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Abstract: *The present work consists of a survey of the properties of the classical Korteweg–de Vries equation, which give rise to local and even global solutions. The known existence and regularity results are extended to cover a wider class of partial differential equations. The explicit time dependence and some global results appear to be new. The results are based on the theory of semigroups and abstract evolution equations.*

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1 Introduction

The Korteweg–de Vries equation is the simplest wave equation involving both a non-linearity and dispersion effects. Several elementary variants appear in the literature, the simplest of which is

$$u_t + uu_x + u_{xxx} = 0. \tag{1}$$

Substituting $u = \frac{\alpha}{\gamma}v + \frac{\beta}{\gamma}$ and $t = \gamma\tau$, a family of equivalent equations

$$v_\tau + \alpha v v_x + \beta v_x + \gamma v_{xxx} = 0 \tag{2}$$

is obtained.

The equation first occured in the study of solitary (water) waves, the profile $\zeta(\chi, t)$ of which, as shown by Korteweg and de Vries in 1895, satisfies the equation

$$\frac{\partial\zeta}{\partial t} = \frac{3}{2}\left(\frac{g}{h}\right)^{\frac{1}{2}}\left(\frac{2}{3}\epsilon\frac{\partial\zeta}{\partial\chi} + \zeta\frac{\partial\zeta}{\partial\chi} + \frac{1}{3}\sigma\frac{\partial^3\zeta}{\partial\chi^3}\right), \tag{3}$$

which is readily seen to be of the form (2). Here χ is a spatial coordinate chosen to be moving (almost) with the wave, t is time, g is acceleration due to gravity, h is the undisturbed depth of water, ϵ is an arbitrary parameter and σ is related to the surface tension T, the density ρ of the liquid and other parameters by $\sigma = \frac{1}{3}h^3 - \frac{Th}{g\rho}$. The historical background and physical meaning of the equation are discussed in more detail by Drazin and Johnson [4, Ch. 1]. Later, the same equation has also occured in plasma physics and in studies of anharmonic non-linear lattices. References to these applications and some others are found in Miura [11].

Of the various approaches to this problem found in literature, we will treat the initial value problem $u(0, x) = \phi(x)$ for the equation (1) on the real line $\mathbb{R}$ as an *abstract evolution equation* in a Banach space and employ the *theory of C_0 semigroups*. Another tool that has been applied in the analysis of the Korteweg–de Vries equation by various authors is the *inverse scattering method*. The book of Drazin and Johnson [4] is written in this spirit, as is the article of Murray [13] (which we will cite for an interesting comparison to our theory). A classical treatment (as opposed to the semigroup approach of more modern mathematics) of the Korteweg–de Vries equation is found e.g. in Bona and Smith [2].

A semigroup theory study of the Korteweg–de Vries equation was first done by Kato [8], who introduced a unified theory applicable to various kinds of quasi-linear partial differential equations. This theory was based on the results for linear evolution equations (the abstract Cauchy problem), a field also strongly contributed to by Kato [6, 7].

Kato considered the problem in a somewhat generalized form

$$u_t + u_{xxx} + a(u)u_x = 0, \tag{4}$$

where $a \in C^\infty(\mathbb{R})$, and obtained the following result:

Theorem 1 (Kato). *The initial value problem for (4) with the initial data $\phi \in H^s(\mathbb{R})$, $s \geq 3$, has a unique solution*

$$u \in C([0,T] : H^s(\mathbb{R})) \cap C^1([0,T] : H^{s-3}(\mathbb{R})) \tag{5}$$

for some $T > 0$, and $u(t, \cdot)$ depends continuously on ϕ in the $H^s(\mathbb{R})$ norm.

Kato also noticed the possibility to obtain similar results with the third order derivative $\frac{\partial^3}{\partial x^3}$ replaced by a more general polynomial $P(\frac{\partial}{\partial x})$ of the differentiation operator.

In later work [9], Kato strengthened the previous theorem to cover the Sobolev spaces of order $\frac{3}{2} < s < 3$, in which case we have a unique solution in the space $C([0,T] : H^s(\mathbb{R})) \cap C^1([0,T] : L^2(\mathbb{R}))$. Furthermore, under an additional assumption, a global existence theorem was obtained.

Definition 2. *An initial value problem on $\mathbb{R}$ for a differential equation (e.g. of type (4)) is said to satisfy* Kato's global growth condition *if there are real numbers $s_1 \geq s_0 > \frac{3}{2}$ and a monotone increasing function $g : [0, \infty[\to [0, \infty[$ such that for any $T > 0$ and any $u \in C([0,T[: H^{s_1})$ satisfying (4), one has*

$$||u(t)||_{s_0} \leq g\left(||u(0)||_{s_0}\right) \qquad \forall t \in [0, T[. \tag{6}$$

Theorem 3 (Kato). *If Kato's global growth condition is satisfied by (4), then Theorem 1 holds for $s \geq s_0$ with $T = \infty$. The proper Korteweg–de Vries equation (1) satisfies this condition with $s_0 = 2$.*

The aim of the present paper is to extend these theorems to cover a wider range of partial differential equations. In doing so, we try to reveal what exactly are the properties of the Korteweg–de Vries equation, which give rise to the above mentioned existence and regularity properties of the solutions. The semigroup approach gives a good ground for this, since the abstract theorems of Kato [8, Th.'s 6, 7] behind the above applications are given in a rather general setting. Therefore, it suffices to seek the range of applicability of these abstract results in order to apply their full strength, but the computations will nevertheless get somewhat involved.

Some preliminary results deal in particular with the characterization of infinitesimal semigroups on $L^2(\mathbb{R})$ and the uniformity of such conditions for operator families, extensions of commutativity from generators to the semigroups, and Sobolev norm estimates. We will cite some results in particular from the theory of semigroups; the reference for many of these will be the book of Pazy [15].

More precisely, we will carry out the extension to Theorem 1 as suggested by Kato [8], with a slight modification, which proves to be necessary. In this spirit we define *Kato's polynomial condition* (Definition 7), which turns out to give a necessary and sufficient condition for a polynomial $P(\frac{\partial}{\partial x})$ of the differentiation operator $\frac{\partial}{\partial x}$ to generate a semigroup on $L^2(\mathbb{R})$. The condition only sets restrictions on the even part of P, and thus the proper Korteweg–de Vries equations satisfies this condition rather trivially.

A wider class of equations, to which the Korteweg–de Vries equation belongs, consists of quasi-linear equations where only odd order derivatives are involved. This class appears every now and then throughout our analysis, and strictly stronger results are obtained than for more general equations. In fact, most of the theorems of Kato concerning the equations (4) extend more or less directly to this class of equations.

In order to find the maximal class of partial differential equations that could be viewed as being of the Korteweg–de Vries type, and also to get a grasp of the full strength of the abstract theorems, we allow for time dependent coefficients in the equations of interest. This is certainly not uncommon in applications, and already in the original context of the Korteweg–de Vries equation (3), time dependence of some of the coefficients could be of interest if the liquid in question is involved in some industrial process, say. For such a generalization (with some more technical assumptions), we are able to prove existence, regularity and continuous dependence results (Theorems 28, 30) analogous to Theorem 1. This result, in particular the explicit time dependence, appears to be new.

Somewhat sharper results (Theorem 32) are obtained for the equations with only odd order derivatives without time dependent coefficients, and a subset of this class of equations even gives us global solutions. Even with these additional restrictions, the coverage of this result appears to be new. Unfortunately, this class of equations does not include the proper Korteweg–de Vries equation, for which Theorem 3 on global existence is nevertheless valid, but derives from rather individual properties of (1) to be discussed shortly.

As a whole, we investigate the properties of the Korteweg–de Vries equation crucial to the existence theory of solutions, and attempt to view the equation in a more general setting as a member of a class of equations with these properties. We derive results analogous to the ones cited above for the range of equations sharing some or all of these properties.

1.1 Notation

The notation is mostly standard, and follows in particular conventions similar to those of Kato [8, 9] and Pazy [15]. An important difference worth pointing out lies in the notation for infinitesimal generators of C_0 semigroups. Here we follow the lines of Pazy to denote by $A \in G(X, M, \omega)$ the fact that A is an infinitesimal generator of a C_0 semigroup of a certain kind, whereas Kato has used a somewhat indirect definition, where the same notation indicates $-A$ being the infinitesimal generator. This latter approach has some advantages in removing minus signs that we cannot avoid in the treatment of abstract evolution equations, but we nevertheless stick to the direct notation. This also affects the sign in the definition of the resolvent operator.

The most frequently occuring symbols are summarized in the following table. All function spaces are understood to be on the real line $\mathbb{R}$, unless the contrary is made explicit; thus $H^s = H^s(\mathbb{R})$, $L^p = L^p(\mathbb{R})$ etc.

$\int$	Lebesgue integral (over $\mathbb{R}$ unless otherwise stated)
$\circ$	Composition of functions in the spatial variable
$\mathbf{1}_S$	Indicator (characteristic function) of the set S
A^*	Adjoint operator of A
$[A, B]$	Commutator $AB - BA$
$B(X), B(X, Y)$	Bounded linear operators of X into itself and into Y
C, C^n	Continuous and n times continuously differentiable functions
D	Spatial differentiation operator $\frac{\partial}{\partial x}$
$\mathcal{D}(A)$	Domain of operator A
$\widehat{f}$	Fourier transform of f
$f', f^{(n)}$	First and nth spatial derivatives of f
$G(X, M, \omega)$	Set of infinitesimal generators of C_0 semigroups $T(t)$ on X for which $\|\|T(t)\|\| \leq Me^{\omega t}$
H^s	Sobolev space of order s of L^2 type, $W^{s,2}$
I	Identity operator
L^p	Lebesgue space of order p, $W^{0,p}$
Λ^s	The isometric isomorphism of H^s onto L^2, $\Lambda^s f(x) = \frac{1}{\sqrt{2\pi}} \int e^{ix\xi}(1+\xi^2)^{\frac{s}{2}} \widehat{f}(\xi) d\xi$
M_f	Operator of (pointwise) multiplication by f
$O(f)$	Functions increasing no faster than f as $x \to \pm\infty$
$\rho(A)$	Resolvent set of operator A
$R(\lambda, A)$	Resolvent operator $(\lambda I - A)^{-1}$
$\Re z$	Real part of z
$W^{s,p}$	Sobolev space of order s of L^p type
$\overline{z}$	Complex conjugate of z
$\|\| \cdot \|\|, (\cdot, \cdot)$	L^2 norm and inner product
$\|\| \cdot \|\|_X, (\cdot, \cdot)_X$	Norm and inner product of the space X
$\|\| \cdot \|\|_s, (\cdot, \cdot)_s$	H^s norm and inner product $(f, g)_s = \int (1+\xi^2)^s \widehat{f}(\xi)\overline{\widehat{g}(\xi)} d\xi$
$\|\| \cdot \|\|_{s,p}, \| \cdot \|_{s,p}$	$W^{s,p}$ norm and seminorm (Definition 21)
$\|\| \cdot \|\|_\infty$	L^∞ (essential) supremum norm
$\|\| \cdot \|\|_{\infty,r}$	$C[-r, r]$ norm $\|\|f\|\|_{\infty,r} = \max_{\|x\| \leq r} \|f(x)\|$
$\|\| \cdot \|\|_{\infty,r,n}$	$C^n[-r, r]$ norm $\|\|f\|\|_{\infty,r,n} = \max_{k \in \{0,\ldots,n\}} \|\|f^{(k)}\|\|_{\infty,r}$

2 Semigroups Generated by Differential Operators

2.1 Polynomials of the Differentiation Operator

In this section we seek conditions under which polynomials $P(D)$ of the differentiation operator D with constant real coefficients generate certain types of semigroups to which the general theorems concerning abstract evolution equations apply. In two of the main propositions the statement of our results will be two-fold: In addition to giving the "useful" result to be applied later, we also construct counter-examples to show that the setting of the results is in some sense the most general possible.

Our first result, Proposition 6, concerns polynomials consisting only of terms with odd powers. In the verification of Proposition 6, we need two results from the theory of semigroups, which are stated below without proof. The proofs may be found in Pazy [15, Th.'s 1.10.8, 2.5.5].

