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Abstract: *Certain convolution and multiplier transformations with operator-valued kernels, acting on spaces of vector-valued functions, are considered. We generalize some of the techniques of Kurtz–Wheeden and Strömberg–Torchinsky from the scalar-valued setting to show that such operators, if bounded on some L^p, $1 < p < \infty$, admit bounded extensions to the atomic Hardy spaces H^p_{at} with $0 < p \leq 1$. In particular, we prove that the assumptions of the one-dimensional Mikhlin-type theorem due to Weis already guarantee the boundedness of the operator not only on L^p but also on H^1_{at}. The results are applied to prove maximal H^p_{at}-regularity of certain differential equations.*

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1 Introduction

Operator-valued Fourier-multipliers acting on vector-valued functions, and the applications of such multiplier theorems to prove maximal L^p-regularity for abstract differential equations, have been studied extensively in recent times. It is now known that the general geometric setting in which strong multiplier theorems are valid is the UMD property (see [17]) of the underlying Banach spaces B_i, and that appropriate R-boundedness (see [7, 20]) of the operator families involved is the right condition for them to give rise to bounded operators from $L^p(\mathbb{R}^n; B_1)$ to $L^p(\mathbb{R}^n; B_2)$ for $p \in (1, \infty)$. A Mikhlin-type theorem for operator-valued multipliers acting between $L^p(\mathbb{R}; B_1)$ and $L^p(\mathbb{R}; B_2)$ was first obtained by Weis [20] in 1999, inspiring numerous results on related problems by various other authors. We refer to Denk, Hieber and Prüss [9] for a collective treatment of and more references on some of the progress up to summer 2001.

However, convolutions with kernels taking values in $\mathcal{B}(B_1; B_2)$ (bounded linear mappings from B_1 to B_2) were already considered in 1962 by Benedek, Calderón and Panzone [2] from a somewhat different point of view. (Recall that convolution operators and Fourier-multipliers are essentially representations in the position space and in the frequency space, respectively, of the same linear transformations.) Benedek *et al.* imposed no geometric conditions on the Banach spaces B_i, neither do their assumptions involve any R-boundedness, which was not invented until much later; what they do assume, in addition to a condition on the convolution kernel, is the *a priori* boundedness of the operator T in question from $L^p(\mathbb{R}^n; B_1)$ to $L^p(\mathbb{R}^n; B_2)$ for some $p \in (1, \infty)$. They then show that the boundedness is actually true for every p in this range. The proof is based on a weak-type L^1-estimate, interpolation, and a duality argument.

Under the same assumptions as those imposed by Benedek *et al.*, we will show that T is actually bounded from $H^1_{\mathrm{at}}(\mathbb{R}^n; B_1)$, the atomic Hardy space of B_1-valued functions to be defined below, to $L^1(\mathbb{R}^n; B_2)$ and also from $L^\infty(\mathbb{R}^n; B_1)$ to $\mathrm{BMO}(\mathbb{R}^n; B_2)$. By interpolation, this also gives another, and in a sense more direct, proof of the theorem in [2].

We then proceed to consider a more restrictive class of operator-valued kernels for integral operators, for which we can derive boundedness from $H^p_{\mathrm{at}}(\mathbb{R}^n; B_1)$ to $H^p_{\mathrm{at}}(\mathbb{R}^n; B_2)$, $0 < p \leq 1$. Such kernels were studied by Kurtz and Wheeden [14] in 1979 in connection with weighted L^p-spaces of scalar-valued functions, and the techniques were later applied by Strömberg and Torchinsky [19] in 1989 to weighted Hardy spaces. Although we will not be concerned with the weighted spaces here, we note that the same techniques provide insight and new results into our non-weighted but vector-valued setting.

The theorems in [14] and in [19] are actually stated for the multipliers, but the proofs are really worked out using convolutions. A lemma central to this argument deduces properties of the kernel from those of the corresponding multiplier, and the same argument also applies in the vector-valued situation.

Using this, we obtain multiplier theorems as a direct consequence of the results proved for convolution operators.

Our theorems fall into two categories. First, there are results of the Benedek–Calderón–Panzone-type, where boundedness on one L^p is assumed, and the boundedness for other p and for the extreme cases is deduced from this. Second, combining these theorems with results such as the Mikhlin–Weis theorem for the $L^p(\mathbb{R}^n; B_i)$-spaces, B_i UMD, we then obtain sufficient conditions also for $H^p_{\mathrm{at}}(\mathbb{R}^n; B)$-boundedness without any *a priori* boundedness assumptions. For $n > 1$, we need to assume a little more than the assumptions of the Mikhlin–Weis theorem to get boundedness from $H^p_{\mathrm{at}}(\mathbb{R}^n; B_1)$ to $H^p_{\mathrm{at}}(\mathbb{R}^n; B_2)$ for $p \leq 1$, but in the one-dimensional setting we obtain a genuine extension of the Mikhlin–Weis theorem to $H^1_{\mathrm{at}}(\mathbb{R}^n; B_i)$ with the same assumptions.

Results involving the properties of the multiplier are convenient in view of applications, since the transformed domain is often the natural place to work in, and multipliers related to certain differential equations have a relatively simple structure to analyze. As an application of our results, we prove maximal regularity of certain equations on B-valued H^p_{at}, $0 < p \leq 1$. This gives, e.g., results similar to those of Cannarsa and Vespri [6] concerning the abstract Cauchy problem, and some other equations are also considered.

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2 Preliminaries

2.1 Miscellaneous concepts

Notation. Lower case n will consistently denote the dimension of the Euclidean space $\mathbb{R}^n$ on which we are working. A positive real number in the subscript will denote the dilation of a function, $\phi_t(x) := t^{-n}\phi(t^{-1}x)$.

An integral without limits refers to integration over the whole space $\mathbb{R}^n$. The Fourier transform of f is denoted by $\hat{f}$ or $\mathcal{F}f$ and the inverse Fourier transform by $\check{f}$. The reflection about the origin is $\widetilde{f}(x) := f(-x)$, and we recall that $\mathcal{F}^2 f = \widetilde{f}$ is always true when the transforms are understood in the sense of (tempered) distributions. In the vector-valued setting, these are defined by $\mathcal{S}'(\mathbb{R}^n; B) := \mathcal{B}(\mathcal{S}(\mathbb{R}^n); B)$, where $\mathcal{S}(\mathbb{R}^n)$ is the conventional scalar-valued Schwartz space with its usual topology.

We use $|\cdot|$ both for the absolute value of a real or complex quantity, the Euclidean norm on $\mathbb{R}^n$, and for the Lebesgue measure of a subset (usually a ball) of $\mathbb{R}^n$. The meaning should, however, be obvious from the context. The norm of a Banach space B is denoted by $|\cdot|_B$; double lines are reserved for norms of function spaces, such as $\|f\|_{L^p(\mathbb{R}^n)}$.

The set of natural numbers is $\mathbb{N} := \{0, 1, 2, \ldots\}$ and that of positive integers is $\mathbb{Z}_+ := \{1, 2, \ldots\} \subsetneq \mathbb{N}$. For $\ell > 0$, we denote by $\lfloor \ell \rfloor$ the greatest integer at most ℓ, and by $\llbracket \ell \rrbracket$ the greatest integer strictly less than ℓ. Thus both functions give the integer part of a non-integer ℓ, but $\llbracket m \rrbracket = m - 1$, $\lfloor m \rfloor = m$ for $m \in \mathbb{Z}_+$.

Constants in some estimates are denoted by c and C, which may be different from one occurrence to another.

A standard partition of unity. We will define one "canonical" partition of unity, which will be considered fixed throughout this paper. The constants in some estimates will depend on the choice of this partition; we shall not bother about any such dependence.

Let $\eta \in \mathcal{D}(\mathbb{R}^n)$ be non-negative, equal to unity in $\bar{B}(0,1)$ and supported in $\bar{B}(0,2)$. Let $\phi(x) := \eta(x) - \eta(2x)$ and $\phi_i(x) := \phi(2^{-i}x)$. Then ϕ_i is supported in the annulus $2^{i-1} \leq |x| \leq 2^{i+1}$, and $\sum_{i=-\infty}^{\infty} \phi_i(x) = 1$ for $x \neq 0$.

Moreover, we have $\eta(x) + \sum_{i=1}^{\infty} \phi_i(x) = 1$ at every x, and substituting $2^{-N}x$ in place of x and using the definition of the ϕ_i, we get $\eta(2^{-N}x) + \sum_{i=N+1}^{\infty} \phi_i(x) = 1$. Comparing the two series it follows that $\eta(2^{-N}x) = \eta(x) + \sum_{i=1}^{N} \phi_i(x)$.

A partition of unity of this kind appears almost everywhere in harmonic analysis nowadays. Kurtz and Wheeden [14] attribute it to Hörmander.

The Fourier-type of Banach spaces. We recall that a Banach space B is said to have *Fourier-type* p, if the Hausdorff–Young inequality

$$\left\| \hat{f} \right\|_{L^{p'}(\mathbb{R};B)} \leq C \|f\|_{L^p(\mathbb{R};B)}, \tag{1}$$

is true for every $f \in (L^1 \cap L^p)(\mathbb{R}; B)$ with some finite C. Obviously every Banach space satisfies this inequality with $p = 1$, and by interpolation the inequality holds for $q \in (1, p)$ if it holds for some $p > 1$. B is said to have a non-trivial Fourier-type, if it has a Fourier-type $p > 1$. Note that once (1) is true, the corresponding inequality with $\mathbb{R}$ replaced by $\mathbb{R}^n$ also holds due to the tensor nature of the Fourier transform.

This notion is due to Peetre [15]. It is proved in [15], e.g., that every space $L^p(\Omega, \Sigma, \mu)$ (of scalar-valued functions) has Fourier-type $\min(p, p')$. Kwapień [13] has shown that B has Fourier-type 2 if and only if it is isomorphic to a Hilbert space.

Because of the significant rôle of the UMD-spaces in the theory of multipliers, it is useful to know that every UMD-space has a non-trivial Fourier-type. This is a consequence of the following results: (This argument was shown to the author by S. Geiss.)

1. A UMD-space does not contain uniformly the spaces $\ell_1^r := (\mathbb{C}^r, |\cdot|_1)$, $r \in \mathbb{Z}_+$.

2. A Banach space B does not contain uniformly ℓ_1^r, $r \in \mathbb{Z}_+$, if and only if B has a non-trivial Rademacher-type.

3. B has a non-trivial Rademacher-type if and only if it has a non-trivial Fourier-type.

The first assertion is easy to prove, since the non-reflexive sequence space ℓ_1 is not UMD (UMD-spaces being even super-reflexive, see [17, p. 205]), and so has infinite UMD-constants $M_p(\ell_1) = \infty$. By approximating ℓ_1-valued martingales by their projections to the r first coordinate directions, it follows readily that the UMD-constant of ℓ_1^r is larger than any preassigned $M > 0$ once r is large enough, i.e., $M_p(\ell_1^r) \to \infty$ as $r \to \infty$, which proves the assertion.

The second and in particular the third claim above are deeper, and we refer to [16, Th.'s 4.4.7 & 5.6.30] and the references cited there, also for the definition of the Rademacher-type. These results are originally due to Pisier and Bourgain, respectively.

2.2 A short review of vector-valued Hardy spaces

There are good reasons why the natural continuation of the family of spaces L^p, $1 < p < \infty$, for $0 < p \leq 1$ should consist of the Hardy spaces H^p, where many important operations of (harmonic) analysis, well-behaved on L^p for $1 < p < \infty$ but blowing up in L^1, retain their good character.

As is well-known in the scalar-valued setting, there exist various equivalent characterizations of the spaces H^p, $0 < p \leq 1$. In the vector-valued situation, not all of these equivalences remain valid, and we must be more careful about the definition. Here we are concerned with the *atomic Hardy spaces*

$$H^p_{\mathrm{at}}(\mathbb{R}^n; B) := \left\{ \sum_{k=0}^{\infty} \lambda_k a_k \ (\text{in } \mathcal{S}'(\mathbb{R}^n; B)) : a_k \text{ an } H^p\text{-atom}, \ \lambda_k \in \mathbb{C}, \ \sum_{k=0}^{\infty} |\lambda_k|^p < \infty, \right\}$$

equipped with the "norm"

$$\|f\|^p_{H^p_{\mathrm{at}}(\mathbb{R}^n;B)} := \inf \sum_{k=0}^{\infty} |\lambda_k|^p ,$$

where the infimum is taken over all atomic decompositions of $f \in H^p_{\mathrm{at}}$ as in the definition of the atomic Hardy space. Of course, this is really a norm only when $p = 1$, but for $0 < p < 1$, $\varrho_{H^p_{\mathrm{at}}}(f, g) := \|f - g\|^p_{H^p_{\mathrm{at}}}$ defines a translation invariant metric on $H^p_{\mathrm{at}}(\mathbb{R}^n; B)$.

Atoms. The definition of the atoms appearing above is the same as in the scalar-valued context: We say that $a \in L^q(\mathbb{R}^n; B)$ is a (p, q, N)-atom, where $0 < p \leq 1 < q \leq \infty$ and $N \in \mathbb{N}$, provided that

1. a is supported in a ball $\bar{B}$,
2. $\|a\|_{L^q} \le |\bar{B}|^{q^{-1}-p^{-1}}$, and
3. $\int x^\alpha a(x)\,\mathrm{d}x = 0$ for all $\alpha \in \mathbb{N}^d$ with $|\alpha|_1 \le N$.

The three requirements above are referred to as the support condition, the size condition and the moment condition, respectively.

We say that a is a (p,q)-atom if it is a (p,q,N)-atom for some $N \in \mathbb{N}$, and that a is an H^p-atom of L^q-type if it is a (p,q,N)-atom for some $N \ge n(p^{-1}-1)$. Finally, a is an H^p-atom, if it is an H^p-atom of some L^q-type, $q > 1$. In the definition of H^p_{at} above, we require that the a_k are (p,q)-atoms for some fixed $q > 1$. The spaces obtained with different values of q coincide and the norms are equivalent.

Dense subsets. It immediately follows from the definition that the convergence of the atomic series, initially taken in the sense of B-valued tempered distributions, actually takes place with respect to the metric of $H^p_{\mathrm{at}}(\mathbb{R}^n;B)$. As a consequence, the space of finite linear combinations of atoms is dense in $H^p_{\mathrm{at}}(\mathbb{R}^n;B)$.

The following density result will also be exploited. We will give a proof for completeness, since the result is perhaps not so familiar in the vector-valued setting.

