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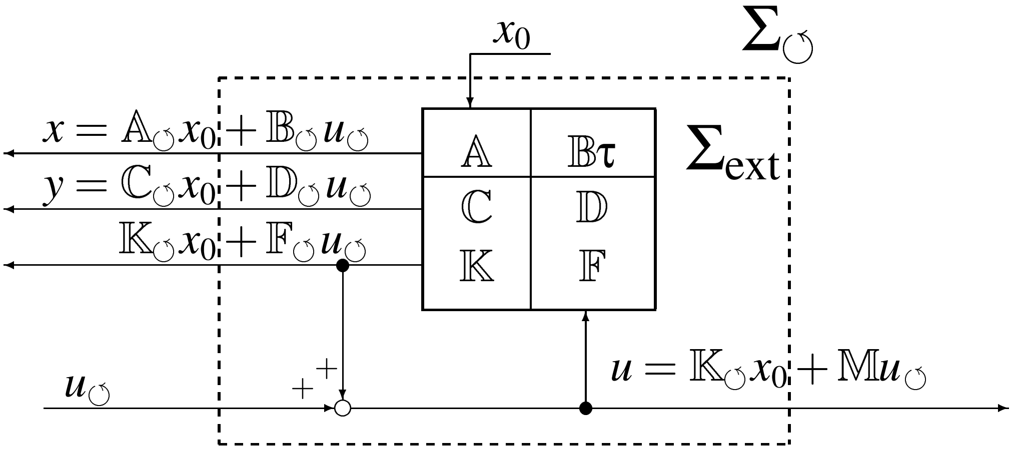
Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations

Volume 1/3 — Time-Invariant Operators & WPLSs

Kalle Mikkola



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Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations

Volume 1/3 — Time-Invariant Operators & WPLSs

Kalle Mikkola

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Kalle Mikkola: *Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations*; Helsinki University of Technology Institute of Mathematics Research Reports A452 (2002).

Abstract: *In this monograph, we solve rather general linear, infinite-dimensional, time-invariant control problems, including the* H^∞ *and LQR problems, in terms of algebraic Riccati equations and of spectral or coprime factorizations. We work in the class of (weakly regular) well-posed linear systems (WPLSs) in the sense of G. Weiss and D. Salamon.*
Moreover, we develop the required theories, also of independent interest, on WPLSs, time-invariant operators, transfer and boundary functions, factorizations and Riccati equations. Finally, we present the corresponding theories and results also for discrete-time systems.

AMS subject classifications: 42A45, 46E40, 46G12, 47A68, 49J27, 49N10, 49N35, 93-02, 93A10, 93B36, 93B52, 93C05, 93C55, 93D15

Keywords: suboptimal H-infinity control, standard H-infinity problem, measurement feedback problem; H-infinity full information control problem, state feedback problem; Nehari problem; LQC, LQR control, H2 problem, minimization; bounded real lemma, positive real lemma; dynamic stabilization, controller with internal loop, strong stabilization, exponential stabilization, optimizability; canonical factorization, (J,S)-spectral factorization, (J,S)-inner coprime factorization, generalized factorization, J-lossless factorization; compatible operators, weakly regular well-posed linear systems, distributed parameter systems; continuous-time, discrete-time; infinite-horizon; time-invariant operators, Toeplitz operators, Wiener class, equally-spaced delays, Popov function; transfer functions, H-infinity boundary functions, Fourier multipliers; Bochner integral, strongly measurable functions, strong Lp spaces; Laplace transform, Fourier transform

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Preface

For H^∞, H^2, LQR and several other linear time-invariant control problems, it is well known that the existences of

(I) a solution of that control problem,

(II) a corresponding coprime or spectral factorization and

(III) a stabilizing solution of the corresponding Riccati equation

are, roughly speaking, all equivalent in the finite-dimensional setting, and from one of them the others can be computed. Therefore, control problems are often solved by computing the solutions of the corresponding Riccati equations.

These results have been extended to infinite-dimensional (semigroup) control systems with bounded input and output operators, and in the eighties and early nineties also to the larger class of Pritchard–Salamon systems and to certain other special cases of our setting.

The main purpose of this monograph is to generalize these results to infinite-dimensional (weakly regular) well-posed linear systems (WPLSs) in the sense of G. Weiss. This is done in Chapters 8–12; see pp. 21–24 for an introduction to WPLSs.

We also develop corresponding discrete-time results (Chapters 13–15 and Sections 11.5 and 12.2), WPLS theory (Chapters 6–7, including regularity, state feedback, output injection and dynamic feedback), and an extensive theory of independent interest for time-invariant operators ("Toeplitz operators") and some of their subclasses (such as extended Callier–Desoer classes and other convolutions with measures), transfer and boundary functions and spectral and coprime factorization (Chapters 2–5 and Sections 6.4–6.5). A more detailed description of some of the main results of this monograph and some historical remarks are provided in Sections 1.1 and 1.2 and in the "notes" parts of each section.

WPLSs cover all linear time-invariant systems that map the initial state and input continuously to the state and output (with inputs and outputs in L^2_{loc}; see pp. 21); in particular all settings mentioned above are covered. Moreover, any transfer function that is bounded and holomorphic on some right half-plane (i.e., that is *well-posed* or *proper*) has a WPLS realization. The input and output operators of a WPLS may be as unbounded as for Pritchard–Salamon systems *independently* of each other as long as the transfer function is well posed, thus allowing roughly twice as much irregularity.

Weak regularity means the existence of a feedthrough operator in a very weak sense; an equivalent condition is that the transfer function has a limit at infinity along the positive real axis. In particular, all I/O maps whose impulse response is a (uniform, strong or weak) L^p function plus some delays (or any vector-valued measure) and several others are weakly regular.

Much of our theory on WPLSs cover the general case, but Riccati equations cannot be defined without feedthrough operators (except in a very weak sense, as in Section 9.7).

We generally allow the input, state and output spaces of WPLSs to be Hilbert spaces of arbitrary dimensions, and some results are given even in a Banach space setting. In addition to exponentially stabilizing controllers and exponentially stabilizing solutions of Riccati equations, we also study stabilizing and strongly stabilizing ones; part of these results may be new even for finite-dimensional systems.

During the last four decades, the literature has become abundant in infinite-dimensional (or *distributed*) systems arising in physics, engineering, economics, mechanics, environmental modeling, biomedical engineering, evolution dynamics, geophysics and other sciences, and the systems can represent semiconductor devices, animal populations, fluid dynamics, microwave circuits, vibration of strings or membranes, heat diffusion, computer hard discs, CD players and many other devices.

For some particular systems and problems, there are now rather mature theories. The purpose of this monograph is to solve the problems in a very general, unifying framework — in the framework of WPLSs.

Our presentation is abstract and theory-oriented; nevertheless, many of our results can be understood without the functional analytic knowledge provided by the appendices. The book is rather self-contained and it can be read without any prior knowledge in system or control theory, although experts are considered as the main audience and some proofs may be demanding.

Acknowledgements: I am very grateful to professor Olof Staffans, who warmly guided me to the world of mathematical control theory, and whose knowledge has been very valuable to me. I also wish to thank professor Stig-Olof Londen for helping me not get lost in Fréchet spaces during my undergraduate studies, and my supervisor, professor Olavi Nevanlinna, and the people at the Institute of Mathematics at Helsinki University of Technology for its warm and stimulating environment.

Professor Ilya Spitkovsky has kindly pointed me to several important results on spectral factorization and Irena Lasiecka and Roberto Triggiani to many results obtained by their school. I have also had inspiring discussions with professors Ruth Curtain, George Weiss and Hans Zwart, doctors Marko Huhtanen and Jarmo Malinen, and others. Part of this work was written with the support of the Academy of Sciences, the Graduate School of Analysis and the Finnish Cultural Foundation. Finally, I wish to thank my friends and loved ones, for making this world such a great place to live in.

Kalle Mikkola

Twice or thrice had I loved thee
Before I knew thy face or name.
So in a voice, so in a shapeless flame,
Angels affect us oft, and worshipped be.
— John Donne (1571–1631)

Chapter 1

Introduction

From the wreck of the past, which hath perish'd,
Thus much I at least may recall,
It hath taught me that what I most cherish'd
Deserved to be dearest of all.

— Lord Byron (1788–1824), "Stanzas to Augusta"

In Section 1.1, we summarize the main contributions of this monograph, avoiding any technicalities. Readers wishing to get a somewhat more accurate picture on the actual results should consult Section 1.2, where we give a glance at each chapter by explaining its contents but yet avoiding most technical details and generality.

Some conventions on notation, proofs and hypotheses are explained in Section 1.3. See the end of the book for symbols, concepts, abbreviations, references and index.

1.1 On the contributions of this book

Our ultimate goal has been to develop the H^∞ Four-Block Problem theory in Chapter 12. This has required us to first develop several other parts of the theory that are of independent interest, such as the Riccati equation theory, the cost minimization theory, the dynamic feedback theory, the WPLS theory or the discrete-time theory, all of which are mainly generalizations of existing theory for finite-dimensional or smooth infinite-dimensional systems.

Our main results include the following:

1. On (generalized) Optimal Control and Riccati equations for WPLSs, we have
   (a) established the relations between different classical coercivity assumptions (Section 10.3), generalized them to WPLSs and applied them to solve the general control problem (Section 8.4).
   (b) formulated Integral Algebraic Riccati Equations to establish the corresponding equivalence in continuous time. This also allowed us to reduce several problems to discrete time, where input and output operators are bounded.
   (c) established the corresponding equivalence for (classical-type) Continuous-time Algebraic Riccati Equations (under weak regularity) (Chapter 9).
      i. The implication from the existence of a solution of the control problem to the existence of a solution of the Riccati Equation was already established by G. Weiss, M. Weiss and O. Staffans under stronger regularity and very strong stabilizability and detectability assumptions.
      ii. We have also shown the existence of a smoother solution under several different additional regularity assumptions (e.g., Section 9.2).
   (d) established the Continuous-time Riccati equations on the domain of the closed-loop semigroup generator for general (possibly irregular) WPLSs (Section 9.7; extension of [FLT]).
   (e) treated all the above for both the exponentially stabilizing controls and for the recently-popular strongly or output-stabilizing controls and others, thus providing new results even for finite-dimensional systems.

2. On specific control problems for WPLSs, we have extended the finite-dimensional results by, in addition to the above, solving
   (a) the H^∞ full-information control problem in terms of the Riccati equation (Chapter 11).

i. In the stable case, the existence of a solution was already shown by O. Staffans, assuming the existence of corresponding spectral factorization; a similar statement applies to the LQR problem below.

(b) the general (measurement feedback, or four-block) H^∞ control problem in terms of two Riccati equations and a spectral radius condition (Chapter 12). We have shown that the recent theory of controllers with an internal loop (cf. [CWW96]) is shown to be intimately connected to a general solution of this problem, and all such solutions are also covered.

(c) the cost minimization (LQR) problem, showing the existence of a solution equivalent to the existence of any solution of the corresponding Riccati equation (the solution need not be stabilizing or even admissible a priori). We have also derived similar generalizations of Strict Bounded and Strictly Positive (Real) Lemmas (Chapter 10).

3. On WPLS system theory, we have

(a) introduced compatibility, which allows one to write any WPLS in a differential form regardless of regularity (Section 6.3).

(b) introduced an infinite-dimensional weakly coprime factorization concept (Sections 6.4 and 6.5) and applied it to establish the stability and uniqueness of a solution of certain Riccati equations and control problems (this is particularly useful when the solution is not required to be exponentially stabilizing). This concept and compatibility have already become the subjects of leading researchers' articles.

(c) characterized the transfer functions (equivalently, impulse responses) having a Pritchard–Salamon realization (thus correcting the errors in [KMR] to which we also provide a counter-example). Similarly, we have characterized transfer functions realizable with bounded input or output operators. (Section 6.9)

(d) generalized the equivalence between exponential dynamic stabilizability and exponential stabilizability and detectability (Theorem 7.2.4).

4. The infinite-dimensional control theory has been limited by several open problems in harmonic and functional analysis and function theory. This has lead us to solve those most intimately connected to our work, e.g., we have

(a) generalized the L^2 Fourier multiplier theorem to the case of functions with values in Hilbert spaces (the separable case was already known) and beyond (Theorem 1.2.2).

(b) generalized similarly the existence result of the boundary function of a H^∞ function (Theorem 1.2.3).

(c) developed a theory of strongly measurable operator valued functions, including the the completeness of L^∞_{strong} (and incompleteness of L^p_{strong}) and its applications including the two above results (Appendix F).

(d) shown the existence of a spectral factorization for convolutions with (Hilbert space) operator-valued measures having a discrete part plus an L^1 part (assuming the invertibility of the Toeplitz operator; see Theorems 1.2.4 and 1.2.5).

(e) extended to the infinite-dimensional case the classical [ClaGoh] H^2 spectral factorization for any Popov function having an invertible Toeplitz operator (Theorem 9.14.6).

Finally, of all the above we also present corresponding infinite-dimensional discrete-time results, which become rather elegant since, in this case, the input and output operators are naturally bounded.

For a control theorist, the generalization of Riccati equation theory to the regular WPLS setting (particularly 1b and 1c above) and the general H^∞ and minimization problems (2.) may rise above the rest.

To observe in detail the other new results in this monograph, the reader should read the "Notes" at the end of each section. There we discuss earlier research in same direction, including any known similar results under less general systems, settings or assumptions.

The size of this book requires some explanations. For the first, the chapters of this monograph are so intimately connected to each other that it would have been impossible to remove a single chapter without destroying, e.g., the proofs in Chapter 12.

If we had limited ourselves only to very smooth systems or to discrete-time systems, the size of this book would have probably fallen by more than half but its contribution even by much more. Indeed, most problems but also most value in our work is in its generality. Certainly, we might have presented our solutions only in terms of factorizations (which are given as an intermediary stage in our proofs), but the Riccati equations are really the form of the classical solutions and something that provides a practical way to solve the problems.

Sometimes the Riccati equations become very complicated for general regular WPLSs, hence we have presented more beautiful corollaries for important special cases, such as for the case where the I/O map is the convolution with a measure. Moreover, the realization of the optimal control in the form of a state feedback or dynamic feedback controller requires the existence of certain factorizations that need not exist in the general case (see Example 11.3.7).

One of the objectives of this book has been to state and prove results of a technical nature that are too long to be published in ordinary research articles but that are necessary building blocks for the final results.

1.2 A summary of this book

We now start a rather self-contained summary, aiming to give the reader a motivation for and a picture of the theory treated in each chapter, by starting with a non-technical description and then presenting some results. We strongly recommend for the reader to read the summaries in this section before diving into the technicalities of the actual chapters.

The results mentioned below are just examples from the theory; here we have usually favored simple, important examples to more general but more complex ones. See the chapters themselves for further definitions, results, details, explanations and references.

Outside the appendices, the letters H, U, W, Y and Z will denote complex Hilbert spaces of arbitrary dimensions unless something else is indicated.

Part I: TI Operator Theory

The appendices and Part I of the book contain results in harmonic and functional analysis (vector-valued functions, shift-invariant operators, transfer functions and boundary functions, the Corona Theorem and spectral factorization among others) that are needed in the control theory of Parts II–IV. Many of the results are also of independent interest. A fast track to WPLSs is to first have a glance at subsections 2.1.1–2.1.7 and then go directly to Part II.

Chapter 2: TI and MTI Operators (MTI $\subset$ TI)

In Chapter 2, we study the theory TI_ω, the space of bounded, shift-invariant operators $\mathrm{L}^2 \to \mathrm{L}^2$, where the L^2 space may have a weight and the functions have their values in a Hilbert space. We also present certain smooth subclasses of TI, particularly MTI, the convolutions with a (vector-valued) measure with no singular continuous part.

Our contributions include the theory of the intersection $\mathrm{TI}_\omega \cap \mathrm{TI}_{\omega'}$ and its causal part for two weights $\omega, \omega' \in \mathbf{R}$ (see 2.1.9–2.1.11 and 3.1.6), necessary and sufficient conditions for losslessness and certain results on static operators and signature operators.

Technically, $\mathrm{TI}_\omega(U,Y)$ is the space of bounded time-invariant linear operators $\mathrm{L}^2_\omega(\mathbf{R};U) \to \mathrm{L}^2_\omega(\mathbf{R};Y)$, where U and Y are Hilbert spaces of arbitrary dimensions, $\omega \in \mathbf{R}$, and

$$\|u\|_{\mathrm{L}^2_\omega} := \Big(\int_{\mathbf{R}} \mathrm{e}^{-2\omega t}\|u(t)\|_U^2\,dt\Big)^{1/2} \tag{1.1}$$

for Bochner-measurable $u : \mathbf{R} \to U$ (thus, $\mathrm{L}^2_0 = \mathrm{L}^2$, $\mathrm{L}^2_\omega := \{\mathrm{e}^{\omega\cdot}u(\cdot) \,|\, u \in \mathrm{L}^2\}$). The *time-invariance* of $\mathbb{D} \in \mathrm{TI}_\omega$ means that $\mathbb{D}\tau(t) = \tau(t)\mathbb{D}$ for all $t \in \mathbf{R}$, where $\tau(t)u := u(\cdot+t)$.

The maps in $\mathrm{TIC}_\omega(U,Y) := \{\mathbb{D} \in \mathrm{TI}_\omega(U,Y) \,|\, \pi_-\mathbb{D}\pi_+ = 0\}$ are called *causal* (or sometimes Toeplitz operators); here $\pi_+u := \chi_{\mathbf{R}_+}u$ and $\pi_-u := \chi_{\mathbf{R}_-}u$ for all functions u, and χ_E is the characteristic function of a set E. The following is well known:

Theorem 1.2.1 *For each* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, *there is a unique function* $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$, *called the* transfer function *(or symbol) of* $\mathbb{D}$, *s.t.* $\widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\hat{u}$ *on* $\mathbf{C}^+_\omega$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$. *The mapping* $\mathbb{D} \mapsto \widehat{\mathbb{D}}$ *is an isometric isomorphism onto.* □

Here $\mathcal{B}(U,Y)$ denotes the space of bounded linear operators $U \to Y$, $\mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ denotes the Banach space of bounded holomorphic functions $\mathbf{C}^+_\omega \to \mathcal{B}(U,Y)$, and $\widehat{u}$ denotes the Laplace transform

$$\widehat{u}(s) := \int_{\mathbf{R}} \mathrm{e}^{-st}u(t)\,dt \qquad (s \in \mathbf{C}^+_\omega := \{s \in \mathbf{C} \mid \operatorname{Re} s > \omega\}) \tag{1.2}$$

of u. Thus, the elements of $\mathrm{TIC}_\infty(U,Y) := \cup_{\omega\in\mathbf{R}}\mathrm{TIC}_\omega(U,Y)$ correspond one-to-one to the holomorphic $\mathcal{B}(U,Y)$-valued functions that are bounded on some right half-plane; such functions are generally called "proper" or "well-posed". The set of the I/O maps of WPLSs is exactly TIC_∞ (see Section 6.1). Transfer functions are studied also in Chapter 3.

In Section 2.2, we study the invertibility of TIC_ω (and TI_ω) operators. In Section 2.3, we develop sufficient conditions for a TIC operator to be static, that is, the multiplication operator induced by an element of $\mathcal{B}(U,Y)$. We also give certain results that will be used in connection with the *signature operators* of optimization problems, Riccati equations and spectral factorizations.

Also Section 2.4 treats signature operators. A main result of this section is that for any $S \in \mathcal{B}(U \times Y)$, the following are equivalent:

(i) $S = \mathbb{E}^* \begin{bmatrix} I_U & 0 \\ 0 & -I_Y \end{bmatrix} \mathbb{E}$ for some $\mathbb{E} \in \mathcal{G}\mathrm{TIC}(U \times Y)$;

(ii) $S = E^* \begin{bmatrix} I_U & 0 \\ 0 & -I_Y \end{bmatrix} E$ for some $E \in \mathcal{G}\mathcal{B}(U \times Y)$.

(Recall that $\mathcal{G}$ denotes the subset of invertible operators.)

Section 2.5 treats the concept "(J,S)-losslessness" (close to "(J,S)-dissipativity"), which is often studied in connection with H^∞ problems and indefinite inner products (losslessness is roughly equivalent to the nonnegativity of the corresponding Riccati operator). There are two widely-used definitions of losslessness whose exact connection has been unknown. We develop necessary and/or sufficient conditions for both concepts and show that they coincide when the input spaces are finite-dimensional.

In Section 2.6, we define the subclass $\mathrm{MTI}(U,Y)$ ("M" for "measures") as the operators $\mathbb{D} \in \mathrm{TI}(U,Y)$ that are of the form

$$(\mathbb{E}u)(t) = \sum_{k=0}^{\infty} T_k u(t-t_k) + \int_{-\infty}^{\infty} f(t-r)u(r)\,dr, \tag{1.3}$$

i.e., of the form $\mathbb{E}u = \mu * u$, where the measure μ consists of a function $f \in \mathrm{L}^1(\mathbf{R};\mathcal{B}(U,Y))$ plus a discrete part with $T_k \in \mathcal{B}(U,Y)$ and $t_k \in \mathbf{R}$ for all $k \in \mathbf{N}$, s.t.

$$\|\mathbb{E}\|_{\mathrm{MTI}} := \|f\|_{\mathrm{L}^1} + \sum_{k\in\mathbf{N}} \|T_k\|_{\mathcal{B}(U,Y)} < \infty. \tag{1.4}$$

The *Wiener class* $\mathrm{MTI}^{\mathrm{L}^1}$ refers to the elements of MTI of form $u \mapsto Tu_0 + f * u$ (i.e., no delays). The class MTIC := MTI $\cap$ TIC (resp. $\mathrm{MTIC}^{\mathrm{L}^1} := \mathrm{MTI}^{\mathrm{L}^1} \cap \mathrm{TIC}$) consists of those elements of MTI (resp. $\mathrm{MTIC}^{\mathrm{L}^1}$) that correspond to measures supported on $\mathbf{R}_+$. In [CD80] and [CZ] among others, the class MTIC (or "$\mathcal{A}(0)$") has been studied for finite-dimensional U and Y.

The basic properties of these classes are listed in Section 2.6. They share most properties of maps with rational transfer functions; in particular, they have the same spectral factorization properties (see Section 5.2). These properties allow us to show (in Part III) that classical conditions for the solvability of standard control problems are necessary and sufficient also for systems whose I/O maps belong to MTIC (such conditions are sufficient but not necessary for general WPLSs); some of this has already been established for less general systems (see, e.g., [CD80] or [CW99]).

Chapter 3: Transfer Functions ($\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$, $\widehat{\mathrm{TIC}} = \mathrm{H}^\infty$)

We study the *Laplace and Fourier transforms* (or *transfer functions* or *symbols*) of TI and TIC maps, that is, (causal and general) time-invariant maps $\mathrm{L}^2 \to \mathrm{L}^2$.

Our main results are two generalizations to unseparable Hilbert spaces, first one of the Fourier multiplier theorem ("$\widehat{\mathrm{TI}}(U,Y) = \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$") and then of the fact that an operator-valued H^∞ function over the right half-plane has a boundary function in strong L^∞ on the imaginary axis as its "strong pointwise limit", in a very natural sense.

We first show that "$\widehat{\mathrm{TI}}(U,Y) = \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$" (Theorem 3.1.3(a1)):

Theorem 1.2.2 *For each $\mathbb{E} \in \mathrm{TI}(U,Y)$, there is a unique (*symbol*) $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ s.t. $\widehat{\mathbb{E}}\widehat{u} = \widehat{\mathbb{E}u}$ a.e. for all $u \in \mathrm{L}^2(\mathbf{R};U)$. This mapping $\mathbb{E} \mapsto \widehat{\mathbb{E}}$ is an isometric isomorphism of* $\mathrm{TI}(U,Y)$ *onto* $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$. □

(The separable case of this claim is well-known. Here $i\mathbf{R}$ is the imaginary axis, and $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ means that $\widehat{\mathbb{E}} : i\mathbf{R} \to \mathcal{B}(U,Y)$ is s.t. $\widehat{\mathbb{E}}u_0 \in \mathrm{L}^\infty(i\mathbf{R};Y)$ for all $u_0 \in U$. It follows that $\|\widehat{\mathbb{E}}\|_{\mathrm{L}^\infty_{\mathrm{strong}}} := \sup_{\|u_0\|_U \le 1} \|\widehat{\mathbb{E}}u_0\|_\infty < \infty$, by Lemma F.1.6.)

Then we go on to show that this *Fourier transform* restricts to an isometric isomorphism of $\mathrm{TI}_a(U,Y) \cap \mathrm{TI}_b(U,Y)$ onto $\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))$, where $\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))$ refers to bounded holomorphic functions $\mathbf{C}_{a,b} \to \mathcal{B}(U,Y)$ and $\mathbf{C}_{a,b} := \{s \in \mathbf{C} \mid a < \mathrm{Re}\, s < b\}$, and that $\widehat{\mathbb{E}}\widehat{u} = \widehat{\mathbb{E}u}$ on $\mathbf{C}_{a,b}$ (both sides of the equation being holomorphic) for all $u \in \mathrm{L}^2_a(\mathbf{R};U) \cap \mathrm{L}^2_b(\mathbf{R};U)$.

In Sections 3.1 and 3.2, we also give further results on the Fourier transform and weaker forms of the two results mentioned above for arbitrary Banach spaces U and Y and L^p in place of L^2 (and "TI^p_ω" in place of TI_ω). These can be considered as extensions of the so called *Fourier multiplier theory*.

In Section 3.3, we establish several results on the boundary functions of holomorphic functions, the most important of which is the following (Theorem 3.3.1(c1)):

Theorem 1.2.3 *For each $f \in \mathrm{H}^\infty(\mathbf{C}_0^+;\mathcal{B}(U,Y))$, there is a* boundary function $f_0 \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ *s.t. f_0u_0 is the nontangential limit of fu_0 a.e. on $i\mathbf{R}$ for all $u_0 \in U$.* $\square$

(The separable case of this theorem was given in [Thomas].) As the observant reader already may have guessed, f_0 is the Fourier transform of $\mathbb{D}$, where $\mathbb{D} \in \mathrm{TI}(U,Y)$ is s.t. $\widehat{\mathbb{D}} = f$. This justifies the use of "$\widehat{\mathbb{D}}$" to denote both the Fourier transform $\widehat{\mathbb{D}} \in \mathrm{L}^\infty_{\mathrm{strong}}(\omega+i\mathbf{R};\mathcal{B}(U,Y))$ and the transfer function (Laplace transform) $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}_\omega^+;\mathcal{B}(U,Y))$ of a map $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$.

Some counter-examples are given to show that Theorem 1.2.3 is not true for general Banach spaces nor with H^2 in place of H^∞.

We also give further results on transfer functions; these results will be needed for the WPLS theory of Parts II and III.

Chapter 4: Corona Theorems and Inverses

In this chapter, we first show that any causal inverses of I/O maps preserve smoothness and then we do the same for causal left inverses (most of this consists of combinations of known results). The latter only holds for finite-dimensional input spaces, but we present partial results on the infinite-dimensional case, on which we shall later build our quasi-coprime factorization theory for WPLSs.

In Theorem 4.1.1, we list the following equivalent conditions for the invertibility of any $\mathbb{D} \in \widetilde{\mathcal{A}}(U,Y)$, where $\widetilde{\mathcal{A}}$ stands for TIC, MTIC, CTIC or for some of their subclasses mentioned above:

(i) $\mathbb{D} \in \mathcal{G}\widetilde{\mathcal{A}}$;

(ii) $\mathbb{D} \in \mathcal{G}\mathrm{TIC}$;

(iii) $\pi_+\mathbb{D}\pi_+ \in \mathcal{G}\mathcal{B}(\pi_+\mathrm{L}^2)$;

(iv) $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{H}^\infty$, i.e., $\widehat{\mathbb{D}}^{-1}$ exists and is bounded on $\mathbf{C}^+$.

In particular, $\widetilde{\mathcal{A}}$ is inverse-closed in TIC. The same holds for the set of maps that are "exponentially $\widetilde{\mathcal{A}}$". For the case $\dim U = \dim Y < \infty$, there are several other equivalent conditions, such as (v) $\inf_{\mathbf{C}^+} |\det(\widehat{\mathbb{D}})| > 0$; (vi) $\mathbb{D}$ is left-invertible in TIC (see the Corona equivalence below for more).

We also give analogous results on TI, MTI, CTI and their (noncausal) subclasses (e.g., $\mathbb{E} \in \mathrm{MTI}$ is invertible in MTI iff $\widehat{\mathbb{E}}$ is boundedly invertible on $i\mathbf{R}$) and further invertibility results.

Then we study the Corona Theorem and its consequences following the methods of M. Vidyasagar. In case $\mathbb{D} \in \widetilde{\mathcal{A}}(U,Y)$, $\dim U < \infty$, we list the following equivalent conditions for the left-invertibility of $\mathbb{D}$:

(i) $\mathbb{V}\mathbb{D} = I$ for some $\mathbb{V} \in \widetilde{\mathcal{A}}(Y,U)$;

(ii) $\mathbb{V}\mathbb{D} = I$ for some $\mathbb{V} \in \mathrm{TIC}(Y,U)$;

(iii) $\widehat{\mathbb{D}}(s)^*\widehat{\mathbb{D}}(s) \geq \varepsilon I$ for all $s \in \mathbf{C}^+$ and some $\varepsilon > 0$;

(iv) $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|u\|_{\mathrm{L}^2_\omega}$ for all $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$, $\omega > 0$ and some $\varepsilon > 0$;

(v) $\mathbb{D}^*\pi_-\mathbb{D} \geq \varepsilon\pi_-$ on L^2 for some $\varepsilon > 0$;

(vi) $\mathbb{D}^{t*}\mathbb{D}^t \geq \varepsilon\pi_{[0,t)}$ for all $t > 0$ and some $\varepsilon > 0$.

(Here $\mathbb{D}^t := \pi_{[0,t)}\mathbb{D}\pi_{[0,t)}$.) It follows that $\mathbb{N} \in \widetilde{\mathcal{A}}(U,Y)$ and $\mathbb{M} \in \widetilde{\mathcal{A}}(U)$ are right coprime over $\widetilde{\mathcal{A}}$, i.e., $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$ for some $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \widetilde{\mathcal{A}}$, iff $\widehat{\mathbb{N}}(s)^*\widehat{\mathbb{N}}(s) + \widehat{\mathbb{M}}(s)^*\widehat{\mathbb{M}}(s) \geq \varepsilon I$ for all $s \in \mathbf{C}^+$ and some $\varepsilon > 0$. Moreover, for most of these classes an equivalent condition is that $\mathbb{D}$ can be complemented to an invertible map $\begin{bmatrix} \mathbb{D} & \mathbb{F} \end{bmatrix}$ over $\widetilde{\mathcal{A}}$. Therefore, for these classes the existence of right or left coprime factors in $\widetilde{\mathcal{A}}$ implies the existence of a doubly-coprime factorization over $\widetilde{\mathcal{A}}$.

The Corona Theorem does not extend to infinite-dimensional U, but we give several partial results for the infinite-dimensional case.

Chapter 5: Spectral Factorization ($\mathbb{E} = \mathbb{Y}^*\mathbb{X}, \quad \mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$)

We study *spectral factorization* ("canonical factorization") in the sense of Israel Gohberg et al. This means factoring the given time-invariant map as the product of a non-causal and a causal invertible time-invariant map (with the inverses having the same properties).

In the frequency domain, spectral factorization equals writing a given operator-valued essentially bounded measurable function on the unit circle as the product $\widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$, where $\widehat{\mathbb{Y}}$ and $\widehat{\mathbb{X}}$ are (the nontangential limits at the circle of) operator-valued bounded, boundedly invertible holomorphic functions on the unit disc; that is, given $\mathbb{E} \in \mathrm{L}^\infty(\partial\mathbf{D};\mathbf{C}^{n\times n})$, finding $\widehat{\mathbb{Y}}, \widehat{\mathbb{Y}}^{-1}, \widehat{\mathbb{X}}, \widehat{\mathbb{X}}^{-1} \in \mathrm{H}^\infty(\mathbf{D};\mathbf{C}^{n\times n})$ such that $\mathbb{E} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ a.e. on $\partial\mathbf{D}$, (in case of unseparable Hilbert spaces in place of $\mathbf{C}^n$, this product must not be interpreted pointwise).

This factorization is an extremely important tool in solving stable control problems, and even the unstable case can often be reduced to the stable one.

For rational transfer functions (equivalently, for finite-dimensional systems), the existence of such a factorization for a bounded time-invariant map $\mathrm{L}^2 \to \mathrm{L}^2$ is equivalent to the invertibility of the Toeplitz operator of this map (The map to be factorized is typically the cost function (or Popov operator) of a control problem.)

Since this necessary Toeplitz invertibility condition is not sufficient for general (non-rational) indefinite maps, the classical conditions for the existence of a solution to a control problem cannot be generalized to general WPLSs, not even if we were not be interested on the regularity of the controller. This makes these factorization results essential for much of the theory, as well as the fact that the regularity implied by these results makes it possible to write down the Riccati equations for the problems and to obtain smooth controllers. Thus, in our most general results in later sections and in some other special cases, we have to use different methods to obtain results, often with fewer equivalent conditions or more complicated formulae.

We also mention that though the spectral factorization need not exist, there are yet "H^2 spectral factors", as shown by Gohberg et al. [ClaGoh] for finite-dimensional Hilbert spaces. We extend this result to the general case in Theorem 9.14.6.

Section 5.1 consists of rather straight-forward derivation of required results from the literature. In Section 5.2, we treat the convolutions with measures consisting of a discrete part plus an (uniformly measurable) L^1 part. Our main contribution is Lemma 5.2.3, by which we can reduce the factorization of such convolutions to the separate factorizations of the discrete and absolutely continuous parts of the measure, which already have been gradually solved during the last three decades.

The positive case of the lemma has already been proved by J. Winkin [Winkin] (for finite-dimensional input and output spaces). Though the existence of a spectral factorization is always guaranteed in the positive case (assuming the invertibility of the corresponding Toeplitz operator), it is important to know the smoothness of the factor, as explained above.

The main corollaries of our lemma are that such convolutions maps have spectral factors, and that these are of the same form as the original maps. This allows one to formulate the solutions to WPLS control problems as in the classical case, though with several technical complications due to the unboundedness of input and output operators (see Section 9.1). These corollaries can be written in the form of the following two theorems:

Theorem 1.2.4 (Positive MTI spectral factorization) *Let U be a Hilbert space, and let $\mathcal{A}$ be one of the classes* TI, MTI, $\mathrm{MTI}^{\mathrm{L}^1}$. *Let $\mathbb{E} \in \mathcal{A}(U)$, and set $\widetilde{\mathcal{A}} := \mathcal{A} \cap \mathrm{TIC}$.*

Then $\mathbb{E} \gg 0$ iff $\mathbb{E}$ has a factorization

$$\mathbb{E} = \mathbb{X}^* \mathbb{X}, \text{ where } \mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}(U). \tag{1.5}$$

Moreover, if $\mathbb{E} \in \mathcal{A}_{\exp}$, then $\mathbb{X}^{\pm 1} \in \widetilde{\mathcal{A}}_{\exp}$. □

(The class $\mathcal{A}_{\exp}$ (resp. $\widetilde{\mathcal{A}}_{\exp}$) consists of "exponentially stable $\mathcal{A}$ (resp. $\mathcal{A}_{\exp}$) maps". By "$\mathbb{E} \gg 0$" (or "$0 \ll \mathbb{E}$") we mean that $\mathbb{E} \geq \varepsilon I$ for some $\varepsilon > 0$.)

If $\mathbb{E} \in \mathrm{MTI} = \mathcal{A}$, then $\widehat{\mathbb{E}}$ and $\widehat{\mathbb{X}}$ are continuous in $i\mathbf{R}$, hence then (1.5) is equivalent to "$\widehat{\mathbb{E}}(it) = \widehat{\mathbb{X}}(it)^* \widehat{\mathbb{X}}(it)$ for all $t \in \mathbf{R}$, $\mathbb{X}, \mathbb{X}^{-1} \in \mathrm{MTIC}(U)$".

The general (indefinite) case is analogous except that for some classes $\mathcal{A}$, our result requires U to be finite-dimensional:

Theorem 1.2.5 (MTI spectral factorization) *Let $\mathbb{E} \in \mathcal{A}(U)$, where $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are as in Theorem 5.2.7. Then the* Toeplitz operator *(or Wiener–Hopf operator) $\pi_+ \mathbb{E} \pi_+ \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+; U))$ is invertible iff $\mathbb{E}$ has a spectral factorization*

$$\mathbb{E} = \mathbb{Y}^* \mathbb{X}, \text{ where } \mathbb{X}, \mathbb{Y} \in \mathcal{G}\widetilde{\mathcal{A}}(U). \tag{1.6}$$

Moreover, if $\mathbb{E} \in \mathcal{A}_{\exp}$, then $\mathbb{X}^{\pm 1}, \mathbb{Y}^{\pm 1} \in \widetilde{\mathcal{A}}_{\exp}$. □

(Note that $\pi_+ \mathbb{E} \pi_+ \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+; U))$ iff $\mathbb{E}\pi_+ + \pi_- \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}; U))$.)

In fact, in the two theorems above, also several other subclasses of MTI can take the place of $\mathcal{A}$ (see Theorems 5.2.8 and 5.2.7). We also state a few other results concerning the spectral factorization of TI maps and some results on other subclasses.

If the assumption "$\mathbb{E} \in \mathcal{A}(U)$" is replaced by "$\mathbb{E} \in \mathrm{TI}(U)$", then the "generalized canonical factors" $\mathbb{X}$ and $\mathbb{Y}$ of $\mathbb{E}$ need no longer be stable in the indefinite case (but their Cayley transforms are invertible in H^2 over the unit disc). For $\dim U < \infty$, this can be found in [CG81] or in [LS] (with the Cayley transforms of $\widehat{\mathbb{X}}^{\pm 1}$ and $\widehat{\mathbb{Y}}^{\pm 1}$ being invertible in H^2 over the unit disc). We show that this theory has an extension for the case where U is an arbitrary Hilbert space (see p. 148 and Theorem 9.14.6).

To emphasize the importance of spectral factorization, we note that one of the main themes of this monograph is the equivalence of the following four conditions for several control problems for an exponentially stable WPLS:

(I) the problem has a (nonsingular) solution;

(II) the Popov Toeplitz operator of the problem is invertible;

(III) the Popov operator of the problem has a spectral factorization;

(IV) the Riccati equation of the problem has a stabilizing solution.

For the case where the WPLS is merely stable, we get almost the same results and the unstable case is somewhat analogous (it can often be reduced to the [exponentially] stable case).

For systems with a I/O map in MTIC (and hence the Popov operator in MTI), the equivalence "(II)$\Leftrightarrow$(III)" follows from either of the two theorems above (the former one covers more classes of I/O maps but is only applicable in minimization problems).

The equivalence "(I)$\Leftrightarrow$(II)" will be established in Chapter 8 and in the sections corresponding to the particular control problems; equivalence "(III)$\Leftrightarrow$(IV)" will be established in Section 9.1 (assuming sufficient regularity of the I/O map and the spectral factor; MTI maps are sufficiently regular for our purposes; hence, for such systems, we have a complete equivalence of (I)–(IV)).

The I/O map of a finite-dimensional system is rational, hence in MTI (if stable). Therefore, in the standard finite-dimensional theory we always have the equivalence of (I)–(IV).

Theorem 1.2.5 is not true for $\mathcal{A} = \mathrm{TI}$, not even when $U = \mathbf{C}^2$ (by Example 8.4.13), and the equivalence "(III)$\Leftrightarrow$(IV)" does not even hold for all regular systems (by Proposition 9.13.1(c1)). For these reasons, some of our results in Chapters 9–12 pose additional regularity assumptions on the system; most of them are satisfied by systems having a MTIC I/O map (cf. Theorem 8.4.9).

Part II: Continuous-Time Control Theory

This part contains the theory of *well-posed linear systems (WPLSs)*: system theory, regularity, spectral and coprime factorization and stabilization (by static feedback, state feedback, output injection or dynamic feedback).

Chapter 6: Well-Posed Linear Systems (WPLS)

Chapters 6 and 7 present an extensive theory on *Well-Posed Linear Systems (WPLSs)*: state-space and frequency-domain theory, stability, regularity, factorization, state feedback, output injection, static and dynamic output feedback and relations to Pritchard–Salamon systems and other special cases.

Some of the results in these chapters are rather straight-forward extensions of existing theory or generalizations of classical results, though yet useful for control problems. The main new contributions of Chapter 6 include the following (in the order of appearance):

1. the relations between the stabilities of different parts of a WPLS (from Lemma 6.1.10 to Example 6.1.14);

2. several, often very technical regularity results needed in the Riccati equation theory;

3. compatibility theory (to write also irregular WPLSs in a differential form as in (1.7));

4. infinite-dimensional quasi- and pseudo-coprime factorization theory and corresponding stabilization theory (Sections 6.4–6.7). This theory serves almost as well as the classical coprime factorization theory for the stabilizability and uniqueness analysis of the solutions of Riccati equations, but these strictly weaker coprimess properties are sometimes more easily verified, and quasi-coprimeness is preserved under discretization in both directions, thus allowing one to reduce several proofs to discrete time.

5. new results on the generators of closed-loop systems (part of Proposition 6.6.18);

6. equivalent conditions for different stability and stabilizability properties (particularly parts of Theorems 6.7.10 and 6.7.15);

7. theory of systems with a smoothing semigroup (Section 6.8, particularly Lemma 6.8.5);

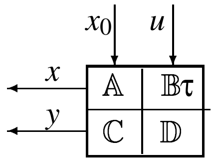
8. the characterization of those transfer functions (equivalently, of I/O maps) that have realizations having a bounded input or output operator or a Pritchard–Salamon realization (Theorems 6.9.1 and 6.9.6);

Also almost all of our results in Chapters 6–12 will be given in a WPLS setting, therefore we motivate these systems briefly below.

Linear time-invariant control systems are usually governed by the equations

$$x'(t) = Ax(t) + Bu(t), \quad y(t) = Cx + Du, \quad x(0) = x_0 \quad (t \geq 0), \tag{1.7}$$

where the *generators* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U, H \times Y)$ of the system are matrices, or more generally, linear operators in Hilbert spaces of arbitrary dimensions, and $u : \mathbf{R}_+ \to U$ is the input, $x : \mathbf{R}_+ \to H$ is the state and $y : \mathbf{R}_+ \to Y$ is the output of the system. If the generators are bounded, then the solution of (1.7) is obviously



$$x = \mathbb{A}x_0 + \mathbb{B}\tau u$$
$$y = \mathbb{C}x_0 + \mathbb{D}u$$

Figure 1.1: Input/state/output diagram of a WPLS $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$

given by the system

$$\begin{cases} x(t) &= \mathbb{A}(t)x_0 + \mathbb{B}\tau(t)u \\ y &= \mathbb{C}x_0 + \mathbb{D}u, \end{cases} \quad \text{where} \tag{1.8}$$

$$\begin{aligned} \mathbb{A}(t) &= e^{At}, & \mathbb{B}\tau(t)u &= \int_0^t \mathbb{A}(t-s)Bu(s)\,ds, \\ \mathbb{C}x_0 &= C\mathbb{A}(\cdot)x_0, & \mathbb{D}u &= C\mathbb{B}\tau(t)u + Du. \end{aligned} \tag{1.9}$$

The formulae (1.8)–(1.9) are actually valid for rather unbounded generators. Therefore, WPLSs are defined by requiring $\mathbb{A}$ to be a strongly continuous semigroup, $\mathbb{D}$ to be time-invariant and causal, $\mathbb{B}$ and $\mathbb{C}$ to be compatible with $\mathbb{A}$ and $\mathbb{D}$, and $\left[\begin{smallmatrix}\mathbb{A}(t) & \mathbb{B}\tau(t)\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ being linear and continuous $H \times \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U) \to H \times \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;Y)$ for each $t \geq 0$, equivalently, that

$$\|x(t)\|_H^2 + \int_0^t \|y(s)\|_Y^2\,ds \leq K_t\big(\|x_0\|_H^2 + \int_0^t \|u(s)\|_U^2\,ds\big) \tag{1.10}$$

for some (equivalently, all) $t > 0$, where K_t depends on t only. An equivalent formulation is given in Definition 6.1.1, where we use the unique natural extensions of $\mathbb{B}$ and $\mathbb{D}$ that allow the inputs to be defined on the whole real line, thus simplifying several formulae.

Abstract linear system theory has been gradually developed since Rudolf Kalman's work in [KFA], by William Helton [Helton76a], Paul Fuhrmann and others until Dietmar Salamon and Anthony Pritchard [PS85] [PS87] formulated the *Pritchard–Salamon systems*, which are formally close to WPLSs. These systems have been extensively studied in eighties and early nineties, but they do not cover all interesting examples. This motivated Salamon to define WPLSs in [Sal87].

The *Lax–Phillips scattering theory* [LP] and the operator-based model theory of Béla Sz.-Nagy and Ciprian Foiaş [SF] gave a remarkable impact to the research already on the seventies, and these theories have been shown to be equivalent to WPLSs (see Chapter 11 of [Sbook]). Thus, also the system theory based on the Lax–Phillips model and extensively developed in Soviet Union by D.Z. Arov and others (independently from WPLSs; see [AN] and its reference list) has exactly the WPLS framework.

Until then, research had been divided by different ways to represent a system, for example:

(1.) in terms of partial differential equations or differential delay equations [Lions] [FLT],

(2.) in terms of the generators $\left[\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right]$ [Helton76a] [Fuhrmann81], (3.) as a frequency domain relationship between inputs and outputs [CG97],

(4.) as a dynamical system (e.g., WPLS) in the sense of Kalman [KFA],

(5.) by fractional representations [Vid] [CD78]

as noted by Ruth Curtain [Curtain97], who emphasized the need for a theory covering both state-space and frequency-domain aspect and unifying all the above representations; the work of Salamon and George Weiss in the late eighties showed that WPLSs satisfy this need. Thereafter WPLSs have become an increasingly popular subject in some parts of control theory, being the most general widely-used class of infinite-dimensional linear systems.

The more specialized approaches still have their advantages in the study of special cases. One of the most important examples of this is the work of Irena Lasiecka, Roberto Triggiani and others (see [LT00a], [LT00b] and references therein), who have solved state feedback problems corresponding to several important PDEs and rather coercive cost functions, by using a more ad hoc approach (of type "(1.)"). At its best, the abstract WPLS approach can complement the others by providing a different insight and an abundance of results including those common for rather general systems and cost functions, thus removing the need to "reinvent the wheel" over and over again.

We study the basic properties, stability, realization theory, dual systems and generators of WPLSs in Section 6.1. For any WPLS, there are generators $B \in \mathcal{B}(U, \mathrm{Dom}(A^*)^*)$ and $C \in \mathcal{B}(\mathrm{Dom}(A), Y)$ satisfying (1.9) in a strong sense (e.g., $\int_0^t \mathbb{A}(s)Bu(-s)\,ds$ converges in $\mathrm{Dom}(A^*)^*$ but its value belongs to the smaller space H and equals $\mathbb{B}u$; also the formula $x' = Ax + Bu$ holds in $\mathrm{Dom}(A^*)^*$ a.e.), as shown by Salamon [Sal89] and Weiss [W89a] [W89b]. Salamon also observed that any TIC_∞ map (or proper transfer function) can be realized as a WPLS.

A WPLS need not have a well-defined feedthrough operator ("D"), but all systems of practical interest seem to have one; such WPLSs are called *regular*. Regularity is treated in Sections 6.2 and 6.3. An equivalent definition of [weak] regularity is that the transfer function has a [weak] limit (necessarily the same $D \in \mathcal{B}(U, Y)$) at infinity along the positive real axis. All weakly regular systems satisfy (1.9) in a weak sense, and the classical formulae such as $\widehat{\mathbb{D}}(s) = D + C(s-A)^{-1}B$ hold if we replace C by its *weak Weiss extension* C_{w}.

Regularity is an extremely important property, because feedthrough operators are of fundamental importance for much of the control theory. For example, optimal control problems are most often solved through Riccati equations that are written in terms of the generators of the system, including the (feedthrough) operator D.

For general WPLSs, equations (1.9) and the classical formulae such as $\widehat{\mathbb{D}}(s) = D + C_{\mathrm{ext}}(s-A)^{-1}B$ still hold in a very weak sense for certain *compatible* pairs (C_{ext}, D); their theory is developed in Section 6.3, which also contains additional results on different forms of regularity, on H^p transfer functions, on the relations between a WPLS and its generators and on reachability and observability.

In Sections 6.4 and 6.5, we define and study coprime, spectral and lossless factorizations. The importance of these factorizations is due to the equivalence on p. 21, with coprime factorization taking the place of spectral factorization in

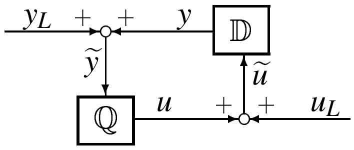


Figure 1.2: Dynamic output feedback controller $\mathbb{Q}$ for $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$

"(III)" in the unstable case, and due to the strong connection between coprime factorization and dynamic stabilization. We also present two weak forms of coprimeness, which are useful in the infinite-dimensional settings, the weaker of them being invariant under (inverse) discretization and hence allowing us to reduce several results to the simpler discrete-time theory.

Thus, the connection between presentations (2.)–(5.) of p. 23 is established in Sections 6.1–6.5. Connection to (1.) is beyond the scope of this book. Instead, we study WPLS theory, with emphasis on Riccati equations and optimal control.

Sections 6.6 and 6.7 treat state feedback, output injection and static output feedback. Since our interest is not limited to exponential stabilization, but we often only require that the controller makes the closed-loop system stable or strongly stable (this has become increasingly popular lately), we meet certain additional difficulties.

In Section 6.8, we study systems whose semigroup is smoothing (e.g., $\mathbb{A}Bu_0 \in H$ a.e. on $\mathbf{R}_+$ for each $u_0 \in U$). In Section 6.9, we show that a transfer function $\widehat{\mathbb{D}}$ has a realization with bounded B iff $\widehat{\mathbb{D}} - \widehat{\mathbb{D}}(+\infty) \in \mathrm{H}^2_{\mathrm{strong}}$ over some right half-plane. We also establish analogous results for realizations with bounded C and for Pritchard–Salamon realizations.

Chapter 7: Dynamic Stabilization

In this chapter, we treat different forms of dynamic stabilization. In *dynamic output feedback* (Section 7.1), the output is fed back to the input through a *Dynamic Feedback Controller*, in order to stabilize and control the plant, as in Figure 1.2.

As one can verify from Figure 1.2, the map from the original input to the output of the plant $\mathbb{D} : u \mapsto y$ becomes $\mathbb{D}(I - \mathbb{Q}\mathbb{D})^{-1} : u_L \mapsto y$.

We have above treated only the I/O maps of the plant and of the controller. We shall also study the problem where the plant and the controller have to be stabilized internally too (see Figure 7.2), but most such results are obtained as corollaries of the I/O theory, since a controller stabilizes a system exponentially iff it I/O-stabilizes the system and both the system and the controller are optimizable and estimatable (this is an extension of the classical concept "exponentially stabilizable and exponentially detectable"), as shown in [WR00], cf. Theorem 7.2.3(c1).

The main new contributions of this section include the relations between external and internal stability of the controlled system (Theorems 7.2.3 and 7.2.4), particularly the extension of the equivalence (1.11); certain results of the internal loop theory required by the H^∞ 4BP theory, including the corollaries on dynamic partial feedback; and the relation between the stabilization of the controlled part and the stabilization of the whole plant in partial feedback (Lemmas 7.3.5 and 7.3.6 and Theorem 7.3.11).

In Chapter 7, we extend most classical results (such as the connection to coprime factorization and Youla parametrization of all stabilizing controllers) to the infinite-dimensional case and present some new results. For example, we extend (see Theorem 7.2.4(c)) the classical equivalence

$$\text{exponentially DF-stabilizable} \iff \text{exponentially stabilizable and detectable} \tag{1.11}$$

to a large subclass on WPLS (including the parabolic systems of Section 9.5).

In Section 7.2, we study the more general *controllers with internal loop*, where $\mathbb{Q}$ need not be well-posed (i.e., proper), as long as the closed-loop system is still well-posed; classical "fractional $\mathrm{H}^\infty/\mathrm{H}^\infty$" controllers fall into this category. For example, if $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ has the *doubly coprime factorization (d.c.f.)* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$, where $\mathbb{M}, \mathbb{N}, \widetilde{\mathbb{M}}, \widetilde{\mathbb{N}} \in \mathrm{TIC}$, $\mathbb{M}, \widetilde{\mathbb{M}} \in \mathcal{G}\mathrm{TIC}_\infty$, and

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = \begin{bmatrix} I_U & 0 \\ 0 & I_Y \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \tag{1.12}$$

for some $\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$, then all stabilizing DF-controllers with internal loop for $\mathbb{D}$ are given by the *Youla parametrization*

$$(\mathbb{T} + \mathbb{M}\mathbb{U})(\mathbb{S} + \mathbb{N}\mathbb{U})^{-1} = (\widetilde{\mathbb{S}} + \widetilde{\mathbb{N}}\widetilde{\mathbb{U}})^{-1}(\widetilde{\mathbb{T}} + \widetilde{\mathbb{M}}\widetilde{\mathbb{U}}), \tag{1.13}$$

where the parameter $\mathbb{U}$ ranges over $\mathrm{TIC}(U)$ (Theorem 7.2.14). The controller (1.13) is well-posed iff $\mathbb{S} + \mathbb{N}\mathbb{U}$ (equivalently, $\widetilde{\mathbb{S}} + \widetilde{\mathbb{N}}\widetilde{\mathbb{U}}$) is invertible in TIC_∞. By shifting stability, we obtain an analogous result on exponential stabilization. We also give a series of results that do not require the plant to have a d.c.f.

Part of the results of Chapter 7 have been established earlier in the works of R. Curtain, R. Rebarber, G. Weiss, M. Weiss and others.

In Section 7.3 we study *dynamic partial output feedback (DPF)*, where the controller can access only a part of the output ("the measurement") and it can affect only part of the input, as in Figure 1.4. (see Figure 7.8 for the I/O part). Consequently, the map $\mathbb{D}_{12} : w \mapsto z$ from the external input w to the actual output z becomes

$$\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q}) := \mathbb{D}_{12} + \mathbb{D}_{11}\mathbb{Q}(I - \mathbb{D}_{21}\mathbb{Q})^{-1}\mathbb{D}_{22} \,:\, w \mapsto z \tag{1.14}$$

when the controller is applied to the system. All stabilizing DPF-controllers for $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ are given by the Youla formula (1.13) applied to $\mathbb{D}_{21}$ in place of $\mathbb{D}$ if $\mathbb{D}$ satisfies standard "stabilizability and detectability" assumptions (by Lemma 7.3.6(b2)).

We list the corollaries of DF-stabilization theory for DPF-stabilization, as above, and present DPF-specific results (with and without internal loop, both I/O theory and state-space theory).

The above results and the further theory developed in Chapter 7 are used in Chapter 12 for the H^∞ *Four-Block Problem (*H^∞ *4BP)*, where one tries to find a stabilizing dynamic partial feedback controller that minimizes the norm of $w \mapsto z$

(or makes it less than a given constant $\gamma > 0$).

Part III: Riccati equations and Optimal control

This part contains a theory on optimal control (both in an abstract setting, and as an application to WPLSs) and Riccati equations, with applications to minimization (LQR and H^2) problems and to the H^∞ full-information and four-block problems.

Chapter 8: Optimal Control ($\frac{d}{du}\mathcal{J} = 0$)

We present an abstract theory on optimization and optimal control in state feedback form (Sections 8.1 and 8.2) and the application of this theory to WPLSs (Sections 8.3 and 8.4) with guidelines to problems finite time interval (Section 8.5) and to systems where the input operator (B) is allowed to be more unbounded than that of WPLSs (Section 8.6). We solve the generalized control problem, whose (possibly indefinite) cost function covers most standard control problems.

Our main contributions include the generalization of the classical coercivity assumption to general WPLSs and cost functions, and the fact that this assumption leads to a solution of the generalized control problem (see Theorems 8.4.3 and 8.3.9); this was already extended to stable WPLSs by O. Staffans. An important part of our theory are also the methods to treate simultaneously all forms of stabilization (i.e., whether one requires the "optimal control" to be, e.g., exponentially, strongly or merely output-stabilizing). These results will then be applied in the derivation of the Riccati equation, LQR and H^∞ theories in the chapters to follow.

We study the critical points of a given cost function and the case where such control corresponds to a stabilizing state feedback pair. Such an "optimal" state feedback pair corresponds to a "stabilizing" solution of the Riccati equation, as shown in Chapter 9. The corresponding special control problems are solved in Chapters 10–12.

Given a WPLS $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ and a cost operator $J = J^* \in \mathcal{B}(Y)$, we consider the *cost function*

$$\mathcal{J}(x_0,u) := \int_0^\infty \langle y(t), Jy(t)\rangle_Y\,dt, \quad \text{where} \quad y := \mathbb{C}x_0 + \mathbb{D}u \quad (x_0 \in H,\ u : \mathbf{R}_+ \to U) \tag{1.15}$$

and u is required to be exponentially stabilizing, strongly stabilizing, stabilizing or something similar, depending on how stable one wishes the closed-loop system to be.

This covers all quadratic (definite or indefinite) costs on the input, state and output (extend $\mathbb{C}$ and $\mathbb{D}$ suitably if necessary, e.g., replace $\mathbb{C}$ by $\left[\begin{smallmatrix}\mathbb{C}\\ 0\end{smallmatrix}\right]$ and $\mathbb{D}$ by $\left[\begin{smallmatrix}\mathbb{D}\\ I\end{smallmatrix}\right]$ to cover cross terms of u and y). In particular, minimization, H^∞ and similar control problems are covered. The solutions of such problems correspond to the controls that are critical points of $\mathcal{J}$, i.e., for which the Fréchet derivative of $\mathcal{J}(x_0,\cdot)$ is zero; we call such controls *J-critical*.

In Section 8.4, we define and study *J-coercivity*, which is a generalization of the standard nonsingularity assumptions of several control problems (including the "J-coercivity" assumptions defined in [S97b]–[S98d], the "Popov Toeplitz invertibility" condition in the stable case and the "no transmission zeros" and "no invariant zeros" conditions in the positive case). We show that any "stabilizable" J-coercive WPLS has a unique J-critical ("optimal") control for each initial state, and that this J-critical control can be presented in WPLS form (this generalizes the corresponding result in [FLT]).

However, the corresponding feedback need not be well-posed without additional assumptions on the system, as illustrated in Examples 8.4.13 and 11.3.7. This leads to some additional difficulties in the Riccati equation theory (the situation is the same even in the case studied in [FLT]). Sections 8.3 and 8.4 also contains a series a further results on J-critical controls and J-coercivity and on the connection of the latter to spectral and coprime factorizations.

The control problems for unstable systems are traditionally reduced to the stable case by preliminary stabilization, when the optimal control is required to be exponentially stabilizing. We show that this is possible for WPLSs too, give a counter-example for other forms of stabilization and develop more complicated tricks to overcome this problem (Theorem 8.4.5).

In the last two section of Chapter 8, we give guidelines on how to extend our optimization and Riccati equation results for problems on finite time interval (Section 8.5) and for more general systems than WPLSs (Section 8.6). These results are not used elsewhere in this monograph.

Chapter 9: Riccati Equations and J-Critical Control

It was shown independently in [WW] and [S97b]–[S98d] that, in the (stable) regular case, the optimal cost operator of certain control problems satisfies a generalized (operator) Riccati equation. We established the converse implication from a stabilizing solution of the Riccati equation to the existence of an optimal control in [Mik97b]. In Chapter 9, we extend both results to the general optimization context of Chapter 8, thus covering also general unstable systems and more singular problems (under weaker regularity assumptions).

We also simplify the equation and the assumptions in several special cases, present a priori sufficient assumptions for the required regularity, and provide weaker results for less regular settings. Moreover, the connection to spectral or coprime factorization and further aspects (such as uniqueness, Riccati inequalities and certain pathologies) are addressed. Possibly ill-posed or irregular optimal controls and corresponding generalized Riccati equations are covered in Section 9.7 (for bounded output operators, a special case of this was solved in [FLT]). We describe below the main results of this chapter.

The existence of a unique regular optimal state feedback operator for a regular WPLS is equivalent to the existence of a (necessarily unique) $\mathcal{U}_*^*$-stabilizing solution of the *Continuous-time Algebraic Riccati Equation (CARE)* and from one the other can be computed (see Theorem 9.9.1; read "optimal" as "J-critical"). This extends most similar results in the literature.

When we optimize over exponentially stabilizing controls or state feedback operators, the term "$\mathcal{U}_*^*$-stabilizing" is equivalent to "exponentially stabilizing" (a WPLS is exponentially stable iff its semigroup $\mathbb{A}$ satisfies $\|\mathbb{A}(t)\|_H \le M e^{\omega t}$ $(t \ge 0)$ for some $\omega < 0$, $M < \infty$). To make things easier, we illustrate this under rather strong assumptions:

Theorem 1.2.6 ($\mathcal{U}_{\mathbf{exp}}$: Unique minimum $\Leftrightarrow$ B_{w}^*-CARE$\Leftrightarrow$ J-coercive) *Assume that the WPLS* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ *and the cost operator* $J = J^* \in \mathcal{B}(Y)$ *are s.t.* $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$, $C \in \mathcal{B}(H,Y)$ *and* $D^*JD \gg 0$. *Then the following are equivalent:*

(i) There is a unique minimizing exponentially stabilizing state feedback operator.

(ii) There is a unique minimizing control over $\mathcal{U}_{\mathrm{exp}}(x_0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \mid x \in \mathrm{L}^2(\mathbf{R}_+;H)\}$ *for each initial state* $x_0 \in H$.

(iii) The Riccati equation

$$(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^*(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) = A^*\mathcal{P} + \mathcal{P}A + C^*JC \qquad (1.16)$$

has a solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ *s.t.* $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ *and the semigroup generated by* $A - BK$ *is exponentially stable, where* $K := -(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)$.

(iv) Σ *is optimizable and* $\mathbb{D}$ *is* J*-coercive over* $\mathcal{U}_{\mathrm{exp}}$.

(v) Σ *is exponentially stabilizable and there is* $\varepsilon > 0$ *satisfying*

$$(ir - A)x_0 = Bu_0 \Rightarrow \langle Cx_0 + Du, J(Cx_0 + Du)\rangle \ge \varepsilon\|x_0\|_H^2 \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}).$$

If (iii) holds, then K *is bounded* ($K \in \mathcal{B}(H,U)$) *and it is the unique minimizing exponentially stabilizing state feedback operator. The minimal cost equals* $\langle x_0, \mathcal{P}x_0\rangle$ *for each* $x_0 \in H$. □

(This is a special case of Corollary 10.2.9 combined with Theorem 9.2.3.)

Thus, the optimal control corresponds to the state feedback $u(t) = Kx(t)$ $(t \ge 0)$, where K is as above. Here B_{w}^* denotes the Weiss extension of $B^* \in \mathcal{B}(\mathrm{Dom}(A^*), U)$. The Riccati equation (1.16) is given on $\mathrm{Dom}(A)$ (see (9.14)). See (1.17) for the more complicated general CARE.

When $\mathcal{J}(x_0, u) = \|Cx\|_2^2 + \|u\|_2^2$, i.e., $C = \left[\begin{smallmatrix}C_1\\ 0\end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix}0\\ I\end{smallmatrix}\right]$, $J = I$, then (1.16) becomes $(B_{\mathrm{w}}^*\mathcal{P})^*B_{\mathrm{w}}^*\mathcal{P} = A^*\mathcal{P} + \mathcal{P}A + C^*C$, the minimizing feedback is given by $u(t) = -B_{\mathrm{w}}^*\mathcal{P}x(t)$ $(t \ge 0)$, and the closed-loop semigroup is generated by $A + BK = A - BB_{\mathrm{w}}^*\mathcal{P}$.

As explained on p. 27, we can have cross terms of u and y in the cost, e.g., replace C by $\left[\begin{smallmatrix}C\\ 0\end{smallmatrix}\right]$ and D by $\left[\begin{smallmatrix}D\\ I\end{smallmatrix}\right]$ to obtain another WPLS and, correspondingly, a "more general" (actually, less general) "standard" form of the Riccati equation, as in, e.g., Remark 9.1.14.

However, the theory of Section 8.3 also allows optimization over various other sets ("$\mathcal{U}_*^*$") of controls than $\mathcal{U}_{\mathrm{exp}}$, e.g., for those which make the state and output strongly stable for each initial state ("$\mathcal{U}_{\mathrm{str}}$"). Correspondingly, the regular optimal state feedback operator (if any) over $\mathcal{U}_{\rm str}$ corresponds to the unique solution of the CARE that is $\mathcal{U}_{\rm str}$-stabilizing, i.e., that stabilizes the state and output strongly for each initial state.

In the literature of infinite-dimensional systems, it has become popular to only require that the output is stable for each initial state and possibly also for each stable external input to the feedback loop. In this case the condition "$\mathcal{U}_*^*$-stabilizing" becomes rather complicated (Definition 9.8.1).

If the system is exponentially detectable, then all the cases mentioned above (and certain others) coincide with exponential stabilization, but this assumption is sometimes too strong. If the system is "coprime stabilizable" (in a suitable, rather weak sense; this assumption always holds when the system is output stable (resp. stable, strongly stable)), then optimization over output-stabilizable (resp. stabilizable, strongly stabilizable) controls corresponds to the "coprime stabilizing" solution of the CARE, and the equivalence of (I)–(IV) on p. 21 holds, see Section 9.1 for details. However, this solution need not be exponentially stabilizing, and the same CARE may also have an exponentially stabilizing solution (see Example 9.13.2; naturally, in a minimization problem the optimal cost becomes higher for stronger stabilizability requirements). Part of these results seem to be new even for finite-dimensional systems.

Very regular systems, such as those of Theorem 1.2.6, are studied in Section 9.2. For them the CARE becomes rather elegant and similar to its finite-dimensional counterparts, as part (iii) of the theorem shows. Such systems cover analytic systems (hence most parabolic-type problems) having rather unbounded input and output operators, as shown in Section 9.5.

In the general case, the optimal control need not correspond to a (well-posed) state feedback operator, as explained in Chapter 8. Nevertheless, such control corresponds to a generalized Riccati equation, as illustrated in Section 9.7 (for WPLSs with a bounded output operator ("C") and a rather coercive cost function, this was shown in [FLT] by F. Flandoli, I. Lasiecka and R. Triggiani). However, since these equations are given on the (unknown) domain of the closed-loop semigroup generator rather than on $\mathrm{Dom}(A)$, it becomes very difficult to solve the Riccati equation and thus obtain the (possibly non-well-posed) feedback operator.

As mentioned above, the existence of a (well-posed) regular state feedback operator for a regular WPLS is equivalent to the CARE having a solution, but in this general case the CARE becomes rather complex: we have to find $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ satisfying

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC & \in \mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A)^*) \\ S = D^*JD + \operatorname*{w-lim}_{s\to+\infty} B_{\rm w}^*\mathcal{P}(s-A)^{-1}B & \in \mathcal{B}(U) \\ SK = -(B_{\rm w}^*\mathcal{P} + D^*JC) & \in \mathcal{B}(\mathrm{Dom}(A), U). \end{cases} \tag{1.17}$$

Obviously, S and K are uniquely determined by $\mathcal{P}$ if S is one-to-one, which corresponds to a unique optimal control. The optimal state feedback is given by $u(t) = K_{\rm w}x(t)$ for a.e. $t \geq 0$. See Definition 9.1.5 for details (and Definition 9.8.1 for noninvertible signature operators).

Note that whereas the special case (1.16) is close the finite-dimensional

CARE, this general form looks almost like the discrete-time Riccati equation; in particular, the *signature operator* S may differ from D^*JD, as observed in [S97b] and [WW]. In the notes to Section 9.8 we explain how the signature properties of the problem are determined by S, not by D^*JD, even when the I/O map is a simple delay. Thus, the situation is analogous to the (finite-dimensional) discrete-time setting, where the signature operator $S := D^*JD + B^*\mathcal{P}B$ takes the role of D^*JD.

We also list several cases in which the CARE can be simplified and cases in which an optimal control is always given by a well-posed regular state feedback pair (and hence corresponds to a CARE; see, e.g., Remark 9.9.14).

The optimal control is given by a well-posed state feedback iff the *Integral Algebraic Riccati Equation (IARE)* has an $\mathcal{U}_*^*$-stabilizing solution, regardless of regularity. While IAREs are not particularly apt for engineering purposes, they provide a link to discrete-time Riccati equations, and this allows us the prove several results whose continuous-time proofs would seem intractable due to the unboundedness of input and output operators. The IAREs also allow us to treat the connection between optimal control and Riccati equations separately from regularity considerations. Naturally, for regular WPLSs, the solutions of the CARE are exactly the solutions of the IARE corresponding to regular feedback. Also these questions are addressed in Section 9.8. Several further properties of Riccati equations are treated in the rest of the chapter. Much of our theory concerning for $\mathcal{U}_*^* \neq \mathcal{U}_{\text{exp}}$ is new even for finite-dimensional systems.

In Section 9.14, we give an extension of the generalized canonical factorization theory to the case of infinite-dimensional input and output spaces (see also p. 148).

Chapter 10: Quadratic Minimization (LQR)

For control problems with a positive Popov operator, one traditionally shows that under certain conditions any solution of the Riccati equation is unique, admissible and exponentially stabilizing. One of our main contributions in this and preceding chapter is the extension of the above fact to WPLSs and partially also to the non-exponentially stabilizing case; this is technically very challenging due to the unbounded input and output operators, which, e.g., make it hard to show when the "optimal feedback" is well posed.

As corollaries, we get several results that formally look like the classical ones. These corollaries include Theorem 1.2.7 below, (b4)&(c1)&(c2) of Theorem 10.1.4, the Strict Bounded and Strictly Positive (Real) Lemmas, and the equivalence between optimizability and exponential stabilizability for systems with a smoothing semigroup (Theorem 9.2.12). We also solve several minimization problems with more general stabilizability or regularity assumptions.

Important new contributions of the chapter also include the connection between different classical coercivity assumptions and their generalizations to WPLSs, including J-coercivity (Section 10.3).

In Section 10.2, we study minimization problems, by which we refer to the minimization of the cost function (1.15). Theorem 1.2.6 is a corollary of that section.

In Section 10.1 we study the special case of the cost function $\|y\|_2^2 + \|u\|_2^2$ and its variants. Under a mild detectability condition, there is at most one nonnegative solution of the CARE, hence we do not have to verify whether a solution is "$\mathcal{U}_*^*$-stabilizing":

Theorem 1.2.7 (LQR: $\min_u \int_0^\infty (\|x\|_H^2 + \|u\|_U^2)$) *Assume that the WPLS $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ is uniformly regular (UR) and estimatable (e.g., that C is bounded and $C^*C \gg 0$). Consider, for some $R \in \mathcal{B}(U)$, $Q \in \mathcal{B}(Y)$ s.t. $R, Q \gg 0$, the cost function*

$$\mathcal{J}(x_0, u) := \int_0^\infty \big(\langle y(t), Qy(t)\rangle_Y + \langle u(t), Ru(t)\rangle_U\big)\, dt \quad (x_0 \in H,\ u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+; U)). \tag{1.18}$$

There is a UR minimizing state feedback operator for Σ iff there is a nonnegative solution $\mathcal{P} \in \mathcal{B}(H)$ satisfying the CARE

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*QC, \\ \quad S = R + D^*QD + \lim_{s \to +\infty} B_{\mathrm{w}}^*\mathcal{P}(s - A)^{-1}B, \\ \quad K = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*QC), \end{cases} \tag{1.19}$$

for some $K \in \mathcal{B}(H_1, U)$, $S \in \mathcal{B}(U)$, $S \gg 0$.

If such a solution exists, then it is the unique nonnegative solution of (1.19), K is a UR exponentially stabilizing state feedback operator for Σ, and K is the unique minimizing state feedback operator over all $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+; U)$ (and over $\mathcal{U}_{\exp}$ and over $\mathcal{U}_{\mathrm{out}}$). □

(This follows from Theorem 10.1.4 and Remark 10.1.5.) For each CARE result in this monograph, including the one above, there is also a "B_{w}^*-CARE" variant that allows us to remove the limit term and simplify the formulation under any of the regularity assumptions of Hypothesis 9.2.2, as illustrated in Theorem 1.2.6.

Without the detectability (estimatability) condition, we observe that a minimizing state feedback operator over $\mathcal{U}_{\exp}$ corresponds to the maximal nonnegative solution of the CARE and a minimizing state feedback operator over $\mathcal{U}_{\mathrm{out}}$ corresponds to the minimal nonnegative solution of the CARE (Theorem 10.1.4). We also derive further results on such and more general minimization problems.

In Section 10.4 we show that the solution of the minimization problem leads to the solution of the H^2 full information and state feedback problems, where one wishes to find a controller ($\mathbb{Q}$; possibly induced by state feedback or dynamic output feedback) that minimizes the norm

$$\|\widehat{\mathbb{D}}\widehat{\mathbb{Q}} + \widehat{\mathbb{D}}_2\|_{\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+; \mathcal{B}(W,Y))}, \tag{1.20}$$

where $\widehat{\mathbb{Q}} : \widehat{w} \to \widehat{u}$ is the frequency-domain control law (determined by $\mathbb{Q}$) to an external (disturbance) input to the control input for the WPLS

$$\left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B} & \mathbb{B}_2 \\ \hline \mathbb{C} & \mathbb{D} & \mathbb{D}_2 \end{array}\right] \quad \text{with generators} \quad \left[\begin{array}{c|cc} A & B & B_2 \\ \hline C & D & 0 \end{array}\right], \tag{1.21}$$

as in Figure 10.1. The above WPLS is obtained by adding a second input to the WPLS Σ, We assume that both $\mathbb{D}$ and $\mathbb{D}_2$ are WR. A stronger problem is to find, for each $w_0 \in W$, a "stabilizing" control u s.t. $\|\widehat{\mathbb{D}}\widehat{u}+\widehat{\mathbb{D}}_2 w_0\|_{\mathrm{H}^2(\mathbf{C}^+;Y)}$ is minimized, see Section 10.4 for details. We show that under minimal assumptions, a minimizing state feedback operator for the original system also solves the H^2 problem and the stronger problem formulated above.

In Section 10.3 we treat most standard assumptions for classical minimization problems and show that they are stronger than or equivalent to positive J-coercivity (over $\mathcal{U}_{\mathrm{exp}}$ or over $\mathcal{U}_{\mathrm{out}}$).

In Section 10.5, we present generalized versions of the Bounded Real Lemma, including the following:

Theorem 1.2.8 (Generalized Strict Bounded Real Lemma) *Assume that* $\gamma > 0$.

If C is bounded and $\dim Y < \infty$*, or if B is bounded, then the following are equivalent:*

(i) Σ is exponentially stable and $\|\mathbb{D}\| < \gamma$*;*

(ii) There is $\mathcal{P} \leq 0$ *s.t.* $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ *and*

$$\begin{bmatrix} A^*\mathcal{P}+\mathcal{P}A-C^*C & (B_{\mathrm{w}}^*\mathcal{P}-D^*C)^* \\ B_{\mathrm{w}}^*\mathcal{P}-D^*C & \gamma^2 I - D^*D \end{bmatrix} \gg 0 \quad on \quad \mathrm{Dom}(A)\times U. \tag{1.22}$$

Moreover, any solution of (ii) satisfies $\mathcal{P} < 0$.

In the Strict Positive Real Lemma, we present analogous conditions for the I/O map to satisfy $\mathbb{D} \in \mathrm{TIC}$ and $\mathrm{Re}\langle \mathbb{D}\cdot,\cdot\rangle \gg 0$ (i.e., $\widehat{\mathbb{D}}+\widehat{\mathbb{D}}^* \geq \varepsilon I$ in $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ for some $\varepsilon > 0$). Naturally, there are also analogous results for unbounded B and C.

In Section 10.6, we present necessary and sufficient conditions for the uniform positivity of the Popov operator ($\mathbb{D}^*J\mathbb{D} \gg 0$), in terms of spectral factorizations and Riccati equations or inequalities. Section 10.7 present additional results for positive Riccati equations (say, with positive signature operator, $S \gg 0$).

Chapter 11: The H^∞ Full-Information Control Problem (FICP)

The H^∞ control problems refer to the minimization of the output of a plant in the presence of a disturbance input. The name "H^∞" comes from the minimization of the (controlled closed-loop system) $\mathrm{L}^2 \to \mathrm{L}^2$ norm from disturbance to output, which equals the H^∞ norm of the corresponding transfer function, by Theorem 1.2.1.

In the FICP, studied in this chapter, we can produce the control signal knowing exactly the state and disturbance of the system, whereas in the Four-Block Problem of Chapter 12 the controllers only input is a separate measurement output.

Our main results state that, given $\gamma > 0$, there is a controller achieving a norm less than γ iff the Riccati equation (1.24) has a nonnegative stabilizing solution. Moreover, if this is the case, then there is such a controller that consists of pure

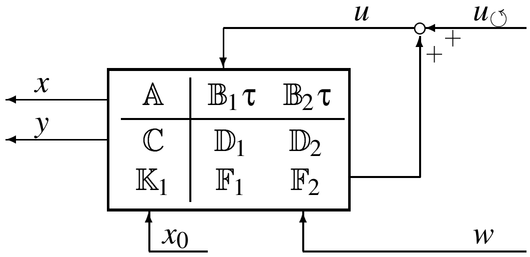


Figure 1.3: The H^∞ FICP

state feedback (with no measurement of the disturbance). This generalizes the classical results to this problem. For WPLSs, O. Staffans had already proved the necessity part of the implication for stable systems having an L^1 impulse response. We also formulate the solution in terms of J-lossless factorizations and solve the corresponding discrete-time problem.

Technically, we study a system of form $\Sigma = \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{C} & \mathbb{D}_1 & \mathbb{D}_2 \end{array}\right]$, with input space $U \times W$ instead of U. If $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W, Y)$ is regular, we can write this as

$$\begin{cases} x' = Ax + B_1 u + B_2 w, \\ y = Cx + D_1 u + D_2 w, \end{cases} \tag{1.23}$$

(if $B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \in \mathcal{B}(U \times W, H_{-1})$ and $C \in \mathcal{B}(\mathrm{Dom}(A), Y)$ are unbounded, then the dynamics (1.23) are satisfied only in the sense described in Theorem 6.2.13).

We have divided the input space in two to model a setting where only part of the input (called the *control*), $u : \mathbf{R}_+ \to U$, is accessible by the controller, whereas the other part represents the *disturbance* (or uncertainties, sensor noise, modeling error) $w : \mathbf{R}_+ \to W$ to the system. The signal y is the *objective* or *error signal* whose norm is to be minimized.

In the optimal H^∞ *State-Feedback Control Problem (SFCP)*, one wishes to find a (pure) state feedback controller of form "$u(t) = Kx(t)$" (with e.g., $K \in \mathcal{B}(H, U)$) such that this feedback stabilizes the system exponentially and minimizes the norm $\|w \mapsto y\|_{\mathcal{B}(\mathrm{L}^2,\mathrm{L}^2)}$. In the optimal H^∞ *Full-Information Control Problem (FICP)*, the controller is allowed to be of form "$u(t) = K(t)x + F_2 w(t)$" (state feedback plus feedforward), as in Figure 1.3. (Here $\left[\begin{array}{c|cc} \mathbb{K} & \mathbb{F}_1 & \mathbb{F}_2 \end{array}\right]$ is the state feedback pair generated by K or $\left[\begin{array}{c|cc} K & 0 & F_2 \end{array}\right]$, and the signal $u_\circlearrowleft$ represents the external disturbances (or external inputs) in the feedback loop. The words "full information" refer to the fact that the controller has access to both the state and the disturbance.)