Proposition 4 (Stone). *A is the infinitesimal generator of a C_0 group of unitary operators on a Hilbert space if and only if A is skew-adjoint.*

Proposition 5 (Crandall–Pazy–Tartar). *Let A be the infinitesimal generator of a C_0 semigroup $T(t)$. Then $T(t)$ is analytic if and only if there are constants $C > 0$ and $\Lambda \geq 0$ such that*

$$||AR(\lambda : A)^{n+1}|| \leq \frac{C}{n\lambda^n} \qquad \text{for } \lambda > n\Lambda, \qquad n \in \mathbb{Z}^+.$$

Proposition 6. *A polynomial of the differentiation operator D generates a C_0 semigroup of unitary operators on H^s, in particular on L^2, if and only if it is of the form $P(D^2)D$, where P is a polynomial. A polynomial of this form does not generate an analytic semigroup on L^2.*

Proof. $A = D^k$ is skew-adjoint, i.e. $A^* = -A$, if and only if k is odd, since

$$\begin{aligned}(Au, v)_s &= \int (1+\xi^2)^s (i\xi)^k \widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi \\ &= (-1)^k \int (1+\xi^2)^s \widehat{u}(\xi)\overline{(i\xi)^k\widehat{u}(\xi)}d\xi = (-1)^k (u, Av)_s.\end{aligned}$$

Now clearly any finite sum (with coefficients) of odd powers of D is also skew-adjoint, whereas no sum containing even powers of D can satisfy this condition. Thus the first assertion of Proposition 6 follows immediately from Proposition 4.

In order to validate the second assertion, we show that the necessary (and sufficient) condition of analyticity given in Proposition 5 fails to hold for the operator $P^2(D)D$. Indeed, let some constants $C > 0$, $\Lambda \geq 0$ be given. We may well assume $\Lambda > 0$, since the condition of Proposition 5 clearly holds for any $\Lambda > \Lambda_0$, if it holds for some Λ_0. We will investigate the operators in the Fourier domain, where D corresponds to multiplication by $i\xi$, and thus

$$\begin{aligned}||P(D^2)DR(\lambda : P(D^2)D)^{n+1}f||^2 &= \int \left| \frac{i\xi P(-\xi^2)}{(\lambda - i\xi P(-\xi^2))^{n+1}} \widehat{f}(\xi) \right|^2 d\xi \\ &= \int \frac{\xi^2 P^2(-\xi^2)}{(\lambda^2 + \xi^2 P^2(-\xi^2))^{n+1}} |\widehat{f}(\xi)|^2 d\xi\end{aligned}$$

To find a contradiction with the condition of Proposition 5, it is sufficient to show that for some $\lambda > n\Lambda$, say $\lambda = 2n\Lambda$, this expression will exceed

$$\frac{C^2}{n^2\lambda^{2n}} \int |\widehat{f}(\xi)|^2 d\xi = \frac{4C^2\Lambda^2}{(2n\Lambda)^{2(n+1)}} \int |\widehat{f}(\xi)|^2 d\xi. \tag{7}$$

We now take $\widehat{f} = \mathbf{1}_{[a,b]}$. For $0 < x < \frac{\lambda^2}{n} = 4n\Lambda^2$, the mapping $x \mapsto \frac{x}{(\lambda^2+x)^{n+1}}$ is increasing, and for large enough ξ, so is $\xi \mapsto \xi^2 P^2(-\xi^2)$. Thus taking $b > a > 0$ large enough, but so that $b^2 P^2(-b^2) < 4n\Lambda^2$ (which is possible for sufficiently large n), we ensure that

$$\int \frac{\xi^2 P^2(-\xi^2)}{\left((2n\Lambda)^2 + \xi^2 P^2(-\xi^2)\right)^{n+1}} |\widehat{f}(\xi)|^2 d\xi \geq \frac{a^2 P^2(-a^2)}{\left((2n\Lambda)^2 + a^2 P^2(-a^2)\right)^{n+1}} (b-a), \tag{8}$$

If the condition of Proposition 5 were to hold, we should now have, comparing equations (7) and (8),

$$\frac{a^2 P^2(-a^2)}{\left(1 + \left(\frac{a^2 P^2(-a^2)}{2n\Lambda}\right)^2\right)^{n+1}} \leq 4C^2\Lambda^2. \tag{9}$$

But we have the limit

$$\lim_{n\to\infty} \left(1 + \left(\frac{\gamma}{n}\right)^2\right)^n = \exp\left(\lim_{n\to\infty} \frac{\gamma^2}{n}\right) = 1,$$

and therefore, for $a^2 P^2(-a^2) > 2C\Lambda$ and sufficiently large n, the inequality in (9) clearly fails, and the proof is complete. □

We now define a condition that turns out to be convenient in generalizing the Korteweg–de Vries equation. A condition of this kind was already proposed by Kato [8] (thus the name) with somewhat larger generality, but the second assertion of the following Proposition 9 shows that the condition as given below is the most general to allow us to apply the theory of semigroups on L^2.

Definition 7. *A polynomial $P(x)$ of real coefficients is said to satisfy* Kato's polynomial condition *if either the sign of the coefficient of the highest even power x^m of x in $P(x)$ is $(-1)^{\frac{m}{2}}$, or if the even powers of x are absent in $P(x)$.*

With this definition at hand, we are ready to formulate a result on polynomials of D, where even powers are allowed. In view of the proof we state here the general characterization of infinitesimal generators of C_0 semigroups in the following proposition, which is found together with proof e.g. in Pazy [15, Th. 1.5.3 (& 1.2.2)].

Proposition 8 (Feller–Miyadera–Phillips). *Let A be a linear operator on X. Then $A \in G(X, M, \omega)$ if and only if*

1. *A is closed and the domain of A is dense in X.*
2. *$]\omega, \infty[\subset \rho(A)$ and*

$$||R(\lambda, A)^n|| \leq \frac{M}{(\lambda - \omega)^n} \qquad \forall \lambda > \omega \qquad \forall n \in \mathbb{Z}^+. \tag{10}$$

Furthermore, any infinitesimal generator of a C_0 semigroup on X is in $G(X, M, \omega)$ for some $M \geq 1$, $\omega \in \mathbb{R}$.

Proposition 9. *Kato's polynomial condition for $-P$ is necessary and sufficient for a polynomial $P(D)$ to generate a C_0 semigroup on L^2. In more detail, we have the following:*

1. *If $-P$ satisfies Kato's polynomial condition, then there exists an ω such that $P(D) \in G(L^2, 1, \omega)$. In this and only in this case, $\Re P(i\xi)$ is bounded above, and one can take any $\omega \geq \max_{\xi\in\mathbb{R}} \Re P(i\xi)$.*

2. *For $\omega < \max_{\xi\in\mathbb{R}} \Re P(i\xi)$, $P(D) \notin G(L^2, M, \omega)$ for any M. In particular, if $-P$ does not satisfy Kato's polynomial condition and thus $\Re P(i\xi)$ is not bounded above, then $P(D)$ does not generate a C_0 semigroup on L^2.*

Proof. The first condition of Proposition 8 is satisfied by any $P(D)$, and we hence concentrate on the validity of the second condition.

$P(D)$ can be separated into its even and odd parts:

$$P(D) = P_1(D^2) + P_2(D^2)D.$$

The Fourier transform of $P(D)$ is the multiplication operator

$$P(i\xi) = P_1(-\xi^2) + iP_2(-\xi^2)\xi,$$

and $(\lambda - P(D))^{-1}$ corresponds to multiplication by $\frac{1}{\lambda - P(i\xi)}$ in the Fourier domain. Here, obviously, $\Re P(i\xi) = P_1(-\xi^2)$. We have

$$\begin{aligned}\int \left| \frac{\widehat{f}(\xi)}{\lambda - P_1(-\xi^2) - iP_2(-\xi^2)\xi} \right|^2 d\xi &\leq \max_{\xi\in\mathbb{R}} \frac{||f||^2}{(\lambda - P_1(-\xi^2))^2 + (P_2(-\xi^2)\xi)^2} \\ &\leq \frac{||f||^2}{(\lambda - \max_{\xi\in\mathbb{R}} P_1(-\xi^2))^2} \qquad \text{for } \lambda > \max_{\xi\in\mathbb{R}} P_1(-\xi^2)\end{aligned}$$

The second condition of Proposition 8 now follows for $M = 1$, $\omega \geq \max_{\xi\in\mathbb{R}} P_1(-\xi^2)$ (upon application of the result about estimating the norm of products) provided that the maximum in question exists. This happens if and only if $-P$ satisfies Kato's polynomial condition, since in that case the term of highest order in $P_1(-\xi^2)$ is $-c(-\xi^2)^{\frac{m}{2}} = -|c|\xi^m$, since c has sign $(-1)^{\frac{m}{2}}$. Here m is even, and thus $P_1(-\xi^2) \to -\infty$ as $\xi \to \pm\infty$. Thus P_1 has a maximum, which is also trivially true, if Kato's polynomial condition is satisfied by the lack of even powers in P. Similarly, we see that $P_1(-\xi^2) \to \infty$ as $\xi \to \pm\infty$ if the condition is not satisfied. This completes the proof of the part 1 of Proposition 9.

Assume then that an $\omega < \max_{\xi\in\mathbb{R}} P_1(-\xi^2)$ and some M are given. By continuity it follows that we can take an interval $[a, b]$ such that $\omega < P_1(-\xi^2)$ for all $\xi \in [a, b]$. Let $\widehat{f} = \mathbf{1}_{[a,b]}$. Then

$$\int \frac{|\widehat{f}(\xi)|^2 d\xi}{\left((\lambda - P_1(-\xi^2))^2 + P_2^2(-\xi^2)\xi^2\right)^n} = \frac{b-a}{\left((\lambda - P_1(-c^2))^2 + P_2^2(-c^2)c^2\right)^n},$$

for some $c \in [a,b]$, and this should not exceed

$$\left(\frac{M}{(\lambda-\omega)^n}\right)^2 \int |\widehat{f}(\xi)|^2 d\xi = \left(\frac{M^{\frac{2}{n}}}{(\lambda-\omega)^2}\right)^n (b-a).$$

For (10) to hold, we should hence have

$$\frac{1}{(\lambda - P_1(-c^2))^2 + P_2^2(-c^2)c^2} \leq \frac{M^{\frac{2}{n}}}{(\lambda-\omega)^2} \underset{n\to\infty}{\longrightarrow} \frac{1}{(\lambda-\omega)^2},$$

and the inequality should also hold in the limit. Taking a common denominator and simplifying, the final inequality is equivalent to

$$2\left(P_1(-c^2)-\omega\right)\lambda \leq P_1^2(-c^2) - \omega^2 + P_2^2(-c^2)c^2. \tag{11}$$

Now $P_1(-c^2) > \omega$ by assumption. Furthermore, for a fixed interval $[a,b]$, the right-hand side of (11) is bounded for $c \in [a,b]$ due to continuity of the polynomials. But we can now choose λ as large as we like, and thus (11) fails, justifying the failure of $P(D)$ to generate a C_0 semigroup on L^2. $\square$

2.2 Perturbations by Multiplication Operators

Our next goal is to show that a result similar to Proposition 9 remains valid, when certain terms involving multiplication operators by suitably bounded functions are added to $P(D)$. Such operators are easily transformed into so-called dissipative operators, for which a variety of results is known. The definition of dissipativeness given below may be stated in more general terms in a Banach space (as in Pazy [15, Def. 4.1]), but the somewhat simpler form in a Hilbert space is sufficient for us.

Definition 10. *A linear operator A in a Hilbert space X is said to be* dissipative *if for every $x \in \mathcal{D}(A)$ we have $\Re(Ax,x)_X \leq 0$.*

A useful condition in terms of dissipativeness that is sufficient for a linear operator to generate a semigroup of contractions is cited in Proposition 11; the proof is found in Pazy [15, Cor. 1.4.4]. (The result we cite is a corollary of the actual Lumer–Phillips theorem.) The device to handle the perturbations by dissipative operators will then be the following Proposition 12. We also require some estimates on the L^2 norms of derivatives, and for this purpose we have Proposition 13. These pave the way for the main result of this section, Proposition 14.

Proposition 11 (Lumer–Phillips). *Let A be a closed linear operator with dense domain in X. If both A and A^* are dissipative, then $A \in G(X,1,0)$.*

Proposition 12 (Trotter–Gustafson). *Let $A \in G(X,1,\omega)$ and B be a dissipative linear operator on X such that $\mathcal{D}(B) \supset \mathcal{D}(A)$ and*

$$||Bx|| \leq \alpha||Ax|| + \beta||x|| \qquad \forall x \in \mathcal{D}(A), \tag{12}$$

where $0 \leq \alpha < 1$ and $\beta \geq 0$. Then $A+B \in G(X,1,\omega)$.

Proof. The case $\omega = 0$ is shown in Pazy [15, Cor. 3.3.3]. The general case follows readily: If $A \in G(X, 1, \omega)$, then $A - \omega I \in G(X, 1, 0)$. If (12) holds, then

$$||Bx|| \leq \alpha||(A - \omega I)x|| + (\alpha\omega + \beta)\, ||x||,$$

and $(A - \omega I) + B \in G(X, 1, 0)$ by the case $\omega = 0$. This is equivalent to $A + B \in G(X, 1, \omega)$. □

Proposition 13. *For $0 < r < s < \infty$ and $u \in H^s$ we have*

$$||D^r u|| \leq ||D^s u||^{\frac{r}{s}} ||u||^{\frac{s-r}{r}} \leq \epsilon||D^s u|| + C(\epsilon, \frac{r}{s})||u||, \tag{13}$$

where $\epsilon > 0$ can be chosen arbitrarily and C is a continuous function of its arguments in the given range. Moreover, if $\deg P = n$, then

$$||D^n u|| \leq c\left(||P(D)u|| + ||u||\right), \tag{14}$$

for some c depending on P.

