\mathrm{i}\xi\cdot x}\,\mathrm{d}x$$

of q_1-q_2 vanishes for all $\xi\in\mathbb{R}^n$. The conclusion is that $q_1=q_2$.

The same kind of argument could be used in connection with electromagnetic inverse problems since the fundamental solutions to Maxwell's free space equations can be composed of the fundamental solutions to the Helmholtz equation. Unfortunately, whatever the choice of the exponentially growing solutions corresponding to u_j, $j=1,2$, is the counterpart of the residual term $\mathcal{R}_{\zeta_1,\zeta_2}$ will not tend to zero as the lengths $|\zeta_j|$ of the adequate complex wave vectors ζ_j tend to infinity. After some modifications in the proof the exponential solutions still yield the desired result (see [38] for details).

In [38] and again in [39] Ola *et al.* proved that IP2 in Section 2.3 has a unique solution if $\mathbb{R}^3\setminus\overline{\Omega}$ is connected and some technical assumptions are satisfied. The article [39] by Ola and Somersalo also contains a proof to the uniqueness of the solution of IP1. These results hold except for a discrete set of magnetic resonance frequences ω that occur when $\sigma=0$.

4.2 Uniqueness of the Inversion after Data Reduction

The Dirichlet-to-Neumann map is formally an integral operator with an appropriate kernel say $K_\Lambda(x,y)$, $x,y\in\partial\Omega$. Assume that U is a non-empty open subset of $\partial\Omega$. One can ask whether the restriction $K_\Lambda|_{U\times U}$ is sufficient to uniquely determine the material parameters in Ω. In [28] Lassas *et al.* studied the question when Ω is replaced by the lower half-space

$$\mathbb{R}^3_- = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 < 0\}.$$

Consider the Schrödinger equation $(\Delta + q(x))\psi(x) = 0$ with $\operatorname{Im} q(x) > 0$ and $\operatorname{supp}(q - q_0) \subset W \subset \mathbb{R}^3_-$ for a known constant $q_0 \in \mathbb{C}$ and a fixed compact W. The kernel K_Λ turns out to be

$$K_\Lambda(x, y) = \frac{\partial}{\partial x_3}\frac{\partial}{\partial y_3} G^D(x, y)\Big|_{x_3=0, y_3=0}$$

where the Dirichlet Green's function G^D is the solution to a certain Lippmann-Schwinger equation (see [28] for details) such that the boundary condition

$$G^D(x, y)|_{x_3=0} = 0$$

is satisfied. Lassas *et al.* proved, using explicitly the exponentially growing solutions as in Section 4.1, that the inhomogeneity $q - q_0$ is uniquely determined by $K_\Lambda|_{U\times U}$.

In electromagnetics the Dirichlet-to-Neumann map is replaced by the admittance map

$$Y : \vec{n} \times \vec{E}|_{\partial\Omega} \mapsto \vec{n} \times \vec{H}|_{\partial\Omega}$$

or its inverse the *impedance map*

$$Z : \vec{n} \times \vec{H}|_{\partial\Omega} \mapsto \vec{n} \times \vec{E}|_{\partial\Omega}.$$

Formally they are integral operators with kernels K_Y and K_Z. Let the scatterer $W_{\mu,\gamma}$ be a subset of a fixed compact set $W \subset \Omega := \mathbb{R}^3_-$. It is also required that in Ω the functions ε, μ and σ are bounded from above by positive numbers, ε and μ are bounded from below by positive numbers and σ is non-negative. Analogously to the result of Lassas *et al.* Paper A shows that the restrictions $K_Y|_{U\times U}$ and $K_Z|_{U\times U}$ uniquely determine the material functions ε, μ and σ in W. The proof is based on plane symmetric Green's functions. They are constructed using the same kind of image principle as in [28]. The exponentially growing solutions are hiding in the proof of the second main theorem in [39]. It solves the uniqueness of IP1.

Paper A also proves a local version of the above mentioned uniqueness theorem. Assume that for each magnetic dipole $\vec{v}$ located at U and tangential to $\partial\Omega$ the tangential component $(\vec{n} \times \vec{E}, \vec{n} \times \vec{H})$ at U of the electromagnetic field generated by the dipole is measured. The local theorem states that such a measurement is sufficient to uniquely determine the material functions ε, μ and σ in W. This uniqueness follows from the global theorem by analytic continuation and reciprocity.