Lemma 1. *The elements of $(H^p_{\mathrm{at}} \cap L^1 \cap L^q)(\mathbb{R}^n;B)$, $q > 1$, with compactly supported Fourier transform are dense in $H^p_{\mathrm{at}}(\mathbb{R}^n;B)$.*

Proof. By the density of finite combinations of atoms, it suffices to approximate a give atom a of $H^p_{\mathrm{at}}(\mathbb{R}^n;B)$ of L^q-type, supported in $\bar{B}(x_0,r)$, as closely as desired by functions of the asserted form. (Observe that every atom automatically belongs to the asserted space $H^p_{\mathrm{at}} \cap L^1 \cap L^q$.) To this end, fix a $\hat{\psi} \in \mathcal{D}(\mathbb{R}^n)$ with $\hat{\psi}(0) = \int \psi(x)\,\mathrm{d}x = 1$. Then $\hat{\psi}(\epsilon\cdot)\hat{a} = \widehat{\psi_\epsilon * a}$ has compact support. Our intention is to prove that $\psi_\epsilon * a \to a$ in $H^p_{\mathrm{at}}(\mathbb{R}^n;B)$ as $\epsilon \downarrow 0$. Note that $\psi_\epsilon * a \in L^1 \cap L^q$, since $\psi_\epsilon \in L^1$ and $a \in L^1 \cap L^q$.

To this end, we recall our standard partition of unity from § 2.1 and write $\psi = \psi\eta(r^{-1}\cdot) + \sum_{i=1}^\infty \psi\phi_i(r^{-1}\cdot) =: \sum_{i=0}^\infty \psi_i$; moreover let $\psi^N = \sum_{i=0}^N \psi_i$. So let us write

$$\psi_\epsilon * a(x) - a(x) = \int \epsilon^{-n}\psi^N(\epsilon^{-1}y)(a(x-y)-a(x))\,\mathrm{d}y + \sum_{i=N+1}^{\infty} \int \epsilon^{-n}\psi_i(\epsilon^{-1}y)(a(x-y)-a(x))\,\mathrm{d}y. \quad (2)$$

All the terms above (as functions of x) have at least the same number of vanishing moments as a; furthermore, the first term is supported in $\bar{B}(x_0, r(1+2^{N+1}\epsilon))$, while the term in the summation indexed by i is supported in $\bar{B}(x_0, r(1+2^{i+1}\epsilon))$. These terms are therefore, up to multiplicative constants, atoms of H^p of L^q-type, and it remains to estimate the constants.

With k denoting either ψ^N or ψ_i, we have, changing the variables from $\epsilon^{-1}y$ to y,

$$\begin{aligned}\left(\int\left|\int k(y)(a(x-\epsilon y)-a(y))\,\mathrm{d}y\right|_B^q\,\mathrm{d}x\right)^{\frac{1}{q}}\\ &\le\int\left(\int|a(x-\epsilon y)-a(y)|_B^q\,\mathrm{d}x\right)^{\frac{1}{q}}|k(y)|\,\mathrm{d}y\\ &\le\begin{cases}\sup_{|y|\le 2^{N+1}}\|a(\cdot-\epsilon y)-a(y)\|_{L^q(\mathbb{R}^n;B)}\|\psi\|_{L^1(\mathbb{R}^n)} & k=\psi^N\\ 2\|a\|_{L^q(\mathbb{R}^n;B)}\|\psi_i\|_{L^1(\mathbb{R}^n)} & k=\psi_i\end{cases}\end{aligned}$$

Taking (2) as an atomic decomposition of $\psi_\epsilon * a - a$, recalling the atomic size condition and the balls in which each of the terms in (2) is supported, we then have the norm estimate

$$\begin{aligned}\|\psi_\epsilon * a-a\|^p_{H^p_{\mathrm{at}}(\mathbb{R}^n;B)}\\ &\le\sup_{|y|\le\delta}\|a(\cdot-y)-a\|^p_{L^q(\mathbb{R}^n;B)}\|\psi\|^p_{L^1(\mathbb{R}^n)}(r(1+\delta))^{n(1-p/q)}\\ &\quad+\sum_{i=N+1}^{\infty}2^p\|a\|^p_{L^q(\mathbb{R}^n;B)}\|\psi_i\|^p_{L^1(\mathbb{R}^n)}(r(1+2^{i-N}\delta))^{n(1-p/q)},\end{aligned}$$

where we took $\epsilon=2^{-N-1}\delta$.

We observe in the summation that $\psi_i=\phi(2^{-i}r^{-1}\cdot)\psi$ is the part of the Schwartz function ψ in the annulus $2^{i-1}r\le|x|\le 2^{i+1}r$, and the integral of this is multiplied by a factor proportional to a power of $r2^i$, i.e., to a power of $|x|$. Since ψ decreases at infinity faster than any polynomial increases, we see that the sum can be made as small as desired when N is chosen large enough.

As for the rest of the estimate, we note that translation is strongly continuous on $L^q(\mathbb{R}^n;B)$, and therefore $\sup_{|y|\le\delta}\|a(\cdot-y)-a\|_{L^q(\mathbb{R}^n;B)}$ can also be forced smaller than any preassigned positive number with an appropriate choice of δ. This completes the proof. $\square$

The vector-valued BMO**.** The definition of the space of vector-valued functions of bounded mean oscillation is the same as that of the scalar-valued one, with absolute values replaced by norms of the underlying space.

The dual space of $H^1_{\mathrm{at}}(\mathbb{R}^n;B)$ can be identified with $\mathrm{BMO}(\mathbb{R}^n;B^*)$ (in the way familiar from the scalar-valued context) if and only if B^* has the Radon–Nikodým property. This result is proved in Blasco [3] for the spaces on the unit-circle (i.e., $\mathbb{R}^n$ replaced by $\mathbb{T}$), but one can also give a proof for the Euclidean spaces. In fact, working with the atomic definition of the Hardy spaces, the continuous embedding $\mathrm{BMO}(\mathbb{R}^n;B^*)\subset H^1_{\mathrm{at}}(\mathbb{R}^n;B)^*$ is almost immediate (without any assumptions on the structure of the Banach spaces), and the converse can be proved with pretty much the same reasoning as in the scalar-valued proof e.g. in [18, pp. 142–6], exploiting the classical duality result saying that $L^p(\mathbb{R}^n; B)^* = L^{p'}(\mathbb{R}^n; B^*)$ when B^* has the Radon–Nikodým property (see [10]).

In the general situation, Blasco [3] has shown that the dual of the B-valued H^1_{at} can be described in terms of a space of certain B^*-valued measures, which he calls $\mathcal{BMO}$, and this coincides with the usual BMO under the assumption of the Radon–Nikodým property.

Remarks. In the scalar-valued setting, another characterization of the H^p-spaces is in terms of a class of tempered distributions, certain maximal functions of which are in L^p (see e.g. Stein [18] for details). We should note that the same proof given there for the equivalence in the scalar-valued situation works also for vector-valued distributions with straightforward modifications, and therefore the maximal characterization gives the same Hardy space $H^p_{\mathrm{max}}(\mathbb{R}^n; B) = H^p_{\mathrm{at}}(\mathbb{R}^n; B)$. This has already been pointed out by Blasco [3] in the case $p = 1$. On the other hand, in [3] it is shown, concerning the Hardy spaces on the torus $\mathbb{T}$, that the space

$$H^1_{\mathrm{con}}(\mathbb{T}; B) := \big\{ f \in L^1(\mathbb{T}; B) : \mathcal{H}f \in L^1(\mathbb{T}; B) \big\}$$

defined in terms of the Hilbert transform (or conjugation) $\mathcal{H}$, is in general different (smaller) than $H^1_{\mathrm{at}}(\mathbb{T}; B)$, and agrees with it if and only if B is UMD.

3 Extremes of the Benedek–Calderón–Panzone theorem

Let $k \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$. We consider the convolution operator

$$Tf(x) := \int k(x-y) f(y) \,\mathrm{d}y,$$

initially defined for $f \in L^\infty_c(\mathbb{R}^n; B_1)$ (i.e., essentially bounded functions with compact support). For such f and x ranging in some compact set K we have

$$\begin{aligned} |Tf(x)|_{B_2} &\le \int_{\operatorname{supp} f} |k(x-y)|_{\mathcal{B}(B_1;B_2)} \,\mathrm{d}y \, \|f\|_{L^\infty} \\ &\le \int_{K - \operatorname{supp} f} |k(y)|_{\mathcal{B}(B_1;B_2)} \,\mathrm{d}y \, \|f\|_{L^\infty}, \end{aligned}$$

where $K - \operatorname{supp} f = \{x - y : x \in K, y \in \operatorname{supp} f\}$ is a bounded set, so the integral is finite, and we see that $Tf \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n; B_2)$.

In the spirit of Benedek, Calderón and Panzone [2], we will assume that k either satisfies the strong operator condition

$$\int_{|x|>c|y|} |(k(x-y) - k(x))u|_{B_2} \,\mathrm{d}x \le C\, |u|_{B_1} \tag{3}$$

for some $c > 1$, $C > 0$, for all $u \in B_1$, or the similar norm condition

$$\int_{|x|>c|y|} |k(x-y) - k(x)|_{\mathcal{B}(B_1;B_2)} \, \mathrm{d}x \leq C, \tag{4}$$

which obviously implies (3).

We then have the following:

Theorem 1. *Suppose the operator T defined above satisfies $\|Tf\|_{L^{p_0}(\mathbb{R}^n;B_2)} \leq C\|f\|_{L^{p_0}(\mathbb{R}^n;B_1)}$ for some $p_0 \in (1,\infty)$. If the kernel k satisfies (3), then T has a bounded extension from $H^1_{\mathrm{at}}(\mathbb{R}^n;B_1)$ to $L^1(\mathbb{R}^n;B_2)$, and thus by interpolation from $L^p(\mathbb{R}^n;B_1)$ to $L^p(\mathbb{R}^n;B_2)$ for $p \in (1,p_0)$. If, in addition, k satisfies (4), then T has a bounded extension from $L^\infty(\mathbb{R}^n;B_1)$ to $\mathrm{BMO}(\mathbb{R}^n;B_2)$, and thus from $L^p(\mathbb{R}^n;B_1)$ to $L^p(\mathbb{R}^n;B_2)$ for all $p \in (1,\infty)$.*

Proof. Let us first study the lower end, i.e., the H^1 case. Consider $f \in L^\infty(\mathbb{R}^n;B_1)$ supported in a ball $\bar{B} = \bar{B}(x_0,r)$ and with zero average. Then

$$Tf(x) = \int_{\bar{B}} k(x-y)f(y)\,\mathrm{d}y = \int_{\bar{B}} [k(x-y) - k(x-x_0)]f(y)\,\mathrm{d}y,$$

and thus

$$\begin{aligned}
&\int_{|x-x_0|>cr} |Tf(x)|_{B_2}\,\mathrm{d}x \\
&\qquad \leq \int_{\bar{B}} \int_{|x-x_0|>cr} |(k(x-y)-k(x-x_0))f(y)|_{B_2}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad = \int_{\bar{B}} \int_{|x|>cr} |(k(x-(y-x_0))-k(x))f(y)|_{B_2}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad\qquad \leq C\int_{\bar{B}} |f(y)|_{B_1}\,\mathrm{d}y \leq C\left|\bar{B}\right|^{1-\frac{1}{r}}\|f\|_{L^{p_0}(\mathbb{R}^n;B_1)},
\end{aligned}$$

where the assumption (3) with $y - x_0$ in place of y and $f(y)$ in place of u was applied in the second to last inequality; note that $|y - x_0| \leq r$ there.

On the other hand,

$$\int_{|x-x_0|\leq cr} |Tf(x)|_{B_2}\,\mathrm{d}x \leq C\left|\bar{B}\right|^{1-\frac{1}{r}}\|Tf\|_{L^{p_0}(\mathbb{R}^n;B_2)} \leq C\left|\bar{B}\right|^{1-\frac{1}{r}}\|f\|_{L^{p_0}(\mathbb{R}^n;B_1)}$$

by the assumed boundedness on L^{p_0}.

Now if a is an atom of H^1 of L^{p_0}-type, supported in $\bar{B}$, then a has zero average like f above, and moreover $\|a\|_{L^{p_0}(\mathbb{R}^n;X)} \leq \left|\bar{B}\right|^{\frac{1}{r}-1}$. It follows that $\|Ta\|_{L^1} \leq C$ uniformly in atoms a of H^1 of L^{p_0}-type, and this implies that T extends to a bounded mapping from $H^1_{\mathrm{at}}(\mathbb{R}^n;B_1)$ to $L^1(\mathbb{R}^n;B_2)$.

The upper end. We now invoke the stronger assumption (4), and our aim is to show boundedness of T from $L^\infty(\mathbb{R}^n;B_1)$ to $\mathrm{BMO}(\mathbb{R}^n;B_2)$. Since $L^\infty(\mathbb{R}^n;B_1)$ lacks convenient dense subspaces, we need to define what we mean by Tf for a general $f\in L^\infty(\mathbb{R}^n;B_1)$. There is a slight problem, since the integrand in the original definition is not in general Bôchner integrable over the whole space. However, the differences

$$\begin{aligned}
Tf(x)-Tf(x_0) &:= \int (k(x-y)-k(x_0-y))f(y)\,\mathrm{d}y \\
&= \left(\int_{|y|>c|x-x_0|}+\int_{|y|\le c|x-x_0|}\right)(k(x-y)-k(x_0-y))f(y)\,\mathrm{d}y
\end{aligned}$$

are well-defined, as the integral over the bounded domain above converges by the local integrability of k, and for the unbounded domain we can invoke the estimate (4). Since a BMO element is only defined up to an additive constant, we are satisfied with the differences.

For a given ball $\bar{B}=\bar{B}(x_0,r)$, we should then estimate the averages $\frac{1}{|\bar{B}|}\int_{\bar{B}}|Tf(x)-c|\,\mathrm{d}x$. We do this in two parts by first decomposing $f=f\mathbf{1}_{c\bar{B}}+f\mathbf{1}_{(c\bar{B})^c}=:f_0+f_1$.

Then

$$\begin{aligned}
\int_{\bar{B}}|Tf_0(x)|_{B_2}\,\mathrm{d}x &\le \left|\bar{B}\right|^{1-\frac{1}{r}}\|Tf_0\|_{L^{p_0}(\mathbb{R}^n;B_2)}\le C\left|\bar{B}\right|^{1-\frac{1}{r}}\|f_0\|_{L^{p_0}(\mathbb{R}^n;B_1)} \\
&\le C\left|\bar{B}\right|^{1-\frac{1}{r}}\left|\bar{B}\right|^{\frac{1}{r}}\|f\|_{L^\infty},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\bar{B}}|Tf_1(x)-Tf_1(x_0)|_{B_2}\,\mathrm{d}x \\
&\qquad\le \int_{\bar{B}}\int_{(c\bar{B})^c}|k(x-y)-k(x_0-y)|_{\mathcal{B}(B_1;B_2)}\,\mathrm{d}y\,\mathrm{d}x\,\|f\|_{L^\infty} \\
&=\int_{\bar{B}}\int_{|y|>cr}|k(y-(x_0-x))-k(y)|_{\mathcal{B}(B_1;B_2)}\,\mathrm{d}y\,\mathrm{d}x\,\|f\|_{L^\infty}\le C\left|\bar{B}\right|\|f\|_{L^\infty}.
\end{aligned}$$

Combining these estimates we find that

$$\|Tf\|_{\mathrm{BMO}(\mathbb{R}^n;B_2)}\le C\|f\|_{L^\infty(\mathbb{R}^n;B_1)},$$

as was to be proved.