Proof. Using Hölder's inequality with respect to the non-negative measure $d\mu = |\widehat{u}(\xi)|^2 d\xi$, we obtain

$$||D^r u||^2 = \int \xi^{2r} d\mu \leq \left(\int \xi^{2r \cdot \frac{s}{r}} d\mu\right)^{\frac{r}{s}} \left(\int 1^{\frac{s}{s-r}} d\mu\right)^{\frac{s-r}{s}} = \left(||D^s u||^{\frac{r}{s}} ||u||^{\frac{s-r}{s}}\right)^2.$$

From the AM-GM inequality we deduce

$$\begin{aligned} ||D^s u||^{\frac{r}{s}} ||u||^{\frac{s-r}{s}} &= \left(\frac{s}{r}\epsilon||D^s u||\right)^{\frac{r}{s}} \left(\left(\frac{s}{r}\epsilon\right)^{\frac{-r}{s-r}} ||u||\right)^{\frac{s-r}{s}} \\ &\leq \frac{r}{s} \cdot \frac{s}{r}\epsilon||D^s u|| + \frac{s-r}{s} \cdot \left(\frac{s}{r}\epsilon\right)^{\frac{-r}{s-r}} ||u|| = \epsilon||D^s u|| + C(\epsilon, \frac{r}{s})||u||. \end{aligned}$$

Now let $P(D) = \sum_{k=0}^{n} a_k D^k$, where $a_n \neq 0$. Then

$$\begin{aligned} ||D^n u|| &\leq \frac{1}{|a_n|}||P(D)u|| + \frac{1}{|a_n|}||u|| + \sum_{k=1}^{n-1} \left|\frac{a_k}{a_n}\right| ||D^k u|| \\ &\leq \frac{1}{|a_n|}||P(D)u|| + \frac{1}{|a_n|}||u|| + \sum_{k=1}^{n-1} \left|\frac{a_k}{a_n}\right| \left(\epsilon||D^n u|| + C(\epsilon, \frac{k}{n})||u||\right). \end{aligned}$$

When ϵ is taken sufficiently small, the terms involving $||D^n u||$ on the right-hand side can be absorbed in the left-hand side, and (14) follows. □

Proposition 14. *Let $a, a', b \in L^\infty$. Then the following claims hold:*

1. *$M_a D + M_b \in G(L^2, 1, \omega)$ for $\omega \geq \omega_1 = \frac{1}{2}||a'||_\infty + ||b||_\infty$.*
2. *If $-P$ satisfies Kato's polynomial condition and $n = \deg P \geq 2$, then $P(D) + M_a D + M_b \in G(L^2, 1, \omega)$ for $\omega \geq \omega_1(||a'||_\infty, ||b||_\infty) + \omega_2(P)$, where ω_1 is as in part 1 and ω_2 depends on P as in Proposition 9.*

Proof. Our first aim is to apply Proposition 11 to deduce part 1. For both M_aD and M_b, we readily obtain the estimates

$$\begin{aligned}|(M_aDu,u)| &= \left|\int aDu\cdot udx\right| = \frac{1}{2}\left|\int aDu^2dx\right| \\ &= \frac{1}{2}\left|-\int a'u^2dx\right| \le \frac{1}{2}||a'||_\infty||u||^2, \\ |(M_bu,u)| &= \left|\int bu^2dx\right| \le ||b||_\infty||u||^2.\end{aligned}$$

Since M_a, M_b are self-adjoint (quite trivially) and D is skew-adjoint (by Proposition 6) on L^2, the same estimates are valid for $(M_aD)^* = -DM_a$ and $M_b^* = M_b$, and thus both $M_aD+M_b-\omega I$ and $(M_aD+M_b-\omega I)^*$ are dissipative for $\omega \ge \frac{1}{2}||a'||_\infty+||b||_\infty$, and by Proposition 11, $M_aD+M_b-\omega I \in G(L^2,1,0)$ for such ω, which is equivalent to $M_aD+M_b \in G(L^2,1,\omega)$. Part 1 is now established.

Now we also have

$$||\left(M_aD+M_b-\omega I\right)u|| \le ||a||_\infty||Du|| + \left(||b||_\infty+\omega\right)||u||. \tag{15}$$

Our intention is to apply Proposition 12 to derive $P(D)+M_aD+M_b \in G(L^2,1,\omega)$ from our previous knowledge that $P(D) \in G(L^2,1,\omega_2)$ (Proposition 9). For this we need to work further on the inequality (15) applying the results of Proposition 13. Indeed, using first (13) and then (14) we have

$$||Du|| \le \epsilon||D^nu|| + C(\epsilon)||u|| \le \epsilon c||P(D)u|| + \left(\epsilon c + C(\epsilon)\right)||u||.$$

Substituting back into (15) and taking ϵ sufficiently small, the condition (12) of Proposition 12 is satisfied. The other conditions of the same proposition hold rather evidently, and we conclude that $P(D)+M_aD+M_b-\omega' I \in G(L^2,1,\omega_2)$ for $\omega' \ge \omega_1$. Thus $P(D)+M_aD+M_b \in G(L^2,1,\omega)$ for $\omega \ge \omega_0 = \omega_1+\omega_2$. This completes the proof. ☐

2.3 Families of Operators and Uniformity

It is common in applications that the operators involved in evolution equations do not remain invariant but typically evolve in time producing a continuum of operators. We thus devote this section to the study of operator families. Due to the extreme usefulness of Kato's polynomial condition above in investigating operators involving differentiation and multiplication by bounded functions, we now wish to extend this definition. The form of Definition 15 below gains justification from the fact that the generation of C_0 semigroups by differential operators was completely characterized in terms of Kato's polynomial condition in Proposition 9. With a straightforward generalization, we obtain a necessary and sufficient condition for our *uniform Kato's polynomial condition* in Proposition 16, and we also seek some sufficient conditions of simpler nature.

Definition 15. *A family of polynomials $\{P_\theta : \theta \in \Theta\}$ of constant real coefficients is said to satisfy* Kato's polynomial condition uniformly *if there is an $\omega \in \mathbb{R}$ such that $-P_\theta \in G(L^2, 1, \omega)$ for all $\theta \in \Theta$ and this fixed ω.*

Proposition 16. *A family of polynomials $\{P_\theta : \theta \in \Theta\}$ of constant real coefficients satisfies Kato's polynomial condition uniformly if and only if the set of maxima of $-\Re P_\theta(i\xi)$ is bounded above uniformly in $\theta \in \Theta$. If each P_θ satisfies Kato's polynomial condition individually, and the degree of the even part of P_θ is bounded uniformly for $\theta \in \Theta$, then any of the following conditions is sufficient for the uniform Kato's polynomial condition:*

1. *The set of zeros, and the set of values of the leading coefficient of $\Re P_\theta(i\xi)$, $\theta \in \Theta$ are bounded uniformly for $\theta \in \Theta$.*

2. *The set $\left\{\sum_{h\in H_\theta} |c_{\theta h}| : \theta \in \Theta\right\}$ is bounded above, and we have the estimate $|d_\theta| \geq \beta \sum_{h\in H_\theta} |c_{\theta h}|$ where d_θ is the coefficient related to the highest even power x^{m_θ} of x in $P_\theta(x)$ and $H_\theta \subset 2\mathbb{N}$ is the set of coefficients of those even powers x^h in $P_\theta(x)$ for which $\operatorname{sgn} c_{\theta h} = -(-1)^{\frac{h}{2}}$.*

3. *The coefficients of even powers in all P_θ are independent of θ.*

Proof. The necessary and sufficient condition stated first is an immediate corollary of Proposition 9. By the same Proposition, Kato's polynomial condition implies the boundedness above of $-\Re P(i\xi)$. Part 1 now follows from the fact that the set of zeros of the derivative of a polynomial, and thus the set where the maxima are obtained, lies in the convex hull of the set of zeros of the polynomial itself, and is therefore bounded by the same bounds. Indeed, let now the set of zeros (and thus the set of points of maximum) be bounded by $\pm Z$, $Z \geq 1$, the values of the leading coefficients by $\pm C$ and the degree by M. We then have

$$\max -\Re P_\theta(i\xi) = \max \pm d_\theta \prod_{k=1}^{m_\theta} (\xi - z_k) \leq C \prod_{k=1}^{M} (Z + Z) = C(2Z)^M,$$

and we conclude that the set of maxima is, indeed, bounded.

We note that the sufficiency of condition 3 follows immeadiately from the sufficiency of condition 2, but emphasizes the fact that Kato's polynomial condition is only a matter of even powers. We then concentrate on condition 2.

When substituting $x = i\xi$ into the even part of $P(x)$, the even powers ξ^h get a sign which is $(-1)^{\frac{h}{2}}$ times the sign of the corresponding coefficient of x^h in $P(x)$, and thus the powers ξ^h, $h \in H_\theta$, referred to in condition 2 are exactly those that get a negative sign in $\Re P(i\xi)$. The leading term, by Kato's polynomial condition and Proposition 9, gets a positive sign, and the signs are reversed in $-\Re P(i\xi)$. Then, denoting $\sigma_\theta = \sum_{h\in H_\theta} |c_{\theta h}|$,

$$
\begin{aligned}
-\Re P_\theta(i\xi) &\le -|d_\theta|\xi^{m_\theta} + \sum_{h\in H_\theta} |c_{\theta h}|\xi^h \\
&\le \begin{cases} \sigma_\theta, & \text{if } |\xi| \le 1 \\ -|d_\theta|\xi^{m_\theta} + \sigma_\theta \xi^{h_\theta}, & \text{if } |\xi| > 1 \text{ and } h_\theta = \max H_\theta \end{cases}
\end{aligned} \tag{16}
$$

Here $\{\sigma_\theta\}$ is bounded by assumption. For the second case we can find the maximum by elementary calculus: Dropping the common subscripts and assuming $d > 0$ for convenience, the derivative with respect to ξ of this expression is $-md\xi^{m-1} + h\sigma\xi^{h-1}$, and in addition to the trivial zero we have a critical point at $\left(\frac{h\sigma}{md}\right)^{\frac{1}{m-h}}$. Evaluating the value of the function at this point we obtain $\left(\frac{h\sigma}{md}\right)^{\frac{h}{m-h}} \sigma \left(1 - \frac{h}{m}\right)$, the last factor of which is bounded by 1, and σ is bounded by assumption. Furthermore, $\frac{h}{m} < 1$, the exponent is non-negative and $\frac{h}{m-h} \le m-1$, and $\frac{\sigma}{d} \le \frac{1}{\beta}$ by assumption, so also the first factor is bounded. By inequality (16), the maximum of $-\Re P_\theta(i\xi)$ is no larger than this bound, and we are done. □

Proposition 16 gives a necessary and sufficient condition of uniform Kato's polynomial condition, but this condition is not easy to use in general. Of course, due to the form of this condition and the limited possibility of solving the maxima of a polynomial analytically, we do not expect any simple form for the condition solely in terms of the coefficients of the polynomials P_θ. A rather simple but by no means exhaustive condition on the coefficients is nevertheless given by condition 2 of Proposition 16 and condition 3 is, of course, quite trivial. However, this is in particular the case with all the equivalent forms (2) of the proper Korteweg–de Vries equation, even if time dependence is allowed in the coefficients.

2.4 Commutativity of Semigroups and Generators

The commutativity of objects is a strong property often of remarkable meaning in manipulating expressions. In this section we show how this property is inherited by semigroups from their infinitesimal generators under appropriate conditions. This result comes into use, since the purely differential operators investigated in this chapter (and shown to generate C_0 semigroups) certainly commute with each other. The tools to achieve this result turn out to be Proposition 17, which was used in a rather different context in Kato's work [6], and a simple construction with the graph norm.

The proof of Proposition 17 can be found in Kato [6, Pr. 2.4] (also in Pazy [15, Th. 5.8]), and it is related to the concept of *admissible subspaces* (see Kato [6, Def. 2.1] or Pazy [15, Def. 4.5.3]), which has significance in the development of the abstract results on evolution equations that we will cite below (Propositions 27, 29), but does not occur explicitly in the present work.

Proposition 17 (Kato). *Let S be an isomorphism of $Y \subset X$ onto X, where X and Y are Banach spaces and the embedding of X into Y is dense and continuous. If A and $A_1 = SAS^{-1}$ are infinitesimal generators of C_0 semigroups $T(t)$ and $T_1(t)$ on X, respectively, then $T_1(t) = ST(t)S^{-1}$ for $t \geq 0$, and $T(t)$ (restricted to Y) is a C_0 semigroup also on Y.*

We shall next state the part of the definition of *generalized Sobolev spaces* relevant to our present work. Definition 18 could be stated without the assumption of A being an infinitesimal generator, but the formulation used here is natural and sufficient in this context. We follow the definition in Engel and Nagel [5, Sect. II.5], but we only need here the first order space, although the concept is readily generalized to a space of any integral order. (We note that the first order Sobolev space H^1 of L^2 type is constructed in the manner of Definition 18 by taking $X = L^2, A = \Lambda^1$.) Some basic properties that we use are stated in the following Proposition 19; for the proof we again refer to the above mentioned text [5]. Thereafter we proceed to state and prove Proposition 20 on the above mentioned commutativity property, which is the main result of this section.