There is no direct method available (even in the finite-dimensional case) to find the exact optimum. Therefore, instead of the optimal problem, the corresponding *suboptimal* H^∞ *problem* is usually treated in the literature. In the suboptimal H^∞ problem, we search for an exponentially stabilizing controller such that $\|w \mapsto y\|_{\mathcal{B}(\mathrm{L}^2,\mathrm{L}^2)} < \gamma$, where $\gamma > 0$ is a given constant; such a controller is called *(γ-)suboptimal.* We extend the classical results by showing that there is a suboptimal state feedback controller iff the Riccati equation condition (iii) below is satisfied. By varying γ we can then find an estimate of the infimal γ and a corresponding (almost optimal) controller (e.g., by a binary search over γ's).

As mentioned above, under standard coercivity assumptions and certain regularity and normalization conditions (see, e.g., Theorem 11.1.4), the following are equivalent:

(i) there is a suboptimal control law $w \mapsto u$, and (A, B_1) is exponentially stabilizable;

(ii) there is a suboptimal state-feedback (plus feedforward) controller $u = K_{\mathrm{w}}x + F_2 x$;

(ii') there is suboptimal pure state-feedback controller $u = K_{\mathrm{w}}x$;

(iii) the Riccati equation

$$\mathcal{P}(B_1 B_1^* - \gamma^{-2} B_2 B_2^*)\mathcal{P} = A^*\mathcal{P} + \mathcal{P}A + C^*C, \tag{1.24}$$

(on $\mathrm{Dom}(A)$) has a nonnegative solution $\mathcal{P} \in \mathcal{B}(H)$ such that $A - (B_1 B_1^* - \gamma^{-2} B_2 B_2^*)\mathcal{P}$ generates an exponentially stable C_0-semigroup.

Moreover, if (iii) holds, then $K := -B_1^*\mathcal{P} \in \mathcal{B}(H, U)$ determines a suboptimal (pure) state-feedback controller for Σ (through $u(t) := Kx(t)$ $(t \geq 0)$). A solution $\mathcal{P}$ of (iii) is unique.

Here we have assumed that B is bounded, $D_2 = 0$ and $D_1^* \begin{bmatrix} C & D_1 \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$; see, e.g., (11.24) and (11.17) for the unsimplified forms of (iii) and K. (Also without the above simplifying assumptions, the suboptimal state feedback operator K is exponentially stabilizing (and uniformly regular, though not necessarily bounded), but we must add a signature condition to (iii); moreover, condition (ii') becomes strictly stronger than the other conditions (which remain equivalent to each other) unless a stronger signature condition is satisfied.)

We present analogous results under different regularity assumptions, and variants for $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$, i.e., where the suboptimal controller needs to be, e.g., merely strongly stabilizing instead of exponentially stabilizing. We also establish the sufficiency of the Riccati equation condition for arbitrary regular WPLSs (see Lemma 11.2.13). In Example 11.3.7(c), we show that, however, this condition is not necessary for general regular WPLSs.

In (i), we have allowed for an arbitrary control law $\mathrm{L}^2(\mathbf{R}_+; W) \mapsto \mathrm{L}^2(\mathbf{R}_+; U)$. If such a control law $\mathbb{Q} : w \mapsto u$ has a transfer function (e.g., $\mathbb{Q} \in \mathrm{TIC}_\infty(W, U)$), then the norm $\|w \mapsto y\|$ equals $\|\mathbb{D}_1\mathbb{Q} + \mathbb{D}_2\|_{\mathrm{TIC}(W,Y)}$, or $\|\widehat{\mathbb{D}}_1\widehat{\mathbb{Q}} + \widehat{\mathbb{D}}_2\|_{\mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(W,Y))}$. By the above equivalence, this problem, the FICP and the SFCP are all equivalent (under simplifying assumptions and suitable regularity). Thus, if there is any suboptimal control law (and (A, B_1) is exponentially stabilizable), then there is actually a causal, linear, stable, time-invariant control law that can be implemented as an exponentially stabilizing state feedback controller (so that $\mathbb{Q} = (I - \mathbb{F}_1)^{-1}\mathbb{F}_2$). Condition (i) can also be formulated as a minimax problem, as explained in Section 11.1 (particularly on pp. 613 and 626).

In Section 11.2, we give proofs and additional variants for the above results, and we extend the (frequency-domain) J-lossless factorization results for the H^∞ FICP given in [Green] and [CG97] to MTIC and similar classes (Theorem 11.2.7).

The *Discrete-Time* H^∞ *FICP* is treated in Section 11.5, and the abstract H^∞ FICP in Section 11.7.

The H^∞ FICP is interesting both for its own merits and for the fact that it can be used to obtain a solution to the H^∞ 4BP presented below.

The methods used for the stable H^∞ FICP also apply to the (one-block) Nehari problem, where one wishes to estimate $d(\mathbb{D}, \mathrm{TIC}^*)$ or the Hankel norm $\|\pi_+\mathbb{D}\pi_-\|$ of some $\mathbb{D} \in \mathrm{TIC}$. Therefore, we take a brief look at this problem in Section 11.8, this includes the following:

Theorem 1.2.9 (Nehari) *Let* $\mathbb{D} \in \mathrm{MTIC}(W,U)$ *and* $\gamma > 0$. *If* $\dim U \times W < \infty$ *or* $\mathbb{D} \in \mathrm{MTIC}_{T\mathbf{Z}}$, *then the following are equivalent:*

(i) There is $\mathbb{U} \in \mathrm{TIC}(U,W)$ *s.t.* $\|\mathbb{D} + \mathbb{U}^*\|_{\mathcal{B}(\mathrm{L}^2)} < \gamma$ *(i.e.,* $d(\mathbb{D}, \mathrm{TIC}^*) < \gamma$*).*

(ii) The Hankel norm $\|\pi_+\mathbb{D}\pi_-\|$ *of* $\mathbb{D}$ *is less than* γ.

(iii) There is $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W)$ *s.t.* $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $\begin{bmatrix} I & \mathbb{D} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} I & \mathbb{D} \\ 0 & I \end{bmatrix} = \mathbb{X} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \mathbb{X}^*$.

(Recall that $\mathbb{D} \in \mathrm{MTIC}_{T\mathbf{Z}}$ means that $\mathbb{D}$ has an L^1 impulse response plus delays of form $\sum_{k=0}^\infty D_k \tau^{-kT}$ for some period $T > 0$.)

The factorization in (iii) is often called a co-spectral factorization. The norm $\|\pi_+\mathbb{D}\pi_-\|$ equals $\rho(\mathbb{B}\mathbb{B}^*\mathbb{C}^*\mathbb{C})^{1/2}$, where $\mathbb{B}\mathbb{B}^*$ and $\mathbb{C}^*\mathbb{C}$ are the reachability and observability Gramians, respectively, of any realization of $\mathbb{D}$ having stable input and output maps.

We do not treat the Nehari Riccati equations, since their theory would require lengthy additions to Chapter 9 due to the noncausality of the corresponding "closed-loop systems".

Chapter 12: H^∞ Four-Block Problem ($\|\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q})\| < \gamma$)

In the H^∞ *Four-Block Problem (*H^∞ *4BP)* (aka. "the standard H^∞ problem" or "the general regulator problem"), one tries to find a DPF-controller that makes the norm $w \mapsto z$ less than a given constant $\gamma > 0$ (see (1.14)), i.e., γ-*suboptimal*.

Consequently, as explained above, the difference to the H^∞ FICP is that now the controller does not have access to the disturbance, only to a part of the output ("the measurement"), as in Figure 1.4 (or in Figure 7.8; see Figures 7.10 and 7.11 for DPF-controllers with internal loop).

Thus, the goal of the engineer is again to minimize the norm from the external disturbance input to the objective output of the system. As in the previous chapter, we again generalize the classical result (previously generalized to Pritchard–Salamon systems by B. van Keulen [Keu]) that there is a γ-suboptimal exponentially stabilizing (measurement feedback) controller iff certain two independent Riccati equations have exponentially stabilizing nonnegative solutions and these (necessarily unique) solutions satisfy the standard spectral radius condition. We formulate the result also in terms of two nested J-lossless factorizations and solve the H^∞ discrete-time Four-Block Problem; in fact these two generalizations of classical results serve as parts of our lengthy proof.

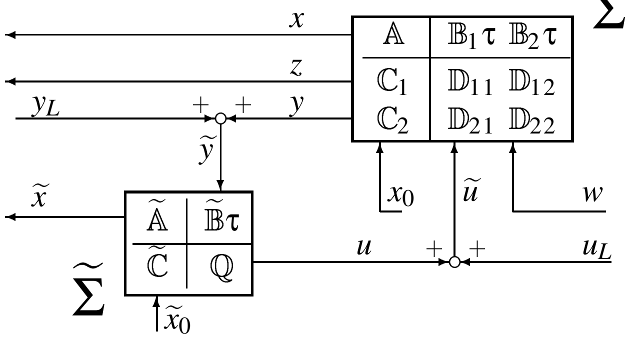


Figure 1.4: DPF-controller $\widetilde{\Sigma}$ for $\Sigma \in \mathrm{WPLS}(U \times W, H, Z \times Y)$

As in Section 7.3, the output "y" is now divided in two, namely "y"$= \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$, where z is the objective output to be minimized and y is a measurement that is fed into the controller. This corresponds to the dynamics

$$\begin{cases} x' = Ax + B_1 u + B_2 w, \\ z = C_1 x + D_{11} u + D_{12} w, \\ y = C_2 x + D_{21} u + D_{22} w, \end{cases} \tag{1.25}$$

with initial state $x_0 \in H$, disturbance input $w \in \mathrm{L}^2(\mathbf{R}_+; W)$, control input $u \in \mathrm{L}^2(\mathbf{R}_+; U)$, objective output $z \in \mathrm{L}^2(\mathbf{R}_+; Z)$ and measurement output $y \in \mathrm{L}^2(\mathbf{R}_+; Y)$ (the controller input). In the case of a general weakly regular system, equations (1.25) hold in the strong sense, see Theorem 6.2.13 for details.

We are then to find a controller $\mathbb{Q} : y \mapsto u$ s.t. the norm $w \mapsto z$ becomes small enough and that the closed-loop connection becomes exponentially stable (that is the main case; we only treat the case where the closed-loop system is merely required to be stable or strongly stable).

(We remind that the order of the subindices corresponding to u and w is often reversed in the literature; this also affects the formulae below.)

In Section 12.1, we present several versions of the standard result that the H^∞ 4BP has a solution iff the two H^∞ Riccati equations have nonnegative exponentially stabilizing solutions satisfying the coupling condition. Since we do not use any simplifying assumptions, our formulae become rather complicated. Therefore, we show here the simplified forms of those formulae (by making additional assumptions):

Theorem 1.2.10 ($\mathbf{H^\infty}$ 4BP) *Let $\gamma > 0$. Make the regularity and nonsingularity assumptions (A1)&(A2) of Theorem 12.1.4.*

Then there is an exponentially stabilizing DPF-controller for Σ (possibly with internal loop) satisfying $\|w \mapsto z\| < \gamma$ iff (1.)–(3.) of Theorem 12.1.4 hold. Under the normalizing conditions

$$D_{12} = 0 = D_{21}, \qquad D_{11}^* \begin{bmatrix} C_1 & D_{11} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix} = D_{22} \begin{bmatrix} B_2^* \\ D_{22}^* \end{bmatrix}, \tag{1.26}$$

conditions (1.)–(3.) can be written as follows:

(1.) **($\mathcal{P}_\mathrm{X}$-CARE)** *There is $\mathcal{P}_\mathrm{X} \in \mathcal{B}(H, \mathrm{Dom}(B_\mathrm{w}^*))$ s.t. $\mathcal{P}_\mathrm{X} \geq 0$ on H, $A +$*

$(\gamma^{-2}B_2(B_2^*)_{\mathrm{w}} - B_1(B_1^*)_{\mathrm{w}})\mathcal{P}_{\mathrm{X}}$ is exponentially stable, and

$$((B_1^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}})^*(B_1^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} - \gamma^{-2}((B_2^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}})^*(B_2^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} = A^*\mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}}A + C_1^*C_1. \tag{1.27}$$

(2.) **($\mathcal{P}_{\mathrm{Y}}$-CARE)** *There is $\mathcal{P}_{\mathrm{Y}} \in \mathcal{B}(H, \mathrm{Dom}(\begin{bmatrix} C_2 \\ C_1 \end{bmatrix}_{\mathrm{w}}))$ s.t. $\mathcal{P}_{\mathrm{Y}} \geq 0$ on H, $A^* + (\gamma^{-2}C_1^*(C_1)_{\mathrm{w}} - C_2^*(C_2^*)_{\mathrm{w}})\mathcal{P}_{\mathrm{Y}}$ is exponentially stable, and*

$$((C_2)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}})^*(C_2)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} - \gamma^{-2}((C_1)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}})^*(C_1)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} = A\mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}}A^* + B_2B_2^*. \tag{1.28}$$

(3.) **(Coupling condition)** $\rho(\mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}}) < \gamma^2$.

Any solutions of (1.) or (2.) are unique. If (1.)–(3.) are satisfied, then all exponentially stabilizing DPF-controllers for Σ satisfying $\|w \mapsto z\| < \gamma$ are the ones parametrized in Theorem 12.1.8, and the regularity claims of Theorem 12.1.4(a)&(b) apply.

In (3.), ρ denotes the spectral radius. One of the alternative regularity assumptions in (A1) is that B is bounded and $\pi_{[0,1)}C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,1); \mathcal{B}(H, Z \times Y))$. For bounded B, the Riccati equation (1.27) takes the classical form

$$\mathcal{P}_{\mathrm{X}}(B_1B_1^* - \gamma^{-2}B_2B_2^*)\mathcal{P}_{\mathrm{X}} = A^*\mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}}A + C_1^*C_1. \tag{1.29}$$

See p. 618 for further simplification and remarks. Analogous remarks apply to (2.); e.g., for bounded C, the Riccati equation (1.28) becomes

$$\mathcal{P}_{\mathrm{X}}(C_2^*C_2 - \gamma^{-2}C_1^*C_1)\mathcal{P}_{\mathrm{X}} = A\mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}}A^* + B_2B_2^*. \tag{1.30}$$

Thus, the classical results become special cases of ours. We also give several results under weaker regularity assumptions (e.g., for the case where $\mathbb{A}B, C_{\mathrm{w}}\mathbb{A}, C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1_{\mathrm{loc}}$; this allows roughly twice as much unboundedness as the assumptions of a Pritchard–Salamon system).

In general we allow for DPF-controllers with internal loop, but we show that such a loop is not needed if $D_{21} = 0$ (i.e., one can use a well-posed controller in that case).

In Section 12.2, we give discrete forms of the results of Chapter 12. For them we need no regularity assumptions (since B and C are always bounded for "discrete-time WPLSs").

In Section 12.3, we study the frequency-domain H^∞ 4BP, where one is only given an I/O map $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$, and one wishes to find a controller (I/O map) $\mathbb{Q} \in \mathrm{TIC}_\infty(Y, U)$ s.t. the closed-loop connection becomes (I/O-)stable and satisfies $\|w \mapsto z\| < \gamma$ (see (1.14) for $\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q}) : w \mapsto z$; we also treat the case where $\mathbb{Q}$ is allowed to have an internal loop). In particular, no state-space or internal stability considerations are required.

Michael Green showed in [Green] (Theorem 4.4) that the frequency-domain 4BP has a solution iff certain two nested spectral factorizations exist (in the rational finite-dimensional case). In Section 12.3 we extend this result and its earlier extensions to maps having a d.c.f. in MTIC (Theorem 12.3.6); we also provide partial results for more general settings. Our proof of Theorem 1.2.10 is based on both the frequency-domain 4BP and the H$^\infty$ FICP. The rest of the chapter consists of proofs and minor results.

Part IV: Discrete-Time Control Theory

Part IV presents the discrete-time counterpart of the theory of Parts I–III. Primarily we list the continuous-time results that hold also for the *discrete-time well-posed linear systems (wpls's)* (cf. Theorem 13.3.13; our notation is much the same in both settings). Most proofs apply mutatis mutandis; we give explicit proofs when this is not the case. Several proofs in Parts I–III are actually reduced to the discrete time. Our main contributions in this part are mainly the same as those in continuous-time (Parts I–III), such as the solutions of the H$^\infty$ problems (Sections 11.5 and 12.2).

Chapter 13: Discrete-Time Maps and Systems (ti & wpls)

In Chapter 13, we present briefly some facts on the discrete counterparts of WPLSs, which we call *discrete-time well-posed linear systems (wpls's)*. They are the systems governed by the difference equations

$$\begin{cases} x_{j+1} = Ax_j + Bu_j, \\ \\ y_j = Cx_j + Du_j, \end{cases} \qquad j \in \mathbf{Z}, \tag{1.31}$$

for some $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U, H \times Y)$. We show that almost all our continuous-time results have discrete-time analogies (see Theorem 13.3.13), and also many further results hold due to the boundedness of the generating operators (A, B, C, D).

In Section 13.1, we study bounded linear time-invariant maps $\ell^2_r(\mathbf{Z};U) \to \ell^2_r(\mathbf{Z};Y)$ ("ti$_r(U,Y)$", where $\|u\|^2_{\ell^2_r(\mathbf{Z};U)} := \sum_k \|r^{-k}u_k\|^2_U$), for $r > 0$, and corresponding transfer functions. The Cayley transform is treated in Section 13.2. (These two sections correspond to Chapters 2 and 3; in particular, we extend the discrete-time Fourier multiplier and H$^\infty$ boundary function theorems for I/O maps over unseparable Hilbert spaces).

In Section 13.3, we study wpls's (this corresponds to Chapter 6; also Chapters 4, 7 and 8 (and partially the rest of this monograph) are treated in Theorem 13.3.13).

In Section 13.4, we show how to obtain wpls's from WPLSs by discretization. This allows us to reduce several WPLS problems to wpls problems, which are often substantially simpler due to the bounded input and output operators.

Discrete-time Riccati equations (DAREs) and spectral and coprime factorization are treated in Chapter 14, minimization problems in Chapter 15, and H^∞ (and Nehari) problems in Sections 11.5 and 12.2.

Chapter 14: Riccati Equations (DARE)

In Chapter 14, we shall present the results of Chapters 9 and 5 (see the above summaries) in their discrete-time forms and supplement this by further results. In particular, we define and study infinite-dimensional *Discrete-time Algebraic Riccati Equations (DAREs).*

We show that for the general cost function, the existence of an optimal control is equivalent for the DARE (1.32) to have a stabilizing solution. Moreover, the optimal controller can be computed from such a solution. We also show that a third equivalent condition is the generalization of the standard coercivity condition combined to exponential stability (Theorem 14.2.7).

Notice that the discrete-time H^∞ control problems are solved in Sections 11.5 and 12.2. The solutions are already known for finite-dimensional problems (see [IOW]).

Given an initial state $x_0 \in H$, we say that "$u \in \mathcal{U}_{\exp}(x_0)$" iff $u \in \ell^2(\mathbf{N};U)$ is such that $x \in \ell^2(H,U)$ (where x is determined by (1.31) with $x_0 = 0$); such controls (u) are sometimes called "exponentially stabilizing" (or "power stabilizing"). (It obviously follows that $y \in \ell^2(\mathbf{N},Y)$.)

One often wants to minimize or, more generally, optimize a cost function (i.e., to find a J-critical control) under the restriction $u \in \mathcal{U}_{\exp}(x_0)$. This problem has a unique solution iff the extended DARE has a solution:

Theorem 1.2.11 *There is a unique J-critical control for each $x_0 \in H$ iff the extended Discrete-time Algebraic Riccati Equation (eDARE)*

$$\begin{cases} K^*SK = A^*\mathcal{P}A - \mathcal{P} + C^*JC, \\ \quad S = D^*JD + B^*\mathcal{P}B, \\ SK = -(D^*JC + B^*\mathcal{P}A), \end{cases} \tag{1.32}$$

has solution $(\mathcal{P},S,K)$ such that $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, S is one-to-one, $K \in \mathcal{B}(H,U)$ and $\sigma(A+BK) \subset \mathbf{D}$. Moreover, any such solution is unique.

If such a solution exists, then the J-critical control is determined by the state feedback $u_j = Kx_j$ and corresponding J-critical cost is given by $\langle x_0, \mathcal{P}x_0\rangle$, where x_0 is the initial state. □

(Here the cost function is of form $\mathcal{J}(x_0,u) := \sum_{k=0}^\infty \langle y, Jy\rangle_Y$ for some $J = J^* \in \mathcal{B}(Y)$. See Theorem 14.1.6 for the proof.)

The above theorem corresponds to $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ (in the discrete-time sense); analogous results hold for other $\mathcal{U}_*^*$'s (cf. Chapter 8). We also present some results in the singular case (where S is not one-to-one and K is not unique) and sufficient conditions for the existence of a unique J-critical control.

Chapter 15: Quadratic Minimization

This chapter is mostly the discrete-time counterpart of Chapter 10; see the above summary for Chapter 10 for corresponding problems and results, such as LQR and H^2 problems, extended minimization, coercivity, real lemmas and maximal solutions of Riccati inequalities/equations. Naturally, several additional discrete-time results are given; the following one solves the extended LQR (minimization) problem:

Corollary 1.2.12 (LQR: min $\sum_{j=0}^{\infty}(\|y_j\|_Y^2+\|u_j\|_U^2)$) *Let $R,Q\gg 0$. Then the following are equivalent:*

(i) there is a $\langle y,Qy\rangle_{\ell^2}+\langle u,Ru\rangle_{\ell^2}$-minimizing control over all $u:\mathbf{N}\to U$ for each $x_0\in H$;

(ii) for each $x_0\in H$ there is $u\in\ell^2(\mathbf{N};U)$ s.t. $y\in\ell^2$;

(iii) the DARE

$$\begin{cases} \mathcal{P}=A^*\mathcal{P}A+C^*QC-K^*SK, \\ S=R+D^*QD+B^*\mathcal{P}B, \\ K=-S^{-1}(D^*QC+B^*\mathcal{P}A), \end{cases} \tag{1.33}$$

has a nonnegative solution $\mathcal{P}$.

If (iii) holds, then the smallest nonnegative solution is minimizing over all $u:\mathbf{N}\to U$.

There is a minimizing control over $\mathcal{U}_{\exp}$ iff the DARE has an exponentially stabilizing solution $\mathcal{P}_+$; such a solution is strictly minimizing over $\mathcal{U}_{\exp}$ and the greatest nonnegative solution of the DARE.

If Σ is exponentially detectable (e.g., $C^*C\gg 0$), then the DARE has at most one nonnegative solution, and such a solution is necessarily strictly minimizing over $\mathcal{U}_{\exp}$. □

In Section 15.5, we show that any strongly stabilizing solution of a positive DARE (or of the corresponding Riccati inequality) is the maximal one. We also study Riccati inequalities in the indefinite case.

Appendices A–F

In the appendices, we present mathematical knowledge that is necessary for a complete understanding of the proofs in the main part of this monograph. The readers unfamiliar with the theory of vector-valued functions might wish to have a glance at the beginnings of Appendices A, B and D before starting to read the main text, but most readers will probably visit the appendices only when in need to clarify some parts of the proofs in the main text.

Most of the appendices consists of vector-valued analogies of "well-known" scalar results, some of which are difficult to find in the literature even in the scalar case, whereas some of our results seem to be new even in the scalar case. Hopefully, the appendices can also serve as a reference for several results that have been commonly used in infinite-dimensional control theory without known references.

In the main text of this monograph, the vector spaces are assumed to be complex ($\mathbf{K} = \mathbf{C}$), but in the appendices, the scalar field $\mathbf{K}$ can be taken to be either $\mathbf{C}$ of $\mathbf{R}$ (in Appendix D and Sections A.4 and F.3, we assume that $\mathbf{K} = \mathbf{C}$, as explicitly stated there; in the other sections in the appendices we always state explicitly any such exceptions).

In Appendix A, we present standard definitions and several facts on algebra, topology and functional analysis, including several useful formulae for the inverses of operators between product spaces.

In Section B.1, we briefly present (Lebesgue) integration, differentiation, measurability and L^p and $\mathcal{C}$ function spaces. In the rest of Appendix B, we extend such concepts for functions with values in Banach spaces (we call such functions *vector-valued*). Our results include the density of finite-dimensional, smooth, compactly carried functions in vector-valued (Lebesgue) L^p spaces (even simultaneously for different p's and weight functions; see Theorem B.3.11), several integral inequalities and equalities (e.g., Theorems B.4.12 and B.4.16), certain product measurability results, differentiation formulae for integrals (Section B.5) and the basic theory of vector-valued Sobolev spaces (Section B.7).

In Appendix C, we briefly introduce vector-valued almost periodic functions.

In Appendix D, we study holomorphic vector-valued functions. This includes (Hardy) H^p spaces, Laplace and Fourier transforms and Poisson integral formulae. We also present some results on convolutions and on vector-valued measures.

In Appendix E, we present the Riesz–Thorin Interpolation Theorem, the Hausdorff–Young Theorem and similar results for vector-valued functions, with applications to control theory.

In Appendix F, we define spaces of *strongly measurable* functions ($f : Q \to \mathcal{B}(B, B_2)$, where $fx : Q \to B$ is (Bochner-)measurable for each $x \in B$) and *weakly measurable* functions ($\Lambda f x$ is measurable for each $x \in B$ and $\Lambda \in B_2^*$). In particular, we define and study $\mathrm{L}^p_{\text{strong}}$ and $\mathrm{L}^p_{\text{weak}}$ spaces (the main applications are contained in the above summary on Chapter 3) and $\mathrm{H}^p_{\text{strong}}$ and $\mathrm{H}^p_{\text{weak}}$ spaces (with applications in system theory). We also develop integration, convolution and Laplace transform theory for strongly or weakly measurable functions.

The completeness of $\mathrm{L}^\infty_{\text{strong}}$ (whereas $\mathrm{L}^p_{\text{strong}}$ is incomplete for $p < \infty$) and some $\mathrm{H}^p \cap \mathrm{H}^p$ type results at the end of Appendix D may be the deepest new results in the appendices, whereas many of the other results are more or less straight-forward generalizations and/or extensions of known facts.

1.3 Conventions

If we spoke a different language, we would perceive a somewhat different world.

— Ludwig Wittgenstein (1889–1951)

Most of the notation is explained at the point where it is used for the first time, and there is an extensive list of references, symbols, terms, abbreviations and acronyms at the end of this book (p. F.3). The correspondence of diagrams of systems to corresponding equations can be observed from Figure 6.1 (p. 155); in particular, inputs correspond to columns and outputs to rows, as in a matrix (and in [Sbook]).

Following the standard convention, in definitions we write *if* instead of *iff* (which means "if and only if"). An asterisk ("$*$") often denotes for something omitted (see the symbol list, p. 1038). By brackets ("$[\dots]$") we denote references (p. 1024) or optional parts; see p. 1037.

For clarity, we have chosen the "Blackboardbold" style to indicate the "integral" operators, e.g., a WPLS is of the form $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$. As a result, we have to use ordinary bold letters ($\mathbf{C}$, $\mathbf{R}$, $\mathbf{Z}$, $\mathbf{N} = \{0, 1, 2, \dots\}$) for standard fields of (complex, real, integer, natural) numbers.

The generators of $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ are denoted by $\left[\begin{array}{c|c} A & B \\ \hline C & \end{array}\right]$ or $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, as in Section 1.2 (alternatively, see Definition 6.1.1, Lemma 6.1.16 and Definition 6.2.3). Similarly, the generators (feedthrough operators) of any other integral maps (always Blackboardbold) will usually be denoted by corresponding ordinary (capital) letters. Note also the bars separating the different parts of the system; this is helpful when the parts consist of larger expressions.

The order of proofs

The "integral" notation (1.8) of a system allows us to treat continuous-time and discrete-time problems in a unified way. This allows us to transfer continuous-time results to discrete time with a minimal effort: it suffices to just list which parts are valid also in discrete time, with same proofs (see Theorem 13.3.13). In particular, within the discrete-time theory (Part IV), any references to continuous-time results refer to corresponding discrete-time variants (ones having undergone the substitutions (13.63)).

However, the proofs of certain results rest on the boundedness of "differential" or difference operators, hence they are given first for discrete time and then extended to continuous time by discretization (because discrete-time systems always have bounded generators). Such results include the uniqueness of the solution of the Riccati equation, the two-Riccati formula of the H^∞ Four-Block Problem and several results on stabilization.

Because of this, to verify the proofs of the whole monograph, one might wish to first read and verify the results in their discrete-time form, and only then in their continuous-time form (see Theorems 13.3.13, 14.1.3, 15.1.1, 11.5.2 and 12.2.2 and their proofs for details and other possible orders). Nevertheless, all results that are valid in both continuous and discrete time are first stated in continuous time, and we give the proofs in their continuous-time forms whenever reasonably possible; in such cases the discrete-time analogies are just references to the continuous-time results and proofs, as in Theorem 13.3.13.

Most readers may read the book "as is", but the reader wishing to have a deeper insight (or to understand all proofs) has to study also the discrete-time part in order to completely absorb the continuous-time part. Conversely, readers interested in discrete-time results only may skip things such as generators and regularity of continuous-time systems, as well as related complicated technical methods.

Trying to balance between the properties needed from a reference manual and those needed for a "chronological" order of proofs, we have grouped some clearly related results together, thus placing some results before those needed in their proofs; we have tried to clarify the order of proofs in those cases.

Proofs

The proofs often contain extra information: remarks, clarifications of ambiguous statements in the theorems, weaker or alternative assumptions, or "counter-examples" showing that our assumptions are not superfluous, etc.

We place a square ("□") at the end of each proof, and at the end of each lemma, proposition, theorem, corollary or remark whose proof is only sketched or replaced by a reference to some other result.

There are several algebraic basic results (e.g., the Schur decomposition of an (operator) matrix) that are often used in control theory without a further mention. We have compiled them to the Operator Matrix Lemma A.1.1, which has helped us make many proofs dramatically shorter, simpler and easier and the results more elegant than in the early versions of this book (you do not want to know...).

Notes

At the end of most sections, there is a "Notes" subsection containing further remarks and external references, including any earlier forms of similar results in the literature (known to us). However, we often refer to a "more up-to-date reference" instead of the first author.

When reading the notes to discrete-time sections, one should also consult the notes to corresponding continuous-time sections. Note also the historical remarks of Section 1.2.

Hypotheses

At the beginning of each chapter, we list any standing hypotheses and assumptions of the chapter or of its parts.

Outside the appendices, any Banach and Hilbert spaces are complex and of arbitrary dimensions unless otherwise stated. In the appendices, the scalar field $\mathbf{K}$ may be either of $\mathbf{R}$ or $\mathbf{C}$ except that in Appendix D and Sections A.4 and F.3 we assume that $\mathbf{K} = \mathbf{C}$, as explicitly stated there.

Part I

TI Operator Theory

Chapter 2

TI and MTI Operators

Gather ye rosebuds while ye may,
Old Time is still a-flying:
And this same flower that smiles today
Tomorrow will be dying.
— Robert Herrick (1591–1674)

Throughout this chapter, H, H_k, U, W, Y, Y_k and Z ($k \in \mathbf{N}$) denote Hilbert spaces of arbitrary dimensions (unless otherwise specified). (Many results of this chapter also hold for Banach spaces and for L^p in place of L^2; see [Sbook] for details.)

In Section 2.1 we shall study the basic theory of $\mathrm{TI}_\omega(U,Y)$, the bounded linear time-invariant operators $\mathrm{L}^2_\omega(\mathbf{R};U) \to \mathrm{L}^2_\omega(\mathbf{R};Y)$ ($\omega \in \mathbf{R}$). Section 2.2 treats the invertibility of TI_ω operators with emphasis on the causal ones, TIC_ω.

Section 2.3 lists sufficient conditions for a TIC operator to be static, that is, to be the multiplication operator induced by an element of $\mathcal{B}(U,Y)$. We also give certain results that will be used in connection with the *signature operators* of optimization problems, Riccati equations and spectral factorizations; such operators are further treated in Section 2.4. Section 2.5 treats the concept (J,S)-*losslessness*.

In Section 2.6 we define the subspace $\mathrm{MTI}_\omega(U,Y)$ (and its subspaces) corresponding to $\mathrm{TI}_\omega(U,Y)$ maps of form $u \mapsto \mu * u$, where μ is a measure consisting of a function $f \in \mathrm{L}^1_\omega(\mathbf{R};\mathcal{B}(U,Y))$ plus a discrete part (this includes the Callier–Desoer class and the Wiener class). We list the basic properties of these classes.

For most readers it suffices just to have a glance at subsections 2.1.1–2.1.7 and possibly also 2.6.3–2.6.4, and then return to this chapter only when pointed by a reference.

The book [RR] is a standard reference for TIC (or "causal shift-invariant") operators; see also [Nikolsky] and [Sbook]. (Due to Lemma 2.1.3, TIC (or TIC_∞) operators are sometimes called "Toeplitz operators"; see p. 56 for the correct definition.)

We shall sometimes refer to Chapter 3, which does not depend on this chapter except on the basic properties of TI and TIC operators.

2.1 Time-invariant operators (TI)

Weep no more, nor sigh, nor groan;
Sorrow calls no time that's gone;
Violets plucked the sweetest rain
Makes not fresh nor grow again.
— John Fletcher (1579–1625)

In this section, we define the class TI of time-invariant bounded linear operators $\mathrm{L}^2(\mathbf{R};*) \to \mathrm{L}^2(\mathbf{R};*)$ and several of its subclasses, and study their basic properties.

We start with notation. For any $\omega \in \mathbf{R}$, $p \in [1,\infty]$ and a measurable set $J \subset \mathbf{R}$, we set

$$\mathrm{L}^p_\omega(J;U) := \big\{ u \in \mathrm{L}^p_{\mathrm{loc}}(J;U) \;\big|\; \big(t \mapsto \mathrm{e}^{-\omega t}u(t)\big) \in \mathrm{L}^p(J;U) \big\} = \mathrm{e}^{\omega\cdot} \cdot \mathrm{L}^p(J;U); \tag{2.1}$$

in particular, $\|u\|_{\mathrm{L}^p_\omega} := \|\mathrm{e}^{-\omega\cdot}u\|_{\mathrm{L}^p}$. Thus, $\mathrm{e}^{\omega\cdot} := \mathrm{e}^{\omega\cdot}$ becomes an isometric isomorphism $\mathrm{L}^p \mapsto \mathrm{L}^p_\omega$, and we have (recall that $\tau^t u := \tau(t)u := u(\cdot + t)$, $\text{Я}u = u(-\cdot)$)

$$\tau(t)\mathrm{e}^{\omega\cdot} = \mathrm{e}^{\omega t}\mathrm{e}^{\omega\cdot}\tau(t), \quad \|\tau(t)u\|_{\mathrm{L}^p_\omega} = \mathrm{e}^{\omega t}\|u\|_{\mathrm{L}^p_\omega}, \ \ (t,\omega \in \mathbf{R},\ u \in \mathrm{L}^p_\omega(\mathbf{R};U)). \tag{2.2}$$

Moreover, $\pi_E u := \chi_E u$, where χ_E is the characteristic function of $E \subset \mathbf{R}$, $\pi_+ := \pi_{\mathbf{R}_+}$ and $\pi_- := \pi_{\mathbf{R}_-} = I - \pi_+$. It follows that

$$\text{Я}^{-1} = \text{Я} = \text{Я}^*,\ \pi_E^2 = \pi_E = \pi_E^*,\ \text{Я}\pi_E\text{Я} = \pi_{-E}, \tag{2.3}$$

$$\tau(t)^* = \tau(-t),\ \tau(t)\tau(s) = \tau(t+s),\ \text{Я}\tau(t) = \tau(-t)\text{Я},\ \pi_E\tau(t) = \tau(t)\pi_{E+t}, \tag{2.4}$$

and that any $L \in \mathcal{B}(U,Y)$ commutes with τ^t, Я and π_E in $\mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$.

We will often use the fact that if $\{u_n\} \subset \mathrm{L}^2_\alpha \cap \mathrm{L}^2_\omega$, $u_n \to u$ in L^2_α, and $u_n \to v$ in L^2_ω, then $u = v$ (a.e.), by Theorem B.3.2. On of its consequences is that $\mathrm{L}^2_\omega(\mathbf{R}_+;U) \subset \mathrm{L}^2_\alpha(\mathbf{R}_+;U)$, continuously, for $\omega \le \alpha$. Finally, we have

$$\|u\|_{\mathrm{L}^2_\omega} = \lim_{r \to \omega+} \|u\|_{\mathrm{L}^2_r} \quad (\omega \in \mathbf{R},\ u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)), \tag{2.5}$$

by the Monotone Convergence Theorem.

Definition 2.1.1 (TI, TIC) *Let* $\omega \in \mathbf{R}$. *We define* $\mathrm{TI}_\omega(U,Y)$ *to be the (closed) subspace of operators* $\mathbb{E} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U);\mathrm{L}^2_\omega(\mathbf{R};Y))$ *that are* time-invariant, *i.e.,* $\tau(t)\mathbb{E} = \mathbb{E}\tau(t)$ *for all* $t \in \mathbf{R}$.

We define $\mathrm{TIC}_\omega(U,Y)$ *to be the (closed) subspace of operators* $\mathbb{D} \in \mathrm{TI}_\omega(U,Y)$ *that are* causal, *i.e.,* $\pi_-\mathbb{D}\pi_+ = 0$, *or, equivalently,* $\mathbb{D}\pi_+\mathrm{L}^2_\omega \subset \pi_+\mathrm{L}^2_\omega$.

We set $\mathrm{TI} := \mathrm{TI}_0$, $\mathrm{TI}_\infty := \cup_{\omega\in\mathbf{R}}\mathrm{TI}_\omega$, $\mathrm{TI}_{\exp} := \cup_{\omega<0}\mathrm{TI}_\omega$, $\mathrm{TIC} := \mathrm{TIC}_0$, $\mathrm{TIC}_\infty := \mathrm{TIC} \cap \mathrm{TI}_\infty$ *and* $\mathrm{TIC}_{\exp} := \mathrm{TIC} \cap \mathrm{TI}_{\exp}$. *For* $\mathbb{E} \in \mathrm{TI}_\omega(U,Y)$ *we set* $\mathbb{E}^t := \pi_{[0,t)}\mathbb{E}\pi_{[0,t)} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$.

We call maps belonging to $\mathrm{TIC}_{\exp}$ exponentially stable, *those belonging to* $\mathrm{TIC} = \mathrm{TIC}_0$ stable, *and those belonging to* $\mathrm{TIC}_\infty \setminus \mathrm{TIC}$ unstable.

We extend any $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *as follows: if* $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)$ *and* $\pi_-u \in \mathrm{L}^2_\omega$, *then we set* $\pi_{(-\infty,T)}\mathbb{D}u := \pi_{(-\infty,T)}\mathbb{D}\pi_{(-\infty,T)}u$ *(*$T \in \mathbf{R}$*) (by causality,* $\mathbb{D}u$ *becomes uniquely defined (a.e.) on* $\mathbf{R}$*).*

By Lemma 2.1.10, we have $\mathrm{TIC} = \mathrm{TIC}_\infty \cap \mathrm{TI}$, $\mathrm{TIC}_\infty = \cup_{\omega\in\mathbf{R}}\mathrm{TIC}_\omega$, $\mathrm{TIC}_{\exp} = \cup_{\omega<0}\mathrm{TIC}_\omega$. See Remark 2.1.9 for the concept $\mathrm{TI}_a \cap \mathrm{TI}_b$ for $a \neq b$.

For any $\mathbb{D} \in \mathrm{TIC}_\omega$ we obviously have $\mathbb{D}\pi_+ = \pi_+\mathbb{D}\pi_+$, $\pi_-\mathbb{D} = \pi_-\mathbb{D}\pi_-$, and $\pi_{(-\infty,t)}\mathbb{D} = \pi_{(-\infty,t)}\mathbb{D}\pi_{(-\infty,t)}$. We will often use these facts as well as the fact that $\mathbf{Я}[\mathrm{L}^2_\omega(\mathbf{R};U)] = \mathrm{L}^2_{-\omega}(\mathbf{R};U)$.

Theorem 2.1.2 (Transfer functions) *Let* $\omega \in \mathbf{R}$. *For each* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$ *there is a unique function* $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$*, called the* transfer function *(or symbol or Laplace transform) of* $\mathbb{D}$*, s.t.* $\widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\hat{u}$ *on* $\mathbf{C}^+_\omega$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$*. The mapping* $\mathbb{D} \mapsto \widehat{\mathbb{D}}$ *is an isometric isomorphism of* $\mathrm{TIC}_\omega(U,Y)$ *onto* $\mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$. □

(The theorem is obtained from Theorem 2.3 of [W91a] by translation by ω (cf. Remark 2.1.6). That theorem also contains a similar claim for L^p ($1 \le p < \infty$) and Banach space U and Y, but in that case the isometric isomorphism onto becomes merely a linear injection into, by Example 3.3.4.)

Recall that $\hat{u}$ denotes the Laplace transform $\hat{u}(s) := \int_{\mathbf{R}} \mathrm{e}^{-st}u(t)\,dt$ of u.

We often identify functions and corresponding multiplication operators, i.e., we consider $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ both as a function and as an operator on $\mathcal{L}[\mathrm{L}^2_\omega(\mathbf{R}_+;U)] = \mathrm{H}^2(\mathbf{C}^+_\omega;U)$ (see Theorem 3.3.1(b)).

A causal time-invariant map $\mathbb{D} : \mathrm{L}^2(\mathbf{R};U) \supset \mathrm{Dom}(\mathbb{D}) \to \mathrm{L}^2(\mathbf{R};Y)$ is called *well-posed* iff $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, i.e., iff there are $\omega \in \mathbf{R}$ and $M < \infty$ s.t. $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \le M\|u\|_{\mathrm{L}^2_\omega}$ for all $u \in \mathrm{Dom}(\mathbb{D})$ and $\mathrm{Dom}(\mathbb{D}) \cap \mathrm{L}^2_\omega$ is dense in $\mathrm{L}^2_\omega(\mathbf{R};U)$.

Thus, if $\widehat{\mathbb{D}} \in \mathrm{H}(\Omega;\mathcal{B}(U,Y))$ for some open $\Omega \subset \mathbf{C}$, then the multiplication map $\hat{u} \mapsto \widehat{\mathbb{D}}\hat{u}$ determines a (necessarily unique) well-posed map $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ iff $\widehat{\mathbb{D}}$ is defined and bounded on some right half-plane (i.e., iff $\widehat{\mathbb{D}} \in \mathrm{H}^\infty_\infty := \cup_{\omega\in\mathbf{R}}\mathrm{H}^\infty(\mathbf{C}^+_\omega;*)$). Therefore, "well-posed" is an extension of the classical concept "proper" (recall that a proper rational function is one that is bounded at infinity). We shall study transfer functions of TIC_∞ maps in detail in Section 3.3 and those of TI_∞ maps in Section 3.1.

We conclude from Theorem 2.1.2 that the extension mentioned at the end of Definition 2.1.1 extends any $\mathbb{D} \in \mathrm{TIC}_\omega$ to a unique $\mathrm{TIC}_{\omega'}$ operator for each $\omega' > \omega$; we identify these two operators. Thus, $\mathrm{TIC}_\omega \subset \mathrm{TIC}_{\omega'}$. The TI_ω operators are not nested in a similar way, but they are also uniquely determined by any $\mathrm{TI}_{\omega'}$ to which they belong; see Remark 2.1.9 for details.

A causal map $\mathbb{D} \in \mathrm{TIC}_\infty$ is determined by its *Toeplitz operator* $\pi_+\mathbb{D}\pi_+$:

Lemma 2.1.3 *For each* $\mathbb{D}_+ \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R}_+;U),\mathrm{L}^2_\omega(\mathbf{R}_+;Y))$ *s.t.* $\tau^{-t}\mathbb{D} = \mathbb{D}\tau^{-t}$ *for* $t \ge 0$*, there is* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$ *s.t.* $\mathbb{D}_+ = \pi_+\mathbb{D}\pi_+$*. This correspondence is an isometric isomorphism.*

Proof: (Recall that we identify functions on $\mathbf{R}_+$ to their zero extensions, hence $\tau^{-t}f$ is zero on $[0,t)$ for each $f : \mathbf{R}_+ \to *$.)

Extend $\mathbb{D}_+$ to $\mathcal{B}(X,\mathrm{L}^2)$, where $X := \cup_{T\in\mathbf{R}}\mathrm{L}^2_\omega([T,+\infty);U)) \subset \mathrm{L}^2_\omega$, by $\mathbb{D}_+\tau^T u := \tau^T\mathbb{D}_+u$ for $T \ge 0$, $u \in \pi_+\mathrm{L}^2_\omega$. One easily verifies that the resulting operator is well-defined and time-invariant. Because this does not alter the norm of $\mathbb{D}_+$, by (2.2), we can extend $\mathbb{D}_+$ to L^2_ω, by density. By continuity, the resulting operator is time-invariant. The converse is obvious. $\square$

Definition 2.1.4 *If $\mathbb{E} \in \mathrm{TI}_\omega(U,Y)$ ($\omega \in \mathbf{R}$), then its* (noncausal) adjoint $\mathbb{E}^*$ *is the* $\mathrm{TI}_{-\omega}(Y,U)$ *map that satisfies*

$$\int_{\mathbf{R}} \langle (\mathbb{E}u)(t), y(t)\rangle\, dt = \int_{\mathbf{R}} \langle u(t), (\mathbb{E}^* y)(t)\rangle\, dt \quad (u \in \mathrm{L}^2_\omega(\mathbf{R};U),\ y \in \mathrm{L}^2_{-\omega}(\mathbf{R};Y)). \tag{2.6}$$

We call $\mathbb{E}^{\mathrm{d}} := \boldsymbol{Я}\mathbb{E}^*\boldsymbol{Я} \in \mathrm{TI}_\omega(Y,U)$ *the* causal adjoint *of* $\mathbb{E} \in \mathrm{TI}_\omega(U,Y)$.

(We used here the identity $\tau(t)\mathbb{E}^* = (\mathbb{E}\tau(-t))^* = (\tau(-t)\mathbb{E})^* = \mathbb{E}^*\tau(t)$.)

Obviously, it is enough to verify (2.6) for $u, y \in C_c^\infty(\mathbf{R};Y)$, by Theorem B.3.11. Note that $\mathbb{E}, \mathbb{E}' \in \mathrm{TI}_\omega \Rightarrow (\mathbb{E}\mathbb{E}')^* = \mathbb{E}'^*\mathbb{E}^*$. See also Lemma 2.1.10(b).

Note that the adjoint is not taken w.r.t. the L^2_ω inner product (which would imply $\mathbb{E}^* \in \mathrm{TI}_\omega(U,Y)$) but w.r.t. the $\mathrm{L}^2 = \mathrm{L}^2_0$ inner product (so that $\mathbb{E}^* \in \mathrm{TI}_{-\omega}(U,Y)$). This way the adjoint does not depend on the choice of ω.

By *duality* we usually mean that one applies known results to the duals (i.e., adjoints) of the operators involved (cf. also Lemma 6.1.4).

Lemma 2.1.5 *Let* $\omega \in \mathbf{R}$. *If* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, *then* $\mathbb{D}^{\mathrm{d}} \in \mathrm{TIC}_\omega(Y,U)$. *Let* $\mathbb{E} \in \mathrm{TI}_\omega(U,Y)$. *Then* $\mathbb{E} \in \mathcal{GB}(\mathrm{L}^2_\omega(\mathbf{R};U), \mathrm{L}^2_\omega(\mathbf{R};Y))$ *iff* $\mathbb{E} \in \mathcal{G}\mathrm{TI}_\omega(U,Y)$. $\square$

(We leave the simple proof to the reader.)

Thus, the inverse (if any) of a time-invariant map is necessarily time-invariant. The inverse of a causal map need not be causal (e.g., $\tau(-1) \in \mathrm{TIC}$, $\tau(1) = \tau(-1)^{-1} \in \mathrm{TI} \setminus \mathrm{TIC}$). However, the "causal adjoint" of a causal map is always causal, as shown in the lemma.

Remark 2.1.6 (Shifting stability) *Let* $\alpha, \omega \in \mathbf{R}$. *Let* $\mathcal{T}_\alpha$ *be the* stability shift *(or scaling operator)* $\mathbb{E} \mapsto e^{\alpha\cdot}\mathbb{E}e^{-\alpha\cdot}$. *Then* $\mathcal{T}_\alpha$ *is an isometric isomorphism of* TI_ω *onto* $\mathrm{TI}_{\omega+\alpha}$ *and of* TIC_ω *onto* $\mathrm{TIC}_{\omega+\alpha}$ *(because* $[\pi_+]\mathrm{L}^2_{\omega+\alpha} = e^{\alpha\cdot}[\pi_+]\mathrm{L}^2_\omega$, *isometrically).*

Obviously, $\mathcal{T}_\alpha\pi_\pm = \pi_\pm\mathcal{T}_\alpha$, $\mathcal{T}_\alpha\tau(t) = \tau(t)\mathcal{T}_\alpha$ *($t \in \mathbf{R}$), and we have*

$$\mathcal{T}_\alpha(\mathbb{E}\mathbb{F}) = (\mathcal{T}_\alpha\mathbb{E})(\mathcal{T}_\alpha\mathbb{F}), \qquad \mathcal{T}_\alpha(\beta\mathbb{E} + \gamma\mathbb{F}) = \beta\mathcal{T}_\alpha\mathbb{E} + \gamma\mathcal{T}_\alpha\mathbb{F}, \tag{2.7}$$

$$(\mathcal{T}_\alpha\mathbb{E})^{-1} = \mathcal{T}_\alpha\mathbb{E}^{-1}, \qquad (\mathcal{T}_\alpha\mathbb{E})^* = \mathcal{T}_{-\alpha}\mathbb{E}^*, \tag{2.8}$$

$$(\mathcal{T}_\alpha\mathbb{E})^d = \mathcal{T}_\alpha\mathbb{E}^d, \qquad \widehat{\mathcal{T}_\alpha\mathbb{E}} = \tau(-\alpha)\widehat{\mathbb{E}}. \tag{2.9}$$

$\square$

(The formula $\widehat{\mathcal{T}_\alpha\mathbb{E}} = \widehat{\mathbb{E}}(\cdot - \alpha)$ refers to Theorem 3.1.3(a1); for $\mathbb{E} \in \mathrm{TIC}_\infty$ it also covers Theorem 2.1.2.)

Note that $\alpha > 0$ decreases stability, i.e., shifts the transfer function to the right. See also Remark 6.1.9.

If $\mathbb{D} \in \mathrm{TI}$ is causal, then, obviously, $\mathbb{D}^*$ is *anti-causal*, i.e., $\pi_+\mathbb{D}^*\pi_- = 0$. If $\mathbb{D}$ is both causal and anti-causal, then we call $\mathbb{D}$ *static*. We identify $D \in \mathcal{B}(U,Y)$ and the multiplication map $M_D : u \mapsto Du$ (note that $M_D \in \mathrm{TIC}(U,Y)$, $\mathrm{M}_D^{-1} = \mathrm{M}_{D^{-1}}$ if either inverse exists, and $\mathrm{M}_D^* = \mathrm{M}_{D^*}$). The static TIC maps are exactly the (multiplication) maps of this form:

Lemma 2.1.7 (Static $\mathbb{D}$) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $\mathbb{D}^* \in \mathrm{TIC}_\infty(Y,U)$. Then $\mathbb{D} \in \mathcal{B}(U,Y)$. Moreover, the imbedding $\mathcal{B} \mapsto \mathrm{TIC}_\omega$ preserves norms and commutes with algebraic operations (for any $\omega \in \mathbf{R}$).*

Thus, $\mathbb{D} \in \mathrm{TIC}_\infty \,\&\, \pi_+\mathbb{D}\pi_- = 0 \Rightarrow \mathbb{D} \in \mathcal{B}$. See Section 2.3 for more on static operators.

Proof: For $\mathbb{D}, \mathbb{D}^* \in \mathrm{TIC}$, this is stated in [RR, Theorem 5.2C, p. 96] (we do not know their proof; a proof consists of Proposition D.1.20 combined with Theorems 2.1.2 and 3.3.1; see [Sbook] for a more general result and its system-theoretic proof).

In the general case, where $\mathbb{D}, \mathbb{D}^* \in \mathrm{TIC}_\omega$ for some $\omega \in \mathbf{R}$, we have $\mathbb{D}_{-\omega} := \mathcal{T}_{-\omega}\mathbb{D} \in \mathrm{TIC}_0$ and $(\mathbb{D}_{-\omega})^* = \mathcal{T}_\omega\mathbb{D}^* \in \mathrm{TIC}_0$, hence $D := \mathbb{D}_{-\omega} \in \mathcal{B}(U,Y)$ and $\mathbb{D} = \mathcal{T}_\omega\mathbb{D}_{-\omega} = D$. $\square$

The *Hankel operator*[1] $\pi_+ \cdot \pi_-$ of a TIC_∞ map determines the map uniquely modulo a static operator:

Corollary 2.1.8 ($\pi_+\mathbb{D}\pi_-$) *Let $\mathbb{D}, \widetilde{\mathbb{D}} \in \mathrm{TIC}_\infty(U,Y)$. Then $\pi_+\mathbb{D}\pi_- = \pi_+\widetilde{\mathbb{D}}\pi_-$ iff $\mathbb{D} = \widetilde{\mathbb{D}} + D$ for some $D \in \mathcal{B}(U,Y)$.* $\square$

(Apply Lemma 2.1.7 to $\mathbb{D} - \widetilde{\mathbb{D}}$.) Note that it suffices that $\pi_+\mathbb{D}\pi_-\phi = \pi_+\widetilde{\mathbb{D}}\pi_-\phi$ for all $\phi \in \mathcal{C}_c^\infty$, by Theorem B.3.11.

As noted above, a TIC_a map is also a (i.e., extends to a unique) TIC_b map for any $b > a$. If the restriction of a TI_a map onto $\mathrm{L}^2_b \cap \mathrm{L}^2_a$ is continuous in the L^2_b norm (to L^2_b), then it extends to a unique TI_b map (and a unique TI_r map for any $r \in [a,b]$), as will be noted in the remark and lemma below. The corresponding technical details are given in Proposition E.1.8.

Remark 2.1.9 ($\mathbf{TI}_a \cap \mathbf{TI}_b$) *Let $a < b$. Assume that $\mathcal{F}$ is the set of simple functions $\mathbf{R} \to U$, or $\mathcal{F} = \mathrm{L}^2_a(\mathbf{R};U) \cap \mathrm{L}^2_b(\mathbf{R};U)$ or $\mathcal{F} = \mathcal{C}_c^\infty(\mathbf{R};U)$.*

If(f) $\mathbb{E} \in \mathcal{F} \to \mathrm{L}^2_a(\mathbf{R};Y)$ is time-invariant, linear and bounded $\mathrm{L}^2_a \to \mathrm{L}^2_a$ and $\mathrm{L}^2_b \to \mathrm{L}^2_b$, or equivalently, $\mathbb{E}$ is linear and there is $M < \infty$ s.t.

$$\|\mathbb{E}\phi\|_{\mathrm{L}^2_a} \le M\|\phi\|_{\mathrm{L}^2_a},\ \ \|\mathbb{E}\phi\|_{\mathrm{L}^2_b} \le M\|\phi\|_{\mathrm{L}^2_b} \text{ and } \mathbb{E}\tau(t)\phi = \tau(t)\mathbb{E}\phi \text{ for all } t \in \mathbf{R},\ \phi \in \mathcal{F}, \tag{2.10}$$

then $\mathbb{E}$ extends to a unique operator $\mathbb{E} \in \mathrm{TI}_r(U,Y)$ for all $r \in [a,b]$, and $\mathbb{E}$ is uniquely defined on $\cup_{r\in[a,b]}\mathrm{L}^2_r(\mathbf{R};U)$, by Proposition E.1.8.

Therefore, if $\mathbb{E}_a \in \mathrm{TI}_a(U,Y)$ and $\mathbb{E}_b \in \mathrm{TI}_b(U,Y)$ are s.t. $\mathbb{E}_a = \mathbb{E}_b$ on $\mathcal{F}$, then we identify $\mathbb{E}_a$ and $\mathbb{E}_b$.

Thus, by "$\mathbb{E} \in \mathrm{TI}_a(U,Y) \cap \mathrm{TI}_b(U,Y)$" we mean such a map (i.e., $\mathbb{E} = \mathbb{E}_a = \mathbb{E}_b$, where $\mathbb{E}_a$ and $\mathbb{E}_b$ are as above).

An alternative characterization is that $\mathbb{E} : \mathrm{L}^2_a(\mathbf{R};U) + \mathrm{L}^2_b(\mathbf{R};U) \to \mathrm{L}^2_a(\mathbf{R};Y) + \mathrm{L}^2_b(\mathbf{R};Y)$ is linear and s.t. $\mathbb{E}|_{\mathrm{L}^2_a} \in \mathcal{B}(\mathrm{L}^2_a, \mathrm{L}^2_a)$, $\mathbb{E}|_{\mathrm{L}^2_b} \in \mathcal{B}(\mathrm{L}^2_b, \mathrm{L}^2_b)$ and $\mathbb{E}\tau^t = \tau^t\mathbb{E}$ for all $t \in \mathbf{R}$ (see Definition E.1.3). $\square$

[1] Sometimes $\pi_-\mathbb{D}\pi_+$ is called the Hankel operator and $\pi_+\mathbb{D}\pi_-$ the anti-Hankel operator of $\mathbb{D}$; our choice is due to [S97b].

If $\mathbb{E}$ is a linear map whose restriction to $\mathrm{Dom}(\mathbb{E}) \cap \mathrm{L}^2_a$ extends to a unique TI_a map (as above), then we identify this TI_a map with $\mathbb{E}$.

By translation-invariance and density, it is enough to verify (2.10) for functions $\phi \in \mathcal{F}$ having their support on, e.g., $\mathbf{R}_+$.

Lemma 2.1.10 ($\mathbf{TI}_a \cap \mathbf{TI}_b$) *Let $\mathbb{E} \in \mathrm{TI}_a(U,Y) \cap \mathrm{TI}_b(U,Y)$, $a < b$. Then the following hold:*

(a1) $\mathbb{E} \in \mathrm{TI}_r(U,Y)$ for all $r \in [a,b]$, and

$$\|\mathbb{E}\|_{\mathrm{TI}_r} =: M_r \leq M_a^{1-\theta_r} M_b^{\theta_r} \leq \max\{M_a, M_b\} \quad (r \in [a,b]), \tag{2.11}$$

where $\theta_r := (r-a)/(b-a)$.

(a2) If $\mathbb{E} \in \mathrm{TIC}_\infty(U,Y)$, then $\mathbb{E} \in \mathrm{TIC}_r(U,Y)$ for all $r \in [a,\infty)$.

(b) $\mathbb{E}^* \in \mathrm{TI}_{-r}(Y,U)$, $\mathbb{E}^{\mathrm{d}} \in \mathrm{TI}_r(Y,U)$ and $\mathbb{E}^{-1} \in \mathrm{TI}_r(Y,U)$ are independent of r.

(c) We have $\mathbb{F} = \mathbb{G} \in \mathrm{TI}_a \cap \mathrm{TI}_b$ if $\mathbb{F} \in \mathrm{TI}_a$, $\mathbb{G} \in \mathrm{TI}_b$, $\mathbb{F}\phi = \mathbb{G}\phi$ for all $\phi \in \mathcal{F}$, and $\mathcal{F}$ is s.t. the translations of $\mathcal{F}$ span a dense subset of both L^2_a and L^2_b (e.g., $\mathcal{F}$ is as in Remark 2.1.9 or $\mathcal{F} = \{\chi_{[0,1]}\}$ or $\mathcal{F} = \{\mathrm{e}^{-\cdot/2}\}$ (by Lemma D.1.25)).

(d) Let $\mathbb{D} \in \mathrm{TIC}_r(U,Y)$, $r \in \mathbf{R}$. Then $\mathbb{D} \in \mathrm{TIC}_{r'}(U,Y)$ and $\|\mathbb{D}\|_{\mathrm{TIC}_{r'}} \leq \|\mathbb{D}\|_{\mathrm{TIC}_r}$ for all $r' \geq r$. Moreover, $\mathbb{D}^* \in \mathrm{TI}_{-r'}(Y,U)$, $\mathbb{D}^{\mathrm{d}} \in \mathrm{TIC}_{r'}(Y,U)$ and $\mathbb{D}^{-1} \in \mathrm{TI}$.

(e) Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, and let $\omega \in \mathbf{R}$. Then $\mathbb{D} \in \mathrm{TIC}_\omega$ iff $\mathbb{D}u \in \mathrm{L}^2_\omega$ for all $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$.

(f) Let $T \in \mathbf{R}$, $\widetilde{\mathbb{E}} \in \mathrm{TI}_a$ and $\mathbb{D} \in \mathrm{TIC}_a$. Let $\omega \in \mathbf{R}$. Then,

$$\|\widetilde{\mathbb{E}}\|_{\mathrm{TI}_\omega} := \|\widetilde{\mathbb{E}}\|_{\mathcal{B}(\mathrm{L}^2_\omega)} = \|\pi_{[T,\infty)}\widetilde{\mathbb{E}}\pi_{[T,\infty)}\|_{\mathcal{B}(\mathrm{L}^2_\omega)} = \lim_{t\to+\infty} \|\widetilde{\mathbb{E}}^t\|_{\mathcal{B}(\mathrm{L}^2_\omega)} \tag{2.12}$$

$$= \sup_{u \in C_c^\infty(\mathbf{R}_+;U),\, \|u\|_{\mathrm{L}^2_\omega} \leq 1} \|\pi_{[T,\infty)}\widetilde{\mathbb{E}}u\|_{\mathrm{L}^2_\omega}, \qquad \textit{and} \tag{2.13}$$

$$\|\mathbb{D}\|_{\mathrm{TIC}_\omega} := \|\mathbb{D}\|_{\mathcal{B}(\mathrm{L}^2_\omega)} = \lim_{t\to+\infty} \|\mathbb{D}^t\|_{\mathcal{B}(\mathrm{L}^2_\omega)} \tag{2.14}$$

$$= \lim_{r\to\omega+} \|\mathbb{D}\|_{\mathrm{TIC}_r} = \sup_{r>\omega} \|\mathbb{D}\|_{\mathrm{TIC}_r} \tag{2.15}$$

(the norms may be infinite for $\omega \neq a$; recall that $\widetilde{\mathbb{E}}^t := \pi_{[0,t)}\widetilde{\mathbb{E}}\pi_{[0,t)}$).

(g) Let $\mathbb{F} \in \mathrm{TI}_r$ for all $r \in (a,b)$. Then $M := \sup_{r\in(a,b)} \|\mathbb{F}\|_{\mathrm{TI}_r} < \infty$ iff $\mathbb{F} \in \mathrm{TI}_a \cap \mathrm{TI}_b$. If this is the case, then $M = \max\{\|\mathbb{F}\|_{\mathrm{TI}_a}, \|\mathbb{F}\|_{\mathrm{TI}_b}\}$.

To give the reader a better intuition on TI_∞ operators, we mention a few results that will be shown later: the Fourier or Laplace transform ("transfer function" or "symbol") of a $\mathrm{TI}_a(U,Y)$ map is in $\mathrm{L}^\infty_{\mathrm{strong}}(a+i\mathbf{R};\mathcal{B}(U,Y))$ (Theorem 3.1.3), that of a $\mathrm{TI}_a \cap \mathrm{TI}_b$ map is also in $\mathrm{H}((a,b);\mathcal{B}(U,Y))$ (Theorem 3.1.6), and that of a TIC_a map also in $\mathrm{H}^\infty(\mathbf{C}^+_a;\mathcal{B}(U,Y))$ (Theorem 2.1.2).

Proof: Recall that simple functions (hence $\mathrm{L}^2_a \cap \mathrm{L}^2_b$) are dense in L^2_r for $r \in [a,b]$, by Theorem B.3.11.

(a1) See Proposition E.1.8 and Remark 2.1.9.

(a2) Now $\pi_-\mathbb{E}\pi_+ = 0$ on any L^2_r ($r \in [a,b]$), by continuity. From (d) we get that $\mathbb{E} \in \mathrm{TIC}_r$ for any $r \geq a$.

(b) It is obvious from (2.6) that $(\mathbb{E}_r)^* = (\mathbb{E}_a)^*$ on $\mathrm{L}^2_{-a} \cap \mathrm{L}^2_{-r}$, hence $\mathbb{E}_r^* = \mathbb{E}_a^*$, (by definition (or by (c)).

Consequently, $\boldsymbol{Я}\mathbb{E}_a^*\boldsymbol{Я} = \boldsymbol{Я}\mathbb{E}_r^*\boldsymbol{Я}$ (on $\mathrm{L}^2_a \cap \mathrm{L}^2_r$).

Finally, if $\mathbb{E}$ is invertible in TI_{r_k} and in TI_{r_k} ($r_k \in [a,b]$) for $k = 1,2$, then $\mathbb{E}^{-1}$ maps $\mathrm{L}^2_{r_1} \cap \mathrm{L}^2_{r_2}$ onto itself (as does $\mathbb{E}$ too); in particular, $\mathbb{E}_{r_1}^{-1} = \mathbb{E}_{r_2}^{-1}$ (on $\mathrm{L}^2_{r_1} \cap \mathrm{L}^2_{r_2}$).

(c) This follows almost directly from Remark 2.1.9 (the Fourier transforms of $\chi_{[0,1]}\}$ and $e^{-\cdot/2}$ are $\neq 0$ a.e.; cf. Lemma D.1.25).

(d) By Theorem 2.1.2, $\mathbb{D}$ extends to a $\mathrm{TIC}_{r'}$ map for $r' \geq r$. The rest follows from (b).

(Note that $\tau(-1) \in \mathrm{TIC} \cap \mathcal{G}\mathrm{TI} \setminus \mathcal{G}\mathrm{TIC}_\infty$, since $\tau(1)$ is noncausal.)

(e) "Only if" is trivial, so assume that $\mathbb{D} \in \mathrm{TIC}_\alpha$, $\alpha \geq \omega$, and $\mathbb{D}u \in \mathrm{L}^2_\omega$ for all $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$. The continuous inclusion $\pi_+\mathrm{L}^2_\omega \subset \mathrm{L}^2_\alpha$ implies that $\mathbb{D}u \in \mathcal{B}(\pi_+\mathrm{L}^2_\omega, \mathrm{L}^2_\alpha)$, hence $\mathbb{D}u \in \mathcal{B}(\pi_+\mathrm{L}^2_\omega, \mathrm{L}^2_\omega)$, by Lemma A.3.6. Thus, (2.10) is satisfied, hence $\mathbb{D} \in \mathrm{TI}_\omega$. By (a2), $\mathbb{E} \in \mathrm{TIC}_\omega$.

(f) Obviously, we can w.l.o.g. assume that $\omega = 0$ (cf. Remark 2.1.6) and $T = 0$ (use time-invariance). By Remark 2.1.9, we have

$$\|\widetilde{\mathbb{E}}\| = \sup_{u \in C_c^\infty(\mathbf{R};U),\, \|u\|_{\mathrm{L}^2_\omega} \leq 1} \|\mathbb{E}u\|_{\mathrm{L}^2_\omega}. \tag{2.16}$$

Assume, that $M < \|\widetilde{\mathbb{E}}\|$. Choose $u \in C_c^\infty(\mathbf{R};U)$ s.t. $\|u\|_2 = 1$ and $\|\widetilde{\mathbb{E}}u\| > M$. By Corollary B.3.8, we have $\|\pi_+\widetilde{\mathbb{E}}\tau^{-t}u\| > M$ for t big enough; take t so big that we also have $\tau^{-t}u \in C_c^\infty(\mathbf{R}_+;U)$ to establish (2.13). Obviously, (2.12) follows from (2.13).

The first two equalities for $\|\mathbb{D}\|$ follow from the above. Also the $\mathbb{D}^t$ claim follows from Corollary B.3.8, so only (2.15) remains to be proved.

Let $\|u\|_{\mathrm{L}^2} = 1$. Then $\|\mathbb{D}u\|_{\mathrm{L}^2_r}/\|u\|_{\mathrm{L}^2_r} \to \|\mathbb{D}u\|_{\mathrm{L}^2}$, by (2.5), hence $\|\mathbb{D}\| \leq \sup_{r>\omega}\|\mathbb{D}\|_{\mathrm{TIC}_r}$. But $\|\mathbb{D}\|_{\mathrm{TIC}_r}$ is decreasing in r, by (d), hence $\|\mathbb{D}\|$ is given by (2.15).

(g) (In fact, we may replace (a,b) by any $D \subset \mathbf{R}$ s.t. $a, b \in \bar{D}$.) This follows from Lemma E.1.9. $\square$

If(f) $\|\mathbb{D}\|_{\mathrm{TI}_\alpha}$ is (finite and) bounded, as $\alpha \to +\infty$, then $\mathbb{D}$ is causal:

Lemma 2.1.11 *Let $\mathbb{D} \in \mathrm{TI}_\omega(U,Y)$, $\omega \in \mathbf{R}$. Then the following are equivalent:*

(i) $\mathbb{D} \in \mathrm{TIC}_\alpha$ for some $\alpha \geq \omega$;

(ii) $\mathbb{D} \in \mathrm{TIC}_\alpha$ for all $\alpha \geq \omega$;

(iii) $\mathbb{D} \in \mathrm{TI}_\alpha$ and $\|\mathbb{D}\|_{\mathrm{TI}_\alpha} \leq \|\mathbb{D}\|_{\mathrm{TI}_\omega}$ for all $\alpha \geq \omega$;

(iv) $\mathbb{D} \in \mathrm{TI}_\alpha$ for all $\alpha \geq \omega$, and $e^{-\alpha\varepsilon}\|\mathbb{D}\|_{\mathrm{TI}_\alpha} \to 0$ for all $\varepsilon > 0$.

Thus, if $\|\mathbb{D}\|_{\mathrm{TI}_\alpha}$ does not grow with an exponential speed, as $\alpha \to +\infty$ (cf. (iv)), then it is bounded and $\mathbb{D}$ is causal.

On the other hand, $\tau^t \in \mathrm{TI}_\omega(U)$ for all $t, \omega \in \mathbf{R}$, but $\tau^t \notin \mathrm{TIC}_\infty$ for $t > 0$ (τ^t is not causal because it maps backwards in time), and $\|\tau^t\|_{\mathrm{TI}_\alpha} = e^{\alpha t}$, by (2.2), so the estimate in (iv) is the "best possible one".

Proof: 1° *"(ii)⇒(i)" and "(iii)⇒(iv)":* This is trivial.

2° *"(i)⇒(ii)&(iii)":* This follows from Lemma 2.1.10(d).

3° *"(iv)⇒(i)":* Assume that (i) does not hold, so that $y := \pi_-\mathbb{D}\pi_+u \neq 0$ for some $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$.

Choose $\varepsilon > 0$ s.t. $r := \|\pi_{(-\infty,-\varepsilon)}y\|_{\mathrm{L}^2_\omega} > 0$. Then, by (2.2),

$$\|y\|_{\mathrm{L}^2_\alpha} := \|e^{-\alpha\cdot}y\|_2 \geq e^{(\alpha-\omega)\varepsilon}\|y\|_{\mathrm{L}^2_\omega} \geq e^{\alpha\varepsilon}\delta', \tag{2.17}$$

where $\delta' := e^{-\omega\varepsilon}r > 0$. Consequently, $\|\mathbb{D}\|_{\mathrm{TI}_\alpha} \geq e^{\alpha\varepsilon}\delta$ ($\alpha \geq \omega$), where $\delta := \delta'/\|u\|_{\mathrm{L}^2_\omega} > 0$, because $\|u\|_{\mathrm{L}^2_\omega} \geq \|u\|_{\mathrm{L}^2_\alpha} > 0$ ($\alpha \geq \omega$).

It follows that $e^{-\alpha\varepsilon}\|\mathbb{D}\|_{\mathrm{TI}_\alpha} \geq \delta \not\to 0$, so that (iv) does not hold. Therefore, (iv) implies (i). □

Lemma 2.1.12 ($\mathbb{D}$ is closed) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $\omega \in \mathbf{R}$. Then $\mathbb{D}$ is closed on $\mathrm{L}^2_\omega(\mathbf{R}_+;U)$ (here we set $\mathrm{Dom}(\mathbb{D}) := \{u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U) \mid \mathbb{D}u \in \mathrm{L}^2_\omega\}$).*

Proof: Choose $\alpha > \omega$ s.t. $\mathbb{D} \in \mathrm{TIC}_\alpha$. If $u_n \to u$ and $\mathbb{D}u_n \to y$ in L^2_ω, then $u_n \to u$ in L^2_α, hence $\mathbb{D}u_n \to \mathbb{D}u$ in L^2_α, hence $y = \mathbb{D}u$ a.e., by Theorem B.3.2. □

We set $\mathrm{L}^2_c(\mathbf{R};U) := \{u \in \mathrm{L}^2(\mathbf{R};U) \mid \operatorname{supp} u \text{ is bounded}\}$. If $\mathbb{D}[\mathrm{L}^2_c] \subset \mathrm{L}^2$, then $\mathbb{D}$ is "almost stable" (cf. Lemma 6.1.11 and Theorem 3.3.1(a4)&(c3)):

Lemma 2.1.13 ($\mathbb{D}\mathrm{L}^2_c \subset \mathrm{L}^2$) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $\omega \in \mathbf{R}$, and let there be $T > 0$ s.t. $\mathbb{D}\pi_{[0,T)}u \in \mathrm{L}^2_\omega$ for $u \in \mathrm{L}^2_c$. Then $M := \|\mathbb{D}\pi_{[0,T)}\|_{\mathcal{B}(\mathrm{L}^2_\omega,\mathrm{L}^2_\omega)} < \infty$, and*

$$\|\mathbb{D}\|_{\mathrm{TIC}_\beta} \leq M_{\beta,\omega,T}M \qquad (\beta > \omega), \tag{2.18}$$

$$\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \leq M_{\alpha,\omega,T}M\|u\|_{\mathrm{L}^2_\alpha} \qquad (\alpha < \omega,\ u \in \mathrm{L}^2_\alpha(\mathbf{R}_+;U)). \tag{2.19}$$

Thus, then $\mathbb{D}[\mathrm{L}^2_c] \subset \mathrm{L}^2_\omega$ and $\mathbb{D}\pi_{[0,t)}u \to \mathbb{D}u$ and $\mathbb{D}^t u \to \mathbb{D}u$ in L^2_ω for all $u \in \mathrm{L}^2_\alpha(\mathbf{R}_+;U) + \mathrm{L}^2_c$, $\alpha < \omega$. Moreover, $(s+r+\omega)^{-1}\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ for all $r > 0$; in particular, $\widehat{\mathbb{D}}(\frac{1-\cdot}{1+\cdot}) \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D};\mathcal{B}(U,Y))$ if $\omega < 1$.

(See Definition 13.2.2 for $\widehat{\heartsuit}$.)

Proof: (In fact, the lemma holds even for $\mathrm{TIC}_{\mathrm{loc}}$ (see Section 8 of [Sbook] for the definition) in place of TIC_∞, with virtually the same proof.)

Except for the $\mathrm{H}^2_{\mathrm{strong}}$ claims, this follows from Lemma 13.1.3 through discretization. (Recall that $\mathrm{L}^2_c(\mathbf{R};U) = \cup_{T>0}\mathrm{L}^2([-T,T);U)$.)

Since $\widehat{\mathbb{D}}(s+r+\omega)^{-1}u_0 \in \mathrm{H}^2_\omega$ for all $u_0 \in U$ (note that $\mathcal{L}^{-1}(s+r+\omega)^{-1}u_0 \in \mathrm{L}^2_{\omega-r/2}$; see also Theorem 3.3.1(b) and Theorem 2.1.2) we have $(s+r+\omega)^{-1}\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$, by Lemma F.3.3(a1).

Set $t := r+\omega$ to observe from Lemma 13.2.1(e2) that $z \mapsto (1+z)[\frac{1-z}{1+z} + t]\widehat{\heartsuit}\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D};\mathcal{B}(U,Y))$, i.e., $[1+t+(t-1)z]\widehat{\heartsuit}\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D};\mathcal{B}(U,Y))$, for all $t > \omega$. If $\omega < 1$, we can take $t = 1$ to obtain that $\widehat{\heartsuit}\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D};\mathcal{B}(U,Y))$. □

Lemma 2.1.14 ($\mathbb{D}^{t*}\mathbb{D}^t \le \gamma^2 I \Rightarrow \|\mathbb{D}\|_{\mathrm{TIC}} \le \gamma$) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $\gamma \in \mathbf{R}$. If*

$$\mathbb{D}^{t*}\mathbb{D}^t \le \gamma^2 I \quad \text{for all } t > 0, \tag{2.20}$$

then $\mathbb{D} \in \mathrm{TIC}(U,Y)$ and $\|\mathbb{D}\|_{\mathrm{TIC}} \le \gamma$. If $J \gg 0$ is s.t. that $\mathbb{D}^{t*}J\mathbb{D}^t \le \gamma^2 I$ for all $t > 0$, then $\mathbb{D} \in \mathrm{TIC}(U,Y)$. □

Proof: (Obviously, the theorem also holds with TI in place of TIC.)

The first claim follows from (2.14). If $J \ge \varepsilon I$, $\varepsilon > 0$, then $\mathbb{D}^{t*}J\mathbb{D}^{t*} \ge \varepsilon\mathbb{D}^{t*}\mathbb{D}^{t*}$, hence also the second claim holds. □

Lemma 2.1.15 ($(\mathbb{D}u)(t) = e^{st}\widehat{\mathbb{D}}(s)u_0$) *Let $\mathbb{D} \in \mathrm{TIC}_\omega$ and let $\mathrm{Re}\, s > \omega$. If $u_0 \in U$ and $u = e^{s\cdot}u_0$, then $(\mathbb{D}u)(t) = e^{st}\widehat{\mathbb{D}}(s)u_0$ for all $t \in \mathbf{R}$.* □

(This is Lemma 6.10 of [S98c], originating from [W91a].) Note that $\pi_- u \in \mathrm{L}^2_\omega$, and that $\mathbb{D}$ was extended as explained in Definition 2.1.1.

Notes

Theorem 2.1.2 is from [W91a]. Corollary 2.1.8, Lemma 2.1.15 and much of Definitions 2.1.1 and 2.1.4 and of our notation are from [S97a] and [S98c]. The case $\omega = 0$ of Lemma 2.1.7 is Theorem 5.2C on p. 96 [RR].

A standard reference for the theory on time-invariant operators is [RR].

The main new contributions of this section are the theory of $\mathrm{TI}_\omega \cap \mathrm{TI}_{\omega'}$ (see 2.1.9–2.1.11 and 3.1.6, and the fact that $\mathbb{D}$ is "almost stable" if $\mathbb{D}\pi_{[0,T)}\mathrm{L}^2 \subset \mathrm{L}^2$ (see Lemma 2.1.13 and the references above it).

2.2 $\mathcal{G}$TIC — invertibility

And that inverted Bowl we call The Sky,
Whereunder crawling coop't we live and die,
Lift not thy hands to It for help – for It
Rolls impotently on as Thou or I.
— Omar Khayyam (1048–1131)

In this section we study the invertibility properties of TI_∞ operators. Invertibility and left invertibility (and right invertibility, by duality) are further studied in Chapter 4. We start from the general (noncausal) case:

Lemma 2.2.1 *Let U,Y be Hilbert spaces, $\omega \in \mathbf{R}$, and $\varepsilon > 0$.*

(a1) Let $\mathbb{E} \in \mathrm{TI}(U,Y)$. Then $\mathbb{E}^*\mathbb{E} \gg 0$ iff $\mathbb{X}\mathbb{E} = I$ for some $\mathbb{X} \in \mathrm{TI}(Y,U)$.

(a2) Let $\mathbb{E} \in \mathrm{TI}_\omega(U,Y)$. Then $\mathbb{E} \in \mathcal{G}\mathrm{TI}_\omega \Leftrightarrow \mathbb{E} \in \mathcal{G}\mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U), \mathrm{L}^2_\omega(\mathbf{R};Y))$.

(b) $\mathbb{X}, \mathbb{E} \in \mathrm{TI}_\omega(\mathbf{C}^n)$ & $\mathbb{X}\mathbb{E} = I \Longrightarrow \mathbb{E}\mathbb{X} = I$.