Interpolation. Marcinkiewicz-type interpolation between the pairs (H^1,L^1) and (L^{p_0},L^{p_0}) as well as between (L^{p_0},L^{p_0}) and (L^∞,BMO) is nowadays quite standard. There exist rather general interpolation results due to Blasco [4] (see Remarks after the proof), but actually for the present case where we have e.g. $H^1_{\mathrm{at}}(\mathbb{R}^n;B_1)$ and $L^{p_0}(\mathbb{R}^n;B_1)$, with the same Banach space B_1 at the one end, and similarly $L^1(\mathbb{R}^n;B_2)$ and $L^{p_0}(\mathbb{R}^n;B_2)$ with the same B_2 at the other, we can simply repeat the proofs for scalar-valued functions, e.g. those in García-Cuerva and Rubio de Francia [12, pp. 307–310] and Duoandikoetxea [11, Th. 6.8]. □

Remarks. Concerning the implication "T is bounded on L^{p_0}" $\Longrightarrow$ "T is bounded on all L^p with $p \in (1,\infty)$", the present proof is more direct than that of Benedek, Calderón and Panzone [2] in the sense that we only work with the given Banach spaces B_1 and B_2 and the given operators $k(x) \in \mathcal{B}(B_1; B_2)$, without considering the dual spaces and adjoint operators. Since we also obtain boundedness in the extreme cases, our result can be regarded as an extension of that in [2] under the assumption (4).

However, the duality argument in [2] is still of some interest, since the boundedness on all L^p (although perhaps not from L^∞ to BMO) can be deduced from assumptions somewhat weaker than (4), namely (3) together with a dual condition

$$\int_{|x|>c|y|} |(k^*(x-y) - k^*(x))v|_{B_1^*} \,\mathrm{d}x \le C\,|v|_{B_2^*}$$

for all $v \in B_2^*$. Here $k^*(x) \in \mathcal{B}(B_2^*; B_1^*)$ is the adjoint operator of $k(x)$ for every $x \in \mathbb{R}^n$.

In [2] it is also shown that the condition (4) together with boundedness on one L^{p_0} implies boundedness on so-called L^P-spaces, $P = (p_1, \ldots, p_n)$, with "mixed" norms.

Concerning interpolation in the vector-valued setting, there is a general theorem of Blasco [4] stating that

$$[H^1_{\mathrm{at}}(\mathbb{R}^n; B_0), L^p(\mathbb{R}^n; B_1)]_\theta = L^q(\mathbb{R}^n; [B_0, B_1]_\theta),$$

where $q^{-1} = (1-\theta) + \theta p^{-1}$, and moreover, provided that $B_0 \cap B_1$ is dense in both B_0 and B_1, and $B_0^* \cap B_1^*$ in both B_0^* and B_1^*, we also have

$$[L^p(\mathbb{R}^n; B_0), L^\infty_0(\mathbb{R}^n; B_1)] = [L^p(\mathbb{R}^n; B_0), \mathrm{BMO}(\mathbb{R}^n; B_1)] = L^q(\mathbb{R}^n; [B_0, B_1]_\theta),$$

where $q^{-1} = (1-\theta)p^{-1}$ and L^∞_0 is the closure of simple functions in L^∞. But as was already pointed out above, and also by Blasco [4], for the case $B_0 = B_1$ the classical arguments for scalar-valued functions can immediately be generalized to give an interpolation result sufficient for the proof above.

4 A class of integral kernels

We say that a function k, with values in $\mathcal{B}(B_1; B_2)$, belongs to the class $KS(q, \ell)$ (kernel condition with strong estimates), where $1 \le q < \infty$ and $\ell > 0$, provided that $k \in C^{\lfloor \ell \rfloor}(\mathbb{R}^n \setminus \{0\}; \mathcal{B}(B_1; B_2))$ and satisfies

$$\left(\frac{1}{R^n}\int_{R<|x|<2R} |D^\alpha k(x)u|^q_{B_2} \,\mathrm{d}x\right)^{\frac{1}{q}} \le cR^{-n-|\alpha|_1}\,|u|_{B_1}$$

for all $R > 0$, $u \in B_1$, and $\alpha \in \mathbb{N}^n$ with $|\alpha|_1 \le \llbracket \ell \rrbracket$, and moreover

$$\left(\frac{1}{R^n}\int_{R<|x|<2R} |(D^\alpha k(x) - D^\alpha k(x-z))u|_{B_2}^q \,\mathrm{d}x\right)^{\frac{1}{q}} \le \begin{cases} c\left(\frac{|z|}{R}\right)^{\ell-\llbracket \ell \rrbracket} R^{-n-\llbracket \ell \rrbracket}\,|u|_{B_1} & \ell \notin \mathbb{Z}_+ \\ c\frac{|z|}{R}\log\frac{R}{|z|}\cdot R^{-n-\llbracket \ell \rrbracket}\,|u|_{B_1} & \ell \in \mathbb{Z}_+ \end{cases}$$

for all $R > 0$, $z \in \mathbb{R}^n$ with $|z| < \frac{1}{2}R$, $u \in B_1$, and $\alpha \in \mathbb{N}^n$ with $|\alpha|_1 = \llbracket \ell \rrbracket$.

We say that $k \in KN(q, \ell)$ (kernel condition with norm estimates), if k satisfies conditions similar to the ones above, but with the u removed and norms of the image space B_2 replaced by the operator norm of $\mathcal{B}(B_1; B_2)$.

The corresponding conditions $KS(\infty, \ell)$ and $KN(\infty, \ell)$ are defined by replacing the L^q-type integrals by essential suprema in the usual way.

These conditions are defined in Strömberg and Torchinsky [19, p. 151] for scalar-valued functions k. In this case the strong estimates and the norm estimates are obviously equivalent. Their notation $\tilde{M}(q, \ell)$ for these conditions emphasizes the relation to similar conditions $M(q, \ell)$ (to be investigated below) for the multiplier m corresponding to the convolution operator on the Fourier side.

The verification of the following two Lemmata is a routine exercise. We only note that the monotonicity in q in Lemma 2 is just Jensen's inequality and that the last assertion of Lemma 3 is an easy consequence of the other assertions, which are somewhat more tedious, but still straightforward.

Lemma 2. *$KS(q, \ell) \subset KS(q_1, \ell_1)$ and $KN(q, \ell) \subset KN(q_1, \ell_1)$ for $q_1 \le q$ and $\ell_1 \le \ell$.*

Lemma 3. *If $k \in KS(q, \ell)$ (respectively $KN(q, \ell)$), then the functions $k_t := t^{-n}k(t^{-1}\cdot)$, $t > 0$ satisfy the condition $KS(q, \ell)$ (resp. $KN(q, \ell)$) with the same constant c. Moreover, if $\psi \in \mathcal{S}(\mathbb{R}^n)$, then $\psi k \in KS(q, \ell)$ (resp. $KN(q, \ell)$), with a constant $cC(\psi, q, \ell)$, and the functions $\psi(t\cdot)k$, $t > 0$, satisfy the condition $KS(q, \ell)$ (resp. $KN(q, \ell)$) uniformly.*

Let us show that the present conditions are stronger than the ones considered in § 3:

Lemma 4. *Each of the conditions $KS(q, \ell)$ (resp. $KN(q, \ell)$) implies the condition (3) (resp. (4)).*

Proof. In view of Lemma 2, it suffices to do this for $q = 1$, $\ell \in (0, 1)$ (thus $\llbracket \ell \rrbracket = 0$). For $k \in KS(1, \ell)$, we have

$$\begin{aligned}\int_{|x|>2|y|} &|(k(x-y) - k(x))u|_{B_2}\,\mathrm{d}x \\ &= \sum_{k=1}^{\infty}\int_{2^k|y|<|x|<2^{k+1}|y|} |(k(x-y) - k(x))u|_{B_2}\,\mathrm{d}x \\ &\le \sum_{k=1}^{\infty} c\left(\frac{|y|}{2^k\,|y|}\right)^{\ell} |u|_{B_1} = c\left(\sum_{k=1}^{\infty} 2^{-k\ell}\right)|u|_{B_1} = c\,|u|_{B_1}\,.\end{aligned}$$

The proof for $k \in KN(1,\ell)$ is similar. $\square$

The significance of the conditions $KS(q,\ell)$ lies in the fact that they provide very satisfactory control over the action of the convolution $k*\cdot$ on atoms of Hardy spaces, as the next result will show. There is hardly anything new in the proof compared to the scalar-valued situation [19, Lemma 11.2], but we nevertheless include the demonstration of this central result for completeness. The formulation of the assertion differs slightly from that in [19], since the definition of the size condition of an atom is different there. The Lemma and its proof again involve our standard partition of unity, § 2.1.

Lemma 5. *Suppose that $k \in KS(q_0,\ell)$, that a is a (p_0,q_0,N_0)-atom supported in $\bar{B} := \bar{B}(x_0,r)$ and that $k_i(x) := \phi(r^{-1}2^{-i-2}x)k(x)$ for $i \in \mathbb{Z}_+$. Then for every i, $c_i(k_i * a)$ is a (p_0,q_0,N_0)-atom supported in $\bar{B}(x_0,r2^{i+4})$, where*

$$c_i := \begin{cases} c2^{i(\ell-n(p_0^{-1}-1))} & \text{if } \ell \le N_0+1 \text{ non-integer},\\ c2^{i(\ell-n(p_0^{-1}-1))}/i & \text{if } \ell \le N_0+1 \text{ integer},\\ c2^{i(N_0+1-n(p_0^{-1}-1))} & \text{if } \ell > N_0+1,\end{cases}$$

with $c = c(p_0,q_0,N_0)$ an appropriate constant.

Proof. It is obvious, after changing the order of integration, that k_i*a has (at least) the same number of vanishing moments as a, and the support condition is also straightforward. Actually, for the proof, it is useful to observe that we even have

$$\begin{aligned}\operatorname{supp}(k_i * a) &= \operatorname{supp}\int_{\bar{B}} k_i(\cdot - y)a(y)\,\mathrm{d}y\\ &\subset \{x : r(2^{i+1}-1) \le |x-x_0| \le r(2^{i+3}+1)\}, \quad (5)\end{aligned}$$

as follows immediately from the triangle inequality applied to $|y-x_0| \le r$ and $r2^{i+1} \le |x-y| \le r2^{i+3}$.

In order to estimate $k_i * a(x) = \int k_i(x-y)a(y)\,\mathrm{d}y$, consider the Taylor expansion of order $N-1$ of the function $\mathbb{R}^n \ni y \mapsto k_i(x-y)u \in B_2$ (where $u \in B_1$) at x_0, where $N = \min(N_0, [\![\ell]\!])$:

$$k_i(x-y)u = \sum_{|\alpha|_1<N} \frac{(y-x_0)^\alpha}{\alpha!}(-1)^{|\alpha|_1}D^\alpha k_i(x-x_0)u$$
$$+\sum_{|\alpha|_1=N}\frac{N}{\alpha!}(y-x_0)^\alpha(-1)^{|\alpha|_1}\int_0^1(1-s)^{N-1}D^\alpha k_i(x-(x_0+s(y-x_0)))u\,\mathrm{d}s.$$

Setting $u := a(y)$ and integrating we obtain

$$\int_{\bar{B}} k_i(x-y)a(y)\,\mathrm{d}y = \sum_{|\alpha|_1=N} c_\alpha \int_0^1 \mathrm{d}s(1-s)^{N-1}$$
$$\times\int_{\bar{B}}\mathrm{d}y(x_0-y)^\alpha\left(D^\alpha k_i((x-x_0)-s(y-x_0)) - D^\alpha k_i(x-x_0)\right)a(y),$$

where the conditions on a concerning moments of orders $|\alpha|_1 < N \le N_0$ made the integrals over the Taylor polynomial vanish, and the conditions concerning the moments of order $|\alpha|_1 = N$ were used to introduce the term $(x_0 - y)^\alpha D^\alpha k_i(x - x_0)a(y)$ without affecting the integrals.

Observe that while strong differentiability of k_i was sufficient to expand $k_i(x-y)u$ and then set $u := a(y)$, we need to know that the linear operators $D^\alpha k_i(x - x_0)$ are closed in order to conclude that

$$\int_{\bar{B}} (x_0 - y)^\alpha D^\alpha k_i(x - x_0)a(y)\,\mathrm{d}y = D^\alpha k_i(x - x_0) \int_{\bar{B}} (x_0 - y)^\alpha a(y)\,\mathrm{d}y = 0,$$

and to get the expression derived above. Since strong differentiability implies that the domain of the operators $D^\alpha k_i(x-x_0)$ is all of B_1, the requirement of closedness already implies boundedness by the Closed Graph Theorem. This is the reason for requiring norm differentiability also in the strong conditions $KS(q, \ell)$.

It now follows from Minkowski's inequality for integrals that

$$\begin{aligned} \|k_i * a\|_{L^{q_0}(\mathbb{R}^n;B_1)} \le \sum_{|\alpha|_1 = N} c_\alpha \int_0^1 \mathrm{d}s(1-s)^{N-1} \int_{\bar{B}} \mathrm{d}y\, |y - x_0|^{|\alpha|_1} \\ \times \Big(\int_{r2^i < |x-x_0| < r2^{i+4}} |D^\alpha k_i((x - x_0) - s(y - x_0))a(y) \\ - D^\alpha k_i(x - x_0)a(y)|_{B_2}^{q_0}\,\mathrm{d}x \Big)^{\frac{1}{q_0}}. \quad (6) \end{aligned}$$

The quantity in large parentheses is now of the form to which the estimates coming from the $K(q_0, \ell)$ condition apply. In order to exploit this efficiently, we need to consider several cases depending on the relation of N_0 and ℓ.

Case 1. Recall that $N < \ell$ by definition. If $N = \llbracket \ell \rrbracket > \ell - 1$, which is the case if ℓ is a non-integer and $N_0 \ge \llbracket \ell \rrbracket \ge \ell - 1$, then the expression to be estimated, with $|\alpha|_1 = N$, is the one in the highest derivative condition in the definition of $K(q_0, \ell)$. Thus it is bounded by

$$c(r2^i)^{n/q_0 - n - \llbracket \ell \rrbracket} \left(\frac{s\,|y - x_0|}{r2^i} \right)^{\ell - \llbracket \ell \rrbracket} |a(y)| \le c r^{n/q_0 - n - N} 2^{i(n/q_0 - n - \ell)} |a(y)|, \qquad (\ell \le N_0 + 1,\ \text{non-integer})$$

where we used $s\,|y - x_0| \le r$.