Definition 18. *Let A be the infinitesimal generator of a C_0 semigroup on a Banach space X such that $A^{-1} \in B(X)$. The space $X_A = \mathcal{D}(A)$ with the norm $||x||_A = ||Ax||_X$ is called the* generalized Sobolev space of first order *related to A.*

Proposition 19. *The space X_A is a Banach space, which is embedded in X densely and continuously, and $A : X_A \to X$ is an isometric isomorphism of X_A onto X.*

Proposition 20. *Let A and B be infinitesimal generators of C_0 semigroups, and let the semigroup generated by B be $T(t)$. If $[A, B] = 0$, then $[A, T(t)] = 0$ for all $t \geq 0$. In particular, if $Q(D)$ is a polynomial of D and $T(t)$ is the C_0 semigroup of generated by a polynomial $P(D)$ (where $-P$ satisfies Kato's polynomial condition), then $[Q(D), T(t)] = 0$ for all $t \geq 0$.*

Proof. We have in particular (Proposition 8) $A \in G(X, M, \omega)$, thus $\widetilde{A} = A - (\omega + \mu)I \in G(X, M, -\mu)$, and then $0 \in \rho(\widetilde{A})$, and $\widetilde{A}^{-1} \in B(X)$. Since the identity operator commutes with everything, we now also have $[\widetilde{A}, B] = 0$.

Now $\widetilde{A} : X_{\widetilde{A}} \to X$ is an isomorphism of $X_{\widetilde{A}}$ onto X, and $B = \widetilde{A}B\widetilde{A}^{-1}$ is the infinitesimal generator of a C_0 semigroup $T(t)$ on X. By Proposition 17 we thus deduce $T(t) = \widetilde{A}T(t)\widetilde{A}^{-1}$, i.e. $[\widetilde{A}, T(t)] = 0$. Then also $[A, T(t)] = 0$.

If $-Q$ satisfies Kato's polynomial condition, the particular case of the Proposition follows immediately from the rest of this Proposition together with Proposition 9. Otherwise, since Kato's polynomial condition only involves the sign of one of the coefficients, we can investigate the negative polynomial, and we still deduce the same conclusion, since clearly $Q(D)$ commutes with $U(t)$ if and only if $-Q(D)$ does so. $\square$

3 Estimates on Sobolev Norms

3.1 General Theory

In our study of equations on the H^s type spaces, the core of inequality manipulations will be in estimating different Sobolev norms. Although we mainly work with the Sobolev spaces H^s of L^2 type, some of our lemmas are more conveniently shown under a somewhat more general setting than the results we actually exploit later. For this we require a characterization of Sobolev spaces different from the Fourier transform procedure which is applicable to L^2 type spaces due to Placherel's theorem. In a general L^p setting, the spaces can be characterized by integrals over difference quotients as in Definition 21 below. This definition and the imbedded result are found (in a more general form) e.g. in Adams [1] and in Lacroix-Sonrier [10, Déf. III.4.9].

Definition 21. *For $p \in [1, \infty[$, $s \in]0, 1[$, the* Sobolev space of order s of L^p type on $\mathbb{R}$ *is defined by the following condition on the seminorm $|\cdot|_{s,p}$:*

$$W^{s,p} = \left\{ u \in L^p : |u|_{s,p}^p = \iint \frac{|u(x_1) - u(x_2)|^p}{|x_1 - x_2|^{1+ps}} dx_1 dx_2 < \infty \right\}. \tag{17}$$

The Sobolev spaces of order $s > 1$ are defined by

$$W^{s,p} = \left\{ u \in W^{\lfloor s \rfloor, p} : D^{\lfloor s \rfloor} u \in W^{s - \lfloor s \rfloor} \right\}.$$

The space $W^{s,p}$ becomes a Banach space with the norm

$$||u||_{s,p}^p = ||u||_{\lfloor s \rfloor, p}^p + |D^{\lfloor s \rfloor} u|_{s - \lfloor s \rfloor, p}^p.$$

Here the Sobolev spaces and norms of integral order are defined in the usual way.

The following standard results (which are special cases of significantly more general theorems, but sufficient for our purposes) often come to use in estimating Sobolev norms. Proposition 22 is a special case of one of the Sobolev imbedding theorems found e.g. in Adams [1, Th. 7.57(c)], and Proposition 23 (Palais [14, Th. 9.5]) is essentially a consequence of these theorems. A simple proof for Proposition 22 in the case $p = 2$ (which has already been applied above) is found in Pazy [15, L. 8.5.1(ii)]. Our formulation of Proposition 23 is similar to that used by Kato [8, L. A1].

Proposition 22 (Sobolev–Lions). *For $p > 1$, $s > \frac{1}{p}$ we have*

$$W^{s,p} \subset L^\infty \quad \textit{and} \quad ||f||_\infty \leq c(s,p) ||f||_{s,p}. \tag{18}$$

Proposition 23 (Palais). *For $s \geq t$ and $s > \frac{1}{p}$,*

$$f \in W^{s,p}, g \in W^{t,p} \quad \Longrightarrow \quad f \cdot g \in W^{t,p}, \quad ||f \cdot g||_{t,p} \leq c(s,t,p) ||f||_{s,p} ||g||_{t,p}. \tag{19}$$

3.2 Kato's Commutator

A technical lemma used in Kato's papers [8, 9] involved a commutator operator of the kind introduced in the following Proposition. A proof given in [8] for a multidimensional case effectively exploited Proposition 23, and Pazy [15] followed the same lines. For a special case of the result, we will present a more elementary proof on the real line $\mathbb{R}$ only, which only applies Plancherel's theorem and standard integration, with the intention of giving new insight into the nature of this result. The use of the assumption $s > \frac{3}{2}$ appears quite explicitly in the proof more than once. Excluding our restriction to the one-dimensional setting, this was the form in which the result was originally presented by Kato [8, L. A2], and it is fully sufficient for our purposes. However, the extended form [9, L. 2.6] is crucially important in proving certain results to be cited that lead to global existence theory, and we will give the short proof of the general case following Kato [8], although we will not explicitly use it.

Proposition 24 (Kato). *For $r > \frac{3}{2}$, $f \in H^r$ and $|s|, |t| \leq r-1$, we have*

$$T = \Lambda^{-s}[\Lambda^{s+t+1}, M_f]\Lambda^{-t} \in B(L^2) \qquad \textit{and} \qquad ||T|| \leq C||f'||_{r-1}.$$

In particular, for $s > \frac{3}{2}$, $f \in H^s$, we have

$$T = [\Lambda^s, M_f]\Lambda^{1-s} \in B(L^2) \qquad \textit{and} \qquad ||T|| \leq C||f'||_{s-1}.$$

Proof. Consider first the particular case of the Proposition. The Fourier transform of T is an integral operator with kernel

$$k(\xi,\eta) = \left((1+\xi^2)^{\frac{s}{2}} - (1+\eta^2)^{\frac{s}{2}}\right)\widehat{f}(\xi-\eta)(1+\eta^2)^{\frac{1-s}{2}},$$

i.e.

$$\widehat{Tu}(\xi) = \int k(\xi,\eta)\widehat{u}(\eta)d\eta.$$

By the mean value theorem and the fact that the derivative is monotone for positive reals, we have the estimate

$$\begin{aligned}\left|\frac{(1+\xi^2)^{\frac{s}{2}} - (1+\eta^2)^{\frac{s}{2}}}{\xi-\eta}\right| &\leq s\max\left\{(1+\xi^2)^{\frac{s-1}{2}}, (1+\eta^2)^{\frac{s-1}{2}}\right\} \\ &\leq s(1+\xi^2)^{\frac{s-1}{2}}\mathbf{1}_{\{|\eta|\leq\frac{|\xi|}{2}\}} + s2^{s-1}(1+\eta^2)^{\frac{s-1}{2}}.\end{aligned} \tag{20}$$

Applying this to the kernel we find that

$$\begin{aligned}|k(\xi,\eta)| \leq s(1+\xi^2)^{\frac{s-1}{2}}|\widehat{g}(\xi-\eta)|(1+\eta^2)^{\frac{s-1}{2}}\mathbf{1}_{\{|\eta|\leq\frac{|\xi|}{2}\}} + s2^{s-1}|\widehat{g}(\xi-\eta)| \\ = k_1(\xi,\eta) + k_2(\xi,\eta),\end{aligned}$$

where $\widehat{g}(\xi) = i\xi\widehat{f}(\xi)$, which is equivalent to $g = f'$.

For the boundedness of T, it now suffices to show that the operators T_1 and T_2 corresponding to the kernels k_1 and k_2, respectively, are bounded. Indeed,

$$\begin{aligned}\int \left| \int k(\xi,\eta)\widehat{u}(\eta)d\eta \right|^2 d\xi &\leq \int \left(\int |k(\xi,\eta)||\widehat{u}(\eta)|d\eta \right)^2 d\xi \\ &\leq 2\int \left[\left(\int k_1(\xi,\eta)|\widehat{u}(\eta)|d\eta \right)^2 + \left(\int k_2(\xi,\eta)|\widehat{u}(\eta)|d\eta \right)^2 \right] d\xi,\end{aligned}$$

where the AM-QM inequality was applied. The previous claim follows by taking the supremum over all $u \in L^2$ of unity norm on both sides and observing that functions whose Fourier transforms are $\widehat{u}$ and $|\widehat{u}|$ are of equal L^2-norm as well as the fact that the supremum taken over all unity-normed functions with real, non-negative Fourier transform (as on the right-hand side) is at most equal to the supremum over all functions of unity norm.

The operator $T_2 = s2^{s-1}M_g$ is readily estimated by

$$\begin{aligned}||T_2 u|| = s2^{s-1}||gu|| &= s2^{s-1}\left(\int |g(x)u(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq s2^{s-1}||g||_\infty \left(\int |u(x)|^2 dx \right)^{\frac{1}{2}} = s2^{s-1}||g||_\infty ||u|| \leq c||g||_{s-1}||u||,\end{aligned}$$

where Proposition 22 was used, since $s-1 > \frac{1}{2}$.

For T_1 we have

$$\begin{aligned}||T_1||^2 &\leq \int\int |k(\xi,\eta)|^2 d\eta d\xi \\ &= \int_{-\infty}^{\infty}\int_{-\frac{|\xi|}{2}}^{\frac{|\xi|}{2}} s^2(1+\xi^2)^{s-1}|\widehat{g}(\xi-\eta)|^2(1+\eta^2)^{1-s}d\eta d\xi \\ &= \left(\int_{-\infty}^{0}\int_{\chi}^{-\frac{\chi}{3}} + \int_{0}^{\infty}\int_{-\frac{\chi}{3}}^{\chi} \right) s^2(1+(\chi+\eta)^2)^{s-1}|\widehat{g}(\chi)|^2(1+\eta^2)^{1-s}d\eta d\chi,\end{aligned} \tag{21}$$

where the change of variable $\chi = \xi - \eta$ was performed.

For $t > \frac{1}{2}$, $x \mapsto (1+x^2)^t$ is a convex mapping, and Jensen's inequality yields

$$\begin{aligned}\left(1+(\chi+\eta)^2\right)^{s-1} &\leq \frac{1}{2}\left[\left(1+(2\chi)^2\right)^{s-1} + \left(1+(2\eta)^2\right)^{s-1} \right] \\ &\leq 2^{2s-3}\left[\left(1+\chi^2\right)^{s-1} + \left(1+\eta^2\right)^{s-1} \right].\end{aligned}$$

Using this estimate in (21), we see that the integrand is majorized by

$$s^2 2^{2s-3}\left(1+\chi^2\right)^{s-1}|\widehat{g}(\chi)|^2\left(1+\eta^2\right)^{1-s} + s^2 2^{2s-3}|\widehat{g}(\chi)|^2,$$

where the variables in both terms are separated. For the first term we have

$$\int \left(1+\chi^2\right)^{s-1} |\widehat{g}(\chi)|^2 d\chi = ||g||_{s-1} \qquad \int \left(1+\eta^2\right)^{1-s} d\eta < \infty,$$

since $1-s<-\frac{1}{2}$, and for the second

$$\int_{\min\{-\frac{\chi}{3},\chi\}}^{\max\{-\frac{\chi}{3},\chi\}} |\widehat{g}(\chi)|^2 d\eta = \frac{4}{3}|\chi||\widehat{g}(\chi)|^2 \leq \frac{4}{3}(1+\chi^2)^{s-1}|\widehat{g}(\chi)|^2,$$

the integral of which with respect to χ over all of $\mathbb{R}$ is again proportional to the norm $||g||_{s-1}$.