(c1) $\mathbb{E} \in \mathrm{TI}(\mathbf{C}^n)$ & $\mathbb{E}^*\mathbb{E} \gg 0 \Longrightarrow \mathbb{E} \in \mathcal{G}\mathrm{TI}(\mathbf{C}^n)$ & $\mathbb{E}\mathbb{E}^* \gg 0$.

(c2) $\mathbb{E} \in \mathrm{TI}(U,\mathbf{C}^n)$ & $\mathbb{E}^*\mathbb{E} \gg 0 \Longrightarrow \dim U \le n$ (and $\mathbb{E} \in \mathcal{G}\mathrm{TI} \Leftrightarrow \dim U = n$).

(c3) $\mathbb{E} \in \mathrm{TI}(U,Y)$ & $\mathbb{E}^*\mathbb{E} \gg 0 \Longrightarrow \dim U \le \dim Y$.

(c4) $\mathbb{E} \in \mathcal{G}\mathrm{TI}_\omega(U,Y) \implies \dim U = \dim Y$.

(d) $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ & $\mathbb{D}^{t*}\mathbb{D}^t \geq \varepsilon I$ *for all* $t > 0 \implies \dim U \leq \dim Y$.

By $\dim H$ we mean the cardinality of an arbitrary Hilbert basis of H. (E.g., $\dim \ell^2(\mathbf{N}) < \dim \ell^2(\mathbf{R})$; cf. Lemma B.3.16.) Naturally, also the shifted versions of the above results hold; (e.g., if some $\mathbb{E} \in \mathrm{TI}_\omega(U,Y)$ is coercive, then $\dim U \leq \dim Y$).

Proof: We use here Theorem 2.1.2 and the separable case of Theorem 3.1.3(a)&(c) (both are known results in that extent). We take $\omega = 0$ w.l.o.g. (see Remark 2.1.6).

(a1) If $\mathbb{E}^*\mathbb{E} \gg 0$, then $\mathbb{X} := (\mathbb{E}^*\mathbb{E})^{-1}\mathbb{E}^* \in \mathrm{TI}(Y,U)$ and $\mathbb{X}\mathbb{E} = I$. Conversely, if $\mathbb{X}\mathbb{E} = I$, then

$$\langle u, \mathbb{E}^*\mathbb{E}u\rangle = \|\mathbb{E}u\|^2 \geq \varepsilon\|u\|^2 \quad \text{for all } u \in \mathrm{L}^2(\mathbf{R},U), \tag{2.21}$$

where $\varepsilon := 1/\|\mathbb{X}\|^2 > 0$, i.e., $\mathbb{E}^*\mathbb{E} \geq \varepsilon I$.

(a2) "Only if" is trivial, so assume that $\mathbb{E} \in \mathrm{TI} \cap \mathcal{GB}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$. Then $\mathbb{E}^{-1}\tau^t = (\tau^{-t}\mathbb{E})^{-1} = (\mathbb{E}\tau^{-t})^{-1} = \tau^t\mathbb{E}^{-1}$ for all $t \in \mathbf{R}$, hence then $\mathbb{E}^{-1} \in \mathrm{TI}(Y,U)$.

(b) $\widehat{\mathbb{X}}, \widehat{\mathbb{Z}} \in \mathrm{L}^\infty(\mathbf{C}^{n\times n})$ and $\widehat{\mathbb{X}}(it)\widehat{\mathbb{Z}}(it) = I$ a.e. on $i\mathbf{R}$, hence $\widehat{\mathbb{Z}}(it)\widehat{\mathbb{X}}(it) = I$ a.e. on $i\mathbf{R}$.

(N.B. This would not hold even for static (constant) operators if the Hilbert space $\mathbf{C}^n$ were replaced by an infinite-dimensional one.)

(c1) Take $\mathbb{X} := (\mathbb{E}^*\mathbb{E})^{-1}\mathbb{E}$ and use (b).

(c2) U is separable since $\mathbb{E}^*D$ is dense in $\mathrm{L}^2(\mathbf{R};U)$ for any D dense in $\mathrm{L}^2(\mathbf{R};\mathbf{C}^n)$ (because $\mathbb{E}^*$ is onto). By Theorem 3.1.3, $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U))$ and $\widehat{\mathbb{E}}^*\widehat{\mathbb{E}} \geq \varepsilon I$ a.e. on $i\mathbf{R}$. If $it \in i\mathbf{R}$ is such that $\widehat{\mathbb{E}}(it)^*\widehat{\mathbb{E}}(it) \geq \varepsilon I$, then $\widehat{\mathbb{E}}(it) \in \mathcal{B}(U,\mathbf{C}^n)$ is coercive, hence $\dim U \leq n$, by Lemma A.3.1(a4).

If $\dim U = n$, then $\exists\mathbb{E}^{-1}$ by (c1), otherwise $\widehat{\mathbb{E}}(it)$ is coercive a.e. and nowhere onto, hence it can't have an inverse.

Remark: There are $\mathbb{E},\mathbb{X} \in \mathcal{B}(\mathrm{L}^2(U),\mathrm{L}^2(\mathbf{C}^n))$ for s.t. $\mathbb{E}^*\mathbb{E} \gg 0$, $\mathbb{X}^*\mathbb{X} \gg 0$, $\mathbb{E} \in \mathcal{GB}(\mathrm{L}^2(U),\mathrm{L}^2(\mathbf{C}^n))$ and $\mathbb{X}$ is not onto, by Lemma B.3.16, for any $n \in \{1,2,3,\ldots\}$ and any separable U. Therefore, time-invariance is not superfluous in (b)–(c4).

(c3) By (c2), we may assume that $\dim Y$ is infinite, so that $\dim \mathrm{L}^2(\mathbf{R};Y) = \dim Y$, by Lemma B.3.16. By Lemma A.3.1(a4), the coercivity of $\mathbb{E}$ implies that $\dim \mathrm{L}^2(\mathbf{R};U) \leq \dim \mathrm{L}^2(\mathbf{R};Y)$. Consequently, $\dim U \leq \dim \mathrm{L}^2(\mathbf{R};U) \leq \dim \mathrm{L}^2(\mathbf{R};Y) = \dim Y$.

(c4) Now $\mathbb{E}^*\mathbb{E} \gg 0$ and $\mathbb{E}\mathbb{E}^* \gg 0$, hence $\dim U = \dim Y$, by (c3).

(d) By Lemma 2.2.4(a), we have $\widehat{\mathbb{D}}^*\widehat{\mathbb{D}} \geq \varepsilon I$ on $\mathbf{C}^+_\omega$, where $\omega \geq 0$ is s.t. $\mathbb{D} \in \mathrm{TIC}_\omega$. Consequently, $\dim U \leq \dim Y$, by Lemma A.3.1(a4) □

For any $\mathbb{E} \in \mathrm{TI}$, we denote below the Toeplitz operator (or Wiener–Hopf operator) $\pi_+\mathbb{E}\pi_+$ of $\mathbb{E}$ by $\mathbf{T}_\mathbb{E}$.

Lemma 2.2.2 (Toeplitz operators) *Let* $\mathbb{E} \in \mathrm{TI}(U)$ *and* $\mathbb{X},\mathbb{Y} \in \mathrm{TIC}(U)$. *Then* $\mathbf{T}_{\mathbb{E}^*} = \mathbf{T}^*_\mathbb{E}$, *and the following is true:*

(a1) If $\mathbf{T}_{\mathbb{E}}$ *is invertible, then so are* $\mathbb{E}$, $\mathbf{T}_{\mathbb{E}^*}$ *and* $\pi_-\mathbb{E}^{\mathrm{d}}\pi_-$.

(a2) $\mathbf{T}_{\mathbb{X}}$ *is invertible iff* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$.

(b) Let $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{TIC}(U)$. *Then* $\mathbf{T}_{\mathbb{E}}$ *invertible iff* $\mathbf{T}_{\mathbb{Y}^*\mathbb{E}\mathbb{X}}$ *is invertible.*

(c1) If $\mathbb{E} \in \mathcal{G}\mathrm{TI}$, *then following are equivalent:*

(i) $\mathbf{T}_{\mathbb{E}}$ *is invertible;*
(ii) $\pi_-\mathbb{E}^{-1}\pi_-$ *is invertible;*
(iii) $\mathbb{E}\pi_+ + \pi_- \in \mathcal{G}\mathcal{B}(\mathrm{L}^2)$*;*
(iv) $\pi_+\mathbb{E} + \pi_- \in \mathcal{G}\mathcal{B}(\mathrm{L}^2)$*;*
(v) $\mathrm{Re}\langle \mathbb{E}\mathcal{H}u, u\rangle \geq \delta\|u\|_2^2$ *for all* $u \in \mathrm{L}^2$ *for some* $\mathcal{H} \in \mathcal{G}\mathcal{B}(\pi_+\mathrm{L}^2)$ *and some* $\delta > 0$.

(c2) **("No equalizing vectors" condition)** *Let* $\mathbb{E} \in \mathrm{MTI}^{\mathrm{L}^1}(\mathbf{C}^n)$, $n \in \mathbf{N}$. *If* $\mathbb{E} \in \mathcal{G}\mathrm{TI}$ *(i.e.,* $\det\widehat{\mathbb{E}}(ir) \neq 0$ *for all* $r \in \mathbf{R} \cup \{\infty\}$*), then* $\mathbf{T}_{\mathbb{E}}$ *is invertible iff* $\mathrm{Ker}(\mathbf{T}_{\mathbb{E}}) = \{0\}$.

(d) We have $\mathbb{E} \gg 0 \Leftrightarrow \mathbf{T}_{\mathbb{E}} \gg 0$*; in particular,* $\mathbb{E} \gg 0$ *implies that* $\mathbf{T}_{\mathbb{E}}$ *is invertible.*

See [DS] for further equivalent conditions for the invertibility of $\mathbb{E}$ (and the existence of a non-TI spectral factorization) in a more general (non-TI) setting.

Proof: (In the lemma, "L^2" denotes $\mathrm{L}^2(\mathbf{R}_+;U)$.)

(a1) By Lemma 4.4 of [S98c], the invertibility of $\mathbf{T}_{\mathbb{E}}$ implies that of $\mathbb{E}$. (The converse does not hold: $\tau(1) \in \mathcal{G}\mathrm{TI}$ but $\mathbf{T}_{\tau(1)}$ is not onto.)

Let $\mathbf{T}_{\mathbb{E}}$ be invertible. Because $\pi_+^* = \pi_+$, we have $(\pi_+\mathbb{E}\pi_+)^* = \pi_+\mathbb{E}^*\pi_+$; hence $\mathbf{T}_{\mathbb{E}^*}^{-1} = \mathbf{T}_{\mathbb{E}}^{-*}$. Moreover, $\pi_-\mathbb{E}^{\mathrm{d}}\pi_- = \boldsymbol{Я}\pi_+\mathbb{E}^*\pi_+\boldsymbol{Я}$ implies the invertibility of $\pi_-\mathbb{E}^{\mathrm{d}}\pi_-$ (on $\pi_-\mathrm{L}^2$).

(a2) If $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, then $\mathbb{X}\pi_+ = \pi_+\mathbb{X}\pi_+ + \pi_-\mathbb{X}\pi_+ = \pi_+\mathbb{X}\pi_+$, by causality, hence then

$$\pi_+\mathbb{X}^{-1}\pi_+\pi_+\mathbb{X}\pi_+ = \pi_+\mathbb{X}^{-1}\mathbb{X}\pi_+ = \pi_+ = I_{\pi_+\mathrm{L}^2} = \pi_+\mathbb{X}\pi_+\pi_+\mathbb{X}^{-1}\pi_+. \tag{2.22}$$

Conversely, if $\pi_+\mathbb{X}\pi_+ = \mathbb{X}\pi_+$ is invertible, then $\mathbb{X} \in \mathcal{G}\mathrm{TI}$, by (a1), and $\pi_+\mathrm{L}^2 = \mathbb{X}\pi_+\mathrm{L}^2$, hence then $\mathbb{X}^{-1}\pi_+\mathrm{L}^2 = \mathbb{X}^{-1}\mathbb{X}\pi_+\mathrm{L}^2 = \pi_+\mathrm{L}^2$, i.e., $\mathbb{X}^{-1}$ is causal.

(b) Now $\mathbf{T}_{\mathbb{X}}$ and $\mathbf{T}_{\mathbb{Y}^*}$ are invertible, by (a2), so the claim follows from equation

$$\mathbf{T}_{\mathbb{Y}^*\mathbb{E}\mathbb{X}} := \pi_+\mathbb{Y}^*\mathbb{E}\mathbb{X}\pi_+ = \pi_+\mathbb{Y}^*\pi_+\mathbb{E}\pi_+\mathbb{X}\pi_+ = \mathbf{T}_{\mathbb{Y}^*}\mathbf{T}_{\mathbb{E}}\mathbf{T}_{\mathbb{X}}. \tag{2.23}$$

(c1) (N.B. (v) holds iff $(\mathbb{E}\mathcal{H}) + (\mathbb{E}\mathcal{H})^* \gg 0$.)

Claims (iii) and (iv) are equivalent to (i) by equations

$$\mathbb{E}\pi_+ + \pi_- = \begin{bmatrix} \pi_+\mathbb{E}\pi_+ & 0 \\ \pi_-\mathbb{E}\pi_+ & \pi_- \end{bmatrix}, \quad \pi_+\mathbb{E} + \pi_- = \begin{bmatrix} \pi_+\mathbb{E}\pi_+ & \pi_+\mathbb{E}\pi_- \\ 0 & \pi_- \end{bmatrix} \tag{2.24}$$

(on $\mathrm{L}^2 = \pi_+\mathrm{L}^2 \times \pi_-\mathrm{L}^2$), respectively. Multiply the former to the left by $\mathbb{E}^{-1}$ to obtain $\pi_- + \mathbb{E}^{-1}\pi_-$ to obtain the equivalence "(iii)$\Leftrightarrow$(ii)". Equivalence "(i)$\Leftrightarrow$(v)" is [DS, Theorem 3] combined with (a1).

(c2) (Nonzero elements of $\mathrm{Ker}(\mathbf{T}_{\mathbb{E}})$ are called equalizing vectors. This "No equalizing vectors" condition was established for rational transfer functions in [Meinsma].)

This proved in [IZ01] (see also Theorem 4.1.1(a)(ii)&(v')). ("Only if" is trivial, "if" follows by noting that the middle element of the standard factorization of $\mathbb{E}$ (see Theorem II.6.3 of [CG81]) must be constant.)

Remark: It is not possible to extend this result to an arbitrary Hilbert space H in place of $\mathbf{C}^n$ by using classical factorization results (collected in Theorem 5.1.6). (And we do not know whether such an extension is true.)

Indeed, Theorem 5.1.6(a) requires the additional assumption that $\mathbf{T}_{\mathbb{E}}$ is a Fredholm operator (note that for self-adjoint $\mathbb{E}$, this additional assumption together with $\mathrm{Ker}(\mathbf{T}_{\mathbb{E}}) = \{0\}$ is equivalent to the invertibility of $\mathbf{T}_{\mathbb{E}}$ for any $\mathbb{E} = \mathbb{E}^* \in \mathrm{TI}$, not merely for $\mathbb{E} \in \mathcal{G}\mathrm{MTI}^{\mathrm{L}^1}$, by Lemma A.3.1(c7)&(c2)(ii)).

(d) This is Lemma 4.4 of [S98c]. □

Lemma 2.2.3 ($\mathcal{G}$TIC) *Let $\mathbb{X} \in \mathrm{TI}(U)$. Then the following are equivalent:*

(i) $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$*;*

(ii) $\mathbb{X}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}$*;*

(iii) $\mathbb{X} \in \mathrm{TIC}$ *and* $\pi_+\mathbb{X}\pi_+$ *is invertible on* $\pi_+\mathrm{L}^2$*;*

(iv) $\mathbb{X} \in \mathcal{G}\mathcal{B}(\mathrm{L}^2)$ *and* $\mathbb{X}\pi_+\mathrm{L}^2 = \pi_+\mathrm{L}^2$*;*

(v) $\mathbb{X} \in \mathcal{G}\mathcal{B}(\mathrm{L}^2)$ *and* $\mathbb{X}^*\pi_-\mathrm{L}^2 = \pi_-\mathrm{L}^2$*;*

(vi) $\mathbb{X} \in \mathrm{TIC}$*,* $\mathbb{X}^*\mathbb{X} \gg 0$ *and* $\pi_+\mathbb{X}\pi_+\mathbb{X}^*\pi_+ \gg 0$ *on* $\pi_+\mathrm{L}^2$*;*

(vii) $\mathbb{X} \in \mathrm{TIC}$*,* $\mathbb{X}\mathbb{X}^* \gg 0$ *and* $\pi_-\mathbb{X}^*\pi_-\mathbb{X}\pi_- \gg 0$ *on* $\pi_-\mathrm{L}^2$.

If $\dim U < \infty$*, then one more equivalent condition is that* $\pi_+\mathbb{X}\pi_+\mathbb{X}^*\pi_+ \gg 0$*; equivalently, we may accept right invertibility in (iii).*

See also Proposition 2.2.5 and Theorem 4.1.1(b).

Proof: 1° *The equivalence:* Because $\mathbb{X} \in \mathrm{TIC} \Rightarrow \mathbb{X}^{\mathrm{d}} \in \mathrm{TIC}$, we obviously have "(i)⇔(ii)". The equivalence "(i)⇔(vi)⇔(vii)" is Lemma 4.11 of [S98c], and "(i)⇔(iii)" is Lemma 2.2.2(a2).

"(i)⇔(iv)": Clearly (i) and (iii) imply (iv) and (iv) implies (iii).

"(ii)⇔(v)": Apply "(i)⇔(iv)" to $\mathbb{X}^{\mathrm{d}} \in \mathcal{G}\mathcal{B}(\mathrm{L}^2)$ to obtain

$$\mathbb{X}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC} \Leftrightarrow \pi_+\mathrm{L}^2 = \mathbb{X}^{\mathrm{d}}\pi_+\mathrm{L}^2 = \boldsymbol{Я}\mathbb{X}^*\boldsymbol{Я}\pi_+\mathrm{L}^2 = \boldsymbol{Я}\mathbb{X}^*\pi_-\mathrm{L}^2. \tag{2.25}$$

2° *Case* $\dim U < \infty$*:* If $\pi_+\mathbb{X}\pi_+\mathbb{X}^*\pi_+ \gg 0$, then there is $\varepsilon > 0$ s.t. $\|\pi_{[t,+\infty)}\mathbb{X}^*\pi_{[t,+\infty)}u\| \geq \varepsilon\|[t,+\infty)u\|$ for $u \in \mathrm{L}^2$ and $t = 0$. By time-invariance, this holds for all $t \in \mathbf{R}$; by continuity, $\|\mathbb{X}^*u\| \geq \varepsilon\|u\|$ for all $u \in \mathrm{L}^2$. If $\dim U < \infty$, then $\mathbb{X}^* \in \mathcal{G}\mathrm{TI}$, by Lemma 2.2.1(a)&(b), hence then (vi) holds. The converse is trivial. (Our favorite counter-example $\mathbb{X} = \tau^{-1} \in \mathrm{TIC} \cap \mathcal{G}\mathrm{TI} \setminus \mathcal{G}\mathrm{TIC}$ shows that left-invertibility is not sufficient.) □

Lemma 2.2.4 ($\mathbb{D}^{t*}\mathbb{D}^t \geq \varepsilon I \Rightarrow \widehat{\mathbb{D}}^*\widehat{\mathbb{D}} \geq \varepsilon I$)

(a) If $\omega \geq 0$, $\mathbb{D}_k \in \mathrm{TIC}_\omega(U,*)$ $(k = 1,2,3,4)$ *and* $\mathbb{D}_1^{t*}\mathbb{D}_2^t \geq \mathbb{D}_3^{t*}\mathbb{D}_4^t$ *for all* $t > 0$, *then*

$$\widehat{\mathbb{D}}_1(s)^*\widehat{\mathbb{D}}_2(s) \geq \widehat{\mathbb{D}}_3(s)^*\widehat{\mathbb{D}}_4(s) \qquad (\operatorname{Re} s > \omega). \tag{2.26}$$

(b) If $\mathbb{D}_k \in \mathrm{TIC}(U,*)$ $(k = 1,2,3,4)$*, then the following are equivalent:*

(i) $\mathbb{D}_1^{t*}\mathbb{D}_2^t \geq \mathbb{D}_3^{t*}\mathbb{D}_4^t$ *for all* $t > 0$*;*
(ii) $\pi_{[-t,0)}\mathbb{D}_1^*\pi_{[-t,0)}\mathbb{D}_2\pi_{[-t,0)} \geq \pi_{[-t,0)}\mathbb{D}_3^*\pi_{[-t,0)}\mathbb{D}_4\pi_{[-t,0)}$ *for all* $t > 0$*;*
(iii) $\mathbb{D}_1^{\mathrm{d}\,t}\mathbb{D}_2^{\mathrm{d}\,t*} \geq \mathbb{D}_3^{\mathrm{d}\,t}\mathbb{D}_4^{\mathrm{d}\,t*}$ *for all* $t > 0$*;*
(iv) $\pi_-\mathbb{D}_1^*\pi_-\mathbb{D}_2\pi_- \geq \pi_-\mathbb{D}_3^*\pi_-\mathbb{D}_4\pi_-$*;*
(v) $\pi_+\mathbb{D}_1^{\mathrm{d}}\pi_+\mathbb{D}_2^{\mathrm{d}*}\pi_+ \geq \pi_+\mathbb{D}_3^{\mathrm{d}}\pi_+\mathbb{D}_4^{\mathrm{d}*}\pi_+$.

If (i)–(v) hold, then $\pi_+\mathbb{D}_1^*\mathbb{D}_2\pi_+ \geq \pi_+\mathbb{D}_3^*\mathbb{D}_4\pi_+$ *and (2.26) holds.*

(c) If $0 \leq J \in \mathcal{B}(Y)$, $0 \leq S \in \mathcal{B}(U)$ *and* $\mathbb{D} \in \mathrm{TIC}(U,Y)$*, then the following are equivalent:*

(i) $\mathbb{D}^{t*}J\mathbb{D}^t \leq \pi_{[0,t)}S$ *for all* $t > 0$*;*
(ii) $\pi_{[-t,0)}\mathbb{D}^*\pi_{[-t,0)}J\mathbb{D}\pi_{[-t,0)} \leq \pi_{[-t,0)}S$ *for all* $t < 0$*;*
(iii) $(\mathbb{D}^{\mathrm{d}})^t J(\mathbb{D}^{\mathrm{d}})^{t*} \leq \pi_{[0,t)}S$ *for all* $t > 0$*;*
(iv) $\pi_-\mathbb{D}^*\pi_-J\mathbb{D}\pi_- \leq \pi_-S$*;*
(v) $\pi_+\mathbb{D}^*\pi_+J\mathbb{D}\pi_+ \leq \pi_+S$.

Recall that $P \geq Q$ means that $P = P^*$, $Q = Q^*$ and $\langle u, Pu\rangle \geq \langle u, Qu\rangle$ for all u. We do not know whether (b)(i)–(v) is implied by (2.26).

Proof: When proving (a) and (b), we assume that $\mathbb{D}_3 = 0 = \mathbb{D}_4$ w.l.o.g. (use substitutions $\mathbb{D}_1 \mapsto \begin{bmatrix}\mathbb{D}_1\\ \mathbb{D}_3\end{bmatrix}$ and $\mathbb{D}_2 \mapsto \begin{bmatrix}\mathbb{D}_2\\ -\mathbb{D}_4\end{bmatrix}$).

(a) Let $s \in \mathbf{C}^+_\omega$ and $u_0 \in U$ be given. Set $u := e^{s\cdot}u_0 \in \mathrm{L}^2_{\mathrm{loc}}$. Then $\pi_-u \in \mathrm{L}^2 \cap \mathrm{L}^2_\omega$, hence $\pi_-\mathbb{D}_k u \in \mathrm{L}^2_\omega(\mathbf{R}_-;Y) \subset \mathrm{L}^2(\mathbf{R}_-;Y)$ $(k = 1,2)$. By Lemma 2.1.15, we have that $(\mathbb{D}_k u)(t) = e^{st}\widehat{\mathbb{D}}_k(s)u_0$ $(t \in \mathbf{R})$. By time-invariance,

$$0 \leq \langle \mathbb{D}_1^t\tau^{-t}u, \mathbb{D}_2^t\tau^{-t}u\rangle = \int_{-\infty}^0 \langle \mathbb{D}_1\pi_{[-t,0)}u, \mathbb{D}_2\pi_{[-t,0)}u\rangle\, dt \quad (t > 0). \tag{2.27}$$

Now $\pi_-\mathbb{D}_k\pi_{(-\infty,t)}u \to 0$ in L^2_ω, hence in L^2 too, as $t \to +\infty$. Therefore, we can let $t \to +\infty$ to obtain that $0 \leq \langle \widehat{\mathbb{D}}_1(s)u_0, \widehat{\mathbb{D}}_2(s)u_0\rangle \int_{-\infty}^0 e^{2t\operatorname{Re}s}\,dt$. Because u_0 was arbitrary, we have $0 \leq \widehat{\mathbb{D}}_1(s)^*\widehat{\mathbb{D}}_2(s)$.

(b) 1° *(i)–(iii):* Equivalence "(i)⇔(ii)" follows from time-invariance (apply τ^{-t} and τ^t to different sides of the inequality), "(ii)⇔(iii)" and "(iv)⇔(v)" by reflection (apply **Я** to both sides), and the implication "(iv)⇒(ii)" by adding $\pi_{[-t,0)}$ to both sides of the inequality in (iv),

2° *(ii)⇒(iv):* Assume (ii). For any $u \in \mathrm{L}^2(\mathbf{R}_-;U)$ and $t > 0$, we have

$$0 \leq \langle \mathbb{D}_1\pi_{[-t,0)}u, \pi_{[-t,0)}\mathbb{D}_2\pi_{[-t,0)}u\rangle = \langle \mathbb{D}_1u, \pi_-\mathbb{D}_2u\rangle - \langle \pi_{(-\infty,t)}\mathbb{D}_1u, \pi_-\mathbb{D}_2\pi_{(-\infty,t)}u\rangle. \tag{2.28}$$

By Corollary B.3.8, $\langle \mathbb{D}_1\pi_{(-\infty,t)}u, \pi_-\mathbb{D}_2\pi_{(-\infty,t)}u\rangle \to 0$ as $t \to +\infty$, hence $\langle \mathbb{D}_1 u, \mathbb{D}_2 u\rangle = \langle \mathbb{D}_1\pi_- u, \pi_-\mathbb{D}_2\pi_- u\rangle$ is real and nonnegative. The second implication is obtained analogously (because $\pi_{[-t,0)}\mathbb{D}_1^*\pi_{[-t,0)}\mathbb{D}_2\pi_{[-t,0)} \ge 0$).

3° *(i)*$\Rightarrow$ $\pi_+\mathbb{D}_1^*\mathbb{D}_2\pi_+ \ge 0$*:* Let $t \to +\infty$ (as in 2°).

(c) We prove below the implication "(v)$\Rightarrow$(i)". The rest of (c) follows from (b) by setting $\mathbb{D}_1 = I$, $\mathbb{D}_2 = S$, $\mathbb{D}_3 = \mathbb{D}$, $\mathbb{D}_4 = J\mathbb{D}$.

Assume (v). Then

$$\pi_+ S \ge \pi_+\mathbb{D}^*\pi_+ J\mathbb{D}\pi_+ = \mathbb{D}^{t\,*} J\mathbb{D}^t + \pi_+\mathbb{D}^*\pi_{[t,\infty)}J\mathbb{D}\pi_+, \tag{2.29}$$

hence $\pi_+ S \ge \mathbb{D}^{t\,*}J\mathbb{D}^t$, hence (i) holds.

(We do not have (v)$\Rightarrow$(i) for $J, S \le 0$ in general. E.g., if $\mathbb{D} = (\tau^{-k})_{k\in\mathbf{N}} \in \mathrm{TIC}(\ell^2(\mathbf{N}))$, then $\pi_+\mathbb{D}^*\pi_+\mathbb{D}\pi_+ = \pi_+$ but $\langle u_k, \mathbb{D}^{t\,*}\mathbb{D}^t u_k\rangle = 0 < \langle u_k, \pi_{[0,t)}u_k\rangle = 1$ for $u_k = \chi_{[0,1)}e_k$, $t \ge k \ge 1$.) $\square$

By the above lemma (and Lemma 2.2.3), we have "$\mathbb{X}^{-1} \in \mathrm{TIC} \Rightarrow \widehat{\mathbb{X}}^*\widehat{\mathbb{X}} \ge \varepsilon I$", but the converse requires a Tauberian condition. Five such conditions are presented below:

Proposition 2.2.5 ($\widehat{\mathbb{X}}^*\widehat{\mathbb{X}} \ge \varepsilon I \Rightarrow \mathbb{X}^{-1} \in \mathrm{TIC}$) *Let $\omega \in \mathbf{R}$ and $\mathbb{X} \in \mathrm{TIC}_{\omega'}(U,Y)$ for all $\omega' > \omega$. If $(\widehat{\mathbb{X}})^*\widehat{\mathbb{X}} \ge \varepsilon^2 I$ on $\mathbf{C}^+_\omega$ (or $\mathbb{X}^{t\,*}\mathbb{X}^t \ge \varepsilon^2 I\pi_{[0,t)}$ for all $t > 0$) for some $\varepsilon > 0$, and any of conditions (1)–(5) below holds, then $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty(U,Y)$, $\mathbb{X}^{-1} \in \mathrm{TIC}_\omega$, $\|\mathbb{X}^{-1}\|_{\mathrm{TIC}_\omega} \le \varepsilon^{-1}$, and $\dim U = \dim Y$.*

(1) $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$;

(2) $\mathbb{X} \in \mathcal{G}\mathrm{TI}_{\omega'}$ (or $\mathbb{X}\mathbb{M} = I$ for some $\mathbb{M} \in \mathrm{TI}_{\omega'}$) for some $\omega' > \omega$;

(3) $\dim U = \dim Y < \infty$;

(4) $\left[\begin{smallmatrix}\mathbb{X} & * \\ * & *\end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}_\infty(U\times W, Y\times Z)$, and $\dim W < \infty$;

(5) $\widehat{\mathbb{X}}(s_0) \in \mathcal{GB}$ or $\widehat{\mathbb{X}}(s_0)$ is onto or $\mathrm{Ker}(\widehat{\mathbb{X}}(s_0)^*) = \{0\}$ for some $s_0 \in \mathbf{C}^+_\omega$; (for uniformly regular $\mathbb{X}$ we allow $s_0 = +\infty$).

If $\mathbb{X} \in \mathrm{TIC}$, and $\mathbb{X}^{t\,*}\mathbb{X}^t \ge \varepsilon^2 I\pi_{[0,t)}$ for all $t > 0$, then also the sixth condition $\mathbb{X}\mathbb{X}^* \gg 0$ implies that $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U,Y)$.

(Note that (1)–(5) are not superfluous: if $\dim U = \infty$, then there is $\mathbb{X} = X \in \mathcal{B}(U)$ s.t. X is left-invertible but not right-invertible.)

Recall that if $\mathbb{X} \in \mathrm{TIC}_\omega$, then $\mathbb{X} \in \mathrm{TIC}_{\omega'}$ for all $\omega' > \omega$.

For $\omega = 0$, Proposition 4.1.7(B)&(C) provide us with several sufficient conditions for $(\widehat{\mathbb{X}})^*\widehat{\mathbb{X}} \ge \varepsilon^2 I$.

Proof: (We take $\omega = 0$ w.l.o.g.)

From (2.26) we observe that if $\mathbb{X}^{t\,*}\mathbb{X}^t \ge \varepsilon^2 I\pi_{[0,t)}$ for all $t > 0$, then $\widehat{\mathbb{X}}^*\widehat{\mathbb{X}} \ge \varepsilon^2 I$ on $\mathbf{C}^+$.

The last claim follows from Lemma 2.2.4(b)(i)&(iv) and Lemma 2.2.3(vii)&(i).

Clearly $\widehat{\mathbb{X}}$ (see Theorem 2.1.2) has the left-inverse $\widehat{\mathbb{T}} = (\widehat{\mathbb{X}}^*\widehat{\mathbb{X}})^{-1}\widehat{\mathbb{X}}^* \in \mathcal{C}_\mathrm{b}(\mathbf{C}^+; \mathcal{B}(Y,U))$ on $\mathbf{C}^+$, i.e., $\widehat{\mathbb{T}}\widehat{\mathbb{X}} \equiv I$ on $\mathbf{C}^+$. Moreover, $\|\widehat{\mathbb{T}}(s)\| \le \varepsilon^{-1}$ for all $s \in \mathbf{C}^+$, by Lemma A.3.1(c1)(1). If $\widehat{\mathbb{X}}(s_0) \in \mathcal{GB}$ for some $s_0 \in \mathbf{C}^+$, then $\widehat{\mathbb{X}}(s_0)^{-1} = \widehat{\mathbb{T}}(s_0)$.

Thus, we only have to show that $\widehat{\mathbb{X}}$ has an inverse on $\mathbf{C}^+$, i.e., that $\widehat{\mathbb{T}}$ is also a right inverse of $\widehat{\mathbb{X}}$, because $\widehat{\mathbb{X}}^{-1}$ is necessarily holomorphic, by Lemma D.1.2(b2), hence $\widehat{\mathbb{X}}^{-1} = \widehat{\mathbb{T}} \in \mathrm{H}^\infty = \widehat{\mathrm{TI}}$ (and $\dim U = \dim Y$, by Lemma 2.2.1(c4)). The invertibility proof depends on the extra assumption:

(1) By definitions, (1) implies (2).

(2) Because $\widehat{\mathbb{T}}, \widehat{\mathbb{X}} \in \mathcal{C}_{\mathrm{b}}(\omega' + i\mathbf{R}; \mathcal{B}(U))$, we have $\mathbb{X} \in \mathcal{G}\mathrm{TI}_{\omega'}$ iff $\widehat{\mathbb{X}} \in \mathcal{G}\mathcal{C}_{\mathrm{b}}(\omega' + i\mathbf{R}; \mathcal{B}(U))$, by Theorem 3.1.3(d). Therefore, (2) implies (5) (note that $\widehat{\mathbb{T}}$ is the transform of a left inverse of $\mathbb{X} \in \mathrm{TI}_{\omega'}$, hence the existence of a right inverse $\mathbb{M}$ is equivalent to the invertibility of $\mathbb{X}$ in $\mathrm{TI}_{\omega'}$).

(3) $\dim U = \dim Y < \infty$ implies that any left inverse of $\widehat{\mathbb{X}}(s)$ (hence $\widehat{\mathbb{T}}(s)$) is an inverse, by Lemma A.1.1(c1).

(4) (The assumptions mean that there are $\mathbb{Y}, \mathbb{Z}, \mathbb{W} \in \mathrm{TIC}_\infty$ s.t. $\begin{bmatrix} \mathbb{X} & \mathbb{Y} \\ \mathbb{Z} & \mathbb{W} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(U \times W, Y \times Z)$ and $\dim W < \infty$.)

By Lemma A.1.1(c1), the left-invertibility of $\widehat{\mathbb{X}}$ implies the invertibility of $\widehat{\mathbb{X}}$ on $\mathbf{C}^+_\omega$.

(5) By the uniqueness of the $\mathcal{B}(U,Y)$ inverse, we have

$$E := \{s \in \mathbf{C}^+ \,\big|\, \widehat{\mathbb{X}}(s)\widehat{\mathbb{T}}(s) = I\} = \{s \in \mathbf{C}^+ \,\big|\, \widehat{\mathbb{X}}(s) \in \mathcal{GB}(U,Y)\}, \tag{2.30}$$

and the latter set is open, by Lemma 6.3.2(d). On the other hand, $\widehat{\mathbb{X}}(s)\widehat{\mathbb{T}}(s) = I$ holds in a closed subset of $\mathbf{C}^+$. Now the connectedness of $\mathbf{C}^+$ and the fact that $E \neq \emptyset$ (because $s_0 \in E$) imply that $E = \mathbf{C}^+$ (if $s_0 = +\infty$ and $\mathbb{X}$ is UR, then $\widehat{\mathbb{X}}(s)$ is invertible for big $s \in \mathbf{R}$, by Lemma A.3.3(A2)).

Remark: If $U = \ell^2(\mathbf{N}) \subset Y$ and $\mathbb{X}$ is the right shift $\tau^{-1} \in \mathcal{B}(U)$, then the assumptions (except (1)–(5)) are satisfied and yet $\mathbb{X} \notin \mathcal{G}\mathrm{TIC}_\infty$. □

Corollary 2.2.6 ($\mathbb{X}^{t*}\mathbb{X}^t \geq \varepsilon I \Rightarrow \exists \mathbb{X}^{-1} \in \mathrm{TIC}$) *Assume that* $\mathbb{X} \in \mathrm{TIC}_\omega(U,Y)$, $\omega \geq 0$, $\varepsilon > 0$ *and* $\mathbb{X}^{t*}\mathbb{X}^t \geq \varepsilon^2 I\pi_{[0,t)}$ *for all* $t > 0$.

If any of (1)–(5) of Proposition 2.2.5 holds, then $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty(U,Y)$, $\mathbb{X}^{-1} \in \mathrm{TIC}$, $\|\mathbb{X}^{-1}\|_{\mathrm{TIC}} \leq \varepsilon^{-1}$ *and* $\dim U = \dim Y$.

Proof: From (2.26) we observe that $\widehat{\mathbb{X}}^*\widehat{\mathbb{X}} \geq \varepsilon^2 I$ on $\mathbf{C}^+$. Thus, $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$, by Proposition 2.2.5. Moreover, $\varepsilon^{-1} I\pi_{[0,t)} \geq ((\mathbb{X}^{-1})^t)^*(\mathbb{X}^{-1})^t$ for all $t > 0$, hence $\|\mathbb{X}^{-1}\|_{\mathrm{TIC}} \leq \varepsilon^{-1}$, by Lemma 2.1.14.

(Again the right shift is a "counter-example" showing that (1)–(5) are not superfluous.) □

If $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U,Y)$ is exponentially stable, then so is $\mathbb{X}^{-1}$; moreover, if $\mathbb{X}$ is generated by an exponentially stable measure, then so is its inverse:

Lemma 2.2.7 ($\mathcal{G}\mathbf{TIC} \cap \mathbf{TI}_{\mathbf{exp}} = \mathcal{G}\mathbf{TIC}_{\mathbf{exp}}$) *Let* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U,Y) \cap \mathrm{TI}_{-r}(U,Y)$, *where* $r > 0$. *Then, for some* $\varepsilon > 0$, *we have* $\mathbb{X}^{-1} \in \mathrm{TIC}_{-\varepsilon}(Y,U)$; *thus,* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_{-\varepsilon}(U,Y) \cap \mathrm{TIC}_{-r}(U,Y)$.

If $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U,Y) \cap \widetilde{\mathcal{A}}_{-r}(U,Y)$*, where* $\widetilde{\mathcal{A}} \subset \mathrm{TIC}$ *is inverse-closed (as in Theorem 4.1.1(g)), then* $\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}(U,Y) \cap \widetilde{\mathcal{A}}_{-r}(U,Y)$ *for some* $\varepsilon > 0$.

Here $\widetilde{\mathcal{A}}$ may be MTIC or any other inverse-closed subclass of TIC, by Theorem 4.1.1(b).

Proof: (We use here Theorem 4.1.1, but this lemma is not used before Chapter 5.)

Let $M := \|\mathbb{X}^{-1}\|^{-1}$. By Lemma D.1.8(c), there is $\varepsilon > 0$ s.t. $\|\widehat{\mathbb{X}}(t+iy) - \widehat{\mathbb{X}}(0+iy)\| < 1/2M$ when $|t| < \varepsilon$ and $y \in \mathbf{R}$. Consequently, $\widehat{\mathbb{X}}(t+iy)^{-1}$ exists and its norm is less than $2M$ for such t and y; in particular, $\widehat{\mathbb{X}}^{-1} \in \mathrm{H}^\infty(\mathbf{C}_{-\varepsilon}; \mathcal{B}(Y,U))$, hence $\mathbb{X}^{-1} \in \mathrm{TI}_{-\varepsilon}(Y,U)$, by Theorem 2.1.2.

The final claim follows from the fact that $\mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}$ is inverse-closed in $\mathrm{TIC}_{-\varepsilon}$, i.e., that $\widetilde{\mathcal{A}}_{-\varepsilon} \cap \mathcal{G}\mathrm{TIC}_{-\varepsilon} = \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}$, by Theorem 4.1.1(g1). $\square$

Local causal invertibility is equivalent to global causal invertibility:

Lemma 2.2.8 ($\mathbb{X}^t \in \mathcal{GB} \Leftrightarrow \mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$) *Let* $\mathbb{X} \in \mathrm{TIC}_\infty(U,Y)$ *and* $-\infty < a < b < \infty$*. Then* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ *iff* $\pi_{[a,b)}\mathbb{X}\pi_{[a,b)}$ *is invertible on* $\pi_{[a,b)}\mathrm{L}^2$.

If $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$*, then the latter inverse is* $\pi_{[a,b)}\mathbb{X}^{-1}\pi_{[a,b)}$.

However, being invertible in a specific TIC_ω is a stronger condition. E.g., if $\widehat{\mathbb{X}}(s) := (s-1)/(s+1)$, then $\mathbb{X} \in \mathrm{TIC}_\omega(\mathbf{C})$ for any $\omega > -1$, but $\mathbb{X}^{-1} \in \mathrm{TIC}_\omega$ for $\omega > 1$ only, hence $\mathbb{X} \in \mathrm{TIC} \cap \mathcal{G}\mathrm{TIC}_\infty \setminus \mathcal{G}\mathrm{TIC}$.

Proof: The latter claim is obvious and the former one follows from Theorem 6.1.9(iv)&(v) of [Sbook], but we give here an alternative proof.

Set $T := b-a$. Take a realization of $\mathbb{X}$ and transform it into a wpls as in Theorem 13.4.4. Because the transformation maps $\mathrm{TIC}_\infty \to \mathrm{tic}_\infty$ through an isomorphism, which is also an algebraic isomorphism, by Theorem 13.4.5(b), it follows from Lemma 13.1.7 that $\mathbb{X}$ is invertible iff $X_\mathrm{d} := \pi_{[0,T)}\mathbb{X}\pi_{[0,T)}$ (the I/O operator of the wpls) is invertible.

By time-invariance the invertibility of $\pi_{[0,T)}\mathbb{X}\pi_{[0,T)}$ is equivalent to the invertibility of $\pi_{[a,b)}\mathbb{X}\pi_{[a,b)}$. $\square$

Notes

It would be interesting to know whether (2.26) with $\omega = 0$ is sufficient for (i)–(v) of Lemma 2.2.4(b) (it is necessary, by (a)) when $\omega = 0$; our guess is that the answer is negative — unfortunately, because a positive answer would provide us with several additional equivalent conditions in Corona Theorems 4.1.6 and 4.1.7. See also Lemma 4.1.10.

Conditions (3) and (5) of Proposition 2.2.5 were used in a preprint of [S98d]. Lemma 2.2.1(b) and the case $\widetilde{\mathcal{A}} = \mathrm{TIC}$ of Lemma 2.2.7 are from [Sbook]. Most of Lemma 2.2.4 is from Section 6 of [S98c]. Also (at least) Lemma 2.2.8 and parts of Lemmas 2.2.2 and 2.2.3 are known, as explained in their proofs. Lemma 2.2.1(c3) is one of the main contributions of this section.

2.3 Static operators

Eppur si muove!

— Galileo Galilei (1564–1642), 1633

In this section, we shall present five important (and apparently new) technical lemmas that will be used in connection with (J,S)-inner factorizations and Riccati equations. Most of the lemmas give some sufficient results for an operator to be static.

If $\mathbb{E} = \mathbb{E}^* \in \mathrm{TI}(U)$, then $\pi_+\mathbb{E}\pi_- = 0$ implies that $\pi_-\mathbb{E}\pi_+ = (\pi_+\mathbb{E}\pi_-)^* = 0$, so that then $\mathbb{E}$ is static, i.e., $\mathbb{E} \in \mathcal{B}(U)$, by Lemma 2.1.7. We now prove a similar claim for "$\mathbb{E} = \mathbb{D}^*J\mathbb{D}$" when $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ is only required to be almost stable ($\mathbb{D}\mathrm{L}^2_\mathrm{c} \subset \mathrm{L}^2$), so that $\mathbb{E}$ need not be defined at all:

Lemma 2.3.1 ($\mathbb{D}^*J\mathbb{D} = S$) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $J = J^* \in \mathcal{B}(Y)$. Assume that $\mathbb{D}u \in \mathrm{L}^2$ (cf. Lemma 2.1.13) and $\langle \mathbb{D}\pi_+v, J\mathbb{D}\pi_-u\rangle = 0$ for all $u,v \in \mathrm{L}^2_\mathrm{c}$. Then there is a unique $S = S^* \in \mathcal{B}(U)$ s.t. $\langle \mathbb{D}v, J\mathbb{D}u\rangle = \langle v, Su\rangle$ for all $u,v \in \mathrm{L}^2_\mathrm{c}$.*

Note that for $\mathbb{D} \in \mathrm{TIC}$ the term $\mathbb{D}^*J\mathbb{D}$ would be well defined and hence the proof of the lemma would be simple.

Proof: In the sequel we shall use the fact that if $u \in \mathrm{L}^2_\mathrm{loc}$ and $\langle v,u\rangle = 0$ for all $v \in \mathrm{L}^2_\mathrm{c}$ (or for all $v \in \mathcal{C}^\infty_\mathrm{c}$), then $u = 0$ (a.e.), by Theorem B.3.11. This implies, that S is unique.

Replace u by $\tau^t u$ to obtain that $\langle \mathbb{D}\pi_{[t,\infty)}v, J\mathbb{D}\pi_{(-\infty,t)}u\rangle = 0$ for all $u,v \in \mathrm{L}^2_\mathrm{c}$. Because $J = J^*$, we have $\langle \mathbb{D}\pi_{(-\infty,s)}v, J\mathbb{D}\pi_{[s,\infty)}u\rangle = 0$ for all $u,v \in \mathrm{L}^2_\mathrm{c}$, hence

$$\langle \mathbb{D}v, J\mathbb{D}\pi_{[s,t)}u\rangle = \langle \mathbb{D}\pi_{[s,t)}v, J\mathbb{D}\pi_{[s,t)}u\rangle \quad (u \in \mathrm{L}^2_\mathrm{c},\ -\infty \le s \le t \le +\infty). \tag{2.31}$$

Set $\mathbb{S}_t := (\mathbb{D}\pi_{[-t,t)})^*J\mathbb{D}\pi_{[-t,t)} \in \mathcal{B}(\mathrm{L}^2([-t,t);U))$ $(t > 0)$. Then $\langle v, \mathbb{S}_t u\rangle = \langle \mathbb{D}v, J\mathbb{D}u\rangle$ for $u,v \in \pi_{[-t,t)}\mathrm{L}^2$, hence for $u \in \pi_{[-t,t)}\mathrm{L}^2$ and $v \in \mathrm{L}^2_\mathrm{c}$, by (2.31). Consequently, $\mathbb{S}_T u = \mathbb{S}_t u \in \pi_{[-t,t)}\mathrm{L}^2$ for all $T > t$, so we can define $\mathbb{S}u := \mathbb{S}_t u$ $(u \in \mathrm{L}^2([-t,t);U))$ (for an arbitrary $t > 0$).

It follows that $\mathbb{S} : \mathrm{L}^2_\mathrm{c} \to \mathrm{L}^2_\mathrm{c}$, $\tau\mathbb{S} = \mathbb{S}\tau$, and $\mathbb{S}_T = \mathbb{S}\pi_{[-T,T)}$. Therefore,

$$\|\mathbb{S}u\|_2^2 = \|\sum_{n\in\mathbf{Z}} \tau^{-n}\mathbb{S}\pi_{[-1,1)}\tau^n u\|^2 = \sum_{n\in\mathbf{Z}} \|\mathbb{S}\pi_{[-1,1)}\tau^n u\|^2 = \sum_{n\in\mathbf{Z}} \|\mathbb{S}\pi_{[-1,1)}\tau^n u\|^2 \tag{2.32}$$

$$\le \|\mathbb{S}_1\|_{\mathcal{B}(\mathrm{L}^2)} \|\sum_{n\in\mathbf{Z}} \|\pi_{[-1,1)}\tau^n u\|^2 = \|\mathbb{S}_1\|_{\mathcal{B}}(\mathrm{L}^2)\|u\|^2 \tag{2.33}$$

for $u \in \mathrm{L}^2_\mathrm{c}$. Consequently, $\mathbb{S}$ can be extended to a $\mathcal{B}(\mathrm{L}^2)$ map that is TI. From (2.31) it follows that $\pi_+\mathbb{S}\pi_- = 0 = \pi_-\mathbb{S}\pi_+$, hence $\mathbb{S} \in \mathcal{B}(U)$, by Lemma 2.1.7. Obviously, $\mathbb{S} = \mathbb{S}^*$. □

Let $\mathbb{D} \in \mathrm{TIC}_\infty$. If $\pi_+\mathbb{D}\pi_- = 0$, then $\mathbb{D}$ is static. In fact, instead of $\pi_+\mathbb{D}\pi_-$ it suffices to observe that $\pi_{[0,t)}\mathbb{D}\pi_{[-\varepsilon,0)} = 0$ for a fixed $\varepsilon > 0$:

Lemma 2.3.2 *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, and $\pi_{[0,t)}\mathbb{D}\pi_{[-\varepsilon,0)} = 0$ for some $\varepsilon > 0$ and all $t > 0$. Then $\mathbb{D} \in \mathcal{B}(U,Y)$.*

Proof: Obviously, $\pi_+\mathbb{D}\pi_{[-\varepsilon,0)} = 0$. Thus,

$$\pi_+\mathbb{D}\pi_{[-n\varepsilon,0)} = 0 \tag{2.34}$$

holds for $n = 1$. Assume now that (2.34) holds for $n = N \in 1+\mathbf{N}$. Then

$$\pi_+\mathbb{D}\pi_{[-(n+1)\varepsilon,-n\varepsilon)} = \tau^1\pi_{[\varepsilon,\infty)}\pi_+\mathbb{D}\pi_{[-n\varepsilon,-(n-1)\varepsilon)}\tau^{-1} = 0. \tag{2.35}$$

By induction, $\pi_+\mathbb{D}\pi_- = 0$, hence $\mathbb{D} \in \mathcal{B}$, by Lemma 2.1.7. □

If $\mathbb{X}, \mathbb{Z} \in \mathrm{TIC}_\infty$ and $\mathbb{X}^* = \mathbb{Z}$, then $\mathbb{X}$ is static. Often it is easier to verify that $\pi_{[0,t)}\mathbb{X}^*\pi_{[0,t)} = \pi_{[0,t)}\mathbb{Z}\pi_{[0,t)}$ for all t; even this is sufficient:

Lemma 2.3.3 *Let $\mathbb{X}, \mathbb{Z} \in \mathrm{TIC}_\infty$ and $\mathbb{X}^t = \mathbb{Z}^{t*}$ for all $t > 0$. Then $\mathbb{X} = \mathbb{Z}^* \in \mathcal{B}$.*

Recall that $\mathbb{X}^t := \pi_{[0,t)}\mathbb{X}\pi_{[0,t)} \in \mathcal{B}(\mathrm{L}^2)$ (for any $\mathbb{X} \in \mathrm{TIC}_\infty$).

Proof: Let $\varepsilon > 0$. Now $\tau^\varepsilon\mathbb{X}^t\tau^{-\varepsilon} = \pi_{[-\varepsilon,t-\varepsilon)}\mathbb{X}\pi_{[-\varepsilon,t-\varepsilon)}$, hence

$$\pi_{[0,t-\varepsilon)}\mathbb{X}\pi_{[-\varepsilon,0)} = \pi_{[0,t-\varepsilon)}\tau^\varepsilon\mathbb{X}^t\tau^{-\varepsilon}\pi_{[-\varepsilon,0)} = \pi_{[0,t-\varepsilon)}\tau^\varepsilon\mathbb{Z}^{t*}\tau^{-\varepsilon}\pi_{[-\varepsilon,0)} \tag{2.36}$$

$$= \pi_{[0,t-\varepsilon)}\mathbb{Z}^*\pi_{[-\varepsilon,0)} = 0, \ \ (t > \varepsilon). \tag{2.37}$$

hence $X := \mathbb{X} \in \mathcal{B}$, by 1°. By time-invariance $\pi_{[-t,t)}\mathbb{Z}\pi_{[-t,t)} = \pi_{[-t,t)}X^*\pi_{[-t,t)} = \pi_{[-t,t)}X^*$ $(t \geq \varepsilon)$, hence $\mathbb{Z} = X^*$, by continuity on some L^2_ω. □

If $J\mathbb{D}$ is static for some static J, then $J\mathbb{D} = JD$ for some static D, at least to some extent:

Lemma 2.3.4 *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, $J \in \mathcal{B}(Y,H)$. If $J\mathbb{D} \in \mathcal{B}(U,H)$, then $\mathbb{D} = \begin{bmatrix} D_1 \\ \mathbb{D}_2 \end{bmatrix} \in \mathrm{TIC}_\infty(U, Y_1 \times Y_2)$, where $Y_1 = Y_2^\perp$, $Y_2 := \mathrm{Ker}(J)$. Moreover, $J\mathbb{D} = J\begin{bmatrix} D_1 \\ 0 \end{bmatrix}$.*

Assume, in addition, that $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$. Then there is $D \in \mathcal{G}\mathcal{B}(U,Y)$ s.t. $J\mathbb{D} = JD$. If $Y = H$, then we can, in addition, require that $\mathbb{D}^{t*}J\mathbb{D}^t = D^*JD\pi_{[0,t)}$ for all $t \geq 0$.

Proof: 1° Let P is be orthogonal projection $Y \to Y_1$, and set $\mathbb{D}_1 := P\mathbb{D}$, $\mathbb{D}_2 = (I-P)\mathbb{D}$. Now $J\mathbb{D}_1 = J\mathbb{D} \in \mathcal{B}(U,H)$, hence $J\widehat{\mathbb{D}}_1$ is constant; because J is one-to-one on Y_1, $D_1 := \widehat{\mathbb{D}}_1 \in \mathcal{B}(U,Y_1)$.

2° Let $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$. Then $\mathbb{D}$ is onto, hence $\mathbb{D}_1$ is onto, hence D_1 is onto hence $D_1^* \in \mathcal{G}\mathcal{B}(Y_1,U_1)$, where $U_1 = \mathrm{Ran}(D_1^*) = \mathrm{Ker}(D_1)^\perp$, by Lemma A.3.1(c1)(iii)&(x)&(c7).

Consequently, $E := D_1|_{U_1} \in \mathcal{G}\mathcal{B}(U_1,Y_1)$, hence $\mathbb{D} = \begin{bmatrix} E & 0 \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(U_1 \times U_2, Y_1 \times Y_2)$, where $U_2 := U_1^\perp$, $\mathbb{D}_{21} := \mathbb{D}_2 Q$, $\mathbb{D}_{22} = \mathbb{D}_2(I-Q)$, and Q is the orthogonal projection $U \mapsto U_1$.

Thus, $\mathbb{D}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(U_2,Y_2)$, by Lemma A.1.1(b2)(1), hence $\dim U_2 = \dim Y_2$, by Lemma 2.2.1(c4), hence $\mathcal{G}\mathcal{B}(U_2,Y_2)$ contains some operator F. Consequently, $D := \begin{bmatrix} E & 0 \\ 0 & F \end{bmatrix} \in \mathcal{G}\mathcal{B}(U,Y)$, $JD = JPD = J\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} = J\begin{bmatrix} D_1 \\ 0 \end{bmatrix} = J\mathbb{D}_1 = J\mathbb{D}$.

3° Case $Y = H$: Apply 2° for $M := \begin{bmatrix} J \\ J^* \end{bmatrix} \in \mathcal{B}(Y,Y^2)$. Then $M\mathbb{D} = MD$ for some $D \in \mathcal{G}\mathcal{B}(U,Y)$, hence $J\mathbb{D} = JD$ and $\mathbb{D}^*J = (J^*\mathbb{D})^* = D^*J$. Consequently, $\mathbb{D}^{t*}J\mathbb{D}^t = \mathbb{D}^{t*}JD\pi_{[0,t)} = D^*JD\pi_{[0,t)}$ $(t > 0)$. □

We finish the section with a technical remark that will allow us to prove certain uniqueness results (modulo a unit $E \in \mathcal{GB}$) on the signature operators of optimization problems and Riccati equations:

Lemma 2.3.5 ($\mathbb{X}^{t*}S\mathbb{X}^t = \mathbb{Z}^{t*}T\mathbb{Z}^t$) *Let $\mathbb{X}, \mathbb{Z} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, $S, T \in \mathcal{B}(U)$ and $\mathbb{X}^{t*}S\mathbb{X}^t = \mathbb{Z}^{t*}T\mathbb{Z}^t$ for all $t > 0$. Then $S = E^*TE$, $S\mathbb{X} = SE^{-1}\mathbb{Z}$, and $S^*\mathbb{X} = S^*E^{-1}\mathbb{Z}$ for some $E \in \mathcal{GB}(U)$.*

In particular, if $\mathrm{Ker}(S) = \{0\}$ or $\mathrm{Ker}(S^*) = \{0\}$ (i.e., S is one-to-one or onto), then $\mathbb{X} = E^{-1}\mathbb{Z}$.

Proof: Set $\mathbb{R} := \mathbb{X}\mathbb{Z}^{-1} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, $\widetilde{S} := \left[\begin{smallmatrix} S \\ S^* \end{smallmatrix}\right], \widetilde{T} := \left[\begin{smallmatrix} T \\ T^* \end{smallmatrix}\right] \in \mathcal{B}(U, U^2)$. Then $\widetilde{S}\mathbb{R}^t = \mathbb{R}^{t\,-*}\widetilde{T}$ for all $t > 0$ (the second row of the equation is the adjoint of the first one), hence $L := \widetilde{S}\mathbb{R} = \mathbb{R}^{-*}\widetilde{T} \in \mathcal{B}(U)$, by Lemma 2.3.3.

By Lemma 2.3.4, we have $\pi_{[0,t)}\widetilde{T} = \mathbb{R}^{t\,*}\widetilde{S}\mathbb{R}^t = R^*\widetilde{S}R\pi_{[0,t)}$ for some $R \in \mathcal{GB}(U)$, hence $\widetilde{T} = R^*\widetilde{S}R$. Set $E := R^{-1}$ to obtain $S = E^*TE$. Finally, $S\mathbb{X} = S\mathbb{R}\mathbb{Z} = SR\mathbb{Z}$, and $S^*\mathbb{X} = S^*\mathbb{R}\mathbb{Z} = S^*R\mathbb{Z}$. $\square$

Notes

The above results are apparently new. Theorem 5.2C of [RR] (case $\omega = 0$ of Lemma 2.1.7) is the only known (to us) result in this direction.

2.4 The signature operator S

To see a World in a grain of sand,
And a Heaven in a wild flower,
Hold Infinity in the palm of your hand,
And Eternity in an hour.
— William Blake (1757–1827)

The signature operator of a control problem is a static operator that describes the definiteness of the problem w.r.t. the input. E.g., in minimization problems, the cost (to be minimized) is usually greater than $\varepsilon\|u\|_{\mathrm{L}^2}^2$ for some $\varepsilon > 0$, where $u : \mathbf{R}_+ \to U$ is the control input. In such problems, the signature operator S is uniformly positive ($S \gg 0$), whereas in indefinite control problems S is indefinite (but yet self-adjoint and invertible if the problem satisfies standard coercivity assumptions).

In this section, we shall see how to write an operator $S = S^* \in \mathcal{GB}(U \times W)$ or an operator $S := \mathbb{E}^* \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \mathbb{E}$, $\mathbb{E} \in \mathcal{G}\mathrm{TI}$, in the form $S = E^* \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] E$ for some $E \in \mathcal{GB}$; we also give some further results on the positive and negative eigenspaces of self-adjoint operators. In Section 2.3, we have derived further similar results.

In Chapters 8–11, the results of this and the previous section will be applied to the signature operators of optimization problems, Riccati equations and spectral factorization.

As before, symbols H, U, W, Y, H_k, Y_k ($k \in \mathbf{N}$) denote Hilbert spaces of arbitrary dimensions.

Lemma 2.4.1 *Define $J_H = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \in \mathcal{B}(H_1 \times H_2)$ and $J_Y = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \in \mathcal{B}(Y_1 \times Y_2)$. The following are equivalent:*

(i) $\dim H_1 \le \dim Y_1$ *and* $\dim H_2 \le \dim Y_2$,

(ii) $J_H = V^* J_Y V$ *for some* $V \in \mathcal{B}(H_1 \times H_2, Y_1 \times Y_2)$,

(iii) $J_H = \mathbb{V}^* J_Y \mathbb{V}$ *for some* $\mathbb{V} \in \mathrm{TI}(H_1 \times H_2, Y_1 \times Y_2)$.

Proof: 1° "(i)⇒(ii)": By Lemma A.3.1(a4)(iii), there are $T_k \in \mathcal{B}(H_k, Y_k)$ satisfying $T_k^* T_k = I_{H_k}$ ($k = 1, 2$). Take $V = \left[\begin{smallmatrix} T_1 & 0 \\ 0 & T_2 \end{smallmatrix}\right]$ to obtain $V^* J_Y V = \left[\begin{smallmatrix} T_1^* T_1 & 0 \\ 0 & -T_2^* T_2 \end{smallmatrix}\right] = J_H$.

2° "(ii)⇒(iii)": Trivial, because $\mathcal{B} \subset \mathrm{TI}$.

3° "(iii)⇒(i)": Define $T := \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] \in \mathcal{B}(H_1, H_1 \times H_2)$ and $\left[\begin{smallmatrix} \mathbb{V}_1 \\ \mathbb{V}_2 \end{smallmatrix}\right] := \mathbb{V}$. From $J_H = \mathbb{V}^* J_Y \mathbb{V} = \mathbb{V}_1^* \mathbb{V}_1 - \mathbb{V}_2^* \mathbb{V}_2$ we get $0 \ll I_{H_1} = T^* J_H T = (\mathbb{V}_1 T)^* (\mathbb{V}_1 T) - (\mathbb{V}_2 T)^* (\mathbb{V}_2 T)$, hence $(\mathbb{V}_1 T)^* (\mathbb{V}_1 T) \gg 0$, so $\dim Y_1 \ge \dim H_1$, by Lemma 2.2.1(c3), because $\mathbb{V}_1 T \in \mathrm{TI}_\infty(H_1, Y_1)$. Similarly, by setting $T := \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right] \in \mathcal{B}(H_2, H_1 \times H_2)$, we get $\dim Y_2 \ge \dim H_2$. □

When we require V or $\mathbb{V}$ to be invertible, the inequalities in (i) become equalities:

Corollary 2.4.2 *Define* $J_H = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \in \mathcal{B}(H_1 \times H_2)$ *and* $J_Y = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \in \mathcal{B}(Y_1 \times Y_2)$. *The following are equivalent:*

(i) $\dim H_1 = \dim Y_1$ *and* $\dim H_2 = \dim Y_2$,

(ii) $J_H = V^* J_Y V$ *for some* $V \in \mathcal{GB}(H_1 \times H_2, Y_1 \times Y_2)$,

(iii) $J_H = \mathbb{V}^* J_Y \mathbb{V}$ *for some* $\mathbb{V} \in \mathcal{G}\mathrm{TI}(H_1 \times H_2, Y_1 \times Y_2)$.

Proof: 1° "(i)⇒(ii)": This follows as in Lemma 2.4.1, with Lemma A.3.1(a5)(iii).

2° "(ii)⇒(iii)": Trivial, because $\mathcal{GB} \subset \mathcal{G}\mathrm{TI}$.

3° "(iii)⇒(i)": This follows from Lemma 2.4.1, because now also $J_Y = (\mathbb{V}^{-1})^* J_H (\mathbb{V}^{-1})$. □

Lemma 2.4.3 *Lemma 2.4.1 and Corollary 2.4.2 also hold with* S *in place of* J_H *if* $S = E^* J_H E$ *for some* $E \in \mathcal{GB}$. □

(This is obvious, since $J_H = E^{-*} S E^{-1}$.)

Any self-adjoint operator on H can be written in the form $E^* \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] E$ w.r.t. some decomposition of H:

Lemma 2.4.4 *Let* $S = S^* \in \mathcal{B}(H)$. *Then* $H = H_+ \oplus H_-$ *s.t.* $H_- = H_+^\perp$ *and* $S = E^* J_1 E$ *for some* $E \in \mathcal{B}(H)$.

If $S = S^* \in \mathcal{GB}(H)$, *then we can have* $E \in \mathcal{GB}(H)$ *above.*

Here $J_1 := \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]_{H_+ \times H_-} := P_+ - P_- \in \mathcal{GB}(H)$, where $P_\pm$ are the orthogonal projections of H onto $H_\pm$ (hence $P_- = P_+^\perp$), hence $J_1 = J_1^* = J_1^{-1}$.

(In fact, condition $J_1 = J_1^* = J_1^{-1}$ is equivalent to the fact that hence $J_1 = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \in \mathcal{GB}(H_+ \times H_-)$ for some closed subspaces H_+ and $H_- = H_+^\perp$ of H, since it implies that $\sigma(J_1) \subset \{-1, 1\}$, and that we can let H_+ and H_- to be the eigenspaces for 1 and -1, by Theorems 12.26 and 12.29 of [Rud73].)

Proof of Lemma 2.4.4: Apply

$$\begin{bmatrix} \widetilde{S}_+ & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{S}_- \end{bmatrix} = \begin{bmatrix} \widetilde{S}_+^{1/2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (-\widetilde{S}_-)^{1/2} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -I \end{bmatrix} \mathrm{diag}(\widetilde{S}_+^{1/2}, 0, (-\widetilde{S}_-)^{1/2}) \tag{2.38}$$

to Lemma A.3.2(f1) to observe that

$$E = \mathrm{diag}(\widetilde{S}_+^{1/2}, 0, (-\widetilde{S}_-)^{1/2}) \begin{bmatrix} P_+ \\ P_0 \\ P_- \end{bmatrix} \in \mathcal{B}(H) \tag{2.39}$$

will do in this lemma. If $H_0 := \mathrm{Ker}(S) = \{0\}$, then the middle row can be omitted; if $S \in \mathcal{GB}$, then $\widetilde{S}_\pm^{1/2} \in \mathcal{GB}(H_\pm)$, so that then $E \in \mathcal{GB}(H)$. □

Now we are ready for the main result (along with Corollary 2.4.2) of this section:

Theorem 2.4.5 *Let* $\mathbb{U}^* J_\gamma \mathbb{U} = S \in \mathcal{B}(U \times W)$ *with* $\mathbb{U} \in \mathcal{G}\mathrm{TI}(U \times W)$ *and* $J_\gamma := \left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right] \in \mathcal{B}(U \times W)$, $\gamma, \gamma' > 0$. *Then* $S = E^* J_{\gamma'} E$ *for some* $E \in \mathcal{GB}(U \times W)$.

Note that here $J_{\gamma'} := P_U - (\gamma')^2 P_W$ is defined by U and W, whereas Lemma 2.4.4 would only provide us with an analogous result for some ("nonfixed") H_+ and H_- in place of U and W.

Proof: Replace $\mathbb{U}$ by $\left[\begin{smallmatrix} I & 0 \\ 0 & \gamma \end{smallmatrix}\right]\mathbb{U}$ to get rid of γ (so that $S = \mathbb{U}^* J_1 \mathbb{U}$). Set $H := U \times W$. We have $S = S^* \in \mathcal{G}\mathrm{TI}(H)$ (because $S = \mathbb{U}^* J_1 \mathbb{U}$), hence $S = S^* \in \mathcal{G}\mathcal{B}(H)$. Thus, $S = T^* J_{H_\pm} T$ for some $T \in \mathcal{G}\mathcal{B}(H, H_+ \times H_-)$, by Lemma 2.4.4.

Therefore, $S = \mathbb{U}^* J_1 \mathbb{U}$ implies $J_{H_\pm} = \mathbb{T}^* J_1 \mathbb{T}$, with $\mathbb{T} := \mathbb{U} T^{-1} \in \mathcal{G}\mathrm{TI}(H_+ \times H_-, U \times W)$. By Corollary 2.4.2, $J_{H_\pm} = V^* J_1 V$ for some $V \in \mathcal{G}\mathcal{B}$, hence $S = T^* J_{H_\pm} T = (VT)^* J_1 VT = E^* J_1 E$, where $E = VT \in \mathcal{G}\mathcal{B}(H)$. $\square$

By allowing $\mathbb{U}$ to be noninvertible, we get the following variant of the above theorem:

Lemma 2.4.6 *Let* $\mathbb{U}^* J_\gamma \mathbb{U} = S \in \mathcal{G}\mathcal{B}(U \times W)$ *with* $\mathbb{U} \in \mathrm{TI}(U \times W, Z \times Y)$ *and* $J_\gamma := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \in \mathcal{B}(Z \times Y)$, $\gamma, \gamma' > 0$. *Then* $S = E^* J_{\gamma'} E$ *for some* $E \in \mathcal{B}(U \times W, Z \times Y)$.

Proof: The proof is virtually that of Theorem 2.4.5 with Lemma 2.4.1 in place of Corollary 2.4.2. $\square$

Notes

For separable U and W, Theorem 2.4.5 was given in the proof of Lemma 5.4 of [S98d]. Because the symbol $\widehat{\mathbb{U}}$ of $\mathbb{U}$ need not be well-defined anywhere on $i\mathbf{R}$ in the unseparable case that proof cannot be generalized (in fact, Theorem 3.1.3(a1), where we show the existence of such a symbol, seems to be known in the separable case only). This is why all Hilbert spaces were required to be separable in [S98d]; we shall remove this restriction in Corollary 11.4.9.

2.5 Losslessness

But who shall dare
To measure loss and gain in this wise?
Defeat may be victory in disguise;
The lowest ebb is the turn of the tide.
— Henry Wadsworth Longfellow (1807–1882)

In this section, we formulate both widely-used forms of losslessness and establish their equivalence for maps with finite-dimensional input spaces. We also give some necessary and/or sufficient conditions.

The importance of this concept is based on the fact that H^∞ problems are solvable iff certain lossless coprime factorizations exist, as shown in [Green] and extended to WPLSs in Theorems 11.2.7 and 12.3.6.

In state-space solutions, one usually relates the optimal control to a nonnegative stabilizing solution of the Riccati equation. Sometimes one replaces the Riccati equation by suitable coprime or spectral factorization of the I/O map; then the nonnegativity condition must be replaced by a losslessness condition, cf. Lemma 9.8.14 and Theorem 6.5 of [S98c].

Definition 2.5.1 *Let $J = J^* \in \mathcal{B}(Y)$, $S = S^* \in \mathcal{B}(U)$ and $\mathbb{N} \in \mathrm{TIC}(U,Y)$. The operator $\mathbb{N}$ is* (J,S)-lossless *iff $\mathbb{N}^*J\mathbb{N} = S$ and $\mathbb{N}^*\pi_-J\mathbb{N} \le \pi_-S$. The operator $\mathbb{N}$ is* frequency-domain (J,S)-lossless *iff $\mathbb{N}^*J\mathbb{N} = S$ and $\widehat{\mathbb{N}}(s)^*J\widehat{\mathbb{N}}(s) \le S$ for $s \in \mathbf{C}^+$.*

The (time-domain) losslessness implies frequency-domain losslessness, by Lemma 2.5.2. For $\dim U < \infty$ also the converse holds, by Proposition 2.5.4.

Both the time-domain and the frequency-domain concept have been widely used (under the name "lossless") in the study of H^∞ problems; see, e.g., [BH88] for losslessness and [Green] for frequency-domain losslessness.

One can interpret losslessness as "no energy is produced nor lost in the system, but some energy may be delayed", if $J = I = S$, but we usually take $J = J_1 = S$ ($J_\gamma := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$, $\gamma \in \mathbf{R}$), which implies the same conclusion on $\mathbb{N}_\frown$, where the direction of the second input and output signals have been reversed; see the proof of Lemma 2.5.3 for details.

The delay mentioned above cannot be negative (i.e., the energy put out before any $t > 0$ cannot exceed the energy put in before that time), because

$$\mathbb{N}^*\pi_-J\mathbb{N} \le S \Leftrightarrow \|\mathbb{N}u\|_{J,\mathrm{L}^2([0,t];Y)} \le \|u\|_{S,\mathrm{L}^2([0,t];U)} \text{ for all } u \in \pi_+\mathrm{L}^2,\ t > 0, \tag{2.40}$$

where $\|u\|_S := \langle u, Su\rangle$, by Lemma 2.2.4(b)(i)&(iv).

Section 6 of [S98c] contains several results on losslessness of stable operators, one of them is given below:

Lemma 2.5.2 *Let $\mathbb{N} \in \mathrm{TIC}(U,Y)$, $J \in \mathcal{B}(Y)$ and $S \in \mathcal{B}(U)$. If $\mathbb{N}^*\pi_-J\mathbb{N} \le \pi_-S$, then $\widehat{\mathbb{N}}(s)^*J\widehat{\mathbb{N}}(s) \le S$ for all $s \in \mathbf{C}^+$.*

Thus, losslessness implies frequency-domain losslessness. □

(This follows from Lemma 2.2.4(b).) See Proposition 2.5.4 for the converse under a Tauberian condition.

Next we give one more "almost equivalent" condition:

Lemma 2.5.3 *Let* $\mathbb{N} \in \mathrm{TIC}(U \times W, Y \times W)$ *and* $\gamma, \gamma' > 0$.
If $\mathbb{N}^* J_\gamma \mathbb{N} = J_{\gamma'}$ *and* $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}(W)$, *then* $\mathbb{N}$ *is* $(J_\gamma, J_{\gamma'})$*-lossless.*

The converse is not true: $\mathbb{N} := \left[\begin{smallmatrix} I & 0 \\ 0 & \tau^t \end{smallmatrix}\right] \in \mathrm{TIC}(U \times W)$ is (J_γ, J_γ)-lossless (for any $\gamma, t > 0$) but $\mathbb{N}_{22}^{-1} \notin \mathrm{TIC}$ (cf. the lemma below). However, the converse is true in the finite-dimensional case, by Proposition 2.5.4(1).

Proof: We assume that $\gamma = 1 = \gamma'$ (and replace $\mathbb{N}$ by $\left[\begin{smallmatrix} I & 0 \\ 0 & \gamma \end{smallmatrix}\right] \mathbb{N} \left[\begin{smallmatrix} I & 0 \\ 0 & \gamma'^{-1} \end{smallmatrix}\right]$ in the general case). We set here $\left[\begin{smallmatrix} \mathbb{Q} & \mathbb{R} \\ \mathbb{S} & \mathbb{T} \end{smallmatrix}\right] := \mathbb{N}$ and $\mathbb{N}_\curvearrowleft := \left[\begin{smallmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & I \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ \mathbb{S} & \mathbb{T} \end{smallmatrix}\right]^{-1}$ (same system with second input and output signals reversed) to clarify the proof. For $\pi = I$ as well as for $\pi = \pi_-$ we have

$$\mathbb{N}_\curvearrowleft^* \pi \mathbb{N}_\curvearrowleft \leq \pi \Leftrightarrow \quad \begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & I \end{bmatrix}^* \pi \begin{bmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} I & 0 \\ \mathbb{S} & \mathbb{T} \end{bmatrix}^* \pi \begin{bmatrix} I & 0 \\ \mathbb{S} & \mathbb{T} \end{bmatrix} \tag{2.41}$$

$$\Leftrightarrow \quad \begin{bmatrix} \mathbb{Q}^* \pi \mathbb{Q} & \mathbb{Q}^* \pi \mathbb{R} \\ \mathbb{R}^* \pi \mathbb{Q} & \mathbb{R}^* \pi \mathbb{R} + I \end{bmatrix} \leq \begin{bmatrix} \mathbb{S}^* \pi \mathbb{S} + I & \mathbb{S}^* \pi \mathbb{T} \\ \mathbb{T}^* \pi \mathbb{S} & \mathbb{T}^* \pi \mathbb{T} \end{bmatrix} \tag{2.42}$$

$$\Leftrightarrow \quad \begin{bmatrix} \mathbb{Q}^* \pi \mathbb{Q} - \mathbb{S}^* \pi \mathbb{S} & \mathbb{Q}^* \pi \mathbb{R} - \mathbb{S}^* \pi \mathbb{T} \\ \mathbb{R}^* \pi \mathbb{Q} - \mathbb{T}^* \pi \mathbb{S} & \mathbb{R}^* \pi \mathbb{R} - \mathbb{T}^* \pi \mathbb{T} \end{bmatrix} \leq \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{2.43}$$

$$\Leftrightarrow \quad \mathbb{N}^* \pi J_1 \mathbb{N} \leq \pi J_1. \tag{2.44}$$

The inequality in (2.44) holds for $\pi = I$, by the assumption $\mathbb{N}^* J_1 \mathbb{N} = J_1$, hence we have $\mathbb{N}_\curvearrowleft^* \mathbb{N}_\curvearrowleft \leq I$ (from (2.41)). But $\mathbb{N}_\curvearrowleft^* (\pi_+ + \pi_-) \mathbb{N}_\curvearrowleft \leq I$ implies that

$$\pi_- \mathbb{N}_\curvearrowleft^* \pi_- \mathbb{N}_\curvearrowleft \pi_- = \pi_- - \pi_- \mathbb{N}_\curvearrowleft^* \pi_+ \mathbb{N}_\curvearrowleft \pi_- \leq \pi_-, \tag{2.45}$$

hence $\pi_- \mathbb{N}^* \pi_- J_1 \mathbb{N} \pi_- \leq \pi_- J_1$. $\square$

By the above lemma, we can establish the Tauberian converse mentioned above:

Proposition 2.5.4 (Lossless $\Leftrightarrow$ f.-d. lossless) *Let* $\mathbb{N} \in \mathrm{TIC}(U \times W, Y \times W)$ *and* $\gamma, \gamma' > 0$. *Assume that (1), (2) or (3) holds, where*

(1) $\dim W < \infty$*;*

(2) $\widehat{\mathbb{N}}_{22}(s_0) \in \mathcal{GB}$ *for some* $s_0 \in \mathbf{C}^+$ *(for uniformly regular* $\mathbb{N}$ *we allow for* $s_0 = +\infty$*);*

(3) $\dim U < \infty$ *and* $\mathbb{N} = \left[\begin{smallmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ 0 & I \end{smallmatrix}\right] \mathbb{X}^{-1}$, *where* $\mathbb{D}_{11}, \mathbb{D}_{12}, \mathbb{X}, \mathbb{X}^{-1} \in \mathrm{TIC}$*;*

Then $\mathbb{N}$ *is* $(J_\gamma, J_{\gamma'})$*-lossless iff* $\mathbb{N}$ *is frequency-domain* $(J_\gamma, J_{\gamma'})$*-lossless.*
Moreover, if $\mathbb{N}$ *is* $(J_\gamma, J_{\gamma'})$*-lossless, then* $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}$ *and* $\|\mathbb{N}_{12} \mathbb{N}_{22}^{-1}\| < \gamma$.

One could easily extend the equivalence for more general J and S by a coordinate transform of their positive and negative eigenspaces (see., e.g., Lemma 2.4.4), but J_γ and $J_{\gamma'}$ satisfy our needs.

Note that this implies that also the corresponding discrete time result holds, by Corollary 13.2.4.

Proof: 1° *Case* $\mathbb{N}$ *f.-d. lossless:* Let $\mathbb{N}$ be frequency-domain $(J_\gamma, J_{\gamma'})$-lossless. The right-bottom corner of the frequency-domain losslessness equation $\widehat{\mathbb{N}}(s)^* J_\gamma \widehat{\mathbb{N}}(s) \le J_{\gamma'}$ for all $s \in \mathbf{C}^+$ shows that $\gamma^2 \widehat{\mathbb{N}}_{22}^* \widehat{\mathbb{N}}_{22} \ge \widehat{\mathbb{N}}_{12}^* \widehat{\mathbb{N}}_{12} + \gamma'^2 I \gg 0$ on $\mathbf{C}^+$, so $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}$, by Proposition 2.2.5 (use its (4) for (3) above). Consequently, $\mathbb{N}$ is $(J_\gamma, J_{\gamma'})$-lossless, by Lemma 2.5.3.