Case 2. If $N = \llbracket \ell \rrbracket = \ell - 1$, which happens when ℓ is an integer and $N_0 \ge \llbracket \ell \rrbracket = \ell - 1$, then we again have the highest derivative condition, but now with a different bound

$$\begin{aligned} c(r2^i)^{n(q_0 - n - \llbracket \ell \rrbracket)} \frac{s\,|y - x_0|}{r2^i} \log \frac{r2^i}{s\,|y - x_0|} |a(y)| \\ \le c r^{n/q_0 - n - N} 2^{i(n/q_0 - n - \ell)} i \cdot |a(y)|, (\ell \le N_0 + 1,\ \text{integer}) \end{aligned}$$

where we used the monotonicity of $x \mapsto x\log x^{-1}$ on $[0, e^{-1}]$ to estimate $x\log x^{-1} \le 2^{-i}\log 2^i = 2^{-i} i$ for $x = \frac{s|y-x_0|}{r2^i} \le 2^{-i}$, and moreover the fact that $\llbracket \ell \rrbracket + 1 = \ell$. (For the case $i = 1$, we have $2^{-1} > e^{-1}$, but this problem is dealt with by simply taking a larger c.)

Case 3. Finally, if $N < \ell - 1$, which is the case when $N_0 < \ell - 1$, and thus $N = N_0$, then the derivatives D^α in the norm to be estimated are not of the highest order in the condition $K(q_0, \ell)$, so we do not have a direct estimate for the difference. However, writing x in place of $x - x_0$ and denoting $z := s(y - x_0)$, $u := a(y)$we have

$$\left(\int |D^\alpha k_i(x) u - D^\alpha k_i(x - z) u|_{B_2}^{q_0} \,\mathrm{d}x \right)^{\frac{1}{q_0}}$$
$$\le \int_0^1 \left(\int \left|(z \cdot D) D^\beta k_i(x - z + zt) u\right|_{B_2}^{q_0} \,\mathrm{d}x \right)^{\frac{1}{q_0}} \mathrm{d}t$$
$$\le c\,|z| \sum_{j=1}^n \int_0^1 \left(\int \left|D^{\beta + e_j} k_i(x - z(1-t)) u\right|_{B_2}^{q_0} \,\mathrm{d}x \right)^{\frac{1}{q_0}} \mathrm{d}t$$

Now $|z(1-t)| < r$, so when x ranges inside the annulus $r(2^{i+1} - 1) \le |x| \le r(2^{i+3} + 1)$, the quantity $\tilde{x} := x - z(1-t)$ satisfies $r2^i \le r(2^{i+1} - 2) \le |\tilde{x}| \le r(2^{i+3} + 2) < r2^{i+4}$, so we can still use the $K(q_0, \ell)$ derivative condition, now for the derivatives of order $|\alpha + e_i|_1 = |\alpha|_1 + 1 = N + 1 < \ell$, to yield the bound

$$c\,|z|\,(r2^i)^{n/q_0 - n - (N+1)}\,|a(y)|_{B_1} \le c r^{n/q_0 - n - N} 2^{i(n/q_0 - n - N - 1)}\,|a(y)|_{B_1}, \qquad (\ell > N_0 + 1).$$

Here $|z| < r$ was again used, and in addition we recalled that $u = a(y)$.

Conclusion. Having handled all the three cases concerning the L^{q_0} norm in (6), we note that the other factors there satisfy the estimates

$$|x - y|^N \le r^N \qquad \text{and}$$
$$\int |a(y)|_{B_1} \,\mathrm{d}y \le \left|\bar{B}\right|^{\frac{1}{q_0'}} \|a\|_{L^{q_0}} \le \left|\bar{B}\right|^{\frac{1}{q_0'}} \left|\bar{B}\right|^{\frac{1}{q_0} - \frac{1}{p_0}} = r^{n(1 - p_0^{-1})}.$$

Note that the estimate of the L^q-norm in (6) gives an estimate independent of s. Thus the integration over s in (6) just produces another constant. Combining these estimates with the three bounds obtained above for different values of ℓ and N_0, we have shown that

$$\|k_i * a\|_{L^{q_0}(\mathbb{R}^n; B_1)} \le \begin{cases} c(r2^i)^{n(q_0^{-1} - p_0^{-1})} 2^{i(n(p_0^{-1} - 1) - \ell)} & \ell \le N_0 + 1 \text{ non-integer} \\ c(r2^i)^{n(q_0^{-1} - p_0^{-1})} 2^{i(n(p_0^{-1} - 1) - \ell)} i & \ell \le N_0 + 1 \text{ integer} \\ c(r2^i)^{n(q_0^{-1} - p_0^{-1})} 2^{i(n(p_0^{-1} - 1) - N_0 - 1)} & \ell > N_0 + 1 \end{cases}$$

This shows that $c_i(k_i * a)$ satisfies the size condition of a (p_0, q_0)-atom in a ball of radius of the order $r2^i$, and the proof is complete. $\square$

In Lemma 5 we obtained estimates for $k_i * a$, where the k_i were parts of k supported in annuli of increasing radii. In the following, we deal with the part of $k*a$ not handled there, namely $k_0 * a$, where k_0 is the part of k around the origin.

Lemma 6. *Let k, k_i, a and $\bar{B} = \bar{B}(x_0, r)$ be as in Lemma 5, and $k_0(x) := \eta(r^{-1}2^{-2}x)k(x) = k(x) - \sum_{j=1}^{\infty} k_i(x)$. Suppose further that $f \mapsto k * f$ is bounded from $L^{q_0}(\mathbb{R}^n; B_1)$ to $L^{q_0}(\mathbb{R}^n; B_2)$. Then there is a constant c_0 such that $c_0(k_0 * a)$ is a (p_0, q_0, N_0)-atom supported on $\bar{B}(x_0, 2^4 r)$.*

Proof. The moment conditions are again inherited from those of a, and the support condition is also straightforward, since $\operatorname{supp} k_0 \subset \bar{B}(0, r2^3)$ and $\operatorname{supp} a \subset \bar{B}(x_0, r)$ so in fact $\operatorname{supp} k_0 * a \subset \bar{B}(x_0, r(2^3+1))$. From (5) we then find that the supports of $k_0 * a$ and $k_i * a$ are disjoint for $i \geq 3$. Thus $k_0 * a(x) = k * a(x) - \sum_{i=1}^{2} k_i * a(x)$ on the support of the left-hand side, and therefore

$$|k_0 * a(x)| \leq |k * a(x)| + \sum_{i=1}^{2} |k_i * a(x)|,$$

whence taking the L^{q_0} norms and using the assumed boundedness of $k * \cdot$ together with the atomic (p_0, q_0) bound for $k_i * a$ coming from Lemma 5, we conclude that

$$\|k_0 * a\|_{L^{q_0}} \leq c\,\|a\|_{L^{q_0}} + c\left|\bar{B}\right|^{q_0^{-1}-p_0^{-1}} \leq c\left|\bar{B}\right|^{q_0^{-1}-p_0^{-1}},$$

where the atomic (p_0, q_0) bound for a was used in the last step. (Note that there is no need to bother about the i-dependence of the bounds for $k_i * a$, since there are only two indices involved now.) Now the bound above is, up to a constant, the size condition required for a (p_0, q_0)-atom with support in the ball $\bar{B}(x_0, 2^4 r) = 2^4\bar{B}$. $\square$

Now a boundedness theorem of the convolution $k * \cdot$ on H^p falls in our hands.

Theorem 2. *Suppose that $k \in KS(q_0, \ell)$ and that the convolution operator $f \mapsto k*f$ is bounded from $L^{q_0}(\mathbb{R}^n; B_1)$ to $L^{q_0}(\mathbb{R}^n; B_2)$. Then it is also bounded from $H^p_{\mathrm{at}}(\mathbb{R}^n; B_1)$ to $H^p_{\mathrm{at}}(\mathbb{R}^n; B_2)$ for each $p \leq 1$ such that*

$$\ell > n(p^{-1} - 1), \qquad i.e., \qquad p > \frac{1}{1 + \ell/n}.$$

Proof. Let a, supported in $\bar{B}(x_0, r)$, be an atom of H^p of L^{q_0}-type. With k_i, $i \in \mathbb{N}$, as in Lemmata 5 and 6, we have

$$k * a(x) = \sum_{j=0}^{\infty} k_j * a(x) = \sum_{j=0}^{\infty} c_j^{-1} b_j(x),$$

where $b_j := c_j(k_j * a)$ are atoms of H^p of L^{q_0}-type according to the two above mentioned Lemmata, and the constants c_j are defined as in those Lemmata. Thus we have, uniformly in H^p atoms a of L^{q_0}-type,

$$\|k * a\|_{H^p_{\mathrm{at}}}^p \leq \sum_{j=0}^{\infty} c_j^{-p}.$$

A moments look at the definitions of the constants c_j in Lemma 5 shows that this series converges to a finite value, in all three cases, if and only if $\ell > n(p^{-1} - 1)$. □

5 Multipliers

5.1 A class of multiplier functions

We now define conditions similar to the $KS(q, \ell)$ and $KN(q, \ell)$ for the multipliers m on the Fourier transform side, and analyze the relation of the conditions satisfied by the multiplier and by the kernel.

We say that a function $m \in L^\infty(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$ belongs to the class $MS(q, \ell)$ (<u>m</u>ultiplier condition with <u>s</u>trong estimates) provided that $m \in C^{\lfloor \ell \rfloor}(\mathbb{R}^n \setminus \{0\}; \mathcal{B}(B_1; B_2))$ and satisfies

$$\left(\frac{1}{R^n} \int_{R<|\xi|<2R} |D^\alpha m(\xi) u|_{B_2}^q \, \mathrm{d}\xi \right)^{\frac{1}{q}} \leq c R^{-|\alpha|_1} |u|_{B_1} \tag{7}$$

for all $R > 0$, $\alpha \in \mathbb{N}^n$ with $|\alpha|_1 \leq \lfloor \ell \rfloor$ and $u \in B_1$, and moreover, if $\ell \notin \mathbb{Z}_+$,

$$\left(\frac{1}{R^n} \int_{R<|\xi|<2R} |(D^\alpha m(\xi) - D^\alpha m(\xi - z)) u|_{B_2}^q \, \mathrm{d}\xi \right)^{\frac{1}{q}} \leq c \left(\frac{|z|}{R} \right)^{\ell - \lfloor \ell \rfloor} R^{-|\alpha|} |u|_{B_1}$$

for all $R > 0$, $z \in \mathbb{R}^n$ with $|z| < \frac{1}{2}R$, $\alpha \in \mathbb{N}^n$ with $|\alpha|_1 = \lfloor \ell \rfloor$ and $u \in B_1$. The condition $MN(q, \ell)$ (<u>m</u>ultiplier condition with <u>n</u>orm estimates) is defined similarly in a way which should be obvious now. Observe that, contrary to the case of the kernel conditions, we only require an estimate for the difference integral when ℓ is a non-integer.

This condition appears in Kurtz and Wheeden [14] for integer values of ℓ, and it was known to be related to the boundedness of multiplier operators even earlier. The classical multiplier theorems in the scalar-valued situation, namely those of Marcinkiewicz and Hörmander–Mikhlin, deduce the boundedness of the corresponding operators on L^p, $1 < p < \infty$, from the conditions $m \in M(1, 1)$ (and dimension $n = 1$) and from $m \in M(2, \ell)$, $\ell > \frac{1}{2}n$, respectively. See [14] for more history and references. The definition of the multiplier condition for general ℓ is taken from Strömberg and Torchinsky [19].

As with the kernel conditions, it is routine (and similar) to check the following:

Lemma 7. *$MS(q,\ell) \subset MS(q_1,\ell)$ and $MN(q,\ell) \subset MN(q_1,\ell_1)$ for $q_1 \le q$ and $\ell_1 \le \ell$.*

Lemma 8. *If $m \in MS(q,\ell)$ (resp. $MN(q,\ell)$), then the functions $m(t\cdot)$, $t > 0$ satisfy the condition $MS(q,\ell)$ (resp. $MN(q,\ell)$) with the same constant. Moreover, if $\psi \in \mathcal{S}(\mathbb{R}^n)$, then $\psi m \in MS(q,\ell)$ (resp. $MN(q,\ell)$) and the functions $\phi(t\cdot)m$ satisfy the condition $MS(q,\ell)$ (resp. $MN(q,\ell)$) uniformly in $t > 0$.*

Why these conditions are useful in view of multiplier theorems should be clear from the following Lemma, which relates the multiplier classes $MS(q,\ell)$ to the kernels satisfying $KS(q,\ell)$ for which we already know some rather satisfying properties.

Lemma 9. *Suppose $m \in L^\infty(\mathbb{R}^n;\mathcal{B}(B_1;B_2))$ satisfies $MS(q,\ell)$, where*

$$1 \le q \le \textit{the Fourier-type of } B_2.$$

If we define

$$m_0(\xi) := \eta(\xi)m(\xi), \quad m_i(\xi) := \phi(2^{-i}\xi)m(\xi), \text{ for } i \in \mathbb{Z}_+, \quad \text{and} \quad k_i := \check{m}_i,$$

then the kernels $k^N := \sum_{i=1}^N k_i$ satisfy the condition $KS(q',\ell - n/q)$ uniformly in N.

Proof. The proof is based on estimating separately the quantities appearing in the condition $KS(q',\tilde{\ell})$, $\tilde{\ell} := \ell - n/q$, for each of the k_i and adding the estimates. Observe that m_i is a bounded function with compact support, and thus the same is true for each $\xi^\alpha m_i(\xi)$, $\alpha \in \mathbb{N}^n$. In particular, these functions are integrable, and it follows that $k_i \in C^\infty(\mathbb{R}^n;\mathcal{B}(B_1;B_2))$ with bounded derivatives of all orders. The same is obviously true for any finite sum of the k_i's.

Establishing proper estimates for the kernel conditions of the k_i's consists of several cases, and we shall not go into the details, which are just the same as in the scalar case worked out in [19, Lemma 11.1]. We point out, however, that the assumption concerning the Fourier type of B_2 comes to use when applying the Hausdorff–Young inequality (1) to pass from estimates on the $L^q(\mathbb{R}^n;B_2)$ norm of $\mathbb{R}^n \ni \xi \mapsto m_i(\xi)u \in B_2$ and its derivatives to the $L^{q'}(\mathbb{R}^n;B_2)$ estimates on $k_i(x)u$. We should also note that the uniformity of the $MS(q,\ell)$ condition of the m_i guaranteed by Lemma 8 is extensively exploited.