Combining all the estimates we find that, indeed,

$$||T|| \leq C||g||_{s-1} = C||f'||_{s-1},$$

and the particular case of the Proposition is established.

For the general case we proceed as above, but in (20) it is sufficient to estimate the maximum simply by the sum of the two values (since thereafter we will apply the strong results of Proposition 23). The majorizing operators T_1, T_2 will then be

$$T_1 = |s+t+1|\Lambda^t M_g \Lambda^{-t}, \qquad T_2 = |s+t+1|\Lambda^{-s} M_g \Lambda^{s},$$

which are essentially the same, since both s and t can attain both positive and negative values in a symmetric range.

Applying Proposition 23, since $r-1 \geq t$ and $r-1>\frac{1}{2}$, we then have

$$||\Lambda^t M_g \Lambda^{-t} u|| = ||g\Lambda^{-t}u||_t \leq C||g||_{r-1}||\Lambda^{-t}u||_t = C||f'||_{r-1}||u||,$$

and the general form of the Proposition follows. □

3.3 Compositions by Smooth Functions

Here we seek conditions to guarantee that compositions of smooth functions with functions of a Sobolev space remain in that Sobolev space with the norm bounded in some sense. This is done in Proposition 25, and a certain kind of continuity property of the composition operation is established in Proposition 26.

Proposition 25. *Let $y \in W^{s,p}$, $s \geq 1$, $p > 1$, and $a \in C^{\lceil s \rceil}$. If $a(0) = 0$, then $a \circ y \in W^{s,p}$ and in every case $(a \circ y)' \in W^{s-1,p}$. Furthermore,*

$$||a \circ y||_{s,p} \leq g\left(||a||_{\infty,cR,\lceil s \rceil}, R, c\right), \qquad R = ||y||_{s,p}$$

where g is monotone increasing in all arguments and $c = c(1,p)$ is the constant in (18).

Proof. We first note that the second assertion is an immediate consequence of the first one, since $a_0(x) = a(x) - a(0)$ satisfies the conditions of the first assertion and $(a \circ y)' = (a_0 \circ y)' \in W^{s-1,p}$ if $a_0 \circ y \in W^{s,p}$. We also note that $||f||_\infty \leq c(t,p)||f||_{t,p}$ for $t > \frac{1}{p}$ (Proposition 22), which implies $||y^{(\ell)}||_\infty \leq c(1,p)||y^{(\ell)}||_{1,p} \leq c(1,p)||y||_{\ell+1,p}$, so $y^{(\ell)} \in L^\infty$ for $\ell = 0, \ldots, \lfloor s \rfloor - 1$, and in particular, $y \in L^\infty$ for any s in the range of the given condition, with $||y||_\infty \leq c(1,p)||y||_{s,p}$.

Using the chain rule, it is readily verified by induction that

$$(a \circ y)^{(m)} = \sum_{k=1}^{m} \left(a^{(k)} \circ y\right) \sum_{\substack{\sum j_\ell = k \\ \sum \ell j_\ell = m}} c_{(j_\ell)_{\ell=1}^m} \prod_{\ell=1}^{m} \left(y^{(\ell)}\right)^{j_\ell} \qquad \forall m \in \mathbb{Z}^+, \tag{22}$$

where $c_{(j_\ell)_{\ell=1}^m}$ are finite numerical constants independent of a and y. Here $a^{(k)} \circ y \in L^\infty$, since $a^{(k)}$ attains a maximum in the compact set with $x \leq ||y||_\infty \leq c(1,p)||y||_{s,p}$. Furthermore, for $\ell \leq \lfloor s \rfloor - 1$, $||y^{(\ell)}||_\infty \leq c(1,p)||y||_{s,p}$, and by the conditions under the second summation in (22), this L^∞ estimate exists for all except possibly one of the factors $y^{(\ell)}$ in the product. For any $\ell \leq m \leq s$, we certainly have $y^{(\ell)} \in L^p$ with the norm bounded by $||y||_{s,p}$, and since an L^p function stays in L^p when multiplied by L^∞ functions, with the norm bounded in an obvious way, the L^p norm of each of the terms appearing in the summation (22) is bounded by a finite expression of the form $||a^{(k)}||_{\infty,||y||_\infty} c(1,p)^{k-1}||y||_{s,p}^k$, and we conclude that $(a \circ y)^{(m)} \in L^p$ for each $m = 1, \ldots, \lfloor s \rfloor$. For $m = 0$, we note that if $a(0) = 0$, then $a(x) = x \int_0^1 a'(xt)dt$, and we have

$$||a \circ y||_{0,p}^p = \int |a \circ y|^p dx \leq \int \left(|y| \cdot ||a'||_{\infty,||y||_\infty}\right)^p dx \leq ||a'||_{\infty,||y||_\infty}^p ||y||_{0,p}^p. \tag{23}$$

Hence $(a \circ y)^{(m)} \in L^p$ for $m = 0, \ldots, \lfloor s \rfloor$, and $a \circ y \in W^{\lfloor s \rfloor,p}$.

For integral s, we are now done. Otherwise, we must show that the expression in (22) is in $W^{s-\lfloor s \rfloor,p}$ for $m = \lfloor s \rfloor$. Note that in this case $\lceil s \rceil = \lfloor s \rfloor + 1$.

For the $a^{(k)} \circ y$ part we have

$$\begin{aligned} &\iint \frac{\left|a^{(k)}(y(x_1)) - a^{(k)}(y(x_2))\right|^p}{|x_1 - x_2|^{1+pt}} dx_1 dx_2 \\ &\leq ||a^{(k+1)}||_{\infty,||y||_\infty}^p \iint \frac{|y(x_1) - y(x_2)|^p}{|x_1 - x_2|^{1+pt}} dx_1 dx_2 \leq ||a^{(k+1)}||_{\infty,||y||_\infty}^p ||y||_{t,p}^p \end{aligned} \tag{24}$$

Assume for a while that $a^{(k)}(0) = 0$. Reasoning as in (23), $a^{(k)} \circ y \in L^p$ for $k \leq \lfloor s \rfloor$, since $a \in C^{\lceil s \rceil} = C^{\lfloor s \rfloor + 1}$, and we have the estimate of the desired form, since $a^{(k+1)}$ exists. By this and (24) we have, according to Definition 21, $a^{(k)} \circ y \in W^{t,p}$ for $t \in]0,1[$, and we can take $t > \frac{1}{p}$. Furthermore, we certainly have $y^{(\ell)} \in W^{s-\ell,p}$, where $s - \ell \geq s - \lfloor s \rfloor$, and for all factors except possibly one in the product in (22) we have $s - \ell \geq 1 > \frac{1}{p}$. Hence we deduce by Proposition 23 that $(a^{(k)} \circ y) \cdot \prod_{\ell=1}^m (y^{(\ell)})^{j_\ell} \in W^{s-\lfloor s \rfloor,p}$.

If $a^{(k)}(0) \neq 0$, let $a_0^{(k)}(x) = a^{(k)}(x) - a^{(k)}(0)$. Then $(a_0^{(k)} \circ y) \cdot \prod_{\ell=1}^m (y^{(\ell)})^{j_\ell} \in W^{s-\lfloor s \rfloor, p}$ by the previous part. We also have $\prod_{\ell=1}^m (y^{(\ell)})^{j_\ell} \in W^{s-\lfloor s \rfloor, p}$ by exactly the same reasoning, and since $W^{s-\lfloor s \rfloor, p}$ is a vector space, we conclude that

$$(a^{(k)} \circ y) \prod_{\ell=1}^m (y^{(\ell)})^{j_\ell} = (a_0^{(k)} \circ y) \prod_{\ell=1}^m (y^{(\ell)})^{j_\ell} + a(0) \prod_{\ell=1}^m (y^{(\ell)})^{j_\ell} \in W^{s-\lfloor s \rfloor, p}.$$

Thus all the terms in (22), with $m = \lfloor s \rfloor$, are in $W^{s-\lfloor s \rfloor, p}$, and we obtain $(a \circ y)^{\lfloor s \rfloor} \in W^{s-\lfloor s \rfloor, p}$. The last assertion of Proposition 25 follows by observing that all the bounds obtained during the course of the proof are of the form indicated by that assertion. ☐

Proposition 26. *Let $y_1, y_2 \in W^{s,p}$, $a \in C^{\lfloor s \rfloor + 2}$, $s \geq 1$ and $p > 1$. Then*

$$||a \circ y_1 - a \circ y_2||_{s,p} \leq g(||a||_{\infty, cR, \lfloor s \rfloor + 2}, R, c) ||y_1 - y_2||_{s,p}, \qquad R = \max\{||y_1||_{s,p}, ||y_2||_{s,p}\}, \tag{25}$$

where g is monotone increasing in all arguments and $c = c(1, p)$ is the constant in (18).

Proof. By Proposition 25, the left-hand side of (25) makes sense, i.e., the corresponding function is in the space indicated by the norm. The remarks concerning the L^∞ norms of various $y^{(\ell)}$ at the beginning of the proof of that Proposition are applicable here, too.

Using the expression (22) we have that

$$\begin{aligned}(a \circ y_1 - a \circ y_2)^{(m)} = &\sum_{k=1}^m \left[a^{(k)} \circ y_1 - a^{(k)} \circ y_2\right] \sum_{\substack{\sum j_\ell = k \\ \sum \ell j_\ell = m}} c_{(j_\ell)_{\ell=1}^m} \prod_{\ell=1}^m \left(y_1^{(\ell)}\right)^{j_\ell} \\ &+ \sum_{k=1}^m \left(a^{(k)} \circ y_2\right) \sum_{\substack{\sum j_\ell = k \\ \sum \ell j_\ell = m}} c_{(j_\ell)_{\ell=1}^m} \sum_{n=1}^m \left(\prod_{\ell=1}^{n-1} \left(y_2^{(\ell)}\right)^{j_\ell}\right) \times \\ &\left[\left(y_1^{(n)}\right)^{j_n} - \left(y_2^{(n)}\right)^{j_n}\right] \left(\prod_{\ell=n+1}^m \left(y_1^{(\ell)}\right)^{j_\ell}\right).\end{aligned} \tag{26}$$

The L^p norms of the first term are readily estimated as in the proof of Proposition 25, since all factors except possibly one in $\prod_{\ell=1}^m \left(y_1^{(\ell)}\right)^{j_\ell}$ have L^∞ norms bounded by $c||y||_{s,p}$ and all factors are in L^p with similar bounds on the norm. The L^∞ norm of the difference term is bounded by

$$||a^{(k)} \circ y_1 - a^{(k)} \circ y_2||_\infty \leq ||a^{(k+1)}||_{\infty, \max_i ||y_i||_\infty} ||y_1 - y_2||_\infty, \tag{27}$$

and the L^∞ norms of y_i are bounded in terms of $||y||_{s,p}$ as before.

In order to estimate the second term of (26) in L^p, we use the factorization

$$\left(y_1^{(n)}\right)^{j_n} - \left(y_2^{(n)}\right)^{j_n} = \left(y_1^{(n)} - y_2^{(n)}\right) \sum_{i=0}^{j_n-1} \left(y_1^{(n)}\right)^i \left(y_2^{(n)}\right)^{j_n-1-i}. \tag{28}$$

Now $||y_1^{(n)} - y_2^{(n)}||_{0,p} \leq c||y_1 - y_2||_{s,p}$ for any $n = 1, \ldots, \lfloor s \rfloor$ and all other factors present in the products in the second term of (26) have L^∞ bounds in terms of $||y_1||_{s,p}^{k_1}||y_2||_{s,p}^{k_2}||a^{(k)}||_{\infty,||y_2||_\infty}$. Indeed, for $n \leq \lfloor s \rfloor - 1$, $y^{(n)} \in L^\infty$ as before, and in the case $n = \lfloor s \rfloor$, the summation over products in (26) reduces to $\left[y_1^{(\lfloor s \rfloor)} - y_2^{(\lfloor s \rfloor)}\right] \in L^p$ and this is only multiplied by $a^{(k)} \circ y \in L^\infty$.

For integral s we now only need to bound $||a \circ y_1 - a \circ y_2||_{0,p}$. This is readily done by raising the following inequality to the power of p and integrating over $\mathbb{R}$:

$$|(a \circ y_1 - a \circ y_2)(x)| \leq ||a'||_{\infty,\max_i ||y_i||_\infty} |(y_1 - y_2)(x)|.$$

Otherwise, we must estimate the $W^{s-\lfloor s \rfloor,p}$ norm of (26) as in the proof of Proposition 25.