2° *Case* $\mathbb{N}$ *lossless:* By Lemma 2.5.2 losslessness implies frequency-domain losslessness; the converse was proved in 1°. By this and 1°, we have $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}$ in either case, so from the equation

$$\mathbb{N}_{12}^* \mathbb{N}_{12} - \gamma^2 \mathbb{N}_{22}^* \mathbb{N}_{22} = -\gamma'^2 I \tag{2.46}$$

(this is the $(2,2)$-block of $\mathbb{N}^* J_\gamma \mathbb{N} = J_{\gamma'}$) we get that $\|\mathbb{N}_{12}\mathbb{N}_{22}^{-1}\| < \gamma$, by Lemma A.3.1(e2). □

We now summarize some of the above results to a condition that is needed for H^∞ problems:

Corollary 2.5.5 *Let* $\mathbb{N} \in \mathrm{TIC}(U_1 \times U_2, Y \times U_2)$ *and* $\gamma, \gamma' > 0$. *Then the following are equivalent:*

(i) $\mathbb{N}$ *is* $(J_\gamma, J_{\gamma'})$*-lossless and* $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(U_2)$*;*

(ii) $\mathbb{N}$ *is frequency-domain* $(J_\gamma, J_{\gamma'})$*-lossless and* $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(U_2)$*;*

(iii) $\mathbb{N}^* J_\gamma \mathbb{N} = J_{\gamma'}$ *and* $\mathbb{N}_{22} \in \mathcal{G}\mathrm{TIC}(U_2)$. □

(This follows from Proposition 2.5.4(2) and Lemma 2.5.3.)

Notes

In the finite-dimensional case, Joseph Ball and William Helton (e.g., [BH88]) have studied losslessness in detail under a chain–scattering formalism. There are also (finite-dimensional) extensions of this concept to time-variant systems (see [Gohberg] or [LKS]) and to nonlinear systems (see [BH92]).

Losslessness was adopted to an infinite-dimensional setting in [S98c], from which Definition 2.5.1 and 2.5.2 are adopted in the same generality as here. See [S98c] for further results and [BH88] for further notions of losslessness and their physical interpretations.

We believe that losslessness is strictly stronger than frequency-domain losslessness. However, it remains an open problem to prove this or the converse. This problem lies close to that of the possible converses to Lemma 2.2.4(a) and Proposition 4.1.7(C); see also Lemma 2.2.4(b)(i)&(iv).

2.6 MTI and its subclasses

According to convention there is a sweet and a bitter, a hot and a cold, and according to convention, there is an order. In truth, there are atoms and a void.

— Democritus (460–370 B.C.), 400 B.C.

In this section we shall define $\mathrm{MTI}(U,Y)$ and $\mathrm{CTI}(U,Y)$ ("M" for measures and "C" for continuity on $i\mathbf{R}\cup\{\infty\}$), which are subspaces of $\mathrm{TI}(U,Y)$, and a number of their subspaces. We also list their basic properties. Recall that U and Y refer to Hilbert spaces of arbitrary dimensions.

The class MTI consists of those elements $\mathbb{E}\in\mathrm{TI}$ that are of form (2.50), that is, a the convolution with a measure consisting of a discrete part plus an L^1 part. Further regularity properties of this class are listed in Proposition 6.3.4, and its spectral factorization properties are treated in Chapter 5 and summarized in Theorem 5.2.7; see also Section 8.4.

When solving several standard control problems in Part III, we can show the sufficiency of classical "necessary and sufficient" conditions, but these conditions are not necessary in general (cf. Example 11.3.7). However, the spectral factorization and regularity properties and algebraic properties of MTI (see Theorems 8.4.9 and 2.6.4) are more or less the same as those of rational functions. This allows us to establish also the necessity of the conditions mentioned above provided that the I/O map of the system belongs to this class.

By *regularity* of a map $\mathbb{D}\in\mathrm{TIC}_\infty(U,Y)$ we mean the existence of the "feedthrough operator" $D:=\widehat{\mathbb{D}}(+\infty):=\lim_{r\to+\infty}\widehat{\mathbb{D}}(r)$. The exact definition and the concepts ULR, UHPR, SLR and SHPR are formulated in Definition 6.2.3 (see also Proposition 6.2.7), but the reader may skip these concepts until they are needed in later chapters.

At this point it suffices for most readers just to have a glance at Definition 2.6.3 and Theorem 2.6.4 and proceed to the next chapter.

By $\mathrm{CTI}(U,Y)$ we mean the maps $\mathbb{E}\in\mathrm{TI}(U,Y)$ that are induced by some $\widehat{\mathbb{E}}\in\mathcal{C}(i\mathbf{R}\cup\{\infty\};\mathcal{B}(U,Y))$ through $\mathbb{E}u:=\widehat{\mathbb{E}}\widehat{u}$ for all $u\in\mathrm{L}^2(\mathbf{R};U)$ (this is a special case of the symbols of Theorem 3.1.3(a1)):

Definition 2.6.1 (CTIC) *We set*[2] $\mathrm{CTI}(U,Y):=\{\mathbb{E}\in\mathrm{TI}\,\big|\,\widehat{\mathbb{E}}\in\mathcal{C}(i\mathbf{R}\cup\{\infty\};\mathcal{B}(U,Y))\}$ *and* $\mathrm{CTIC}(U,Y):=\mathrm{CTI}(U,Y)\cap\mathrm{TIC}(U,Y)$. *We call* CTIC *the (time domain)* half-plane algebra. *Moreover, we set*

$$\mathrm{CTI}^{\mathcal{BC}}(U,Y):=\{\mathbb{E}\in\mathrm{CTI}\,\big|\,\widehat{\mathbb{E}}(s)-\widehat{\mathbb{E}}(\infty)\in\mathcal{BC}\ \text{for all}\ s\in i\mathbf{R}\},\tag{2.47}$$

$$\mathrm{CTIC}^{\mathcal{BC}}(U,Y):=\mathrm{CTI}^{\mathcal{BC}}\cap\mathrm{CTIC}.\tag{2.48}$$

We equip all these spaces with the TI *norm (i.e., with the* $\mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$ *operator norm). If* $\mathbb{E}\in\mathrm{CTI}$*, we call* $E:=\widehat{\mathbb{E}}(\infty)$ *the* feedthrough operator *of* $\mathbb{E}$.

[2] We equip $i\mathbf{R}\cup\{\infty\}$ with its one-point-compactification topology; equivalently, the topology induced by the topology of $\partial\mathbf{D}$ through the Cayley transform (this implies that "$+i\infty=-i\infty$"). Similarly, $\overline{\mathbf{C}^+}\cup\{\infty\}$ is equipped with the one-point-compactification topology, and that topology is induced by the topology of $\overline{\mathbf{D}}$ through the Cayley transform.

As above, we always denote the "feedthrough" operators by the same letters as the corresponding TI operators. From Lemma 2.6.2 we observe that the above concept of feedthrough operators coincides with that of Definition 6.2.3.

One easily verifies that all the four spaces defined above are closed subspaces of TI (because the set $\mathcal{BC}$ of compact (see p. 871) linear mappings is closed in $\mathcal{B}$, by Lemma A.3.4(B1)).

We call the function $\phi_{\mathrm{Cayley}} : s \mapsto (1-s)/(1+s)$ the *Cayley transform*. It maps $\mathbf{C}^+$ one-to-one and onto $\mathbf{D}$, and $\overline{\mathbf{C}^+} \cup \{\infty\}$ one-to-one and onto $\overline{\mathbf{D}}$, and it is the inverse of itself. See Lemma 13.2.1 for details.

Lemma 2.6.2 *The space* $\mathrm{CTI}(U,Y)$ *consists of those* $\mathbb{E} \in \mathrm{TI}(U,Y)$*, for which* $s \mapsto \widehat{\mathbb{E}} \circ \phi_{\mathrm{Cayley}}^{-1}$ *belongs to the set* $\mathcal{C}(\partial\mathbf{D};\mathcal{B}(U,Y))$*, equivalently, for which* $\widehat{\mathbb{E}} \in \mathcal{C}(i\mathbf{R};\mathcal{B}(U,Y))$ *and* $\widehat{\mathbb{E}}$ *has the same limit at* $\pm i\infty$*.*

The space $\mathrm{CTIC}(U,Y)$ *consists of those* $\mathbb{D} \in \mathrm{TIC}(U,Y)$*, for which* $s \mapsto \widehat{\mathbb{D}} \circ \phi_{\mathrm{Cayley}}^{-1}$ *belongs to the (uniform)* disc algebra $\mathcal{C}(\overline{\mathbf{D}};\mathcal{B}(U,Y)) \cap \mathrm{H}(\mathbf{D};\mathcal{B}(U,Y))$*, equivalently,* $\widehat{\mathrm{CTIC}} = \mathcal{C}(\overline{\mathbf{C}^+} \cup \{\infty\};\mathcal{B}(U,Y)) \cap \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$*; a third equivalent condition is that* $\widehat{\mathbb{D}} \in \widehat{\mathrm{CTIC}}$ *iff* $\widehat{\mathbb{D}} \in \mathcal{C}(\overline{\mathbf{C}^+};\mathcal{B}(U,Y)) \cap \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$ *and* $\widehat{\mathbb{D}}$ *has a uniform limit (on* $\mathbf{C}^+$*) at infinity (i.e.,* $\mathbb{D}$ *is uniformly half-plane-regular).*

Moreover, $\mathbb{D} \in \mathrm{CTIC}^{\mathcal{BC}}$ *implies that* $\mathbb{D}(s) - D \in \mathcal{BC}$ *for all* $s \in \overline{\mathbf{C}^+} \cup \{\infty\}$*.*

Finally, if $\mathbb{E} \in \widehat{\mathrm{CTI}}(U,Y)$*, then* $\widehat{\mathbb{E}^*} = \widehat{\mathbb{E}}^* \in \widehat{\mathrm{CTI}}(U,Y)$*.*

Proof: 1° CTI: The above characterizations of CTI are obviously equivalent to the fact that $\widehat{\mathbb{E}} \in \mathcal{C}(i\mathbf{R} \cup \{\infty\};\mathcal{B}(U,Y))$, which in turn always defines a TI map (by, e.g., Theorem 3.1.3(a)).

2° CTIC: Let $\mathbb{D} \in \mathrm{CTIC}(U,Y)$. Let $f \in \mathcal{C}(i\mathbf{R} \cup \{\infty\};\mathcal{B}(U,Y))$ and $F \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$ be its Fourier and Laplace transforms, respectively. By [Rud73, Theorem 11.30(b)], for each $\Lambda \in Y^*$ and $u_0 \in U$, there is $g_{\Lambda,u_0} \in \mathrm{L}^\infty(i\mathbf{R})$ s.t. $\Lambda F u_0 \in \mathrm{H}^\infty(\mathbf{C}^+)$ is the Poisson integral $P_r * g_{\Lambda,u_0}(i\cdot)$ [Lemma D.1.8]. Consequently, $\mathcal{L}\Lambda\mathbb{D}\phi u_0 = g_{\Lambda,u_0}\widehat{\phi}u_0$ on $i\mathbf{R}$ for any $\phi \in \mathrm{L}^2(\mathbf{R}_+)$, hence $g_{\Lambda,u_0} = \Lambda f u_0$ a.e. on $i\mathbf{R}$.

It follows that $\Lambda F u_0 = \Lambda(P_r * f)u_0$ on $\mathbf{C}^+$, still for arbitrary Λ and u_0, hence $F(ir+\cdot) = P_r * f(i\cdot)$ on $\mathbf{C}^+$.

From Lemma D.1.8(a2) it now follows that $F(ir+t) = (P_r * f)(t) - D = P_r * (f-D)(t) \to 0$ as $|ir+t| \to \infty$; in particular, $\widehat{\mathbb{D}} \in \mathcal{C}(\overline{\mathbf{C}^+};\mathcal{B}(U,Y))$ (and in H^∞). Now the other two characterizations of CTIC follow from this. (If $\widehat{\mathbb{D}} \in \mathcal{C}(\overline{\mathbf{C}^+};\mathcal{B}(U,Y)) \cap \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$ is UHPR, then $\widehat{\mathbb{D}}$ has a uniform limit as $\overline{\mathbf{C}^+} \ni s$ (not necessarily $\mathbf{C}^+ \ni s$ as for UHPR) and $|s| \to \infty$, by Lemma 6.3.6(a1); thus, then $\widehat{\mathbb{D}}$ is continuous at ∞ too.)

3° $\mathrm{CTIC}^{\mathcal{BC}}$: Also this claim follows from the Poisson integral formula (for $\widehat{\mathbb{D}} - D$).

4° $\widehat{\mathbb{E}}^*$: The equation

$$\int_{i\mathbf{R}} \langle \widehat{f}, \widehat{\mathbb{E}}^* \widehat{g} \rangle_U \, dm = \int_{i\mathbf{R}} \langle \widehat{\mathbb{E}}\widehat{f}, \widehat{g} \rangle_Y \, dm = \int_{i\mathbf{R}} \langle \widehat{f}, \widehat{\mathbb{E}^*}\widehat{g} \rangle_U \, dm \tag{2.49}$$

shows that $(\widehat{\mathbb{E}}^* - \widehat{\mathbb{E}^*})\widehat{g} \in \mathrm{L}^2(i\mathbf{R};U)$ is zero a.e., by, e.g., Theorem B.4.12, hence $\widehat{\mathbb{E}}^*$ is the unique $\mathrm{L}^\infty_{\mathrm{strong}}$ function (equivalence class) corresponding to $\mathbb{E}^*$ (see

Theorem 3.1.3). Obviously, $\mathbb{E}^* \in \mathcal{C}(i\mathbf{R} \cup \{\infty\}; \mathcal{B}(Y,U)) = \widehat{\mathrm{CTI}}(Y,U)$. $\square$

Given the system $x' = Ax + Bu$, $y = Cx + Du$, $x(0) = 0$, we have $y = f * u + Du$, where $f := Ce^{A\cdot}B$, provided that $A, B, C, D \in \mathcal{B}$.

Even for somewhat unbounded A, B, C, D, the map $\mathbb{D} : u \mapsto y$ is still of form $\mathbb{D}u = \mu * u$ for a measure μ consisting of a $\mathrm{L}^1_{\mathrm{loc}}$ part plus a feedthrough ($D\delta_0$, where δ_0 is the unit mass at zero (the "delta function")); this will be denoted by $\mathrm{MTIC}_\infty^{\mathrm{L}^1}$ below. If we allow for delays, we end up with (2.50) or "the measures with no continuous singular part", MTIC_∞.

The above motivates the definition below. For most readers, it suffices just to note the definitions of MTI, MTIC, $\mathrm{MTIC}^{\mathrm{L}^1}$ and $\mathrm{MTIC}_{\mathbf{S}}$ for $\mathbf{S} = T\mathbf{Z}$, $T > 0$ and then proceed to the remarks below the definition:

Definition 2.6.3 (MTI, $\mathbf{MTI_d}$, $\mathbf{MTI^{L^1}}$) *We define* $\mathrm{MTI}(U,Y)$ *to be the space of operators* $\mathbb{E} \in \mathrm{TI}(U,Y)$ *of the form* $\mathbb{E}u = (f + \sum_{k=0}^\infty T_k \delta_{t_k}) * u \quad (u \in \mathrm{L}^2(\mathbf{R};U))$, *i.e.,*

$$(\mathbb{E}u)(t) = \sum_{k=0}^{\infty} T_k u(t - t_k) + \int_{-\infty}^{\infty} f(t-r)u(r)\,dr, \tag{2.50}$$

where $f \in \mathrm{L}^1(\mathbf{R}; \mathcal{B}(U,Y))$, $T_k \in \mathcal{B}(U,Y)$ *for all k, and the* MTI *norm ("uniform total variation norm")*

$$\|\mathbb{E}\|_{\mathrm{MTI}} := \|f\|_{\mathrm{L}^1} + \sum_k \|T_k\| \tag{2.51}$$

is finite; here $\delta_t * := \delta_0(\cdot - t)* = \tau(-t)$ *is the delay of time* $t \in \mathbf{R}$.

For $\mathbb{E} \in \mathrm{MTI}$ *of the above form and* $\mathbf{S} \subset \mathbf{R}$ *we set*

$$\Pi_{\mathrm{L}^1}\mathbb{E} := f*, \quad \Pi_{\mathbf{S}}(\mathbb{E}) := (\sum_{t_k \in \mathbf{S}} T_k \delta_{t_k})*, \quad \mathrm{supp}_{\mathrm{d}}(\mathbb{E}) := \{t \in \mathbf{R} \,|\, \Pi_{\{t\}}\mathbb{E} \neq 0\}. \tag{2.52}$$

Thus, $\Pi_{\mathbf{S}}$ *is the projection to the measure carried on* $\mathbf{S}$*, and* $\mathrm{supp}_{\mathrm{d}}(\mathbb{E})$ *is the set of the nonzero atoms of* $\mathbb{E}$.

By the Wiener class ($\mathrm{MTI}^{\mathrm{L}^1}$) $\mathrm{MTI}^{\mathrm{L}^1}$ *we mean the subspace of those* $\mathbb{E} \in \mathrm{MTI}$ *for which* $\mathrm{supp}_{\mathrm{d}}(\mathbb{E}) \subset \{0\}$*, and by the* discrete measure class $\mathrm{MTI_d} := \Pi_{\mathbf{R}}\mathrm{MTI}$ *the subspace of those* $\mathbb{E} \in \mathrm{MTI}$ *for which the* L^1 *part is zero (hence* $\mathrm{MTI} = \mathrm{MTI_d} \oplus (\mathrm{L}^1 *)$*).*

For the causal versions ($f = \pi_+ f$ *and* $t_k \geq 0$ *for all k) of these spaces, we add the letter C at the end:*

$$\mathrm{MTIC} := \mathrm{MTI} \cap \mathrm{TIC}, \ \mathrm{MTIC}^{\mathrm{L}^1} := \mathrm{MTI}^{\mathrm{L}^1} \cap \mathrm{TIC}, \ \textit{and} \ \mathrm{MTIC_d} = \mathrm{MTI_d} \cap \mathrm{TIC}. \tag{2.53}$$

We define $\mathrm{MTI}^{\mathcal{BC}}(U,Y) \subset \mathrm{MTI}(U,Y)$ *to be the subspace of operators of form (2.50) s.t.* $f \in \mathrm{L}^1(\mathbf{R}_+; \mathcal{BC}(U,Y))$ *and* $T_k \in \mathcal{BC}(U,Y)$ *for* $t_k \neq 0$*, and we set* $\mathcal{A}^{\mathcal{BC}} := \mathcal{A} \cap \mathrm{MTI}^{\mathcal{BC}}$ *when* $\mathcal{A}$ *is any of the spaces defined above.*

If $\mathbf{S} \subset \mathbf{R}$*, we set* $\mathrm{MTI}_{\mathbf{S}} := \{\mathbb{E} \in \mathrm{MTI} \,|\, \mathrm{supp}_{\mathrm{d}}(\mathbb{E}) \subset \mathbf{S}\}$ *and* $\mathcal{A}_{\mathbf{S}} := \mathcal{A} \cap \mathrm{MTI}_{\mathbf{S}}$ *when* $\mathcal{A}$ *is any of the spaces defined above.*

We define MTI_ω *(*$\omega \in \mathbf{R}$*) and its subspaces (with subscript* ω*) to be the* *corresponding spaces for which*

$$\|\mathbb{E}\|_{\mathrm{MTI}_\omega} := \|e^{-\omega\cdot} f\|_1 + \sum_{k\in\mathbf{N}} \|e^{-\omega t_k} T_k\|_{\mathcal{B}(U,Y)} < \infty. \tag{2.54}$$

Finally, we define the strong equivalents of MTI_ω *and its subspaces by setting* $\mathrm{SMTI}_\omega(U,Y) := \mathcal{B}(U, \mathrm{MTI}_\omega(\mathbf{C},Y))$*, and analogously for any other space in place of* MTI_ω.

We make below a series of remarks concerning the above definitions. See Theorem 2.6.4 for more on the properties of these classes; note in particular (by (h2)) that $\mathrm{MTI}_\omega \subset \mathrm{SMTI}_\omega \underset{c}{\subset} \mathrm{TI}_\omega$ ($\omega \in \mathbf{R}$). More results on the regularity of $\mathrm{MTIC}^{\mathrm{L}^1}$, $\mathrm{SMTIC}^{\mathrm{L}^1}$ and their weak equivalent are given in Proposition 6.3.4.

The set $\mathrm{MTI}_{\mathrm{d}}$ is known as the *Almost Periodic Wiener algebra* (APW) by Karlovich, Spitkovsky et al. and as the *Wiener–Pitt algebra* (WP) by Babadzhanyan and Rabinovich [BR]. For $\dim U, \dim Y < \infty$, the set MTIC is the well-known set denoted by $\mathcal{A}$ or $\mathcal{A}(0)$ (or sometimes by $\mathrm{LA}^+(0)$) in the texts of Callier, Desoer, Winkin and others (see, e.g., [CD80], [CW99], [CZ]).

The so called *causal Wiener class* $\mathrm{MTIC}^{\mathrm{L}^1}$ consists of operators of the form $\mathbb{E} = f + T_1\delta_0$, where $f \in \pi_+\mathrm{L}^1$, i.e., of (I/O) maps $\mathbb{E}$ having an L^1 impulse response plus a feedthrough operator. By Lemma D.1.23, transfer functions that are holomorphic around $i\mathbf{R} \cap \{\infty\}$ belong to $\mathrm{MTI}^{\mathrm{L}^1}$. The Wiener class (or "Wiener algebra") $\mathrm{MTI}^{\mathrm{L}^1}$ is denoted by W or $\mathcal{W}$ in [CG81] and [CG97], and $\mathrm{MTIC}^{\mathrm{L}^1}$ is denoted by $\mathcal{A}$ in [CG97].

The MTI norm ("measure norm", i.e., total variation) and the SMTI norm are stronger than the TI norm ($\mathcal{B}(\mathrm{L}^2)$ norm), by Theorem 2.6.4(h2); in fact, they are strictly stronger, even in the scalar case, because if $f \in \mathrm{L}^1(\mathbf{R}+)$, then $\|f*\|_{\mathrm{TI}} = \|\widehat{f}\|_{\mathrm{L}^\infty}$ (by Theorem 3.1.3) and $\|\widehat{f}\|_{\mathrm{L}^\infty}$ may be less than $\|f\|_1$.

Each $\mathbb{E} \in \mathrm{MTI}$ corresponds (linearly and isometrically, in particular, $\|\mathbb{E}\|_{\mathrm{MTI}} = \int_{\mathbf{R}} d|\mu|$) to a Borel measure $\mu = f\,dt + \sum_{k=0}^\infty T_k\delta_{t_k}$ ($\mathcal{B}(U,Y)$-valued, countably additive and of bounded uniform total variation) consisting of a discrete (atomic) part and of an absolutely continuous part (i.e., one without a singular continuous part) through the formula $\mathbb{E}u = \mu * u$ for all $u \in \mathrm{L}^2$ (equivalently, through $\widehat{\mathbb{E}} = \widehat{\mu}$). This correspondence is an isometric isomorphism (onto if $\dim U < \infty$ or $\dim Y < \infty$, as one easily deduces from the results on pages 99 and 82 of [DU]). However, if $\dim U = \infty$, then some absolutely continuous $\mathcal{B}(U)$-valued Borel measures are not differentiable a.e. (see p. 219 & p. 217(3) of [DU]).

The set $\mathrm{supp}_{\mathrm{d}}(\mathbb{E})$ is denoted by $\sigma(\widehat{\mathbb{E}})$ (or $\sigma(\widehat{\mu})$) in the theory of almost-periodic functions (see p. 2188 of [RSW]). Clearly the set L^1 is an ideal of MTI (we identify a measure, its derivative and the corresponding TI operator when there is no risc of misconception).

For $T \in \mathbf{R}$, the class $\mathrm{MTI}_{T\mathbf{Z}}$ corresponds to MTI "measures with equally-spaced atoms" with space T. Because $\mathrm{MTI}_{T\mathbf{Z}}$ is isomorphic to $\ell^1(\mathbf{Z})$ and L^1* is an ideal of $\mathrm{MTI}_{T\mathbf{Z}}$, several factorization and invertibility problems on $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$ can be reduced to those on ℓ^1 and to those on $\mathrm{MTI}^{\mathrm{L}^1}$. Due to this fact (the earliest application of which we have seen in the thesis [Winkin]), we shall use the class $\mathrm{MTI}_{T\mathbf{Z}}$ extensively throughout this monograph.

As the definition shows, by the subscript $\mathcal{BC}$ we mean that $\widehat{\mathbb{E}} - E$ is $\mathcal{BC}$-valued. In the case of $\mathrm{MTI}^{\mathcal{BC}}$ and its subclasses, this means that f and T_k (for k s.t. $t_k \neq 0$) are $\mathcal{BC}$-valued. The class $\mathrm{CTIC}^{\mathcal{BC}}$ consists $\widehat{\mathbb{E}} = E + \hat{g}$, where $E \in \mathcal{B}$, $\hat{g} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{BC}) \cap \mathcal{C}(\overline{\mathbf{C}^+} \cup \{\infty\}; \mathcal{BC})$, and $\hat{g}(\infty) = 0$.

All well-posed fractions of exponential WTIC functions have a d.c.f. if $\dim U, \dim Y < \infty$, by Theorem 2.1 of [CD80]. This follows from the fact that a function in H^∞_ω for some $\omega < 0$ has only a finite number of zeros on $\mathbf{C}^+$.

The MTI spaces defined above are ULR and have beautiful factorization properties (see Theorems 2.6.4 and 4.1.6), which make them perfect candidates for stabilization theory. Moreover, most of them also admit spectral factorization (see Theorem 8.4.9), hence our optimization theory becomes more complete for such maps than for general TI maps.

Next we note that these classes are closed under composition, causal adjoints and inverses, and we list some of their further properties:

Theorem 2.6.4 (MTI, SMTI, CTI) *Let $\mathcal{A}$ be one of the classes* CTI, $\mathrm{CTI}^{\mathcal{BC}}$, MTI, $\mathrm{MTI}^{\mathcal{BC}}$, $\mathrm{MTI_d}$, $\mathrm{MTI}^{\mathcal{BC}}_\mathrm{d}$, $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}$, $\mathrm{MTI}_\mathbf{S}$ *and* $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$, *where* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$. *Set* $\widetilde{\mathcal{A}} := \mathcal{A} \cap \mathrm{TIC}$.

Let $\mathbb{E} \in \mathcal{A}(U,Y)$, $\mathbb{F} \in \mathcal{A}(Y,Z)$ *and* $\omega, \omega' \in \mathbf{R}$. *Then the following hold:*

(a1) $\mathcal{A}(U,Y)$ *is a Banach space,* $\mathcal{A}(U)$ *is a Banach algebra and* $\mathcal{B} \underset{a}{\subset} \mathcal{A} \underset{a}{\subset} \mathrm{TI}$ *(cf. Definition 6.2.4); in particular,* $\mathcal{A}\mathcal{A} = \mathcal{A}$ *and* $\widetilde{\mathcal{A}}\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}} \underset{a}{\subset} \mathrm{TIC}$.

(a2) If $\mathcal{A} \neq \mathrm{CTI}, \mathrm{CTI}^{\mathcal{BC}}$, *then* $\mathbb{E}\mathbb{F}$ *is equal to the convolution of* $\mathbb{E}$ *and* $\mathbb{F}$, *and* $\widehat{\mathbb{E}}$ *is equal to the Laplace transform of* $\mathbb{E}$ *taken in* MTI *sense (as in (D.22)).*

(b1) $\mathbb{E}^*, \mathbb{E}^\mathrm{d} \in \mathcal{A}$ *and* $\mathbb{E}\mathbb{F} \in \mathcal{A}$.

(b2) If $\mathbb{E}, \mathbb{F} \in \widetilde{\mathcal{A}}$, *then* $\mathbb{E}^\mathrm{d} \in \widetilde{\mathcal{A}}$ *and* $\mathbb{E}\mathbb{F} \in \widetilde{\mathcal{A}}$.

(b3) We have $\{\mathbb{E}^* \mid \mathbb{E} \in \mathcal{A}_\omega\} = \mathcal{A}_{-\omega}$ *and* $\|\mathbb{E}\|_{\mathcal{A}_\omega} = \|\mathbb{E}^*\|_{\mathcal{A}_{-\omega}}$.

(c1) $\mathbb{E} \in \mathcal{G}\mathrm{TI}(U,Y) \Leftrightarrow \mathbb{E} \in \mathcal{G}\mathcal{A}(U,Y)$.

(c2) $\mathbb{E} \in \mathcal{G}\mathrm{TIC}(U,Y) \Leftrightarrow \mathbb{E} \in \mathcal{G}\widetilde{\mathcal{A}}(U,Y)$.

(d) If $\mathbb{F} = F + f* \in \mathrm{MTI}^{\mathrm{L}^1}(Y,Z)$ *and* $\mathbb{G} = G + g* \in \mathrm{MTI}^{\mathrm{L}^1}(U,Y)$, *then* $\mathbb{F}\mathbb{G} = FG + (Fg + fG + f * g)* \in \mathrm{MTI}^{\mathrm{L}^1}(U,Z)$.

(e1) $\widehat{\mathbb{E}} \in \mathcal{C}_\mathrm{bu}(i\mathbf{R}; \mathcal{B}(U,Y))$ *and* $\|\widehat{\mathbb{E}}(s)\| \le \|\mathbb{E}\|_\mathrm{TI} \le \|\mathbb{E}\|_\mathcal{A}$ *for all* $s \in i\mathbf{R}$.

(e2) If $\mathbb{E} \in \widetilde{\mathcal{A}}$, *then* $\widehat{\mathbb{E}} \in \mathcal{C}_\mathrm{bu}(\overline{\mathbf{C}^+}; \mathcal{B}(U,Y)) \cap \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U,Y))$ *and* $\|\widehat{\mathbb{E}}(s)\| \le \|\mathbb{E}\|_\mathrm{TI} \le \|\mathbb{E}\|_\mathcal{A}$ *for all* $s \in \overline{\mathbf{C}^+}$.

(f) We have $\widetilde{\mathcal{A}} \subset \mathrm{ULR}$, $\mathrm{MTI}^{\mathrm{L}^1} \subset \mathrm{CTI}$ *and* $\mathrm{MTIC}^{\mathrm{L}^1} \subset \mathrm{CTIC} \subset \mathrm{UHPR}$.

(g1) Assume that $\mathcal{A} \neq \mathrm{CTI}, \mathrm{CTI}^{\mathcal{BC}}$. *Then* $\mathcal{A}_\omega = \mathrm{e}^{\omega\cdot}\mathcal{A}\mathrm{e}^{-\omega\cdot}$ *and* $\widetilde{\mathcal{A}}_\omega = \mathrm{e}^{\omega\cdot}\widetilde{\mathcal{A}}\mathrm{e}^{-\omega\cdot}$. *Moreover,* $\widetilde{\mathcal{A}}_\omega \underset{c}{\subset} \widetilde{\mathcal{A}}_{\omega'}$ *for* $\omega' \ge \omega$.

(g2) Assume that $\mathcal{A} \neq \mathrm{CTI}, \mathrm{CTI}^{\mathcal{BC}}$. *Then* $\mathcal{A}^*_\omega = \mathcal{A}_{-\omega}$, $\mathcal{A}^\mathrm{d}_\omega = \mathcal{A}_\omega$. *and* $\widetilde{\mathcal{A}}^*_\omega = \widetilde{\mathcal{A}}_{-\omega}$.

(h1) Replace the letters MTI *by* SMTI *everywhere in this theorem. Then (a1)–(a2) and (d)–(g1) still hold except that* $\widehat{\mathbb{E}}$ *need not be strongly continuous in (e1)–(e2) and instead of (f) we only have that* $\widetilde{\mathcal{A}} \subset \mathrm{SLR}$ *and* $\mathrm{SMTIC}^{\mathrm{L}^1} \subset \mathrm{SHPR}$.

(h2) We have $\|\mathbb{G}\|_{\mathrm{TI}_\omega} \le \|\mathbb{G}\|_{\mathrm{SMTI}_\omega} \le \|\mathbb{G}\|_{\mathrm{MTI}_\omega} \le \infty$ *for all* $\mathbb{G} \in \mathrm{TI}_\omega$.

(i1) If $\mathbb{D} \in \mathrm{MTIC}_\infty(U,Y)$*, then* $\|\pi_{[0,t)}(\mathbb{D}-D)\pi_{[0,t)}\|_{\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))} \to 0$*, as* $t \to 0+$.

(i2) If $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;\mathcal{B}(U,Y))$*, then* $\|u \mapsto (f*u)(t)\|_{\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U),Y)} \to 0$*, as* $t \to 0+$.

(i3) If $\mathbb{D} \in \mathrm{SMTIC}_\infty(U,Y)$*, and* $u = \chi_{\mathbf{R}_+}u_0$*, then* $\mathbb{D}u \in \mathcal{C}(\mathbf{R}_+;Y)$ *and* $(\mathbb{D}u)(0) = Du_0$.

(j) If $\dim U < \infty$ *or* $\dim Y < \infty$*, then* $\mathcal{A}^{\mathcal{BC}} = \mathcal{A}$ *(since then* $\mathcal{BC}(U,Y) = \mathcal{B}(U,Y)$*).*

Obviously, the formula in (d) holds more generally too. See Lemma D.1.12 for adjoints and Laplace transforms of elements of MTI. Note also that almost all above results hold for Banach spaces U and Y too, as shown in the statements to which we refer in the theorem and its proof.

Chapter 4, Proposition 6.3.4 and Lemmas F.2.2–F.2.4 also describe properties of MTI and SMTI classes. The corresponding spectral and coprime factorization properties are given in Sections 5.2 and 8.4.

Proof: (c1)&(c2)&CTI-claims: By Theorem 4.1.1 and Lemma 4.1.3, $\mathcal{A}$ [and $\widetilde{\mathcal{A}}$] is inverse closed and [causal] adjoint closed, as indicated in (b1)–(c2). (We do not know whether the SMTI classes are inverse closed.)

The other claims about CTI and its subclasses are easy to prove (e.g., $\widehat{\mathrm{CTI}}(U,Y) = \mathcal{C}(i\mathbf{R}\cup\{\infty\};\mathcal{B}(U,Y))$ has the properties stated above; see Lemma 2.6.2 for the half-plane-regularity of CTIC maps). Therefore, for the rest of the proof, we assume that $\mathcal{A} \ne \mathrm{CTI}$ and $\mathcal{A} \ne \mathrm{CTI}^{\mathcal{BC}}$.

(a2) This follows from Lemma D.1.12(c3).

(a1)&(d) It is clear that $\mathcal{A}$ is a vector space. The isometric isomorphism

$$\mathrm{MTI}(U,Y) \ni (\sum_{r\in\mathbf{R}} T_r\delta_r + f)* \mapsto ((T_r)_{r\in\mathbf{R}}, f) \in \ell^1(\mathbf{R};\mathcal{B}(U,Y)) \times \mathrm{L}^1(\mathbf{R};\mathcal{B}(U,Y)) \tag{2.55}$$

shows that MTI is a Banach space (since L^1 and ℓ^1 are Banach spaces). It is easy to verify that also the image of $\mathcal{A}$ under (2.55) is a Banach space (recall that $\mathcal{BC}$ is closed in $\mathcal{B}$), hence $\mathcal{A}$ is a Banach space.

The composition of MTI operators corresponds to the convolution of the corresponding measures, by Lemma D.1.12(c1), hence $\mathcal{A}(U)$ is a Banach algebra (with a unit) and (b1) holds.

Now we have shown that $\mathcal{A}\mathcal{A} = \mathcal{A} \underset{a}{\subset} \mathrm{TI}$. Since $\mathrm{TICTIC} = \mathrm{TIC} \underset{a}{\subset} \mathrm{TI}$, we have $\widetilde{\mathcal{A}}\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}$. Obviously, $\mathcal{B} \underset{a}{\subset} \mathcal{A}$.

(b1)&(b2) These follow from Lemma 4.1.3 and (a1).

(b3) One easily verifies this.

(e1)&(e2) This follows from Lemma D.1.12(a1)&(c2)&(a1')&(c').

(f) By Lemma D.1.12(b'), we have $\widetilde{\mathcal{A}} \subset \mathrm{ULR}$; by (e), Lemma D.1.11(a1)&(a1') and Lemma 2.6.2, we have $\mathrm{MTIC}^{\mathrm{L}^1} \subset \mathrm{UHPR}$, $\mathrm{MTIC}^{\mathrm{L}^1} \subset \mathrm{CTIC}$ and $\mathrm{MTI}^{\mathrm{L}^1} \subset \mathrm{CTI}$.

(g1) By Lemma D.1.12(d), we have $\mathcal{A}_\omega = \mathrm{e}^{\omega\cdot}\mathcal{A}\mathrm{e}^{-\omega\cdot}$ and $\widetilde{\mathcal{A}}_\omega = \mathrm{e}^{\omega\cdot}\widetilde{\mathcal{A}}\mathrm{e}^{-\omega\cdot}$.

Since $\mathrm{e}^{-\omega\cdot} \le 1$ on $\mathbf{R}_+$, it is easy to verify from the definition that $\widetilde{\mathcal{A}}_\omega \subset \widetilde{\mathcal{A}}_{\omega'}$. (See Remark 2.1.6 for further results.)

(g2) This follows from Lemma 4.1.3.

(h1) This follows as above, by using Lemmas F.2.2 and F.2.3. (We do not know about the excluded claims.)

(h2) We omit the simple proof of $\|\mathbb{G}\|_{\mathrm{SMTI}_\omega} \le \|\mathbb{G}\|_{\mathrm{MTI}_\omega}$; the other inequality is given in (e2) and (h1).

(i1) This follows from the fact that the MTI norm of the part of $\mathbb{D}$ lying on $(0,t)$ decreases to zero. We leave the simple details to the reader.

(i2) This holds because $\|(f*u)(t)\|_Y \le \|\pi_{[0,t)}f\|_2\|\pi_{[0,t)}u\|$, by Lemma D.1.7, and $\|\pi_{[0,t)}f\|_2 \to 0$ as $t \to 0$, by Corollary B.3.8.

(i3) The proof is analogous to that of (i1) and omitted (note that $\mathbb{D}\cdot u_0 \in \mathrm{MTIC}(\mathbf{C},Y)$).

(j) See Lemma A.3.4(B1). $\square$

We finish this section by a minor technical result that allows us to to approximate $\mathcal{BC}$-valued TI maps by finite-dimensional operators:

Lemma 2.6.5 *Let $\mathbb{D} \in \mathcal{A}(U,Y)$ and $\widehat{\mathbb{D}}(+\infty) = 0$, where $\mathcal{A} = \mathrm{CTI}^{\mathcal{BC}}$ or $\mathcal{A} = \mathrm{MTI}^{\mathcal{BC}}$. Then there are countable orthogonal sequences $\{u_n\}_{n=1}^\infty \subset U$ and $\{y_n\}_{n=1}^\infty \subset Y$ s.t. when P_n [P_n'] is the orthogonal projection of U [Y] onto $\mathrm{span}(u_1,\ldots,u_n)$ [$\mathrm{span}(y_1,\ldots,y_n)$], we have $P_n'\mathbb{D}P_n \to \mathbb{D}$, $P_n'\mathbb{D} \to \mathbb{D}$, and $\mathbb{D}P_n \to \mathbb{D}$ in $\mathcal{A}(U,Y)$, as $n \to \infty$.*

If $U = Y$, we can choose $\{u_n\}$ so that $P_n\mathbb{D}P_n \to \mathbb{D}$ in $\mathcal{A}(U)$.

Of course, if $\mathbb{D} = (\sum_k T_k\delta_{t_k} + f)*$, then we can replace $P_n'fP_n$ (i.e., $\begin{bmatrix} P_n'fP_n & 0 \\ 0 & 0\end{bmatrix}$) by some smooth or simple function with a compact support for each n without losing the convergencies (see Theorem B.3.11).

Proof: *Case* CTI*:* 1° Let $n \in \mathbf{N}$. For each $s \in i\mathbf{R}\cup\{\infty\}$, $\widehat{\mathbb{D}}(s) \in \mathcal{BC}$, hence there are $P_{n,s}$ and $P_{n,s}'$ as in Lemma A.3.4(B2), so that $\|\widehat{\mathbb{D}}(s) - P_{n,s}'\widehat{\mathbb{D}}(s)P_{n,s}\| < 1/n$. The set

$$G_s := \{z \in i\mathbf{R}\cup\{\infty\} \,\big|\, \|\widehat{\mathbb{D}}(z) - P_{n,s}'\widehat{\mathbb{D}}(z)P_{n,s}\| < 1/n\} \tag{2.56}$$

is open, for each s. By the compactness of $\widehat{\mathbb{D}}(i\mathbf{R}\cup\{\infty\})$, there is a finite subset $\{s_1,\ldots,s_{m_n}\} \subset i\mathbf{R}\cup\{\infty\}$, so that $i\mathbf{R}\cup\{\infty\} \subset G_{s_1}\cup\ldots\cup G_{s_{m_n}}$. Let $u_{n,1},\ldots,u_{n,k_n}$ [$y_{n,1},\ldots,y_{n,k_n'}$] be an orthogonal base of $\cup_{j=1}^{m_n}\mathrm{Ran}(P_{n,s_j})$ [$\cup_{j=1}^{m_n}\mathrm{Ran}(P_{n,s_j}')$]. Thus, $\|\widehat{\mathbb{D}}(z) - \widetilde{P}_n'\widehat{\mathbb{D}}(z)\widetilde{P}_n\| < 1/n$ for all $z \in i\mathbf{R}\cup\{\infty\}$ for the corresponding projections.

2° Let $u_1,u_2,\ldots$ be the sequence $u_{1,1},u_{1,2},\ldots,u_{1,k_1},u_{2,1},u_{2,2},\ldots$ and let $y_1,y_2,\ldots$ be the sequence $y_{1,1},y_{1,2},\ldots,y_{1,k_1'},y_{2,1},y_{2,2},\ldots$ to obtain $\|\widehat{\mathbb{D}} - P_n'\widehat{\mathbb{D}}P_n\| \to 0$. Clearly $\|\widehat{\mathbb{D}} - P_n'\widehat{\mathbb{D}}\|$ and $\|\widehat{\mathbb{D}} - \widehat{\mathbb{D}}P_n\|$ are even smaller.

If $U = Y$, we may modify our proof by choosing only the u_k's, using the last claim in Lemma A.3.4(B2), or we can as well use the sequence $u_1,y_1,u_2,y_3,u_3,\ldots$.

Case MTI*:* 1° Let $\mathbb{D}(s) = [\sum_{k=0}^\infty T_k\delta_{r_k} + f]*$, where $T_k \in \mathcal{BC}(U,Y)$ for all k and $f \in \mathrm{L}^1(\mathbf{R};\mathcal{BC}(U,Y))$. Let $j \in \mathbf{N}+1$.

Then there are $m \in \mathbf{N}+1$, $\mathbb{E} = [\sum_{k=0}^m T_k'\delta_{r_k} + \widetilde{f}]*$ s.t. $m \in \mathbf{N}+1$, $T_k' \in \mathcal{BC}(U,Y)$ for all k, $f = \sum_{k=0}^m S_k\chi_{E_k}$, where each $S_k \in \mathcal{BC}(U,Y)$ is finite-dimensional, and $\|\mathbb{D} - \mathbb{E}\|_{\mathrm{MTI}} < 1/j$ (replace each T_k by a finite-dimensional

T_k' (see Lemma A.3.4(B1)), use the density of simple functions in L^1 (Theorem B.3.11(a1)) and choose suitable $m < \infty$). Let

$$U_j := \operatorname{span}\Big(\operatorname{Ker}(T_1')^\perp \cup \ldots \cup \operatorname{Ker}(T_m')^\perp \cup \operatorname{Ker}(S_1)^\perp \cup \ldots \cup \operatorname{Ker}(S_m)^\perp\Big) \subset U, \tag{2.57}$$

$$Y_j := \operatorname{span}\big(\operatorname{Ran}(T_1') \cup \ldots \cup \operatorname{Ran}(T_m') \cup \operatorname{Ran}(S_1) \cup \ldots \cup \operatorname{Ran}(S_m)\big) \subset Y. \tag{2.58}$$

Let $\widetilde{P}_j$ $[\widetilde{P}_j']$ be the orthogonal projection of U $[Y]$ onto U_j $[Y_j]$. Then $\mathbb{E} = \widetilde{P}_j'\mathbb{E}\widetilde{P}_j$, hence

$$\|\mathbb{D} - \widetilde{P}_j'\mathbb{D}\widetilde{P}_j\| = \|\mathbb{D} - \widetilde{P}_j'\mathbb{D}\widetilde{P}_j + \widetilde{P}_j'\mathbb{E}\widetilde{P}_j - \mathbb{E}\| \le \|\mathbb{D} - \mathbb{E}\| + \|\widetilde{P}_j'(\mathbb{E} - \mathbb{D})\widetilde{P}_j\| < 2/j. \tag{2.59}$$

2° Let $u_1, \ldots, u_{k_1}$ span U_1, add then vectors $u_{k_1+1}, \ldots, u_{k_2} \subset U_2$ so that $u_1, \ldots, u_{k_2}$ span the space $\operatorname{span}(U_1 \cup U_2)$ etc. If $\widetilde{P}_j$ and $\widetilde{P}_j'$ are the projections defined above for some m, then $\operatorname{Ran}(\widetilde{P}_j) \subset \operatorname{Ran}(P_n)$ and $\operatorname{Ran}(\widetilde{P}_j') \subset \operatorname{Ran}(P_n')$ for some n (and every subsequent one), hence $\|\mathbb{D} - P_n'\mathbb{D}P_n\| \le \|\mathbb{D} - \widetilde{P}_j'\mathbb{D}\widetilde{P}_j\| < 1/j$. Therefore, $\|\mathbb{D} - P_n'\mathbb{D}P_n\| \to 0$, as $n \to \infty$.

If $U = Y$, we can replace U_j and Y_j by $\operatorname{span}(U_j \cup Y_j)$ to obtain the required projections. □

Notes

The class MTIC was introduced into control theory in [CD78] and [CD80]. An excellent reference on the class is [CZ]. The class $\mathrm{MTIC}_{T\mathbf{Z}}$ is treated in [Winkin] and [CW99], and $\mathrm{MTI}_{\mathrm{d}}$ in [BKRS] and [RSW], among others. All these assume that U and Y are finite-dimensional; in that case probably all of Theorem 2.6.4 is well known. General vector-valued measures (with MTIC as a special case) are treated in [DU] and in [Dinculeanu].

Chapter 3

Transfer Functions ($\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$, $\widehat{\mathrm{TIC}} = \mathrm{H}^\infty$)

Though earth and man were gone,
And suns and universes ceased to be,
And Thou were left alone,
Every existence would exist in Thee.
— Emily Brontë (1818–1848)

In this chapter, we shall study the transfer functions of TI and TIC maps, i.e., the functions $\widehat{\mathbb{E}}$ for which $y = \mathbb{E}u$ corresponds to $\widehat{y} = \widehat{\mathbb{E}}\widehat{u}$, when $\mathbb{E} \in \mathrm{TI}(U,Y)$, u is an input signal and y is the corresponding output signal.

For $\mathbb{E} \in \mathrm{TIC}$, this was given in Theorem 2.1.2 (with $\widehat{y} = \widehat{\mathbb{E}}\widehat{u}$ on $\mathbf{C}^+$, i.e., in $\mathrm{H}^2(\mathbf{C}^+;Y)$); for general $\mathbb{E} \in \mathrm{TI}$, we only know that $\widehat{y} = \widehat{\mathbb{E}}\widehat{u}$ a.e. $\mathbf{C}^+$, i.e., that $\widehat{y} = \widehat{\mathbb{E}}\widehat{u}$ in $\mathrm{L}^2(i\mathbf{R};Y)$; this mapping $\mathbb{E} \mapsto \widehat{\mathbb{E}}$ will be established in Theorem 3.1.3.

Thus, $\widehat{\mathrm{TIC}} = \mathrm{H}^\infty$ and $\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$. In Theorem 3.1.6, we shall show that $\widehat{\mathrm{TI}_a \cap \mathrm{TI}_b} = \mathrm{H}^\infty(\{a < \mathrm{Re}\,\cdot < b\}; \mathcal{B}(U,Y))$. We also provide some further results on these three forms of transfer functions and weaker forms of the them for arbitrary Banach spaces U and Y and L^p in place of L^2 (and "TI^p" in place of TI). These can be considered as extensions of so called *Fourier multiplier theory*.

In Section 3.3, we establish several results on the boundary functions and poles of holomorphic functions.

By H, U and Y we denote arbitrary Hilbert spaces unless something else is indicated.

3.1 Transfer functions of TI ($\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$)

transfer, *n.:*
A promotion you receive on the condition that you leave town.

To be able to prove that "$\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$", we first recall the definition of $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$:

Definition 3.1.1 ($\mathbf{L}^\infty_{\mathbf{strong}}$) *Let U, Y and W be Hilbert spaces, and let Q be a set with a complete positive measure. A function $F : Q \to \mathcal{B}(U,Y)$ is said to be* strongly measurable, *if Fu_0 is Bochner measurable for each $u_0 \in U$.*

We define $\mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(U,Y))$ to be the space of (equivalence classes of) strongly measurable functions $Q \to \mathcal{B}(U,Y)$ with norm

$$\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} := \sup_{\|u\|\le 1} \|Fu\|_{\mathrm{L}^\infty} < \infty. \tag{3.1}$$

We define the multiplication in $\mathrm{L}^\infty_{\mathrm{strong}}$ by the formula $[F][G] := [FG]$ for any $F \in \mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(U,Y))$, $G \in \mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(Y,W))$.

Moreover, $[G] \in \mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(Y,U))$ is the adjoint *of $[F] \in \mathrm{L}^\infty_{\mathrm{strong}}$ (i.e. $[F]^* = [G]$) if $\langle Fu,y\rangle = \langle u,Gy\rangle$ a.e. for all $u \in U$ and $y \in Y$.*

All this is well-defined, by Section F.1. Naturally, functions $F,G \in \mathrm{L}^\infty_{\mathrm{strong}}$ are equivalent ($G \in [F]$) iff $\|F-G\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 0$, i.e., iff $Fu = Gu$ a.e. for all $u \in U$. (By $[F]$ we denote the equivalence class of F.)

As above, we write $F \in \mathrm{L}^\infty_{\mathrm{strong}}$ instead of $[F] \in \mathrm{L}^\infty_{\mathrm{strong}}$ when there is no danger of confusion. See Example 3.1.4 for adjoints and Section F.1 for further theory on $\mathrm{L}^\infty_{\mathrm{strong}}$ and strongly (and weakly) measurable operator-valued functions.

Recall that we treat the imaginary axis $i\mathbf{R}$ as $\mathbf{R}$ for measure-theoretic and differentiability aspects etc.; in particular, $m(iE) := m(E)$ for any measurable $E \subset \mathbf{R}$, where m is the one-dimensional Lebesgue measure, and $f \in \mathcal{C}^k(i\mathbf{R};B)$ iff $g \in \mathcal{C}^k(\mathbf{R};B)$, where $g(\cdot) := f(i\cdot)$. The same applies to the applies to any other vertical axis $\omega + i\mathbf{R}$, where $\omega \in \mathbf{R}$. We recall the following from Section F.1:

Lemma 3.1.2 *The space $\mathrm{L}^\infty_{\mathrm{strong}}(\Omega;\mathcal{B}(U,Y))$ is a Banach space and the space $\mathrm{L}^\infty_{\mathrm{strong}}(\Omega;\mathcal{B}(U))$ is a Banach algebra for any measurable $\Omega \subset \mathbf{R}$.* □

(This follows easily from Lemma F.1.3(b) and Theorem F.1.9(s1).)

Now we are able to state that $\mathrm{L}^\infty_{\mathrm{strong}} = \widehat{\mathrm{TI}}$:

Theorem 3.1.3 ($\widehat{\mathbf{TI}} = \mathbf{L}^\infty_{\mathbf{strong}}$) *For any Hilbert spaces U and Y, the following hold:*

(a1) **$\widehat{\mathbf{TI}} = \mathbf{L}^\infty_{\mathbf{strong}}$:** *For each $\mathbb{E} \in \mathrm{TI}(U,Y)$ there is a unique function $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ called the* Fourier transform *(or symbol) of $\mathbb{E}$, s.t. $\widehat{\mathbb{E}}\widehat{f} = \widehat{\mathbb{E}f}$ on $i\mathbf{R}$ for all $f \in \mathrm{L}^2(\mathbf{R};U)$. (We also call the mapping $\mathbb{E} \mapsto \widehat{\mathbb{E}}$ the Fourier transform.)*

The Fourier transform is an isometric isomorphism of $\mathrm{TI}(U,Y)$ *onto $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$, and it commutes with adjoints and compositions; in* *particular, this mapping is an isometric B^*-algebra isomorphism of* $\mathrm{TI}(U)$ *onto* $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U))$.

(a2) Each $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ *has a representative* $F : i\mathbf{R} \to \mathcal{B}(U,Y)$ *s.t.* $\|F(ir)\| \le \|\widehat{\mathbb{E}}\|_{\mathrm{L}^\infty_{\mathrm{strong}}} := \sup_{u\in U}\|Fu\|_{\mathrm{L}^\infty}$ *for all* $r \in \mathbf{R}$.

(b) If $\mathbb{D} \in \mathrm{TIC}(U,Y)$*, then the boundary function (see Theorem 3.3.1(c1)) of its Laplace transform coincides with its Fourier transform; we identify the two.*

(c) If $\dim U < \infty$*, then* $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y)) = \mathrm{L}^\infty(i\mathbf{R};\mathcal{B}(U,Y))$.

(d) Let $[F] = \widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$*. Assume that either* F *is piecewise continuous or* U *is separable. Then* $\|[F]\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \operatorname{ess\,sup}\|F\|_{\mathcal{B}(U,Y)}$ *and* $[F^*] = [F]^*$*. Moreover,* $\mathbb{E}^*\mathbb{E} \ge 0$ *iff* $F^*F \ge 0$ *a.e.*

Furthermore, $\mathbb{E} \in \mathcal{G}\mathrm{TI}$ *iff* $F(ir) \in \mathcal{G}\mathcal{B}$ *for a.e.* $r \in \mathbf{R}$ *and* $[F^{-1}] \in \mathrm{L}^\infty_{\mathrm{strong}}$*. If* F *is piecewise continuous, then a third equivalent condition is that* $F(ir) \in \mathcal{G}\mathcal{B}$ *for a.e.* $r \in \mathbf{R}$ *and* F^{-1} *is essentially bounded; and a fourth that* $F(ir) \in \mathcal{G}\mathcal{B}$ *for all* $r \in \mathbf{R}$ *and* F^{-1} *is bounded.*

(e1) Let $\mathbb{E} \in \mathrm{TI}(U)$*. Then* $\mathbb{E} \ge 0$ *iff* $\langle \widehat{\mathbb{E}}u, u\rangle \ge 0$ *a.e. for all* $u \in U$ *(iff* $\widehat{\mathbb{E}} \ge 0$ *a.e., provided that either* $\widehat{\mathbb{E}}$ *is piecewise continuous or* U *is separable).*

(e2) Let $\mathbb{E}_k \in \mathrm{TI}(U,*)$ *(*$k = 1,2,3,4$*). Then* $\mathbb{E}_1^*\mathbb{E}_2 \ge \mathbb{E}_3^*\mathbb{E}_4$ *iff* $\langle \widehat{\mathbb{E}}_1 u, \widehat{\mathbb{E}}_2 u\rangle \ge \langle \widehat{\mathbb{E}}_3 u, \widehat{\mathbb{E}}_4 u\rangle$ *a.e. for all* $u \in U$ *(iff* $\widehat{\mathbb{E}}_1^*\widehat{\mathbb{E}}_2 \ge \widehat{\mathbb{E}}_3^*\widehat{\mathbb{E}}_4$ *a.e., provided that either* $\widehat{\mathbb{E}}_k$ *is piecewise continuous for* $k = 1,2,3,4$*, or* U *is separable).*

Of course, we have $\widehat{\mathrm{TI}_\omega}(U,Y) = \mathrm{L}^\infty_{\mathrm{strong}}(\omega + i\mathbf{R};\mathcal{B}(U,Y))$ in a similar way (because $\mathrm{L}^2_\omega = \mathrm{e}^{\omega\cdot}\mathrm{L}^2$ and hence $\widehat{\mathrm{L}^2_\omega} = \tau(-\omega)\widehat{(\mathrm{L}^2)}$); see also Remark 2.1.6.

All above results also hold for $\partial\mathbf{D}$ mutatis mutandis, by Theorem 13.2.3. See Theorem 3.2.4 for a result weaker than (a1) in the case of separable Banach spaces U and Y. Note also that $\mathcal{C}_{\mathrm{b}}(i\mathbf{R};\mathcal{B}(U,Y))$ is a closed subspace of $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$.

Given $\mathbb{E} \in \mathrm{TI}(U,Y)$ and $u_0 \in U$, we have $\widehat{\mathbb{E}}u_0 = \widehat{\phi}^{-1}\mathcal{L}\mathbb{E}\phi u_0$ a.e. on $i\mathbf{R}$, where, e.g., $\phi(t) = \mathrm{e}^{-t^2/2}$ (see Lemma D.1.25), because the Fourier transform is one-to-one on L^2. The following proof of (a1) is based on this.

Proof of Theorem 3.1.3: (a1) Let $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$. By Theorem F.1.7(b) and The Plancherel Theorem, $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y)) \subset \mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$, isometrically, through $f \mapsto \mathbb{E}f := \mathcal{L}^{-1}\widehat{\mathbb{E}}\widehat{f}$. Moreover, $\mathbb{E}\tau^t f = \mathcal{L}^{-1}\widehat{\mathbb{E}}\mathrm{e}^{t\cdot}\widehat{f}\mathcal{L}^{-1}\mathrm{e}^{t\cdot}\widehat{\mathbb{E}}\widehat{f} = \tau^t\mathbb{E}f$ for all $f \in \mathrm{L}^2$. Thus, $\mathcal{L}^{-1}\mathrm{L}^\infty_{\mathrm{strong}} \subset \mathrm{TI}(U,Y)$, isometrically.

Therefore, we only have to show the converse. Thus, below we assume that $\mathbb{E} \in \mathrm{TI}(U,Y)$, find a candidate $\widehat{\mathbb{E}}$ and then show that $\mathcal{L}\mathbb{E}f = \widehat{\mathbb{E}}\widehat{f}$ for all $f \in \mathrm{L}^2$.

(We note that the proof does require that U and Y are Hilbert spaces, because we want all $\mathrm{L}^2(\mathbf{R};Y)$ functions to have a Fourier(–Plancherel) transform and we want to be able to extend $\mathcal{B}(U_0,Y)$ operators to $\mathcal{B}(U,Y)$ operators whenever U_0 is a subspace of Y.)

1° Define $\phi(t) := \mathrm{e}^{-t^2/2}$. Then $\widehat{\phi}(ir) = \sqrt{2\pi}\mathrm{e}^{-r^2/2} > 0$ for $r \in \mathbf{R}$, by Lemma D.1.25; in particular $\widehat{\phi}(ir)^{-1}$ is everywhere defined and continuous.

2° By L we shall denote the operator defined in Lemma B.5.3.

3° Set $A_t := \{u \mid it \in \mathrm{Leb}(L\mathcal{L}\mathbb{E}\phi u)\} \subset U$. Then, for a fixed $u \in U$, we have $u \in A_t$ for a.e. t. By (B.56), A_t is a subspace of U.

If $u \in U$ and $\Lambda \in Y^*$, then $g \mapsto \Lambda\mathbb{E}gu$ is in $\mathrm{TI}(\mathbf{C})$ and of norm $\le \|\mathbb{E}\|\,\|\Lambda\|\,\|u\|$, hence, by the corresponding scalar result [BL, Theorem 6.1.2, p. 132], there is $T_{\Lambda,u} \in \mathrm{L}^\infty(i\mathbf{R})$ s.t. $\|T_{\Lambda,u}\| \le \|\mathbb{E}\|\,\|\Lambda\|\,\|u\|$ and

$$T_{\Lambda,u}\widehat{g} = \mathcal{L}\Lambda\mathbb{E}gu = \Lambda\mathcal{L}\mathbb{E}gu \text{ for } g \in \mathrm{L}^2(\mathbf{R}). \tag{3.2}$$

If follows that for all $\Lambda \in Y^*$ s.t. $\|\Lambda\| \le 1$ we have

$$|\Lambda\widehat{\phi}(ir)^{-1}L(\mathcal{L}\mathbb{E}\phi u)(ir)| \le |\widehat{\phi}(ir)^{-1}LT_{\Lambda,u}(ir)\widehat{\phi}(ir)| \le \|\mathbb{E}\|\,\|u\|, \tag{3.3}$$

hence

$$\|\widehat{\phi}(ir)^{-1}(L\mathcal{L}\mathbb{E}\phi u)(ir)\|_Y \le \|\mathbb{E}\|\,\|u\|. \tag{3.4}$$

Let $F(it) \in \mathcal{B}(U,Y)$ be a linear extension of $A_t \ni u \mapsto \widehat{\phi}(it)^{-1}(L\mathcal{L}\mathbb{E}\phi u)(it) \in Y$ (that mapping is linear, by (B.56)) satisfying $\|F(it)\| \le \|\mathbb{E}\|$ (by (3.4), this is possible, e.g., extend to the closure of A_t by continuity, and extend by zero on $A_t^\perp$).

Because for $u \in U$ we have $F(it)u = \widehat{\phi}(it)^{-1}(L\mathcal{L}\mathbb{E}\phi u)(it)$ for $t \in A_t$, hence for a.e. $t \in \mathbf{R}$, the function F is in $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$, in particular, $F \in \mathcal{B}(\mathrm{L}^2(i\mathbf{R};U),\mathrm{L}^2(i\mathbf{R};Y))$, by Theorem F.1.7(b).

For $\widehat{f} \in \mathrm{L}^2(i\mathbf{R})$ and $u \in U$ we have

$$\Lambda F(it)\widehat{f}(it)u = \widehat{f}(it)\widehat{\phi}(it)^{-1}(L\mathcal{L}\mathbb{E}\phi u)(it) = \widehat{f}(it)T_{\Lambda,u}(it) = \Lambda(\mathcal{L}\mathbb{E}fu)(it) \text{ a.e.} \tag{3.5}$$

for all $\Lambda \in Y^*$ (the first inequality follows from the definition of $F(it)$ and from the fact that $u \in A_t$ for a.e. t, the second and third inequality follow from (3.2)). Because Ff and $\mathcal{L}\mathbb{E}fu$ are $\mathrm{L}^2(i\mathbf{R};Y)$ functions, hence measurable, it follows from (3.5) that $(Ff)(it)u = (\mathcal{L}\mathbb{E}fu)(it)$ for (a.e.) $t \in \mathbf{R}$, by Lemma B.2.6.

We can extend $Ffu = Tfu$ to $Fg = Tg$ for arbitrary simple functions g by linearity, then for arbitrary $g \in \mathrm{L}^2(i\mathbf{R};U)$, by density (Theorem B.3.11) and continuity.

The second paragraph of (a1) is easy to prove, e.g., by an application to functions of the form $\chi_E u$ and $\chi_E y$, one easily notes that $\widehat{\mathbb{E}^*} = \widehat{\mathbb{E}}^*$ for an arbitrary $\mathbb{E} \in \mathrm{L}^\infty_{\mathrm{strong}}$ (in particular, $\mathrm{L}^\infty_{\mathrm{strong}}$ is a B^*-algebra).

(a2) The representative F constructed in the proof of (a1) obviously satisfies the conditions of (a2).

(b) This will be proved in Theorem 3.3.1(c1).

(c) See Lemma F.1.3(d).

(d) The claim on $\mathbb{E}^*\mathbb{E} \ge 0$ follows from (e1) and the claim $[F^*] = [F]^*$, because $\widehat{\mathbb{E}}^*\widehat{\mathbb{E}} = [F^*][F] = [F^*F]$.

The rest follows from Lemma F.1.3(f1)&(f2). In the proof of the $\mathcal{G}$TI-equivalence, we also need the fact $\mathbb{E} \in \mathcal{G}\mathrm{TI} \Leftrightarrow [F] \in \mathcal{G}\mathrm{L}^\infty_{\mathrm{strong}}$, from (a1) (the extra condition imposed on "G" in (f2) follows from (a2) above).

For the last remark we note that if F is piecewise continuous and F^{-1} exists a.e. and F^{-1} is bounded, then F^{-1} exists everywhere, by Lemma A.3.3(A3), and $[F^{-1}] \in \mathrm{L}^\infty_{\text{strong}}$.

(Instead of piecewise continuity, it suffices to assume that $i\mathbf{R}$ is divided into an at most countable number of intervals of positive measure, and F is continuous on each of them.)

(e) For clarity, we only prove (e1) (which is a special case of (e2)); the same proof of applies for (e2) with slight changes.

1° If $\langle\widehat{\mathbb{E}}u,u\rangle_U \ge 0$ a.e. for all $u \in U$, then $\langle\widehat{\mathbb{E}}f,f\rangle_{\mathrm{L}^2(i\mathbf{R};U)} \ge 0$ for all simple functions $f \in \mathrm{L}^2$, hence for all $f \in \mathrm{L}^2$.

Conversely, if $u \in U$ and $\langle\widehat{\mathbb{E}}u\chi_E, u\chi_E\rangle_{\mathrm{L}^2(i\mathbf{R};U)} \ge 0$ for all finite-measurable sets E, then $\langle\widehat{\mathbb{E}}u,u\rangle \ge 0$ a.e.

2° If $\widehat{\mathbb{E}}$ is piecewise continuous, then the $\widehat{\mathbb{E}} \ge 0$ claim is obvious. If U is separable and $\langle\widehat{\mathbb{E}}u,u\rangle \ge 0$ a.e. for all $u \in U$, then there is a null set N s.t. $\langle\widehat{\mathbb{E}}u,u\rangle \ge 0$ on N^c for all u in a countable, dense subset of U, hence for all $u \in U$. The converse is trivial. □

In the case of unseparable Y (and discontinuous $\mathrm{L}^\infty_{\text{strong}}$ functions), there are some peculiarities, e.g., even the element $0 \in \mathrm{L}^\infty_{\text{strong}}$ may have a representative $F : \mathbf{R} \to \mathcal{B}(U,Y)$ with $\operatorname{ess\,sup}\|F\|_{\mathcal{B}(U,Y)} = \infty$ and F^* nonmeasurable:

Example 3.1.4 **[$\|[F]\|_{\mathrm{L}^\infty_{\text{strong}}} = 0$ & $\|F\|_{\mathrm{L}^\infty} = \infty$]** Let $\{e_r\}_{r\in\mathbf{R}}$ be the natural base of $U := \ell^2(\mathbf{R})$, and let $d_0 \in Y$ be s.t. $\|d_0\| = 1$; here Y can be any unseparable Hilbert space.

For all $f : \mathbf{R} \to \mathbf{R}$, we define $F_f : i\mathbf{R} \to \mathcal{B}(U)$ by $F_f(ir)u := f(r)u_r d_0$ (where $u = (u_r)_{r\in\mathbf{R}}$), so that $\|F_f(ir)\|_{\mathcal{B}(U,Y)} = |f(r)|$ for all $r \in \mathbf{R}$. Consequently, $F_f u = 0$ a.e. for all $u \in U$, because $u_r = 0$ for a.e. r; in particular, $[F_f] = [0] \in \mathrm{L}^\infty_{\text{strong}}$, i.e., $\|[F_f]\|_{\mathrm{L}^\infty_{\text{strong}}} = 0$, even though $\|F\|_{\mathrm{L}^\infty} := \|\|F\|_{\mathcal{B}(U,Y)}\|_{\mathrm{L}^\infty(\mathbf{R})} = \|f\|_\infty$ may be infinite (and F may be nonmeasurable). Moreover,

$$\langle F_f(ir)u,y\rangle = \langle f(r)u_r d_0,y\rangle = \langle f(r)u_r,y_0\rangle = \langle u, f(r)y_0 e_r\rangle \quad \text{for all } u \in U,\ y \in Y, \tag{3.6}$$

where $y_0 := \langle d_0,y\rangle =: \Lambda y$, hence, $F_f^* = f(r)e_r\Lambda$, in particular, $F_f(ir)^*d_0 = f(r)e_r$ and $\|F_f(ir)^*\|_{\mathcal{B}(U,Y)} = |f(r)| = \|F_f(ir)\|$ ($r \in \mathbf{R}$). The following holds:

(a) If $f(r) = r$, then $F_f(ir)^*d_0 = re_r$, hence $\|F_f^*d_0\|_\infty = \infty$. Consequently,

$$\text{“}\|[F_f^*]\|''_{\mathrm{L}^\infty_{\text{strong}}} := \sup_{y_0\in Y}\operatorname*{ess\,sup}_{r\in\mathbf{R}}\|F_f(ir)^*y_0\|_U = \infty. \tag{3.7}$$

(b) If $g(r) \equiv 1$, then “$\|[F_g^*]\|_{\mathrm{L}^\infty_{\text{strong}}}$”$= 1$. However, even in this case, we have $[F_f] \notin \mathrm{L}^\infty_{\text{strong}}$, because F_g^* is not even strongly measurable: obviously, the function $r \mapsto F_g(ir)^*d_0 = e_r \in U$ is not almost separably-valued, hence neither measurable (nor is the function F_f of (a) or any other F_f except those with $f = 0$ a.e.).

◁

Thus, in general, the adjoint of a representative of some $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\text{strong}}$ need not be in the class of $\widehat{\mathbb{E}}^*$, not even in the class of any $\mathrm{L}^\infty_{\text{strong}}$ function, even if this representative were bounded (see (b) above).

However, there is a unique $\widehat{\mathbb{F}} \in \mathrm{L}^\infty_{\mathrm{strong}}$ s.t. $\langle u, E^* y\rangle = \langle Eu, y\rangle = \langle u, Fy\rangle$ a.e. for all u, y whenever E and F are representatives of $\widehat{\mathbb{E}}$ and $\widehat{\mathbb{F}}$, respectively, by Theorem 3.1.3(a1). Note that, if Fu is strongly measurable, then F^* is weakly measurable in the sense that $\langle F^* y, u\rangle$ is measurable for all $u \in U$ and $y \in Y$.

Sometimes TI operators (and WPLSs) are studied over Banach spaces and general L^p (see, e.g., [Sbook] and several articles of G. Weiss). Many of our results generalize to that setting with ease but some do not at all. The emphasis of this book is in L^2 signals over Hilbert spaces, because this allows one to formulate the standard control problems. However, we give here certain results in a wider setting for future reference.

Theorem 3.1.5 ($\mathbf{TI}^p_\omega$) *Let X and Y be Banach spaces, $p, q \in [1, \infty]$ and $\omega \in \mathbf{R}$. Define*

$$\mathrm{TI}^{p,q}_\omega(X,Y) := \{\mathbb{E} \in \mathcal{B}(\mathrm{L}^p_\omega(\mathbf{R};X), \mathrm{L}^q_\omega(\mathbf{R};Y)) \bigm| \mathbb{E}\tau(t) = \tau(t)\mathbb{E} \text{ for all } t \in \mathbf{R}\}, \tag{3.8}$$

$\mathrm{TI}^p_\omega := \mathrm{TI}^{p,p}_\omega$ and $\mathrm{TI} := \mathrm{TI}^{2,2}_\omega$. Then $\mathrm{TI}^{p,q}_\omega$ is a closed subspace of $\mathcal{B}(\mathrm{L}^p_\omega, \mathrm{L}^q_\omega)$. Moreover, $\|\mathbb{E}f\|_{\mathrm{W}^{n,q}_\omega} \le \|\mathbb{E}\|_{\mathrm{TI}^{p,q}} \|f\|_{\mathrm{W}^{n,p}_\omega}$ and $\mathbb{E}\partial^n f = \partial^n \mathbb{E} f$ for all $f \in \mathrm{W}^{n,p}_\omega(\mathbf{R};U)$, $\mathbb{E} \in \mathrm{TI}^{p,q}_\omega$.

Finally, $\mathbb{E}[\mathcal{S}(\mathbf{R};X)] \subset \mathrm{W}^{\infty,q}_\omega \subset \mathcal{C}^\infty(\mathbf{R};Y)$ for all $\mathbb{E} \in \mathrm{TI}^{p,q}(X,Y)$, and $\mathbb{E}[\mathcal{C}_0(\mathbf{R};X)] \subset \mathcal{C}_{\mathrm{b}}(\mathbf{R};Y)$ (even $\mathbb{E}[\mathcal{C}_0(\mathbf{R};X)] \subset \mathcal{C}_0(\mathbf{R};Y)$ if Y is a Hilbert space) for all $\mathbb{E} \in \mathrm{TI}^\infty(X,Y)$.

(We have set $\mathrm{W}^{\infty,p}_\omega := \cap_{k\in\mathbf{N}} \mathrm{W}^{k,p}_\omega$. Note also that the definition of TI^∞ in [Sbook] differs from that of ours: it requires the L^∞ functions to vanish at infinity.)

The TI^p operators correspond to Fourier multipliers (see Section 3.2; note that the multiplier does not determine the operator uniquely in case $p = \infty$). Analogously, $\mathrm{TI}^{p,\infty}(U,Y) = \mathcal{B}(\mathrm{L}^p(\mathbf{R};B), B_2)*$, in particular, $\mathrm{TI}^{p,\infty}(\mathbf{C}) = \mathrm{L}^{p'}(\mathbf{R})*$, where $p^{-1} + p'^{-1} = 1$, for $p < \infty$. but since the case $q \ne p$ is only rarely treated, we shall not consider it further (and we omit the proofs).

Proof: 1° Obviously, $\mathrm{TI}^{p,q}_\omega$ is a closed subspace of $\mathcal{B}(\mathrm{L}^p_\omega, \mathrm{L}^q_\omega)$. By Lemma B.7.8, a function $f \in \mathrm{L}^p_\omega(\mathbf{R};X)$ is in $\mathrm{W}^{1,p}_\omega$ iff $(\tau(h)f - f)/h$ converges in L^p_ω as $h \to 0$. Thus, for $f \in \mathrm{W}^{1,p}_\omega$ we have (here $\overset{X}{\lim}$ means a limit in the space X)

$$\begin{aligned}\mathbb{E}\partial f &= \mathbb{E} \overset{\mathrm{L}^p_\omega}{\lim_{h\to 0}} (\tau(h)f - f)/h = \overset{\mathrm{L}^q_\omega}{\lim_{h\to 0}} \mathbb{E}(\tau(h)f - f)/h \\ &= \overset{\mathrm{L}^q_\omega}{\lim_{h\to 0}} (\tau(h)\mathbb{E}f - \mathbb{E}f)/h = \partial(\mathbb{E}f),\end{aligned}$$

hence $\partial(\mathbb{E}f)$ exists and is equal to $\mathbb{E}\partial f$. Thus, $\mathbb{E}[\mathrm{W}^{1,p}_\omega] \subset \mathrm{W}^{1,q}_\omega$.

By induction, $\mathbb{E}\partial^n \subset \partial^n\mathbb{E}$ and $\mathbb{E}\mathrm{W}^{n,p}_\omega \subset \mathrm{W}^{n,q}_\omega$ for any $n \in \mathbf{N}$. Consequently, $\|\mathbb{E}f\|_{\mathrm{W}^{n,q}_\omega} \le \|\mathbb{E}\|_{\mathrm{TI}^{p,q}} \|f\|_{\mathrm{W}^{n,p}_\omega}$

2° $\mathbb{E}[\mathcal{S}(\mathbf{R};X)] \subset \mathcal{C}^\infty(\mathbf{R};Y)$*:* Let $\mathbb{E} \in \mathrm{TI}^p(X,Y)$. Obviously, $\mathcal{S} \subset \mathrm{W}^{\infty,p}$. By 1°, $\mathbb{E}[\mathcal{S}(\mathbf{R};X)] \subset \mathrm{W}^{\infty,q}$. By Corollary B.7.7, we have $\mathrm{W}^{\infty,q} \subset \mathcal{C}^\infty$.

3° $\mathbb{E}[\mathcal{C}_0(\mathbf{R};X)] \subset \mathcal{C}_{\mathrm{b}}(\mathbf{R};Y)$*:* Let $\mathbb{E} \in \mathrm{TI}^\infty(X,Y)$. Then $\mathbb{E}[\mathcal{S}(\mathbf{R};X)] \subset \mathcal{C}^\infty \cap \mathrm{L}^\infty \subset \mathcal{C}_{\mathrm{b}}(\mathbf{R};Y)$, hence $\mathbb{E}[\mathcal{C}_0(\mathbf{R};X)] \subset \mathcal{C}_{\mathrm{b}}(\mathbf{R};Y)$, by continuity (by Theorem B.3.11(c), $\mathcal{C}_0$ is the closure of $\mathcal{S}$ in L^∞).

4° $\mathbb{E}[\mathcal{C}_0(\mathbf{R};X)] \subset \mathcal{C}_0(\mathbf{R};Y)$*:* Assume that Y is a Hilbert space. Set $T\phi := (\mathbb{E}Я\phi)(0)$ for all $\phi \in \mathcal{C}_0(\mathbf{R};X)$. One easily verifies that $T \in M := \mathcal{B}(\mathcal{C}_0(\mathbf{R};X),Y)$ and $\mathbb{E} = T*$, where $(T*\phi)(t) := T\tau^{-t}Я\phi$ ($t \in \mathbf{R}$, $\phi \in \mathcal{C}_0(\mathbf{R};X)$). By Lemma D.1.14(d), we have $T*\phi \in \mathcal{C}_0$ for all $\phi \in \mathcal{C}_0$, hence $\mathbb{E}[\mathcal{C}_0(\mathbf{R};X)] \subset \mathcal{C}_0(\mathbf{R};Y)$.

Remark — This is not true for general Y: Let X be any Banach space (e.g., $X = \mathbf{C}$). Let $Y := \ell^\infty(\mathbf{N};\mathrm{L}^\infty)$, where $\mathrm{L}^\infty := \mathrm{L}^\infty(\mathbf{R};X)$. One easily verifies that $\mathbb{E} \in \mathrm{TI}^\infty(X,Y)$, where $(\mathbb{E}f)_n := \tau^{-n}f$ ($f \in \mathrm{L}^\infty$). Obviously, $f \neq 0 \Longrightarrow \mathbb{E}f \notin \mathcal{C}_0(\mathbf{R};\mathrm{L}^\infty)$. (Note also that $\pi_-\mathbb{E}\pi_+ = 0$.) □

If $\mathbb{E} \in \mathrm{TI}_a \cap \mathrm{TI}_b(U,Y)$, $a < b$, then $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(r+i\mathbf{R};\mathcal{B}(U,Y))$ for all $r \in (a,b)$. Actually, $\widehat{\mathbb{E}}$ can be redefined so that it becomes holomorphic on $\mathbf{C}_{a,b} := (a,b)+i\mathbf{R} = \{s \in \mathbf{C} \,|\, a < \mathrm{Re}\, s < b\}$:

Theorem 3.1.6 ($\widehat{\mathbf{TI}_a \cap \mathbf{TI}_b} = \mathbf{H}^\infty(\{a < \mathrm{Re}\cdot < b\},\mathcal{B})$) *Let U and Y be Hilbert spaces and $-\infty < a \leq b < \infty$.*

(a) Let $\mathbb{E} \in \mathrm{TI}_a(U,Y) \cap \mathrm{TI}_b(U,Y)$. Then there is a unique $\widehat{\mathbb{E}} \in \mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))$, s.t.

$$\mathcal{L}\mathbb{E}u = \widehat{\mathbb{E}}\widehat{u} \quad \text{on } \mathbf{C}_{a,b}, \tag{3.9}$$

for all $u \in \mathrm{L}^2_a(\mathbf{R};U) \cap \mathrm{L}^2_b(\mathbf{R};U)$. Moreover, $\|\widehat{\mathbb{E}}\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))} = \max\{\|\mathbb{E}\|_{\mathrm{TI}_a}, \|\mathbb{E}\|_{\mathrm{TI}_b}\}$, $\widehat{\mathbb{E}}$ has "strong" nontangential boundary functions (again denoted by $\widehat{\mathbb{E}}$) on $a+i\mathbf{R}$ and $b+i\mathbf{R}$, and $\mathcal{L}\mathbb{E}u = \widehat{\mathbb{E}}\widehat{u}$ a.e. in $r+i\mathbf{R}$ for each $r \in [a,b]$ and $u \in \mathrm{L}^2_r(\mathbf{R};U)$.