Due to the dilation invariance of the conditions $MS(q,\ell)$ and $KS(q',\tilde{\ell})$ the proof can be reduced to showing the appropriate $KS(q',\tilde{\ell})$ conditions when $R = 1$, and there we have, quoting [19],

$$\sum_{i=0}^{\infty} \left(\int_{1<|\xi|<2} |D^\alpha k_i(x)u|_{B_2}^{q'} \, \mathrm{d}x \right)^{\frac{1}{q'}} = c\, |u|_{B_1},$$

when $|\alpha|_1 < \tilde{\ell} := \ell - n/q$, and

$$\sum_{i=0}^{\infty} \left(\int_{1<|x|<2} |(D^\alpha k_i(x) - D^\alpha k_i(x-z))u|_{B_2}^{q'} \, \mathrm{d}x \right)^{\frac{1}{q'}} \leq \begin{cases} c\,|z|^{\tilde{\ell} - [\![\tilde{\ell}]\!]}\, |u|_{B_1} & \tilde{\ell} \notin \mathbb{Z}_+, \\ c\,|z| \log \frac{1}{|z|} \cdot |u|_{B_1} & \tilde{\ell} \in \mathbb{Z}_+. \end{cases}$$

for $|\alpha|_1 = [\![\tilde{\ell}]\!]$.

From these, the uniform estimates for the $k^N := \sum_{i=1}^N k_i$ follow by the inequality of Minkowski. □

5.2 Multiplier operators in the lower extreme

Before stating our multiplier theorem, it is useful to make some remarks on the definition of these operators. Formally, given $m \in L^\infty(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$, the corresponding multiplier operator is $T = T_m := \mathcal{F}^{-1} m \mathcal{F}$, where the function m is identified with a pointwise multiplication operator in an obvious way.

One often initially defines (cf. e.g. [8]) T as an operator acting on the rather restricted function class $\mathcal{F}\mathcal{D}(\mathbb{R}^n; B_1)$ (Fourier transforms of test functions, or equivalently, Schwartz functions whose Fourier transforms have compact support). This has the advantage that one can even consider multipliers m which are only locally integrable; indeed, for $\psi \in \mathcal{F}\mathcal{D}(\mathbb{R}^n; B_1)$, we have $\hat{\psi} \in \mathcal{D}(\mathbb{R}^n; B_1)$ and so $m\hat{\psi}$ is integrable with compact support. The inverse Fourier transform is then defined even in the usual L^1 sense as a convergent Bôchner integral, and it gives a bounded, infinitely differentiable function.

However, a result of Clément and Prüss [8, Prop. 1] shows that for $m \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$ to be a multiplier between any of the spaces $L^p(\mathbb{R}^n; B_1)$ and $L^p(\mathbb{R}^n; B_2)$, with $1 < p < \infty$, it is necessary that the set of operators $\{m(t) : t \text{ Lebesgue-point of } m\}$ be R-bounded, thus uniformly bounded, so it is very reasonable to restrict the study to multipliers $m \in L^\infty(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$.

We can then define T on all functions $f \in L^q(\mathbb{R}^n; B_1)$, with $1 \leq q \leq$ the Fourier-type of B_1 (thus at least on $L^1(\mathbb{R}^n; B_1)$), by the formula $Tf := \mathcal{F}^{-1}(m\hat{f})$, interpreted as follows: The Fourier transform $\hat{f}$ is a well-defined function in $L^{q'}(\mathbb{R}^n; B_1)$, and thus we have $m\hat{f} \in L^{q'}(\mathbb{R}^n; B_2) \subset \mathcal{S}'(\mathbb{R}^n; B_2)$. The inverse Fourier transform can then be taken in the sense of (tempered) distributions, and we end up with a $Tf \in \mathcal{S}'(\mathbb{R}^n; B_2)$.

The problem is then to determine conditions under which this distribution coincides with an element of one of the function spaces we have studied, for all f in (a dense subset of) an appropriate space, and moreover the conditions under which the map T is bounded between these spaces. Note that $(L^1 \cap L^p)(\mathbb{R}^n; B)$ is dense in $L^p(\mathbb{R}^n; B)$ for all $p \in (1, \infty)$, and moreover $(L^1 \cap H^p_{\mathrm{at}})(\mathbb{R}^n; B)$ (even $(L^1 \cap L^q \cap H^p_{\mathrm{at}}(\mathbb{R}^n; B))$ for any $q > 1$) is dense in $H^p_{\mathrm{at}}(\mathbb{R}^n; B)$ for $p \in (0, 1]$ (Lemma 1), and all these dense subsets remain dense even if we impose the additional restriction that the Fourier transforms of the functions have compact support.

Thus for all these spaces, it suffices to prove the boundedness of T on such subsets, where the initial definition of T is valid, and then obtain the extension to the whole space by continuity. However, in the upper extreme with the "large" spaces L^∞ and BMO, which will be considered in §§ 5.3–5.4, no convenient dense subsets exist and we must restrict ourselves to a subspace in order to even have T properly defined. For this reason, we wanted to make the initial definition on a function class as large as possible.

Now we state the multiplier theorem.

Theorem 3. *Suppose that $m \in L^\infty(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$, and that the corresponding multiplier operator T is bounded from $L^{q_0}(\mathbb{R}^n; B_1)$ to $L^{q_0}(\mathbb{R}^n; B_2)$ for some $q_0 \in (1, \infty)$. Suppose further that $m \in MS(q, \ell)$ for some q such that*

$$1 \leq q \leq \text{the Fourier-type of } B_2, \qquad q \leq q_0' \qquad \text{and} \qquad \ell > n/q.$$

Then T extends to a bounded mapping from $H^p_{\mathrm{at}}(\mathbb{R}^n; B_1)$ to $H^p_{\mathrm{at}}(\mathbb{R}^n; B_2)$ for all p in the range

$$1 \geq p > \frac{1}{1/q' + \ell/n}.$$

Observe that $1/q' + \ell/n > 1/q' + 1/q = 1$ under the assumptions, so that the asserted range of p is non-empty. Also note that only the Fourier-type of the image space B_2 is relevant, and moreover the theorem always contains the case $q = 1$, without any geometric conditions on the Banach spaces in question.

Proof. By Lemma 9, the kernels k^N defined there satisfy the kernel condition $KS(q', \ell - n/q) =: KS(q', \tilde{\ell})$ uniformly in N. By the monotonicity of the kernel conditions (Lemma 2), they therefore satisfy $KS(q_0, \tilde{\ell})$ (uniformly) as well, since $q' \geq q_0$. Moreover, k^N corresponds to the multiplier $\eta(2^{-N}\cdot)m$, so $T_N f := k^N * f = \check{\eta}_{2^{-N}} * Tf$ is bounded from $L^{q_0}(\mathbb{R}^n; B_1)$ to $L^{q_0}(\mathbb{R}^n; B_2)$ uniformly in N (by the assumed boundedness of T and an easy estimate for the convolution operator with an integrable kernel).

Thus by Theorem 1, the operators T_N are uniformly bounded from the Hardy space $H^p_{\mathrm{at}}(\mathbb{R}^n; B_1)$ to $H^p_{\mathrm{at}}(\mathbb{R}^n; B_2)$ for p satisfying

$$1 \geq p > \frac{1}{1 + \tilde{\ell}/n} = \frac{1}{1 + (\ell - n/q)/n} = \frac{1}{1/q' + \ell/n}.$$

For $\psi \in (H^p_{\mathrm{at}} \cap L^1 \cap L^{q_0})(\mathbb{R}^n; B_1)$ with compactly supported Fourier transform we have (at least in the sense of distributions) $\widehat{T_N \psi}(\xi) = \eta(2^{-N}\xi)m(\xi)\hat{\psi}(\xi) = m(\xi)\hat{\psi}(\xi) = \widehat{T\psi}(\xi)$, whenever N is large enough so that $\bar{B}(0, 2^N)$ contains the support of $\hat{\psi}$. Thus for such ψ we have

$$\|T\psi\|_{H^p_{\mathrm{at}}(\mathbb{R}^n; B_2)} = \lim_{N \to \infty} \|T_N \psi\|_{H^p_{\mathrm{at}}(\mathbb{R}^n; B_2)} \leq C \|\psi\|_{H^p_{\mathrm{at}}(\mathbb{R}^n; B_1)},$$

and the assertion follows from the density in H^p_{at} of the functions of the type considered (Lemma 1). $\square$

The formulation of our multiplier theorem in the strongest possible setting is somewhat technical. In the applications in § 6, the multipliers will satisfy the conditions $MS(q,\ell)$ (in fact, $MN(q,\ell)$, and even more) for arbitrarily large ℓ. It is therefore convenient to state the following obvious consequence of the Theorem:

Corollary 1. *If $m \in L^\infty(\mathbb{R}^n;\mathcal{B}(B_1;B_2))$ satisfies $MS(1,\ell)$ for arbitrarily large ℓ, and the corresponding operator T is bounded from $L^{q_0}(\mathbb{R}^n;B_1)$ to $L^{q_0}(\mathbb{R}^n;B_2)$ for some $q_0 \in (1,\infty)$, then T extends to a bounded mapping from $H^p_{\mathrm{at}}(\mathbb{R}^n;B_1)$ to $H^p_{\mathrm{at}}(\mathbb{R}^n;B_2)$ for all $p \in (0,1]$.*

Next we examine consequences of Theorem 3 in spaces with additional geometric structure.

The Hilbert space case. It might be illuminating to see what this theorem gives in a Hilbert space setting, where the boundedness from $L^2(\mathbb{R}^n;\mathcal{H}_1)$ to $L^2(\mathbb{R}^n;\mathcal{H}_2)$ already follows (by Plancherel's formula) from the assumption $m \in L^\infty(\mathbb{R}^n;\mathcal{B}(B_1;B_2))$. Moreover, the Fourier-type of $\mathcal{H}_2$ is of course 2. With q_0 and the Fourier-type equal to 2, we could in principle take $1 \le q \le 2$, but let us for simplicity just consider $q = 2$. This gives

Corollary 2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and suppose that the multiplier $m \in L^\infty(\mathbb{R}^n;\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2))$ satisfies the condition $MS(2,\ell)$ with $\ell > \frac{1}{2}n$. Then the corresponding operator T extends to a bounded mapping from $H^p_{\mathrm{at}}(\mathbb{R}^n;\mathcal{H}_1)$ to $H^p_{\mathrm{at}}(\mathbb{R}^n;\mathcal{H}_2)$ for $1 \ge p > (1/2+\ell/n)^{-1}$, in particular from $H^1_{\mathrm{at}}(\mathbb{R}^n;\mathcal{H}_1)$ to $H^1_{\mathrm{at}}(\mathbb{R}^n;\mathcal{H}_2)$.*

By interpolation, we also get boundedness from $L^p(\mathbb{R}^n;\mathcal{H}_1)$ to $L^p(\mathbb{R}^n;\mathcal{H}_2)$ for all $p \in (1,2)$. Boundedness from $L^p(\mathbb{R}^n;\mathcal{H}_1)$ to $L^p(\mathbb{R}^n;\mathcal{H}_2)$ for $p \in (2,\infty)$ can be achieved if we also assume that $m^* \in MS(2,\ell)$, where the pointwise values of $m^* \in L^\infty(\mathbb{R}^n;\mathcal{B}(\mathcal{H}_2;\mathcal{H}_1))$ are given by the adjoint operators of corresponding pointwise values of m (cf. Remarks in § 3; in a Hilbert space setting, the duality argument is particularly easy to make). A stronger assumption which implies both of these is the condition $m \in MN(2,\ell)$. This is the Hilbert space version of the Hörmander–Mikhlin theorem, but we also automatically get boundedness on some of the spaces H^p, for p not too small.

The UMD-space case with R-boundedness. A possibly more interesting application of Theorem 3 appears in combination with the n-dimensional version of the Mikhlin–Weis theorem [9, Th. 3.25].

The statement of the theorem involves explicitly the concept of R-boundedness. This notion was first used by Bourgain [5, Lemma 7] without giving the concept a name. Since then the notion has been formalized, and many of its properties are explored in Clément, de Pagter, Sukochev and Witvliet [7] and in Weis [20]; we also refer to these papers for the definition. The *R-bound* of a collection $\mathcal{T} \subset \mathcal{B}(B_1;B_2)$ is denoted by $\mathcal{R}(\mathcal{T})$. We recall here that an R-bounded collection $\mathcal{T}$ is always uniformly bounded, and $\sup_{T\in\mathcal{T}} |T|_{\mathcal{B}(B_1;B_2)} \le$

$\mathcal{R}(\mathcal{T})$, but the converse is only true under special conditions on the geometry of B_1 and B_2 (see [1, Prop. 1.13]).

Now let us state the result:

Theorem 4. *Let B_1 and B_2 be UMD-spaces, and suppose $m \in C^n(\mathbb{R}^n \setminus \{0\}; \mathcal{B}(B_1; B_2))$ satisfies*

$$\mathcal{R}\left(\{|\xi|^{|\alpha|_1} D^\alpha m(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}\}\right) < \infty \qquad \text{for all } \alpha \in \{0,1\}^n. \tag{8}$$

Then the corresponding operator T is bounded from $L^p(\mathbb{R}^n; B_1)$ to $L^p(\mathbb{R}^n; B_2)$ for every $p \in (1, \infty)$.

If moreover $m \in MS(q, \ell)$, where $1 \leq q \leq$ the Fourier-type of B_2, and $\ell > n/q$, then T is also bounded from $H^p_{\mathrm{at}}(\mathbb{R}^n; B_1)$ to $H^p_{\mathrm{at}}(\mathbb{R}^n; B_2)$ for $1 \geq p > (1/q' + \ell/n)^{-1}$.

Proof. The first paragraph of the Theorem is a restatement of the n-dimensional Mikhlin–Weis theorem [9, Th. 3.25]. This gives us boundedness on the spaces L^p as stated, so the assumptions of Theorem 3 are satisfied, and we get boundedness on H^p_{at} with p in the asserted range. $\square$

A few words about the conditions in Theorem 4 are in order. First, with $\ell \in (n/q, n+1)$, we have $\lfloor \ell \rfloor \leq n$, and thus the differentiability conditions required by the condition $MS(q, \ell)$ are already included in the assumption $m \in C^n(\mathbb{R}^n \setminus \{0\})$ of the Mikhlin–Weis theorem, so no extra smoothness is required. Moreover, the R-boundedness conditions (8) already imply some of the inequalities (7) in the $MS(q, \ell)$ condition, namely those with $\alpha \in \{0,1\}^n$. In general, there are other α's with $|\alpha|_1 \leq \lfloor \ell \rfloor$, so a part of the condition $m \in MS(q, \ell)$ is not implied by the R-boundedness assumptions and needs to be included as a separate assumption.

However, as we will next show, in the one-dimensional setting, all the assumptions we need are already included in the R-boundedness conditions, and thus we obtain a genuine extension of the one-dimensional Mikhlin–Weis theorem to $H^1_{\mathrm{at}}(\mathbb{R}; B_1)$:

Corollary 3. *Let B_1 and B_2 be UMD-spaces, and suppose $m \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(B_1; B_2))$ satisfies*

$$\mathcal{R}\left(\{m(\xi) : \xi \neq 0\}\right) < \infty \qquad \text{and} \qquad \mathcal{R}\left(\{\xi m'(\xi) : \xi \neq 0\}\right) < \infty.$$

Then the corresponding operator T is bounded from $L^p(\mathbb{R}; B_1)$ to $L^p(\mathbb{R}; B_2)$ for $p \in (1, \infty)$ and from $H^1_{\mathrm{at}}(\mathbb{R}; B_1)$ to $H^1_{\mathrm{at}}(\mathbb{R}; B_2)$.