To this end, we investigate the numerator of the integrand appearing in the definition of the Sobolev seminorm in (17):

$$\begin{aligned}
&\left|(a^{(k)} \circ y_1 - a^{(k)} \circ y_2)(x_1) - (a^{(k)} \circ y_1 - a^{(k)} \circ y_2)(x_2)\right| \\
&= \left|a^{(k+1)}(\eta_1) \cdot (y_1 - y_2)(x_1) - a^{(k+1)}(\eta_2) \cdot (y_1 - y_2)(x_2)\right| \\
&\quad (\eta_i \text{ between } y_1(x_i) \text{ and } y_2(x_i)) \\
&= \left|a^{(k+1)}(\eta_1) \cdot [(y_1 - y_2)(x_1) - (y_1 - y_2)(x_2)]\right. \\
&\quad + \left.\left[a^{(k+1)}(\eta_1) - a^{(k+1)}(\eta_2)\right] \cdot (y_1 - y_2)(x_2)\right| \\
&\leq ||a^{(k+1)}||_{\infty,\max_i ||y_i||_\infty} |(y_1 - y_2)(x_1) - (y_1 - y_2)(x_2)| \\
&\quad + ||a^{(k+2)}||_{\infty,\max_i ||y_i||_\infty} |\eta_1 - \eta_2| \cdot ||y_1 - y_2||_\infty.
\end{aligned} \tag{29}$$

By the choice of η_1 and η_2 we further have

$$|\eta_1 - \eta_2| \leq |y_1(x_1) - y_1(x_2)| + |y_2(x_1) - y_2(x_2)|.$$

We take the power of p, apply the power-mean inequality on the right-hand side of (29) to convert the power of a sum into a sum of powers, divide by $|x_1 - x_2|^{1+pt}$, $t \in]0,1[$, and integrate over $\mathbb{R}^2$. This yields

$$\begin{aligned}
\left|a^{(k)} \circ y_1 - a^{(k)} \circ y_2\right|_{t,p}^p \leq\, & ||a^{(k+1)}||_{\infty,\max_i ||y_i||_\infty}^p |y_1 - y_2|_{t,p}^p \\
& + ||a^{(k+2)}||_{\infty,\max_i ||y_i||_\infty}^p \left(|y_1|_{t,p}^p + |y_2|_{t,p}^p\right) ||y_1 - y_2||_\infty,
\end{aligned}$$

and we find that $a^{(k)} \circ y_1 - a^{(k)} \circ y_2 \in W^{t,p}$, $k = 1, \ldots, \lfloor s \rfloor$, with the norm bounded by a similar form as the right-hand side of (25). Since the rest of the first term on the right-hand side of (26), for $m = \lfloor s \rfloor$, is in $W^{s-\lfloor s \rfloor,p}$, we then take $t > \frac{1}{p}, t \geq s - \lfloor s \rfloor$ and conclude (Proposition 23) that all of the first term of (26) is in $W^{s-\lfloor s \rfloor,p}$ with the norm bounded as in (25).

We then have to show the same for the second term on the right-hand side of (26), but this can be done in a rather straightforward manner, since exactly the same reasoning as in the proof of Proposition 25 can be used, except that one factor $y^{(\ell)}$ is replaced by a difference term $y_1^{(\ell)} - y_2^{(\ell)}$ (by applying the factorization (28)), and this is in the same space and has a norm bounded in the desired way. This completes the proof. $\square$

4 Local Existence Theorems

4.1 Existence of Solutions for a General Class of Equations

We now have sufficient knowledge of the operators and norms involved to apply the general results on abstract evolution equations to the Korteweg–de Vries type equations. The only machinery lacking is the statement of those results, which is done in the following. The proofs may be found in Kato [8, Th.'s 6, 7], and for the existence theorem under essentially similar assumptions also in Pazy [15, Th. 6.4.6].

We present Kato's results in a slightly simplified form sufficient for our purposes. On the other hand, we have also made some refinements following the formulation of these theorems in a later paper of Kato [9], but these are rather straightforward. The very last assertion of Proposition 27 follows by observing the restrictions placed on the interval of existence $[0, T_1]$ in the proof of the rest of Proposition 27 [8, Th. 6]. In the original formulation of Proposition 29 [8, Th. 7], the continuous dependence is not guaranteed to hold for every $T_2 < T_1$, but only for some T_2. However, this T_2 only depends on the Lipschitz constant L' in (38) and the radius r of the ball B_r considered in the theorems. With this knowledge, it is easy to extend the continuous dependence to any given $T_2 < T_1$ in a finite number of steps by the standard argument.

The abstract theorems of Kato would also allow us to consider "forced" equations with a non-homogenous term $f(t) = f(t, x)$ added (with appropriate conditions; see Kato [8]), and it would be possible also to establish continuous dependence of the solution not only on the "data" (the initial value ϕ), but also on the "model" (the coefficients of the differential equation). However, no new ideas would be involved, so we do not consider these extensions here any further, but proceed to state the above mentioned results.

Proposition 27 (Kato). *Let the following assumptions be satisfied:*

1. *X and Y are reflexive Banach spaces, and $Y \subset X$ is embedded in X densely and continuously. There is an isometric isomorphism $S : Y \to X$.*

2. *$-A(t, v) \in G(X, 1, \omega)$ for all $t \in [0, T]$, $v \in B_r = \{y \in Y : ||y||_Y < r\}$ and a fixed $\omega \in \mathbb{R}$.*

3. *$[S, A(t,v)]S^{-1} \in B(X)$ and is uniformly bounded for $t \in [0,T]$, $v \in B_r$.*
4. *For all $t \in [0,T]$, $v, v_1, v_2 \in B_r$, $A(t,v) \in B(Y,X)$ is continuous in t in the operator norm and satisfies uniformly in t the Lipschitz condition*

$$||A(t,v_1) - A(t,v_2)||_{B(Y,X)} \leq L||v_1 - v_2||_X. \tag{30}$$

Then the evolution equation

$$\frac{du}{dt} + A(t,u)u = 0 \qquad \forall t \in [0,T], \qquad u(0) = \phi \in B_r \tag{31}$$

has, for some $T_1 = \in]0,T]$, a unique solution

$$u \in C([0,T_1] : B_r) \cap C^1([0,T_1] : X). \tag{32}$$

T_1 has a lower bound depending continuously on $||\phi||_Y$.

Theorem 28. *Let $P(t,D) = \sum_{k=0}^{n} c_k(t)D^k$ be a polynomial in the differentiation operator D, the coefficients of which are functions of t. Let $\{P(t,\cdot) : t \in [0,T]\}$ satisfy Kato's polynomial condition uniformly, and $c_k \in C[0,T]$. Let $s \geq n \geq 2$ and $a, b \in C([0,T] : C^{\lceil s \rceil}[-cr, cr])$ where $c = c(1,2)$ is the constant in (18) and $r > 0$. Then the initial value problem*

$$\begin{aligned} \frac{du}{dt} + P(t,D)u + a(t,u)Du + b(t,u)u = 0 \quad \forall t \in [0,T], \\ u(0) = \phi \in H^s, \quad ||\phi||_s < r \end{aligned} \tag{33}$$

has, for some $T_1 \in]0,T]$, a unique solution

$$u \in C([0,T_1] : H^s) \cap C^1([0,T_1] : H^{s-n}). \tag{34}$$

T_1 has a lower bound depending continuously on $||\phi||_s$.

Proof. By Proposition 27, it suffices to verify the conditions of that Proposition for an appropriate choice of the operators and spaces. We will actually use Proposition 27 to establish the somewhat weaker result with H^{s-n} in (34) replaced by L^2. The exact claim of the Theorem then follows by solving for $\frac{du}{dt}$ in (33) and noting that the other terms are in H^{s-n} for each t and $u \in H^s$ satisfying the equation. Indeed, $P(t,D)u \in H^{s-n}$ and $Du \in H^{s-1}$ quite obviously, and Proposition 25 implies that $a_0(t) \circ u \in H^s \subset H^{s-1}$, where $a_0(t,x) = a(t,x) - a(t,0)$, and thus $a_0(t) \circ u \cdot Du \in H^{s-1}$ by Proposition 23. Now $a(t) \circ u \cdot Du \in H^{s-1} \subset H^{s-n}$ follows from the vector space property, since clearly $a(t,0)Du \in H^{s-1}$. A similar reasoning, of course, applies to $b(t) \circ u \cdot u$.

We now verify the conditions of Proposition 27 concerning the operator $A(t,v) = P(t,D) + M_{a(t,v)}D + M_{b(t,v)}$.

1. $X = L^2$ and $Y = H^s$ are reflexive Banach spaces, with $H^s \subset L^2$ densely and continuously embedded. $\Lambda^s : H^s \to L^2$ is an isometric isomorphism.

2. By Proposition 14, each $-A(t,v)$ individually is in $G(L^2,1,\omega)$, where $\omega = \omega_1(||(a(t)\circ v)'||_\infty, ||(b(t)\circ v)||_\infty) + \omega_2(P(t))$. The assumption of uniform Kato's polynomial condition ensures that $\omega_2(P(t))$ is uniformly bounded in t (Definition 15, Propositions 14, 9). We also verify readily that

$$\begin{aligned} ||a(t)'\circ v\cdot v'||_\infty &\le ||a(t)'||_{\infty,||v||_\infty}||v'||_\infty \\ &\le ||a(t)||_{\infty,c||v||_s,1}c||v||_s, \le \max_{t\in[0,T]}||a(t)||_{\infty,cr,\lceil s\rceil}cr. \end{aligned} \tag{35}$$

Similar reasoning holds for $||b(t)\circ v||_\infty$, and ω_1 was monotone increasing in both arguments (Proposition 14). Thus we find a fixed ω such that $-A(t,v)\in G(L^2,1,\omega)$ for all $t\in[0,T]$, $v\in B_r$.

3. Since D and thus $P(t,D)$ commute with Λ^s for each t, we have

$$[\Lambda^s, A(t,v)]\Lambda^{-s} = [\Lambda^s, M_{a(t,v)}]\Lambda^{-s}D + [\Lambda^s, M_{b(t,v)}]\Lambda^{-s} \tag{36}$$

Using Propositions 24 and 25 we find that

$$\begin{aligned} ||[\Lambda^s, M_{a(t,v)}]\Lambda^{-s}Du|| &\le ||[\Lambda^s, M_{a(t,v)}]\Lambda^{1-s}||\cdot||\Lambda^{-1}Du|| \\ \le C||(a(t)\circ v)'||_{s-1}||u|| &\le g(||a(t)||_{\infty,c||v||_s,\lceil s\rceil}, ||v||_s, c) \\ &\le g(\max_{t\in[0,T]}||a(t)||_{\infty,cr,\lceil s\rceil}, r, c). \end{aligned}$$

The same reasoning applies to the second term on the right-hand side of (36) with the only exception in the second estimate where we only have $||\Lambda^{-1}u||\le||u||$. This establishes condition 3.

4. We certainly have $D\left(A(t,v)\right)\supset H^s$, with $A(t,v): H^s\to L^2$ a bounded operator, since the differentiation operators are bounded from H^s to L^2 in an obvious way and the multiplicant functions $a(t,v)$ and $b(t,v)$ are in L^∞, since, for fixed t, they are continuous functions in $v\in B_r$, which varies by Proposition (18) over a finite interval only. The continuity in t follows from the continuity in t of the time dependent coefficients in the maximum norm, since this clearly gives a bound for the operator norm of a point-wise multiplication operator on L^2.

   In the verification of the Lipschitz condition (30), we only need to bother about the part $M_{a(t,v)}D + M_{b(t,v)}$, since the rest is independent of v. We have

$$\begin{aligned} & ||\left(A(t,v_1)-A(t,v_2)\right)u|| \\ & \le ||a(t,v_1)-a(t,v_2)||\cdot||u'||_\infty + ||b(t,v_1)-b(t,v_2)||\cdot||u||_\infty \\ & \le C\left(||a(t)'||_{\infty,cr} + ||b(t)'||_{\infty,cr}\right)||v_1-v_2||\cdot||u||_s, \end{aligned} \tag{37}$$

   where a result derived by integrating over $\mathbb{R}$ the inequalities of type $|f(v_1(x))-f(v_2(x))|^2\le\max|f'|^2\cdot|v_1(x)-v_2(x)|^2$ was applied to produce the second inequality. By the continuity of a' and b' in t in the spatial maximum norm on a compact interval, we deduce that a bound uniform in t can be found for the maximum norms of derivatives in (37).

By Proposition 27, and the observations in the beginning of the proof, the verification of Theorem 28 is complete. □

Proposition 29. *In addition to the conditions of Proposition 27, assume the commutator Lipschitz condition*

$$||[S, A(t, v_1) - A(t, v_2)]S^{-1}||_X \leq L'||v_1 - v_2||_Y \tag{38}$$

for $v_1, v_2 \in B_r$. Then, if $\phi_n \to \phi$ in Y and $T_2 < T_1$, we have a unique solution u_n of (31) with ϕ replaced by ϕ_n, such that u_n is in the same class as u in (32) for all sufficiently large n, and $u_n(t) \to u(t)$ in Y uniformly in $t \in [0, T_2]$.