(b) Conversely, if $\widehat{\mathbb{E}} \in \mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))$, then there is a unique $\mathbb{E} \in \mathrm{TI}_a(U,Y) \cap \mathrm{TI}_b(U,Y)$ s.t. $\widehat{\mathbb{E}}|_{r+i\mathbf{R}}$ is its transform for some (hence all) $r \in (a,b)$. If this is the case, then $\mathbb{E}$ and $\widehat{\mathbb{E}}$ are as in (a).

The above boundary functions exist in the sense that $\widehat{\mathbb{E}}u_0 \in \mathrm{H}^\infty$ has the boundary function $\widehat{\mathbb{E}}u_0$ on $a+i\mathbf{R}$ in the sense of (1.) and (2.) of Theorem 3.3.1(a), for each $u_0 \in U$. It follows that $\widehat{\mathbb{E}}\widehat{u} \in \mathrm{H}^2(\mathbf{C}_{a,b};Y)$ has the boundary function $\widehat{\mathbb{E}}\widehat{u} \in \mathrm{L}^2$ on $a+i\mathbf{R}$ in the sense of (1.), (2.) and (4.)–(6.) of Theorem 3.3.1(a), for each $u \in \mathrm{L}^2_a \cap \mathrm{L}^2_b(\mathbf{R};U)$, by Proposition D.1.21(a). The "mirror images" of these two claims hold at $b+i\mathbf{R}$.

Note also that both sides of (3.9) are holomorphic on $\mathbf{C}_{a,b}$, by Proposition D.1.21(a). By the last claim in (a), $\widehat{\mathbb{E}}|_{r+i\mathbf{R}}$ is the Fourier transform of $\mathbb{E} \in \mathrm{TI}_r$ for all $r \in [a,b]$; this justifies our notation (and that of Theorem 2.1.2).

Proof: (The proofs of Theorems 3.1.6 and 3.1.7 use implicitly Theorem 3.3.1(a); naturally, the converse is not true. Part of (a) could also be obtained from Theorem 3.1.7 as a corollary, but we prefer to the simpler proof below.)

(a) 1° *We have $\widehat{\mathbb{E}} \in \mathrm{H}(\mathbf{C}_{a,b};\mathcal{B}(U,Y))$:* Let ϕ be as in Lemma D.1.25. We define the function $\widehat{\mathbb{E}}u_0 := \widehat{\phi}^{-1}\mathcal{L}\mathbb{E}\phi u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$ for each $u_0 \in U$ (since $\mathcal{L}\mathbb{E}\phi u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$, by Proposition D.1.21(a), we have $\widehat{\mathbb{E}}(s)u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$).

For any fixed $s \in \mathbf{C}_{a,b}$, the operator $\widehat{\mathbb{E}}(s) : U \to Y$ is obviously linear; by the norm inequality in Lemma D.1.10(a), it is also bounded (for this fixed s). By Lemma D.1.1(b), it follows that $\widehat{\mathbb{E}} \in \mathrm{H}(\mathbf{C}_{a,b}; \mathcal{B}(U,Y))$.

2° $\|\widehat{\mathbb{E}}\|_\infty \leq M$: Let $F(r+i\cdot)$ be a representative of the Fourier transform (see Theorem 3.1.3(a)) of $\mathbb{E}$, for each $r \in [a,b]$, so that

$$\max_{r\in[a,b]} \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}(r+i\mathbf{R};\mathcal{B}(U,Y))} = \max\{\|\mathbb{E}\|_{\mathrm{TI}_a}, \|\mathbb{E}\|_{\mathrm{TI}_b}\} =: M \tag{3.10}$$

by (2.11). Then, given $r \in (a,b)$, we have $F\widehat{\phi}u_0 = \widehat{\mathbb{E}}\widehat{\phi}u_0$ a.e. on $r+i\mathbf{R}$, hence $Fu_0 = \widehat{\mathbb{E}}u_0$ a.e. on $r+i\mathbf{R}$, i.e., $F = \widehat{\mathbb{E}}$ as an element of $\mathrm{L}^\infty_{\mathrm{strong}}(r+i\mathbf{R};\mathcal{B}(U,Y))$; in particular, $\sup\|\widehat{\mathbb{E}}(r+i\mathbf{R})\|_{\mathcal{B}(U,Y)} = \|\mathbb{E}\|_{\mathrm{TI}_r} \leq M$ (since $\widehat{\mathbb{E}}$ is continuous in $\mathbf{C}_{a,b}$, by 1°). Because r was arbitrary, we have $\|\widehat{\mathbb{E}}\|_{\mathcal{B}(U,Y)} \leq M$ on $\mathbf{C}_{a,b}$.

3° *Identity (3.9) holds:* Set $\widehat{\mathbb{E}} := F$ on $a+i\mathbf{R}$ and on $b+i\mathbf{R}$, so that $\mathcal{L}\mathbb{E}u = \widehat{\mathbb{E}}\widehat{u}$ a.e. in $r+i\mathbf{R}$ for each $r \in [a,b]$ and $u \in \mathrm{L}^2_r(\mathbf{R};U)$ (recall from 2° that $\widehat{\mathbb{E}} = F$ on $r+i\mathbf{R}$ as elements of $\mathrm{L}^\infty_{\mathrm{strong}}$). If $u \in \mathrm{L}^2_a(\mathbf{R};U) \cap \mathrm{L}^2_b(\mathbf{R};U)$, then both sides of (3.9) are holomorphic, by Proposition D.1.21(a), hence then the equality holds on the whole $\mathbf{C}_{a,b}$.

4° *Boundary functions:* By Proposition D.1.21(c), the function $\widehat{\mathbb{E}}\widehat{u}$ is the nontangential boundary function of itself at $a+i\mathbf{R}$ and at $b+i\mathbf{R}$, in the sense of (1.), (2.) and (4.)–(6.) of Theorem 3.3.1(a1). Set $u = \phi u_0$ for an arbitrary $u_0 \in U$ and divide by $\widehat{\phi}^{-1}$ to obtain that $\widehat{\mathbb{E}}u_0$ is the nontangential boundary function of itself at $a+i\mathbf{R}$ and at $b+i\mathbf{R}$, in the sense of (2.) (and (1.)) of Theorem 3.3.1(a1).

(b) The function $\widehat{\mathbb{E}}|_{r+i\mathbf{R}}$ defines a unique $\mathbb{E}_r \in \mathrm{TI}_r(U,Y)$, and

$$\|\mathbb{E}_r\|_{\mathrm{TI}_r} \leq \|\widehat{\mathbb{E}}\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))} =: M, \tag{3.11}$$

for each $r \in (a,b)$, by Theorem 3.1.3.

Let $u \in \mathcal{F} := \mathrm{L}^2_a(\mathbf{R};U) \cap \mathrm{L}^2_b(\mathbf{R};U)$ and $r \in (a,b)$ be arbitrary. Then $\widehat{\mathbb{E}}_r\widehat{u} = \widehat{\mathbb{E}}\widehat{u}$ a.e. on $r+i\mathbf{R}$. But $\widehat{\mathbb{E}}\widehat{u} \in \mathrm{H}^2(\mathbf{C}_{a,b};Y)$, hence $\widehat{\mathbb{E}}\widehat{u} = \widehat{f}$ for some unique $f \in \mathrm{L}^2_a \cap \mathrm{L}^2_b$, by Proposition D.1.21(c). By uniqueness, $\mathbb{E}_r u = f$ a.e., hence (3.9) holds with $\mathbb{E}_r$ in place of $\mathbb{E}$. Because $r \in (a,b)$ was arbitrary, we have $\mathbb{E}_r u = \mathbb{E}_{r'} u$ a.e. for any $r, r' \in (a,b)$. Because $u \in \mathcal{F}$ was arbitrary, it follows from Lemma 2.1.10(c) that $\mathbb{E}_r = \mathbb{E}_{r'} \in \mathrm{TI}_r \cap \mathrm{TI}_{r'}$, for any $r, r' \in (a,b)$.

Fix some $r \in (a,b)$ and set $\mathbb{E} := \mathbb{E}_r$. By the above, (3.9) holds for all $u \in \mathcal{F}$, and $\mathbb{E} \in \mathrm{TI}_{r'}(U,Y)$ and $\|\mathbb{E}\|_{\mathrm{TI}_{r'}} \leq M$ for all $r' \in (a,b)$. By Lemma 2.1.10(g), it follows that $\mathbb{E} \in \mathrm{TI}_a \cap \mathrm{TI}_b$. Thus, $\mathbb{E}$ and $\widehat{\mathbb{E}}$ are as in (a), so that we obtain the rest of (b) from (a) (if $\widehat{\mathbb{E}}$ is the Fourier transform of $\widetilde{\mathbb{E}} \in \mathrm{TI}_r$ for some $r \in (a,b)$, then $\widetilde{\mathbb{E}} = \mathbb{E}$, by uniqueness). □

We observe that we have achieved an alternative proof of Theorem 2.1.2: it is a corollary of Theorem 3.1.6 and Lemma 2.1.11. A similar weaker claim (the transform being a contractive linear isometry into; also this claim is given in [W91a]) is true also when U and Y are general Banach spaces and L^2 is replaced by L^p, $1 \leq p < \infty$, as one obtains from the following (weaker) generalization of the above theorem:

Theorem 3.1.7 ($\widehat{\mathbf{TI}_a \cap \mathbf{TI}_b} \subset \mathbf{H}^\infty(\{a < \mathbf{Re}\cdot < b\}, \mathcal{B})$) *Assume, for this theorem, that X and Y are Banach spaces, $1 \le p < \infty$, and $-\infty < a \le b < \infty$. Let $\mathbb{E} \in \mathrm{TI}^p_a(X,Y) \cap \mathrm{TI}^p_b(X,Y)$. Then there is a unique $\widehat{\mathbb{E}} \in \mathrm{H}^\infty(\mathbf{C}_{(a,b)}; \mathcal{B}(X,Y))$ s.t.*

$$\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u} \quad \text{on } \mathbf{C}_{a,b} \tag{3.12}$$

for all simple $u \in \mathrm{L}^2(\mathbf{R};X)$. Moreover, (3.12) holds for all $u \in \mathrm{L}^p_a(\mathbf{R};X) \cap \mathrm{L}^p_b(\mathbf{R};X)$. Finally, $\|\widehat{\mathbb{E}}\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))} \le \max\{\|\mathbb{E}\|_{\mathrm{TI}_a}, \|\mathbb{E}\|_{\mathrm{TI}_b}\}$.

The converse is not true; indeed, for $Y := \ell^\infty(\mathbf{N})$, there is $\widehat{\mathbb{E}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(Y))$ (continuous to the boundary) s.t. $\widehat{\mathbb{E}}\widehat{u} \notin \mathcal{L}[\mathrm{L}^2(\mathbf{R};Y)]$ for some $u \in \mathrm{L}^2(\mathbf{R};Y)$, by Example 3.3.4. Moreover, $\|\widehat{\mathbb{E}}\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))}$ can be arbitrarily small compared to $\max\{\|\mathbb{E}\|_{\mathrm{TI}_a}, \|\mathbb{E}\|_{\mathrm{TI}_b}\}$, by the third remark of Example 3.3.4.

Recall from Proposition D.1.21(a) that $\widehat{u}, \widehat{\mathbb{E}u} \in \mathrm{H}(\mathbf{C}_{a,b};*)$ for all $u \in \mathrm{L}^p_a(\mathbf{R};X) \cap \mathrm{L}^p_b(\mathbf{R};X)$. Note also that if $u \in \mathrm{L}^2(\mathbf{R};X)$ is simple, then $u \in \mathrm{L}^q_r(\mathbf{R};X)$ for all $q \in [1,\infty]$, $r \in \mathbf{R}$.

We observe from Definition E.1.3 and Proposition E.1.8 that Remark 2.1.9 also applies in this general case, in particular, $\mathrm{TI}^p_a \cap \mathrm{TI}^p_b$ is well defined.

If Y is separable, then one can obtain an analogous theorem in case $p = \infty$ for $u \in \mathcal{C}_{0,a}(\mathbf{R};X) \cap \mathcal{C}_{0,b}(\mathbf{R};X)$ by slightly modifying the proof below and that of Theorem 3.2.4. However, such functions are not dense in $\mathrm{L}^\infty_a \cap \mathrm{L}^\infty_b$ and the counter-example of Section 3 of [W91a] shows that even Theorem 2.1.2 (which is a corollary Theorem 3.1.7, as explained before the theorem) is false for $p = \infty$ (also when $Y = \mathbf{C} = U$), hence so is Theorem 3.1.7.

Proof of Theorem 3.1.7: *Part I — Preparations:*

I.1° Defining $\widehat{\mathbb{E}} \in \mathrm{H}(\mathbf{C}_{a,b}; \mathcal{B}(X,Y))$: This goes as in the proof of Theorem 3.1.6: Let ϕ be as in Lemma D.1.25. We define the function $\widehat{\mathbb{E}}u_0 := \widehat{\phi}^{-1}\mathcal{L}\mathbb{E}\phi u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$ for each $u_0 \in X$ (since $\mathcal{L}\mathbb{E}\phi u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$, by Proposition D.1.21(a), we have $\widehat{\mathbb{E}}(s)u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$).

For any fixed $s \in \mathbf{C}_{a,b}$, the operator $\widehat{\mathbb{E}}(s) : X \to Y$ is obviously linear; by the norm inequality in Lemma D.1.10(a), it is also bounded (for this fixed s). By Lemma D.1.1(b), it follows that $\widehat{\mathbb{E}} \in \mathrm{H}(\mathbf{C}_{a,b}; \mathcal{B}(X,Y))$.

I.2° Defining M: By Proposition E.1.8, we have

$$M := \sup_{r \in [a,b]} \|\mathbb{E}\|_{\mathrm{TI}^p_r} = \max\{\|\mathbb{E}\|_{\mathrm{TI}^p_a}, \|\mathbb{E}\|_{\mathrm{TI}^p_b}\}. \tag{3.13}$$

Part II — Separable X and Y:

II.1° $\|\widehat{\mathbb{E}}\|_\infty \le M$, and (3.14) holds with $\widehat{\mathbb{E}}$ in place of F: Choose S as in Theorem 3.2.4. By Theorem 3.2.4, for each $r \in [a,b]$ we can choose $F(r+\cdot) : i\mathbf{R} \to \mathcal{B}(X,S^*)$ s.t. $\|F(r+it)\|_{\mathcal{B}(X,S^*)} \le M$ for all $t \in \mathbf{R}$ and

$$(F\widehat{u})\Lambda = \mathcal{L}\Lambda\mathbb{E}u \text{ a.e. on } r + i\mathbf{R} \text{ for all } \Lambda \in S \text{ and all finite-dimensional } u \in \mathrm{L}^2_r(\mathbf{R};X) \cap \mathrm{L}^p_r(\mathbf{R};X). \tag{3.14}$$

This defines a function $F : \overline{\mathbf{C}_{a,b}} \to \mathcal{B}(X,S^*)$.

By $I.1°$, we have $\widehat{\mathbb{E}}\widehat{\phi}u_0 = \mathcal{L}\mathbb{E}\phi u_0 \in \mathrm{H}(\mathbf{C}_{a,b};Y)$ for all $u_0 \in X$. By Proposition D.1.21(a), we have $\mathbb{E}\phi u_0 \in \mathrm{L}^1_r(\mathbf{R};Y)$ for all $r \in (a,b)$, hence $\mathcal{L}\Lambda\mathbb{E}\phi u_0 =$

$\Lambda\mathcal{L}\mathbb{E}\phi u_0$, hence $\mathcal{L}\Lambda\mathbb{E}\phi u_0 = \Lambda\widehat{\mathbb{E}}\widehat{\phi} u_0$, for each $u_0 \in X$. Combine this with (3.14) to observe that (for an arbitrary fixed $r \in (a,b)$)

$$(\widehat{\mathbb{E}}\widehat{\phi}u_0)\Lambda = (\mathcal{L}\mathbb{E}\phi u_0)\Lambda = (F\widehat{\phi}u_0)\Lambda \quad \text{a.e. on } \ r+i\mathbf{R}, \tag{3.15}$$

hence $(\widehat{\mathbb{E}}u_0)\Lambda = (Fu_0)\Lambda$ a.e. on $r+i\mathbf{R}$, for all $\Lambda \in S$ and $u_0 \in X$, hence $(\widehat{\mathbb{E}}u_0)\Lambda = (Fu_0)\Lambda$ a.e. on $r+i\mathbf{R}$. Choose a null set N_k for Λ_k for each $k \in \mathbf{N}$ to observe that $(\widehat{\mathbb{E}}u_0)\Lambda = (Fu_0)\Lambda$ on $(r+i\mathbf{R}) \setminus N$, where $N := \cup_{k\in\mathbf{N}} N_k$, for all $\Lambda \in \{\Lambda_k\}$, hence for all $\Lambda \in S$, by linearity.

Thus, $F(r+it)u_0 = \widehat{\mathbb{E}}(r+it)u_0$ as elements of S, i.e., $F(r+it)u_0 = \widehat{\mathbb{E}}(r+it)u_0 \in Y$, for all t s.t. $r+it \notin N$, hence $Fu_0 = \widehat{\mathbb{E}}u_0$ a.e. on $r+i\mathbf{R}$. Since $r \in (a,b)$ was arbitrary, we conclude that (3.14) holds with $\widehat{\mathbb{E}}$ in place of F, for all $r \in (a,b)$. We also conclude that

$$\|\widehat{\mathbb{E}}(r+it)\|_{\mathcal{B}(X,Y)} = \|\widehat{\mathbb{E}}(r+it)\|_{\mathcal{B}(X,S^*)} = \|F(r+it)\|_{\mathcal{B}(X,S^*)} \le M \tag{3.16}$$

for a.e. $t \in \mathbf{R}$, hence $\|\widehat{\mathbb{E}}(r+it)\|_{\mathcal{B}(X,Y)} \le M$ for all $t \in \mathbf{R}$, for each $r \in (a,b)$. Thus, $\|\widehat{\mathbb{E}}\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(X,Y))} \le M$.

II.2° (3.12) and the "moreover" claim hold: By Theorem 3.2.4(b2), condition (3.14) actually holds for all finite-dimensional $u \in \mathrm{L}^1_r \cap \mathrm{L}^p_r$ (since then $u \in \mathrm{L}^2_r$ or $p \le 2$), when $r \in (a,b)$.

But $\mathrm{L}^p_a \cap \mathrm{L}^p_b \subset \mathrm{L}^1_r \cap \mathrm{L}^p_r$, by Proposition D.1.21(a1), hence (3.14) holds for all finite-dimensional $u \in \mathrm{L}^p_a \cap \mathrm{L}^p_b$. But $\mathcal{L}\Lambda\mathbb{E}u = \Lambda\mathcal{L}\mathbb{E}u$, and S separates points (since it is norming), hence $\mathcal{L}\mathbb{E}u = \widehat{\mathbb{E}}\widehat{u}$ a.e. on $r+i\mathbf{R}$, for such u and all $r \in (a,b)$. By continuity, we have $\mathcal{L}\mathbb{E}u = \widehat{\mathbb{E}}\widehat{u}$ everywhere on $\mathbf{C}_{a,b}$ for such u

Given a general $u \in \mathrm{L}^p_a \cap \mathrm{L}^p_b(\mathbf{R};X)$, there are finite-dimensional $\{u_n\} \subset \mathcal{C}^\infty_{\mathrm{c}}(\mathbf{R};X)$ s.t. $u_n \to u$ in L^p_a and in L^p_b, by Theorem B.3.11(b2). Then $\mathbb{E}u_n \to \mathbb{E}u$ in L^p_a and in L^p_b. By Proposition D.1.21(a2), it follows that $\mathcal{L}\Lambda\mathbb{E}u_n \to \mathcal{L}\Lambda\mathbb{E}u$ and $\widehat{\mathbb{E}}\widehat{u}_n \to \widehat{\mathbb{E}}\widehat{u}$ pointwise on $\mathbf{C}_{a,b}$. Therefore, $\mathcal{L}\mathbb{E}u = \widehat{\mathbb{E}}\widehat{u}$.

Part III — the general case:

III.1° $\|\widehat{\mathbb{E}}\|_\infty \le M$: Let X_0 be any closed separable subspace of X. Choose Y_0 as in Lemma 3.2.6, so that $\mathbb{E}_{X_0,Y_0} := \mathbb{E}_{|\mathrm{L}^p(\mathbf{R};X_0)} \in \mathrm{TI}^p(X_0,Y_0)$.

By Part II, $\widehat{\mathbb{E}}\widehat{u} \in \mathrm{H}(\mathbf{C}_{a,b};Y)$ for all $u \in \mathrm{L}^p_a \cap \mathrm{L}^p_b(\mathbf{R};X_0)$. Take $\widehat{u} = \phi x_0$ for arbitrary $x_0 \in X_0$ to observe that $\widehat{\mathbb{E}}(s_0)x_0 \in Y_0$. Thus, $\widehat{\mathbb{E}}_{X_0} := \widehat{\mathbb{E}}_{|X_0} \in \mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(X_0,Y_0))$.

We deduce from this and Part II that $\|\widehat{\mathbb{E}}x_0\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};Y)} \le M$ for all $x_0 \in X_0$. Since X_0 was arbitrary, we have $\widehat{\mathbb{E}} \in \mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(X,Y))$ and $\|\widehat{\mathbb{E}}\|_{\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(X,Y))} \le M$.

III.2° (3.12) and the "moreover" claim hold: Given $u \in \mathrm{L}^p_a \cap \mathrm{L}^p_b(\mathbf{R};X)$, choose a closed separable subspace X_0 of X s.t. $u(t) \in X_0$ for a.e. $t \in \mathbf{R}$, so that $u \in \mathrm{L}^p(\mathbf{R};X_0)$ (after redefinition on a null set). Choose Y_0 as in *III.1°*, so that $\widehat{\mathbb{E}}_{X_0,Y_0} \in \mathrm{TI}^p(X_0,Y_0)$. Now we observe from Part II that (3.12) and the "moreover" claim hold.

III.3° Uniqueness: Assume that also some $\widehat{\mathbb{F}} \in \mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(X,Y))$ is as in the theorem (in place of $\widehat{\mathbb{E}}$). Then $(\widehat{\mathbb{E}} - \widehat{\mathbb{F}})\widehat{f}u_0 = 0$ on $\mathbf{C}_{a,b}$ for all simple

$f \in \mathrm{L}^2(\mathbf{R})$ and all $u_0 \in U$. Given $s_0 \in \mathbf{C}_{a,b}$, take $f := \chi_{[0,r]}$ (so that $f \in \mathrm{H}(\mathbf{C})$, $\widehat{f}(s) = (1-\mathrm{e}^{-rs})/s$), where $r > 0$ is s.t. $r\operatorname{Re} s_0 \notin 2\pi\mathbf{N}$, so that $\widehat{f}(s_0) \neq 0$ to observe that $(\widehat{\mathbb{E}} - \widehat{\mathbb{F}})(s_0) = 0$. □

Notes

The $\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$ result of Theorem 3.1.3(a1) is well known in the case of separable Hilbert spaces, see, e.g., Theorem 1 of [FS], which also provides an analogous result on unbounded closed operators. Lemma 13.1.5 provides the discrete-time counterpart of our result and Appendix F provides further results on $\mathrm{L}^\infty_{\mathrm{strong}}$.

By Example 3.3.4, not all $\mathrm{L}^\infty_{\mathrm{strong}}$ functions (not even all $\widehat{\mathrm{CTIC}}$ functions) correspond to TI operators when U and Y are allowed to be Banach spaces.

We conjecture that all $\mathrm{TI}^p(U,Y)$ maps have $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ transfer functions also when U and Y are general Banach spaces and $1 \le p < \infty$. However, we cannot prove this; see Section 3.2 for weaker analogies.

The claims on $\mathrm{W}^{n,p}_\omega$ in Theorem 3.1.5 are from [Sbook]. The special case $\mathbb{E} \in \mathrm{TIC}$ of Theorems 3.1.6 and 3.1.7 is essentially contained in Theorem 2.3 of [W91a].

See Appendix F and the notes on p. 1023 for $\mathrm{L}^\infty_{\mathrm{strong}}$.

3.2 $\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\mathrm{strong}}$ for Banach spaces (Fourier Multipliers)

I had a feeling once about mathematics – that I saw it all. Depth beyond depth was revealed to me – the Byss and the Abyss. I saw – as one might see the transit of Venus or even the Lord Mayor's Show – a quantity passing through infinity and changing its sign from plus to minus. I saw exactly why it happened and why tergiversation was inevitable – but it was after dinner and I let it go.

— Winston Churchill, (1874–1965)

In this section, we study Banach space Fourier multiplier theory; we shall mainly give Banach and TI^p equivalents of Theorem 3.1.3, $1 \le p < \infty$, with some partial results and guidelines for the case $p = \infty$, to which a complete extension would be false. Some of these results were used in the proof of Theorem 3.1.7.

We start by recalling the scalar case from [BL]:

Lemma 3.2.1 ($\mathbf{TI}^p(\mathbf{C}) \subset \mathbf{TI}^2(\mathbf{C})$) *Let $\mathbb{E} \in \mathrm{TI}^p_\omega(\mathbf{C})$, where $p \in [1,\infty)$ and $\omega \in \mathbf{R}$. Then $\mathbb{E}|_{C_c^\infty}$ has a unique extension to $\mathrm{TI}^2_\omega(\mathbf{C})$, and this extension coincides with $\mathbb{E}$ on $\mathrm{L}^p_\omega \cap \mathrm{L}^2_\omega$.*

In particular, there is a unique $\widehat{\mathbb{E}} \in \mathrm{L}^\infty(\omega + i\mathbf{R})$ s.t. $\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u}$ for all $u \in \mathrm{L}^2_\omega(\mathbf{R}) \cap \mathrm{L}^p_\omega(\mathbf{R})$. In fact, we have $\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u}$ for all $u \in \mathrm{L}^2_\omega(\mathbf{R})$; if $p \in [1,2]$, then $\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u}$ for all $u \in \mathrm{L}^p_\omega(\mathbf{R})$ too. Moreover, $\|\widehat{\mathbb{E}}\|_{\mathrm{L}^\infty(\omega+i\mathbf{R})} \le \|\mathbb{E}\|_{\mathrm{TI}^p_\omega}$.

The above also holds with $\mathrm{TI}^{C_0}_\omega$ in place of TI^p_ω and $C_{0,\omega}$ in place of L^p_ω (for $p = \infty$).

We conclude that $\mathrm{TI}^p(\mathbf{C}^n, \mathbf{C}^m) \subset \mathrm{TI}^2(\mathbf{C}^n, \mathbf{C}^m)$ for all $n, m \in \mathbf{N}$ (apply the lemma to each component of $\mathbb{E}$). Naturally, the space $\mathrm{TI}^{C_0}_\omega$ mentioned above refers to

$$\mathrm{TI}^{C_0}_\omega(U,Y) := \{\mathbb{E} \in \mathcal{B}(C_{0,\omega}(\mathbf{R};U), C_{0,\omega}(\mathbf{R};Y)) \bigm| \mathbb{E}\tau(t) = \tau(t)\mathbb{E} \text{ for all } t \in \mathbf{R}\}, \tag{3.17}$$

where $C_{0,\omega}(\mathbf{R},U) := e^{\omega\cdot} C_0(\mathbf{R};U)$ (with norm $\|f\|_{C_{0,\omega}} := \|e^{-\omega\cdot} f\|_{C_0} := \|e^{-\omega\cdot} f\|_\infty$ for all $f \in C_{0,\omega}$). Since C_c^∞ is dense in $C_{0,\omega}$ (by Theorem B.3.11(c), since $e^{-\omega\cdot} C_c^\infty = C_c^\infty$), this is in accordance with standard Fourier multiplier theory (and this allows a density argument unlike L^∞_ω would do). We observe from Theorem 3.1.5 that $\mathbb{E}|_{C_{0,\omega}} \in \mathrm{TI}^{C_0}_\omega(U,Y)$ for all $\mathbb{E} \in \mathrm{TI}^\infty_\omega(U,Y)$.

As shown in Example 3.2.3, we may have $\widehat{\mathbb{E}} = 0$ (equivalently, $\mathbb{E}|_{C_c^\infty} = 0$, hence $\mathbb{E}|_{C_0} = 0$) for $\mathbb{E} \in \mathrm{TI}^\infty \setminus \{0\}$.

Proof of Lemma 3.2.1: (W.l.o.g. we assume that $\omega = 0$.)

1° This follows easily from Theorem 6.1.2 of [BL] (and Theorem B.3.11 and Lemma A.3.10) if we require that $u \in \mathcal{S}(\mathbf{R})$. In particular, if $u \in \mathrm{L}^p \cap \mathrm{L}^2$, then there are $\{u_n\} \subset C_c^\infty(\mathbf{R})$ s.t. $u_n \to u$ in L^p_ω and in L^2_ω, thus, then $\mathbb{E}_2 u \leftarrow \mathbb{E}_2 u_n = \mathbb{E}u_n \to \mathbb{E}u$, as $n \to \infty$, where $\mathbb{E}_2$ is the extension of $\mathbb{E}|_{C_c^\infty}$ to $\mathrm{TI}^2(\mathbf{C})$. Therefore $\mathbb{E} = \mathbb{E}_2$ on $\mathrm{L}^p \cap \mathrm{L}^2$.

2° Assume that $p \in [1,2]$ and that $u \in \mathrm{L}^p(\mathbf{R})$. Choose $\{u_n\} \subset C_c^\infty(\mathbf{R})$ s.t. $u_n \to u$ in $\mathrm{L}^p(\mathbf{R})$. Since $\mathbb{E}u_n \to \mathbb{E}u$ in $\mathrm{L}^p(\mathbf{R})$, we have $\widehat{\mathbb{E}u_n} \to \widehat{\mathbb{E}u}$ in $\mathrm{L}^q(i\mathbf{R})$, where p and q are as in Theorem E.1.7. But $\widehat{\mathbb{E}u_n} = \widehat{\mathbb{E}}\widehat{u}_n \to \widehat{\mathbb{E}}\widehat{u}$ a.e. on $i\mathbf{R}$, hence $\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u}$ a.e. on $i\mathbf{R}$.

3° For general p, we have $\mathbb{E} \in \mathrm{TI}^2$, hence $\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u}$ a.e. on $i\mathbf{R}$ for all $u \in \mathrm{L}^2(\mathbf{R})$, by 2°.

4° Parts 1° and 3° apply to $\mathrm{TI}_0^{C_0}$ too, since for any $f \in C_0 \cap \mathrm{L}^2$, there is $\{f_n\} \subset C_c^\infty$ s.t. $f_n \to f$ in L^2 and in C_0 (multiply the convolution from Lemma 2.18 of [Adams] with suitable ϕ from Lemma B.3.10). □

Corollary 3.2.2 *Let $\mathbb{E} \in \mathrm{TI}^p_\omega(X,Y)$, where X and Y are arbitrary Banach spaces and $1 \le p < \infty$. Then $\Lambda\mathbb{E}f \in \mathrm{L}^2_\omega(\mathbf{R}) \cap \mathrm{L}^p_\omega(\mathbf{R})$ for each $\Lambda \in Y^*$ and each finite-dimensional $f \in \mathrm{L}^2_\omega(\mathbf{R};X) \cap \mathrm{L}^p_\omega(\mathbf{R};X)$ (for $f \in \mathrm{L}^2_\omega(\mathbf{R};X) \cap C_{0,\omega}(\mathbf{R};X)$ we can allow for $p = \infty$).*

Proof: (We assume that $p < \infty$; the case $p = \infty$ is analogous.) Let $f \in \mathrm{L}^2_\omega(\mathbf{R};X_0) \cap \mathrm{L}^p_\omega(\mathbf{R};X_0)$, where X_0 is a finite-dimensional subspace of X.

If $X_0 = \operatorname{span}\{x_0\}$ for some $x_0 \in X$, then $\Lambda\mathbb{E}P^*_{x_0} \in \mathrm{TI}^p_\omega(\mathbf{C})$, where $P_{x_0}\alpha := \alpha x_0$ ($\alpha \in \mathbf{C}$), hence then the claim follows from Lemma 3.2.1.

For a general n-dimensional X_0, we can apply the above to the n one-dimensional elements of $\Lambda\mathbb{E}P^*_{X_0}$, where P_{X_0} is an isomorphism $X_0 \to \mathbf{C}^n$. □

In case $p = \infty$, the operator $\widehat{\mathbb{E}}$ does no longer define $\mathbb{E}$ uniquely:

Example 3.2.3 [$\mathbf{0 \ne \mathbb{E} \in TIC^\infty(C)}$ but $\widehat{\mathbb{E}} = \mathbf{0}$] Let $\Lambda \in \mathrm{L}^\infty(\mathbf{R})^*$ be a "Banach limit at $-\infty$". Define $\mathbb{E} \in \mathrm{TIC}^\infty(\mathbf{C})$ by $(\mathbb{E}f)(t) := \Lambda f$ ($t \in \mathbf{R}$), so that $\|\mathbb{E}\|_{\mathrm{TIC}^\infty} = 1$. (Note that $\mathbb{E}f$ is a constant function for each $f \in \mathrm{L}^\infty(\mathbf{R})$.)

Then $\mathbb{E}f = 0$ for all $f \in C_0$ and for all $f \in \mathrm{L}^p \cap \mathrm{L}^\infty$ ($1 \le p < \infty$); in particular, $\widehat{\mathbb{E}} \equiv 0$ in the sense of Lemma 3.2.1, and 0 is the unique continuous extension of $\mathbb{E}|_{\mathcal{S}}$ to $\mathcal{B}(\mathrm{L}^p(\mathbf{R}))$ (or equivalently, to $\mathrm{TI}^p(\mathbf{C})$), for any $p \in [1,\infty)$. ◁

For $\mathbb{E} \in \mathrm{TI}^p$, $p < \infty$ this cannot happen since $\mathbb{E}|_{\mathcal{S}}$ determines $\mathbb{E}$ uniquely.

In the above example, $\mathbb{E}f$ coincides with the unique extension (namely 0) of $\mathbb{E}|_{\mathcal{S}}$ to $\mathrm{TI}^p(\mathbf{C})$ for all $f \in \mathrm{L}^p \cap \mathrm{L}^\infty$, but we do not know whether this is the case in general. At least the same cannot happen for $\mathbb{E} \in \mathrm{TI}^p$, $p < \infty$: when $p,q \in [1,\infty)$, $\mathbb{E} \in \mathrm{TI}^p(X,Y)$ and $f \in \mathrm{L}^p \cap \mathrm{L}^q(\mathbf{R};X)$, there is $\{f_n\} \subset C_c^\infty \subset \mathcal{S}$ s.t. $f_n \to f$ in both L^p and L^q, by Theorem B.3.11, hence then the unique extension (if any) of $\mathbb{E}|_{\mathcal{S}}$, to TI^q necessarily coincides with $\mathbb{E}$ on $\mathrm{L}^p \cap \mathrm{L}^q$.

(Also when extending an element of TI^p_ω to TI^p_α for some $\omega, \alpha \in \mathbf{R}$, we may have similar problems only in the case that $p = \infty$, due to same density arguments as above. See also Proposition E.1.8.)

Proof of Example 3.2.3: Define $\Lambda_n \in \mathrm{L}^\infty(\mathbf{R})^*$ by $\Lambda_n f := n^{-1}\int_{-n}^0 f\,dm$, for $n \in \mathbf{N}+1$, and set $\Lambda f := \lim_{n\to+\infty} \Lambda_n f$ for $f \in X$, where $X \subset \mathrm{L}^\infty(\mathbf{R})$ is the set of those $f \in \mathrm{L}^\infty(\mathbf{R})$ for which the limit exists. Use the Hahn–Banach theorem to extend Λ to $\mathrm{L}^\infty(\mathbf{R})^*$ with $\|\Lambda\|_{\mathcal{B}(\mathrm{L}^\infty)} = 1$ (since $\Lambda 1 = 1$). Obviously, $\tau^t f - f \in X$

and $\Lambda(\tau^t f - f) = 0$ for all f, hence Λ is time-invariant, hence so is $\mathbb{E}$. (Note that $\mathbb{E}f = \Lambda * \text{Я} f$ if we use the standard definition $\Lambda * g := \Lambda(\tau^{-t}\text{Я} g)$.)

Since $\mathbb{E}f \equiv 0$ whenever $\pi_{(-\infty,T)}f = 0$ for some $T \in \mathbf{R}$, we have $\mathbb{E}|_{\mathcal{C}_0} = 0$, by continuity, and $\pi_-\mathbb{E}\pi_+ = 0$, hence $\mathbb{E} \in \mathrm{TIC}^\infty$. By the Hölder Inequality, we have $f \in X$ and $\Lambda f = 0$ for any $f \in \mathrm{L}^p \cap \mathrm{L}^\infty$, $p \in [1,\infty)$.

Remark: One could also obtain an analogous operator in $\mathrm{TIC}^\infty(X,Y)$ for general Banach spaces X and Y by starting with $\Lambda_n f := n^{-1}\int_{-n}^0 Lf\,dm$ for some $L \in X^*$. □

Now we present a weak "generalization" of "$\widehat{\mathrm{TI}} = \mathrm{L}^\infty_{\text{strong}}$" (Theorem 3.1.3) for Banach spaces; recall that Theorem 3.1.7 is an application of this theorem:

Theorem 3.2.4 ($\mathbb{E} \in \mathrm{TI}^p(X,Y) \implies \widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\text{weak}^*}(i\mathbf{R};\mathcal{B}(X,S^*))$) *Let X and $Y \neq \{0\}$ be separable Banach spaces and $1 \le p < \infty$.*

By Lemma A.3.9, we can choose a sequence $\{\Lambda_k\} \subset Y^*$ s.t. $\|\Lambda_k\| = 1$ for all $k \in \mathbf{N}$ and $\|y\|_Y = \sup_{k\in\mathbf{N}}|\Lambda_k y|$.

Set $S := \mathrm{span}(\{\Lambda_k\}) \subset Y^*$. Then $(Iy)\Lambda := \Lambda y$ defines a (natural) linear isometry $I : Y \to S^*$, hence we can consider Y as a subspace of S^* and I as the inclusion $Y \subset S^*$.

If $\mathbb{E} \in \mathrm{TI}^p(X,Y)$, then there is $\widehat{\mathbb{E}} : i\mathbf{R} \to \mathcal{B}(X,S^*)$ s.t. $\|\widehat{\mathbb{E}}(it)\|_{\mathcal{B}(X,S^*)} \le \|\mathbb{E}\|_{\mathrm{TI}^p(X,Y)}$ for all $t \in \mathbf{R}$ and

$$(\widehat{\mathbb{E}}\widehat{f})\Lambda = \mathcal{L}\Lambda\mathbb{E}f \text{ a.e. on } i\mathbf{R} \text{ for all } \Lambda \in S \text{ and all finite-dimensional } f \in \mathrm{L}^2(\mathbf{R};X)\cap\mathrm{L}^p(\mathbf{R};X). \tag{3.18}$$

Moreover, the following hold:

(a) Furthermore, in (3.18) we can allow Λ to be any $\Lambda \in \bar{S} \subset Y^*$ (recall that $(\overline{S})^* = S^*$); if $p \le 2$, then, simultaneously, any $f \in \mathrm{L}^1(\mathbf{R};X)\cap\mathrm{L}^p(\mathbf{R};X)$ can be allowed.

(b1) For a fixed $\widehat{\mathbb{E}}$, equation (3.18) characterizes $\mathbb{E} \in \mathrm{TI}^p(X,Y)$ uniquely.

(b2) For a fixed $\mathbb{E} \in \mathrm{TI}^p(X,Y)$, equation (3.18) characterizes $\widehat{\mathbb{E}}$ uniquely in the sense that if $\widehat{\mathbb{E}}, \widehat{\mathbb{F}} : i\mathbf{R} \to \mathcal{B}(X,S^*)$ satisfy (3.18), then $\widehat{\mathbb{E}}x = \widehat{\mathbb{F}}x$ a.e. for all $x \in X$.

(c) For each $x \in X$ and $\Lambda \in S$, we have $(\widehat{\mathbb{E}}x)\Lambda \in \mathrm{L}^\infty(i\mathbf{R})$.

(d) For a fixed $x \in X$, the function $\widehat{\mathbb{E}}x : i\mathbf{R} \to S^*$ is (Bochner) measurable iff $\widehat{\mathbb{E}}x \in Y$ a.e. However, if $\widehat{\mathbb{E}}x \in Y$ a.e. for all $x \in X$, then $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\text{strong}}(i\mathbf{R};\mathcal{B}(X,Y))$.

As noted in the second remark of Example 3.3.4, the Plancherel Theorem does not hold for general Banach spaces. However, if $f \in \mathrm{L}^2(\mathbf{R};X)$ and f has a finite-dimensional range, or in $f \in \mathrm{L}^1(\mathbf{R};X)$, then f has a well-defined Fourier transformation (in L^2 or $\mathcal{C}_0$, respectively), by Lemma D.1.11(a1) or Lemma A.3.4(Q1).

By Corollary 3.2.2, we have $\Lambda\mathbb{E}f \in \mathrm{L}^2(\mathbf{R})$ for each $\Lambda \in Y^*$ and each finite-dimensional $f \in \mathrm{L}^2 \cap \mathrm{L}^p(\mathbf{R};X)$, hence $\widehat{\Lambda\mathbb{E}f} \in \mathrm{L}^2(i\mathbf{R})$ and $\widehat{f} \in \mathrm{L}^2(\mathbf{R};X)$ are well defined in (3.18). If $p \le 2$ and $f \in \mathrm{L}^1 \cap \mathrm{L}^p$, then $\widehat{f} \in \mathcal{C}_0(i\mathbf{R};X)$ and $\Lambda\mathbb{E}f \in \mathrm{L}^p(\mathbf{R})$, hence then $\widehat{\Lambda\mathbb{E}f} \in \mathrm{L}^q(i\mathbf{R};Y)$ is well defined in (3.18), where $p^{-1}+q^{-1}=1$ (see Theorem E.1.7).

Naturally, we can shift the above theorem to obtain a result on TI^p_ω for any $\omega \in \mathbf{R}$. By slightly modifying the proof, we obtain an analogous claim for TI^{C_0} too (with a density argument similar to that in 4° of the proof of Lemma 3.2.1).

Proof of Theorem 3.2.4: (N.B. one can deduce from the proof and Lemma 3.2.1 that the operator $\Lambda\mathbb{E}$ has a unique continuous extension to $\mathrm{L}^2(\mathbf{R};X_0)$ for any finite-dimensional subspace X_0 of X; and (3.18) holds for all elements $\mathrm{L}^2(\mathbf{R};X_0)$ (for this extended $\Lambda\mathbb{E}$).)

The construction of the normed space $S \subset Y^*$ is straightforward. Clearly $\|Iy\| \le \|y\|_{Y^{**}} = \|y\|_Y$; the converse follows from $\|Iy\|_{S^*} \ge \sup_k |\Lambda_k y| = \|y\|_Y$. Moreover, any functional $T \in S^*$ has a (necessarily norm-preserving) unique extension $\overline{T} \in (\overline{S})^*$, by, e.g., Lemma A.3.10. Thus only the claims concerning $\widehat{\mathbb{E}}$ are left to be proved.

1° *The functions $T_{\Lambda,x} \in \mathrm{L}^\infty(i\mathbf{R})$:* We denote $\|\mathbb{E}\|_{\mathrm{TI}^p(X,Y)}$ by $\|\mathbb{E}\|$. If $x \in X$ and $\Lambda \in Y^*$, then $\|g \mapsto \Lambda\mathbb{E}gx\|_{\mathrm{TI}^p(\mathbf{C})} \le \|\mathbb{E}\|\,\|\Lambda\|\,\|x\|$, hence, by Lemma 3.2.1 and Theorem 3.1.3(a1), there is $T_{\Lambda,x} \in \mathrm{L}^\infty(i\mathbf{R})$ s.t. (we choose the representative $T_{\Lambda,x} := LT_{\Lambda,x}$ from Lemma B.5.3)

$$\sup_{i\mathbf{R}} |T_{\Lambda,x}| \le \|\mathbb{E}\|\,\|\Lambda\|\,\|x\| \quad \text{and} \tag{3.19}$$

$$T_{\Lambda,x}\widehat{g} = \mathcal{L}\Lambda\mathbb{E}gx \text{ (a.e.) for all } g \in \mathrm{L}^2(\mathbf{R}) \cap \mathrm{L}^p(\mathbf{R}). \tag{3.20}$$

The mapping $Y^* \times X \ni (\Lambda,x) \mapsto T_{\Lambda,x} \in \mathrm{L}^\infty(i\mathbf{R})$ is bilinear (because $\mathcal{L}$ is linear), and its norm is at most $\|\mathbb{E}\|$, by (3.19). It follows from (B.56) (and the choice $T_{\Lambda,x} := LT_{\Lambda,x}$) that

$$it \in \mathrm{Leb}(T_{\Lambda,x}) \cap \mathrm{Leb}(T_{\Lambda,x'}) \implies T_{\Lambda,\alpha x+\beta x'}(it) = \alpha T_{\Lambda,x}(it) + \beta T_{\Lambda,x'}(it) \tag{3.21}$$

whenever $t \in \mathbf{R},\ \Lambda \in Y^*,\ x,x' \in X,\ \alpha,\beta \in \mathbf{C}$.

2° *The construction of $\widehat{\mathbb{E}}$:* Let $\{x_j\} \subset X$ be dense. Set

$$A := \cap_{j,k\in\mathbf{N}} \mathrm{Leb}(T_{\Lambda_k,x_j}), \qquad X_0 := \mathrm{span}(\{x_j\}_{j\in\mathbf{N}}) \subset X. \tag{3.22}$$

Then $m(i\mathbf{R}\setminus A) = 0$. For $it \in i\mathbf{R}\setminus A$, we set $\widehat{\mathbb{E}}(it) = 0$. For $it \in A$, $x \in X_0$ and $\Lambda \in S$, we define $(\widehat{\mathbb{E}}(it)x)\Lambda := T_{\Lambda,x}(it)$. It follows from (B.56) that $\widehat{\mathbb{E}}(it)x : S \to \mathbf{C}$ is linear; it is bounded by $\|\mathbb{E}\|\,\|x\|$, by (3.19), hence $\widehat{\mathbb{E}}(it)x \in S^*$ (for fixed $x \in X_0$ and $it \in A$).

By (3.21), mapping $x \mapsto \widehat{\mathbb{E}}(it)x \in S^*$ is linear, and by (3.19) it is bounded by $\|\mathbb{E}\|$, hence $\widehat{\mathbb{E}}(it) \in \mathcal{B}(X_0,S^*)$ (for fixed $it \in A$). By Lemma A.3.10, $\widehat{\mathbb{E}}(it)$ can be extended to an element of $\mathcal{B}(X,S^*)$ without affecting its norm, hence $\|\widehat{\mathbb{E}}(it)\|_{\mathcal{B}(X,S^*)} \le \|\mathbb{E}\|$ for all $it \in A$, hence for all $it \in i\mathbf{R}$.

3° *The verification of (3.18) for finite-dimensional $f \in \mathrm{L}^2 \cap \mathrm{L}^p(\mathbf{R};X)$:* Let $\Lambda \in S$ and $g \in \mathrm{L}^2(\mathbf{R}) \cap \mathrm{L}^p(\mathbf{R})$. Then

$$(\widehat{\mathbb{E}}\widehat{g}x)\Lambda = \widehat{g}(\widehat{\mathbb{E}}x)\Lambda = \widehat{g}T_{\Lambda,x} = \mathcal{L}\Lambda\mathbb{E}gx \quad \text{a.e. on } i\mathbf{R} \tag{3.23}$$

(the first equality holds everywhere, the second on A; the third equality holds in L^2, by (3.20), hence also pointwise a.e.) when $x \in X_0$; by continuity, this holds whenever $x \in X$ (the right-hand-side converges in L^2, hence a.e.; the left-hand-side converges pointwise everywhere when $x \to x'$ for some $x' \in X$). Thus, (3.18) holds for $f = gx$ with $g \in \mathrm{L}^2 \cap \mathrm{L}^p(\mathbf{R})$ and $x \in X$ arbitrary, hence whenever $f \in \mathrm{L}^2(\mathbf{R};X) \cap \mathrm{L}^p(\mathbf{R};X)$ is finite-dimensional, by linearity.

(a) 4° *The verification of (3.18) for finite-dimensional $f \in \mathrm{L}^p(\mathbf{R};X)$:* Assume that $1 \le p \le 2$. Then (3.20) also holds for any $g \in \mathrm{L}^p(\mathbf{R})$, by Lemma 3.2.1, hence (3.18) holds whenever $f \in \mathrm{L}^p(\mathbf{R};X)$ is finite-dimensional, by linearity, as in 3°.

For a general $f \in \mathrm{L}^1(\mathbf{R};X) \cap \mathrm{L}^p(\mathbf{R};X)$, there are finite-dimensional $f_n \in \mathcal{C}_c^\infty(\mathbf{R};X)$ $(n \in \mathbf{N})$ s.t. $f_n \to f$ in L^1 and in L^p, as $n \to +\infty$, by Theorem B.3.11(b1). It follows that $\Lambda\mathbb{E}f_n \to \Lambda\mathbb{E}f$ in L^p, hence $\mathcal{L}\Lambda\mathbb{E}f_n \to \mathcal{L}\Lambda\mathbb{E}f$ in L^q, hence a.e., where $p^{-1} + q^{-1} = 1$. But $\widehat{f_n} \to \widehat{f}$ in $\mathcal{C}_0$, hence $\widehat{\mathbb{E}}\widehat{f_n}\Lambda \to \widehat{\mathbb{E}}\widehat{f}\Lambda$ everywhere, hence $\mathcal{L}\Lambda\mathbb{E}f = \widehat{\mathbb{E}}\widehat{f}\Lambda$ a.e.

(Note that if X is a Hilbert space, then $\mathcal{L} \in \mathcal{B}(\mathrm{L}^p(\mathbf{R};X), \mathrm{L}^q(\mathbf{R};X))$, by Theorem E.1.7, hence then any $f \in \mathrm{L}^p(\mathbf{R};X)$ will do (then we can have $\widehat{f_n} \to \widehat{f}$ in L^q, hence a.e. on $i\mathbf{R}$ in the above proof).)

5° *Case $\Lambda \in \bar{S}$:* The extension for $\Lambda \in \bar{S}$ follows by continuity (because S and $\bar{S} \subset Y^*$ have the same dual, by Lemma A.3.10).

(c) 6° Let $g := \phi$, where $\phi \in \mathrm{L}^2(\mathbf{R}) \cap \mathrm{L}^p(\mathbf{R})$ is as in Lemma D.1.25. Divide (3.23) by $\widehat{\phi}^{-1}$ to obtain that $(\widehat{\mathbb{E}}x)\Lambda = T_{\Lambda,x} \in \mathrm{L}^\infty(i\mathbf{R})$ for any $x \in X$ (hence $\widehat{\mathbb{E}}x : i\mathbf{R} \to S^*$ is "weakly*-measurable").

(b1) 7° If $\widehat{\mathbb{E}} = 0$, then $\Lambda\mathbb{E}f = 0$ a.e. for all $\Lambda \in S$ and all simple $f \in \mathrm{L}^p(\mathbf{R};X)$, by (3.18), hence then $\mathbb{E}f = 0$ a.e. for all simple $f \in \mathrm{L}^p(\mathbf{R};X)$ by Lemma B.2.6, hence $\mathbb{E}f = 0$ for all $f \in \mathrm{L}^p(\mathbf{R};X)$, by density. Thus, if $\mathbb{E}$ and $\mathbb{F}$ correspond to some $\widehat{\mathbb{E}}$ as in the theorem, then $(\mathbb{F} - \mathbb{E}) = 0$ as elements of $\mathrm{TI}^p(X,Y)$.

(b2) 8° If $\mathbb{E} = 0$, then $(\widehat{\mathbb{E}}x)\Lambda = \widehat{\phi}^{-1}\mathcal{L}\Lambda\mathbb{E}\phi x = 0$ a.e. on $i\mathbf{R}$ for all $x \in X$ and $\Lambda \in S$, where ϕ is as in Lemma D.1.25; thus, then $\widehat{\mathbb{E}}x = 0$ a.e. for each $x \in X$ (choose a null set N_k for Λ_k for each k to obtain that $\widehat{\mathbb{E}}x = 0$ on $i\mathbf{R} \setminus \cup_{k \in \mathbf{N}} N_k$).

Thus, $\widehat{\mathbb{E}}, \widehat{\mathbb{F}} : i\mathbf{R} \to \mathcal{B}(X, S^*)$ satisfy (3.18) (at least for one-dimensional L^2 functions f and all $\Lambda \in S$), then $(\widehat{\mathbb{F}} - \widehat{\mathbb{E}})x = 0$ a.e. for each $x \in X$, as required.

(d) 9°

9.1° If $x \in X$ and $\widehat{\mathbb{E}}x \in Y$ a.e., then $\widehat{\mathbb{E}}x$ is almost separably-valued, and from the measurability of $\Lambda\widehat{\mathbb{E}}x$ for each $\Lambda \in S$ (see 6°) one can deduce that $\Lambda\widehat{\mathbb{E}}x$ is measurable for each $\Lambda \in Y^*$ (see, e.g., [Thomas, Corollary 2.9]), hence then $\widehat{\mathbb{E}}x$ is Bochner measurable.

(N.B. Whenever $\widehat{\mathbb{E}}x$ is almost separably-valued, equivalently, whenever $\widehat{\mathbb{E}}x \in Y_1$ a.e., where Y_1 is any separable subspace of S^*, then $\widehat{\mathbb{E}}x : i\mathbf{R} \to S^*$ is Bochner-measurable, by the above reasoning.)

9.2° Conversely, assume that $x \in X$ and that $\widehat{\mathbb{E}}x : i\mathbf{R} \to S^*$ is measurable. Then $f := \widehat{\mathbb{E}}(i\cdot)x$ is bounded, by 2°, hence $f \in \mathrm{L}^\infty(\mathbf{R};S^*)$. By (3.18) and Lemma D.1.11(e1), we have

$$(\mathcal{L}f\chi_A)\Lambda = \mathcal{L}(f\chi_A\Lambda) = 2\pi\Lambda\mathbb{E}(i\cdot)g_A(i\cdot)x \tag{3.24}$$

a.e. whenever $A \subset \mathbf{R}$, $m(A) < \infty$ and $\widehat{g}_A := \chi_A$ (note that $\mathcal{L}f\chi_A \in \mathcal{C}_0(i\mathbf{R};S^*)$).

Choose $N \subset \mathbf{R}$ s.t. $m(N) = 0$ and equality holds on $\mathbf{R} \setminus N$ in (3.24) when

$\Lambda \in \{\Lambda_k\}$. By linearity, the same holds for all $\Lambda \in S$ on $\mathbf{R} \setminus N$, hence $(\mathcal{L} f \chi_A)(it) = 2\pi\Lambda\mathbb{E}(it)g_A(it)x \in Y$ as elements of S^*, for each $t \in \mathbf{R} \setminus N$, hence a.e. Because A was arbitrary, we have $\widehat{\mathbb{E}}(i\cdot)x = f(\cdot) \in Y$ a.e., by Lemma D.1.22.

9.3° Finally, assume that $\widehat{\mathbb{E}}x \in Y$ a.e. for each $x \in X$ Then $\widehat{\mathbb{E}}x \in \mathrm{L}^\infty(i\mathbf{R};Y)$, as shown above. Let $\{x_k\} \subset X$ be dense, and let $\widehat{\mathbb{E}}x_k \in Y$ on N^c for each $k \in \mathbf{N}$, where $m(N) = 0$. By continuity, then $\widehat{\mathbb{E}}x \in Y$ on N^c for all $x \in X$, so we can redefine $\widehat{\mathbb{E}} = 0$ on N to make $\widehat{\mathbb{E}}$ $\mathcal{B}(X,Y)$-valued without affecting its properties stated in the theorem. Thus, $\widehat{\mathbb{E}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(X,Y))$. □

As another application of the above theorem, we deduce an implication that will be needed for Theorem 4.1.1:

Lemma 3.2.5 ($\mathbb{E} \in \mathrm{MTI}(X,Y) \cap \mathcal{G}\mathrm{TI}(X,Y) \implies \widehat{\mathbb{E}} \in \mathcal{G}\mathcal{C}_{\mathrm{b}}(i\mathbf{R};\mathcal{B}(X,Y))$) *Let X and Y be Banach spaces. Let $\mathbb{E} \in \mathrm{TI}(X,Y)$ and $\mathbb{V} \in \mathrm{TI}(Y,X)$. Assume, in addition, that $\mathbb{E} \in \mathrm{MTI}(X,Y)$.*

If $\mathbb{V}\mathbb{E} = I$, then $\|\widehat{\mathbb{E}}x\| \geq \|x\|/\|\mathbb{V}\|$ a.e. for all $x \in X$. If, in addition, $\mathbb{E}\mathbb{V} = I$ (i.e., $\mathbb{V} = \mathbb{E}^{-1}$), then $\widehat{\mathbb{E}} \in \mathcal{G}\mathcal{C}_{\mathrm{b}}(i\mathbf{R};\mathcal{B}(X,Y))$ and $\|\widehat{\mathbb{E}}^{-1}\| \leq \|\mathbb{E}^{-1}\|$.

(In fact, $\mathbb{E}$ could be replaced by a more general measure.)

Proof: W.l.o.g. we assume that $X \neq \{0\} \neq Y$.

1° *The separable case, $\mathbb{V}\mathbb{E} = I$:*
Choose $S \subset X^*$ and $\widehat{\mathbb{V}} : i\mathbf{R} \to \mathcal{B}(Y,S^*)$ for $\mathbb{V} \in \mathrm{TI}(Y,X)$ as in Theorem 3.2.4. Let $g \in \mathrm{L}^1(\mathbf{R};X) \cap \mathrm{L}^2(\mathbf{R};X)$, so that $f := \mathbb{E}g \in \mathrm{L}^1 \cap \mathrm{L}^2$ and $\widehat{f} = \widehat{\mathbb{E}}\widehat{g}$, by Lemma D.1.12(c2)&(c1). Equation (3.18) with $f = \mathbb{E}g$ in place of an arbitrary $f \in \mathrm{L}^1 \cap \mathrm{L}^2$ becomes (use Theorem 3.2.4(a))

$$(\widehat{\mathbb{V}}\widehat{\mathbb{E}}\widehat{g})\Lambda = \mathcal{L}\Lambda\mathbb{V}\mathbb{E}g = \mathcal{L}\Lambda I g \text{ a.e. for all } \Lambda \in S. \tag{3.25}$$

Because $g \in \mathrm{L}^1 \cap \mathrm{L}^2$ was arbitrary, the uniqueness claim of Theorem 3.2.4 applied to $I \in \mathrm{TI}(X)$ implies that for an arbitrary $x \in X$ we have $Ix = \widehat{\mathbb{V}}\widehat{\mathbb{E}}x$ a.e. (in S^*, hence in X), in particular,

$$\|\widehat{\mathbb{E}}x\| \leq \|x\|/\|\widehat{\mathbb{V}}\| \leq \|x\|/\|\mathbb{V}\| \tag{3.26}$$

a.e., hence everywhere, by the continuity of $\widehat{\mathbb{E}}$.

2° *The separable case, $\mathbb{V} = \mathbb{E}^{-1}$:*
It is enough to show that the range of $\widehat{\mathbb{E}}(it) \in \mathcal{B}(X,Y)$ is dense for all $t \in \mathbf{R}$, because then $\widehat{\mathbb{E}}(it) \in \mathcal{G}\mathcal{B}(X,Y)$ for all $t \in \mathbf{R}$, by 1° and Lemma A.3.4(D1), consequently, $\widehat{\mathbb{E}} \in \mathcal{G}\mathcal{C}(i\mathbf{R};\mathcal{B}(X,Y))$ (the inverse is continuous by Lemma A.3.3(A)), and the bound $\|\widehat{\mathbb{E}}^{-1}\| \leq \|\mathbb{V}\|$ ($\mathbb{V} = \mathbb{E}^{-1}$) follows from Lemma A.3.4(D1) (a.e., hence everywhere), thus, $\widehat{\mathbb{E}}^{-1} \in \mathcal{C}_{\mathrm{b}}(i\mathbf{R};\mathcal{B}(Y,X))$ (note also that $\widehat{\mathbb{V}}y = \widehat{\mathbb{V}}\widehat{\mathbb{E}}\widehat{\mathbb{E}}^{-1}y = \widehat{\mathbb{E}}^{-1}y \in X$ a.e. for each $y \in Y$).

Therefore, it is sufficient to assume that there is $a \in \mathbf{R}$ s.t. the range $Y_a := \widehat{\mathbb{E}}(ia)[X]$ is not dense in Y and derive a contradiction — that shall we do.

By Lemma A.3.14, there are $y_0 \in Y$ and $\Lambda_0 \in Y^*$ s.t. $\|y_0\| = 1 = \|\Lambda_0\|$, $\Lambda_0 Y_a = \{0\}$, and $\|\Lambda_0 y_0\| > 1/2$. Because $\Lambda_0\widehat{\mathbb{E}} : i\mathbf{R} \to X^*$ is continuous and

$\Lambda_0\widehat{\mathbb{E}}(ia) = 0$, there is $\delta > 0$ s.t.

$$\|\Lambda_0\widehat{\mathbb{E}}(it)\|_{X^*} < \delta' := 1/999(1+\|\mathbb{E}^{-1}\|) \text{ when } |t-a| < \delta^2/2, \tag{3.27}$$

i.e, when $it \in J := i(a-\delta^2/2, a+\delta^2/2) \subset i\mathbf{R}$.

Set $g := (\Lambda_0 y_0)^{-1}(\mathcal{L}^{-1}\chi_J)y_0 \in \mathrm{L}^2(\mathbf{R};Y)$. Then $\mathcal{L}\Lambda_0 g = \chi_J$, so we can choose a simple function $f \in \mathrm{L}^2(\mathbf{R};X)$ so that $\|f - \mathbb{E}^{-1}g\|_2$ is $< \delta$ and small enough to guarantee that $\delta/2 > \|\mathcal{L}\Lambda_0\mathbb{E}(f - \mathbb{E}^{-1}g)\|_2$, i.e., that

$$\delta/2 > \|\mathcal{L}\Lambda_0\mathbb{E}f - \chi_J\|_2 = \|\Lambda_0\widehat{\mathbb{E}}\widehat{f} - \chi_J\|_2. \tag{3.28}$$

From $\|\chi_J\|_2 = \delta$ it follows that

$$\|\mathbb{E}^{-1}g\|_2 \le \|\mathbb{E}^{-1}\|\,|\Lambda_0 y_0|^{-1}(2\pi)^{-1/2}\|\chi_J\|_2 \le \delta\|\mathbb{E}^{-1}\|, \tag{3.29}$$

hence $\|f\|_2 \le \delta(1+\|\mathbb{E}^{-1}\|)$, so

$$\delta^2/99 > (\delta')^2 2\pi\|f\|_2^2 = (\delta')^2\int_J \|\widehat{f}\|_X^2\,dm \ge \int_J |\Lambda_0\widehat{\mathbb{E}}\widehat{f}|^2\,dm. \tag{3.30}$$

But from (3.28) we obtain that

$$\|\Lambda_0\widehat{\mathbb{E}}\widehat{f}\|_2 \ge \|\chi_J\| - \delta/2 = \delta/2, \tag{3.31}$$

which together with (3.30) leads to a contradiction, as desired.

3° *The general case,* $\mathbb{V}\mathbb{E} = I$*:*
Let $x_0 \in X$ be arbitrary. Set $X_0 := \mathrm{span}(x_0)$, $Y_0 := \{0\}$, and find closed, separable subspaces $X' \subset X$ and $Y' \subset Y$ as in Lemma 3.2.6.

Set $\mathbb{E}' := \mathbb{E}_{|\mathrm{L}^2(\mathbf{R};X')} \in \mathrm{TI}(X',Y')$, $\mathbb{V}' := \mathbb{V}_{|\mathrm{L}^2(\mathbf{R};Y')} \in \mathrm{TI}(Y',X')$. Clearly $\mathbb{V}'\mathbb{E}' = I \in \mathrm{TI}(X')$ and $\mathbb{E}'\mathbb{V}' = I \in \mathrm{TI}(Y')$, hence, by part I,

$$\|\widehat{\mathbb{E}'}x_0\| \ge \|x_0\|/\|\mathbb{V}'\| \ge \|x_0\|/\|\mathbb{V}\|, \tag{3.32}$$

as required (if $A \mapsto \mu(A) \in \mathcal{B}(X,Y)$ is the measure generating $\mathbb{E}$, then by the definition of convolution, $A \mapsto \mu(A)_{|X'} \in \mathcal{B}(X',Y')$ generates $\mathbb{E}'$, in particular $\widehat{\mathbb{E}'}x = \widehat{\mathbb{E}}x$). Because $x_0 \in X$ was arbitrary, the claim follows.

4° *The general case,* $\mathbb{V} = \mathbb{E}^{-1}$*:*
Let $y_0 \in Y$ and $t \in \mathbf{R}$ be arbitrary. Choose $X_0 = \{0\}$ and $Y_0 = \mathrm{span}(y_0)$, and proceed as in 3°. By part I, $\widehat{\mathbb{E}'}(it) \in \mathcal{GB}(X',Y')$, hence there is $x_0 \in X$ s.t. $y_0 = \widehat{\mathbb{E}'}(it)x_0 = \widehat{\mathbb{E}}(it)x_0$.

Because $y_0 \in Y$ was arbitrary, $\widehat{\mathbb{E}}(it)$ is onto, hence $\widehat{\mathbb{E}}(it)$ invertible and $\|\widehat{\mathbb{E}}(it)^{-1}\| \le \|\mathbb{V}'\| \le \|\mathbb{V}\|$, by 3° and Lemma A.3.4(D1). Because $t \in \mathbf{R}$ was arbitrary, the proof is complete. □

We have already used the following lemma several times to reduce certain results to the separable case:

Lemma 3.2.6 ($\mathbf{TI}^p(X,Y) \to \mathbf{TI}^p(X_0,Y_0)$) *Let X and Y be Banach spaces and $1 \le p < \infty$. Let $\mathbb{E} \in \mathrm{TI}^p(X,Y)$.*

Then, for each closed separable subspace X_0 of X, there is a closed separable subspace Y_0 of Y s.t. $\mathbb{E}\mathrm{L}^p(\mathbf{R};X_0) \subset \mathrm{L}^p(\mathbf{R};Y_0)$, i.e., that $\mathbb{E}_{|\mathrm{L}^p(\mathbf{R};X_0)} \subset \mathrm{TI}^p(X_0,Y_0)$.

If, in addition, $\mathbb{V} \in \mathrm{TI}^p(Y,X)$*, then any separable subsets* $X_0 \subset X$ *and* $Y_0 \subset Y$ *are contained, respectively, in closed separable subspaces* $X' \subset X$ *and* $Y' \subset Y$*, s.t.* $\mathbb{E}_{|\mathrm{L}^p(\mathbf{R};X')} \subset \mathrm{TI}^p(X',Y')$ *and* $\mathbb{V}_{|\mathrm{L}^p(\mathbf{R};Y')} \subset \mathrm{TI}^p(Y',X')$.

Analogously, if $\mathbb{E}_k \in \mathrm{TI}^p(X_k, X_{k+1})$ *for* $k = 1,\ldots,n$*,* $n \in \mathbf{N}+1$*,* $X_{n+1} = X_1$*, and the sets* $X_k' \subset X_k$ *are separable* $(k = 1,\ldots,n)$*, then there are closed separable subspaces* $X_k'' \subset X_k$ $(k = 1,\ldots,n)$ *s.t.* $X_k' \subset X_k''$ *and* $\mathbb{E}_k{}_{|\mathrm{L}^p(\mathbf{R};X_k')} \in \mathrm{TI}^p(X_k', X_{k+1}')$ $(k = 1,\ldots,n)$*. Moreover, we can require that* $X_k'' = X_j''$ *whenever* $X_k = X_j$ *for some* k, j*.*

We might as well have stated the lemma for a general $\mathbb{E} \in \mathcal{B}(\mathrm{L}^p(\mathbf{R};X), \mathrm{L}^q(\mathbf{R};Y))$, where $p,q \in [1,\infty)$ (finding Y_0 s.t. $\mathbb{E}\mathrm{L}^p(\mathbf{R};X_0) \subset \mathrm{L}^q(\mathbf{R};Y_0)$; the latter claims can be generalized analogously), with the same proof (with only slight changes if $q \neq p$).

Furthermore, we could instead of separability require the density of a subset a some greater cardinality than that of $\mathbf{N}$ (in X_0 and in Y_0).

Proof: 1° *Finding* Y_0*:*
Let $\{f_k\} \subset \mathrm{L}^p(\mathbf{R})$ and $\{x_k\} \subset X$ be dense subsets (where k ranges over $\mathbf{N}$). Then the set D of finite linear combinations of functions of the form $f_k x_j$ is dense in $\mathrm{L}^p(\mathbf{R};X)$, because simple functions can obviously be approximated by such functions.

Choose a separably-valued representative of each $\mathbb{E}f$ $(f \in D)$, and let Y_0 be the closed span of $\cup_{f\in D}(\mathbb{E}f)[\mathbf{R}]$. Then Y_0 is separable as a countable union of separable sets, and $\mathbb{E}f \in \mathrm{L}^p(\mathbf{R};Y)$ (as an equivalence class) for all $f \in D$, hence, by the continuity of $\mathbb{E}$, for all $f \in \mathrm{L}^p(\mathbf{R};X_0)$.

2° *Finding* X' *and* Y'*:*
Replace first X_0 and Y_0 by their closed spans. Choose then a closed separable subspace Y_0'' of Y, as Y_0 was chosen in 1° for the pair $(\mathbb{E}, X_0)$, then set $Y_1 := \operatorname{span} Y_0 \cup Y_0''$.

For each $k \in \{2,3,4,\ldots\}$ choose $X_k \subset X$ for the pair $(\mathbb{V}, Y_{k-1})$, and then choose $Y_k \subset Y$ for the pair $(\mathbb{E}, X_k)$, as in 1°.

Set $X'' := \cup_k X_k$, $Y'' := \cup_k Y_k$, $X' := \overline{X''}$, $Y' = \overline{Y''}$. If $f \in \mathrm{L}^p(\mathbf{R};X')$ is simple and has its values in X'', then $f \in \mathrm{L}^p(\mathbf{R};X_k)$ for some $k \in \mathbf{N}$, hence then $\mathbb{E}f \in \mathrm{L}^p(\mathbf{R};Y_k) \subset \mathrm{L}^p(\mathbf{R};Y')$. But such functions are dense in $\mathrm{L}^p(\mathbf{R};X'')$, by Theorem B.3.11(a1)&(a3), hence, by continuity, $\mathbb{E}f \in \mathrm{L}^p(\mathbf{R};Y')$ for any $f \in \mathrm{L}^p(\mathbf{R};Y')$ (since $\mathrm{L}^p(\mathbf{R};Y')$ is closed in $\mathrm{L}^p(\mathbf{R};Y)$). Similarly, $\mathbb{V}[\mathrm{L}^p(\mathbf{R};Y')] \subset \mathrm{L}^p(\mathbf{R};X')$.

By Lemma B.2.3(a)&(c), X' and Y' are separable.

3° *Requiring that* $X' = Y'$ *(assuming that* $X = Y$*):* When choosing X_k in 2°, replace the one from 1° with its union with Y_{k-1}. Analogously, when choosing Y_k in 2°, replace the one from 1° with its union with X_k.

4° *Finding* X_k' $(k = 1,\ldots,n)$*:* Use the method of 2°–3° suitably modified. □

Notes

The classical scalar Fourier multiplier result, Lemma 3.2.1, is essentially contained in Theorem 6.1.2 of [BL] (which also gives further results). When U and Y are Hilbert spaces of arbitrary dimensions, we have $M_2(U,Y) =$

$\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y)$, by Theorem 3.1.3(a1), $M_\infty(U,Y) = \mathcal{B}(\mathcal{C}_0(i\mathbf{R};U),Y)$, by Lemma D.1.14, and $M_p(U,Y)^* = M_q(Y^*,U^*)$ (this last claim also holds when U and Y are Banach spaces), where $p^{-1}+q^{-1}=1$, by a proof analogous to that in the scalar case. This gives us no information on M_p for $p \neq 2,\infty$, since interpolation would require that, e.g., $M_p(U,Y) = M_q(Y^*,U^*)$. (Here $M_p(U,Y) := \widehat{\mathrm{TI}^p}(U,Y)$ for $p<\infty$, $M_\infty(U,Y) := \widehat{\mathrm{TI}_0^{\mathcal{C}}}(U,Y)$; the elements of these sets are called *Fourier multipliers*.)

On p. 135 of [BL], it is claimed that Theorem 6.1.2 has an obvious analogy in this general setting and that "the proofs are the same with trivial changes", but we cannot share this view for the following reasons: 1. the case $p=2$ is far from obvious (in fact, Theorem 3.1.3(a1) seems to be a new result); 2. the proof of $M_p = M_{p'}$ on p. 133 of [BL] would only yield $M_p^* = M_{p'}$ even for finite-dimensional Hilbert spaces, hence the original proof seems to cover only the case $p=\infty$; 3. we do not know any similar results from the literature (even in the finite-dimensional result mentioned below Lemma 3.2.1 one would require a different proof for a sharp norm bound).

There are several results on vector-valued Fourier multipliers in the literature; see, e.g., [BL], [Prüss93] or [Zimmermann] for sufficient conditions of a bounded function $i\mathbf{R} \to \mathcal{B}$ to be a Fourier multiplier.

Example 3.2.3 resembles the example of Section 3 of [W91a]. Another Banach limit is given in Exercise 4 of [Rud73].

3.3 H^2 and H^∞ boundary functions in L^2 and $\mathrm{L}^\infty_{\mathrm{strong}}$

> Boundary, *n.:*
> *In political geography, an imaginary line between two nations, separating the imaginary rights of one from the imaginary rights of the other.*
> — Ambroce Bierce (1842–1914), "The Devil's Dictionary"

In this section, we establish several results on the boundary functions of holomorphic functions, the most important of which is the connection $\mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y)) \to \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$. We also give other related results that will be needed for WPLS theory of Parts II and III, and we construct counter-examples for analogous results for more general settings.

At the end of this section, we shall show that the set of singular points ("poles") of the pointwise inverse of a transfer function may have limit points unlike in the case of finite-dimensional input and output spaces; we then use this to construct a "completely unstable" transfer function.

As before, H, U and Y are assumed to be arbitrary Hilbert spaces unless something else is indicated.

Recall the transfer functions of TIC_∞ operators from Theorem 2.1.2; recall also that we use the Lebesgue measure (of $\mathbf{R}^1$) on the imaginary axis $i\mathbf{R} := \{ir \,|\, r \in \mathbf{R}\}$ and on its translations $\omega + i\mathbf{R}$ ($\omega \in \mathbf{R}$). For any $r > 0$, the circle $\partial r\mathbf{D}$ is identified with $[0, 2\pi)$, hence $m(\partial r\mathbf{D}) = 2\pi$, where $\partial\mathbf{D} := \{e^{it} \,|\, t \in [0, 2\pi)\} = \{s \in \mathbf{C} \,|\, |s| = 1\}$.

The following theorem is the main result of this section. We use often claims (a2)&(a1) for the boundary functions of H^2 functions and claim (c1) for the boundary functions of operator-valued H^∞ functions, whereas the others are used just a few times:

Theorem 3.3.1 (H^p boundary functions) *Let $\omega \in \mathbf{R}$ and $1 \le p \le \infty$. Let B be a Banach space and let H, U and Y be Hilbert spaces. Then the following hold:*

(a1) Let $f \in \mathrm{H}^p(\mathbf{C}^+_\omega; B)$. Assume that there is $f_0 \in \mathrm{L}^p(\omega + i\mathbf{R}; Y)$ s.t. any one of (1.)–(6.) holds:

(1.) $\lim_{t\to\omega+} \Lambda f(ir + t) = \Lambda f_0(ir + \omega)$, *for almost every $r \in \mathbf{R}$, whenever $\Lambda \in B^*$;*

(2.) f converges to f_0 nontangentially at every Lebesgue point of f_0, hence a.e.;

(3.) f is the Poisson integral *of f_0, i.e.,*

$$f(\omega + t + ir) = \frac{t}{\pi} \int_{\mathbf{R}} \frac{f(\omega + i\rho)\, d\rho}{t^2 + (r - \rho)^2} \quad (t > 0,\ r \in \mathbf{R}). \tag{3.33}$$

(4.) $\int_E f(i\cdot + t)\, dm \to \int_E f_0(i\cdot + \omega)\, dm$ for all bounded measurable $E \subset \mathbf{R}$;

(5.) $\int_{-R}^{R} g f(i\cdot + t)\, dm \to \int_{-R}^{R} g f_0(i\cdot + \omega)\, dm$ for all $R > 0$ and $g \in \mathrm{L}^\infty(\mathbf{R}; \mathcal{B}(B, *))$;

(6.) $f(i\cdot+t) \to f_0(i\cdot+\omega)$ *in* L^p, *as* $t \to \omega+$;

Then f_0 *is unique, (1.)–(5.) hold (and (6.) if* $p < \infty$*), and*

$$\|f_0\|_p = \|f\|_{\mathrm{H}^p_\omega} = \lim_{r\to\omega+} \|f(i\cdot+r)\|_p \geq \|f(i\cdot+t)\|_p \quad (t > \omega). \tag{3.34}$$

If this is the case, then we call f_0 *the* (vector) L^p boundary function *of* f *and denote* $f(ir+\omega) := f_0(ir+\omega)$ $(r \in \mathbf{R})$ *and write* $f \in \mathrm{H}^p(\mathbf{C}^+_\omega;B) \cap \mathrm{L}^p(\omega+i\mathbf{R};B)$. *Any of (1.)–(6.) characterizes* f_0 *uniquely (in* L^p*, that is, a.e.).*

(a2) Every $f \in \mathrm{H}^p(\mathbf{C}^+_\omega;H)$ *has a* L^p *boundary function (recall that* U, H *and* Y *are assumed to be Hilbert spaces).*

(a3) If $f \in \mathrm{H}^p(\mathbf{C}^+_\omega;B)$, *then* $f|_{\mathbf{C}^+_{\omega'}} \in \mathrm{H}^p(\mathbf{C}^+_{\omega'};B)$ *has the* L^p *boundary function* $f|_{\omega'+i\mathbf{R}}$ *for any* $\omega' > \omega$.