Proof. The assumptions are just those of the Mikhlin–Weis theorem [20, Theorem 3.4], and the assertion concerning the L^p-boundedness is nothing but the statement of that theorem. For the H^1_{at}-boundedness, we need to verify (according to Theorem 4) that $m \in MS(q, \ell)$ with $1 \leq q \leq$ the Fourier-type of B_2, and $\ell > 1/q$.

Since every UMD-space has a non-trivial Fourier-type, let us fix a $q > 1$, but not exceeding the Fourier-type of B_2. Then $1 > 1/q$, so let us take

$\ell = 1$, and it suffices to check that $m \in MS(q, 1)$. Since the dimension $n = 1$, this means checking the inequality (7) with $\alpha = 0$ and $\alpha = 1$. But these inequalities (even their norm versions) immediately follow from the assumed R-bounds (even the corresponding uniform bounds would more than suffice). $\square$

Of course, depending on the Fourier-type of B_2, we also obtain boundedness for some of the $H^p_{\mathrm{at}}(\mathbb{R}; B_i)$, with $p < 1$ but greater than the bound given in Theorem 3.

It is not surprising that one obtains stronger Fourier multiplier theorems in spaces with a larger Fourier-type. A very intimate connection of a non-trivial Rademacher-type (which is equivalent to a non-trivial Fourier-type, see § 2.1) to the boundedness of multipliers in the periodic case is contained in Arendt and Bu [1, Prop. 1.12]. It is interesting, however, that there it is the type of the Banach space B_1 in the domain of the multiplier operator that matters, whereas our result shows dependence on the Fourier-type of the space B_2.

5.3 The upper extreme and interpolation

Let us consider the norm conditions for multipliers now. Again, as in the proof of Lemma 9, we can simply repeat the argument in [19, Lemma 11.1], now working directly with the operators $k_i(x)$ and $m_i(x)$ without the evaluation point $u \in B_1$. Thus the norms of B_2 are replaced by the operator norms of $\mathcal{B}(B_1; B_2)$, and we employ the Hausdorff–Young inequality between $L^1(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$ and $L^\infty(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$ (for which no assumption concerning the Fourier-type is required). This leads to a result analogous to Lemma 9 in which $m \in MN(1, \ell)$ implies $k^N \in KS(\infty, \ell - n)$ uniformly in N.

However, all we really want to have is the norm condition (4) of Benedek, Calderón and Panzone. Indeed, we know from Theorem 3 and Corollary 1 that the strong conditions already allow us to reach everything we could hope for concerning the boundedness of operators in the lower extreme, i.e., between the H^p_{at}-spaces, and for the upper extreme we simply require the condition (4), as we know from Theorem 1. But we already know from Lemma 4 that each of the conditions $KN(q, \ell)$, $q \geq 1$, $\ell > 0$, implies the condition (4). We can hence formulate the analogue of Lemma 9 as follows:

Lemma 10. *Suppose $m \in L^\infty(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$ satisfies $MN(1, \ell)$ for $\ell > n$, and the m_i's, k_i's and k^N's are defined as in Lemma 9. Then k^N, $N \in \mathbb{N}$, satisfy (4) uniformly.*

This again gives us a multiplier result. The conclusion we obtain in the extreme case of L^∞ and BMO is perhaps somewhat unsatisfactory, since the lack of a convenient dense subspace, like the one exploited at the end of the proof of Theorem 3, prevents us from repeating the arguments leading to the final result there. Nevertheless, as we shall see below, the result we obtain is strong enough to get the L^p-boundedness for $1 < p < \infty$ by interpolation.

Proposition 1. *Suppose $m \in L^\infty(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$ and the corresponding operator T is bounded from $L^{q_0}(\mathbb{R}^n; B_1)$ to $L^{q_0}(\mathbb{R}^n; B_2)$ for some $q_0 \in (1, \infty)$. Suppose further that $m \in MN(1, \ell)$ for some $\ell > n$. Then the operators T_N, $N \in \mathbb{N}$, with multipliers $\eta(2^{-N}\cdot)m$ are uniformly bounded from $L^\infty(\mathbb{R}^n; B_1)$ to $\mathrm{BMO}(\mathbb{R}^n; B_2)$.*

We then give the promised multiplier result for L^p. This is analogous to Theorem 1 in that the boundedness of an operator on all L^p is deduced from its boundedness on one, but now we make the assumptions on the multiplier m acting on the Fourier transform side.

Corollary 4. *Suppose m and T are as in the assumptions of Proposition 1. Then T has a bounded extension from $L^p(\mathbb{R}^n; B_1)$ to $L^p(\mathbb{R}^n; B_2)$ for every $p \in (1, \infty)$.*

Proof. Since the norm condition $MN(1, \ell)$ implies the corresponding strong condition $MS(1, \ell)$, T is bounded from $H^1_{\mathrm{at}}(\mathbb{R}^n; B_1)$ to $H^1_{\mathrm{at}}(\mathbb{R}^n; B_2)$ by Theorem 3. Hence the operators $f \mapsto T_N f = \check{\eta}_{2^{-N}} * Tf$ are also bounded between these spaces, uniformly in N.

By Proposition 1, the T_N's are also uniformly bounded from $L^\infty(\mathbb{R}^n; B_1)$ to $\mathrm{BMO}(\mathbb{R}^n; B_2)$. It follows thus from interpolation that these operators are uniformly bounded from $L^p(\mathbb{R}^n; B_1)$ to $L^p(\mathbb{R}^n; B_2)$ for every $p \in (1, \infty)$.

Now the same argument as at the end of the proof of Theorem 3 can be used to settle the matters, since the set of those $f \in L^p(\mathbb{R}^n; B_1)$ whose Fourier transform is compactly supported is dense in $L^p(\mathbb{R}^n; B_1)$. $\square$

5.4 Duality arguments

We briefly consider another way of reaching the upper extreme by means of the duality of $H^1_{\mathrm{at}}(\mathbb{R}^n; B)$ and $\mathrm{BMO}(\mathbb{R}^n; B^*)$, which holds given that B^* has the Radon–Nikodým property, which will be assumed here.

Duality arguments between $L^p(\mathbb{R}^n; B)$ and $L^{p'}(\mathbb{R}^n; B^*)$ in connection with the operator-valued convolution operators were already used in Benedek, Calderón and Panzone [2], and also commented on in Remarks in § 3. They exploited the fact that a convolution operator as in § 3 with kernel $k \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$ is bounded from $L^p(\mathbb{R}^n; B_1)$ to $L^{p'}(\mathbb{R}^n; B_2)$ if and only if the convolution operator with kernel k^* (adjoint taken pointwise) is bounded from $L^{p'}(\mathbb{R}^n; B_2^*)$ to $L^{p'}(\mathbb{R}^n; B_1^*)$. One can also show that a multiplier operator with multiplier $m \in L^\infty(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$ is bounded from $L^p(\mathbb{R}^n; B_1)$ to $L^p(\mathbb{R}^n; B_2)$ if and only if the operator with multiplier m^* (adjoint taken pointwise) is bounded from $L^{p'}(\mathbb{R}^n; B_2^*)$ to $L^{p'}(\mathbb{R}^n; B_1^*)$. Such duality arguments are fairly standard and easy to make in the setting of the L^p spaces, $1 < p < \infty$, since we can exploit, e.g., the very convenient dense subspace $\hat{\mathfrak{D}}$ of Schwartz functions with compactly supported Fourier transform.

We shall concentrate on the duality in the extremes of H^1_{at} and L^1 at one end and L^∞ and BMO at the other. We only consider the case of multipliers here. Convolution operators can be treated in a similar and even somewhat simpler fashion, since one avoids the problem of having to switch between the non-transformed and transformed domains. When dealing with multipliers, this is essential, and the Fourier-type of the underlying Banach spaces will be of significance.

We will use the duality equality $\int \left\langle \hat{f}(x), g(x) \right\rangle \, \mathrm{d}x = \int \langle f(x), \hat{g}(x) \rangle \, \mathrm{d}x$, which is easily verified for $f \in L^1(\mathbb{R}^n; B^*)$, $g \in L^1(\mathbb{R}^n; B)$, and can be extended by continuity to $f \in L^q(\mathbb{R}^n; B^*)$, $g \in L^q(\mathbb{R}^n; B)$ given that B and B^* have Fourier-type q. We denote by $\mathcal{F}L^q(\mathbb{R}^n; B)$ the image of $L^q(\mathbb{R}^n; B)$ under the Fourier transform. Since $\mathcal{F}^2 f = \widetilde{f} := f(-\cdot)$, at least in the sense of distributions, this is equivalently the set of those f whose Fourier transform belongs to $L^q(\mathbb{R}^n; B)$. Let T_m be the multiplier operator with multiplier m.

We start with the following duality lemma:

Lemma 11. *Let $q \in (1, 2]$ and suppose that*

$$g \in (L^q \cap \mathcal{F}L^q)(\mathbb{R}^n; B_1), \qquad f \in (L^q \cap \mathcal{F}L^q)(\mathbb{R}^n; B_2^*),$$

and that

$$T_m \in \mathcal{B}(L^q(\mathbb{R}^n; B_1); L^q(\mathbb{R}^n; B_2)), \qquad T_{m^*} \in \mathcal{B}(L^q(\mathbb{R}^n; B_2^*); L^q(\mathbb{R}^n; B_1^*)),$$

where B_i and B_i^*, $i = 1, 2$, have Fourier-type q. Then

$$\int \langle T_{m^*} f(x), g(x) \rangle \, \mathrm{d}x = \int \left\langle \widetilde{f}(x), T_m \widetilde{g}(x) \right\rangle \, \mathrm{d}x$$

Proof. Using the duality equality for the Fourier transform and the definition of the multiplier operators, we compute

$$\begin{aligned} \mathrm{LHS} &= \int \langle T_{m^*} f(x), \mathcal{F}[\check{g}](x) \rangle \, \mathrm{d}x = \int \langle \mathcal{F}[T_{m^*} f](\xi), \check{g}(\xi) \rangle \, \mathrm{d}\xi \\ &= \int \left\langle m^*(\xi) \hat{f}(\xi), \check{g}(\xi) \right\rangle \, \mathrm{d}\xi = \int \left\langle \hat{f}(\xi), m(\xi) \check{g}(\xi) \right\rangle \, \mathrm{d}\xi \\ &= \int \left\langle \hat{f}(\xi), \mathcal{F}[T_m \widetilde{g}](\xi) \right\rangle \, \mathrm{d}\xi = \int \left\langle \mathcal{F}[\hat{f}](x), T_m \widetilde{g}(x) \right\rangle \, \mathrm{d}x = \mathrm{RHS}\,. \end{aligned}$$

□

Now we can deduce properties of the operator T_{m^*} in the upper extreme from those of T_m in the lower extreme:

Lemma 12. *Suppose, in addition to the assumptions of Lemma 11, that*

$$T_m \in \mathcal{B}(H^1_{\mathrm{at}}(\mathbb{R}^n; B_1); L^1(\mathbb{R}^n; B_2)) \quad (\textit{resp. } \mathcal{B}(H^1_{\mathrm{at}}(\mathbb{R}^n; B_1); H^1_{\mathrm{at}}(\mathbb{R}^n; B_2))\ ),$$

and that B_1^* has the Radon–Nikodým property. Then T_{m^*} maps

$$(L^\infty \cap L^q \cap \mathcal{F}L^q)(\mathbb{R}^n; B_2^*) \subset L^\infty(\mathbb{R}^n; B_2^*)$$
$$(\textit{resp. } (\mathrm{BMO} \cap L^q \cap \mathcal{F}L^q)(\mathbb{R}^n; B_2^*) \subset \mathrm{BMO}(\mathbb{R}^n; B_2^*)\ )$$

boundedly into $\mathrm{BMO}(\mathbb{R}^n; B_1^*)$.

On the other hand, if B_2^{**} *has the Radon–Nikodým property and if* T_{m^*} *satisfies the assumptions for* T_m *above, then* T_m *satisfies the conclusions for* T_{m^*} *(with obvious modifications concerning the underlying Banach space* B_i, B_i^* *in the statement).*

It is perhaps useful to note that the required Radon–Nikodým properties hold if B_1 and B_2 are reflexive (see [10]). Moreover, if the spaces are UMD, then also a non-trivial Fourier-type q required in the conditions of Lemma 11 (and thus also the present one) is guaranteed to exist (cf. § 2.1).

Proof. We use the duality of $H^1_{\mathrm{at}}(\mathbb{R}^n; B_1)$ and $\mathrm{BMO}(\mathbb{R}^n; B_1^*)$ and the density in $H^1_{\mathrm{at}}(\mathbb{R}^n; B_1)$ of $g \in (H^1_{\mathrm{at}} \cap L^q)(\mathbb{R}^n; B_1)$ with compactly supported Fourier transform. In order to apply the duality equation from Lemma 11, we observe that $g \in H^1_{\mathrm{at}}(\mathbb{R}^n; B_1) \subset L^1(\mathbb{R}^n; B_1)$ implies $\hat{g} \in L^\infty(\mathbb{R}^n; B_1)$, and this combined with the assumption that $\hat{g}$ has compact support shows that $\hat{g} \in L^q(\mathbb{R}^n; B_1)$. We then obtain

$$
\begin{aligned}
\|T_{m^*} f\|_{\mathrm{BMO}(\mathbb{R}^n; B_1^*)} &\cong \|T_{m^*} f\|_{H^1_{\mathrm{at}}(\mathbb{R}^n; B_1)^*} \\
&= \sup_{\substack{g \in (H^1_{\mathrm{at}} \cap L^q)(\mathbb{R}^n; B_1) \\ \operatorname{supp} \hat{g} \text{ compact}, \|g\|_{H^1_{\mathrm{at}}} \le 1}} \left| \int \langle T_{m^*} f(x), g(x) \rangle \, \mathrm{d}x \right| \\
&\le \begin{cases} \sup \left\| \widetilde{f} \right\|_{L^\infty(\mathbb{R}^n; B_2^*)} \|T_m \widetilde{g}\|_{L^1(\mathbb{R}^n; B_2)} \le \left\| \widetilde{f} \right\|_{L^\infty(\mathbb{R}^n; B_2^*)} \|T_m\|_{\mathcal{B}(H^1_{\mathrm{at}}; L^1)}, \\ \sup \left\| \widetilde{f} \right\|_{\mathrm{BMO}(\mathbb{R}^n; B_2^*)} \|T_m \widetilde{g}\|_{H^1(\mathbb{R}^n; B_2)} \le \left\| \widetilde{f} \right\|_{\mathrm{BMO}(\mathbb{R}^n; B_2^*)} \|T_m\|_{\mathcal{B}(H^1_{\mathrm{at}}; H^1_{\mathrm{at}})}, \end{cases}
\end{aligned}
$$

where the first and second cases correspond to the two alternative assumptions of the Lemma.