Theorem 30. *Let $a, b \in C([0, T] : C^{\lfloor s \rfloor + 2}[-cr, cr])$ (with maximum norm of appropriate derivatives) and let the assumptions of Theorem 28 be satisfied. Then, if $\phi_n \to \phi$ in H^s and $T_2 < T_1$, we have a unique solution u_n of (33) with ϕ replaced by ϕ_n, such that u_n is in the same class as u in (34) for all sufficiently large n, and $u_n(t) \to u(t)$ in H^s uniformly in $t \in [0, T_2]$.*

Proof. It clearly suffices to verify the condition (38) of Proposition 29. It follows as in verification of condition 3 in Theorem 28 that the operators $P(t, D)$, which are independent of v in $A(t, v)$ and commute with Λ^s, cancel, and it suffices to establish the bounds for the terms involving multiplication operators:

$$\begin{aligned} ||[\Lambda^s, M_{a(t,v_1)-a(t,v_2)}]\Lambda^{1-s}\Lambda^1 Du|| &\leq c||\,(a(t, v_1) - a(t, v_2))'\,||_{s-1}||u|| \\ &\leq g(\max_{t\in[0,T]} ||a(t)||_{\infty,cr,\lfloor s \rfloor+2}, r, c)||v_1 - v_2||_s||u|| \end{aligned}$$

where Proposition 24 was used for the first inequality and Proposition 26 for the second, with the fact $||v_1||_s, ||v_2||_s < r$. An analogous estimate holds for the similar term involving b, and the Lipschitz condition (38) is established. □

4.2 Improved Results for Equations with Only Odd Order Derivatives

We next prove a stronger existence theorem with a larger range of valid s as the order of the Sobolev space H^s, but to achieve this, following closely the ideas of Kato [9], we are forced to make more restrictive assumption on the polynomial $P(D)$ in the evolution equation. Indeed, we now want to apply Proposition 6 to obtain a C_0 group of unitary operators $U(t)$ and are therefore restricted to odd polynomials $P(D)$. We also give away the time dependence of coefficients in order to use the time t as the group parameter for a particular transformation that we require.

In view of the proof of Theorem 32, we cite one more result of significance in the theory of abstract evolution equations: Proposition 31 by Darmois [3] strengthens the theorem of Kato (Proposition 27) used above, and its applicability in this context was pointed out by Kato [9] in the proof of Theorem 32 below for the case (4).

Proposition 31 (Darmois). *Proposition 27 is also true with $A(t,v) \in B(Y,X)$ only strongly continuous in $t \in [0,T]$ in the condition 4.*

Theorem 32. *Let $P(D)$ be an odd polynomial in D. Let $s > \frac{3}{2}$ and $a, b \in C([0,T] : C^{\lceil s \rceil}[-cr, cr])$ where $c = c(1,2)$ is the constant in (18) and $r > 0$. Then the initial value problem*

$$\begin{aligned} \frac{du}{dt} + P(D)u + a(t,u)Du + b(t,u)u = 0 \quad \forall t \in [0,T], \\ u(0) = \phi \in H^s, \quad ||\phi||_s < r \end{aligned} \tag{39}$$

has, for some $T_1 \in]0,T]$, a unique solution

$$u \in C([0,T_1] : H^s) \cap C^1([0,T_1] : L^2), \tag{40}$$

which depends continuously on the initial value ϕ in the sense of Theorem 30.

Proof. Let $U(t)$ be the group of unitary operators on L^2 or H^s generated by $P(D)$ according to Proposition 6 and perform the transformation $v(t) = U(t)u(t)$ resulting in the equation

$$\begin{aligned} \frac{dv}{dt} + A(t,v)v = 0, \qquad v(0) = \phi, \\ A(t,y) = U(t)\left(M_{a(t,\widetilde{y})}D + M_{b(t,\widetilde{y})}\right)U(-t), \qquad \widetilde{y} = U(-t)y. \end{aligned} \tag{41}$$

Our intention is to verify the conditions of Proposition 27 for this transformed equation, in which the operator $A(t,v)$ is only of the first order in D, unlike in the original problem (39), where the degree of the polynomial P can be arbitrarily large. In the verification of condition 4 of Proposition 27 we make use of the sufficiency of the weaker condition guaranteed by Proposition 31. The space where we seek the solutions of (41) will be the same solution space (40) as for the original problem, and the fact that the transformation operator $U(t)$ is unitary and strongly continuous (C_0) on both L^2 and H^s, we then have u in the same space as v.

1. Verification of the first condition is exactly as in Theorem 28.

2. By Proposition 14, $\widetilde{A}(t,\widetilde{y}) = M_{a(t,\widetilde{y})}D + M_{b(t,\widetilde{y})} \in G(L^2, 1, \omega)$ for $\omega \geq \frac{1}{2}||(a(t) \circ \widetilde{y})'||_\infty + ||b(t) \circ \widetilde{y}||_\infty$, and the supremum norms may be bounded as in (35), noting that $||\widetilde{y}||_s = ||y||_s$, since $U(t)$ is unitary. The fact that $A(t,y) = U(t)\widetilde{A}(t,\widetilde{y})U(-t) \in G(L^2, 1, \omega)$ then follows by observing that the conditions of Proposition 8 hold for $U\widetilde{A}U^*$ if they hold for $\widetilde{A}$, when U is unitary. Finally, we note that the sign in $\pm A(t,y)$ is quite irrelevant here, since it can be absorbed in the multiplication operators by $-M_a = M_{-a}$.

3. Since D and thus $P(D)$ commutes with Λ^s, it follows that $P(D) = \Lambda^s P(D) \Lambda^{-s}$, and according to Proposition 6, $P(D)$ is the infinitesimal

generator of the C_0 group $U(t)$ on L^2. Proposition 17 with $A = A_1 = P(D)$ together then implies $U(t) = \Lambda^s U(t) \Lambda^{-s}$. Thus $U(t)$ and Λ^s commute, and we have

$$[\Lambda^s, A(t,y)]\Lambda^{-s} = U(t)[\Lambda^s, \widetilde{A}(t,\widetilde{y})]\Lambda^{-s}U(-t).$$

Since $U(t)$ is unitary, it is sufficient to bound the norm of

$$[\Lambda^s, \widetilde{A}(t,\widetilde{y})]\Lambda^{-s} = [\Lambda^s, M_{a(t,\widetilde{y})}]\Lambda^{-s}D + [\Lambda^s, M_{b(t,\widetilde{y})}]\Lambda^{-s},$$

but this is exactly the same as in (36) and the calculations used in part 3 of Theorem 28 can be repeated.

4. The fact that $A(t,y) \in B(H^s, L^2)$ and the Lipschitz condition (30) can be seen essentially by repeating the argument in part 4 of the proof of Theorem 28 and using the fact that $U(t)$ is unitary. The strong continuity in t in the $B(H^2, L^2)$ norm as required by Proposition 31 results from the following:

   (a) $D : H^s \to L^2$ is bounded.

   (b) $U(t)$ is a C_0, i.e. strongly continuous, unitary group on L^2.

   (c) The operator norms of the point-wise multiplication operators on L^2 are bounded by the L^∞ norms of the multiplicant functions $a(t, U(-t)y)$, $b(t, U(-t)y)$, for which we have

   $$\begin{aligned}||a(t_1, U(-t_1)y) - a(t_2, U(-t_2)y)||_\infty \\ \le ||a(t_1, U(-t_1)y) - a(t_2, U(-t_1)y)||_\infty \\ + ||a(t_2, U(-t_1)y) - a(t_2, U(-t_2)y)||_\infty,\end{aligned}$$

   and the first term on the right will approach zero due to the continuity of a in t in the maximum norm, whereas for the second term we invoke Proposition 26 and recall the strong continuity of $U(t)$.

5. Using the commutativity of Λ^s and $U(t)$ as in part 3 of the proof, the verification of the Lipschitz condition (38) of Proposition 29 reduces to the situation in the proof of Theorem 30, and the same argument can be repeated.

These conditions verified, Propositions 27 and 31 then imply the claim. □

Having proved Theorem 32, we now make further reductions on the class of equations under consideration, so that we totally drop the explicit dependence on t and also the extra term $b(u)u$, to get an equation

$$u_t + \sum_k c_k u^{(2k+1)} + a(u)u' = 0. \tag{42}$$

Then we are in the situation considered by Kato [9], except for having a general odd polynomial $P(D)$ instead of just D^3. However, the results of Kato were established for the transformed equation (41) (with $a(t,\widetilde{y}) = a(\widetilde{y})$ and $b(t,\widetilde{y}) = 0$), and thus these theorems are immediately applicable. We only need to note that an arbitrary odd polynomial $P(D)$ generates a C_0 group $U(t)$ of unitary operators on L^2 and H^s by Proposition 6 as before, and the commutativity properties required in some algebraic manipulations with polynomial differentiation operators and the Λ^s follow from Propositions 20 and 17 (the latter in the manner indicated in the part 3 of the proof of Theorem 32).

With the remarks above, the results of Kato [9, Th.'s I(c), II] extend more or less directly to the following Theorem 33. The proof of this result is where Kato uses the general form of Proposition 24. We also note that since we do not now have the explicit time dependence, the differential equation (42) may be considered on all of $\mathbb{R}^+$.

Theorem 33 (Kato). *Let $a \in C^\infty$. Then for equations of type (42), for which a local solution of the initial value problem (39) is guaranteed on some proper interval $[0, T_1]$ by Theorem 32, the following hold:*

1. *The value T_1 can be chosen independently of the order s of the Sobolev space H^s in the following sense: If u is a solution satisfying (40) and the initial value $\phi \in H^r$ for some $r > \frac{3}{2}$, then u also satisfies (40) with s replaced by r. In particular, if $\phi \in H^\infty = \cap_s H^s$, then u satisfies (40) with $s = \infty$.*

2. *If such an equation satisfies Kato's global growth condition, then T_1 can be taken arbitrarily large, and we have a global solution*

$$u \in C(\mathbb{R}^+ : H^s) \cap C^1(\mathbb{R}^+ : L^2), \tag{43}$$

whenever the initial value $\phi \in H^s$ where $s \geq s_0$ and s_0 is the order for which Kato's global growth condition is satisfied.

This completes our survey of the local theory for Korteweg–de Vries type equations, and also paves the way for the global results.

5 Global Theory

To obtain global solutions of an evolution equation, the basic idea is to show that a local solution can be extended arbitrarily far in a somehow controlled manner. Usually this control is established by means of *a priori* estimates of the solution, which play a significant part in the theory of partial differential equations. (See e.g. Bona and Smith [2, Sect. 4].) As asserted in Theorem 33, Kato's global growth condition, when satisfied, is a sufficient *a priori* bound to derive global existence of the solution. It is thus the matter of this section to seek more explicit conditions for that to hold.

5.1 Conservation Laws and *A Priori* Bounds

The existence of conservation laws plays a central role in establishing *a priori* bounds on the solutions of a differential equation. The definition similar to the following was given by Miura, Gardner and Kruskal [12]; strictly speaking, our definition concerns what they called a *local conservation law.*

Definition 34. *An identity of the form $T_t + X_x = 0$, which is satisfied by each solution u of a given evolution equation in a given function class, is called a* conservation law. *Here the* conserved density T *and the* flux $-X$ *are functionals of u of the form $f \circ (u, u', \dots, u^n)$, where $f : \mathbb{R}^{n+1} \to \mathbb{R}$. We also require that $X(u)(x,t) \to 0$ as $x \to \pm\infty$ to ensure that the conserved density is related to a* constant of motion $\int T dx$, *i.e.* $\frac{d}{dt} \int T dx = 0$.

The following Proposition 35 establishes the existence of three conserved densities for a class of Korteweg–de Vries type equations. With the original physical interpretation of the proper Korteweg–de Vries equation, these correspond to the conservation laws of mass, horizontal momentum and energy (Drazin and Johnson [4, Sect 5.1]). For the proper Korteweg–de Vries equation (1) and a similar equation $u_t + u^2 u_x + u_{xxx} = 0$, a countable infinity of conservation laws has been shown to exist by Miura, Gardner and Kruskal [12], but nothing beyond the three conserved densities of Proposition 35 can be shown in the general case; in fact, among the equations

$$u_t + u^p u_x + u_{xxx} = 0 \qquad p \in \mathbb{Z}^+$$

the two above mentioned cases $p = 1, 2$ are the only ones possessing more than three polynomial conservation laws (Miura [11]).

Proposition 35. *The generalized Korteweg–de Vries equation (42) with $a \in C$ has the conserved densities*

$$u, \qquad \frac{1}{2}u^2 \qquad \text{and} \qquad \frac{1}{2}\sum_k (-1)^k c_k (u^{(k)})^2 + a^{(-2)} \circ u, \tag{44}$$

where antiderivatives of a are taken to have a zero at the origin.