(a4) If $f \in \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(\mathbf{C}^n,Y))$, $n \in \mathbf{N}$, *then* $f \in \mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(\mathbf{C}^n,Y)) \cap \mathrm{L}^p(\omega+i\mathbf{R};\mathcal{B}(\mathbf{C}^n,Y))$*; thus, then* f *has a* L^p *boundary function.*

(b) **(Paley–Wiener Theorem)** *The Laplace transform* $\mathrm{L}^2_\omega(\mathbf{R}_+;Y) \ni h \mapsto \hat{h} \in \mathrm{H}^2(\mathbf{C}^+_\omega;Y)$ *and the Fourier transform* $\mathrm{L}^2_\omega(\mathbf{R};Y) \ni g \mapsto \hat{g} \in \mathrm{L}^2(\omega+i\mathbf{R};Y)$ *are isomorphisms times* $\sqrt{2\pi}$, *and the former can be considered as the restriction of the latter to* $\pi_+\mathrm{L}^2_\omega$. *Finally, for all* $h \in \mathrm{L}^2_\omega(\mathbf{R}_+;Y)$ *and* $f,g \in \mathrm{L}^2(\mathbf{R};Y)$ *we have that*

$$\langle \hat{f},\hat{g}\rangle_{\mathrm{L}^2} = 2\pi\langle f,g\rangle_{\mathrm{L}^2}, \quad \|\hat{g}\|_{\mathrm{L}^2} = \sqrt{2\pi}\|g\|_{\mathrm{L}^2}, \quad \|\hat{h}\|_{\mathrm{H}^2_\omega} = \sqrt{2\pi}\|h\|_{\mathrm{L}^2_\omega}. \tag{3.35}$$

(c1) For every $f \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ *there is a unique (in* $\mathrm{L}^\infty_{\mathrm{strong}}$*)* (operator) boundary function $f_0 \in \mathrm{L}^\infty_{\mathrm{strong}}(\omega+i\mathbf{R};\mathcal{B}(U,Y))$ *s.t. for all* $u_0 \in U$ *the function* f_0u_0 *is the boundary function (see (a1)) of* $fu_0 \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;Y)$. *It follows that* f *is the strong Poisson integral ("$\not{\int}$") of* f_0 *and* $\|f_0\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \|f\|_{\mathrm{H}^\infty_\omega}$ *(we can choose* f_0 *s.t.* $\sup\|f_0\| = \|f\|_{\mathrm{H}^\infty_\omega}$*).*

We denote $f(\omega+ir) := f_0(\omega+ir)$, *where* f_0 *is the function constructed in the proof. For each* $\hat{u} \in \mathrm{H}^2(\mathbf{C}^+_\omega;U)$ *we have* $(f\hat{u})(\omega+ir) = f(\omega+ir)\hat{u}(\omega+ir)$ a.e. *on* $\omega+i\mathbf{R}$*; in particular,*

$$\mathbb{D} \in \mathrm{TIC}_\omega(U,Y) \implies \widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\hat{u} \text{ a.e. on } \omega+i\mathbf{R}. \tag{3.36}$$

The mapping $\mathrm{H}^\infty_\omega \ni f \mapsto f_0 \in \mathrm{L}^\infty_{\mathrm{strong}}$ *is an isometry to a closed subspace of* $\mathrm{L}^\infty_{\mathrm{strong}}$.

Finally, for any $\mathbb{D} \in \mathrm{TIC}_\omega$*, the boundary function of* $\widehat{\mathbb{D}} \in \mathrm{H}^\infty$ *coincides with the Fourier transform of* $\mathbb{D}$ *(from Theorem 3.1.3).*

(c2) If U *is separable, then we can choose* f_0 *in (c1) s.t.* $f(\omega+ir+\cdot) \to f_0(\omega+ir)$ *nontangentially in the strong topology of* $\mathcal{B}(U,Y)$*, for a.e.* $r \in i\mathbf{R}$*, and* $\|f(\omega+ir+\cdot)\| \to \|f_0(\omega+ir)\|$ *nontangentially for a.e.* $r \in i\mathbf{R}$.

(c3) Let $\mathbb{D} \in \mathrm{TIC}_\infty(\mathbf{C}^n,Y)$ *and* $\mathbb{D}[\mathrm{L}^2_{\mathrm{c}}] \subset \mathrm{L}^2_\omega$. *Then* $f := \widehat{\mathbb{D}} \in \mathrm{H}(\mathbf{C}^+_\omega;\mathcal{B}(\mathbf{C}^n,Y))$ *and* $\widehat{\mathbb{D}}$ *has a "*$\mathrm{L}^2_{\mathrm{loc}}$ *boundary function"* $f_0 =: \widehat{\mathbb{D}}$ *s.t.* $f(i\cdot+t) \to f_0(i\cdot+\omega)$ *in* $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};B)$*, as* $t \to \omega+$*, and* $(\cdot+1-\omega)^{-1}\widehat{\mathbb{D}} \in \mathrm{L}^2(\omega+i\mathbf{R};\mathcal{B}(\mathbf{C}^n,Y))$*, and*

(1.), (2.), (4.) and (5.) of (a1) are satisfied. Consequently, $\widehat{\mathbb{D}}\widehat{u} = \widehat{\mathbb{D}u}$ *a.e. on* $i\mathbf{R}$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R}_+;\mathbf{C}^n)$, $\alpha < \omega$.

If $\omega = 0$, *then* $\widehat{\mathbb{D}} \circ \phi_{\mathrm{Cayley}} \in \mathrm{H}^2(\mathbf{D};\mathcal{B}(\mathbf{C}^n,Y))$.

(d1) For every $f \in \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ *there is a* boundary function f_0 *whose values are (possibly unbounded) operators with domains in U and ranges in Y, s.t.* $f_0 u_0 \in \mathrm{L}^p(\omega + i\mathbf{R}, Y)$ *is the boundary function of* fu_0 *(see (a1)) for any* $u_0 \in U$.

In particular, $\sup_{u_0 \in U} \|f_0 u\|_p = \|f\|_{\mathrm{H}^p_{\mathrm{strong}}(\omega+i\mathbf{R};\mathcal{B}(U,Y))}$. *If U is separable, then*
$\mathrm{Dom}(f_0(\omega + ir))$ *is dense for a.e.* $r \in \mathbf{R}$.

(d2) Let $f \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$. *Then* $f \in \mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ *iff the boundary function* $f_0 \in \mathrm{L}^\infty_{\mathrm{strong}}$ *(see (c1)) is in* $\mathrm{L}^p_{\mathrm{weak}}(\omega + i\mathbf{R};\mathcal{B}(U,Y))$ *too.*

If this is the case, then $\|f_0\|_{\mathrm{L}^p_{\mathrm{weak}}} = \|f\|_{\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B})}$. *Claim (d2) also holds with "strong" in place of "weak".*

(d3) If $f \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y)) \cap \mathrm{L}^p(\omega + i\mathbf{R};\mathcal{B}(U,Y))$, *then* $f \in \mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$.

(d4) Conversely, if $f \in \mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ *and U is separable, then f has a unique "strong operator boundary function"* $f_0 \in \mathrm{L}^p_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ *s.t.* $f_0 u_0$ *is the boundary function of* fu_0 *for any* $u_0 \in U$ *(in the sense of (a1), and the same null set applies for all* u_0*). Moreover,* $\|\|f_0\|_{\mathcal{B}(U,Y)}\|_{\mathrm{L}^p(i\mathbf{R})} = \|f\|_{\mathrm{H}^p}$.

(e) Results analogous to (a1)–(d4) (except that (c3) must be replaced by Lemma 13.1.3(d)) hold for $R\mathbf{D} := \{z \in \mathbf{C} \mid |z| < R\}$ $(R > 0)$ *in place of* $\mathbf{C}^+_\omega$ *(write the convergence "(1.)" as* $f_0(z) = \lim_{r\to R-} f(rz)$ *a.e., where* $|z| = R$*). The Poisson integral on* $R\mathbf{D}$ *is given by*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} f(Re^{it})\,dt \quad (r > 0,\ \theta \in \mathbf{R}). \tag{3.37}$$

See Proposition D.1.21(c) and Theorem 3.1.6 for the boundary functions of $\mathrm{H}^2(\mathbf{C}_{a,b};U)$ and $\mathrm{H}^\infty(\mathbf{C}_{a,b};\mathcal{B}(U,Y))$ functions, respectively.

Note that the operator boundary function of (c1) need not be a vector boundary function (i.e., the limits need not converge in the operator norm), not even in the separable case of (d3), by Example 1. on p. 92 of [RR], where $f(z) := (x_k)_{k\in\mathbf{N}} \mapsto (z^k x_k)_{k\in\mathbf{N}}$, so that $f \in \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(\ell^2(\mathbf{N})))$.

Part (c1) (which seems to be new in the unseparable case) is the best we can say for unseparable U; e.g., it may be that the strong or weak limit of $f(ir+t)$ exists for no $ir \in i\mathbf{R}$ as $t \to 0+$ (i.e., there are U and $f \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U))$ s.t. for each ir, there is $u_0 \in U$ s.t. $f(ir+t)u_0$ does not have even a weak limit as $t \to 0+$), as shown in p. 133 [Thomas]). See [Thomas] for further results for the separable case and a counter-example for those results in the unseparable case.

The proof of Theorem 3.1.6 also shows how we could reduce (c1) to Theorem 3.1.3(a1), but we have preferred to give a direct proof below, because this proof is much simpler than that of Theorem 3.1.3(a1). The proof of (c1) lies heavily on the boundedness of the function, thus it cannot be used for H^p, $p < \infty$.

The equation $\widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\widehat{u}$ in (3.36) shows that the boundary function coincides with the Fourier transform of $\mathbb{D}$ given in Theorem 3.1.3. Naturally, the boundary function is in L^∞ if $\dim U < \infty$.

The result (c1) seems to be useful only for proving results such as those in Lemma 6.3.6, because many important properties of a $\mathbb{D} \in \mathrm{TIC}$ are not shared pointwise (a.e.) by an arbitrary representative of the Fourier transform $\widehat{\mathbb{D}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$. (E.g., if $\mathbb{D} \in \mathcal{G}\mathrm{TIC}$, then $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{L}^\infty_{\mathrm{strong}}$, but $\widehat{\mathbb{D}}(ir)$ may be noninvertible for all $r \in \mathbf{R}$.) An analogous remark applies to Theorem 3.1.3(a1).

The boundary function in (d1) in not $\mathcal{B}(U,Y)$-valued in general (see Example 3.3.6), unless $p = \infty$ (note that $\mathrm{H}^\infty_{\mathrm{strong}} = \mathrm{H}^\infty$, by the Closed Graph Theorem, hence (c1) applies to $p = \infty$).

For general $f \in \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ (or H^p), we have $f \in \mathrm{H}^p_{\mathrm{strong}} \cap \mathrm{H}^\infty(\mathbf{C}^+_{\omega+\varepsilon};\mathcal{B}(U,Y))$ ($\varepsilon > 0$), by Lemma F.3.2(a), hence we can apply (d2) with $\omega+\varepsilon$ in place of ω.

Proof of Theorem 3.3.1: W.l.o.g., we state the proofs for the case $\omega = 0$ (because $\mathrm{L}^2_\omega \ni u \mapsto e^{-\omega t}u \in \mathrm{L}^2$ is an isometric isomorphism and $\widehat{e^{-\omega t}u}(s) = \widehat{u}(s+\omega)$ for all $s \in \mathbf{C}^+$).

(a1)&(a2) (See p. 967 for nontangential limits.)

1° *Implications "(3.)⇒(2.)⇒(1.)", "(3.)⇒(5.)", and "(6.)⇒(5.)⇒(4.)" (any p) and "(3.)⇒(6.)" ($p < \infty$), and (3.34) for general B:* These follow from Lemma D.1.8(a3)&(a1) ("C_0" of (a1)) except for "(6.)⇒(5.)" (note that $\chi_{[-R,R]}g \in \mathrm{L}^q$, where $q^{-1} + p^{-1} = 1$, and use Hölder) and "(5.)⇒(4.)" (take $g := \chi_E$, $R := \sup|E|$).

2° *Uniqueness:* By Lemma B.2.6, (1.) characterizes f_0 uniquely (note that we may have a different null set for each Λ); so does also (4.) by Theorem B.4.12(e). By 1°, so do (2.), (3.), (5.) and (6.) too.

3° *(a2) when H is a separable Hilbert space:* Now $f_0 \in \mathrm{L}^p$ satisfying (2.) and (3.) (hence (1.)–(5.), by 1°) exists, by pp. 81, 85 and 90 of [RR]. By Lemma D.1.8(a1), we have $\|f_0\|_p = \|f\|_{\mathrm{H}^p}$

4° *(a2) when H is a Hilbert space:* Replace H by the closed span of $f[\mathbf{C}^+]$, which is a separable Hilbert space, by Lemma B.2.3(f)&(a). By 3°, $f_0 \in \mathrm{L}^p$ satisfying (1.)–(5.) (and (6.) for $p < \infty$) and $\|f_0\|_p = \|f\|_{\mathrm{H}^p}$ exists.

(We note one could prove (a2) whenever Y is a reflexive Banach space (an alternative condition is that Y^{**} is separable) and $1 < p \le \infty$, by using the (scalar) techniques of Chapter 11 of [Rud86], and the fact that Y is a Radon–Nikodym space [DU].)

5° *(a1) when B is a Banach space:* Assume any of (1.)–(6.). Let $g \in \mathrm{H}^p$ be the Poisson integral of f_0. Now Λf_0 is the boundary function of Λf, hence $\Lambda g = \Lambda f$, for any $\Lambda \in B^*$, by 3°. Having thus established (3.), we get the claim from 1° and 2°.

(a3) By the continuity of f, condition (1.) of (a1) is satisfied.

(a4) By Lemma A.1.1(a4), $f = \sum_{k=1}^n f_k P_k$ for some $f_1,\dots,f_n \in \mathrm{H}^p(\mathbf{C}^+_\omega;Y)$, where P_k is the kth canonical projection $\mathbf{C}^n \to \mathbf{C}$. Let $\widetilde{f_k} \in \mathrm{H}^p \cap \mathrm{L}^p$ be the boundary function of f_k ($k = 1,\dots,n$) (see (a2)). Obviously, $\|\sum_{k=1}^n f_k(ir+t)P_k - \sum_{k=1}^n \widetilde{f_k}(ir)P_k\|_{\mathcal{B}(\mathbf{C}^n;Y)} \to 0$ for a.e. $r \in \mathbf{R}$, as $t \to \omega+$, hence $\sum_{k=1}^n \widetilde{f_k}P_k \in$

$\mathrm{L}^p(\omega+i\mathbf{R};\mathcal{B}(\mathbf{C}^n;Y))$ is the boundary function of f.

(b) See p. 91 of [RR]. (And alternative reference to (a) and (b) is A.6.18–21 of [CZ].)

(c1) (We take $\omega=0$ w.l.o.g.; cf. Remark 2.1.6.) To be exact, we construct here a *function* $f_0 : i\mathbf{R}\to\mathcal{B}(U,Y)$ satisfying the conditions in the lemma, and then we show that the equivalence class of f_0 is a unique member of $\mathrm{L}^\infty_{\mathrm{strong}}$.

For each $r\in\mathbf{R}$, we define $U_r := \{u\in U \mid \exists \lim_{t\to 0+} f(ir+t)u =: y_{r,u}\}$. For a fixed r, the map $U_r\ni u\mapsto y_{r,u}$ is clearly linear and $\|u\mapsto y_{r,u}\|_{\mathcal{B}(U_r,Y)}\le\|f\|_\infty$, hence $u\mapsto y_{r,u}$ has a norm-preserving extension $f_0(ir)\in\mathcal{B}(U,Y)$ (note that $\|f_0(ir)\|\le\|f\|_{\mathrm{H}^\infty}$; we extend it to $\overline{U_r}$ by continuity (see Lemma A.3.10) and set $f_0(ir)=0$ on $U_r^\perp$). For any $u\in U$, we have $f_0(ir)u=\lim_{n\to\infty}f(ir+1/n)u$ a.e. r (because $u\in U_r$ a.e. $r\in\mathbf{R}$ by (b)), hence $f_0(i\cdot)u$ is measurable. Since u was arbitrary and f_0 is bounded, $f_0\in\mathrm{L}^\infty_{\mathrm{strong}}$.

Let $\widehat{u}\in\mathrm{H}^2$. Because the closed span U_u of $\widehat{u}[\mathbf{C}^+]$ is separable, we can take a null set $N_u\subset\mathbf{R}$ s.t. $f(ir+t)u_0\to f_0(ir)u_0$ for all $u_0\in U_u$ and all $r\notin N_u$ (choose a null set for each u_0 in a countable dense subset of U_u, and let N_u be the union of these sets; by Lemma A.3.4(H1) with "$F(s)=f(ir+1/s)-f_0(ir)$" we get the convergence for all $u_0\in U_u$ and any fixed $r\notin N_u$). Choosing null set N for $f\widehat{u}\in\mathrm{H}^2$ as in (a), we now have that

$$\begin{aligned}(f\widehat{u})(ir) &= \lim_{t\to 0+} f(ir+t)\widehat{u}(ir+t)\\ &= \lim_{t\to 0+}[f(ir+t)[\widehat{u}(ir+t)-\widehat{u}(ir)]+f(ir+t)\widehat{u}(ir)]\\ &= 0+f(ir)\widehat{u}(ir)\end{aligned}\tag{3.38}$$

for $r\notin N\cup N_u$. The equality $\|f_0\|_\infty=\|f\|_{\mathrm{H}^\infty}$ follows from the fact that $\|f_0u_0\|_\infty=\|fu_0\|_{\mathrm{H}^\infty}$ for all $u_0\in U$ by (b).

Moreover, if $f_1 : i\mathbf{R}\to\mathcal{B}(U,Y)$ also satisfies $f_1(i\cdot)u_0=\lim_{t\to 0+}f(i\cdot+t)u_0$ a.e. for $u_0\in U$, then $(f_1-f_0)u_0=0$ a.e. for $u_0\in U$, hence $f_1=f_0$ as a member of $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$. However, considered as functions, they may differ everywhere (and we may have $\|f_1(ir)\|_{\mathcal{B}(U,Y)}\ge r$ for all $r\in\mathbf{R}$ even though $f_1=0$ in $\mathrm{L}^\infty_{\mathrm{strong}}$ (i.e., $f_1u_0=0$ a.e. on $i\mathbf{R}$, for all $u_0\in U$)). The mapping $f\mapsto f_0$ is clearly linear, hence it is an isometry to a subspace of $\mathrm{L}^\infty_{\mathrm{strong}}$. The claims about TIC_ω now follow from Theorem 6.2.1.

Finally, if $G\in\mathrm{L}^\infty_{\mathrm{strong}}$ is the Fourier transform of $\mathbb{D}\in\mathrm{TIC}$ (see Theorem 3.1.3), then $F\widehat{f}u_0=\widehat{\mathbb{D}fu_0}=G\widehat{f}u_0=\widehat{f}Gu_0$ a.e. on $i\mathbf{R}$ for all $f\in\mathrm{L}^2$ and all $u_0\in U$, hence $Fu_0=Gu_0$ a.e. for all $u_0\in U$, i.e., $F=G$ in $\mathrm{L}^\infty_{\mathrm{strong}}$.

(c2) This is Theorem B on p. 85 of [RR], provided that also Y is separable. In the general case, the closed span Y_0 of $\{\widehat{\mathbb{D}}(z)u_0 \mid u_0\in U, z\in\mathbf{C}^+_\omega\}$ is separable, hence $\widehat{\mathbb{D}}\in\mathrm{H}^\infty(U,Y_0)$, and we can apply the result mentioned above.

(c3) (Take $\omega=0$ w.l.o.g. Set $U:=\mathbf{C}^n$.) By Lemma 2.1.13, we have $g:=(\cdot+1)^{-1}\widehat{\mathbb{D}}\in\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U,Y))$. By Lemma F.3.2(e), $\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U,Y))=\mathrm{H}^2(\mathbf{C}^+;\mathcal{B}(U,Y))$. By (a4), we have $g\in\mathrm{H}^2(\mathbf{C}^+;\mathcal{B}(U,Y))\cap\mathrm{L}^2(i\mathbf{R};\mathcal{B}(U,Y))$, in particular, g has a L^2 boundary function g_0. Set $f_0:=(\cdot+1)g_0$, so that $f_0\in(\cdot+1)\mathrm{L}^2(i\mathbf{R};\mathcal{B}(U,Y))\subset\mathrm{L}^2_{\mathrm{loc}}(i\mathbf{R};\mathcal{B}(U,Y))$ and f and f_0 inherit (1.), (2.)

and (5.) (hence also (4.)) from g and g_0; in particular, $\widehat{\mathbb{D}}$ converges to f_0 nontangentially at every $ir \in \mathrm{Leb}(f_0) = \mathrm{Leb}(g_0)$ (see (B.54)).

Let $\alpha < 0$ and $u \in \mathrm{L}^2_\alpha(\mathbf{R}_+;U)$. Since $\mathbb{D} \in \mathcal{B}(\pi_+\mathrm{L}^2_\alpha, \pi_+\mathrm{L}^2)$, by (2.19), the function $\widehat{\mathbb{D}}\widehat{u}$ converges to $\widehat{\mathbb{D}u}$ nontangentially a.e., by (a2). Since $\widehat{\mathbb{D}} \to f_0$ and $\widehat{u} \to \widehat{u}$ a.e. nontangentially, we must have $f_0\widehat{u} = \widehat{\mathbb{D}u}$ a.e.

The claim on $\omega = 0$ follows from Lemma 2.1.13.

(d1) (We set $\omega = 0$ w.l.o.g.) For $ir \in i\mathbf{R}$ we set

$$\mathrm{Dom}(f_0(ir)) := \{u_0 \in U \,\big|\, \lim_{t\to 0+} f(ir+t)u_0 =: f_0(ir)u_0 \text{ exists } \}. \tag{3.39}$$

Obviously, $\mathrm{Dom}(f_0(ir))$ is a subspace of U and $f_0(ir)$ is linear. Let $u_0 \in U$ be arbitrary. Then $fu_0 \in \mathrm{H}^p(\mathbf{C}^+;U)$, hence $f_0(ir)u_0$ is defined a.e. and equal to the boundary function described in (a). In particular,

$$\sup_{u_0\in U} \|f_0u_0\|_p = \sup_{u_0\in U} \|fu_0\|_{\mathrm{H}^p} =: \|f\|_{\mathrm{H}^p_{\text{strong}}(i\mathbf{R};\mathcal{B}(U,Y))}. \tag{3.40}$$

(Note that f_0 is unique in the sense that f_0u_0 is uniquely defined a.e. for each $u_0 \in U$.)

Let U be separable. Let $\{u_k\} \subset U$ be dense. For each k, there is a null set $N_k \subset \mathbf{R}$ s.t. $u_k \in \mathrm{Dom}(f_0(ir))$ for $r \in \mathbf{R}\setminus N_k$; consequently, $\{u_k\} \subset \mathrm{Dom}(f_0(ir))$ for all $r \in \mathbf{R}\setminus N$, where $N := \cup_k N_k$.

(d2) We prove the $\mathrm{H}^p_{\text{strong}}$ claim; add $\Lambda \in Y^*$ $[\|\Lambda\| \le 1]$ everywhere to obtain the weak result.

1° *"Only if":* Let $u_0 \in U$. Then $fu_0 \in \mathrm{H}^p_{\text{strong}}$, hence it converges a.e. to a boundary function $f_{u_0} \in \mathrm{L}^p(\omega + i\mathbf{R};Y)$, by (a), with $\|f_{u_0}\|_p = \|fu_0\|_{\mathrm{H}^p}$. But $f_{u_0} = f_0u_0$ a.e., by (c1), hence

$$\|f\|_{\mathrm{L}^p_{\text{strong}}} = \sup_{\|u_0\|\le 1} \|f_0u_0\|_p = \sup_{\|u_0\|\le 1} \|fu_0\|_{\mathrm{H}^p} = \|f_0\|_{\mathrm{H}^p_{\text{strong}}}. \tag{3.41}$$

By (a), fu_0 is the Poisson integral of f_0u_0 for each $u_0 \in U$, i.e., f is the strong Poisson integral of f_0.

2° *"If":* This follows from Lemma D.1.8(a1)&(a3).

(d3) This follows from (d2) and Lemma D.1.8(a1)&(a3).

(d4) By Theorem B on p. 85 of [RR], f has a boundary function f_0 in the strong operator topology (any f in the Nevanlinna class N has, and H^p functions belong to the Nevanlinna class, by (4-1) on p. 75 of [RR]). By the theorem, there is a null set $N \subset i\mathbf{R}$ s.t. $fu_0 \to f_0u_0$ nontangentially at every point of $i\mathbf{R}\setminus N$ (i.e., $f \to f_0$ nontangentially "in the strong operator topology" at every point of $i\mathbf{R}\setminus N$).

By Theorem C on p. 90, $f \mapsto f_0$ is an isometry $\mathrm{H}^p \to$"$\mathrm{L}^p_{\mathcal{B}}(i\mathbf{R})$", and "$\mathrm{L}^p_{\mathcal{B}}(i\mathbf{R})$" means "weakly measurable functions" with finite norm $\|\|\cdot\|_{\mathcal{B}(U,Y)}\|_{\mathrm{L}^p(i\mathbf{R})}$; the strong measurability of f_0 follows from that of f_0u_0 (by (a1)) for each $u_0 \in H$, hence $f_0 \in \mathrm{L}^p_{\text{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$.

(In [RR], we should have $U = Y$, but we may replace f by $\left[\begin{smallmatrix} 0 & 0 \\ f & 0 \end{smallmatrix}\right] \in \mathrm{H}^p(\mathbf{C}^+;\mathcal{B}(U\times Y))$. Replace Y first by its separable subspace if necessary (see Lemma B.2.4).

(e) This can be proved in the same way as (a)–(d) were proved (use the

$\mathrm{H}(\mathbf{D};*)$ results of [RR]). □

If a boundary function is zero on a set of positive measure, then both this function and the original function are zero almost everywhere:

Lemma 3.3.2 *Let f be a holomorphic function, and let f_0 be its boundary function in the sense of some of (a1)–(d4) of Theorem 3.3.1. Then the following are equivalent:*

(i) $f \not\equiv 0$ on $\mathbf{C}^+_\omega$;

(ii) $f \neq 0$ a.e. on $\mathbf{C}^+_\omega$;

(iii) $f_0 \neq 0$ (on a subset of positive measure of $\omega + i\mathbf{R}$);

(iv) $f_0 \neq 0$ a.e. (on $\omega + i\mathbf{R}$).

For (e), the above claims hold with substitutions $\mathbf{C}^+_\omega \mapsto R\mathbf{D}$ and $\omega + i\mathbf{R} \mapsto \partial R\mathbf{D} = \{z \in \mathbf{C} \mid |z| = R\}$.

Proof: Trivially (iv)⇒(iii). By Lemma D.1.2(e), we have (i)⇔(ii) (if $f \in \mathrm{H}(\Omega;B)$ and $E := f^{-1}[\{0\}]$ has a positive measure, then $K \cap E$ is infinite for some compact $K \subset \Omega$, by Lemma A.2.3, hence Ω (and K) contains a limit point of $E(\cap K)$, hence $f = 0$). Thus, it suffices to show establish (i)⇒(iv), (iii)⇒(i).

1° *Case $f \in \mathrm{H}^p(\mathbf{C}^+)$, $p \in [1,\infty]$:* This is well known (use e.g., Theorem 17.18 of [Rud86] and Cayley transform (see Lemma 13.2.1(e2)&(d)) for (i)⇒(iv), and the Poisson formula for (iii)⇒(i)).

2° *Case $f \in \mathrm{H}^p(\mathbf{C}^+;B)$, $p \in [1,\infty]$:* Now Λf_0 is the boundary function of Λf for each $\Lambda \in B^*$, hence this follows from 1° and Lemma B.2.6.

3° *Case $f \in \mathrm{H}^p(\mathbf{C}^+;\mathcal{B}(U,Y))$, $p \in [1,\infty]$:* Now $f_0 u_0$ is the boundary function of $f u_0$ for each $u_0 \in U$, hence this follows from 2°.

4° *Other cases:* For $r\mathbf{D}$, $r > 0$, in place of $\mathbf{C}^+$, the above proof applies mutatis mutandis. For $\mathbf{C}^+_\omega$ in place of $\mathbf{C}^+$, we obtain this by shifting f and f_0. □

We shall later need the following lemma:

Lemma 3.3.3 ($\langle \widehat{v}, F\widehat{u} \rangle_{\mathrm{L}^2(i\mathbf{R};\mathbf{C}^n)} = 0$ for all $u,v \implies F = 0$) *Let $\mathbb{D},\mathbb{N} \in \mathrm{TIC}_\infty(\mathbf{C}^n,Y)$, $\mathbb{D}[\mathrm{L}^2_{\mathrm{c}}],\mathbb{N}[\mathrm{L}^2_{\mathrm{c}}] \subset \mathrm{L}^2$ and $J \in \mathcal{B}(Y)$. Set $F := \widehat{\mathbb{N}}^* J \widehat{\mathbb{D}}$. Then $(1+\cdot)^{-2}F \in \mathrm{L}^1(i\mathbf{R};\mathcal{B}(\mathbf{C}^n))$.*

If $\langle \widehat{v}, F\widehat{u} \rangle_{\mathrm{L}^2(i\mathbf{R};\mathbf{C}^n)} = 0$ for all $u,v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+;\mathbf{C}^n)$, then $F = 0$ a.e.

Proof: 1° Set $U := \mathbf{C}^n$. By Theorem 3.3.1(c3), we have $\widehat{\mathbb{D}},\widehat{\mathbb{N}} \in (1+\cdot)\mathrm{L}^2(i\mathbf{R};\mathcal{B}(\mathbf{C}^n))$, hence $F \in (1+\cdot)^2\mathrm{L}^1(i\mathbf{R};\mathcal{B}(\mathbf{C}^n))$.

2° *Assumptions:* To obtain a contradiction, we assume that $\langle \widehat{v}, F\widehat{u} \rangle_{\mathrm{L}^2(i\mathbf{R};U)} = 0$ for all $u,v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+;U)$ (hence for all $u,v \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$ and all $\omega < 0$, by Lemma 2.1.13), but F is not zero a.e. on $i\mathbf{R}$. Then there is $r \in \mathbf{R}$ s.t. $ir \in \mathrm{Leb}(F)$ and $F(ir) \neq 0$.

W.l.o.g. we assume that $U = \mathbf{C}$, $0 \in \mathrm{Leb}(F)$ and $a := F(0)/2 > 0$ (choose $v_0, u_0 \in U$ and $\alpha \in \partial\mathbf{D}$ s.t. $\alpha v_0^* F(ir)u_0 > 0$ and replace F by $\alpha v_0^* F(\cdot - ir)u_0$; replace then u by $e^{ir\cdot} uu_0$ and v by $e^{ir\cdot} vv_0$ at the end of the proof).

3° Obviously, $|(1+ir)\widehat{f}_{t,0}(ir)| < 4t$ when $|r| > 1$ and $t \in (0,1)$, where $\widehat{f}_{t,0}(s) := 2t^{3/2}(s+t)^{-2}$. Consequently,

$$\int_{\pm[1,\infty)i} |f_{t,0}(ir)|^2 |F(ir)|\, dr < a/5 \tag{3.42}$$

for all $t \in (0, \delta_1)$, where $\delta_1 := a\|(\cdot+1)^{-2}F\|_1/20 > 0$.

4° $\varepsilon, \delta > 0$: Choose $\varepsilon > 0$ s.t. $\mathrm{Re}(2R)^{-1}\int_{-R}^{R} F(ir)\, dr > a$ for all $R \in (0,\varepsilon)$. Choose $\delta > 0$ s.t. $\int_{r\in(\varepsilon,1)} \|\widehat{f}_{t,0}(ir)\|^2 \|F(ir)\|\, dr < a/3$ for all $t \in (0,\delta]$ (see Lemma D.1.24(a) and note that $F \in \mathrm{L}^1([-1,1]i)$).

5° Choose $t \in (0, \max\{\delta, a/2\})$ s.t. $\int_{(-\varepsilon,\varepsilon)i} |f_{t,0}|^2\, dm > 2/3$ (see Lemma D.1.24(a)). From, e.g., Theorem 1.19 of [Rud86] we obtain functions

$$g_n := \sum_{k=1}^{m_n} R_{n,k}\chi_{(-r_{n,k},r_{n,k})i} \quad (n \in \mathbf{N}) \tag{3.43}$$

s.t. $R_{n,k} > 0$, $r_{n,k} \in (0,\varepsilon)$ $(n,k \in \mathbf{N})$, $0 \le g_1 \le g_2 \le \cdots \le g_n \le \cdots$ and $g_n(ir) \to g := |f_{t,0}(ir)|^2 \chi_{i(-\varepsilon,\varepsilon)} = (t^2+r^2)^{-1}\chi_{i(-\varepsilon,\varepsilon)}$ monotonously, for each $r \in \mathbf{R}$. Consequently, $\|g_n\|_1 = \sum_k R_{n,k}(2r_{n,k}) \to \|g\| > 2/3$. By The Monotone Convergence Theorem and 2°, we have

$$\mathrm{Re}\int_{-\varepsilon}^{\varepsilon} |f_{t,0}(ir)|^2 F(ir)\, dr = \lim_{n\to\infty}\int_{-\varepsilon}^{\varepsilon} g_n(ir)\,\mathrm{Re}\, F(ir)\, dr \ge \limsup_{n\to\infty}\sum_k R_{n,k}(2r_{n,k})a > 2a/3. \tag{3.44}$$

By this, (3.42) and 4°, we have $\mathrm{Re}\int_{i\mathbf{R}} |f_{t,0}|^2 F\, dm = \mathrm{Re}\langle f_{t,0}, F f_{t,0}\rangle > a(2/3 - 1/5 - 1/3) > 0$, a contradiction, as required. □

According to Theorem 2.3 of [W91a] (or to Theorem 3.1.7), the first part of Theorem 2.1.2 holds for arbitrary Banach spaces U and Y and for any $p \in [1,\infty)$. Even in that case, $\mathcal{L}: \mathrm{TIC} \to \mathrm{H}^\infty$ is a contractive algebra isomorphism of $\mathrm{TIC}(U,Y)$ *into* $\mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U,Y))$. However, the isomorphism is not isometric, nor onto:

Example 3.3.4 (A counter-example for Banach spaces: $\mathrm{H}^\infty \not\subset \widehat{\mathrm{TI}}$) Let $U = \mathbf{C}$, $Y = \ell^\infty(\mathbf{N})$, $e_k := \chi_{\{k\}}$ $(k \in \mathbf{N})$ (i.e., $e_0 = \{1,0,0,\ldots\}$, $e_1 := \{0,1,0,0,\ldots\}$ etc.) and $\widehat{\mathbb{D}}(s)u_0 := (e^{-ks}u_0)_{k\in\mathbf{N}}$, so that $\|\widehat{\mathbb{D}}(s)\| \le 1$ for all $s \in \overline{\mathbf{C}^+}$.

By Lemma D.1.1, $\widehat{\mathbb{D}}$ is holomorphic, because $e^{-k\cdot}$ is holomorphic for each $k \in \mathbf{N}$. Thus, $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(\mathbf{C},Y))$. However, $\widehat{\mathbb{D}}\widehat{f} \notin \mathcal{L}[\mathrm{L}^2(\mathbf{R}_+;Y)]$ for $f := \chi_{[0,1)} \in \mathrm{L}^2(\mathbf{R}_+)$, hence $\widehat{\mathbb{D}}$ is not the transfer function of any $\mathbb{D} \in \mathrm{TI}(\mathbf{C},Y)$.

Furthermore, $\widehat{\mathbb{V}}: s \mapsto (k^{-1/2}(s+1)^{-1}\widehat{\mathbb{D}}(s))_{k\in\mathbf{N}}$ satisfies $\widehat{\mathbb{V}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(\mathbf{C},Y))$ and $\widehat{\mathbb{V}}$ is continuous on $\overline{\mathbf{C}^+} \cup \{\infty\}$ ("$\widehat{\mathbb{V}} \in \widehat{\mathrm{CTIC}}$") but still $\widehat{\mathbb{V}}$ is not the transfer function of any $\mathbb{V} \in \mathrm{TI}(\mathbf{C},Y)$ (thus, "CTIC $\not\subset$ TIC" in this sense; cf. Definition 2.6.1).

Remarks: 1. We can use $\mathrm{TIC}(Y)$ in the example in place of $\mathrm{TIC}(\mathbf{C},Y)$.
2. Fourier transform does not map $\mathrm{L}^2(\mathbf{R};Y)$ into (nor onto) $\mathrm{L}^2(i\mathbf{R};Y)$.

3. $\mathrm{TIC} \to \mathrm{H}^\infty$ is not an isometry (nor onto). $\triangleleft$

Proof: (Note that we just use the Hilbert space definitions of Chapter 2 extended to allow for Banach spaces as input and output spaces.)

1° $\widehat{\mathbb{D}}\widehat{f} \notin \mathcal{L}[\mathrm{L}^2(\mathbf{R}_+;Y)]$ *for* $f := \chi_{[0,1)}$*:* By $P_k \in Y^*$ we denote the projection of $P_k : (y_k)_{k\in\mathbf{N}} \mapsto y_k$, for any $k \in \mathbf{N}$. Then P_k commutes with the Laplace transform (because $P_k \in Y^*$).

Choose $f \in \mathrm{L}^2([0,1])$ s.t. $\|f\|_2 = 1$ (e.g., $f = \chi_{[0,1]}$). If $g \in \mathrm{L}^2(\mathbf{R};Y)$ were s.t. $\widehat{g} = \widehat{\mathbb{D}}\widehat{f}$ on $\mathbf{C}^+$, then $P_k\widehat{g} = P_k(\widehat{\mathbb{D}}\widehat{f}) = \mathrm{e}^{-ks}\widehat{f}$, hence we would have $P_k g = \tau(-k)f$, and, consequently,

$$\|g\|_2^2 = \sum_k \int_k^{k+1} \|g\|_Y^2\, dm = \sum_k \int_k^{k+1} |g_k|^2\, dm = \sum_k \|f\|_2^2 = \infty. \tag{3.45}$$

Therefore, $\widehat{\mathbb{D}}$ does not map $\mathcal{L}\mathrm{L}^2(\mathbf{R}_+;U)$ into $\mathcal{L}\mathrm{L}^2(\mathbf{R};Y)$, hence $\widehat{\mathbb{D}}$ does not define a $\mathrm{TI}(U,Y)$ map.

2° *Constructing* $\widehat{\mathbb{V}}$ *with required properties:* Let $(a_k)_{k\in\mathbf{N}}$ be a sequence in $\mathbf{C}$ s.t. $a_k \to 0$ as $k \to \infty$. Define $T \in \mathcal{B}(\ell^\infty)$ by $(y_k)_{k\in\mathbf{N}} := (a_k y_k)_{k\in\mathbf{N}}$. Then also $\widehat{\mathbb{F}}(s) = T\widehat{\mathbb{D}}$ is in $\mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$, but, unlike for $\widehat{\mathbb{D}}$, we have $\widehat{\mathbb{F}} \in \mathcal{C}(\overline{\mathbf{C}^+};\mathcal{B}(U,Y))$, and $\widehat{\mathbb{F}}(ir+t) \to (a_k\mathrm{e}^{-irk})_{k\in\mathbf{N}} =: \widehat{\mathbb{F}}(ir) \in \mathcal{C}_{\mathrm{bu}}(i\mathbf{R};\mathcal{B}(U,Y))$ as $t \to 0+$, which will be shown in 3° below.

The function $\widehat{\mathbb{V}}(s) := \frac{1}{s+1}\widehat{\mathbb{F}}(s)$ is continuous on $\overline{\mathbf{C}^+} \cup \{\infty\}$ (we use the symbol $\widehat{\mathrm{CTIC}}$ of this kind of functions in Section 2.6), because it has the limit 0 at ∞ (because $\|\widehat{\mathbb{V}}(s)\| \le \frac{1}{|s|+1} \to 0$ as $|s| \to \infty$ on $\overline{\mathbf{C}^+}$). Let $F(t) := \mathrm{e}^{-t}\chi_{\mathbf{R}_+}(t)$, so that $\widehat{F}(s) = \frac{1}{s+1}$. If $\widehat{h} = \widehat{\mathbb{V}}\widehat{f} = \widehat{F}T\widehat{g}$, then $h = F * Tg$, in particular $P_k h = F * P_k T g = a_k F * g_k = a_k F * \tau(-k)f = \tau(-k)a_k F * f$, hence

$$\|h\|_2^2 = \sum_k \int_k^{k+1} \|h\|_Y^2\, dm \ge \sum_k \int_k^{k+1} |h_k|^2\, dm \ge \sum_k \int_0^1 |a_k F * f|^2\, dm \ge \sum_k |a_k|^2 r = \infty, \tag{3.46}$$

if $\sum_k |a_k|^2 = \infty$ (e.g., $a_k = k^{-1/2}$) and $r := \int_0^1 |F * f|^2 > 0$ (e.g., $f = \chi_{[0,1]}$). Thus, also $\widehat{\mathbb{V}}$ does not map $\mathcal{L}\mathrm{L}^2(\mathbf{R}_+;U)$ into $\mathcal{L}\mathrm{L}^2(\mathbf{R}_+;Y)$.

3° $\widehat{\mathbb{F}}$ is continuous on $\overline{\mathbf{C}^+}$: Given $s_0 \in \overline{\mathbf{C}^+}$ and $\varepsilon > 0$, choose first K s.t. $|a_k| < \varepsilon/2$ for $k \ge K$. Use then continuity to choose $\delta > 0$ s.t. $|\mathrm{e}^{-sk} - \mathrm{e}^{-s_0 k}| < \varepsilon/\sup_k |a_k|$ when $s \in \overline{\mathbf{C}^+}$ and $|s - s_0| < \delta$. Now $\|\widehat{\mathbb{F}}(s) - \widehat{\mathbb{F}}(s_0)\|_Y := \sup_k |\widehat{\mathbb{F}}(s)_k - \widehat{\mathbb{F}}(s_0)_k| < \varepsilon$ for such s, QED.

Remarks: 1. *We can use* $\mathrm{TIC}(Y)$ *in the example in place of* $\mathrm{TIC}(\mathbf{C},Y)$*:* just use $\widehat{\mathbb{D}}P_0$ and $\widehat{\mathbb{V}}P_0$ in place of $\widehat{\mathbb{D}}$ and $\widehat{\mathbb{V}}$, where $P_0(\alpha_k)_{k\in\mathbf{N}} := \alpha_0$.

2. *The Fourier transform does not map* $\mathrm{L}^2(\mathbf{R};Y)$ *into (nor onto)* $\mathrm{L}^2(i\mathbf{R};Y)$*; in fact, it does not even map* $\mathrm{L}^1 \cap \mathrm{L}^\infty \to \mathrm{L}^2$*:* Since $\|\widehat{g}\|_2 = \|\widehat{f}\|_2 = \sqrt{2\pi}\|f\|_2 = \sqrt{2\pi}$ and $\|g\|_2 = \infty$, we observe that the Fourier transform does not map $\mathrm{L}^2(\mathbf{R}_+;Y)$ onto $\mathrm{L}^2(i\mathbf{R};Y)$ (since g is the only possible inverse transform of $\widehat{g}$, by 1°). By replacing the roles of g and $\widehat{g}$, one can show that the Fourier transform does not map $\mathrm{L}^2(\mathbf{R}_+;Y)$ into $\mathrm{L}^2(i\mathbf{R};Y)$; not even $\mathrm{L}^1 \cap \mathrm{L}^\infty$ into $\mathrm{L}^2(i\mathbf{R};Y)$. (Indeed, if $f \in \mathcal{S}(\mathbf{R})$, then $\widehat{f} \in \mathcal{S}(i\mathbf{R})$, hence then $\widehat{g} := \widehat{\mathbb{D}}\widehat{f} \in \mathrm{L}^2(i\mathbf{R};Y)$ is "rapidly decreasing" (though not differentiable), hence $\widehat{g} \in \mathrm{L}^p$ for all $p \in [1,\infty]$.)

3. TIC $\to$ H$^\infty$ *is not an isometry (nor onto):* The operator $\widehat{\mathbb{V}}_n := \sum_{k=0}^n P_k \widehat{\mathbb{D}} \in \mathrm{H}^\infty$ is obviously the transfer function of a TIC$(\mathbf{C}, Y)$ map and $\|\widehat{\mathbb{V}}_n\| = \|\widehat{\mathbb{D}}\| = 1$ for $n \in \mathbf{N}$, but $\|\mathbb{V}_n\|_{\mathrm{TIC}} \to \infty$ (by the above computations), hence the Laplace transform TIC $\to$ H$^\infty$ is not an isometry (nor onto, by 1°). □

In fact, the above example also shows that Theorem 3.3.1(c1) does not hold for Banach spaces:

Example 3.3.5 (A counter-example for Banach spaces: $\widehat{\mathbb{D}} \in \mathrm{H}^\infty$ has no strong nor $\mathrm{L}^\infty_{\mathrm{strong}}$ boundary function) Let $U = \mathbf{C}$, $Y = \ell^\infty(\mathbf{N})$, and let $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U,Y))$ be as in Example 3.3.4, i.e., $\widehat{\mathbb{D}}(s) = (\mathrm{e}^{-ks})_{k \in \mathbf{N}}$.

Then $\widehat{\mathbb{D}}$ does not have a boundary function in the sense of Theorem 3.3.1(c1); in fact, $\widehat{\mathbb{D}}(ir + t)u_0$ does not converge in Y, as $t \to 0+$, for any nonzero $u_0 \in U$. Moreover, the componentwise boundary function of $\widehat{\mathbb{D}}$ is not an element of $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R}; \mathcal{B}(U;Y))$. ◁

Recall from Example 3.3.4 that the above $\widehat{\mathbb{D}}$ does not correspond to any TIC (not even to any TI) operator.

Proof: 1° Clearly $(\widehat{\mathbb{D}}(ir+t)u_0)_k \to \mathrm{e}^{-irk} u_0$ as $t \to 0+$, for every $u_0 \in \mathbf{C} = U$, $k \in \mathbf{N}$, so the componentwise boundary function $ir \mapsto \widehat{\mathbb{D}}(ir) := (\mathrm{e}^{-irk})_{k \in \mathbf{N}}$ is the only possible boundary function.

However, $\sup_{t \in (0,\delta)} \sup_{k \in \mathbf{N}} |\mathrm{e}^{-(ir+t)k} u_0 - \mathrm{e}^{-irk} u_0| = |u_0| > 0$ for any $\delta > 0$ (take $k > 2/\delta$ and $t = \pi/2k$), hence there is no limit of $\widehat{\mathbb{D}}(ir+t)u_0$ as $t \to 0+$, i.e., there is no strong boundary function.

2° Moreover, $\widehat{\mathbb{D}}(ir) \notin \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R}; \mathcal{B}(U,Y))$: if $\widehat{\mathbb{D}}(ir)$ were in $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R}; \mathcal{B}(U,Y))$, then $\widehat{\mathbb{D}}(i\cdot)$ would be an L^∞ function by Theorem 3.1.3(c) (because $U = \mathbf{C}$), hence then we would have $\widehat{\mathbb{D}}(i\cdot)P_0 \in \mathrm{L}^\infty(i\mathbf{R}; \mathcal{B}(Y))$, where P_0 is the projection to the 0th component ($P_0 : (y_k)_{k \in \mathbf{N}} \mapsto y_0$).

However, $r \mapsto \widehat{\mathbb{D}}(ir)P_0$ is a (semi)group of bounded linear operators on $\mathcal{B}(Y)$. Because it is not uniformly continuous, it is not uniformly measurable, by [HP, Theorem 10.2.1]. (A second proof: if $\widehat{\mathbb{D}}$ were $\mathrm{L}^\infty(i\mathbf{R}; Y)$, its Poisson integral would converge pointwise a.e. A third proof: the function $\widehat{g} = \widehat{\mathbb{D}}\widehat{f}$ in Example 3.3.4, would be L^1, hence g would be $\mathcal{C}_0$, but it does not vanish at infinity.) □

Output maps (hence also the causal adjoints of input maps) of WPLSs correspond to strong H^2 functions. Unfortunately, such functions need not have boundary functions with values in $\mathcal{B}$ (cf. Theorem 3.3.1(d)), not even for separable U and Y. For simplicity, we give our counter-example on the unit disc $\mathbf{D}$:

Example 3.3.6 (A counter-example: $F \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D}; \mathcal{B}(\ell^2))$ has no $\mathcal{B}(\ell^2)$-valued boundary function) Let $\{z_n\} \subset \partial\mathbf{D}$ be dense. Choose $r \in (0, 1/2)$. Define $z \mapsto F_n(z) := (z - z_n)^{-r} \in \mathrm{H}^2(\mathbf{D}; \mathbf{C})$ ($n \in \mathbf{N}$). Define the "diagonal operator" $F \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D}; \mathcal{B}(\ell^2))$ by $Fe_n := F_n e_n$ ($n \in \mathbf{N}$), where the vectors $e_n = \chi_{\{n\}}$ form the canonical base of $\ell^2(\mathbf{N}) =: U =: Y$. Indeed, $\|F_n\|_{\mathrm{H}^2} = \|F_0\|_{\mathrm{H}^2} =: M$ $(n \in \mathbf{N})$, hence

$$\|F \sum_{k=0}^{n} \alpha_k e_k\|_{\mathrm{H}^2}^2 \leq M^2 \sum_{k=0}^{n} |\alpha_k|^2. \tag{3.47}$$

Thus, F can be extended to the whole ℓ^2 so that $\|F\|_{\mathrm{H}^2_{\mathrm{strong}}} \leq M$. Moreover, if F is defined on $\partial\mathbf{D}$ so that $F_{|\partial\mathbf{D}} u$ is a.e. the boundary function of Fu for each $u \in U$, then $Fe_n = F_n$ on $\overline{\mathbf{D}} \setminus N$ for each $n \in \mathbf{N}$, where N is a null set.

Given $z \in \partial\mathbf{D} \setminus N$ and $M' > 0$, there is n s.t. $M' < |F_n(z)| = \|F(z)e_n\|_U$ (take n s.t. $|z - z_n|$ is small enough), hence $\|F(z)\|_{\mathcal{B}(U)} \geq M'$. Consequently, the operator $F(z)$ is unbounded (and possibly not defined for all $u \in U$) on $\partial\mathbf{D} \setminus N$.

(Note that Fu has a boundary function a.e. for each $u \in U$, the problem is that these boundary functions for $u \in U$ are not due to any (single) $\mathcal{B}(U)$*-valued* function; the values of the boundary function must be unbounded operators, as above.) $\triangleleft$

See also Example F.3.6.

In the scalar case the Poisson integral $P * f$ of any $f \in \mathrm{L}^p(i\mathbf{R}; B)$ is a harmonic function on the half-plane. This function is analytic iff $f \in \mathrm{H}^p$; a third equivalent condition for $f \in \mathrm{L}^1$ is that the Fourier transform of f is zero on R_-. All this holds also in the vector-valued case:

Lemma 3.3.7 ($f \in \mathrm{H}^p \Leftrightarrow P * f \in \mathrm{H}$) *Let* $f_0 \in \mathrm{L}^p(\omega + i\mathbf{R}; B)$, $p \in [1, \infty]$. *Then* f_0 *is the boundary function of some* $f \in \mathrm{H}^p(\omega + i\mathbf{R}; B)$ *iff the Poisson integral of* f_0 *is analytic on* $\mathbf{C}^+_\omega$. *For* $p = 1$ *a third equivalent condition is that*

$$\widetilde{f}(t) := \int_{\mathbf{R}} f_0(\omega + ir) \mathrm{e}^{irt} \, dr = 0 \quad \textit{for all } t < 0 \tag{3.48}$$

(if this is the case, then $f(\omega + \cdot) = \mathcal{L}\widetilde{f}/2\pi$*).*

Analogously, $f_0 \in \mathrm{L}^p(\partial\mathbf{D}; B)$ *is the boundary function of some* $f \in \mathrm{H}^p(\mathbf{D}; B)$ *iff the Poisson integral of* f_0 *is analytic on* $\mathbf{D}$*; a third equivalent condition is that* $\widehat{f}(n) := \frac{1}{2\pi} \int_{\partial\mathbf{D}} \mathrm{e}^{-inr} f_0(\mathrm{e}^{ir}) \, dr = 0$ *for* $n = -1, -2, \ldots$ *(if this is the case, then* $f = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$*).*

Recall that if B is a Hilbert space, then any H^p function has an L^p boundary function. Note also that $\int_{\mathbf{R}} f(\omega + ir) \mathrm{e}^{\cdot ir} \, dr \in C_0(\mathbf{R}; B)$, by Lemma D.1.11(a1)&(a3).

Proof: We prove the $\omega + i\mathbf{R}$ claims; the $\mathbf{D}$ claims (which can be scaled for $\mathbf{D}_r$) follow analogously from Theorem 17.13 of [Rud86] (note that $\mathrm{L}^p \subset \mathrm{L}^1$ on $\partial\mathbf{D}$).

1° If the Poisson integral f of f_0 is analytic, then $f \in \mathrm{H}^p$ and f_0 is the boundary function of f, by Lemma D.1.8(a3). Conversely, if f_0 is the boundary function of some $f \in \mathrm{H}^p$, then f is the Poisson integral of f_0, by Theorem 3.3.1(a1).

2° *Case* $p = 1$*:* For $B = \mathbf{C}$, the third condition is equivalent, by Lemma II.3.7 and Theorem II.3.8 of [Garnett]. From this and 1° it follows for general B, that the Poisson integral of Λf is analytic for all $\Lambda \in B^*$ iff

$\widetilde{f}(t) := \int_{\mathbf{R}} \Lambda f(\omega + ir)e^{tir}\,dr = 0$ for all $t < 0$. This and Lemma D.1.1(a) imply that the Poisson integral of f is analytic iff $\int_{\mathbf{R}} f(\omega + ir)e^{irt}\,dr = 0$ for all $t < 0$. By Lemma F.3.7(a3), we have $f(\cdot) = \mathcal{L}e^{\omega\cdot}\widetilde{f}/2\pi$ $\square$

Next we extend the standard formula $\widehat{\mathbb{D}^{\mathrm{d}}}(s) = \widehat{\mathbb{D}}(\bar{s})^*$ for causal adjoint transfer functions to the vector-valued case (including noncausal TI maps):

Lemma 3.3.8 ($\widehat{\mathbb{D}^{\mathrm{d}}}(s) = \widehat{\mathbb{D}}(\bar{s})^*$) *Let $u \in \mathrm{L}^2(\mathbf{R};U)$ and $\mathbb{E} \in \mathrm{TI}(U,Y)$. Then $\widehat{\boldsymbol{\mathcal{R}}u}(ir) = \widehat{u}(-ir)$ for $r \in \mathbf{R}$. Moreover, $\widehat{\mathbb{E}^*}(ir) = \widehat{\mathbb{E}}^*(ir)$, $\widehat{\boldsymbol{\mathcal{R}}\mathbb{E}\boldsymbol{\mathcal{R}}}(ir) = \widehat{\mathbb{E}}(-ir)$ and $\widehat{\mathbb{E}^{\mathrm{d}}}(ir) = \widehat{\mathbb{E}}^*(-ir)$ for $r \in \mathbf{R}$.*

If $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, then $\widehat{\mathbb{D}^{\mathrm{d}}}(s) = \widehat{\mathbb{D}}(\bar{s})^*$ for $s \in \mathbf{C}^+_\omega$.

See Definition 3.1.1 (and Theorem 3.1.3(d)) for $\widehat{\mathbb{E}}^*$.

Proof: 1° TI: Obviously, $\widehat{\boldsymbol{\mathcal{R}}u}(s) = \widehat{u}(-s)$ wherever either integral converges. It follows that

$$\mathcal{L}(\boldsymbol{\mathcal{R}}\mathbb{E}\boldsymbol{\mathcal{R}}u)(ir) = \mathcal{L}(\mathbb{E}\boldsymbol{\mathcal{R}}u)(-ir) = \widehat{\mathbb{E}}(-ir)\widehat{u}(ir) \quad (r \in \mathbf{R}). \tag{3.49}$$

Therefore, $\mathcal{L}(\boldsymbol{\mathcal{R}}\mathbb{E}\boldsymbol{\mathcal{R}})(ir) = \widehat{\mathbb{E}}(-ir)$ for all $r \in \mathbf{R}$. By linearity and continuity, we obtain that $\langle \widehat{\mathbb{E}}\widehat{u}, \widehat{v}\rangle = \langle \widehat{u}, \widehat{\mathbb{E}}^*\widehat{v}\rangle$ for all $u, v \in \mathrm{L}^2$ (see Definition 3.1.1), hence $\widehat{\mathbb{E}^*} = \widehat{\mathbb{E}}^*$, from which it follows that $\widehat{\mathbb{E}^{\mathrm{d}}}(ir) = \widehat{\mathbb{E}}^*(-ir)$ for $r \in \mathbf{R}$.

2° TIC: Finally, let $\mathbb{D} \in \mathrm{TIC}$ (the TIC_ω result is obtained by shifting) and $f = \widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$, and define $h(\cdot) := f(\bar{\cdot})^* \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(Y,U))$. Let $\mathbb{F} \in \mathrm{TIC}(Y,U)$ be defined by $\widehat{\mathbb{F}} = h$, and let h_0 be the boundary function of h.

Then $\langle f_0(-ir)u_0, y_0\rangle = \lim_{t\to 0+}\langle f(-ir+t)u_0, y_0\rangle = \lim_{t\to 0+}\langle u_0, f(-ir+t)^*y_0\rangle = \langle u_0, h_0(ir)y_0\rangle$ a.e. on $i\mathbf{R}$ for $u_0 \in U$, $y_0 \in Y$, hence $\boldsymbol{\mathcal{R}}[f_0]^* = [h_0] \in \mathrm{L}^\infty_{\text{strong}}$; in particular, $[h_0] = \widehat{\mathbb{F}}$, by 1°. Therefore, h is an H^∞ function with the boundary function $\widehat{\mathbb{F}}$, but, by Theorem 3.3.1(c1), the transfer function of $\mathbb{F}$ is the only such function. $\square$

During the rest of this section, we shall study the poles of the inverses of transfer functions and use the results to construct a “completely” unstable transfer function.

Let $\Omega \subset \mathbf{C}$ is open and $m, n \in \mathbf{N}$. If $0 \neq f \in \mathrm{H}(\Omega;\mathbf{C})$, then the set of zeros of f (i.e., of poles of f^{-1}) does not have limit points in Ω (see, e.g., Theorem 10.18 of [Rud86]). It follows that if $f \in \mathrm{H}(\Omega;\mathcal{B}(\mathbf{C}^n))$ is invertible at some $s_0 \in \Omega$, then f is invertible on a set whose complement does not have limit points in Ω (this complement is the set of zeros of $\det f$). The same applies to the left-invertibility of $f \in \mathrm{H}(\Omega;\mathcal{B}(\mathbf{C}^n,\mathbf{C}^m))$ (because if $L \in \mathbf{C}^{n\times m}$ is s.t. $Lf(s_0) = I$, then $Lf \in \mathrm{H}(\Omega;\mathcal{B}(\mathbf{C}^n)))$.

These facts are extensively used in control theory. However, if $\dim U = \infty$ and $f \in \mathrm{H}(\Omega;\mathcal{B}(U))$, then the set of “poles” of f^{-1} may be any closed subset of Ω, even if f were bounded, when, e.g., $\Omega = \mathbf{C}^+_\omega$ ($\omega \in \mathbf{R}$) or $\Omega = \mathbf{D}_r$ ($r > 0$):

Lemma 3.3.9 (Poles of $\widehat{\mathbb{D}} \in \mathrm{H}^\infty$) *Let $\dim U = \infty$. Let $K \subset \mathbf{C}$ be closed and let $s_0 \in \mathbf{C}\setminus K$. Then there is $\widehat{\mathbb{D}} \in \mathrm{H}(\{s_0\}^c;\mathcal{B}(U))$ s.t. $\widehat{\mathbb{D}}^{-1} \in \mathrm{H}(K^c;\mathcal{B}(U))$, $\widehat{\mathbb{D}} \in$*

$\mathcal{G}\mathrm{H}^\infty(K_\varepsilon^c;\mathcal{B}(U))$ for any $\varepsilon > 0$, where $K_\varepsilon := \{s \in \mathbf{C} \,|\, d(s, K \cup \{s_0\}) \le \varepsilon\}$, and the set of singularities of $\widehat{\mathbb{D}}^{-1}$ is K in the sense that $\lim_{K^c \ni s \to s_1} \|\widehat{\mathbb{D}}^{-1}(s)\| = +\infty$ for each $s_1 \in \partial K$.

Thus, when $\omega < \omega'$ and $K \subset \{z \,|\, \operatorname{Re} z < \omega'\}$ is closed, there is $\mathbb{D} \in \mathrm{TIC}_\omega(U) \cap \mathcal{G}\mathrm{TIC}_{\omega'}(U)$ s.t. the set of singularities of $\widehat{\mathbb{D}}^{-1}$ is K. One observes from the proof that $\widehat{\mathbb{D}}(s) \notin \mathcal{G}\mathcal{B}(U)$ for any $s \in K$, since the poles of the "components" of $\widehat{\mathbb{D}}$ are dense in K.

We might call these singularities "poles" (at least those on ∂K), since this would be in accordance with the definition "$\lim_{\Omega \ni s \to s_1} \|f(s)^{-1}\| \to +\infty$" (for scalar functions this is equivalent to the standard one, by Theorem 10.21(c) of [Rud86]).

(Unless s_1 is an isolated point of K, there is a sequence $\{z_n\} \subset K^c$ s.t. $z_n \to s_1$ and $\|(z_n - s_1)^N \widehat{\mathbb{D}}(s_1)\| \to +\infty$, as $n \to +\infty$, for any $N \in \mathbf{N}$ (e.g., choose them so that $d(z_n, K) < (z_n - s_1)^N/n$ and use (3.51)), hence one might argue that the points of ∂K (or K) should nevertheless be called essential singularities.

Proof: By Lemma A.3.1(a2), we can assume that $U = \ell^2(A)$ for some infinite set A. Set $K_0 := K \cup \{s_0\}$, $\gamma := d(s_0, K) > 0$.

1° *Functions $\{f_a\}_{a \in A}$:* Choose $\phi : A \to K$ s.t. $\phi[A]$ is dense in K. Set

$$f_a(s) := \frac{s - \phi(a)}{s - s_0} = 1 + \frac{s_0 - \phi(a)}{s - s_0} \quad (s \in \mathbf{C} \setminus \{s_0\},\ a \in A). \tag{3.50}$$

Then $|f_a(s)|, |f_a^{-1}(s)| < 1 + \gamma/\varepsilon =: M_\varepsilon$ for all $\varepsilon > 0$ and all $s \in K_\varepsilon^c$. Moreover, $f_a \in \mathrm{H}(\{s_0\}^c)$ and $f_a^{-1} \in \mathrm{H}(K^c)$.

2° *Function f:* For each s, we define $f(s) \in \ell^\infty(A)$ by $f(s)_a := f_a(s)$. Now $\Lambda_a f \in \mathrm{H}(\{s_0\}^c)$ for all $a \in A$, where $\Lambda_a u := u_a$, hence $f \in \mathrm{H}(\{s_0\}^c; \ell^\infty(A))$, by Lemma D.1.1(a).

Analogously, $f^{-1} \in \mathrm{H}(K^c; \ell^\infty(A))$, where $f^{-1}(s) = (f_a(s)^{-1})_{a \in A} \in \ell^\infty$ (this is obviously the inverse of $f(s)$ on K_0^c).

But $\ell^\infty(A)$ (as multiplication operators) is a Banach subalgebra of $\mathcal{B}(\ell^2(A))$. Therefore, $f \in \mathrm{H}(\{s_0\}^c; \mathcal{B}(U))$, $f^{-1} \in \mathrm{H}(K^c; \mathcal{B}(U))$, by Lemma D.1.2(b1). By 1°, we have $\|f\|, \|f^{-1}\| \le M_\varepsilon$ on K_ε^c for each $\varepsilon > 0$.

3° *Singularities of f^{-1}:* Let $s \in K^c$. For each $\delta > d(s, K)$ there is $a \in A$ s.t. $|s - \phi(a)| < \delta$ and hence

$$|f_a^{-1}(s)| = \frac{|s - s_0|}{|s - \phi(a)|} > \frac{\gamma - \delta}{\delta}, \tag{3.51}$$

therefore $\|f^{-1}(s)\|_{\mathcal{B}(U)} \ge \frac{\gamma - d(s,K)}{d(s,K)}$. Thus, $\|f^{-1}(s)\| \to \infty$ as $d(s, K) \to 0$. Consequently, $\widehat{\mathbb{D}} := f$ has the required properties. □

By using the above techniques, we can construct a "completely unstable" $\mathbb{D} \in \mathrm{TIC}_\infty$ whenever U and Y are infinite-dimensional (note that $\dim Y \ge \dim U$ iff there is a left-invertible $\mathbb{D} \in \mathrm{TIC}_\omega(U, Y)$, by Lemma 2.2.1(c3))

Example 3.3.10 ($\mathbb{D}u \notin \mathrm{L}^2$ for all nonzero $u \in \mathrm{L}^2$) Let $\dim Y \ge \dim U = \infty$. If $\omega > 0$, then there is a left-invertible $\mathbb{D} \in \mathrm{TIC}_\omega(U, Y)$ s.t. $\mathbb{D}u \notin \mathrm{L}^2$ for all nonzero

$u \in \mathrm{L}^2(\mathbf{R}_+;U)$. $\triangleleft$

In fact, given $\gamma > 0$, we can choose an ULR $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ s.t. $\mathbb{D}$ satisfies the above conditions for any $\omega > \gamma$, and $\widehat{\mathbb{D}} \in \mathrm{H}((\gamma + i/(\mathbf{N}+1))^c; \mathcal{B}(U,Y))$, and $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\Omega_{\omega',\omega}; \mathcal{B}(U,Y))$ whenever $\omega' < \gamma < \omega$, where $\Omega_{\omega',\omega} := \{s \in \mathbf{C} \,|\, \mathrm{Re}\, s \in [\omega',\omega]^c\}$ (take $s_n := \gamma + i/(n+2)$ in the proof below).

Therefore, in that case, we have $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^-_\alpha; \mathcal{B}(U,Y))$ for each $\alpha < \gamma$, hence this (left) part of $\widehat{\mathbb{D}}$ is the transfer function (Fourier transform) of some $\widetilde{\mathbb{D}} \in \cap_{\alpha<\gamma} \mathrm{TI}_\alpha(U,Y)$, which is strictly anti-causal, i.e., $\pi_+\widetilde{\mathbb{D}}\pi_- = 0 \neq \pi_-\widetilde{\mathbb{D}}\pi_+$ (apply Theorem 3.1.6(b) to $\widehat{\mathbb{D}}$ and Lemma 2.1.11(iv)&(i) to $\boldsymbol{Я}\widetilde{\mathbb{D}}\boldsymbol{Я}$).

Proof: W.l.o.g., we assume that $\dim Y = \dim U$ (replace then $\mathbb{D} \in \mathrm{TIC}_\omega(U)$ by $T\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$ for any left-invertible $T \in \mathcal{B}(U,Y)$). By Lemma A.3.1(a2) and Lemma B.2.2, we may assume that $U = \ell^2(A)$ and $Y = \ell^2(A \times \mathbf{N})$ for some infinite set A.

Choose $K_0 := \{s_n\}_{n=-1}^\infty \subset \gamma + i\mathbf{R}$ s.t. $s_n \to s_0$, as $n \to \infty$, and $\gamma := d(s_{-1}, K) > 0$, where $K := \{s_n\}_{n=0}^\infty$.

Set $f_j(s) := (s - s_j)/(s - s_{-1})$ $(s \in \mathbf{C} \setminus \{s_{-1}\},\ j \in \mathbf{N})$, so that $|f_j(s)|, |f_j(s)^{-1}| < 1 + \gamma/\varepsilon =: M_\varepsilon$ for all $\varepsilon > 0$ and all $s \in \mathbf{K}^c_\varepsilon$, where $K_\varepsilon := \{s \in \mathbf{C} \,|\, d(s,K_0) \leq \varepsilon\}$ (as in 1° of the proof of Lemma 3.3.9).

Define $\widehat{\mathbb{D}}(s) \in \mathcal{B}(U,Y)$ by $(\widehat{\mathbb{D}}(s)u)_{a,j} := 2^{-j} f_j^{-1} u_a$ $(u \in U)$. Then

$$\|\widehat{\mathbb{D}}(s)u\|_Y^2 = \sum_{a\in A,\ j\in\mathbf{N}} \|2^{-j} f_j^{-1}(s) u_a\|^2 \leq \sum_{a\in A} 2\|f^{-1}(s)\|_{\ell^\infty} \|u_a\|^2, \tag{3.52}$$

hence $\|\widehat{\mathbb{D}}(s)\|_{\mathcal{B}(U,Y)} \leq 2M_\varepsilon$ for $s \in K^c_\varepsilon$.

Define $\widehat{\mathbb{E}}(s) \in \mathcal{B}(Y,U)$ by $(\widehat{\mathbb{E}}(s)y)_a := f_0(s) y_{a,0}$ $(y \in U)$. Then $\|\widehat{\mathbb{E}}(s)\| \leq M_\varepsilon$ for $s \in K^c_\varepsilon$. Obviously, $\widehat{\mathbb{E}}(s)\widehat{\mathbb{D}}(s) = I$ for $s \in K^c_0$. It follows from Lemma D.1.1(a) that $\widehat{\mathbb{D}}$ and $\widehat{\mathbb{E}}$ are holomorphic on $\mathbf{C}^+_\omega$, hence $\widehat{\mathbb{D}}, \widehat{\mathbb{E}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U))$, hence $\mathbb{D}, \mathbb{E} \in \mathrm{TIC}_\omega(U)$.

As in the proof of Lemma 3.3.9, we can verify that $\widehat{\mathbb{D}}$ has the properties claimed above. We proof the claim "$\mathbb{D}u \in \mathrm{L}^2 \Rightarrow u = 0$":

Let $u \in \mathrm{L}^2(\mathbf{R}_+;U) \setminus \{0\}$. Choose $a \in A$ s.t. $u_a \not\equiv 0$. By Lemma D.1.2(e), there is $j \in \mathbf{N}$ s.t. $\widehat{u}_a(s_j) \neq 0$. It follows that $\|f_j \widehat{u}_a\| \to \infty$ as $s \to s_j$ (see 3° of the proof of Lemma 3.3.9).

But $(\widehat{\mathbb{D}}\widehat{u})_{a,j} = 2^{-j} f_j^{-1} \widehat{u}_a$, and $(y \mapsto y_{a,j}) \in \mathcal{B}(Y,\mathbf{C})$, hence $\widehat{\mathbb{D}}\widehat{u} \notin \mathrm{H}(\mathbf{C}^+;Y)$, in particular, $\mathbb{D}u \notin \mathrm{L}^2$. $\square$

On the other hand, if $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ is optimizable (see Definition 6.7.3) and $\dim U < \infty$, then there is $\delta > 0$ s.t. every $\lambda \in \sigma(A) \cap \mathbf{C}^+_{-\delta} =: K$ is an isolated eigenvalue of finite multiplicity, by [JZ99]. In particular, then $\widehat{\mathbb{D}} \in \mathrm{H}(\mathbf{C}_{-\delta} \setminus K; \mathcal{B}(U,Y))$

Notes

Most of (a), (b), (c2) and (d4) (and (e)) of Theorem 3.3.1 is well known (see, e.g., [RR] for the separable case). A comprehensive study on the separable case of (c1) is given in [Thomas]. Also Lemmas 3.3.2 and 3.3.8 are probably well known.

The monographs [Duren] and [Hoffman] are classical references for H^p spaces and their boundary functions. The monograph [RR] by Marvin Rosenblum and James Rovnyak is a classical reference on case on separable Hilbert spaces; it also contains further results and extensions to the Nevanlinna class.

Chapter 4

Corona Theorems and Inverses

If I have not seen so far it is because I stood in giant's footsteps.

In Theorem 4.1.1, we show that MTI, CTI (resp. MTIC, CTIC) and most of their subclasses are inverse closed in TI (resp. in TIC), and provide several equivalent conditions for the invertibility of such maps. We also give some extensions and related results. Then we show that these classes are adjoint closed (Lemma 4.1.3).

Thereafter, we study the Corona Theorem and its consequences, giving equivalent conditions for left-invertibility (use duality for right-invertibility) in $\mathcal{A}(U,Y)$, where $\dim U < \infty$ and $\mathcal{A}$ equals TIC, $\mathrm{MTIC}^{\mathrm{L}^1}$, CTIC or some of certain other classes. We also list some consequences of these results to coprime factorization, following M. Vidyasagar [Vid].

The Corona Theorem does not extend to infinite-dimensional U (see Lemma 4.1.10), but we give several partial results for the infinite-dimensional case. A casual reader probably wants to just read (main) Theorems 4.1.1 and 4.1.6 and then go on to the next section.

Recall that U, H and Y denote Hilbert spaces of arbitrary dimensions unless something else is indicated.

We start by showing that several useful subclasses of TIC (and those of TI) are inverse closed (this means the equivalence (ii)$\Leftrightarrow$(i) in (b) and (a) below):

Theorem 4.1.1 (Inverse-closed classes) *Let $p \in [1,\infty]$. $\mathcal{A}$ be one of the classes*

$$\mathrm{TI},\ \mathrm{CTI},\ \mathrm{CTI}^{\mathcal{BC}}, \tag{4.1}$$

$$\mathrm{MTI},\ \mathrm{MTI}^{\mathcal{BC}},\ \mathrm{MTI}_{\mathrm{d}},\ \mathrm{MTI}_{\mathrm{d}}^{\mathcal{BC}},\ \mathrm{MTI}^{\mathrm{L}^1},\ \mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}},\ \mathcal{B}+(\mathrm{L}^1\cap\mathrm{L}^p)*. \tag{4.2}$$

Then $\mathcal{A}$ is inverse-closed in TI*; in particular, $\widetilde{\mathcal{A}} := \mathcal{A}\cap\mathrm{TIC}$ is inverse-closed in* TIC*. In fact, we can say more:*

(a) For $\mathbb{E} \in \mathcal{A}(U,Y)$ the conditions (i)–(iv) are equivalent (and they are equivalent to (iv'), unless $\mathcal{A} = \mathrm{TI}$):

(i) $\mathbb{E} \in \mathcal{GA}$;
(ii) $\mathbb{E} \in \mathcal{G}\mathrm{TI}$;
(iii) $\mathbb{E} \in \mathcal{GB}(\mathrm{L}^2)$;

(iv) $\widehat{\mathbb{E}} \in \mathcal{G}\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$*;*

(iv') $\widehat{\mathbb{E}} \in \mathcal{G}\mathcal{C}_{\mathrm{b}}(i\mathbf{R};\mathcal{B}(U,Y))$*, i.e.,* $\widehat{\mathbb{E}}^{-1}$ *exists and is bounded on* $i\mathbf{R}$.

If $\dim U = \dim Y < \infty$*, then also the left-invertibility conditions in Lemma 4.1.9 are equivalent to (i) as well as to (v) (and to (v'), unless* $\mathcal{A} = \mathrm{TI}$*):*

(v) $\operatorname{ess\,inf}_{i\mathbf{R}} |\det(\widehat{\mathbb{E}})| > 0$.

(v') $\inf_{i\mathbf{R}} |\det(\widehat{\mathbb{E}})| > 0$*;*

(b) Let $\widetilde{\mathcal{A}} := \mathcal{A} \cap \mathrm{TIC}$*. For* $\mathbb{D} \in \widetilde{\mathcal{A}}(U,Y)$ *the conditions (i)–(iv) are equivalent, where*

(i) $\mathbb{D} \in \mathcal{G}\widetilde{\mathcal{A}}$*;*

(ii) $\mathbb{D} \in \mathcal{G}\mathrm{TIC}$*;*

(iii) $\pi_+\mathbb{D}\pi_+ \in \mathcal{G}\mathcal{B}(\pi_+\mathrm{L}^2)$*;*

(iv) $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{H}^\infty$*, i.e.,* $\widehat{\mathbb{D}}^{-1}$ *exists and is bounded on* $\mathbf{C}^+$.

See Lemma 2.2.3 for further equivalent conditions. If $\dim U = \dim Y < \infty$*, then also the left-invertibility conditions in Theorem 4.1.6(a) are equivalent to (i) as well as to (v):*

(v) $\inf_{\mathbf{C}^+} |\det(\widehat{\mathbb{D}})| > 0$.

(c) If $\mathbb{E} \in \mathcal{G}\mathrm{MTI}$*, then the discrete part of* $\mathbb{E}^{-1}$ *is the inverse of the discrete part of* $\mathbb{E}$*. Moreover, if* $\mathrm{supp}_{\mathrm{d}}(\mathbb{E}) \subset \mathbf{S} \subset \mathbf{R}$ *and* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$*, then* $\mathrm{supp}_{\mathrm{d}}(\mathbb{E}^{-1}) \subset \mathbf{S}$.

In particular, classes $\mathrm{MTI}_{\mathbf{S}}$ *and* $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$ *(resp.* $\mathrm{MTIC}_{\mathbf{S}}$ *and* $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$*) are inverse closed in* TI *(resp.* TIC*).*

(d) Let $\mathcal{D}(U,Y)$ *be either* $\ell^1 := \ell^1(\mathbf{Z};\mathcal{B}(U,Y))$ *or* $\ell^1_{\mathcal{BC}} := \{a \in \ell^1 \,|\, a_j \in \mathcal{BC}(U,Y)$ *for all* $j \neq 0\}$*, with norm* $\|(a_j)_{j\in\mathbf{Z}}\|_{\ell^1} := \sum_{j\in\mathbf{Z}} \|a_j\|_{\mathcal{B}(U,Y)}$ *and convolution as the group operation. Set* $\widehat{\mathcal{D}}(U,Y) := \{\widehat{a} := \sum_{j\in\mathbf{Z}} a_j z^j \,\big|\, (a_j) \in \mathcal{D}\}$ *and* $\widehat{\mathcal{D}^+}(U,Y) := \{\widehat{a} \in \widehat{\mathcal{D}} \,\big|\, a_j = 0 \text{ for } j < 0\}$ *(these are the* Z-transforms *of* $\mathcal{D}$ *and* $\mathcal{D}^+ := \mathcal{D} \cap \mathrm{tic}$*, cf. Theorem 5.1.3).*

Let $a \in \mathcal{D}(U,Y)$ *and* $\mathbb{E} := \sum_{j\in\mathbf{Z}} a_j \tau^j$*. Then the following are equivalent:*

(i) $a \in \mathcal{G}\mathcal{D}(U,Y)$*;*

(iii) $(a*) \in \mathcal{G}\mathcal{B}(\ell^2(\mathbf{Z};\mathrm{L}^2([0,1);U)))$*;*

(iv) $\widehat{a} \in \mathcal{G}\mathcal{C}(\partial\mathbf{D};\mathcal{B}(U,Y))$*;*

(v) $\mathbb{E} \in \mathcal{G}\mathrm{MTI}_{\mathrm{d},\mathbf{Z}}$*;*

(v') $\mathbb{E} \in \mathcal{G}\mathrm{MTI}$.

If, in addition, $a \in \mathcal{D}^+(U,Y)$*, then the following are equivalent:*

(i) $a \in \mathcal{G}\mathcal{D}^+(U,Y)$*;*

(iii) $\pi_{\mathbf{N}}(a*)\pi_{\mathbf{N}} \in \mathcal{G}\mathcal{B}(\ell^2(\mathbf{N};\mathrm{L}^2([0,1);U)))$*;*

(iv) $\widehat{a} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{D};\mathcal{B}(U,Y))$*;*

(iv') $\widehat{a} \in \mathcal{GC}(\overline{\mathbf{D}};\mathcal{B}(U,Y))$;
(v) $\mathbb{E} \in \mathcal{G}\mathrm{MTIC}_{\mathrm{d},\mathbf{Z}}$;
(v') $\mathbb{E} \in \mathcal{G}\mathrm{MTIC}$.

(e) **(Banach spaces)** *Assume, in addition, that $\mathcal{A} \neq \mathrm{TI}$, $\mathcal{A} \neq \mathrm{CTI}$ and $\mathcal{A} \neq \mathrm{CTI}^{\mathcal{BC}}$. If U and Y are arbitrary Banach spaces, then (a)–(d) still hold (with* TI *and its subspaces defined exactly as in the Hilbert space case).*

(f) **(Banach algebras)** *Assume, in addition, that $\mathcal{A} \neq \mathrm{TI}$, $\mathcal{A} \neq \mathrm{CTI}$ and $\mathcal{A} \neq \mathrm{CTI}^{\mathcal{BC}}$. If we replace everywhere $\mathcal{B}(U,Y)$ by an arbitrary Banach algebra A, then parts (a)–(d) hold to the following extent:*

(a) (i)⇔(iv');
(b) (i)⇔(iv);
(c) Completely as stated;
(d) (i)⇔(iv)⇔(v)⇔(v') for $\mathcal{D}$;
(i)⇔(iv)⇔(iv')⇔(v)⇔(v') for $\mathcal{D}^+$.