The other way round. Now, if we suppose that T_{m^*} is bounded from $H^1_{\mathrm{at}}(\mathbb{R}^n; B_2^*)$ to $L^1(\mathbb{R}^n; B_1^*)$ (respectively $H^1_{\mathrm{at}}(\mathbb{R}^n; B_1^*)$), then by the first part of the proof $T_{m^{**}}$ is bounded from

$$
\begin{aligned}
&(L^\infty \cap L^q \cap \mathcal{F}L^q)(\mathbb{R}^n; B_1^{**}) \subset L^\infty(\mathbb{R}^n; B_1^{**}) \\
&\qquad\qquad (\text{resp. } (\mathrm{BMO} \cap L^q \cap \mathcal{F}L^q)(\mathbb{R}^n; B_1^{**}) \subset \mathrm{BMO}(\mathbb{R}^n; B_1^{**})\ )
\end{aligned}
$$

to $\mathrm{BMO}(\mathbb{R}^n; B_2^{**}) \cong H^1_{\mathrm{at}}(\mathbb{R}^n; B_2^*)^*$.

But considering B_i embedded in B_i^{**} in the usual way, we have the identity $m^{**}(\xi)|_{B_1} = m(\xi) \in \mathcal{B}(B_1; B_2)$; and thus we conlude that T_m is bounded between the same function spaces as $T_{m^{**}}$, but with the second duals B_i^{**} replaced by B_i's as the underlying Banach spaces. □

We can then formulate a multiplier result valid at least on some subsets of $\mathrm{BMO}(\mathbb{R}^n; B_1)$:

Proposition 2. *Let $m \in L^\infty(\mathbb{R}^n; \mathcal{B}(B_1; B_2))$, and suppose the corresponding multiplier operator T_m is bounded from $L^{q_0}(\mathbb{R}^n; B_1)$ to $L^{q_0}(\mathbb{R}^n; B_2)$ for some $q_0 \in (1, \infty)$. Suppose further that*

$$m \in MS(q_1, \ell_1), \qquad \textit{where} \qquad \begin{cases} 1 \leq q_1 \leq \text{ \textit{the Fourier-type of} } B_2, \\ q_1 \leq q_0', \qquad \ell_1 > n/q_1. \end{cases}$$

and that the pointwise adjoint $m^* \in L^\infty(\mathbb{R}^n; \mathcal{B}(B_2^*; B_1^*))$ satisfies

$$m^* \in MS(q_2, \ell_2), \qquad \textit{where} \qquad \begin{cases} 1 \leq q_2 \leq \text{ \textit{the Fourier-type of} } B_1^*, \\ q_2 \leq q_0, \qquad \ell_2 > n/q_2. \end{cases}$$

Then T_m is bounded from $L^q(\mathbb{R}^n; B_1)$ to $L^q(\mathbb{R}^n; B_2)$ for all $q \in (1, \infty)$, and moreover it maps boundedly (in the BMO*-norm)*

$$(\mathrm{BMO} \cap L^p \cap \mathcal{F}L^p)(\mathbb{R}^n; B_1) \subset \mathrm{BMO}(\mathbb{R}^n; B_1) \qquad \textit{into} \qquad \mathrm{BMO}(\mathbb{R}^n; B_2)$$

for every $p > 1$ for which all of the spaces B_i, B_i^* have Fourier-type p.

Note that both of the MS-conditions above are implied by the single norm condition

$$m \in MN(q_1, \ell_1), \quad \text{where} \qquad \begin{cases} 1 \leq q_1 \leq \text{ Fourier-types of } B_2,\ B_1^*, \\ q_1 \leq \min(q_0, q_0'), \quad \ell_1 > n/q_1, \end{cases}$$

since $m(\xi)$ and $m^*(\xi)$ have equal norms.

Proof. By Theorem 3, T_m is bounded from $H^1_{\mathrm{at}}(\mathbb{R}^n; B_1)$ to $H^1_{\mathrm{at}}(\mathbb{R}^n; B_2)$ and by interpolation from $L^q(\mathbb{R}^n; B_1)$ to $L^q(\mathbb{R}^n; B_2)$ for all $q \in (1, q_0]$. Then by standard duality, T_{m^*} is bounded from $L^r(\mathbb{R}^n; B_2^*)$ to $L^r(\mathbb{R}^n; B_1^*)$ for all $r \in [q_0', \infty)$.

Applying Theorem 3 to T_{m^*}, starting from boundedness from $L^{q_0'}(\mathbb{R}^n; B_2^*)$ to $L^{q_0'}(\mathbb{R}^n; B_1^*)$, we get that T_{m^*} is bounded from $H^1_{\mathrm{at}}(\mathbb{R}^n; B_2^*)$ to $H^1_{\mathrm{at}}(\mathbb{R}^n; B_1^*)$, and thus by interpolation from $L^q(\mathbb{R}^n; B_2^*)$ to $L^q(\mathbb{R}^n; B_1^*)$ for all $q \in (1, \infty)$. Then by standard duality, T_m is bounded from $L^q(\mathbb{R}^n; B_1)$ to $L^q(\mathbb{R}^n; B_2)$ for all $q \in (1, \infty)$.

Now, from the fact that T_{m^*} is bounded from $H^1_{\mathrm{at}}(\mathbb{R}^n; B_2^*)$ to $H^1_{\mathrm{at}}(\mathbb{R}^n; B_1^*)$, it now follows from Lemma 12 that T_m is bounded in the BMO-norm in the way asserted in the statement of the Proposition. ☐

6 Applications to maximal regularity

We consider applications of the multiplier theorems to the problem of maximal regularity of a number of differential equations. This is only a brief account, and one can certainly consider many other problems as well. We only state results in the lower extreme, i.e., the Hardy spaces, since our multiplier theorems in the upper extreme are not so appealing, although partial maximal regularity results on appropriate subspaces can be obtained there, too, in the same spirit as the corresponding multiplier theorems.

6.1 The Cauchy problem $u' + Au = f$, $u(0) = 0$

Given that $-A$ generates a bounded analytic semigroup $\exp(-tA)$, $t \geq 0$, this problem is known to have a unique "mild" solution $u \in L^p_{\mathrm{loc}}(\mathbb{R}_+; B)$ for every $f \in L^p(\mathbb{R}_+; B)$ (cf. Weis [20]). We say that the problem has maximal L^p-regularity if for every $f \in L^p(\mathbb{R}_+; B)$ the solution u is a.e. differentiable with values in $\mathcal{D}(A)$, and moreover, for some $C < \infty$, we have

$$\|u'\|_{L^p(\mathbb{R}_+;B)} + \|u\|_{L^p(\mathbb{R}_+;B)} + \|Au\|_{L^p(\mathbb{R}_+;B)} \leq C \|f\|_{L^p(\mathbb{R}_+;B)} .$$

Of course, it suffices to verify the above boundedness for u and, say, Au, since the similar estimate for u' is then obtained directly from $u' = f - Au$. Maximal regularity on H^p can be defined similarly in an obvious way.

We now wish to apply the multiplier theory to this problem. To this end, we observe that, at least in the sense of distributions, the original Cauchy problem is equivalent to

$$(\mathbf{i}\xi + A)\hat{u}(\xi) = \hat{f}(\xi),$$

where we have extended f to the whole line by setting $f(t) := 0$ for $t < 0$ in order to take the Fourier transform. Given that $(\mathbf{i}\xi + A)$ is invertible for all $\xi \in \mathbb{R}$, we find that

$$\hat{u}(\xi) = (\mathbf{i}\xi + A)^{-1}\hat{f}(\xi) \qquad \text{and} \qquad \hat{Au}(\xi) = A(\mathbf{i}\xi + A)^{-1}\hat{f}(\xi).$$

so the problem of maximal regularity involves showing that

$$\tilde{m}(\xi) := (\mathbf{i}\xi + A)^{-1} \qquad \text{and} \qquad m(\xi) := A(\mathbf{i}\xi + A)^{-1}$$

give rise to a bounded multiplier operator on the appropriate spaces.

A necessary condition for this to be the case on any of the spaces L^p, $1 < p < \infty$, is, according to Clément and Prüss [8, Prop. 1], the R-boundedness of $\{m(\xi) : \xi \text{ a Lebesgue point of } m\}$, which in this case is $\{m(\xi) : \xi \in \mathbb{R}\}$, and similarly for $\tilde{m}$. Note, however, that the R-boundedness of $\{\tilde{m}(\xi) : \xi \in \mathbb{R}\}$ already follows from the assumption for m and the invertibility of A: Since

$$(\mathbf{i}\xi + A)^{-1} = \sum_{k=0}^{\infty} (\mathbf{i}\xi)^k A^{-(k+1)}, \qquad \text{for } |\xi| < \left\|A^{-1}\right\|^{-1}_{\mathcal{B}(B)},$$

the R-bound of $\{\tilde{m}(\xi)\}$ with ξ sufficiently small is bounded in terms of the operator norm of A^{-1} (cf. [20, Prop. 2.6]), and for $|\xi| > c$, we have

$$\begin{aligned}\mathcal{R}(\{(\mathbf{i}\xi + A)^{-1} : |\xi| > c\}) &\leq \frac{2}{c}\mathcal{R}(\{\mathbf{i}\xi(\mathbf{i}\xi + A)^{-1} : |\xi| > c\}) \\ &\leq \frac{2}{c}\left(1 + \mathcal{R}(\{A(\mathbf{i}\xi + A)^{-1} : |\xi| > c\})\right),\end{aligned}$$

where Kahane's contraction principle (see [9, Lemma 3.5]) was used in the first inequality.

Let us now examine the derivatives of m required in the multiplier conditions. We have $D^j m(\xi) = (-\mathbf{i})^j j! A(\mathbf{i}\xi + A)^{-1-j}$, so

$$\begin{aligned}&\mathcal{R}(\{\xi^j D^j m(\xi) : \xi \neq 0\})\\ &\leq j!\mathcal{R}(\{A(\mathbf{i}\xi + A)^{-1} : \xi \neq 0\})\mathcal{R}(\{\mathbf{i}\xi(\mathbf{i}\xi + A)^{-1} : \xi \neq 0\})^j \leq j!C(1+C)^j,\end{aligned}$$

where $C = \mathcal{R}(\{A(\mathbf{i}\xi + A)^{-1} : \xi \in \mathbb{R}\})$. We applied the product rule of R-bounds [9, Prop. 3.4] in the first step. With $\tilde{m}$ we can compute in a completely analogous fashion, just omitting the A in front.

Thus, if A is invertible and $m(\xi)$ satisfies the necessary condition for maximal L^p-regularity, then $m(\xi)$ satisfies arbitrarily many of the conditions of the type in the assumptions of the Mikhlin–Weis theorem, and consequently the conditions $M(q, \ell)$ for any q and arbitrarily large ℓ. This means the following:

Proposition 3. *For the B-valued Cauchy problem*

$$u'(t) + Au(t) = f(t), \text{ for } t > 0, \qquad u(0) = 0, \tag{9}$$

with $-A$ an invertible generator of a bounded analytic semigroup, the following are equivalent:

- *(9) has maximal L^p-regularity for some $p \in (1, \infty)$,*
- *(9) has maximal L^p-regularity for all $p \in (1, \infty)$ and maximal H^p_{at}-regularity for all $p \in (0, 1]$.*

If B is UMD, then these conditions are further equivalent to

- *$\{A(\mathbf{i}\xi + A)^{-1} : \xi \in \mathbb{R}\}$ is R-bounded.*

Remarks. In Cannarsa and Vespri [6], the maximal regularity of the abstract Cauchy problem is considered on finite intervals $[0, T]$ instead of $\mathbb{R}_+$. They show that if (9) has maximal L^q-regularity for some q, then it has also maximal L^p-regularity for all $p \in (q, \infty)$, and if B is reflexive, for all $p \in (1, \infty)$. Cannarsa and Vespri also allow for the possibility $q = 1$ in the assumptions of their theorem, and they also consider non-zero initial conditions $u(0) = x$. Their proof does not exploit any multiplier theorems, but it is worked out in the concrete setting with the operator A and the semigroup e^{-tA}. Their argument actually goes via the upper extreme, i.e., they first prove maximal regularity between L^∞ and BMO, and then use interpolation.

A characterization of maximal L^p-regularity of the Cauchy problem (9) for B UMD in terms of R-boundedness was obtained in Weis [20]. We also refer to this work for comments on the assumption of the invertibility of A; Weis does not use this assumption, but his definition of maximal regularity is slightly weaker.

We note that the R-boundedness of operator collections such as the one in Proposition 3 is nowadays often expressed in terms of the notion of *R-sectoriality*, due to Clément and Prüss [8]: An operator A is called R-sectorial if $\{A(t+A)^{-1} : t > 0\}$ is R-bounded, and this already implies, by a well-known power series argument, R-boundedness of some of the sets

$$\{A(z+A)^{-1} : |\arg z| \le \pi - \theta\} \tag{10}$$

with $\theta < \pi$. The *R-angle* ϕ_A^R of A is then defined as the infimum of the $\theta \in (0,\pi)$ for which the set (10) is R-bounded. For the R-boundedness condition in Proposition 3, we have $|\arg(\mathbf{i}\xi)| = \frac{1}{2}\pi$, and thus an R-angle $\phi_A^R < \frac{1}{2}\pi$ would imply the condition.

6.2 The Laplace equation $-\triangle u + Au = f$

On the Fourier transform side this is $(|\xi|^2 + A)\hat{u}(\xi) = \hat{f}(\xi)$, so $\hat{u}(\xi) = (|\xi|^2 + A)^{-1}\hat{f}(\xi)$ provided that $[0,\infty)$ belongs to the resolvent of A, and $\hat{Au}(\xi) = A(|\xi|^2 + A)^{-1}\hat{f}(\xi)$. Thus proving maximal regularity amounts to showing that the multipliers

$$\tilde{m}(\xi) := (|\xi|^2 + A)^{-1} \qquad \text{and} \qquad m(\xi) := A(|\xi|^2 + A)^{-1}$$

give rise to a bounded operator on the appropriate space. We will assume that $\{m(\xi) : \xi \in \mathbb{R}^n\} = \{A(t+A)^{-1} : t \ge 0\} \subset \mathcal{B}(B)$ is R-bounded, since otherwise m cannot be a multiplier on any of the $L^p(\mathbb{R}^n; B)$, $p \in (1,\infty)$. As in § 6.1, the R-boundedness of $\{\tilde{m}(\xi) : \xi \ge 0\}$ follows from this combined with the invertibility of A.