5 Conclusion

The work considers the scattering of a time-harmonic electromagnetic wave from a bounded obstacle in a free space but the methods and results are also applicable for other wave phenomena and, *mutatis mutandis*, in time domain. The purpose of this study is twofold. On one hand, it is shown that an inverse problem is uniquely solvable from local surface data. On the other hand, the direct problem can be solved up to an arbitrary precision in a bounded computational domain. The aim is at better understanding of the principles that certain extensively used computational methods are based on. This know-how is a prerequisite for further development of the algorithms. Nevertheless, the approach is quite theoretical and abstract.

An earlier result stating that the electromagnetic material parameters are uniquely determined by a global surface measurement is improved by proving that a local measurement on a tiny piece of the surface is sufficient to guarantee the uniqueness. The theorem is important, for instance, in geophysics since global measurements surrounding the whole Earth are impossible.

To work reliably integral equations reguire smooth and simple boundaries. In connection with complicated geometries one has to use the finite element or finite difference methods in the vicinity of the scatterer. Hence the unbounded exterior domain has to be modelled by a bounded domain on the exterior boundary of which one has to impose some absorbing boundary condition to prevent spurious reflections. This work contains a thorough study of the perfectly matched layer (PML) boundary condition. The absorbing layer is constructed by complex stretching of the metric tensor. If the field quantities are represented by differential forms Maxwell's equations preserve their original form and the conventional electromagnetic tools like the Stratton-Chu representation formula can be used also in the stretched geometry. The main result states that in the presence of PML the truncated boundary value problem has a unique solution for all frequencies and the solution tends to the physical scattering solution exponentially as the thickness of the absorbing layer increases. Our presentation is coordinate invariant, hence applicable in arbitrary geometries. The authors hope that the tools and methods used in this treatise turn out to be useful when developing better PML's.

Finally, we emphasize that the perfectly matched layer was only a starting point of our work. The studies go far beyond towards a general complexified scattering theory.

References

[1] S. Agmon, A. Douglis and L. Nirenberg: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. Comm. Pure Appl. Math. **17** (1964) 35–92.

[2] M. S. Agranovich: *Elliptic boundary problems.* Encyclopaedia Math. Sci., 79, Partial differential equations, IX, 1–144, 275–281, Springer, 1997.

[3] Alvin Bayliss and Eli Turkel: Radiation boundary conditions for wave-like equations. Comm. Pure Appl. Math. **33** (1980) 707–725.

[4] Jean Pierre Bérenger: A perfectly matched layer for the absorption of electromagnetic waves. J. Comp. Phys. **114** (1994) 185–200.

[5] Jean Van Bladel: *Electromagnetic Fields.* Hemisphere Publishing Corporation, New York 1985.

[6] Alain Bossavit: Simplicial finite elements for scattering problems in electromagnetism. Comp. Meth. in Appl. Mech. and Eng. **76** (1989) 299–316.

[7] A. Bossavit: *Computational Electromagnetics. Variational formulation, complementarity, edge elements.* Academic Press Inc. 1998.

[8] A. P. Calderón: On an Inverse Boundary Value Problem, Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matemàtica, 65–73, Rio de Janeiro 1980.

[9] Manfredo Perdigão do Carmo: *Riemannian Geometry.* Birkhäuser, Boston 1992.

[10] W. C. Chew: *Waves and Fields in Inhomogeneous Media.* Van Nostrand, New York 1990 (Reprinted by IEEE Press 1995).

[11] W. C. Chew and W. H. Weedon: A 3D perfectly matched medium from modified Maxwell's equations with stretched coordinates. Microwave Opt. Tech. Lett. **7** (1994) 599–604.

[12] W. C. Chew, J. Jin and E. Michielssen: Complex Coordinate System as a Generalized Absorbing Boundary Condition, IEEE Antennas and Propagation Society International Symposium in Montreal, 1997 Digest — Volume 3, 2060–2063, 1997.

[13] F. Collino and P. Monk: Optimizing the Perfectly Matched Layer. SIAM J. Sci. Comput. **19** (1998) 2061–2090.

[14] F. Collino and P. Monk: The perfectly matched layer in curvilinear coordinates. Comput. Methods Appl. Mech. Engrg. **164** (1998) 157–171.

[15] David Colton and Rainer Kress: *Integral Equation Methods in Scattering Theory.* John Wiley & Sons, New York 1983.

[16] David Colton and Rainer Kress: *Inverse Acoustic and Electromagnetic Scattering Theory.* Springer-Verlag, Berlin 1992.

[17] D. McDuff and D. Salamon: *Introduction to symplectic topology.* Oxford University Press, 1995.

[18] Björn Engquist and Andrew Majda: Absorbing boundary conditions for the numerical simulation of waves. Math. Comp. **31** (1977) 629–651.