Proof. The first conserved density is obtained directly by integrating (42) over $\mathbb{R}$. For the second we first multiply (42) by u and then integrate. The second term vanishes, since $\int u^{(2n+1)} u dx = (-1)^n \int u^{(n+1)} u^{(n)} dx$ and the integrand is now an exact spatial derivative. So is the case for $a(u)u \cdot u'$ resulting from the third term, and denoting by α the integral function of $x \mapsto a(x)x$ with $\alpha(0) = 0$, we have that $a(u)u \cdot u' = (\alpha \circ u)'$, and $u(x) \to 0$ as $x \to \pm\infty$ while $\alpha(u) \to 0$ as $u \to 0$.

For the third conserved density, we note that $a(u)u' = (a^{(-1)} \circ u)'$, where we use the freedom to choose the antiderivative in such a fashion that $a^{(-1)}(0) = 0$. We first multiply (42) by $c_n u^{(2n)}$ and integrate to obtain

$$\begin{aligned} \int \Big[(-1)^n u^{(n)} u_t^{(n)} &+ \sum_k c_k (-1)^{k+n} u^{(k+n)} u^{(k+n+1)} \\ &+ \big(a^{(-1)}(u) u^{(2n)}\big)' - a^{(-1)}(u) u^{(2n+1)} \Big] dx = 0, \end{aligned} \tag{45}$$

where the left-over terms resulting from integration by parts of polynomials of spatial derivatives of u have been discarded immediately with implicit understanding that u is in a Sobolev space of sufficiently high order for such terms to vanish at infinity. Now the second and third terms of the integrand are also exact spatial derivatives, the first of which is at once seen to vanish in integration and also the other after noting that $a^{(-1)} \circ u(x) \to 0$ as $x \to \pm\infty$, since $u(x) \to 0$ and $a^{(-1)}$ is continuous and has a zero at the origin.

We then multiply (42) by $a^{(-1)}(u)$ yielding

$$a^{(-1)}(u)u_t + \sum_k c_k a^{(-1)}(u)u^{(2k+1)} + a^{(-1)}(u)a(u)u' = 0, \tag{46}$$

where the last term is found to vanish in integration after writing it in the form $a^{(-1)}(u)a(u)u' = (a^{(-1)} \cdot a) \circ u \cdot u' = \frac{1}{2}\left((a^{(-1)})^2\right)' \circ u \cdot u' = \frac{1}{2}\left((a^{(-1)})^2 \circ u\right)'$ and using, as above, the facts that $u(x) \to 0$ as $x \to \pm\infty$ and $a^{(-1)}(u) \to 0$ as $u \to 0$.

The unwanted terms seen to vanish, we then apply $\sum_n c_n \cdot$ to (45) and add it to (46) integrated over $\mathbb{R}$ to deduce, after observing the obvious exact time derivatives,

$$\frac{d}{dt}\int \left[\frac{1}{2}\sum_n c_n(-1)^n (a^{(n)})^2 + a^{(-2)} \circ u\right] dx = 0,$$

which establishes the remaining conservation law. $\square$

The conserved densities at hand, the following Proposition 36 gives an *a priori* bound for the class of equations considered in the previous Proposition 35, with a growth condition imposed on a. In doing so, it generalizes (and is inspired by) some of the ideas used by Bona and Smith [2, Pr. 2] to derive a similar result for the regularized proper Korteweg–de Vries equation $u_t + uu_x + u_{xxx} - \epsilon u_{xxt} = 0$. However, we do not here consider the regularization term ϵu_{xxt}, which was related to the method of Bona and Smith to derive existence and continuous dependence theorems for the proper Korteweg–de Vries equation via convergence of the solutions of the regularized equation as $\epsilon \to 0$.

Proposition 36. *Let $2m+1$ be the order of the highest spatial derivative with non-zero coefficient in (42) and let $|a(x)|$ increase no faster than $|x|^r$ with $r < 4m$ as $x \to \pm\infty$. Then any solution u of (42) satisfies*

$$||u(t)||_m \leq g(||u(0)||_m),$$

where $g : [0,\infty[\to [0,\infty[$ is monotone increasing.

Proof. Let us denote $u(0) = \phi$. From Proposition 35 it follows that $\frac{1}{2}||u||^2 = \frac{1}{2}||\phi||^2$ at any instant and

$$\frac{1}{2}\sum_{k=1}^{m}(-1)^k c_k ||u^{(k)}||^2 + \int a^{(-2)} \circ u dx = \frac{1}{2}\sum_{k=1}^{m} c_k ||\phi^{(k)}||^2 + \int a^{(-2)} \circ \phi dx,$$

which implies

$$|c_m| \cdot ||u^{(m)}||^2 \leq \sum_{k<m} |c_k| \cdot ||u^{(k)}||^2 + 2\int |a^{(-2)} \circ u| dx$$
$$+ \sum_k |c_k| \cdot ||\phi^{(k)}||^2 + 2\int |a^{(-2)} \circ \phi| dx. \tag{47}$$

In (47), it is immediately clear that the third term on the right hand side has a bound of the desired form. Using Proposition 13 on the first term, we find that $||u^{(k)}||^2 \leq ||u^{(m)}||^{\frac{2k}{m}} ||u||^{\frac{2(m-k)}{m}} = ||u^{(m)}||^{\frac{2k}{m}} ||\phi||^{\frac{2(m-k)}{m}}$, where clearly $\frac{2k}{m} < 2$. We then investigate the second term on the right of (47) by making use of the equality (note that $a^{(-1)}(0) = a^{(-2)}(0) = 0$)

$$a^{(-2)}(x) = x\int_0^1 a^{(-1)}(xt)dt = x\int_0^1 xt \int_0^1 a(xts)dsdt$$

and the resulting inequality $|a^{(-2)}(x)| \leq \frac{1}{2}x^2 ||a||_{\infty,|x|}$ to yield

$$2\int |a^{(-2)} \circ u| dx \leq ||a||_{\infty,||u||_\infty} \int u^2 dx = ||a||_{\infty,||u||_\infty} ||u||^2.$$

(Note the similarity with the estimate in (23).) Using the assumption on the growth of a and applying the inequality $|u^2(x)| = |\left(\int_{-\infty}^{x} - \int_x^\infty\right) uu'dx| \leq \left(\int u^2 dx \int (u')^2 dx\right)^{\frac{1}{2}}$ used by Bona and Smith [2, Pr. 2] as well as Proposition 13, we find that

$$||a||_{\infty,||u||_\infty} \leq c_1 ||u||_\infty^r + c_2 \leq c_1 ||u'||^{\frac{r}{2}} ||u||^{\frac{r}{2}} + c_2 \leq c_1 ||u^{(m)}||^{\frac{r}{2m}} ||u||^{\frac{r(2m-1)}{2m}} + c_2. \tag{48}$$

Substituting $t = 0$, i.e. $u = \phi$, we find that the fourth term on the right of (47) has a bound of the desired form. For the second term we note that we assumed in the statement of the Proposition that $\frac{r}{2m} < 2$ in (48).

All the estimates so far imply that

$$||u^{(m)}||^2 \leq \sum_s g_s(||\phi||_m) ||u^{(m)}||^s, \tag{49}$$

where s in the summation ranges over finitely many values and $0 \leq s < 2$, and g_s are monotone increasing functions. Now the left-hand side of (49) increases strictly faster than the right-hand side for fixed $||\phi||_m$ as $||u^{(m)}|| \to \infty$, which violates the inequality, and thus $||u^{(m)}||$ must be bounded in terms of a (monotone increasing) function of $||\phi||_m$. That the same is true for $||u||_m$ follows from Proposition 13 and the fact that $||u|| = ||\phi||$. □

5.2 Global Solutions

We are now only a step away from the global solutions. Proposition 36 shows that the class of equations considered satisfy Kato's global growth condition when m is sufficiently large. Unfortunately, this is not the case with the proper Korteweg–de Vries equation (1), so that Proposition 36 is not strong enough to deduce the existence of global solutions to (1). The fact that these nevertheless exist (Theorem 3) under certain circumstances must be viewed as a rather individual property of (1), which seems to be shared only with another equation of fairly similar form [11], as already mentioned above.

We will shortly present the part of results of Miura, Gardner and Kruskal [12] that is needed to establish Proposition 38 yielding Kato's global growth condition for the two special equations. A result similar to Proposition 38 is found in Bona and Smith [2, Pr. 6] giving a more general result for the proper Korteweg–de Vries equation.

Proposition 37 (Miura–Gardner–Kruskal). *The two equations*

$$u_t + uu_x + u_{xxx} = 0, \qquad v_t + v^2 v_x + v_{xxx} = 0 \tag{50}$$

have a countable infinity of conservation laws. In particular, the following densities are conserved

$$\frac{1}{4}u^4 - 3uu'^2 + \frac{9}{5}u''^2, \qquad \frac{1}{6}v^6 - 5v^2v'^2 + 3v''^2.$$

Proposition 38 (Bona–Smith). *The solutions u of either of the equations (50) satisfy*

$$||u(t)||_2 \leq g(||u(0)||_2),$$

with g monotone increasing.

Proof. Let $v(0) = \phi$. Using Proposition 37, we have

$$3||v''||^2 \leq \frac{1}{6}||v||_\infty^4||v||^2 + 5||v||_\infty^2||v'||^2 + 3||\phi''||^2 + \frac{1}{6}||\phi||_\infty^4||\phi||^2 + 5||\phi||_\infty^2||\phi'||^2.$$

Here we can use the bounds $||v||_\infty \leq c||v||_1$ by Proposition 22 and the obvious inequalities $||v||, ||v'|| \leq ||v||_1$ etc., to get

$$||v''||^2 \leq g_1(||v||_1) + g_2(||\phi||_2),$$

and then apply Proposition 36 to give $||v||_1 \leq ||\phi||_1 \leq ||\phi||_2$. The desired bound follows for $||v||_2$, since it holds for $||v||_1$ and $||v''||$. Note in applying Proposition 36 that now $a(x) = x^2$, $m = 1$ and indeed $2 < 4{\cdot}1$. The reasoning for u in (50) is quite identical. ☐

Theorem 39. *The following equations have a global solution in the space (43) for the initial value problem $u(0) = \phi \in H^s$:*

$$\begin{cases} u_t + \sum_{k=0}^m c_k u^{(2k+1)} + a(u)u' = 0, \\ c_m \neq 0, \quad s \geq m \geq 2, \quad a \in O(|x|^r), \quad r < 4m \end{cases} \tag{51}$$

$$u_t + u''' + u^p u' = 0 \quad s \geq 2, \quad p = 1, 2. \tag{52}$$

Proof. For the first class of equations, the claim follows from Theorem 33 and Proposition 36; for the second, use Theorem 33 with Proposition 38. ☐

5.3 Irregularities and Nonexistence Results

Now that we have the global existence theorem, a natural question to ask is whether our assumptions are necessary to get this result. In particular, one might inquire whether it is possible to replace the Sobolev spaces that have played a crucial part in our analysis by some function space with sufficient smoothness assumptions, but no restrictions on the integrability over infinite intervals. The answer to this question turns out to be “no”, even if the initial data is C^∞ smooth, and the answer, essentially the results of Murray [13], shows, on the other hand, that even certain discontinuous initial function can yield a smooth solution. In fact, these two facts are very closely related, owing to the reversibility in time of the equation (1), as will be seen in Corollary 41. The proofs of the results are found in Murray [13, Th. 1.1].

Proposition 40 (Murray). *For initial data $\phi = c\mathbf{1}_{[-r,r]}$, $c \neq 0$, $r > 0$, there is a solution $u(t) \in C^\infty$ for $t > 0$ of the proper Korteweg–de Vries equation (1) satisfying the initial condition $u(0) = \phi$ in the weak sense*

$$\lim_{t \downarrow 0} \int_\Delta u(t,x)dx = \int_\Delta \phi(x)dx$$

for any interval $\Delta \subset \mathbb{R}$.

Corollary 41 (Murray). *A C^∞ initial data ϕ does not guarantee even a continuous global solution u to (1).*

Proof. If u satisfies (1), then so does

$$w(t,x) = u(T-t,-x), \qquad t < T. \tag{53}$$

Now let u be the C^∞ solution of (1) with initial value $c\mathbf{1}_{[-r,r]}$, which exists by Proposition 40. Then w is a solution of (1) with $w(0) \in C^\infty$, but w becomes discontinuous in a finite time T, and thus no unique global solution can exist in any space of continuous functions. □

We hence know that the Sobolev space setting cannot be weakened to mere smoothness of any order, at least to obtain global solutions. We also note that the functions $c\mathbf{1}_{[-r,r]}$ in Proposition 40 are in H^s only when $s < \frac{1}{2}$.

Finally, we refer to Drazin and Johnson [4, Sect. 1.1, Fig. 1.2] for a discussion of the fact that in the context of non-linear wave equations, the lack of continuous global solutions may even arise from the physical nature of the phenomenon and not only computational difficulties with the model.

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