One can easily verify by induction that the αth partial derivative of a smooth radial function $g(x) = f(\frac{1}{2}|x|^2)$ is given by

$$D^\alpha g(x) = \sum_{j \le |\alpha|_1} f^{(j)}(|x|^2/2) \sum_{\substack{L \subset \{1,\dots,|\alpha|_1\} \\ \#L = 2j - |\alpha|_1}} c(\alpha, L) \prod_{\ell \in L} x_{\alpha(\ell)},$$

where $\alpha(\ell) := 1$ for $0 < \ell \le \alpha_1$, $\alpha(\ell) := 2$ for $\alpha_1 < \ell \le \alpha_1 + \alpha_2$ etc., and $c(\alpha, L) \in \mathbb{N}$ only depend on α and L.

Since the derivatives of the resolvent $(t+A)^{-1}$ have the same form as if A were just a number and the resolvent thus just an ordinary rational function, we can also apply the above formula to $A(|\xi|^2 + A)^{-1} = \frac{1}{2}A(t + \frac{1}{2}A)^{-1}$, $t = \frac{1}{2}|\xi|^2$, to get

$$D^\alpha A(|\xi|^2 + A)^{-1} = \sum_{j \le |\alpha|_1} (-2)^j j! A(|\xi|^2 + A)^{-1-j} \sum_{\#L = 2j - |\alpha|_1} c(\alpha, L) \prod_{\ell \in L} \xi_{\alpha(\ell)}.$$

To show that the set $\{|\xi|^{|\alpha|_1} D^\alpha m(\xi) : \xi \in \mathbb{R}^n\}$ is R-bounded, we need to consider the finite number of R-bounds

$$\begin{aligned}\mathcal{R}\left(\{|\xi|^{|\alpha|_1} A(|\xi|^2 + A)^{-1-j} \prod_{\ell\in L} \xi_{\alpha(\ell)} : \xi \in \mathbb{R}^n\}\right) \\ \leq \mathcal{R}\left(\{|\xi|^{|\alpha|_1} A(|\xi|^2 + A)^{-1-j} |\xi|^{2j-|\alpha|_1} : \xi \in \mathbb{R}^n\}\right) \\ \leq \mathcal{R}\left(\{A(|\xi|^2 + A)^{-1} : \xi \in \mathbb{R}^n\}\right) \mathcal{R}\left(\{|\xi|^2 (|\xi|^2 + A)^{-1} : \xi \in \mathbb{R}^n\}\right)^j \\ \leq C(1+C)^j,\end{aligned}$$

where $C = \mathcal{R}(\{A(|\xi|^2 + A)^{-1} : \xi \in \mathbb{R}^n\})$. The first inequality employed Kahane's contraction principle and the second one the product rule of R-bounds (see e.g. [9, 3.4–3.5]). For $\tilde{m}(\xi)$, the computations are just the same, omitting the A in front.

With the same reasoning as with the Cauchy problem, the above gives the following:

Proposition 4. *For the B-valued Laplace equation*

$$-\triangle u(x) + Au(x) = f(x), \tag{11}$$

with $[0, \infty)$ in the resolvent of A, the following are equivalent:

- *(11) has maximal L^p-regularity for some $p \in (1, \infty)$,*
- *(11) has maximal L^p-regularity for all $p \in (1, \infty)$ and maximal H^p_{at}-regularity for $p \in (0, 1]$.*

If B is UMD, then these conditions are further equivalent to

- *$\{A(t + A)^{-1} : t \geq 0\}$ is R-bounded.*

Note that now the R-boundedness condition is exactly the requirement of R-sectoriality of A.

6.3 The fractional-order equation $D^\alpha u + Au = f$ where $\alpha \in (0, 2)$

Before proceeding, we emphasize that α in this subsection denotes a positive real number and not a multi-index like elsewhere. The fractional derivative D^α is defined (cf. [21, §12.8]), for $0 < \alpha < 1$ and $t > 0$ by

$$(D^\alpha u)(t) := D(g_{1-\alpha} * u\mathbf{1}_{\mathbb{R}_+})(t), \qquad g_\beta(t) := \frac{t^{\beta-1}}{\Gamma(\beta)}\mathbf{1}_{\mathbb{R}_+}(t),$$

and $D^{1+\alpha} := DD^\alpha$. We assume the initial condition $u(0) = 0$, and if $\alpha > 1$ also the additional condition $u'(0) = 0$.

In Clément and Prüss [8, p. 85], it is shown that this problem admits a unique solution $u \in L^p(\mathbb{R}_+; \mathcal{D}(A))$ for every $f \in L^q(\mathbb{R}_+; B)$, $q \in (1, \infty)$, given that A is invertible and R-sectorial with R-angle $\phi_A^R < \pi(1-\frac{1}{2}\alpha)$. (The requirement of this condition gives the natural restriction for the range of the order of the derivative $\alpha \in (0,2)$.) Moreover, this solution satisfies

$$\|D^\alpha u\|_{L^q(\mathbb{R}_+;B)} + \|u\|_{L^q(\mathbb{R}_+;B)} + \|Au\|_{L^q(\mathbb{R}_+;B)} \leq C\,\|f\|_{L^q(\mathbb{R}_+;B)}. \tag{12}$$

Let us make these same assumptions and see what happens in $H^p_{\mathrm{at}}(\mathbb{R}_+;B)$.

Assume for the moment that $\alpha \in (0,1)$, and consider the modified kernel $g^\mu_\beta(t) := g_\beta(t)e^{-\mu t}$, $\mu > 0$. Then $g^\mu_\beta \in L^1(\mathbb{R}_+)$, and the convolution $g^\mu_\beta * u \in L^q(\mathbb{R}_+;B)$ given that $u \in L^q(\mathbb{R}_+;B)$. Let us now fix a $q > 1$ such that B has Fourier-type q. Then the Fourier transforms of u and $g^\mu_{1-\alpha} * u$ (where u is extended to the whole line by $u(t) := 0$ for $t < 0$) are well-defined $L^{q'}(\mathbb{R};B)$-functions such that

$$\mathcal{F}[g^\mu_{1-\alpha} * u](\xi) = \hat{g^\mu_{1-\alpha}}(\xi)\hat{u}(\xi). \tag{13}$$

The Fourier transform of g^μ_β is

$$\int_0^\infty \frac{t^{\beta-1}}{\Gamma(\beta)} e^{-(\mu+\mathbf{i}\xi)t}\,\mathrm{d}t = \frac{1}{(\mu+\mathbf{i}\xi)^\beta}; \tag{14}$$

for $\xi = 0$, this is the definition of the Γ-function and a simple change of variable, and for general $\mu + \mathbf{i}\xi \in \mathbb{C}_+ := \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$, the result can be deduced from this special case with the help of Cauchy's theorem, by investigating the integrand of the analytic function $z \mapsto z^{\beta-1}\exp(-z)$ over the path

$$[\epsilon,\rho] \cup \rho e^{\mathbf{i}[0,\vartheta]} \cup [\epsilon,\rho]e^{\mathbf{i}\vartheta} \cup \epsilon e^{\mathbf{i}[0,\vartheta]},$$

where $\vartheta = \arg(\mu+\mathbf{i}\xi) \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, and we consider the limit $\epsilon \to 0$, $\rho \to \infty$.

Let us then observe that

$$\begin{aligned}\left|g_\beta * u(t) - g^\mu_\beta * u(t)\right|_B &\leq \int_0^t g_\beta(s)\,|u(t-s)|_B\,(1-e^{-\mu s})\,\mathrm{d}s \\ &\leq g_\beta * |u|_B\,(t)\cdot(1-e^{-\mu t}),\end{aligned}$$

and thus

$$\begin{aligned}\left\|g_\beta * u - g^\mu_\beta * u\right\|_{L^p(0,T;B)} &\leq \|g_\beta\|_{L^1(0,T)}\,\|u\|_{L^p(\mathbb{R};B)}\,(1-e^{-\mu T}) \\ &\leq \frac{T^\beta}{\Gamma(\beta+1)}\,\|u\|_{L^p(\mathbb{R};B)}\,\mu T.\end{aligned}$$

Since this bound is only slowly increasing as a function of T, it is quite easy to see that $g^\mu_\beta * u \to g_\beta * u$ (as $\mu \to 0$) in the sense of the B-valued tempered distributions $\mathcal{S}'(\mathbb{R};B)$. Then also $D(g^\mu_\beta * u) \to D(g_\beta * u)$ in the same sense, and moreover the distributional Fourier transforms also converge, since $\mathcal{F}$ is continuous on $\mathcal{S}'(\mathbb{R};B)$. But we know from (13) and (14) above, together with the formula for the transform of a derivative, that

$$\mathcal{F}[D(g^\mu_{1-\alpha} * u)](\xi) = \frac{\mathbf{i}\xi}{(\mu+\mathbf{i}\xi)^{1-\alpha}}\hat{u}(\xi) = \frac{\mathbf{i}\xi}{\mu+\mathbf{i}\xi}(\mu+\mathbf{i}\xi)^\alpha\hat{u}(\xi) \underset{\mu\downarrow 0}{\longrightarrow} (\mathbf{i}\xi)^\alpha\hat{u}(\xi),$$

and it is easy to see that the pointwise convergence above also gives convergence in $\mathcal{S}'(\mathbb{R};B)$. Thus we conclude that

$$\mathcal{F}[D^\alpha u](\xi) = (\mathbf{i}\xi)^\alpha \hat{u}(\xi).$$

For $\alpha > 1$, we just invoke the formula for the Fourier transform of the distributional derivative (of usual integer order) again to give the above formula also for $\alpha > 1$.

Thus, given $f \in L^q(\mathbb{R}_+;B)$, we know from Clément and Prüss [8] that our fractional-order equation has a solution $u \in L^q(\mathbb{R}_+;\mathcal{D}(A))$, and by the considerations above, the Fourier transform of this solution satisfies

$$(\mathbf{i}\xi)^\alpha \hat{u}(\xi) + A\hat{u}(\xi) = \hat{f}(\xi).$$

and so the multipliers corresponding to the present problem are

$$\tilde{m}(\xi) = ((\mathbf{i}\xi)^\alpha + A)^{-1} \qquad \text{and} \qquad m(\xi) = A((\mathbf{i}\xi)^\alpha + A)^{-1}.$$

Now the assumed R-sectoriality of A implies in particular that $\{A((\mathbf{i}\xi)^\alpha + A)^{-1} : \xi \in \mathbb{R}\}$ is R-bounded, which is a necessary condition for m to be a multiplier on any L^p, $p \in (1,\infty)$. From the maximal regularity result in [8] quoted above it follows that m is indeed a multiplier on these spaces. Moreover, from (12) we see that $\|u\|_{L^p(\mathbb{R}_+;B)} \leq C\|f\|_{L^p(\mathbb{R}_+;B)}$, so $\tilde{m}(\xi) = ((\mathbf{i}\xi)^\alpha + A)^{-1}$ is also a multiplier.

We then investigate the multiplier conditions satisfied by m and $\tilde{m}$. For iterated derivatives of a composition of functions, one can show by induction that

$$(f \circ g)^{(j)} = \sum_{i=1}^{j} (f^{(i)} \circ g) \sum_{\substack{\sum i_h = i \\ \sum h i_h = j}} c_{(i_h)_{h=1}^j} \prod_{h=1}^{j} (g^{(h)})^{i_h},$$

where the $c_{(i_h)_{h=1}^j}$ are numerical constants depending only on the parameters indicated. We apply this to $f(t) = A(t + A)^{-1}$, $g(t) = (\mathbf{i}t)^\alpha$, bearing in mind what was said above about the derivatives of the resolvent in § 6.2.

We obtain

$$\begin{aligned} &D^j m(\xi) \\ &= \sum_{i=1}^{j} (-1)^i i! A((\mathbf{i}\xi)^\alpha + A)^{-1-j} \sum_{(i_h)_{h=1}^j} c_{(i_h)_{h=1}^j} \prod_{h=1}^{j} \left(\mathbf{i}^a a \cdots (a-h+1)\xi^{a-h}\right)^{i_h} \\ &\qquad\qquad = \sum_{i=1}^{j} \sum_{(i_h)_{h=1}^j} c(i,j,(i_h)_{h=1}^j, a) A((\mathbf{i}\xi)^\alpha + A)^{-1} \mathbf{i}^{ai} \xi^{ai-j}, \end{aligned}$$

where the conditions $\sum_{h=1}^j i_h = i$ and $\sum_{h=1}^j h i_h = j$ were used in the last step.

The R-bound of $\{\xi^j D^j m(\xi) : \xi \neq 0\}$ is now bounded by a finite number of R-bounds of the form

$$\begin{aligned}\mathcal{R}&\left(\{\xi^j A((\mathbf{i}\xi)^\alpha + A)^{-1-j}\mathbf{i}^{ai}\xi^{ai-j} : \xi \neq 0\}\right)\\ &= \mathcal{R}\left(\{A((\mathbf{i}\xi)^\alpha + A)^{-1}[(\mathbf{i}\xi)^\alpha((\mathbf{i}\xi)^\alpha + A)^{-1}]^i : \xi \neq 0\}\right) \leq C(1+C)^i,\end{aligned}$$

where $C = \mathcal{R}(\{A((\mathbf{i}\xi)^\alpha + A)^{-1} : \xi \in \mathbb{R}\})$.

For $\tilde{m}$ we obtain almost the same results, the only exception being the lack of the operator A in front of the expressions, and we then use the R-boundedness of the collection $\{((\mathbf{i}\xi)^\alpha + A)^{-1} : \xi \in \mathbb{R}\}$.

Thus m and $\tilde{m}$ satisfy infinitely many of the conditions considered in our multiplier theorems, and so they also give rise to bounded operators on all of the spaces $H^p_{\mathrm{at}}(\mathbb{R}; B)$. Thus we have the result:

Proposition 5. *Let $\alpha \in (0, 2)$, let B be a UMD-space and A an invertible and R-sectorial operator $A : \mathcal{D}(A) \subset B \to B$ with R-angle $\phi^R_A < \pi(1 - \frac{1}{2}\alpha)$. Then for every f in the dense subspace $(L^q \cap H^p_{\mathrm{at}})(\mathbb{R}_+; B)$ of $H^p_{\mathrm{at}}(\mathbb{R}_+; B)$, where $1 < q \leq$ the Fourier-type of B and $p \in (0, 1]$, the fractional-order equation*

$$D^\alpha u(t) + Au(t) = f(t), \text{ for } t > 0, \quad \alpha \in (0,2), \quad \begin{cases} u(0) = 0, \\ u'(0) = 0 \quad \text{if } \alpha > 1, \end{cases} \tag{15}$$

has a unique solution $u \in (L^q \cap H^p_{\mathrm{at}})(\mathbb{R}_+; \mathcal{D}(A))$ which satisfies (12) and

$$\|D^\alpha u\|_{H^p_{\mathrm{at}}(\mathbb{R}_+;B)} + \|u\|_{H^p_{\mathrm{at}}(\mathbb{R}_+;B)} + \|Au\|_{H^p_{\mathrm{at}}(\mathbb{R}_+;B)} \leq C \|f\|_{H^p_{\mathrm{at}}(\mathbb{R}_+;B)}.$$

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