The above replacement means, e.g., that

$$\mathrm{MTI} := \{\mathbb{E} = \sum_j a_j \delta_{t_j} * + f * \,\big|\, \|\mathbb{E}\|_{\mathrm{MTI}} := \sum_j \|a_j\|_A + \|f\|_{\mathrm{L}^1(\mathbf{R};A)} < \infty\}, \tag{4.3}$$

$$\mathrm{MTIC} := \{\mathbb{E} = \sum_j a_j \delta_{t_j} * + f * \in \mathrm{MTI} \,\big|\, f \in \mathrm{L}^1(\mathbf{R}_+;A) \;\&\; t_j \geq 0 \; \textit{for all} \; j\}, \tag{4.4}$$

and their subclasses are defined analogously, $\widetilde{\mathcal{A}} := \mathcal{A} \cap \mathrm{MTIC}$ in (b), etc.

(g1) **(Exponential stability)** *Let $\omega \in \mathbf{R}$. By Remark 2.1.6 (see also Lemma D.1.12(d)), the (stability-shifted) class $\mathcal{A}_\omega := \mathrm{e}^{\omega \cdot}\mathcal{A}\mathrm{e}^{-\omega \cdot}$ (resp. $\widetilde{\mathcal{A}}_\omega := \mathrm{e}^{\omega \cdot}\widetilde{\mathcal{A}}\mathrm{e}^{-\omega \cdot}$) is inverse-closed in TI_ω (resp. TIC_ω), and claims analogous to (a)–(f), (i) and (j) apply (with $i\mathbf{R}$ replaced by $\omega + i\mathbf{R}$ etc.).*

(g2) We can replace $\widetilde{\mathcal{A}}$ in (b) by $\widetilde{\mathcal{A}}_{\exp} := \cup_{\omega<0}\widetilde{\mathcal{A}}_\omega$. Thus, $\mathbb{D} \in \mathcal{G}\mathrm{TIC} \Leftrightarrow \mathbb{D} \in \mathcal{G}\widetilde{\mathcal{A}}_{\exp}$ for $\mathbb{D} \in \widetilde{\mathcal{A}}_{\exp}$.

(h) Everywhere in (a)–(d), we have $\widehat{\widehat{\mathbb{E}}^*} = \widehat{\mathbb{E}}^*$, $(a*)^* = (\text{Я}a^*)*$, $\widehat{\text{Я}a^*} = \widehat{a}^*$, $(\pi_+\mathbb{E}\pi_+)^* = \pi_+\mathbb{E}^*\pi_+$ and $(\pi_{\mathbf{N}}(a*)\pi_{\mathbf{N}})^* = \pi_{\mathbf{N}}(\text{Я}a^*)\pi_{\mathbf{N}}$, and the adjoint in TI *is the same as the adjoint in $\mathcal{B}(\mathrm{L}^2)$).*

Consequently, if $\mathbb{E} = \mathbb{E}^*$, then $\mathbb{E}$ is invertible iff any of its forms is left-invertible (or right-invertible).

(i) **($\dim U < \infty$)** *Assume that $\dim U = \dim Y < \infty$ or that the operator under study is self-adjoint.*

Then, in (a)–(e), we may replace $\mathcal{G}$ by $\mathcal{G}_{\mathrm{left}}$ or by $\mathcal{G}_{\mathrm{right}}$ where $\mathcal{G}_{\mathrm{left}}\mathcal{X} := \{x \in \mathcal{X} \,\big|\, yx = I \text{ for some } y \in \mathcal{X}\}$, $\mathcal{G}_{\mathrm{right}}\mathcal{X} := \{x \in \mathcal{X} \,\big|\, xy = I \text{ for some } y \in \mathcal{X}\}$, except that in (b)(iii) and latter (d)(iii) only $\mathcal{G}_{\mathrm{right}}$ can be allowed.

In particular, Theorem 4.1.6 and Lemma 4.1.9 provide additional equivalent conditions.

(j) $(\boldsymbol{\mathcal{B}}+(\mathbf{H}^{p}_{[\mathrm{strong}]}\cap\mathbf{H}^{\infty}))$ *The classes* $\mathcal{B}+(\mathrm{H}^p\cap\mathrm{H}^\infty)(\mathbf{C}^+;\mathcal{B})$ *and* $\mathcal{B}+(\mathrm{H}^p_{\mathrm{strong}}\cap\mathrm{H}^\infty)(\mathbf{C}^+;\mathcal{B})$ *are inverse-closed in* H^∞.

Analogously, one can prove the inverse-closedness of several other similar classes. Note from (g2) that $\mathrm{TIC}_{\mathrm{exp}}$, $\mathrm{MTIC}_{\mathrm{exp}}$ etc. are inverse-closed in TIC.

For maps $\mathbb{E}$ form $\mathrm{MTI}^{\mathrm{L}^1}$ or CTI (resp. $\mathrm{MTIC}^{\mathrm{L}^1}$ or CTIC), a pointwise inverse of $\widehat{\mathbb{E}}$ on $i\mathbf{R}\cup\{\infty\}$ (resp. on $\overline{\mathbf{C}^+}\cup\{\infty\}$) is necessarily bounded (because it is continuous on a compact set); hence in that case pointwise invertibility is one more equivalent condition in (a) (resp. in (b)).

Parts (e) and (f) are not needed in this monograph, they are stated only for future reference.

Proof: We start by some preparations (1°–3°):

1° *Any of the conditions in (a)–(d) implies that* $\dim U=\dim Y$*:* For (a)(ii), this follows from Lemma 2.2.1(c4). By Lemma 2.1.5 and Theorem 3.1.3(a1), conditions (a)(ii)–(a)(iv) are equivalent. The other conditions in (a)–(c) obviously imply some of (a)(ii)–(a)(iv). For (d), the situation is analogous.

2° *In (a)–(d), we can w.l.o.g. assume that* $U=Y$ *(*$=\mathbf{C}^n$ *if* $\dim U<\infty$*):* The set $\mathcal{GB}(Y,U)$ is nonempty, by 1°. Each of the conditions in (a)–(d), is invariant under the (left) multiplication by an operator in $\mathcal{GB}(Y,U)$. For the same reason, we can take $U=\mathbf{C}^n$ if $\dim U<\infty$ (by Lemma A.3.4(Q1)).

3° We shall also use the fact that if $\dim U<\infty$, then the left invertibility of $\mathbb{E}$ or $\mathbb{D}$ is equivalent to its invertibility, by Lemma 2.2.1(b).

Case TI: For $\mathcal{A}=\mathrm{TI}$, claims (i) and (ii) coincide. In 1° we observed that (ii)–(iv) are equivalent.

"(i)⇔(v)": If (i) holds and $\dim U<\infty$, then $\operatorname{ess\,inf}|\det(\widehat{\mathbb{E}})|=\operatorname{ess\,sup}|1/\det(\widehat{\mathbb{E}}^{-1})|<\infty$; the converse follows from the determinant formula of an inverse matrix.

Other cases: If $\widehat{\mathbb{E}}\in\mathcal{C}_{\mathrm{b}}$ and $\widehat{\mathbb{E}}$ exists, then it is continuous, by Lemma A.3.4(A2), so the equivalence ("i.e.") stated in (iv') holds.

For $\mathbb{E}\in\mathrm{MTI}\cup\mathrm{CTI}$, we have $\widehat{\mathbb{E}}\in\mathcal{C}_{\mathrm{bu}}$, by Theorem 2.6.4(e1), hence (i)⇒(iv') and (v)⇔(v'). Therefore, for $\mathcal{A}\neq\mathrm{TI}$, it is enough to assume all the other (equivalent) conditions (ii)–(iv') and prove (i). That shall be done below.

Case CTI: For $\mathcal{A}=\mathrm{CTI}$ we have (iv')⇒(i), by Lemma A.3.4(A2).

Case $\mathrm{MTI}_{\mathbf{d}}$*:* For $\mathcal{A}=\mathrm{MTI}_{\mathrm{d}}$, (iv) implies (i) by [Gri, Theorem 4] (to be exact, the theorem says that (iv) implies that $\mathbb{E}^{-1}$ is a measure. One easily verifies that the discrete part of $\mathbb{E}^{-1}$ is also an inverse of $\mathbb{E}$, hence equal to $\mathbb{E}^{-1}$).

Case MTI: Let $\mathbb{E}\in\mathrm{MTI}\cap\mathcal{G}\mathrm{TI}$, so that $\widehat{\mathbb{E}}\in\mathcal{GC}_{\mathrm{bu}}$. Let $\widehat{\mathbb{E}}_{\mathrm{d}}$ be the discrete part of $\widehat{\mathbb{E}}$ and set $\widehat{f}:=\widehat{\mathbb{E}}-\widehat{\mathbb{E}}_{\mathrm{d}}\in\mathcal{L}\mathrm{L}^1$.

Because $\widehat{f}$ vanishes at infinity, by Lemma D.1.11(b), $\|\widehat{\mathbb{E}}^{-1}\widehat{\mathbb{E}}_{\mathrm{d}}-I\|=\|\widehat{\mathbb{E}}^{-1}\widehat{f}\|<1/2$ outside $i[-T,T]$ for some $T>0$, in particular, $\widehat{\mathbb{E}}_{\mathrm{d}}$ is boundedly invertible outside $i[-T,T]$. We fix such a $T>0$ and let $M:=\sup_{|t|>T}\|\widehat{\mathbb{E}}(it)\|$

By the almost-periodicity of $\widehat{\mathbb{E}}_{\mathrm{d}}$ [Lemma C.1.2(h2)], there is $R>T$ s.t. $\|\widehat{\mathbb{E}}_{\mathrm{d}}(it)-\widehat{\mathbb{E}}_{\mathrm{d}}(i(t-R))\|<1/2M$ for all $t\in\mathbf{R}$, hence for $t\in[-T,T]$ the operator $\widehat{\mathbb{E}}_{\mathrm{d}}^{-1}(i(t-R))\widehat{\mathbb{E}}_{\mathrm{d}}(it)\in\mathcal{B}$ has an inverse of norm≤ 2, so the operator $\widehat{\mathbb{E}}_{\mathrm{d}}(it)$ is boundedly invertible; $\mathbb{E}_{\mathrm{d}}^{-1}\in\mathrm{MTI}_{\mathrm{d}}$ by case $\mathrm{MTI}_{\mathrm{d}}$ above.

Now $\widehat{\mathbb{E}}_{\mathrm{d}}^{-1}\widehat{\mathbb{E}} = I + \widehat{\mathbb{E}}_{\mathrm{d}}^{-1}\widehat{f}$ is boundedly invertible, hence $\mathbb{E}^{-1}$ is a bounded Borel measure, by [Gri, Theorem 5]. Because L^1 is an ideal of bounded Borel measures (see, e.g., [Gri, p. 159]), we have $\widehat{g} = \widehat{\mathbb{E}}_{\mathrm{d}}^{-1}\widehat{f}$ for some $g \in \mathrm{L}^1$.

By [Gri, Theorem 5], $\mu := (I + g*)^{-1}$ is a bounded Borel measure, so $I = \mu + g * \mu$ implies that $\mu - I = -g * \mu \in \mathrm{L}^1$, hence $\mu \in \mathrm{MTI}^{\mathrm{L}^1}$.

Case WTI: By case MTI, the inverse of $E + g* \in \mathrm{MTI}^{\mathrm{L}^1} \cap \mathcal{G}\mathrm{TI}$ is $E^{-1} + f*$ for some $f \in \mathrm{L}^1$. (Alternatively, use Lemma 4.1.2(a1)&(b).)

Case $\mathcal{B} + (\mathrm{L}^1 \cap \mathrm{L}^p)*$: Because $\mathrm{L}^1 * (\mathrm{L}^1 \cap \mathrm{L}^p) \subset \mathrm{L}^1 \cap \mathrm{L}^p$, by Lemma D.1.7, $\mathcal{A}$ is a subclass of $\mathrm{MTI}^{\mathrm{L}^1}$. By Lemma 4.1.2(a1)&(b), class $\mathcal{A}$ is inverse-closed in $\mathrm{MTI}^{\mathrm{L}^1}$, hence in TI.

$\mathcal{BC}$ cases: Use again Lemma 4.1.2(a1)&(b) (with $\mathcal{X} \mapsto \{\mathbb{E} - \Pi_{\{0\}}\mathbb{E} \,\big|\, \mathbb{E} \in \mathrm{MTI}^{\mathcal{BC}}\}$, $\mathcal{A} \mapsto \{\mathbb{E} - \Pi_{\{0\}}\mathbb{E} \,\big|\, \mathbb{E} \in \mathrm{MTI}\}$ in case $\mathcal{A} = \mathrm{MTI}^{\mathcal{BC}}$, and analogously for $\mathrm{MTI}_{\mathrm{d}}^{\mathcal{BC}}$ and $\mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}$).

(b) *Case TI:* Conditions (i) and (ii) again coincide, "(ii)⇔(iii)" is contained in Lemma 2.2.3 and "(ii)⇔(iv)" follows from Theorem 6.2.1. Equivalence "(iv)⇔(v)" follows again from the determinant formula of an inverse matrix.

Other cases: As the rest is proved above, it is enough to assume (ii)–(iv) and prove (i), because (i)⇒(ii) follows from $\widetilde{\mathcal{A}} \subset \mathrm{TIC}$. But if $\mathbb{D} \in \mathcal{A} \cap \mathcal{G}\mathrm{TIC}$, then $\mathbb{D}^{-1} \in \mathcal{A}$, by (a), hence $\mathbb{D}^{-1} \in \mathcal{A} \cap \mathrm{TIC}$.

(c) The first claim was proved in case MTI of the proof of (a). For the second claim, let $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$. Define the projection $\Pi_{\mathbf{S}} \in \mathcal{B}(\mathrm{MTI}, \mathrm{MTI}_{\mathrm{d}})$ by

$$\Pi_{\mathbf{S}}((\sum_k T_k \delta_{t_k} + f)*) := (\sum_{t_k \in \mathbf{S}} T_k \delta_{t_k}) * . \tag{4.5}$$

One easily verifies that $\mathbb{E} \in \Pi_{\mathbf{S}}\mathrm{MTI}$ implies $\mathbb{E}(\Pi_{\mathbf{S}}\mathbb{F}) = \Pi_{\mathbf{S}}(\mathbb{E}\mathbb{F})$ for any $\mathbb{F} \in \mathrm{MTI}$. Thus, if $\mathbb{E} \in \Pi_{\mathbf{S}}\mathrm{MTI} \cap \mathcal{G}\mathrm{MTI}$ and $\mathbb{F} = \mathbb{E}^{-1}$, then $\mathbb{E}\mathbb{F} = \Pi_{\mathbf{S}}(\mathbb{E}\mathbb{F}) = \mathbb{E}\Pi_{\mathbf{S}}\mathbb{F}$, hence $\Pi_{\mathbf{S}}\mathbb{F} = \mathbb{E}^{-1} = \mathbb{F}$, so the last claim of the theorem holds for $\mathbb{E} \in \mathcal{G}\mathrm{MTI}_{\mathrm{d}}$; in the general case follows by applying this for the (invertible, by Lemma 5.2.3(a)) discrete part of $\mathbb{E}$.

(d) (Note that $\mathcal{D}(U)$ is a Banach algebra with respect to convolution; see Section 13.1 for details. In the definition of $\mathbb{E}$ we refer to $\tau^t \in \mathrm{TI}(U,Y)$ (not $\mathrm{ti}(U,Y)$), hence $\mathbb{E} \in \mathrm{TI}(U,Y)$.) As above, we assume that $Y = U$.

We prove the claims on $\mathcal{D}$; those for $\mathcal{D}_+$ can be proved analogously (recall that $\mathbb{E} \in \mathrm{MTIC} \Leftrightarrow \mathrm{supp}_{\mathrm{d}}(\mathbb{E}) \subset \mathbf{R}_+$).

Clearly $\widehat{\mathcal{D}}(U,Y) \subset \mathcal{C}(\partial\mathbf{D}; \mathcal{B}(U,Y))$. The equivalence (i)⇔(v), follows from Theorem 13.4.5(m), (v)⇔(v') from (c) (with $S = \mathbf{Z}$), and the other equivalences by applying (a) to $\mathbb{E}$ and then using Theorem 13.4.5(m) to convert the conditions on $\mathbb{E}$ to those on a.

(e) (Example 3.3.4 shows an $\mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(\mathbf{C}, \ell^\infty))$ (even "$\widehat{\mathrm{CTIC}}$") function that does not correspond to a TI operator; it does not even map $\mathcal{L}\chi_{[0,1]}$ into $\mathcal{L}\mathrm{L}^2$. Moreover, we do not know whether $\mathcal{C}_{\mathrm{b}}$ is inverse-closed in $\mathrm{L}^\infty_{\mathrm{strong}}$ when U is not a Hilbert space nor separable (cf. Lemma F.1.3(f1)&(f2)). Therefore we have concentrated on MTI only (MTI maps L^2 into L^2, by Lemma D.1.12(c2)).)

The proof goes as in the Hilbert space case above, but it is most easily obtained as follows:

(e) on (a): The implication (i)$\Leftrightarrow$(iv') is given in (f). The implication (i)$\Rightarrow$(ii) holds because $\mathcal{A} \subset \mathrm{TI}$, and equivalences (ii)$\Leftrightarrow$(iii) and (iv')$\Leftrightarrow$(v')$\Leftrightarrow$(v) can be proved as above. Implication (ii)$\Rightarrow$(iv') follows from Lemma 3.2.5, and implication (iv)$\Leftrightarrow$(iv') follows from Theorem F.1.9(s4).

(e) on (b): The implication (i)$\Leftrightarrow$(iv) is given in (f). The implication (ii)$\Rightarrow$(iv) follows from Theorem 2.3 of [W91a]. The proof of Lemma 2.2.2(a2) again establishes (ii)$\Leftrightarrow$(iii), and equivalence (iv)$\Leftrightarrow$(v) follows again from the determinant formula of an inverse matrix.

(f) *(f) on (a):* We have (i)$\Rightarrow$(iv'), by Lemma D.1.12(a1)&(c)&(a3). The proof of the major part of the proof of (a) (starting from "case $\mathrm{MTI_d}$") shows that (iv') implies (i).

(f) on (b): We have (i)$\Rightarrow$(iv), by Lemma D.1.12(a1')&(c')&(a3'). The converse can be deduced from part (a) as in part "Other cases" of the proof of (b).

(f) on (c) and (d): The proof of part (c) applies without changes. The proof (d) also applies, because the last paragraph of Theorem 13.4.5(m) is valid in the Banach algebra case too, with the same proof (*mutatis mutandis*).

(g1) This is obvious, because the stability shift is an isometric isomorphism and commutes with compositions (and shifts the Laplace and Fourier transforms by the formula $\widehat{e^{\omega\cdot}\mathbb{E}e^{-\omega\cdot}}(\cdot) = \widehat{\mathbb{E}}(\omega+\cdot)$).

(g2) Combine (b) with Lemma 2.2.7.

(h) For (a)(iv), this follows from Theorem 3.1.3(a). The TI and $\mathcal{A}$ adjoints mean L^2 adjoints, by definition. Part (c) follows from (a). See Lemmas 3.3.8 and 13.1.8 for $\widehat{\mathbb{E}}^*$ and $\widehat{a}^*$ and p. 782 for $(a*)^*$.

(i) 1° If $\mathbb{E} = \mathbb{E}^*$, then it follows from (h) that $\mathbb{V}\mathbb{E} = I$ implies that $\mathbb{E}\mathbb{V}^* = (\mathbb{E}\mathbb{V})^* = I$; consequently, $\mathbb{V}^* = \mathbb{V}\mathbb{E}\mathbb{V}^* = \mathbb{V}$ is the inverse of $\mathbb{E}$ (by (h) $\mathbb{E} = \mathbb{E}^*$ iff $\widehat{\mathbb{E}} = \widehat{\mathbb{E}}^*$).

2° Let $\dim U = \dim Y < \infty$. We have $\mathcal{G}_{\mathrm{left}}\mathrm{TI}(U) = \mathcal{G}\mathrm{TI}(U)$, by Lemma 2.2.1(b), hence $\mathcal{G}_{\mathrm{left}}\mathrm{TIC}(U) = \mathcal{G}\mathrm{TIC}(U)$. Obviously, (i)–(iv') of (a) imply (ii) (for (iii) this follows from Lemma 2.2.1(a)); the converses follows from the original (a). The noncausal part of (d) is obtained analogously (alternatively, apply Theorem 13.4.5(m)).

The claims on $\mathcal{G}_{\mathrm{right}}$ follow analogously (alternatively, by taking adjoints).

Part (b) is analogous to (a) except for (iii), which follows from Lemma 2.2.3. Parts (c) and (d) can deduced from (a) and (b).

(j) For $p = \infty$ this is trivial, so assume that $p < \infty$. Set $\mathcal{A} := \{\mathbb{D} \in \mathrm{H}^\infty \cap \mathrm{ULR} \,\big|\, D = 0\}$. Then $\mathcal{B} + \mathcal{A} = \mathrm{H}^\infty \cap \mathrm{ULR}$ is inverse-closed in H^∞, by (c); $\mathrm{H}^p_{\mathrm{strong}} \subset \mathcal{A}$, by Proposition 6.3.3(a), and $\mathrm{H}^\infty \cdot (\mathrm{H}^\infty \cap \mathrm{H}^p_{\mathrm{strong}}) \subset (\mathrm{H}^\infty \cap \mathrm{H}^p_{\mathrm{strong}})$, by Lemma F.3.5(c). Consequently, $\mathcal{B} + \mathrm{H}^p_{\mathrm{strong}} \cap \mathrm{H}^\infty$ is inverse-closed in $\mathcal{B} + \mathcal{A}$, hence in H^∞, by (a1)&(a2)&(b) of Lemma 4.1.2. For H^p, the proof is analogous and hence omitted. □

We list here a few basic results on inverse-closedness, some of which were already used:

Lemma 4.1.2 (Inverse-closedness) *Let* $\omega \in \mathbf{R}$.

(a1) Assume that $\mathcal{B}+\mathcal{X} \underset{a}{\subset} \mathcal{B}+\mathcal{A} \underset{a}{\subset} \mathrm{TI}_\omega$ *(recall that this requires closedness under composition), and that* $\mathcal{B}\cap\mathcal{A} = \{0\} = \mathcal{B}\cap\mathcal{X}$.

If $ax \in \mathcal{X}$ *for all* $a \in \mathcal{A}$, $x \in \mathcal{X}$, *then* $\mathcal{B}+\mathcal{X}$ *is inverse-closed in* $\mathcal{B}+\mathcal{A}$.

(a2) Part (a1) also holds with TIC_ω, H^∞_ω, L^∞_ω *or* $\mathrm{L}^\infty_{\mathrm{strong},\omega}$ *in place of* TI_ω *and/or with* xa *in place of* ax.

(b) If $\mathcal{P}$ *is inverse-closed in* $\mathcal{Q}$ *and* $\mathcal{Q}$ *is inverse-closed in* $\mathcal{R}$, *then* $\mathcal{P}$ *is inverse-closed in* $\mathcal{R}$.

(c) If $\mathcal{Z}$ *is inverse-closed in* TI_ω, *then so is* $\mathcal{Z}^*$. *If* $\mathcal{Z}$ *is inverse-closed in* TIC_ω, *then so is* $\mathcal{Z}^{\mathrm{d}}$.

Proof: (Recall that $\mathcal{P}$ is inverse-closed in $\mathcal{Q}$ iff $\mathcal{P}$ is a subgroup of $\mathcal{Q}$ and $\mathcal{P}\cap\mathcal{G}\mathcal{Q} = \mathcal{G}\mathcal{P}$, i.e., $x \in \mathcal{P} \& x^{-1} \in \mathcal{Q} \Rightarrow x^{-1} \in \mathcal{P}$ for all $x \in \mathcal{P}$.)

(a1) Let $X+x \in \mathcal{B}+\mathcal{X}$ have the inverse $A+a \in \mathcal{B}+\mathcal{A}$. Then $I = (A+a)(X+x) = AX + aX + Ax + ax$, hence $\mathcal{B} \ni I - AX = aX + Ax + ax \in \mathcal{A}$, hence $AX = I$. Analogously, $XA = I$, hence $a = -(A+a)xX^{-1} \in \mathcal{X}$.

(N.B. usually this formula also provides a norm estimate for the inverse, e.g., $\|(X+x)^{-1} - X^{-1}\|_{\mathrm{H}^2} \le \|(X+x)^{-1}\|_{\mathrm{H}^\infty}\|x\|_{\mathrm{H}^2}\|X^{-1}\|_{\mathcal{B}}$ for any $X \in \mathcal{B}$, $x \in \mathrm{H}^\infty \cap \mathrm{H}^2$.)

(a2) Same proof applies to TI, H^∞_ω etc. too.

(b) This is obvious (if $p \in \mathcal{P}\cap\mathcal{G}\mathcal{R}$, then $p^{-1} \in \mathcal{Q}$, hence then $p^{-1} \in \mathcal{P}$).

(c) If $\mathcal{Z}\cap\mathcal{G}\mathrm{TI}_\omega = \mathcal{G}\mathcal{Z}$ and $y \in \mathcal{Z}^* \cap \mathcal{G}\mathrm{TI}_\omega$, then $y^* \in \mathcal{Z}\cap\mathcal{G}\mathrm{TI} = \mathcal{G}\mathcal{Z}$, hence $y^{-1} = ((y^*)^{-1})^* \in \mathcal{Z}^*$. The case for $\mathcal{Z}^{\mathrm{d}}$ is analogous. $\square$

Now we will show the rather obvious fact that most (noncausal) classes mentioned in Theorem 4.1.1 are also adjoint-closed (see Definition 2.1.4 for the definition of the adjoint):

Lemma 4.1.3 (Adjoint-closed classes) *Let* $\mathcal{A}$ *be one of the classes* TI, CTI, $\mathrm{CTI}^{\mathcal{BC}}$, MTI, $\mathrm{MTI}^{\mathcal{BC}}$, $\mathrm{MTI}_{\mathrm{d}}$, $\mathrm{MTI}^{\mathcal{BC}}_{\mathrm{d}}$, $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}$ *and* $\mathcal{B}+(\mathrm{L}^1\cap\mathrm{L}^p)*$, *and let* U *and* Y *be Hilbert spaces of arbitrary dimensions. Then we have the following:*

(a) The class $\mathcal{A}$ *is adjoint-closed in* TI*: if* $\mathbb{E} \in \mathcal{A}(U,Y)$, *then* $\mathbb{E}^*, \mathbb{E}^{\mathrm{d}} \in \mathcal{A}(Y,U)$.

(b) If $\omega \in \mathbf{R}$ *and* $\mathbb{E} \in \mathcal{A}_\omega(U,Y) := e^{\omega\cdot}\mathcal{A}(U,Y)e^{-\omega\cdot}$, *then* $\mathbb{E}^* \in \mathcal{A}_{-\omega}(Y,U)$ *and* $\mathbb{E}^{\mathrm{d}} \in \mathcal{A}_\omega(Y,U)$.

(c) If $\mathbb{E} \in \mathcal{G}\mathrm{MTI}$, *then the discrete part of* $\mathbb{E}^*$ *is the adjoint of the discrete part of* $\mathbb{E}$. *Furthermore,* $\mathrm{supp}_{\mathrm{d}}(\mathbb{E}^*) = -\mathrm{supp}_{\mathrm{d}}(\mathbb{E})$ *and* $\mathrm{supp}_{\mathrm{d}}(\mathbb{E}^{\mathrm{d}}) = \mathrm{supp}_{\mathrm{d}}(\mathbb{E})$, *and classes such as* $\mathrm{MTI}_{\mathbf{S}}$ *and* $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$ *are adjoint closed in* TI *when* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$.

(d) Set $\widetilde{\mathcal{A}} := \mathcal{A}\cap\mathrm{TIC}$, $\widetilde{\mathcal{A}}_\omega := e^{\omega\cdot}\widetilde{\mathcal{A}}e^{-\omega\cdot}$, *where* $\omega \in \mathbf{R}$. *Then* $\mathbb{D} \in \widetilde{\mathcal{A}}_\omega(U,Y) \Leftrightarrow \mathbb{D}^{\mathrm{d}} \in \widetilde{\mathcal{A}}_\omega(Y,U)$.

Set $\widetilde{\mathcal{A}} := \mathcal{A}_{\mathbf{S}}\cap\mathrm{TIC}$, $\widetilde{\mathcal{A}}_{\omega,S} := e^{\omega\cdot}\widetilde{\mathcal{A}}_S e^{-\omega\cdot}$, *where* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$. *Then* $\mathbb{D} \in \widetilde{\mathcal{A}}_{\omega,S}(U,Y) \Leftrightarrow \mathbb{D}^{\mathrm{d}} \in \widetilde{\mathcal{A}}_{\omega,S}(Y,U)$.

One could, of course, extend this result for Banach spaces with ease; although the Banach adjoint of a $\mathrm{TI}(U,Y)$ operator is an element of $\mathrm{TI}(Y^*,U^*)$.

Proof: (a) 1° *We have* $\mathbb{E}^* \in \mathcal{A}(Y,U)$*:* Because $\tau(t)^* = \tau(-t)$, the claim holds for $\mathcal{A} = \mathrm{TI}$. For CTI this is given in Lemma 2.6.2; the case $\mathcal{A} = \mathrm{CTI}^{\mathcal{B}C}$ follows. For MTI and its subclasses the claim follows from the fact that $(\mu *)^* = (\text{Я}\mu^*)*$, by Lemma D.1.12(d).

2° *We have* $\mathbb{E}^{\mathrm{d}} \in \mathcal{A}(Y,U)$*:* By Lemma D.1.12(d), $(\mu *)^{\mathrm{d}} = (\mu^*)*$, hence this holds for MTI and its subclasses; for TI this is obvious.

(b) Because $(\mathrm{e}^{\omega\cdot})^* = \mathrm{e}^{\omega\cdot}$, it follows from (a) that

$$(\mathrm{e}^{\omega\cdot}\mathcal{A}(U,Y)\mathrm{e}^{-\omega\cdot})^* = \mathrm{e}^{-\omega\cdot}\mathcal{A}(Y,U)\mathrm{e}^{\omega\cdot} =: \mathcal{A}_{-\omega}(Y,U). \tag{4.6}$$

(c) This follows from Lemma D.1.12(d).

(d) This follows from (b) and Lemma D.1.12(d). □

The Carleson Corona Theorem states that for $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathbf{C}^{n\times 1})$ to have a left inverse $\widehat{\mathbb{V}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathbf{C}^{1\times n})$, the (clearly necessary) condition $\widehat{\mathbb{D}}(s)^*\widehat{\mathbb{D}}(s) \geq \varepsilon I$ (for all $s \in \mathbf{C}^+$ and some $\varepsilon > 0$) is also sufficient.

For general $\mathbb{D} \in \mathrm{TIC}(U,Y)$, the left-inverse $(\widehat{\mathbb{D}}^*\widehat{\mathbb{D}})^{-1}\widehat{\mathbb{D}}^* \in \mathcal{C}_{\mathrm{b}}(\mathbf{C}^+;\mathcal{B}(Y,U))$ of $\widehat{\mathbb{D}}$ is not holomorphic, the left-inverse $(\mathbb{D}^*\mathbb{D})^{-1}\mathbb{D}^* \in \mathrm{TI}(Y,U)$ of $\mathbb{D}$ is not causal, and, surprisingly, there need not be any (H^∞, i.e., TIC) inverse in general, as Serge Treil has shown (see Lemma 4.1.10). However, assuming $\dim U < \infty$, such an inverse is guaranteed, as shown in (Corona) Theorem 4.1.6.

We will use the Corona Theorem to find certain left inverses and coprime factorizations. We start by showing that the theorem (part (ii) below) is equivalent to two other problems (see, e.g., Chapters 10 & 11 of [Rud73] for basic results on Banach algebras and maximal ideal spaces).

Lemma 4.1.4 *Assume that $\mathcal{A}$ is a commutative (complex) Banach algebra, $1_{\mathcal{A}}$ be its identity, $\mathfrak{M}$ be its maximal ideal space, and $\mathfrak{M}_0 \subset \mathfrak{M}$. Then the following are equivalent:*

(i) $\mathfrak{M}_0$ is dense in $\mathfrak{M}$.

(ii) **(Corona Theorem)** *Let $f_1,\dots,f_n \in \mathcal{A}$. There are vectors $g_1,\dots,g_n \in \mathcal{A}$ s.t.*

$$f_1g_1 + \cdots + f_ng_n = 1_{\mathcal{A}} \tag{4.7}$$

iff there is $\varepsilon > 0$ s.t. $|\widehat{f_1}(M)| + \cdots + |\widehat{f_n}(M)| \geq \varepsilon$ for all $M \in \mathfrak{M}_0$.

(iii) Let $\mathbb{D} \in \mathcal{A}^{n\times m}$. Then there is $\mathbb{V} \in \mathcal{A}^{m\times n}$ s.t. $\mathbb{V}\mathbb{D} = I$ iff

$$\widehat{\mathbb{D}}(M)^*\widehat{\mathbb{D}}(M) \geq \varepsilon I \ \text{ for all } \ M \in \mathfrak{M}_0. \tag{4.8}$$

In (ii) and (iii), we might write "if" instead of "iff", because the converse holds for any $\mathfrak{M}_0 \subset \mathfrak{M}$ (take $\varepsilon := 1/\sup\|\widehat{\mathbb{V}}\|^2_{\mathcal{B}(\mathbf{C}^n,\mathbf{C}^m)}$).

The Corona Theorem means in general a proof that the *corona* $\mathfrak{M}\setminus\overline{\mathrm{M}_0}$ is empty. By the lemma, this is the case iff (iii) holds, hence we can call results of form (iii) corona theorems.

Proof: By Lemmas 8.1.28 and 8.1.34 of [Vid, pp. 339–340], claim (i) implies (ii) and (iii). Claim (ii) is the case $m = 1$ of (iii), and (ii)⇒(i) is established in, e.g., pp. 201–203 of [Duren]. $\square$

Now we list certain Banach algebras for which the set $\mathfrak{M}_0 := \mathbf{C}^+$ is dense in their maximal ideal spaces (with the standard identification $\mathbf{C}^+ \ni s \mapsto \{f \in \mathcal{A} \,|\, \hat{f}(s) = 0\} \in \mathfrak{M}_0$):

Lemma 4.1.5 (Maximal ideals) *The extended closed half-plane $\overline{\mathbf{C}^+} \cup \{\infty\}$ with the one-point-compactification topology is the maximal ideal space of* $\mathrm{MTIC}^{\mathrm{L}^1}(\mathbf{C})$ *and* $\mathrm{CTIC}(\mathbf{C})$*, and $\mathbf{C}^+$ is dense also in the maximal ideal spaces of* $\mathrm{TIC}(\mathbf{C})$, $\mathrm{MTIC}(\mathbf{C})$ *and* $\mathrm{MTIC}_{\mathrm{d}}(\mathbf{C})$.

The maximal ideal space of $\mathrm{CTI}(\mathbf{C})$, $\ell^1(\mathbf{N})$ *and* $\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}(\mathbf{C})$ *(for $T > 0$) is the closed unit disc $\overline{\mathbf{D}}$; the latter space through identification $\sum_{k=0}^{\infty} \alpha_k \delta_{Tk} \mapsto \sum_{k=0}^{\infty} \alpha_k z^k$. Similarly, the maximal ideal space of $\ell^1(\mathbf{Z})$ and* $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}(\mathbf{C})$ *is the unit circle $\{z \in \mathbf{C} \,|\, |z| = 1\}$.*

The extended imaginary axis $i\mathbf{R} \cup \{\infty\}$ with the one-point-compactification topology is the maximal ideal space of $\mathrm{MTI}^{\mathrm{L}^1}(\mathbf{C})$*, and $i\mathbf{R} \cup \{\infty\}$ dense also in the maximal ideal spaces of* $\mathrm{MTI}(\mathbf{C})$ *and* $\mathrm{MTI}_{\mathrm{d}}(\mathbf{C})$.

Theorem 4.1.6(a) will show that $\overline{\mathbf{C}^+} \cup \{\infty\}$ (resp. $i\mathbf{R} \cup \{\infty\}$) is dense in the maximal ideal space of $\mathrm{MTIC}_{\mathbf{S}}$ (resp. $\mathrm{MTI}_{\mathbf{S}}$), whenever $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$. The same holds for $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$ and $\mathrm{MTIC}_{\mathrm{d},\mathbf{S}}$ unless $\mathbf{S} = T\mathbf{Z}$ for some $T \in \mathbf{R}$ (case $T = 0$ is trivial, case $T \neq 0$ is given in the above lemma).

Proof: The maximal ideal spaces of $\mathrm{MTI}^{\mathrm{L}^1}(\mathbf{C})$ and $\mathrm{MTIC}^{\mathrm{L}^1}(\mathbf{C})$ are given on pages 107 and 112 of [GRS], respectively, and those of $\mathrm{CTIC}(\mathbf{C})$ and $\mathrm{CTI}(\mathbf{C})$ in [Rud73, Example 11.13(c)&(a)] (to be exact, [Rud73] shows that the closed unit disc is the maximal ideal space of the disc algebra (via the Z transform); $\mathrm{CTIC}(\mathbf{C})$ is isomorphic to the disc algebra through the Cayley transform).

For TIC (i.e., H^∞) this is the Corona theorem [Duren, Chapter 12].

Case MTIC is shown on p. 145–150 (cf. [Vid, p 342]).

Cases $\ell^1(\mathbf{N})$ and $\ell^1(\mathbf{Z})$ are given on [GRS, p. 118] and in [Rud73, Example 11.13(b)], respectively. The cases $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$ and $\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$ follow from the isomorphism $\sum_k \alpha_k \delta_{Tk} \mapsto \{\alpha_k\}$.

For $\mathrm{MTIC}_{\mathrm{d}}$ this is follows from Theorem 5.2 of [BKRS] as follows: Clearly $\mathfrak{M}_0 := \mathbf{C}^+$ is a subset of $\mathfrak{M} := \mathfrak{M}(\mathrm{MTIC}_{\mathrm{d}}(\mathbf{C}))$. Let now $\mathbb{D} \in \mathrm{MTIC}_{\mathrm{d}}(\mathbf{C}^{n \times m})$ and $\widehat{\mathbb{D}}^* \widehat{\mathbb{D}} \geq \varepsilon I$ (this condition is necessary, because a left $\mathrm{MTIC}_{\mathrm{d}}$ inverse is a left TIC inverse). Then there is $\mathbb{V} \in \mathrm{TIC}(\mathbf{C}^n, \mathbf{C}^m)$ s.t. $\mathbb{V}\mathbb{D} = I$, hence $\|\pi_- \mathbb{D} u\|_2 \geq \varepsilon_1 \|\pi_- u\|_2$, for all $u \in \mathrm{L}^2$, where $\varepsilon_1 := \|\pi_- \mathbb{V} \pi_-\|$, i.e., $\pi_- \mathbb{D}^* \pi_- \mathbb{D} \pi_- \geq \varepsilon_1^2 \pi_-$. By the $*$-isomorphism introduced in Theorem 8 of [Karlovich93], this is equivalent to the condition (iii) of Theorem 5.2 of [BKRS] (for $A := \varepsilon_1^{-1} \mathbb{D}^*$), hence Theorem 5.2(viii) provides $\mathbb{V} \in \mathrm{MTI}_{\mathrm{d}}(\mathbf{C}^{n \times m})$ s.t. $A\mathbb{V}^* = I$, i.e., $\varepsilon_1^{-1} \mathbb{V}\mathbb{D} = I$, so we obtain the density of $\mathbf{C}^+$ from Lemma 4.1.4.

Cases $\mathrm{MTI}(\mathbf{C})$ and $\mathrm{MTI}_{\mathrm{d}}(\mathbf{C})$ follow from Lemma 4.1.4, because $\widehat{\mathbb{D}}^* \widehat{\mathbb{D}} \geq \varepsilon I$ on $i\mathbf{R}$ implies that $\mathbb{D}^* \mathbb{D} \geq \varepsilon I$, hence the left inverse $\mathbb{V} := (\mathbb{D}^* \mathbb{D})^{-1} \mathbb{D}^*$ belongs to $\mathcal{A}$, by Theorem 4.1.1. $\square$

We are now ready to state several equivalent conditions for left-invertibility:

Theorem 4.1.6 (Corona Theorem) *Let $\mathcal{A}$ be one of the classes* TIC, CTIC, MTIC, $\mathrm{MTIC_d}$, $\mathrm{MTIC}^{\mathrm{L}^1}$, $\mathrm{MTIC_{\mathbf{S}}}$ *and* $\mathrm{MTIC_{d,\mathbf{S}}}$, *where* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$. *Let* $\dim U < \infty$. *Then (a) and (b) hold:*

(a) For $\mathbb{D} \in \mathcal{A}(U,Y)$ the following are equivalent:

(i) $\mathbb{V}\mathbb{D} = I$ for some $\mathbb{V} \in \mathcal{A}(Y,U)$;
(ii) $\mathbb{V}\mathbb{D} = I$ for some $\mathbb{V} \in \mathrm{TIC}(Y,U)$;
(iii) $\widehat{\mathbb{D}}(s)^*\widehat{\mathbb{D}}(s) \geq \varepsilon I$ for all $s \in \mathbf{C}^+$ and some $\varepsilon > 0$;
(iv) $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|u\|_{\mathrm{L}^2_\omega}$ for all $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$, $\omega > 0$ and some $\varepsilon > 0$;
(v) $\mathbb{D}^*\pi_-\mathbb{D} \geq \varepsilon\pi_-$ on L^2 for some $\varepsilon > 0$;
(vi) $\mathbb{D}^{t*}\mathbb{D}^t \geq \varepsilon\pi_{[0,t)}$ for all $t > 0$ and some $\varepsilon > 0$.

(See Proposition 4.1.7 for additional equivalent conditions.)

(b) Let $\mathbb{N} \in \mathcal{A}(U,Y)$, $\mathbb{M} \in \mathcal{A}(U)$. Then $\mathbb{N}$ and $\mathbb{M}$ are r.c. over $\mathcal{A}$ iff $\widehat{\mathbb{N}}(s)^*\widehat{\mathbb{N}}(s) + \widehat{\mathbb{M}}(s)^*\widehat{\mathbb{M}}(s) \geq \varepsilon I$ for all $s \in \mathbf{C}^+$ and some $\varepsilon > 0$.

Let $\mathcal{A}$ be TIC, CTIC, $\mathrm{MTIC}^{\mathrm{L}^1}$, $\mathrm{MTIC}_{T\mathbf{Z}}$ *or* $\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$, *where* $T \in \mathbf{R}$. *Then also (c) and (d) hold:*

(c) Let $\mathbb{D} \in \mathcal{A}(U,Y)$. Then the conditions (i)–(vi) are equivalent to

(vii) $\mathbb{D}$ can be complemented to $\begin{bmatrix}\mathbb{D} & \mathbb{E}\end{bmatrix} \in \mathcal{G}\mathcal{A}(U \times Y_0, Y)$, where Y_0 is a closed subspace of Y.

If $Y = U \times Z$, where also Z is a Hilbert space, we may require $Y_0 = Z$ in (vii).

(d) Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ have a r.c.f. $\mathbb{N}\mathbb{M}^{-1}$ (or a l.c.f. $\mathbb{M}^{-1}\mathbb{N}$) with $\mathbb{N},\mathbb{M} \in \mathcal{A}$. Then $\mathbb{D}$ has a d.c.f. over $\mathcal{A}$.

(See Definitions 6.4.4 and 6.4.1 for *.c.f. and *.c.) By taking (causal) adjoints, one gets the dual claims, e.g., the equation $\mathbb{D}\mathbb{V} = I$ has a solution $\mathbb{V} \in \mathcal{A}$ iff $\widehat{\mathbb{D}}\widehat{\mathbb{D}}^* \geq \varepsilon I$ on $\mathbf{C}^+$. If $\mathbb{D} = \mathbb{D}^*$ or $\dim U = \dim Y < \infty$, then $\mathbb{D}$ is left-invertible iff $\mathbb{D}$ is invertible. by Theorem 4.1.1(h).

Part (a) of the theorem fails (at least for $\mathcal{A} = \mathrm{TIC}$) when $\dim U = \infty$ and $\dim Y \geq \dim U$, by Lemma 4.1.10.

Condition (iii) can obviously be rephrased as $\|\widehat{\mathbb{D}}u_0\| \geq \varepsilon^{1/2}\|u_0\|$ on $\mathbf{C}^+$ for all $u_0 \in U$. Similarly, $\mathbb{M}^*\mathbb{M} + \mathbb{N}^*\mathbb{N} \geq \varepsilon I$ on $\mathbf{C}^+$ for some $\varepsilon > 0$ iff $\|\mathbb{M}u_0\| + \|\mathbb{N}u_0\| \geq \varepsilon'\|u_0\|$ on $\mathbf{C}^+$ for all $u_0 \in U$ and some $\varepsilon' > 0$.

If $\mathbb{N}$ and $\mathbb{M}$ in (d) are exponentially r.c., i.e., $\mathbb{N}_\alpha := e^{\alpha\cdot}\mathbb{N}e^{-\alpha\cdot}$ and $\mathbb{M}_\alpha$ are r.c. over $\mathcal{A}$ for some $\alpha < 0$ (equivalently, $\left[\begin{smallmatrix}\mathbb{N}\\ \mathbb{M}\end{smallmatrix}\right]^* \left[\begin{smallmatrix}\mathbb{N}\\ \mathbb{M}\end{smallmatrix}\right] \geq \varepsilon I$ on $\mathbf{C}^+_\alpha$), then we can take all maps in the d.c.f. to be exponentially stable (by choosing a d.c.f. for $\mathbb{N}_\alpha\mathbb{M}_\alpha^{-1}$ and shifting it back to TIC_α).

Proof: (Note that $\dim Y \geq \dim U$ is necessary for (ii), hence for all equivalent conditions.)

I The case $\dim Y < \infty$ *of parts (a) and (c):* W.l.o.g. (the statements are invariant under isomorphisms (i.e., $\mathcal{GB}(\mathbf{C}^m, U)$ and $\mathcal{GB}(\mathbf{C}^n, Y)$ mappings) we set $U = \mathbf{C}^m$ and $Y = \mathbf{C}^n$.

(a) 1° "(iii)⇒(i)": This follows from Lemmas 4.1.4 and 4.1.5, except for the case $\mathcal{A} = \mathrm{MTIC}_{T\mathbf{Z}}$, which is deduced from the case $\mathcal{A} = \mathrm{MTIC}$ as follows:

Let $\mathbb{D} = (\mu + f)* \in \mathrm{MTIC}_{\mathbf{S}}$. The equivalence $r + s \in \mathbf{S} \Leftrightarrow r \in \mathbf{S}$, valid for $s \in \mathbf{S}$, implies that for $\nu \in \mathrm{MTI}$ we have

$$\Pi_{\mathbf{S}}\nu = 0 \Longrightarrow \Pi_{\mathbf{S}}(\nu' * \mu) = 0. \tag{4.9}$$

If $(\nu + g)* \in \mathrm{MTIC}$ is a left-inverse of $\mu + f$, and we set $\nu' := \Pi_{\mathbf{S}}\nu$, $\nu'' := \nu - \nu'$, then

$$I\delta_0 = (\nu' * \mu + \nu'' * \mu) + (\nu' * f + \nu'' * f + g * \mu), \tag{4.10}$$

hence $\nu' * \mu = I\delta_0 \in \Pi_{\mathbf{S}}\mathrm{MTIC}$. Thus, $(\nu' + h) * (\mu + f) = I\delta_0$, if we set $h := -(\nu' * f) * (\nu + g) \in \mathrm{L}^1$.

2° The other implications: "(i)⇒(ii)" is trivial, because $\mathcal{A} \subset \mathrm{TIC}$, "(ii)⇒(v)" follows from equation $\|\pi_-\mathbb{V}\pi_-\|\|\pi_-\mathbb{D}u\|_2 \ge \|\pi_- u\|_2$ (i.e., we can take $\varepsilon := \|\pi_-\mathbb{V}\pi_-\|^{-1/2}$), "(v)⇒(vi)⇒(iv)⇒(iii)" is given in Proposition 4.1.7(C)&(A).

(c) The implication (vii)⇒(i) is trivial, so we study the converse.

The case $\mathcal{A} = \mathrm{TIC}$ is contained in the Tolokonnikov's lemma [Nikolsky, App. 3.10, p. 293].

By [Vid, Theorem 8.1.68], a complex Banach algebra with a contractible maximal ideal space is Hermite ($\overline{\mathbf{C}^+} \cup \{\infty\}$ is homeomorphic to $\overline{\mathbf{D}}$, hence contractible); see [Vid, p. 348] or [Lin] for details. Thus, $\mathrm{MTIC}^{\mathrm{L}^1}(\mathbf{C})$, $\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}(\mathbf{C})$ and $\mathrm{CTIC}(\mathbf{C})$ are Hermite rings, by Lemma 4.1.5.

By [Vid, Theorem 8.1.59], the ring $\mathcal{A}(\mathbf{C})$ is Hermite iff every left-invertible $\mathbb{D} \in \mathcal{A}(\mathbf{C}^m, \mathbf{C}^n)$ can be complemented, so cases $\mathcal{A} = \mathrm{MTIC}^{\mathrm{L}^1}$, $\mathcal{A} = \mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$ and $\mathcal{A} = \mathrm{CTIC}$ follow from this (hence also $\mathrm{TIC}(\mathbf{C})$ (and $\mathrm{H}^\infty(\mathbf{C})$) is Hermite).

Finally, let $\mathbb{D} = \mathbb{D}_1 + \mathbb{D}_2 \in \mathrm{MTIC}_{T\mathbf{Z}}$, where $\mathbb{D}_1 = \Pi_{\mathbf{R}}\mathbb{D}$ is the discrete part of $\mathbb{D}$, be left-invertible over $\mathrm{MTIC}_{T\mathbf{Z}}$. Then so is $\mathbb{D}_1 \in \mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$, hence $\begin{bmatrix}\mathbb{D}_1 & \mathbb{E}_1\end{bmatrix}^{-1} = \begin{bmatrix}\mathbb{F}_1 \\ \mathbb{G}_1\end{bmatrix} \in \mathcal{G}\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}(\mathbf{C}^n)$ for some $\mathbb{E}_1, \mathbb{F}_1, \mathbb{G}_1 \in \mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$, by (c). Therefore,

$$\mathbb{H} := \begin{bmatrix}\mathbb{F}_1 \\ \mathbb{G}_1\end{bmatrix}\mathbb{D} = \begin{bmatrix}I + \mathbb{F}_1\mathbb{D}_2 \\ \mathbb{G}_1\mathbb{D}_2\end{bmatrix} \in \mathrm{MTIC}^{\mathrm{L}^1}, \tag{4.11}$$

is left-invertible over $\mathrm{MTIC}_{T\mathbf{Z}}$, hence over TIC, hence over $\mathrm{MTIC}^{\mathrm{L}^1}$, by (a), so $\mathbb{H}$ can be complemented $\begin{bmatrix}\mathbb{H} & \mathbb{K}\end{bmatrix} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1}$. Thus,

$$\begin{bmatrix}\mathbb{D} & \begin{bmatrix}\mathbb{F}_1 \\ \mathbb{G}_1\end{bmatrix}^{-1}\mathbb{K}\end{bmatrix} = \begin{bmatrix}\mathbb{F}_1 \\ \mathbb{G}_1\end{bmatrix}^{-1}\begin{bmatrix}\mathbb{H} & \mathbb{K}\end{bmatrix} \in \mathcal{G}\mathrm{MTIC}_{T\mathbf{Z}}. \tag{4.12}$$

If $Y = U \times Z$, then $\dim U \times Y_0 = \dim U \times Z$, hence $\dim Y_0 = \dim Z$ (because $\dim U < \infty$), equivalently, there is a map (isomorphism) $T \in \mathcal{GB}(Z, Y_0)$. Just replace $\mathbb{E}$ by $\mathbb{E}T$ (i.e., $\begin{bmatrix}\mathbb{D} & \mathbb{E}\end{bmatrix}$ by $\begin{bmatrix}I & 0 \\ 0 & T\end{bmatrix} \in \mathcal{GB}$) to the right to replace Y_0 by Z.

II The general case of parts (a) and (c) for $\mathcal{A} = \mathrm{TIC}$*:*

The implication "(iii)⇒(vii)" (with norm estimates for $\mathbb{E}$ and $\begin{bmatrix}\mathbb{D} & \mathbb{E}\end{bmatrix}^{-1}$) is Tolokonnikov's Lemma [Nikolsky, Lemma A.3.10, p. 293] (which uses the solution of "(iii)⇒(ii)", i.e., the Fuhrmann–Vasyunin Theorem [Nikolsky, Theorem A.3.11, p. 293]). The above proofs of the other implications apply to the general case too (note that (vii) implies (i)).

In [Nikolsky], Y was assumed to be separable, but that is no loss of generality:

Because $\widehat{\mathbb{D}} : \mathbf{C}^+ \times U \to Y$ is continuous and $\mathbf{C}^+ \times U$ is separable, so is $\widehat{\mathbb{D}}[\mathbf{C}^+][U]$. Let Y_1 be the (separable) closed span of $\widehat{\mathbb{D}}[\mathbf{C}^+][U]$, so that we can write $\mathbb{D} =: \begin{bmatrix}\widetilde{\mathbb{D}} \\ 0\end{bmatrix} \in \mathrm{TIC}(U, Y_1 \times Y_1^\perp)$.

Take $Y_0 \subset Y_1$ and $\widetilde{\mathbb{E}}$ s.t. $\begin{bmatrix}\widetilde{\mathbb{D}} & \widetilde{\mathbb{E}}\end{bmatrix} \in \mathcal{G}\mathrm{TIC}(U \times Y_0, Y_1)$. Then $\begin{bmatrix}\widetilde{\mathbb{D}} & \widetilde{\mathbb{E}} & 0 \\ 0 & 0 & I\end{bmatrix} \in \mathrm{TIC}(U \times Y_0 \times Y_1^\perp, Y)$ is an invertible (by Lemma A.1.1(b1)) complementation of $\mathbb{D}$.

III The general case of parts (a) and (c) for $\mathcal{A} \neq \mathrm{TIC}$:

(a) "(iii)⇒(i)": Let $y_1, \ldots, y_m$ span DU. Let $y_{m+1}, y_{m+2}, \ldots$ be chosen as the sequence "$y_1, y_2, \ldots$" from Lemma 2.6.5 (applied to $\mathbb{D} - D$; use also Theorem 2.6.4(j)). Then $P_n'\mathbb{D} \to \mathbb{D}$ in $\mathcal{A}(U,Y)$ (hence in $\mathrm{TIC}(U,Y)$ too, hence $P_n'\widehat{\mathbb{D}} \to \widehat{\mathbb{D}}$ in $\mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U,Y))$), as $n \to \infty$; here P_n' is the orthogonal projection of Y onto $Y_n := \mathrm{span}\{y_1, \ldots, y_m\}$. The rest depends on the statement:

Take n large enough to have $(P_n'\widehat{\mathbb{D}})^*(P_n'\widehat{\mathbb{D}}) \geq \varepsilon/2I$ on $\mathbf{C}^+$. Let $\mathbb{V}_n \in \mathcal{A}(Y_n, U)$ s.t. $\mathbb{V}_n(P_n'\widehat{\mathbb{D}}) = I$. Choose $\mathbb{V} := \mathbb{V}_n P_n' \in \mathcal{A}(Y,U)$ to get (i). The other implications are obtained as in 2° of I(a) above.

(c) "(iii)⇒(vii)": Take P_n' as above and write $\mathbb{D}$ as $\begin{bmatrix}\mathbb{D}_1 \\ \mathbb{D}_2\end{bmatrix} \in \mathcal{A}(U, Y_n \times Y_n^\perp)$. Choose then $\mathbb{E}_1 \in \mathcal{A}(Y_n, U)$ s.t. $\begin{bmatrix}\mathbb{D}_1 & \mathbb{E}_1\end{bmatrix} \in \mathcal{G}\mathcal{A}(Y_n, U \times Y_0)$, where Y_0 is a subspace of Y_n. Now $\mathbb{E} = \begin{bmatrix}\mathbb{E}_1 & 0 \\ 0 & I\end{bmatrix} \in \mathcal{A}(Y_0 \times Y_n^\perp)$ complements $\mathbb{D}$ to $\begin{bmatrix}\mathbb{D}_1 & \mathbb{E}_1 & 0 \\ \mathbb{D}_2 & 0 & I\end{bmatrix}$, which is invertible, by Lemma A.1.1(b1).

(b) This follows from (a) by setting $\mathbb{D} := \begin{bmatrix}\mathbb{N} \\ \mathbb{M}\end{bmatrix}$.

(d) Use (c) to complement $\begin{bmatrix}\mathbb{M} \\ \mathbb{N}\end{bmatrix}$ to an invertible matrix

$$\begin{bmatrix}\mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X}\end{bmatrix} =: \begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}^{-1} \in \mathcal{G}\mathcal{A}(\mathbf{C}^m \times \mathbf{C}^n). \tag{4.13}$$

This gives a d.c.f. of $\mathbb{D}$ over $\mathcal{A}$. □

The above Corona Theorem says that, when $\dim U < \infty$, the condition "$\widehat{\mathbb{D}}^*\widehat{\mathbb{D}} \geq \varepsilon I_U$ on $\mathbf{C}^+$" is equivalent to the left-invertibility of any $\mathbb{D} \in \mathrm{TIC}(U, *)$. In the case of an infinite-dimensional U, condition "$\widehat{\mathbb{D}}^*\widehat{\mathbb{D}} \geq \varepsilon I$ on $\mathbf{C}^+$" is equivalent only to the following kind of *pseudo–left-invertibility* (recall Lemma 4.1.10):

Proposition 4.1.7 (∞-dimensional partial Corona theorem) *Let* $\varepsilon > 0$.

(A) Assume that $\mathbb{D} \in \mathrm{TIC}(U,Y)$. *Then conditions (i)–(v) below are equivalent.*

Furthermore, any of (i')–(v) is invariant under the replacement $\omega > 0 \mapsto \omega \geq 0$. *Finally, if (i) holds, then (a1)–(d) hold for any* $\alpha \geq \omega \geq 0$ *and* $T, \beta \in \mathbf{R}$.

(B) Assume that $\mathbb{D} \in \mathrm{TIC}_{\omega'}(U,Y)$ *for all* $\omega' > 0$. *Then (i)–(v) below are equivalent.*

Furthermore, (iv') and (iv") are invariant under the replacement $\omega > 0 \mapsto \omega \geq 0$. *Finally, if (i) holds, then (a1)–(c2) hold for any* $\alpha \geq \omega > 0$ *and* $T, \beta \in \mathbf{R}$.

(C) Assume that $\mathbb{D} \in \mathrm{TIC}_{\omega'}(U,Y)$ *for all* $\omega' > 0$. *Then (vi) of Theorem 4.1.6 implies (i)–(v). If* $\mathbb{D} \in \mathrm{TIC}$, *then conditions (v) and (vi) of Theorem 4.1.6 are equivalent (and imply (i)–(v) below).*

(i) There is $\widehat{\mathbb{V}} : \mathbf{C}^+ \to \mathcal{B}(Y,U)$ *s.t.* $\widehat{\mathbb{V}}\widehat{\mathbb{D}} \equiv I$ *and* $\|\widehat{\mathbb{V}}\| \leq \varepsilon^{-1}$.

(i') For each $\omega > 0$ *there is* $\mathbb{V} \in \mathrm{TI}_\omega(Y,U)$ *s.t.* $\|\mathbb{V}\|_{\mathrm{TI}_\omega} \leq \varepsilon^{-1}$ *and* $\mathbb{V}\mathbb{D} = I_{\mathrm{TI}_\omega}$.

(iii) $\widehat{\mathbb{D}}(s)^*\widehat{\mathbb{D}}(s) \geq \varepsilon^2 I$ *for all* $s \in \mathbf{C}^+$.

(iv) $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|u\|_{\mathrm{L}^2_\omega}$ *(*$u \in \mathrm{L}^2_\omega(\mathbf{R};U)$*) for all* $\omega > 0$.

(iv') $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|u\|_{\mathrm{L}^2_\omega}$ *(*$u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U)$*) for all* $\omega > 0$.

(iv") $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|u\|_{\mathrm{L}^2_\omega}$ *(*$u \in \mathcal{C}^\infty_c(\mathbf{R}_+;U)$*) for all* $\omega > 0$.

(v) $(\mathbb{D}_{-\omega})^*\mathbb{D}_{-\omega} \geq \varepsilon^2 I$ *for all* $\omega > 0$.

(a) Condition (i) holds with some $\widehat{\mathbb{V}} \in \mathcal{C}_b(\mathbf{C}^+;\mathcal{B}(Y,U))$ *s.t.* $\sup_{s\in\mathbf{C}^+}\|\widehat{\mathbb{V}}(s)\| \leq \varepsilon^{-1}$.

(b1) $\mathbb{D}u \in \mathrm{L}^2_\omega \Leftrightarrow u \in \mathrm{L}^2_\omega$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.

(b2) $\mathbb{D}u \in \pi_{[T,\infty)}\mathrm{L}^2_\omega \Leftrightarrow u \in \pi_{[T,\infty)}\mathrm{L}^2_\omega$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.

(b3) There is $M < \infty$ *s.t.* $\varepsilon\|u\|_{\mathrm{L}^2_\omega} \leq \|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \leq M\|u\|_{\mathrm{L}^2_\omega}$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.

(c1) We have $\mathbb{D}\mathbb{F} \in \mathrm{TIC}_\omega \Leftrightarrow \mathbb{F} \in \mathrm{TIC}_\omega$. *when* $\mathbb{F} \in \mathrm{TIC}_\infty(H,U)$.

(c2) We have $\mathbb{D}\mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2_\omega) \Leftrightarrow \mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2_\omega)$ *when* X *is a normed space and* $\mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2_\beta(\mathbf{R}_+;U))$.

(c3) Let $f \in \mathrm{H}^\infty(\mathbf{C}^+;U)$ *and* $\phi \in \mathrm{H}^\infty(\mathbf{C}^+;\mathbf{C}) \setminus \{0\}$. *Then* $\widehat{\mathbb{D}}\phi^{-1}f \in \mathrm{H}^\infty \Leftrightarrow \phi^{-1}f \in \mathrm{H}^\infty$.

(d) $\mathbb{D} \in \mathcal{G}\mathrm{TIC} \Leftrightarrow \mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \mathbb{D}\mathbb{D}^* \gg 0 \Leftrightarrow \widehat{\mathbb{D}}(s) \in \mathcal{G}\mathcal{B}(U,Y)$ *for some* $s \in \mathbf{C}^+$ *(*$\Leftrightarrow DD^* \gg 0$ *provided that* $\mathbb{D} \in \mathrm{UR}$*).*

See also the comments following Lemma 6.5.2. Recall that $\mathrm{L}^2_\infty := \cup_{\omega\in\mathbf{R}}\mathrm{L}^2_\omega$, and note that $\mathbb{D}_{-\omega} := \mathrm{e}^{-\omega\cdot}\mathbb{D}\mathrm{e}^{\omega\cdot} \in \mathrm{TIC}$ for $\omega \geq 0$.

Proof: (A) The equivalence is given in (B).

Assume that some, hence all of (i)–(v) holds. By (B), (iv') and (iv") hold also after the replacement; trivially, so do (i) and (iii) too. Since the proofs of implications (iv")⇒(iv)⇒(v)⇒(i') also hold for $\omega = 0$, all of (i)–(v) hold after the replacement.

(C) This follows from Lemma 2.2.4(a)&(b).

(B) "(i)⇔(iii)⇒(a)": This follows from Lemma A.3.1(c1)(1) when we set $\widehat{\mathbb{V}} := (\widehat{\mathbb{D}}^*\widehat{\mathbb{D}})^{-1}\widehat{\mathbb{D}}^* \in \mathcal{C}_{\mathrm{b}}(\mathbf{C}^+;\mathcal{B}(Y,U))$.

"(i')⇒(v)": This follows from $\|\mathbb{V}\|\,\|\mathbb{D}u\| \geq \|\mathbb{V}\mathbb{D}u\|$ with $\varepsilon = 1/\|\mathbb{V}\|$.

"(v)⇒(i')": Set $\mathbb{V}' := ((\mathbb{D}_{-\omega})^*\mathbb{D}_{-\omega})^{-1}\mathbb{D}_{-\omega}^*$. Then $\|\mathbb{V}'\|_{\mathrm{TI}} \leq \varepsilon^{-1}$, by (c1)(1) of Lemma A.3.1, and $\mathbb{V}'\mathbb{D}_{-\omega} = I$. Set $\mathbb{V} := \mathrm{e}^{-\omega\cdot}\mathbb{V}'\mathrm{e}^{\omega\cdot}$ to obtain (i').

"(iii)⇔(v)": Let $\omega > 0$. Because $\widehat{\mathbb{D}}_{-\omega}(s) = (s+\omega)$ $(s \in \mathbf{C}^+_{-\omega})$, condition (iii) holds iff $(\widehat{\mathbb{D}}_{-\omega})^*\widehat{\mathbb{D}}_{-\omega} - \varepsilon^2 I \geq 0$ on $i\mathbf{R}$ for all $\omega > 0$. Because $\widehat{\mathbb{D}}_{-\omega}$ is continuous on $i\mathbf{R}$, the latter is equivalent to (v), by Theorem 3.1.3(d).

"(iv')⇒(iv")⇔(iv)": Assume (iv"). By time-invariance and (2.2), the inequality holds for any $u \in \mathcal{C}_{\mathrm{c}}^\infty$. By density (recall that $\mathbb{D} \in \mathrm{TIC}_\omega$), (iv) holds. The other two implications are trivial.

"(iv)⇒(iv')": Assume (iv). If $u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U) \setminus \mathrm{L}^2_\omega$, then, for each $n \in \mathbf{N}$, $\infty > \|u\|_{\mathrm{L}^2_{\alpha_n}} > n$ for some $\alpha_n > \omega$, by the monotone convergence theorem. Consequently, $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \geq \|\mathbb{D}u\|_{\mathrm{L}^2_{\alpha_n}} \geq n\varepsilon$. Because n was arbitrary, we have $\mathbb{D}u \notin \mathrm{L}^2_\omega$. Thus, we have proved (b1) for $u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U)$. Obviously, this and (iv) imply (iv').

"(iv)⇔(v)": Now $(\mathbb{D}_{-\omega})^*\mathbb{D}_{-\omega} \geq \varepsilon^2 I$ iff $\|\mathbb{D}_{-\omega}u\|_2 \geq \varepsilon\|u\|_2$ for all $u \in \mathrm{L}^2$, which holds iff $\|\mathbb{D}v\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|v\|_{\mathrm{L}^2_\omega}$ for all $v \in \mathrm{L}^2_\omega = \mathrm{e}^{\omega\cdot}\mathrm{L}^2$.

$\omega \geq 0$*:* By (2.5), we can allow $\omega = 0$ in (iv') and in (iv").

(b1) By (iv') this holds for π_+u in place of u. But $\pi_-u \in \mathrm{L}^2_\omega$, hence $\mathbb{D}\pi_-u \in \mathrm{L}^2_\omega$ too.

(b2) This follows from (b1) and causality, except that we have to show that $u \in \pi_{[T,\infty)}\mathrm{L}^2_\omega$ assuming that $u \in \mathrm{L}^2_\omega$ and $\mathbb{D}u \in \pi_{[T,\infty)}\mathrm{L}^2_\omega$.

For $T = 0$ we have $\|u\|_{\mathrm{L}^2_r} \leq \varepsilon^{-1}\|\mathbb{D}u\|_{\mathrm{L}^2_r} \to 0$, as $r \to +\infty$, hence $\pi_-u = 0$. Use time-invariance for $T \neq 0$.

(b3) Set $M := \|\mathbb{D}\|_{\mathrm{TIC}}$. Assume that at least one of the norms is finite. Then, by (b1), $u \in \mathrm{L}^2_\omega$, hence (b3) follows from (iv).

(c1) If $\mathbb{D}\mathbb{F} \in \mathrm{TIC}_\omega$ (i.e., $\|\mathbb{D}\mathbb{F}u\|_{\mathrm{L}^2_\omega} \leq M\|u\|_{\mathrm{L}^2_\omega}$ for all $u \in \mathcal{C}_{\mathrm{c}}^\infty(\mathbf{R}_+;U)$), then $\|\mathbb{F}\|_{\mathrm{TIC}_\omega} \leq \varepsilon^{-1}\|\mathbb{D}\mathbb{F}\|_{\mathrm{TIC}_\omega}$, by (c2) and (2.13) (with $X := \pi_+\mathcal{C}_{\mathrm{c}}^\infty$ under the L^2_ω norm). Conversely, $\|\mathbb{D}\mathbb{F}\|_{\mathrm{TIC}_\omega} \leq \|\mathbb{D}\|_{\mathrm{TIC}}\|\mathbb{F}\|_{\mathrm{TIC}_\omega}$.

(c2) "⇐": This is trivial. "⇒": If $\beta > \omega$, then

$$\|\mathbb{C}x\|_{\mathrm{L}^2_\omega} \leq \varepsilon^{-1}\|\mathbb{D}\mathbb{C}x\|_{\mathrm{L}^2_\omega} \leq \varepsilon^{-1}\|\mathbb{D}\mathbb{C}\|_{\mathcal{B}(X,\mathrm{L}^2_\omega)}\|x\|_X \quad (x \in X), \tag{4.14}$$

by (b3). Therefore, $\|\mathbb{C}\| \leq \varepsilon^{-1}\|\mathbb{D}\mathbb{C}\|$.

(c3) Trivially, $\phi^{-1}f \in \mathrm{H}^\infty \Rightarrow \widehat{\mathbb{D}}\phi^{-1}f \in \mathrm{H}^\infty$. Conversely, If $\widehat{\mathbb{D}}\phi^{-1}f \in \mathrm{H}^\infty$, then $\phi^{-1}f = \widehat{\mathbb{V}}\widehat{\mathbb{D}}\phi^{-1}f$ is bounded, by (i), hence in H^∞, because $\phi^{-1}f$ is holomorphic, by Lemma D.1.2(h) (the zeros of ϕ are isolated, by Lemma D.1.2(e)). (In fact, the same equivalence holds whenever $\phi \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U))$ is s.t. it is invertible outside a set having no limit points in $\mathbf{C}^+$.)

(d) This follows from Proposition 2.2.5. □

We also need a "left invertibility over TIC" concept that is invariant under (inverse) discretization (see Theorem 13.4.5(g)) but still has at least the properties of Lemma 4.1.8(b1)–(d). Therefore, we define quasi–left invertibility as follows:

Lemma 4.1.8 (Quasi–left invertibility) *Assume that* $\mathbb{D} \in \mathrm{TIC}(U,Y)$ *is s.t.*

$$\mathbb{D}u \notin \mathrm{L}^2 \quad \textit{for all} \quad u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U) \setminus \mathrm{L}^2. \tag{4.15}$$

Then $\mathbb{D}$ *is called* quasi-left-invertible (over TIC)*, and there is* $\varepsilon > 0$ *s.t. (a)–(e) hold for any* $\omega \geq 0$ *and any normed space* X*.*

- *(a)* $\mathbb{D}^*\mathbb{D} \geq \varepsilon I$.
- *(b1)* $\mathbb{D}u \in \mathrm{L}^2 \Leftrightarrow u \in \mathrm{L}^2$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.
- *(b3) We have* $\varepsilon\|u\|_{\mathrm{L}^2} \leq \|\mathbb{D}u\|_{\mathrm{L}^2} \leq \|\mathbb{D}\|_{\mathrm{TIC}}\|u\|_{\mathrm{L}^2}$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.
- *(c1) We have* $\mathbb{D}\mathbb{F} \in \mathrm{TIC} \Leftrightarrow \mathbb{F} \in \mathrm{TIC}$ *(and* $\|\mathbb{F}\|_{\mathrm{TIC}} \leq \varepsilon^{-1}\|\mathbb{D}\mathbb{F}\|_{\mathrm{TIC}}$*) when* $\mathbb{F} \in \mathrm{TIC}_\infty(H,U)$.
- *(c2) We have* $\mathbb{D}\mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2) \Leftrightarrow \mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2)$ *(and* $\|\mathbb{C}\| \leq \varepsilon^{-1}\|\mathbb{D}\mathbb{C}\|$*) when* X *is a normed space and* $\mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2_\omega(\mathbf{R}_+;U))$.
- *(d) If also* $\mathbb{F} \in \mathrm{TIC}(*,U)$ *is quasi-left-invertible, then so is* $\mathbb{D}\mathbb{F}$.
- *(e) For each* $u_0 \in U \setminus \{0\}$*, we have* $\widehat{\mathbb{D}}u_0 \neq 0$ *on* $\mathbf{C}^+$ *and* $\|\widehat{\mathbb{D}}(ir)u_0\|_Y \geq \varepsilon\|u_0\|_U$ *for a.e.* $r \in \mathbf{R}$.
- *(f) If* $\mathbb{E},\mathbb{F} \in \mathrm{TIC}$ *and* $\mathbb{D} = \mathbb{E}\mathbb{F}$*, then* $\mathbb{F}$ *is quasi-left-invertible.*
- *(g) Pseudo–left invertibility implies quasi–left invertibility, but the converse does not hold even for* $U = \mathbf{C} = Y$.
- *(h) Quasi–left invertibility (over* TIC*) implies left invertibility over* TI *(though not over* TIC*), but the converse does not hold even for* $U = \mathbf{C} = Y$.

From (e) we observe that a quasi-left-invertible $\widehat{\mathbb{D}}$ must have no zeros on $\overline{\mathbf{C}^+}$, but unlike for pseudo-left-invertible transfer functions, $|\widehat{\mathbb{D}}|$ may be arbitrarily small (on $\mathbf{C}^+$, not on $i\mathbf{R}$) and even zero at $+\infty$ (take $\widehat{\mathbb{D}}(s) = e^{-s}$). We do not know whether (e) is equivalent to quasi–left invertibility. A sufficient condition is obviously that for each $\omega \geq 0$, there is $\varepsilon_\omega > 0$ s.t. $\widehat{\mathbb{D}}(s)^*\widehat{\mathbb{D}}(s) \geq \varepsilon_\omega$ when $0 < \operatorname{Re} s < \omega$.

Proof: We first show the existence of a number $\varepsilon > 0$ s.t. (a) is satisfied. Then we show that also (b1)–(g) are satisfied with the same ε.

(a) We assume that $\mathbb{D}^*\mathbb{D} \not\geq \varepsilon$ for any $\varepsilon > 0$, and construct an $u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U) \setminus \mathrm{L}^2$ s.t. $\mathbb{D}u \in \mathrm{L}^2$.

By assumption, $\mathbb{D}^*\mathbb{D} - 2^{-n} \not\geq 0$ ($n \in \mathbf{N}$). By Theorem 3.1.3(e1), for any $n \in \mathbf{N}$, there are $u_n \in U$ and $E_n \subset i\mathbf{R}$ s.t. $\|u_n\|_U = 1$, $m(E_n) > 0$ and $\langle([\widehat{\mathbb{D}}]^*[\widehat{\mathbb{D}}] - 2^{-n}I)u_n, u_n\rangle < 0$, i.e., $\|\widehat{\mathbb{D}}(s)u_n\| < 2^{-n}$, for $s \in E_n$.

Choose distinct points $ir_k \in E_k$ ($k \in \mathbf{N}$) as in Lemma D.1.24. Let $ir_\infty \in i\mathbf{R} \cup \{\infty\}$ be a limit point of $\{ir_k\}$. We assume that $r_\infty \in \mathbf{R}$ (case $r_\infty = \infty$ is analogous but easier (require, e.g., that $r_0 > 1$, $|r_{k+1}| > 3|r_k|$, and work as below), hence omitted).

W.l.o.g., we assume that $r_\infty = 0$ (replace $\widehat{\mathbb{D}}$ by $\widehat{\mathbb{D}}(\cdot - ir_\infty) \in \mathrm{H}^\infty$) and $|r_k| > 3|r_{k+1}|$ for all $k \in \mathbf{N}$ (choose a subsequence if necessary).

For each $k \in \mathbf{N}$, we set

$$\varepsilon_k := \min\{|r_k|/2, 2^{-k}\} \tag{4.16}$$

and find $f_k := f_{t_k,r_k} \in \mathrm{L}^2$ for $\varepsilon = \varepsilon_k$ and $E = E_k$ as in Lemma D.1.24 (with $p = 2$). Set $v_n := \sum_{k=1}^n f_k u_k \in \mathrm{L}^2$ ($n \in \mathbf{N}$).

Given $\omega > 0$, there is $N \in \mathbf{N}+1$ s.t. $2^{-N} < \omega$, and hence $\|f_k\|_{\mathrm{L}^2_\omega} < \varepsilon_k \le 2^{-k}$ for all $k > N$; in particular, $v_n \to u$ in L^2_ω for some $u \in \mathrm{L}^2_\omega$, as $n \to \infty$, by Lemma A.3.4(L1) (the limit (equivalence class) u is independent of ω, by Lemma D.1.4(b3)).

Analogously, we see that $\widehat{u}(s)$ converges absolutely on $\{s \in \mathbf{C} \,\big|\, |s| \ge \varepsilon\}$, for any $\varepsilon > 0$; in particular, $\widehat{u} \in \mathrm{H}(\mathbf{C}^+; U)$ has a unique continuous extension $\widehat{u} \in \mathcal{C}(\overline{\mathbf{C}^+} \setminus \{0\}; U)$. If we had $u \in \mathrm{L}^2$, then we would have $\widehat{u} \in \mathrm{L}^2(i\mathbf{R}; U)$, by Theorem 3.3.1(b)&(a1)(1.).

However, intervals $I_n := i(r_n - \varepsilon_n, r_n + \varepsilon_n) \subset i(r_n/2, 3r_n/2)$ ($n \in \mathbf{N}$) are disjoint, and hence $\|\widehat{f}_k\|^2_{\mathrm{L}^2(I_n)} \le 2\varepsilon_n \varepsilon_k^2$, so that

$$\|\widehat{u}\|_{\mathrm{L}^2(i\mathbf{R};U)} \ge \sum_{n=1}^\infty \|\widehat{u}\|_{\mathrm{L}^2(I_n;U)} \ge \sum_{n=1}^\infty \left(\|\widehat{f}_n\|_{\mathrm{L}^2(I_n)} - \sum_{n \ne k=1}^\infty \|\widehat{f}_k\|_{\mathrm{L}^2(I_n)} \right) \tag{4.17}$$

$$\ge \sum_{n=1}^\infty (1 - \varepsilon_n - (2\varepsilon_n)^{1/2} \sum_{k=1}^\infty \varepsilon_k \ge \sum_{n=1}^\infty (1 - \varepsilon_n - (2\varepsilon_n)^{1/2}) = \infty. \tag{4.18}$$

Therefore, $u \notin \mathrm{L}^2$. It only remains to show that $\mathbb{D}u \in \mathrm{L}^2$. But

$$\|\widehat{\mathbb{D}}\widehat{f}_n u_n\|_{\mathrm{L}^2(i\mathbf{R};U)} \le \|\widehat{\mathbb{D}}\widehat{f}_n u_n\|_{\mathrm{L}^2(E_n;U)} + \|\widehat{\mathbb{D}}\widehat{f}_n u_n\|_{\mathrm{L}^2(i\mathbf{R}\setminus E_n;U)} \tag{4.19}$$

$$< \sqrt{2\pi}\varepsilon_n + \varepsilon_n\|\mathbb{D}\| \le 2^{-n}(\sqrt{2\pi} + \|\mathbb{D}\|) = M2^{-n}, \tag{4.20}$$

where $M := \sqrt{2\pi} + \|\mathbb{D}\|$, because $\|\widehat{\mathbb{D}}\widehat{f}_n u_n\|_Y \le |\widehat{f}_n|\varepsilon_n$ on E_n, $\|\widehat{\mathbb{D}}\widehat{f}_n u_n\|_Y \le |\widehat{f}_n|\|\mathbb{D}\|$ a.e. on $i\mathbf{R}$, and $\|\widehat{f}\|_2 = \sqrt{2\pi}$, $\|\widehat{f}\|_{\mathrm{L}^2(i\mathbf{R}\setminus E_n)} < \varepsilon_n$.