[19] M. J. Grote and J. E. Keller: Exact non-reflecting boundary condition for the time dependent wave equation. SIAM J. Appl. Math. **55** (1995) 280–297.

[20] Kiyosi Itô (ed.): *Encyclopedic Dictionary of Mathematics, 2nd ed., vol. I–II.* MIT Press, Cambridge, MA 1993.

[21] Seppo Järvenpää: *Implementation of PML Absorbing Boundary Condition for Solving Maxwell's Equations with Whitney Elements.* Ph.D. Thesis, University of Helsinki, Helsinki 2001.

[22] D. S. Katz, E. T. Thiele and A. Taflove: Validation and extension to three dimensions of the Bérenger PML absorbing boundary condition for FD-TD meshes. IEEE Microwave Guided Wave Lett. **4** (1994) 268–270.

[23] L. Kettunen, K. Forsman and A. Bossavit: Formulation of the eddy current problem in multiply connected regions in terms of h. Internat. J. Numer. Methods Engrg. **41** (1998) 935–954.

[24] A. Kirsch and P. Monk: An analysis of the coupling of finite-element and Nyström methods in acoustic scattering. IMA J. Numerical Anal. **14** (1994) 523–544.

[25] P. R. Kotiuga: *Hodge Decompositions and Computational Electromagnetics.* Ph.D. Thesis, McGill University, Montreal 1984.

[26] Lassas, M., Sarkola, E. and Somersalo, E.: The MEI method and double surface radiation conditions. Helsinki University of Technology, Institute of Mathematics Research Reports A357 (1996).

[27] Matti Lassas and Erkki Somersalo: On the existence and convergence of the solution of PML equations. Computing **60** (1998) 229–241.

[28] Matti Lassas, Margaret Cheney and Günther Uhlmann: Uniqueness for a wave propagation inverse problem in a half-space. Inverse Problems **14** (1998) 679–684.

[29] Matti Lassas and Erkki Somersalo: Analysis of the PML equations in general convex geometry. To appear in Proc. Roy. Soc. Edinburgh Sect. A

[30] Matti Lassas, Jukka Liukkonen and Erkki Somersalo: Complex Riemannian metric and absorbing boundary conditions. J. Math. Pures Appl. **80** (2001) 739–768.

[31] Rolf Leis: *Initial Boundary Value Problems in Mathematical Physics.* John Wiley & Sons, Chichester 1986.

[32] Ismo V. Lindell: *Methods for Electromagnetic Field Analysis.* Clarendon Press, Oxford 1992.

[33] Jukka Liukkonen: Uniqueness of electromagnetic inversion by local surface measurements. Inverse Problems **15** (1999) 265–280.

[34] Kenneth K. Mei, Rafael Pous, Zhaoqing Chen, Yao-Wu Liu and Mark D. Prouty: Measured equation of invariance: a new concept in field computations. IEEE Trans. Antenn. Propagat. **42** (1994) 320–328.

[35] A. I. Nachman: Reconstructions from boundary measurements. Annals of Mathematics **128** (1988) 531–576.

[36] A. I. Nachman: Global uniqueness for a two-dimensional inverse boundary value problem. Annals of Mathematics **143** (1996) 71–96.

[37] Mikio Nakahara: *Geometry, Topology and Physics.* IOP Publishing, Bristol 1998.

[38] Petri Ola, Lassi Päivärinta and Erkki Somersalo: An inverse boundary value problem in electrodynamics. Duke Math. J. **70** (1993) 617–653.

[39] Petri Ola and Erkki Somersalo: Electromagnetic inverse problems and generalized Sommerfeld potentials. SIAM J. Appl. Math. **56** (1996) 1129–1145.

[40] L. Paquet: Problèmes mixtes pour le système de Maxwell. Annales Facultè des Sciences Toulouse, Vol. IV (1982) 103–141.

[41] P. Petropoulos: Reflectionless sponge layers as absorbing boundary conditions for the numerical solution of Maxwell equations in rectangular, cylindrical, and spherical coordinates. SIAM J. Appl. Math. **60** (2000) 1037–1058.

[42] R. Picard: On a structural observation in generalized electromagnetic theory. J. Math. Anal. Appl. **110** (1985) 247–264.

[43] Georges de Rham: *Varietés différentiables.* Hermann, Paris 1960.

[44] T. B. A. Senior and J. L. Volakis: *Approximate Boundary Conditions in Electromagnetics*. IEE, London 1995.

[45] Samuli Siltanen: *Electrical Impedance Tomography and Faddeev Green's Functions*. Ph.D. Thesis, Helsinki University of Technology, Helsinki 1999.

[46] J. Sjöstrand and M. Zworski: Complex scaling and distribution of scattering poles. J. Amer. Math. Soc. **4** (1991) 729–769.

[47] J. Sjöstrand and M. Zworski: The complex scaling method for scattering by strictly convex obstacles. Ark. Mat. **33** (1995) 135–172.

[48] J. Sjöstrand and M. Zworski: Asymptotic distribution of resonances for convex obstacles. Acta Math. **183** (1999) 191–253.

[49] Michael M. Taylor: *Partial Differential Equations. Basic Theory.* Springer, New York 1996.

[50] F. L. Teixeira and W. C. Chew: Analytical derivation of a conformal perfectly matched absorber for electromagnetic waves. Micro. Opt. Tech. Lett. **17** (1998) 231–236.

[51] F. L. Teixeira and W. C. Chew: Differential forms, metrics, and the reflectionless absorption of electromagetic waves. Journal of Electromagnetic Waves and Applications **13** (1999) 665–686.

[52] E. Turkel and A. Yefet: Absorbing PML boundary layers for wave-like equations. Absorbing boundary conditions. Appl. Numer. Math. **27** (1998) 533–557.

[53] K. F. Warnick and D. V. Arnold: Green forms for anisotropic, inhomogeneous media. J. Electromagn. Waves Appl. **11** (1997) 1145–1164.

[54] K. F. Warnick, R. H. Selfridge and D. V. Arnold: Teaching electromagnetic field theory using differential forms. IEEE Trans. Edu. **40** (1997) 53–68.

[55] Hermann Weyl: Die natürlichen Randwertaufgaben im Außenraum für Srahlungsfelder beliebiger Dimension und beliebigen Ranges. Math. Z. **56** (1952) 105–119.

Paper A

Jukka Liukkonen: Uniqueness of electromagnetic inversion by local surface measurements. Inverse Problems **15** (1999) 265–280.

Inverse Problems homepage at *www.iop.org/Journals/IP*

Paper B

Jukka Liukkonen: Generalized Electromagnetic Scattering in a Complex Geometry. J. Math. Anal. Appl. **254** (2001) 498–514.

Paper C

Matti Lassas, Jukka Liukkonen and Erkki Somersalo: Complex Riemannian metric and absorbing boundary conditions. J. Math. Pures Appl. **80** (2001) 739–768.

(continued from the back cover)

A435 Timo Salin
On Quenching with Logarithmic Singularity, May 2001

A434 Ville Havu
An Analysis of Asymptotic Consistency Error in a Parameter Dependent Model Problem, Mar 2001

A433 Ville Havu and Juhani Pitkäranta
Analysis of a bilinear finite element for shallow shells I: Consistency error, Jan 2001

A432 Tuomas Hytönen
Optimal Wrap-Around Network Simulation, Dec 2000

A431 Ville Turunen
Pseudodifferential calculus on compact Lie groups, Jun 2001

A430 Ville Havu and Juhani Pitkäranta
Analysis of a bilinear finite element for shallow shells I: Approximation of Inextensional Deformations, May 2000

A428 Jarmo Malinen
Discrete Time H^∞ Algebraic Riccati Equations, Mar 2000

A427 Marko Huhtanen
A Hermitian Lanczos method for normal matrices, Jan 2000

A426 Timo Eirola
A Refined Polar Decomposition: $A = UPD$, Jan 2000

A425 Hermann Brunner, Arvet Pedas and Gennadi Vainikko
Piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels, Dec 1999

A424 Olavi Nevanlinna
Resolvent conditions and powers of operators, Dec 1999

A423 Jukka Liukkonen
Generalized Electromagnetic Scattering in a Complex Geometry, Nov 1999

A422 Marko Huhtanen
A matrix nearness problem related to iterative methods, Oct 1999

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS RESEARCH REPORTS

The list of reports is continued inside. Electronical versions of the reports are available at *http://www.math.hut.fi/reports/* .

A440 Ville Turunen
Pseudodifferential calculus on compact Lie groups and homogeneous spaces, Sep 2001

A439 Jyrki Piila and Juhani Pitkäranta
On corner irregularities that arise in hyperbolic shell membrane theory, Jul 2001

A438 Teijo Arponen
The complete form of a differential algebraic equation, Jul 2001

A437 Viking Högnäs
Nonnegative operators and the method of sums, Jun 2001

A436 Ville Turunen
Pseudodifferential calculus on compact homogeneous spaces, Jun 2001

ISBN 951-22-5744-0
ISSN 0784-3143
Espoo, 2001