Consequently, $\mathbb{D}v_n \to y$ in L^2, for some $y \in \mathrm{L}^2$, by Lemma A.3.4(L1). Choose $\omega > 0$. Because $v_n \to u$ in L^2_ω, we have $\mathbb{D}v_n \to \mathbb{D}u$ in L^2_ω, hence $\mathbb{D}u = y$ a.e., in particular, $\mathbb{D}u \in \mathrm{L}^2$.

(b3) We have $|\mathbb{D}u\|_2 \ge \varepsilon\|u\|_2$ for all $u \in \mathrm{L}^2([T,+\infty); U)$ and all $T \in \mathbf{R}$, by (a) and time-invariance, hence for all $u \in \mathrm{L}^2(\mathbf{R}; U)$, because both sides of the inequality are continuous on L^2. Because $\mathrm{L}^2_\omega(\mathbf{R}_-; U) \subset \mathrm{L}^2$, we have $\mathrm{L}^2_\omega(\mathbf{R}; U) + \mathrm{L}^2_\infty(\mathbf{R}_+; U) \subset \mathrm{L}^2(\mathbf{R}_-; U) + \mathrm{L}^2_\infty(\mathbf{R}_+; U)$, hence $\|\mathbb{D}u\|_2 = \infty$ whenever $u \in \mathrm{L}^2_\omega(\mathbf{R}; U) + \mathrm{L}^2_\infty(\mathbf{R}_+; U)$ and $\|u\|_2 = \infty$. Take $M := \|\mathbb{D}\|_{\mathrm{TIC}}$ to obtain (b3).

(b1) This follows from (b3).

(c1)&(c2) See the proof of Proposition 4.1.7(c1)&(c2).

(d) This is obvious (note also that now $\|\mathbb{D}\mathbb{F}u\|_2 \ge \varepsilon\varepsilon_{\mathbb{F}}\|v\|_2$ for all $v \in \mathrm{L}^2$).

(e) Inequality $\|\widehat{\mathbb{D}}(ir)u_0\|_Y \ge \varepsilon\|u_0\|_U$ follows from Theorem 3.1.3(e1) (cf. the beginning of the proof).

Assume then that $\widehat{\mathbb{D}}(s_0)u_0 = 0$ for some $u_0 \in U \setminus \{0\}$ and $s_0 \in \mathbf{C}^+$. Then

$\widehat{\mathbb{D}}(\cdot)(s-s_0)^{-1}u_0 \in \mathrm{H}(\mathbf{C}^+;U)$, by Lemma D.1.2(j) and a simple computation, although $u := (\cdot - s_0)^{-1}u_0 \in \mathrm{H}^2(\mathbf{C}^+_{\mathrm{Re}\,s_0+1};U) \setminus \mathrm{H}^2(\mathbf{C}^+;U)$.

(f) If $u \in \mathrm{L}^2$, then $\|\mathbb{F}u\|_2 \geq \varepsilon\|\mathbb{E}\|^{-1}\|u\|_2$. If $u \notin \mathrm{L}^2$, then $\mathbb{D}u \notin \mathrm{L}^2$, hence then $\mathbb{F}u \notin \mathrm{L}^2$.

(g) Implication follows from Proposition 4.1.7(b3); note that $\mathbb{D} := \tau^{-1} \in \mathrm{TIC}(U)$ is quasi-left-invertible but not p.r.c. (since $\widehat{\mathbb{D}}(s) = e^{-s}$).

(h) By (a), quasi-left-invertible maps are left invertible over TI (but not necessarily over TIC, by (g)).

Let $\widehat{\mathbb{D}}(s) := (s-1)/(s+1) \in \mathrm{H}^\infty(\mathbf{C}^+)$. Then $\mathbb{D}^*\mathbb{D} = I$, hence $\mathbb{D}^*$ is the inverse of $\mathbb{D}$ in $\mathrm{TI}(\mathbf{C})$, but $\mathbb{D}$ is not quasi-left-invertible, by (e) (alternatively, because $\widehat{u} := (s-1)^{-1} \in \mathrm{H}^2(\mathbf{C}^+_2) \setminus \mathrm{H}^2(\mathbf{C}^+)$ and $\widehat{\mathbb{D}}\widehat{u} = (s+1)^{-1} \in \mathrm{H}^2$). □

The noncausal case is simple:

Lemma 4.1.9 (Noncausal corona theorem) *If $\mathcal{A}$ is one of the classes* TI*,* CTI*,* MTI*,* $\mathrm{MTI_d}$*,* $\mathrm{MTI}^{\mathrm{L}^1}$*,* $\mathrm{MTI_S}$*,* $\mathrm{MTI_{d,S}}$*, then $\mathcal{A}(U,Y)$ is left inverse closed in* TI*, and $\mathbb{E} \in \mathcal{A}(U,Y)$ is left invertible iff $\mathbb{E}^*\mathbb{E} \geq \varepsilon I$ on* L^2 *(iff $\widehat{\mathbb{E}}^*\widehat{\mathbb{E}} \geq \varepsilon I$ a.e. on $i\mathbf{R}$, provided that U is separable or $\mathcal{A} \neq$* TI*).*

If $\mathcal{A}$ is one of the classes TI*,* CTI*,* $\mathrm{MTI}^{\mathrm{L}^1}$*,* $\mathrm{MTI}_{T\mathbf{Z}}$*,* $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$*, then any left invertible element $\mathbb{E} \in \mathcal{A}(\mathbf{C}^m, \mathbf{C}^n)$ can be complemented to an invertible operator.*

Proof: 1° *Left invertibility:* By Lemma A.3.1(c1)(ii)&(v), "$\mathbb{E}^*\mathbb{E} \geq \varepsilon I$" is necessary. (It is equivalent to "$\widehat{\mathbb{E}}^*\widehat{\mathbb{E}} \geq \varepsilon I$ a.e. on $i\mathbf{R}$", by Theorem 3.1.3(d), provided that U is separable of $\widehat{\mathbb{E}}$ is continuous.)

Conversely, if $\mathbb{E}^*\mathbb{E} \gg 0$, then the formula $\mathbb{V} := (\mathbb{E}^*\mathbb{E})^{-1}\mathbb{E}^* \in \mathcal{A}$ (by Theorem 4.1.1 and Lemma 4.1.3(a)) provides a left-inverse for $\mathbb{E}$.

2° *Complementation:* (Clearly left invertibility is necessary.) Classes CTI, $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$ and $\mathrm{MTI}_{T\mathbf{Z}}$ can be handled by using the methods of the proof of Theorem 4.1.6(c) (based on the fact that they have contractible maximal ideals); a left-invertible $\mathbb{E} \in \mathrm{TI}(\mathbf{C}^m, \mathbf{C}^n)$ can be complemented as follows:

By Lemma 6.4.7, there is $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$ s.t. $\mathbb{X}^*\mathbb{X} = \mathbb{E}^*\mathbb{E}$. By [CG97, Lemma 2.2] (and Theorem 3.1.3(a)&(c)), we can complement the isometric $\mathbb{N} := \mathbb{E}\mathbb{X}^{-1} \in \mathrm{TI}$ to a unitary $\begin{bmatrix}\mathbb{N} & \mathbb{F}\end{bmatrix} \in \mathcal{G}\mathrm{TI}$, hence $\begin{bmatrix}\mathbb{E} & \mathbb{F}\end{bmatrix} = \begin{bmatrix}\mathbb{N} & \mathbb{F}\end{bmatrix}\left[\begin{smallmatrix}\mathbb{X} & 0\\ 0 & I\end{smallmatrix}\right] \in \mathcal{G}\mathrm{TI}$. □

Lemma 4.1.10 (No ∞-dim. Corona Theorem) *Let U be infinite-dimensional and* $\dim Y \geq \dim U$*. Then there is $\mathbb{D} \in \mathrm{TIC}(U,Y)$ s.t. $\widehat{\mathbb{D}}^*\widehat{\mathbb{D}} \geq I$ on $\mathbf{C}^+$, but $\mathbb{V}\mathbb{D} \neq I$ for all $\mathbb{V} \in$* TIC.

Consequently, $\mathbb{D}$ cannot be complemented to an invertible element of TIC (because the proof of implication "(vii)⇒(i)" of Theorem 4.1.6 applies to this case too). Cf. also Proposition 4.1.7.

Proof: A counter-example is constructed in [Treil89], below its Theorem 1, assuming U and Y to be separable (we may need to multiply the counter-example by a positive constant). In the general case, write U as $U_1 \times U_2$

and Y as $U \times Y_2$ (modulo an isometric isomorphism), where U_1 is separable (this is possible because $\dim Y \geq \dim U$, by Lemma 2.2.1(c3)). Let $\widehat{\mathbb{F}}$ be as in the counter-example, and set $\mathbb{D} := \left[\begin{smallmatrix} \mathbb{F} & 0 \\ 0 & I \end{smallmatrix}\right]$. Then $\widehat{\mathbb{D}}^* \widehat{\mathbb{D}} \geq I$ on $\mathbf{C}^+$, but if $\mathbb{V}\mathbb{D} = I = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \in \mathcal{B}(U_1 \times U_2)$, then $\mathbb{V}_{11}\mathbb{F} = I$, hence then $\mathbb{V} \notin \mathrm{TIC}$. □

Notes

Several sufficient conditions for the invertibility of a measure with values in a Banach algebra are given in [Gri]. Much of Theorem 4.1.1(f) is based on those results. The article [Gri] also provides further results and treats very general measures.

The monograph [Vid] is an excellent classical reference for the connection between dynamic stabilization, coprime factorization and the Corona Theorem. It contains the principal ideas of Lemmas 4.1.4 and 4.1.5 and Theorem 4.1.6 and applications for several classes.

The complementation result Theorem 4.1.6(c) for TIC is due to V.A. Tolokonnikov [Tolokonnikov]; see pp. 288–298 of [Nikolsky] for a presentation in English, further results, norm estimates and historical remarks. These results, being more recent than those of [Vid], do not seem to be widely known.

The original Corona Theorem is due to L. Carleson [Carleson]. A matrix-valued Corona Theorem ($\mathcal{B}(\mathbf{C}^n, \mathbf{C}^{nm})$ only) was given in [Fuhrmann68]. An extension of the matrix-valued Corona Theorem with an arbitrary $\mathbb{D} \in \mathrm{TIC}(\mathbf{C}^n, \mathbf{C}^m)$ in place of $1_{\mathcal{A}}$ in (4.7) is given in [Anderson].

Chapter 5

Spectral Factorization ($\mathbb{E} = \mathbb{Y}^*\mathbb{X}$, $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* S \mathbb{X}$)

God created spectral factorizations; the rest is made by man.

— Frank Callier, in a discussion of the importance of spectral factorizations, indefinite inner spaces and Riccati equations, MTNS'98.

This chapter treats the spectral factorization (or canonical factorization) of MTI maps. Spectral factorization will be used in later chapters for the solution of several control problems.

In Section 5.1, we apply the early factorization theory of Israel Gohberg and Yuri Leiterer (not being a prophet, we cannot refer directly to [God]) to $\mathrm{MTI}^{\mathrm{L}^1}$ and $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$ in continuous time and to ℓ^1 in discrete time. In Section 5.2, we adopt several $\mathrm{MTI}_{\mathrm{d}}$ factorization results to our setting and show that the factorization of MTI maps can be reduced to that of $\mathrm{MTI}^{\mathrm{L}^1}$ and $\mathrm{MTI}_{\mathrm{d}}$ maps. We thus obtain both positive and indefinite spectral factorization results for several MTI classes.

We also state a few other results concerning the spectral factorization of TI maps. By H, U and Y we again denote Hilbert spaces of arbitrary dimensions. (The results based on [GL73a] could be modified for arbitrary Banach spaces.)

Also Section 6.4 contains related results, but we have chosen its current place since that section is a prerequisite for Sections 6.6–6.7 and Chapter 7.

5.1 Auxiliary spectral factorization results

Grief can take care of itself; but to get the full value of a joy you must have somebody to divide it with.

— Mark Twain (1835–1910)

In this section we apply the spectral factorization theory of Gohberg and Leiterer to $\mathrm{MTI}^{\mathrm{L}^1}$ (Theorem 5.1.2), ℓ^1 (Theorem 5.1.3) and $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$ (Corollary 5.1.4). In Section 5.2, we shall then refine these and other results to cover further classes and to provide more information on the factors.

First we define a spectral factorization:

Definition 5.1.1 (SpF) *A factorization* $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ *is a* spectral factorization *of* $\mathbb{E} \in \mathrm{TI}(U)$ *if* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{TIC}(U)$.

For $\dim U < \infty$, this could be rephrased in the familiar form "if $\widehat{\mathbb{X}}, \widehat{\mathbb{Y}} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{C}^+;\mathbf{C}^{n\times n})$ and $\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ a.e. on $i\mathbf{R}$, then $\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ is a *spectral factorization* of $\widehat{\mathbb{E}}$" ($\in \mathrm{L}^\infty(\mathbf{C}^+;\mathbf{C}^{n\times n})$), by Theorems 2.1.2 and 3.3.1.

Even for a general U, the identity $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ can be written as "$\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ on $i\mathbf{R}$" when $\mathbb{E}, \mathbb{X}, \mathbb{Y} \in \mathrm{MTI}$, but for general $\mathbb{E} \in \mathrm{TI}$ we must be satisfied with the equality "$\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ in $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U))$", which need not imply pointwise equality anywhere (for separable U an equivalent formulation is that "$\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ a.e. on $i\mathbf{R}$"); see Theorem 3.1.3 for details. However, in this chapter we mainly study MTI maps, for which we have continuity and pointwise equality everywhere on $i\mathbf{R}$ regardless of U.

As the first spectral factorization result, we apply Theorem 5.1.6 to the Wiener class:

Theorem 5.1.2 ($\mathbf{MTIC}^{\mathbf{L}^1}$ spectral factorization) *Let* $\mathbb{E} \in \mathrm{MTI}^{\mathrm{L}^1}(U)$*, i.e.,* $\widehat{\mathbb{E}} = E + \hat{f}$*, where* $E \in \mathcal{B}(U)$ *and* $f \in \mathrm{L}^1(\mathbf{R};\mathcal{B}(U))$.

Then the Toeplitz operator $\pi_+\mathbb{E}\pi_+$ *is invertible iff* $\mathbb{E}$ *has a factorization* $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ *with* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1}$.

If, in addition, $\mathbb{E} \in \mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}(U)$ *(i.e.,* $f \in \mathrm{L}^1(\mathbf{R};\mathcal{BC}(U))$*), then* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1,\mathcal{BC}}$.

Proof: By Lemma 5.1.7, we may apply Theorem 5.1.6 to obtain the above factorizations in the same way as in the proof of Theorem 5.1.3.

Note that $E + f \in \mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}(U)$ implies that $P_-^0((E+\widehat{f})\circ\phi_{\mathrm{Cayley}}^{-1}) = \widehat{\pi_- f}\circ\phi_{\mathrm{Cayley}}^{-1} \in \mathfrak{C}_\infty$, and that $P_+((E+\widehat{f})\circ\phi_{\mathrm{Cayley}}^{-1})$ can be seen to be a Fredholm operator as in the proof of Theorem 5.1.3 (alternatively, it follows easily from the fact that E must be invertible, by, e.g., Proposition 6.3.1(c)). See also Lemma 5.1.5. The parametrization of all factors is given in Lemma 6.4.5(i). $\square$

Next we apply Theorem 5.1.6 to the *discrete Wiener class* $\widehat{\ell^1}$. We will use the following notation (as in Theorem 4.1.1 and in Section 13.1):

$$\begin{aligned}
\ell^1 &:= \ell^1(\mathbf{Z};\mathcal{B}(U)) := \{(a_j)_{j\in\mathbf{Z}} \,\big|\, a_j \in \mathcal{B}(U) \text{ and } \|(a_j)_{j\in\mathbf{Z}}\|_{\ell^1} := \sum_{j\in\mathbf{Z}} \|a_j\|_{\mathcal{B}(U)} < \infty\},\\
\ell^1_{\mathcal{BC}} &:= \{a \in \ell^1 \,\big|\, a_j \in \mathcal{BC}(U,Y) \text{ for all } j \neq 0\},\\
\ell^1_{\pm} &:= \{a \in \ell^1 \,\big|\, a_j = 0 \text{ for all } \pm j < 0\},\\
\ell^1_{\mathcal{BC},\pm} &:= \{a \in \ell^1_{\mathcal{BC}} \,\big|\, a_j = 0 \text{ for all } \pm j < 0\}.
\end{aligned} \tag{5.1}$$

We equip these spaces with *convolution multiplication*

$$(a_j)_{j\in\mathbf{Z}} * (b_k)_{k\in\mathbf{Z}} := \big(\sum_j a_j b_{n-j}\big)_{n\in\mathbf{Z}}. \tag{5.2}$$

As in Section 13.1, one can verifies that ℓ^1 with convolution multiplication is a Banach algebra, and the five other classes defined above are closed subalgebras. The *Z-transform* of $a = (a_j)_{j\in\mathbf{Z}} \in \ell^1$ is

$$\widehat{a} := \sum_{j\in\mathbf{Z}} a_j z^j \in \mathcal{C}(\overline{\mathbf{D}}) \cap \mathrm{H}^\infty(\mathbf{D}), \tag{5.3}$$

and $\widehat{a*b} = \widehat{a}\widehat{b}$. The class $\widehat{\ell^1}$ (obviously isomorphic to the Banach algebra ℓ^1) is sometimes called the discrete Wiener class. The canonical projection $\pi^+ : \ell^2(\mathbf{Z};U) \mapsto \ell^2(\mathbf{N};U)$ obviously satisfies $\widehat{\pi^+}\sum_{j\in\mathbf{Z}} z^j x_j := \sum_{j\in\mathbf{N}} z^j x_j$. Recall from Lemma D.1.15 that $\mathrm{L}^2(\partial\mathbf{D};U) = \{\sum_{j\in\mathbf{Z}} z^j x_j \,\big|\, \sum \|z_j\|_U^2 < \infty\}$ and that $\mathrm{H}^2(\mathbf{D};U) = \widehat{\pi^+}[\mathrm{L}^2(d\mathbf{D};U)]$.

Theorem 5.1.3 (Discrete $\boldsymbol{\ell^1}$ spectral factorization) *Let $\mathbb{E} \in \ell^1(\mathbf{Z};\mathcal{B}(U))$, i.e., $\widehat{\mathbb{E}} = \sum_{j=-\infty}^{\infty} z^j E_j$, where $E_j \in \mathcal{B}(U)$ for all j and $\sum_j \|E_j\| < \infty$.*

Then the Toeplitz operator $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+} \in \mathcal{B}(\mathrm{H}^2(\mathbf{D};U))$ is invertible iff $\widehat{\mathbb{E}}$ has a spectral factorization $\widehat{\mathbb{E}} = \widehat{\mathbb{E}}_-\,\widehat{\mathbb{E}}_+$ with $\mathbb{E}_+ \in \mathcal{G}\ell^1_+$ and $\mathbb{E}_- \in \mathcal{G}\ell^1_-$. If, in addition, $\mathbb{E} \in \ell^1_{\mathcal{BC}}$ (i.e., $E_j \in \mathcal{BC}(U)$ for $j \neq 0$), then $\widehat{\mathbb{E}}_+ \in \mathcal{G}\ell^1_{\mathcal{BC},+}$ and $\widehat{\mathbb{E}}_- \in \mathcal{G}\ell^1_{\mathcal{BC},-}$.

Proof: 1° *$\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ is invertible iff $\widehat{\mathbb{E}}$ has a spectral factorization:*
The first claim follows from (a) and (c) of Theorem 5.1.6, as soon as we have verified the assumptions of the Theorem.

One easily verifies that assumptions (1) and (2) of Theorem 5.1.6 hold (for both $\widehat{\ell^1}$ and $\widehat{\ell^1_{\mathcal{BC}}}$), where we have set $P_+\sum_{j=-\infty}^{\infty} z^j E_j := \sum_{j=0}^{\infty} z^j E_j$.

(3a) One easily deduces from [HP, p. 97], that the Laurent series of a holomorphic (around $\partial\mathbf{D}$) function converges absolutely on $\partial\mathbf{D}$. Conversely, the holomorphic function $\sum_{j\in\mathbf{Z}} r^{-|j|} E_j z^j$ converges to $\widehat{\mathbb{E}}$ in $\widehat{\ell^1}$, as $r \to 1-$. Finally, $\widehat{\ell^1}$ (equivalently, ℓ^1 as a convolution algebra) is inverse closed by Theorem 4.1.1(d).

(a) If $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ is invertible, then it is a Fredholm operator, hence the assumptions of (a) are satisfied in this case, and the converse follows from the implication (i)$\Rightarrow$(ii) of (c).

2° *The case $\mathbb{E} \in \ell^1_{\mathcal{BC}}$:* Below we show that the assumptions of (b) are satisfied, so that the form of $\widehat{\mathbb{E}}_\pm$ follows from (b).

Assumptions (1) and (2) were handled above.

Because $z^n \in \mathcal{R}(\partial\mathbf{D})$ for $n \in \mathbf{Z}$, the assumption (3b) is satisfied. Similarly, we see that $P^0_- \widehat{\mathbb{E}}$ is in $\mathfrak{C}_\infty$ when $\widehat{\mathbb{E}} \in \ell^1_{\mathcal{BC}}$, so it only remains to be shown that $P_+\widehat{\mathbb{E}}(z)$ is a Fredholm operator for all $z \in \overline{\mathbf{D}}$.

The invertibility of $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ implies the invertibility of $\widehat{\mathbb{E}}$ on $\partial\mathbf{D}$, by Lemma 5.1.5; in particular, $\widehat{\mathbb{E}}(1) \in \mathcal{GB}(U)$.

But $\widehat{\mathbb{E}}(z) = E_0 + \widehat{\mathbb{F}}(z)$, where $\widehat{\mathbb{F}}(z) := \sum_{j \neq 0} E_j z^j \in \mathcal{BC}(U)$ for all $z \in \overline{\mathbf{D}}$, hence $\widehat{\mathbb{E}}(z) = \widehat{\mathbb{E}}(1) + \widehat{\mathbb{F}}(z) - \widehat{\mathbb{F}}(1) \in \mathcal{GB}(U) + \mathcal{BC}(U)$, and $\mathcal{GB} + \mathcal{BC}$ operators are Fredholm operators, by Lemma A.3.4(B4).

Finally, $\widehat{\mathbb{E}}, \widehat{\mathbb{E}}_-, \widehat{\mathbb{E}}_-^{-1} \in \ell^1_{\mathcal{BC}}$ implies that $\widehat{\mathbb{E}}_+, \widehat{\mathbb{E}}_+^{-1} \in \ell^1_{\mathcal{BC}}$. □

Due to isomorphism, the above is equivalent to the following:

Corollary 5.1.4 ($\mathbf{MTI}_{\mathbf{d},T\mathbf{Z}}$ spectral factorization) *Let $T \in \mathbf{R}$, and let $\mathbb{E} \in \mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$, i.e., $\mathbb{E} = \sum_{k \in \mathbf{Z}} E_j \delta_{jT}$, where $E_j \in \mathcal{B}(U)$ for all j and $\|\mathbb{E}\|_{\mathrm{MTI}} := \sum_j \|E_j\| < \infty$.*

Then the Toeplitz operator $\pi_+\mathbb{E}\pi_+ \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))$ is invertible iff $\mathbb{E}$ has a spectral factorization $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ with $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}(U)$. If, in addition, $\mathbb{E} \in \mathrm{MTI}^{\mathcal{BC}}_{\mathrm{d},T\mathbf{Z}}$ (i.e., $E_j \in \mathcal{BC}(U)$ for $j \neq 0$), then $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{MTIC}^{\mathcal{BC}}_{\mathrm{d},T\mathbf{Z}}(U)$.

Proof: This is Theorem 5.1.3 rephrased according to the isomorphism stated in Theorem 13.4.5 (note that $\widetilde{I}\widehat{\pi^+}(E_j)_{j \in \mathbf{Z}}\widehat{\pi^+} = \pi_+\mathbb{E}\pi_+$, $\widetilde{I}[\ell^1_+] = \mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$, and $\widetilde{I}[\ell^1_-] = \{\mathbb{Y}^* \,|\, \mathbb{Y} \in \mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}\}$). □

The rest of this sections consists only of results that are needed for the proofs of the above results.

We start the proofs with an auxiliary lemma:

Lemma 5.1.5 *Let $\widehat{\mathbb{E}} \in \mathcal{C}(\partial\mathbf{D}; \mathcal{B}(H))$ and set $\widehat{\mathbb{F}} := \widehat{\mathbb{E}} \circ \phi_{\mathrm{Cayley}} \in \mathcal{C}(i\mathbf{R} \cup \{\infty\}; \mathcal{B}(H))$. Then $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ is invertible iff $\widehat{\pi_+}\widehat{\mathbb{F}}\widehat{\pi_+}$ is invertible. Moreover, if $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ is invertible, then $\widehat{\mathbb{E}} \in \mathcal{GC}(\partial\mathbf{D}; \mathcal{B}(H))$.*

Proof: The equivalence follows from Theorem 13.2.3(a1)&(b1)&(c). If $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ is invertible on $\mathrm{H}^2(\mathbf{D};H)$, then $\widehat{\mathbb{E}}$ is invertible on $\mathrm{L}^2(\partial\mathbf{D};H)$, by discretized Lemma 6.4.6, hence then $\widehat{\mathbb{E}}$ is invertible in $\mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D};H)$, by Theorem 3.1.3(a1), hence in $\mathcal{C}(\partial\mathbf{D}; \mathcal{B}(H))$, by Theorem F.1.9(s4) (applied to $Q := [0, 2\pi)$). □

The following "raw result" from [GL-Crit] and [GL73a] is the basis for the above factorization results:

Theorem 5.1.6 *Let H be a Hilbert space. Let $\mathcal{R}(\partial\mathbf{D})$ be the set of rational scalar functions with poles outside $\partial\mathbf{D}$. Let $\mathfrak{C} \subset \mathcal{C}(\partial\mathbf{D}; \mathcal{B}(H))$ be a Banach algebra with a norm $\|\cdot\|_{\mathfrak{C}}$ s.t.*

(1) $\sup_{\partial\mathbf{D}} \|\widehat{\mathbb{E}}(\cdot)\|_{\mathcal{B}(H)} \le c\|\widehat{\mathbb{E}}\|_{\mathfrak{C}}$ *for all* $\widehat{\mathbb{E}} \in \mathfrak{C}$ *for some* $c > 0$*, and*

(2) $\mathfrak{C}$ *is the direct sum* $\mathfrak{C}^+ \oplus \mathfrak{C}_0^-$*, where* $\mathfrak{C}^+ = \mathfrak{C} \cap \mathrm{H}^\infty(\mathbf{D};\mathcal{B})$*,* $\mathfrak{C}^- = \mathfrak{C} \cap \mathrm{H}^\infty(\overline{\mathbf{D}}^c;\mathcal{B})$*, and* $\mathfrak{C}_0^- = \{f \in \mathfrak{C}^- \mid f(\infty) = 0\}$.

Let $P_+ : \mathfrak{C} \to \mathfrak{C}^+$ *and* $P_-^o := I - P_+ : \mathfrak{C} \to \mathfrak{C}_0^-$ *be the corresponding projections. Let* $\widehat{\mathbb{E}} \in \mathcal{G}\mathfrak{C}$*. Then we have the following:*

(a) Let functions holomorphic on a neighborhood of $\partial\mathbf{D}$ *be a dense subset of* $\mathfrak{C}$*, and let* $\mathfrak{C}$ *be inverse closed in* $\mathcal{C}$ *(i.e., if* $\widehat{\mathbb{E}} \in \mathfrak{C} \cap \mathcal{GC}(\partial\mathbf{D};\mathcal{B}(H))$*, then* $\widehat{\mathbb{E}}^{-1} \in \mathfrak{C}$*).*

Then $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ *is a Fredholm operator on* $\mathrm{H}^2(\mathbf{D};H)$ *iff* $\widehat{\mathbb{E}}$ *has a factorization of the form*

$$\widehat{\mathbb{E}} = \widehat{\mathbb{E}}_- G \widehat{\mathbb{E}}_+, \quad \widehat{\mathbb{E}}_+ \in \mathcal{G}\mathfrak{C}^+,\ \widehat{\mathbb{E}}_- \in \mathcal{G}\mathfrak{C}^-,\ G(z) = P_0 + \sum_{j=1}^n z^{\kappa_j} P_j, \tag{5.4}$$

where $n \in \mathbf{N}$*,* P_j $(j = 1,\dots,n)$ *are disjoint one-dimensional projections,* $P_0 = I - \sum_j P_j$*, and* $\kappa_j \in \mathbf{Z} \setminus \{0\}$*,*

(b) Let the rational functions $\sum_{j=1}^n r_j T_j$ $(r_j \in \mathcal{R}(\partial\mathbf{D}),\ T_j \in \mathcal{B}(H)$ *for all* $j)$ *be a dense subset of* $\mathfrak{C}$*.*

Let $P_+\widehat{\mathbb{E}}(z)$ *be a Fredholm operator for all* $z \in \overline{\mathbf{D}}$*, and let* $P_-^o\widehat{\mathbb{E}} \in \mathfrak{C}_\infty$*. Then* $\widehat{\mathbb{E}}$ *has the factorization (5.4) with* $\widehat{\mathbb{E}}_- - I, \widehat{\mathbb{E}}_-^{-1} - I \in \mathfrak{C}_\infty^- := \mathfrak{C}_\infty \cap \mathrm{H}^\infty(\overline{\mathbf{D}}^c;\mathcal{B})$*.*

Here the set $\mathfrak{C}_\infty$ *is the closure (in* $\mathfrak{C}$*) of rational* $\mathcal{BC}$*-valued operators*

$$\sum_{j=1}^n r_j T_j, \quad r_j \in \mathcal{R}(\partial\mathbf{D}),\ T_j \in \mathcal{BC} \text{ for all } j. \tag{5.5}$$

(c) Let all the assumptions of (a) or those of (b) be satisfied, and let $\widehat{\mathbb{E}} = \widehat{\mathbb{E}}_- G \widehat{\mathbb{E}}_+$ *be the resulting factorization. Then the following are equivalent:*

(i) $G = I$*,*
(ii) $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ *is invertible on* $\mathrm{H}^2(\mathbf{D};H)$*,*
(iii) $\widehat{\pi_+}(\widehat{\mathbb{E}} \circ \phi_{\mathrm{Cayley}})\widehat{\pi_+}$ *is invertible on* $\mathcal{L}\pi_+\mathrm{L}^2(\mathbf{R};H)$*.*

Moreover, if (i) holds and we set $\widehat{\mathbb{X}} := \widehat{\mathbb{E}}_+ \circ \phi_{\mathrm{Cayley}}$*,* $\widehat{\mathbb{Y}}(s) := (\widehat{\mathbb{E}}_- \circ \phi_{\mathrm{Cayley}})(-\bar{s})^*$*, then* $\widehat{\mathbb{X}}, \widehat{\mathbb{Y}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(H))$*,* $\widehat{\mathbb{Y}}^*\widehat{\mathbb{X}} = \widehat{\mathbb{E}} \circ \phi_{\mathrm{Cayley}}$ *on* $i\mathbf{R}$*, and all spectral factorizations of* $\widehat{\mathbb{E}}$ *are given by* $\widehat{\mathbb{E}} = (\widehat{\mathbb{E}}_- T)(T\widehat{\mathbb{E}}_+)$*,* $T \in \mathcal{GB}(U)$ *(i.e.,* $\widehat{\mathbb{E}} \circ \phi_{\mathrm{Cayley}} = (T\widehat{\mathbb{Y}})^*(T\widehat{\mathbb{X}})$*).*

We remark that the original results in [GL73a] and [GL-Crit] are given in a more abstract and general form.

Do not mix $\widehat{\pi^+}\widehat{\mathbb{E}}$ with $P_+\widehat{\mathbb{E}}$ ((the restriction of) P_+ is an operator on $\widehat{\mathrm{MTI}}$, i.e., it operatres $\widehat{\mathbb{E}}$, whereas $\widehat{\pi^+}$ is an operator on H^2 (so are $\widehat{\mathbb{E}}$ and $P_+\widehat{\mathbb{E}}$ too)).

If H is finite-dimensional, then it is possible to formulate the theorem without a reference to Fredholm operators; see Theorems II.3.1 and II.4.1 of [CG81].

Proof: (a) This is Theorem 2 of [GL-Crit] (use (13.29) and note that "*PA*" in [GL-Crit] refers to composition, i.e., $PAu := P(Au)$, as stated on p. 102 of [GL72]; P does not operate directly on A).

(b) This is Theorem 2.1, p. 40, of [GL73a] (cf. pp. 38–39 of [GL73a]).

(c) 1° By Lemma 5.1.5, (ii) and (iii) are equivalent. Because $\widehat{\mathbb{E}}_- \circ \phi_{\text{Cayley}} \in \mathrm{H}^\infty(\mathbf{C}^-;\mathcal{B}(H))$, we have $\widehat{\mathbb{Y}} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(H))$; clearly $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(H))$ too. If (i) holds, then $(\widehat{\mathbb{E}} \circ \phi_{\text{Cayley}})(ir) = \widehat{\mathbb{Y}}(-\overline{ir})^*\widehat{\mathbb{X}}(ir) = \widehat{\mathbb{Y}}(ir)^*\widehat{\mathbb{X}}(ir)$ for $r \in \mathbf{R}$, and the uniqueness claim follows from Lemma 6.4.5(a).

Thus, only (i)$\Leftrightarrow$(iii) is left to be proved.

2° "(i)$\Leftrightarrow$(iii)": Set $\widehat{\mathbb{G}} := D \circ \phi_{\text{Cayley}} \in \mathcal{C}(i\mathbf{R} \cup \{\infty\};\mathcal{B}(H))$. Then $\mathbb{G} \in \mathrm{TI}(H)$, $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{TIC}(H)$, and $\mathbb{F} = \mathbb{Y}^*\mathbb{G}\mathbb{X}$.

Because $\pi_+\mathbb{X}\pi_+ = \mathbb{X}\pi_+$ and $\pi_+\mathbb{Y}^*\pi_+ = \pi_+\mathbb{Y}^*$ are invertible on $\pi_+\mathrm{L}^2$ (the inverses are $\pi_+\mathbb{X}^{-1}\pi_+$ and $\pi_+\mathbb{Y}^{-*}\pi_+$), the equation

$$\pi_+\mathbb{F}\pi_+ = \pi_+\mathbb{Y}^*\mathbb{G}\mathbb{X}\pi_+ = \pi_+\mathbb{Y}^*\pi_+\mathbb{G}\pi_+\mathbb{X}\pi_+ \tag{5.6}$$

implies that $\pi_+\mathbb{G}\pi_+$ is invertible iff $\pi_+\mathbb{F}\pi_+$ is. By Lemma 5.1.5, this can be paraphrased as "$\widehat{\pi^+}G\widehat{\pi^+}$ is invertible on $\mathrm{H}^2(\mathbf{D};H)$ iff $\widehat{\pi^+}\widehat{\mathbb{E}}\widehat{\pi^+}$ is". Of course, $G = I$ is sufficient, so we study the necessity:

Let $\widehat{\pi^+}G\widehat{\pi^+}$ be invertible. If $u_j = P_ju_j \in H$ and $\kappa_j > 0$, then $u := 1u_j \notin \widehat{\pi^+}G\widehat{\pi^+}\mathrm{H}^2$, because $P_j\widehat{\pi^+}G\widehat{\pi^+} = \widehat{\pi^+}s_j^{\kappa}\widehat{\pi^+}P_j$; similarly no κ_j can be negative, hence $n = 0$ and $G = P_0 = I$. $\square$

(The proofs in [GL-Crit] and [GL73a] go as follows: first it is assumed that $\mathbb{E}$ is holomorphic around the unit circle, then this is applied to "rational finite-dimensional $\mathbb{E}$'s", then the density of such operators in $\mathrm{MTI}^{\mathrm{L}^1}$ is combined with the fact that any element near the identity has a spectral factorization.)

The above theorem can be applied to the Wiener class:

Lemma 5.1.7 *The class* $\mathfrak{C} := \{\widehat{\mathbb{E}} \circ \phi_{\text{Cayley}}^{-1} \,|\, \mathbb{E} \in \mathrm{MTI}^{\mathrm{L}^1}(U)\}$ *with* $\|\widehat{\mathbb{E}} \circ \phi_{\text{Cayley}}^{-1}\|_{\mathfrak{C}} := \|\mathbb{E}\|_{\mathrm{MTI}^{\mathrm{L}^1}}$ *satisfies the assumptions of (1), (2), (a) and (b) (on $\mathfrak{C}$) of Theorem 5.1.6 when we define the projection* $P_+ \in \mathcal{B}(\mathfrak{C},\mathfrak{C}^+)$ *by*

$$P_+ : (E + \widehat{f}) \circ \phi_{\text{Cayley}}^{-1} \mapsto E + \widehat{\pi_+ f} \circ \phi_{\text{Cayley}}^{-1} \tag{5.7}$$

for $E \in \mathcal{B}(U)$, $f \in \mathrm{L}^1(\mathbf{R};U)$ *(i.e., for* $E + \widehat{f} \in \widehat{\mathrm{MTI}^{\mathrm{L}^1}}(U)$*). Moreover, in this case,* $\mathfrak{C}_\infty$ *corresponds to maps* $\mathbb{E} \in \mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}(U)$ *whose feedthrough operator is compact.*

Proof: We remark that now $\mathfrak{C}$ denotes the Cayley transforms of functions in $\widehat{\mathrm{MTI}^{\mathrm{L}^1}}$, with their original norm, exactly as in [GL73a, Theorem 4.3].

Conditions (1) and (2) can be proved as in [CG81, pp. 62–63] (which treats the case $B = \mathbf{C}$).

(a) The functions holomorphic on a neighborhood of $\partial\mathbf{D}$ are contained in $\mathfrak{C}$, by Lemma D.1.23. The density follows from that the rational functions (case (b) below). Inverse-closedness was shown in Theorem 4.1.1(a) (use the Cayley transform).

(N.B. Although [GL73a, p. 44] suggests that the strongly measurable (*sil'no izmerimyj*) Wiener class would do, this strong measurability must mean Bochner measurability (with respect to the uniform operator norm), not measurability with respect to the strong topology, because (in (b)) the closure of the Fourier inverse transforms of rational functions (with poles off $\mathbf{R} \cup \{\infty\}$) is $\mathrm{L}^1(\mathbf{R};B)$, not $\mathrm{L}^1_{\mathrm{strong}}(\mathbf{R};B)$; a similar remark applies to (a).)

(b) 1° To show that the rational functions are dense in $\widehat{\mathrm{MTI}^{\mathrm{L}^1}}(B)$, we work as follows:

Now for $\widehat{\mathbb{E}} = E + \widehat{f} \in \widehat{\mathrm{MTI}^{\mathrm{L}^1}}(B)$ we may replace $\widehat{f}$ by $\sum_{k=1}^n T_k \hat{\chi}_{E_k}$, with $T_k \in \mathcal{B}(B)$ for all k, by the density of simple functions in L^1 (see Theorem B.3.11(a1)), and then replace each $\hat{\chi}_{E_k}$ by some rational function, by [CG81, pp. 62–63], to end up with a rational function close to $\widehat{\mathbb{E}}$, as required.

2° $\mathfrak{C}_\infty$: If $E \in \mathcal{BC}(U)$ and $f \in \mathrm{L}^1(\mathbf{R};\mathcal{BC}(U))$, then the above approximation provides (use Theorem B.3.11(a1) with $B := \mathcal{BC}(U)$) a rational function of form (5.5); conversely, if $\widehat{\mathbb{E}} \circ \phi_{\mathrm{Cayley}}$ is of the form (5.5), then $\widehat{\mathbb{E}} - E \in \mathcal{L}\mathrm{L}^1(\mathbf{R};\mathcal{BC}(U))$, by Lemma D.1.23. □

Notes

During the third quarter of the last century, Budjanu, Gohberg and several others developed an extensive theory on the factorization of $\mathrm{MTI}^{\mathrm{L}^1}$ maps and of maps in certain other TI classes for the purposes of singular integral equation theory. Some of the articles (mainly in Russian) also treat a more general factorization where $\pi_+\mathbb{E}\pi_+$ and $\mathbb{Y}$ need not be invertible (and the word "spectral" or "canonical" is dropped).

Soon this theory became popular also amidst control theorists, and today many articles in the infinite-dimensional control theory are based principally on spectral factorization. Also the factorization theory is still being rapidly developed.

In case $\dim U < \infty$, the most important results can be found in English in the book [CG81]; a somewhat more up-to-date book on the subject is [LS]. Both books also contain the case ("generalized spectral factorization") where the discrete-time ($\mathcal{Z}$-transformed) factors are allowed to be in $\mathcal{G}\mathrm{H}^2$ over the unit disc instead of $\mathcal{G}\mathrm{H}^\infty$ (cf. Example 8.4.13). These books also have extensive reference lists.

The applications of this theory given in this section are rather straightforward, and most of them have finite-dimensional analogies in the literature.

5.2 MTI spectral factorization ($\mathbb{E}, \mathbb{Y}, \mathbb{X} \in$ MTI)

Each Man is in his Spectre's power
Until the arrival of that hour,
When his Humanity awake,
And cast his Spectre into the Lake.
— William Blake (1757–1827)

The purpose of this section is to establish part "(II)⇔(III)" of the equivalence on page 21 for several systems and problems. We build up a series of lemmas on MTI ending up with two existence theorems on spectral factorization, both of which cover several MTI subclasses (and TI in the positive case).

Our strategy is the following: We first show that if a map $\mathbb{E} \in \mathrm{MTI}(U)$ has an invertible *Toeplitz operator* $\pi_+\mathbb{E}\pi_+$ (on $\mathrm{L}^2(\mathbf{R}_+;U)$), then also the discrete (atomic) part of the map has an invertible Toeplitz operator (Lemma 5.2.3(b)). Then we adopt several $\mathrm{MTI_d}$ factorization results to our setting and show that by factorizing first the discrete part of a MTI map using these results and then the "remainder $\mathrm{MTI}^{\mathrm{L}^1}$ part" by using the results of Section 5.1, one obtains a spectral factorization of the original map (Theorems 5.2.7 and 5.2.8).

We start by listing some basic facts on spectral factorization:

Lemma 5.2.1 (Spectral Factorization) *Let* $\mathbb{E} \in \mathrm{TI}(U)$.

(a) *Then* $\mathbb{E} \gg 0$ *iff* $\mathbb{E}$ *has the spectral factorization* $\mathbb{E} = \mathbb{X}^*\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$.

If this is the case, then all spectral factorizations of form $\mathbb{E} = \mathbb{Z}^*\mathbb{Z}$ *are given by* $\mathbb{E} = (L\mathbb{X})^*(L\mathbb{X})$, *where* $L \in \mathcal{GB}(U)$ *is unitary.*

Assume now that $\mathbb{E} \in \mathrm{TI}(U)$ *has a spectral factorization* $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ *for some* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{TIC}(U)$. *Then we have the following:*

(b) *The Toeplitz operator* $\pi_+\mathbb{E}\pi_+$ *is invertible on* $\pi_+\mathrm{L}^2$, *and* $\pi_+\mathbb{X}^{-1}\pi_+\mathbb{Y}^{-*}\pi_+$ *is its inverse.*

(c) *If* $\mathbb{E} \in \mathrm{TI}_{-\omega} \cap \mathrm{TI}_\omega$ *for some* $\omega > 0$, *then* $\mathbb{Y}, \mathbb{X} \in \mathcal{G}\mathrm{TIC}_{\exp}(U) \cap \mathrm{TIC}_{-\omega}(U)$.

(d) *If* $\mathbb{E} = \mathbb{E}^*$, *then* $\mathbb{Y} = \mathbb{X}^* S$ *for some* $S = S^* \in \mathcal{GB}(U)$*; thus, then* $\mathbb{E} = \mathbb{X}^* S\mathbb{X}$.

If, in addition, $\mathbb{E} \in \mathrm{TI}_\omega(U)$ *for some* $\omega \neq 0$, *then* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_{\exp}(U)$.

(e) *The map* $\mathbb{E}^{\mathrm{d}} := \text{Я}\mathbb{E}\text{Я} \in \mathrm{TI}(U)$ *has the* co-spectral factorization $\mathbb{E}^{\mathrm{d}} = \mathbb{X}^{\mathrm{d}}(\mathbb{Y}^{\mathrm{d}})^*$
(obviously, $\mathbb{X}^{\mathrm{d}}, \mathbb{Y}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}(U)$*).*

(f) *All spectral factorizations of* $\mathbb{E}$ *are given by* $\mathbb{E} = (L^{-*}\mathbb{Y})^*(L\mathbb{X})$, *where* $L \in \mathcal{GB}(U)$.

Theorems 5.2.7 and 5.2.8 below list classes that are closed w.r.t. spectral factorization and for which the converse of (b) holds (i.e., the Toeplitz operator $\pi_+\mathbb{E}\pi_+$ is invertible iff $\mathbb{E}$ has a spectral factorization).

The uniqueness result in (f) says that $X := \widehat{\mathbb{X}}(+\infty) \in \mathcal{GB}(U)$ can be chosen arbitrarily, and it determines $\mathbb{X}$, $\mathbb{Y}$ and $Y = EX^{-1} \in \mathcal{GB}(U)$ (we have $E, X, Y \in \mathcal{GB}(U)$, by Proposition 6.3.1(a3)).

Proof: (Part of this is given in Lemma 4.3 of [S98c].)

(a) This is Lemma 4.3(iv) of [S98c], but its proof needs clarification for the unseparable case: use Theorem 3.7, p. 54 and Theorem 3.4, p. 50 of [RR] to obtain $\mathbb{E} = \mathbb{Z}^*\mathbb{Z}$, where $\mathrm{Ran}(\mathbb{Z}) = \mathrm{L}^2(\mathbf{R}; U_2)$ for some closed $U_2 \subset U$ (because $\mathbb{Z}$ is coercive and "outer"), find $E \in \mathcal{GB}(U_2, U)$ (such an E exists, by Lemma 2.2.1(c4)) and set $\mathbb{X} := E\mathbb{Z}$.

(We would obtain an alternative, control-theoretic proof of (a) (with $\mathbb{D}^*J\mathbb{D}$ in place of $\mathbb{E}$, $\mathbb{D} \in$ TIC) as in Theorem 14.3.2.)

(b) Using the fact that $\pi_+\mathbb{Y}^{-*}\pi_+ = \pi_+\mathbb{Y}^{-*}$ etc. (by causality), one easily verifies that $\pi_+\mathbb{X}^{-1}\pi_+\mathbb{Y}^{-*}\pi_+$ is the inverse of $\pi_+\mathbb{Y}^*\mathbb{X}\pi_+$ on $\pi_+\mathrm{L}^2$ (i.e., that their product is π_+).

(c) Because $\mathbb{X} = \mathbb{Y}^{-*}\mathbb{E} \in \mathrm{TI}_{-\omega}$, we have $\mathbb{X}^{-1} \in \mathrm{TI}_{-\varepsilon'>0}$ for some $\varepsilon' > 0$, by Lemma 2.2.7, hence $\mathbb{X} \in \mathcal{G}\mathrm{TI}_{-\varepsilon} \cap \mathrm{TI}_{-\omega} \cap \mathrm{TIC} = \mathcal{G}\mathrm{TIC}_{-\varepsilon}(U,Y) \cap \mathrm{TIC}_{-\omega}(U,Y)$ for some $\varepsilon > 0$. The same holds for $\mathbb{Y}$, because $\mathbb{Y} = \mathbb{X}^{-*}\mathbb{E}^*$.

(d) Now $\mathbb{E} = \mathbb{Y}^*\mathbb{X} = \mathbb{E}^* = \mathbb{X}^*\mathbb{Y}$, hence $\mathbb{Y} = S\mathbb{X}$ for some $S \in \mathcal{GB}(U)$, by (f). The latter claim is obtained from Proposition 5.2.2 (because $\mathbb{E} \in \mathrm{TI}_\omega \Rightarrow \mathbb{E}^* \in \mathrm{TI}_{-\omega}$).

(e) Obviously $\mathbb{E}^{\mathrm{d}} = \mathbb{Y}^*\mathbb{X} \Rightarrow \mathbb{E} = \mathbb{X}^{\mathrm{d}}(\mathbb{Y}^{\mathrm{d}})^*$, and, by Lemma 2.2.3, $\mathbb{X} \in \mathcal{G}\mathrm{TIC} \Leftrightarrow \mathbb{X}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}$.

(f) Let $\mathbb{E} = \mathbb{Y}_0^*\mathbb{X}_0$ also be a spectral factorization. Then $L := \mathbb{Y}_0^{-*}\mathbb{Y}^* = \mathbb{X}_0\mathbb{X}^{-1} \in$ TIC and $L^* = \mathbb{Y}\mathbb{Y}_0^{-1} \in$ TIC, hence $L \in \mathcal{B}(U)$, by Lemma 2.1.7. Obviously, $L = \mathbb{X}_0\mathbb{X}^{-1}$ is invertible. □

In part (c) above, we stated that the spectral factorization of an "exponentially stable" map is exponentially stable. Below we shall prove this claim and the fact that the same holds with MTI or something analogous in place of TI:

Proposition 5.2.2 (Exponentially stable SpF) *Let $\mathcal{A} \subset$ TI be inverse-closed and adjoint-closed (cf. Theorem 4.1.1 and Lemma 4.1.3), and set $\widetilde{\mathcal{A}} := \mathcal{A} \cap$ TIC and*

$$\mathcal{A}_r := \{e^{r\cdot}\mathbb{E}e^{-r\cdot} \,\big|\, \mathbb{E} \in \mathcal{A}\}, \quad \widetilde{\mathcal{A}}_r := \{e^{r\cdot}\mathbb{D}e^{-r\cdot} \,\big|\, \mathbb{D} \in \widetilde{\mathcal{A}}\}, \quad r \in \mathbf{R}. \tag{5.8}$$

Assume that $\omega > 0$, $\mathbb{E} \in \mathcal{A}_{-\omega}(U) \cap \mathcal{A}_\omega(U)$ (e.g., $\mathbb{E} = \mathbb{E}^* \in \mathcal{A}_{-\omega}(U)$), and $\mathbb{E}$ has the spectral factorization $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$, where $\mathbb{Y}, \mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}(U)$.

Then $\mathbb{Y}, \mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}(U,Y) \cap \widetilde{\mathcal{A}}_{-\omega}(U,Y)$ for some $\varepsilon > 0$; in particular, $\mathbb{X}^{\pm 1}$ and $\mathbb{Y}^{\pm 1}$ are exponentially stable.

In particular, if $\mathbb{E} = \mathbb{E}^* \in$ MTI is "exponentially MTI", i.e., $\mathbb{E} \in \mathrm{MTI}_{-\varepsilon}$ for some $\varepsilon > 0$, then its (possible) spectral factors are "exponentially MTI".

Proof: Because $\mathbb{X} = \mathbb{Y}^{-*}\mathbb{E} \in \mathcal{A}_{-\omega}$ (recall that $\{\mathbb{F}^* \,\big|\, \mathbb{F} \in \mathcal{A}_\omega\} = \mathcal{A}_{-\omega}$, by Lemma 4.1.3(b)), we have $\mathbb{X}^{-1} \in \mathcal{A}_{-\varepsilon}$ for some $\varepsilon > 0$, by Lemma 2.2.7. But $\mathcal{A}_r \cap \widetilde{\mathcal{A}} \subset \widetilde{\mathcal{A}}_r$ for $r \in \mathbf{R}$, hence $\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}(U,Y) \cap \widetilde{\mathcal{A}}_{-\omega}(U,Y)$. Similarly, $\mathbb{Y} = \mathbb{X}^{-*}\mathbb{E}^* \in \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}(U,Y) \cap \widetilde{\mathcal{A}}_{-\omega}(U,Y)$. □

Now we turn our attention to MTI maps. Our first task is to show that the invertibility of the Toeplitz operator of $\mathbb{E} \in \mathrm{MTI}$ implies that of its discrete part:

Lemma 5.2.3 (MTI Toeplitz) *For $\mathbb{E} \in \mathrm{MTI}$ ($= \mathrm{MTI_d} + \mathrm{L}^1*$) we write $\mathbb{E}_\mathrm{d} := \Pi(\mathbb{E})$ for the discrete part (of the form $\sum_{k=1}^\infty L_k\delta(\cdot - t_k)*$)*

Let $\mathbb{E} \in \mathrm{MTI}(U,Y)$.

- *(a) If $\mathbb{E} \in \mathcal{G}\mathrm{TI}$ then $\mathbb{E}_\mathrm{d} \in \mathcal{G}\mathrm{MTI_d}$.*
- *(b) If $\mathbb{E} \in \mathrm{MTI}(U,Y)$ and $\pi_+\mathbb{E}\pi_+ \in \mathcal{GB}(\mathrm{L}^2)$, then $\pi_+\mathbb{E}_\mathrm{d}\pi_+ \in \mathcal{GB}(\mathrm{L}^2)$ (and $\mathbb{E}, \mathbb{E}_\mathrm{d} \in \mathcal{G}\mathrm{TI}$)*
- *(c) If $\mathbb{E} \in \mathrm{MTI}(U)$ and $\mathbb{E} \geq 0$, then $\mathbb{E}_\mathrm{d} \geq 0$.*

Proof: (a) This is contained in Theorem 4.1.1.

(b) We will prove that if the Toeplitz operator $\mathbf{T}_\mathbb{E} := \pi_+\mathbb{E}\pi_+$ of $\mathbb{E} \in \mathrm{MTI}$ is coercive, i.e., $\|\mathbf{T}_\mathbb{E} u\| \geq \varepsilon u$ for all $u \in \pi_+\mathrm{L}^2$, then so is $\mathbf{T}_{\mathbb{E}_\mathrm{d}}$ (with the same $\varepsilon > 0$).

Claim (b) follows from this, because T is invertible iff T and T^* are coercive, by Lemma A.3.1(c3). (The last two claims follow from (a) and Lemma 2.2.2(a1).)

Define $\mathbb{F} \in \mathrm{MTI_d}$ and $f \in \mathrm{L}^1$ by $\mathbb{F} := \mathbb{E}_\mathrm{d}$ and $f* := \mathbb{E} - \mathbb{F}$ (so that $\mathbb{E} = \mathbb{F}u + f*u$ for all $u \in \mathrm{L}^2$).

Let $\delta > 0$ be arbitrary. Let $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ be otherwise arbitrary but $\|u\|_2 = 1$. By Lemma D.1.11(b), there is $T_\delta > 0$ s.t.

$$T > T_\delta \Longrightarrow \|\widehat{f}(\cdot)\widehat{u}(\cdot - iT)\|_2 < \delta. \tag{5.9}$$

Because $\widehat{\mathbb{F}}(i\cdot)$ is almost periodic, by Lemma C.1.2(h2), there is $T > T_\delta$ s.t. $\|\widehat{\mathbb{F}}(it) - \widehat{\mathbb{F}}(i(t-T))\| < \delta$ for all $t \in \mathbf{R}$. Therefore (recall that $\widehat{\pi_+}\widehat{u} := \widehat{\pi_+ u}$, hence $\|\widehat{\pi_+}\| = \|\pi_+\| = 1$; note also that $\|e^{iT\cdot}u\|_2 = \|u\|_2 = 1$ and $\mathcal{L}(e^{iT\cdot}u) = \widehat{u}(\cdot - iT)$)

$$\begin{aligned}
\|\pi_+\mathbb{F}u\|_2 = \;& \|\pi_+ e^{iT\cdot}\mathbb{F}u\|_2 && = \|\widehat{\pi_+}\mathcal{L}(e^{iT\cdot}\mathbb{F}u)\|_2 \\
= \;& \|\widehat{\pi_+}(\widehat{\mathbb{F}u})(\cdot - iT)\|_2 && = \|\widehat{\pi_+}\widehat{\mathbb{F}}(\cdot - iT)\widehat{u}(\cdot - iT)\|_2 \\
\geq \;& \|\widehat{\pi_+}\widehat{\mathbb{F}}\widehat{u}(\cdot - iT)\|_2 - \|\widehat{\pi_+}[\widehat{\mathbb{F}}(\cdot - iT) - \widehat{\mathbb{F}}]\widehat{u}(\cdot - iT)\|_2 \\
\geq \;& \|\widehat{\pi_+}\widehat{\mathbb{F}}\widehat{u}(\cdot - iT)\|_2 - \|\delta\widehat{u}\|_2 && \geq \|\widehat{\pi_+}\widehat{\mathbb{E}}\widehat{u}(\cdot - iT)\|_2 - \|\widehat{\pi_+}\widehat{f}\widehat{u}(\cdot - iT)\|_2 - \delta \\
\geq \;& \|\widehat{\pi_+}\widehat{\mathbb{E}}\widehat{u}(\cdot - iT)\|_2 - \delta - \delta && = \|\pi_+\mathbb{E}e^{iT\cdot}u\|_2 - 2\delta \\
= \;& \|\pi_+\mathbb{E}\pi_+e^{iT\cdot}u\|_2 - 2\delta && = \varepsilon - 2\delta.
\end{aligned}$$

Because $\delta > 0$ was arbitrary, $\|\pi_+\mathbb{F}u\|_2 \geq \varepsilon = \varepsilon\|u\|_2$.

(c) To obtain a contradiction, assume that $\mathbb{E} \in \mathrm{MTI}(U)$, $\mathbb{E} \geq 0$ and $\mathbb{E}_\mathrm{d} \not\geq 0$. Then $\widehat{\mathbb{E}} \geq 0$ on $i\mathbf{R}$ but there is $u_0 \in U$ s.t. $g := \langle u_0, \widehat{\mathbb{E}}_\mathrm{d}(\cdot)u_0\rangle_U$ satisfies $g(ir_0) \not\geq 0$ for some $r_0 \in \mathbf{R}$.

Set $\delta := d(g(ir_0), [0,+\infty))0$. Let $f \in \mathrm{L}^1(\mathbf{R};\mathcal{B}(U))$ be the one for which $\mathbb{E} = \mathbb{E}_\mathrm{d} + f*$. By Lemma D.1.11(b), there is $T_\delta > |r_0|$ s.t. $\|\widehat{f}(ir)\|_{\mathcal{B}(U)} < \delta$ for $|r| > T_\delta$.

Because $\widehat{\mathbb{E}}_{\rm d}(i\cdot)$ is almost periodic, by Lemma C.1.2(h2), there is $T > 2T_\delta$ s.t. $\|\widehat{\mathbb{E}}_{\rm d}(it) - \widehat{\mathbb{E}}_{\rm d}(i(t-T))\| < \delta/2$ for all $t \in \mathbf{R}$. Then the distance from

$$\langle u_0, \widehat{\mathbb{E}}(ir_0+T)u_0\rangle = g(ir_0+T) + \langle u_0, \widehat{f}(ir_0+T)u_0\rangle \tag{5.10}$$

to $[0,+\infty)$ is greater than $\delta - \delta/2 > 0$, a contradiction, as required. □

Next we state the spectral factorization results of Yuri Karlovich [Karlovich91] and others for matrix-valued atomic measures:

Lemma 5.2.4 *Let* $\mathbb{E} = \mathrm{MTI}_{\rm d}(\mathbf{C}^n)$. *Then* $\mathbb{E}$ *has a spectral factorization* $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ *with* $\mathbb{Y}, \mathbb{X} \in \mathcal{G}\mathrm{MTIC}_{\rm d}(\mathbf{C}^n)$ *iff* $\pi_+\mathbb{E}\pi_+ \in \mathcal{G}(\pi_+\mathrm{L}^2)$.

Moreover, if $\mathrm{supp}_{\rm d}(\mathbb{E}) \subset \mathbf{S}$, *where* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$, *then* $\mathrm{supp}_{\rm d}(\mathbb{X}), \mathrm{supp}_{\rm d}(\mathbb{Y}) \subset \mathbf{S}$.

In particular, if the atoms of $\widehat{\mathbb{E}}$ are at points nT, $n \in \mathbf{Z}$ for some $T > 0$, then so are those of $\mathbb{X}$.

(Note from Lemma 5.2.1(f) that the factorization is unique modulo a multiplicative constant.)

Proof: The first claim is a rephrasement of the equivalence "4)⇔6)" of Theorem 7 of [Karlovich93] (use the fact that $\mathbb{E}\pi_+ + \pi_-$ is invertible on $\mathrm{L}^2(\mathbf{R};\mathbf{C}^n)$ iff $\pi_+\mathbb{E}\pi_+$ is invertible on $\mathrm{L}^2(\mathbf{R}_+;\mathbf{C}^n)$, by Lemma A.1.1(b1)&(b2)).

The $\mathbf{S}$-claim follows from [RSW, Theorem 6.1]. □

We shall use the following lemma in the positive case:

Lemma 5.2.5 (MTI$_{\rm d}$ SpF when $\|I - \mathbb{E}\| < 1$) *Assume that* $\mathbb{E} \in \mathrm{MTI}_{\rm d}(U)$ *and* $\|I - \mathbb{E}\|_{\mathrm{TI}(U)} < 1$. *Then* $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ *for some* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{MTIC}_{\rm d}(U)$.

Moreover, if $\mathrm{supp}_{\rm d}(\mathbb{E}) \subset \mathbf{S}$, *where* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$, *then* $\mathrm{supp}_{\rm d}(\mathbb{X}), \mathrm{supp}_{\rm d}(\mathbb{Y}) \subset \mathbf{S}$.

Proof: 1° $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ *for some* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{MTIC}_{\rm d}(U)$*:* Condition $\|I - \mathbb{E}\|_{\mathrm{TI}(U)} < 1$ is equivalent to condition $\sup_{t\in\mathbf{R}} \|I - \widehat{\mathbb{E}}(it)\|_{\mathcal{B}(U)} < 1$, by Theorem 3.1.3(d) and Theorem 2.6.4(e1).

By Theorem I of [BR], it follows that $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ for some $\mathbb{X}, \mathbb{Y} \in \mathrm{MTIC}_{\rm d}(U)$.

2° $\mathrm{supp}_{\rm d}(\mathbb{X}), \mathrm{supp}_{\rm d}(\mathbb{Y}) \subset \mathbf{S}$*:* Assume that $\mathrm{supp}_{\rm d}(\mathbb{E}) \subset \mathbf{S}$ and $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$. We shall examine the proof of Theorem I so as to show that $\mathrm{supp}_{\rm d}(\mathbb{X}), \mathrm{supp}_{\rm d}(\mathbb{Y}) \subset \mathbf{S}$. (Note that the Fourier transform of [BR] has an extra reflection; this is just a matter of notation.)

By Lemma C.1.2(f5), we have $\|A - I\|_{\mathcal{B}(\overline{\mathrm{AP}})} \le \|A - I\|_\infty < 1$. From Lemma C.1.2(f4) we observe that $\|\Pi_{[0,+\infty)}\|_{\mathcal{B}(\overline{\mathrm{AP}})} \le 1$. Consequently,

$$\mathbb{X}_+\phi := \sum_{k\in\mathbf{N}} (\Pi_{[0,+\infty)}(I-A)\Pi_{[0,+\infty)})^k\phi \tag{5.11}$$

converges in $\overline{\mathrm{AP}}(\mathbf{R};U)$, for all $\phi \in U$, where $\overline{\mathrm{AP}}$ denotes the Besicovich space (see Lemma C.1.2(f4)). Since the atoms of $(\Pi_{[0,+\infty)}(I-A)\Pi_{[0,+\infty)})^k$ belong to $\mathbf{S} \cup \{0\} = \mathbf{S}$, for each $k \in \mathbf{N}$, also the atoms of $\mathbb{X}_+\phi$ belong to $\mathbf{S}$. Because $\phi \in U$

was arbitrary, the atoms of $\mathbb{X}_+$ belong to $\mathbf{S}$. By p. 18 of [BR], $\mathbb{X}_+ \in \widehat{\mathrm{MTIC}_\mathrm{d}}$, hence $\mathbb{X}_+ \in \widehat{\mathrm{MTIC}_{\mathrm{d},\mathbf{S}}}$,

By analogous proofs, one shows that $\mathbb{Y}_+ \in \widehat{\mathrm{MTIC}_{\mathrm{d},\mathbf{S}}}$ and $\mathbb{X}_- \in \widehat{\mathrm{MTIC}_{\mathrm{d},\mathbf{S}}}^*$. From the formula $A = (I+\mathbb{X}_-)\mathbb{Y}_+$ of p. 51 of [GL-Identity] (which is the basis of the proof of [BR]), we observe that $\mathbb{Y}_+ = \mathbb{X}$, $I+\mathbb{X}_- = \mathbb{Y}^*$, where $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ is a spectral factorization of $\mathbb{E}$. This completes the proof (recall Lemma 5.2.1(f)). □

Indeed, the positive case is obtained as a corollary:

Corollary 5.2.6 *Let* $\mathbb{E} \in \mathrm{MTI}_\mathrm{d}(U)$. *Then* $\mathbb{E} \gg 0 \Leftrightarrow \mathbb{E} = \mathbb{X}^*\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{MTIC}_\mathrm{d}(U)$.

Moreover, if $\mathrm{supp}_\mathrm{d}(\mathbb{E}) \subset \mathbf{S}$, *where* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$, *then* $\mathrm{supp}_\mathrm{d}(\mathbb{X}), \mathrm{supp}_\mathrm{d}(\mathbb{Y}) \subset \mathbf{S}$.

Proof: Set $\mathbb{F} := \mathbb{E}/\|\mathbb{E}\|_{\mathrm{TI}} \in \mathrm{MTI}_\mathrm{d}(U)$. Then $I \geq \mathbb{F} \gg 0$, hence $\|I-\mathbb{F}\| < 1$, by Lemma A.3.1(b9). By Lemma 5.2.5, it follows that $\mathbb{F} = \mathbb{Y}^*\mathbb{Z}$ for some $\mathbb{Y}, \mathbb{Z} \in \mathrm{MTIC}_{\mathrm{d},\mathbf{S}}$. Use Lemma 5.2.1(a)&(f) to observe that $\mathbb{F} = \widetilde{\mathbb{X}}^*\widetilde{\mathbb{X}}$ for some $\widetilde{\mathbb{X}} \in \mathrm{MTIC}_{\mathrm{d},\mathbf{S}}$. Set $\mathbb{X} := \|\mathbb{E}\|_{\mathrm{TI}}^{1/2}\widetilde{\mathbb{X}} \in \mathrm{MTIC}_{\mathrm{d},\mathbf{S}}$. □

Now we can combine the above results to two theorems:

Theorem 5.2.7 (MTI spectral factorization) *Let* $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$, *and let either (1.) or (2.) hold, where*

(1.) $\mathcal{A}$ *is one of the classes* $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}$, $\mathrm{MTI}_{T\mathbf{Z}}$, $\mathrm{MTI}^{\mathcal{BC}}_{T\mathbf{Z}}$, $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$, *and* $\mathrm{MTI}^{\mathcal{BC}}_{\mathrm{d},T\mathbf{Z}}$;

(2.) $\dim U < \infty$ *and* $\mathcal{A}$ *is one of the classes* MTI, MTI_d, $\mathrm{MTI}_\mathbf{S}$, *and* $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$.

Let $\mathbb{E} \in \mathcal{A}(U)$, *and set* $\widetilde{\mathcal{A}} := \mathcal{A} \cap \mathrm{TIC}$. *Then the Toeplitz operator* $\mathbf{T}_\mathbb{E} := \pi_+\mathbb{E}\pi_+ \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))$ *is invertible iff* $\mathbb{E}$ *has a spectral factorization*

$$\mathbb{E} = \mathbb{Y}^*\mathbb{X}, \text{ where } \mathbb{X}, \mathbb{Y} \in \mathcal{G}\widetilde{\mathcal{A}}(U). \tag{5.12}$$

Let $\mathbb{E}$, $\mathbb{X}$ *and* $\mathbb{Y}$ *be as above. Then* $\mathbb{E}_\mathrm{d} = \mathbb{Y}_\mathrm{d}^*\mathbb{X}_\mathrm{d}$ *is also a spectral factorization (in* MTI_d*), where* $()_\mathrm{d}$ *denotes the discrete part. Moreover, if* $\mathrm{supp}_\mathrm{d}(\mathbb{E}) \subset \mathbf{S}$, *then* $\mathrm{supp}_\mathrm{d}(\mathbb{X}) \subset \mathbf{S}$ *and* $\mathrm{supp}_\mathrm{d}(\mathbb{Y}) \subset \mathbf{S}$.

If, in addition, $\omega > 0$ *and* $\mathbb{E} \in \mathcal{A}_{-\omega} \cap \mathcal{A}_\omega$, *then* $\mathbb{Y}, \mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}(U,Y) \cap \widetilde{\mathcal{A}}_{-\omega}(U,Y)$ *for some* $\varepsilon > 0$; *in particular,* $\mathbb{X}^{\pm 1}$ *and* $\mathbb{Y}^{\pm 1}$ *are exponentially stable.*

If we merely know that $\mathbb{E} \in \mathrm{TI}$, then it is no longer guaranteed that the "canonical factors" $\mathbb{X}$ and $\mathbb{Y}$ are stable, see the notes on p. 148 for details.

However, positive results can be given also for certain other cases, see Theorems 5.2.8 and 9.2.14.

Proof: 1° *The discrete part* $\mathbb{E}_\mathrm{d} = \mathbb{W}^*\mathbb{Z}$:
If (5.12) holds, then $\mathbf{T}_\mathbb{E}^{-1} = \mathbf{T}_{\mathbb{X}^{-1}}\mathbf{T}_{\mathbb{Y}^{-1}}$, so assume that $\mathbf{T}_\mathbb{E}$ is invertible. Then so is $\mathbf{T}_{\mathbb{E}_\mathrm{d}}$, by Lemma 5.2.3(b), so we can factor $\mathbb{E}_\mathrm{d}$ as $\mathbb{W}^*\mathbb{Z}$, $\mathbb{W}, \mathbb{Z} \in \mathcal{G}\mathrm{MTIC}_\mathrm{d}$ ($\mathbb{Z}, \mathbb{W} \in \mathcal{G}\mathrm{MTIC}^{\mathcal{BC}}_\mathrm{d}$, if $\mathcal{A} = \mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}$, $\mathcal{A} = \mathrm{MTI}^{\mathcal{BC}}_{T\mathbf{Z}}$, or $\mathcal{A} = \mathrm{MTI}^{\mathcal{BC}}_{\mathrm{d},T\mathbf{Z}}$, because

in those cases $\mathbb{E}_{\mathrm{d}} \in \mathrm{MTI}_{\mathrm{d}}^{\mathcal{BC}}$) by Corollary 5.1.4 in case (1.) and by Lemma 5.2.4 in case (2.).

Moreover, if $\mathrm{supp}_{\mathrm{d}}(\mathbb{E}) \subset \mathbf{S}$, then $\mathrm{supp}_{\mathrm{d}}(\mathbb{Z}), \mathrm{supp}_{\mathrm{d}}(\mathbb{W}) \subset \mathbf{S}$, by Lemma 5.2.4 in case (2.); in case (1.) either $\mathbf{S} = \{0\}$, in which case $\mathbb{E}, \mathbb{Z}, \mathbb{W} \in \mathcal{B}(U)$, or $\mathbf{S}$ contains a set of the form $T'\mathbf{Z}$ for some $T' > 0$, in which case $\mathrm{supp}_{\mathrm{d}}(\mathbb{Z}), \mathrm{supp}_{\mathrm{d}}(\mathbb{W}) \subset T'\mathbf{Z} \subset \mathbf{S}$, by Corollary 5.1.4.

If $\mathbb{E} - \mathbb{E}_{\mathrm{d}} = 0$, then we can take $\mathbb{X} := \mathbb{Z}$, $\mathbb{Y} := \mathbb{W}$ to obtain the required result, but in the general case we proceed as follows.

2° *The absolutely continuous part* $\mathbb{E}_{\mathrm{ac}} := \mathbb{E} - \mathbb{E}_{\mathrm{d}}$*:*
Because L^1* is an ideal of MTI, and $\mathbb{E}_{\mathrm{ac}} \in \mathrm{L}^1*$, we have $\mathbb{W}^{-*}\mathbb{E}_{\mathrm{ac}}\mathbb{Z}^{-1} = g*$ for some $g \in \mathrm{L}^1$, hence

$$\mathbb{W}^{-*}(\mathbb{E}_{\mathrm{d}} + \mathbb{E}_{\mathrm{ac}})\mathbb{Z}^{-1} = I + g \tag{5.13}$$

can be factorized as $\mathbb{T}^*\mathbb{S}$ with $\mathbb{T}, \mathbb{S} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1}$ (with $\mathbb{T}, \mathbb{S} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1,\mathcal{BC}}$, if ($\mathbb{E}_{\mathrm{ac}}$ and hence) $g* \in \mathrm{L}^1(\mathbf{R}; \mathcal{BC}(U))*$), by Theorem 5.1.2 and the invertibility of $\mathbf{T}_{\mathbb{W}^{-*}\mathbb{E}\mathbb{Z}^{-1}}$, which follows from Lemma 2.2.2(b).

Thus, we have $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$, where $\mathbb{Y} := \mathbb{T}\mathbb{W} \in \mathcal{G}\mathrm{MTIC}$ and $\mathbb{X} := \mathbb{S}\mathbb{Z} \in \mathcal{G}\mathrm{MTIC}$. Moreover, $\mathbb{Y}_{\mathrm{d}} = \mathbb{T}_{\mathrm{d}}\mathbb{W}_{\mathrm{d}} = \mathbb{W}_{\mathrm{d}} = \mathbb{W}$ and $\mathbb{X}_{\mathrm{d}} = \mathbb{S}_{\mathrm{d}}\mathbb{Z}_{\mathrm{d}} = \mathbb{Z}_{\mathrm{d}} = \mathbb{Z}$, hence $\mathbb{E}_{\mathrm{d}} = \mathbb{Y}_{\mathrm{d}}^*\mathbb{X}_{\mathrm{d}}$.

The last paragraph of the theorem follows from Proposition 5.2.2. □

The assumption that the input space U must be finite-dimensional, is probably superfluous even in case (2.); there may be expected results in this direction in the near future. We have written this work based more on hypotheses (see Hypothesis 8.4.7) than on classes, in order for the reader to easily incorporate any new factorization results to this work.

In the uniformly positive case (which can be used for minimization problems, positive and bounded real lemmas and analogous), we do not need any dimensionality restrictions:

Theorem 5.2.8 (Positive MTI SpF) *Let U be a Hilbert space, let $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$, and let $\mathcal{A}$ be one of the classes* TI, MTI, $\mathrm{MTI}_{\mathrm{d}}$, $\mathrm{MTI}_{\mathbf{S}}$, $\mathrm{MTI}_{\mathrm{d},\mathbf{S}}$, $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}^{\mathrm{L}^1,\mathcal{BC}}$, $\mathrm{MTI}_{T\mathbf{Z}}$, $\mathrm{MTI}_{T\mathbf{Z}}^{\mathcal{BC}}$, $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}$, *and* $\mathrm{MTI}_{\mathrm{d},T\mathbf{Z}}^{\mathcal{BC}}$.

Let $\mathbb{E} \in \mathcal{A}(U)$, and set $\widetilde{\mathcal{A}} := \mathcal{A} \cap$ TIC. *Then $\mathbb{E} \gg 0$ iff $\mathbb{E}$ has a spectral factorization*

$$\mathbb{E} = \mathbb{X}^*\mathbb{X}, \text{ where } \mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}(U). \tag{5.14}$$

Let $\mathbb{E}$ and $\mathbb{X}$ be as above. Then $\mathbb{E}_{\mathrm{d}} = \mathbb{X}_{\mathrm{d}}^*\mathbb{X}_{\mathrm{d}}$ is also a spectral factorization (in $\mathrm{MTI}_{\mathrm{d}}$*), where $\cdot_{\mathrm{d}}$ denotes the discrete part. Moreover, if* $\mathrm{supp}_{\mathrm{d}}(\mathbb{E}) \subset \mathbf{S}$, *then* $\mathrm{supp}_{\mathrm{d}}(\mathbb{X}) \subset \mathbf{S}$.

If, in addition, $\omega > 0$ and $\mathbb{E} \in \mathcal{A}_{-\omega}$, then $\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_{-\varepsilon}(U,Y) \cap \widetilde{\mathcal{A}}_{-\omega}(U,Y)$ for some $\varepsilon > 0$.

If $\mathbb{E} \geq 0$, then $\mathbb{E} \gg 0$ iff $\mathbb{E}$ is invertible; equivalently, iff $\pi_+\mathbb{E}\pi_+$ is invertible (by Lemma A.3.1(b1) and Lemma 2.2.2(d)).

Proof: 1° If (5.14) holds, then, $\mathbb{E} \gg 0$, by, e.g., Lemma 5.2.1(a).

2° Let $\mathbb{E} \gg 0$. Then $\pi_+\mathbb{E}\pi_+ \gg 0$, by Lemma 2.2.2(d), hence the claims follow from Theorem 5.2.7 (because $\mathbb{E} = \mathbb{X}^* S\mathbb{X}$, by Lemma 5.2.1(d), and clearly $S \gg 0$, so $\mathbb{E} = (S^{1/2}\mathbb{X})^*(S^{1/2}\mathbb{X})$, $\mathbb{X} \in \widetilde{\mathcal{A}}$) when we note that the assumption $\dim U < \infty$ in the proof of Theorem 5.2.7 can be removed in this positive case, by using, in case (2.), Corollary 5.2.6 instead of Lemma 5.2.4 and Lemma 5.2.3(c) instead of (b).

3° The exponential (measure) stability is obtained as in Theorem 5.2.7. □

We have presented our spectral factorization results for several subclasses of MTI in order to get more specific information on the smoothness of the spectral factors. For example, the uniform half-plane-regularity of $\mathrm{MTIC}^{\mathrm{L}^1}$ maps allows us to even simplify the Riccati equations, provided that the Popov operator belongs to $\mathrm{MTI}^{\mathrm{L}^1}$.

Notes

Except for (e) and the $\mathrm{TIC}_{\mathrm{exp}}$ claims, Lemma 5.2.1 is contained in Lemma 4.3 of [S98c]. Proposition 5.2.2 was established in the finite-dimensional positive $\mathrm{MTI}_{T\mathbf{Z}}$ case in Lemma 3.3 of [Winkin], by using analytic extensions.

For finite-dimensional U, most of Theorem 5.2.8 (for $\mathrm{MTI}^{\mathrm{L}^1}$, $\mathrm{MTI}_{\mathrm{d}}$ and $\mathrm{MTI}_{T\mathbf{Z}}$) is contained in Theorem 3.1M of [Winkin] (and in [CW99]). Also Lemma 5.2.3(c) and our strategy to start with a factorization of the discrete part are from [Winkin]. (The space U is assumed to be finite-dimensional and $\mathbb{E}$ is assumed to be uniformly positive everywhere in [Winkin].)

As obvious from the proof, Lemma 5.2.4 is essentially contained in [RSW] (originating in [Karlovich91] and the joint articles of Yuri Karlovich and Ilya Spitkovsky et al.).

Lemma 5.2.5 and Corollary 5.2.6 are essentially contained in [BR] except for the claims on **S**. Our proofs use the ideas of the proof of Theorem 6.1 of [RSW]. The exact assumption in [BR] is "$|\langle\widehat{\mathbb{E}}u_0, u_0\rangle| \ge \varepsilon\|u_0\|_U^2$", hence slightly more general than "$\mathbb{E} \gg 0$" (but does not allow for, e.g., $\mathbb{E} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$). See [Karlovich93] (particularly Theorems 14 and 15) for similar (not analogous) factorization results for [semi-]almost periodic functions (with values in $\mathbf{C}^{n\times n}$), for further equivalent conditions and for factorizations of functions with non-invertible Toeplitz operators.

For any $\mathbb{E} \in \mathrm{TI}$ s.t. $\mathbb{E} \ge 0$, the invertibility of the Toeplitz operator is equivalent to the existence of a spectral factorization, by Theorem 5.2.8. For a general $\mathbb{E} \in \mathrm{TI}$ s.t. $\pi_+\mathbb{E}\pi_+$ is invertible, we need an extra assumption, as in Theorems 5.2.7, Theorem 9.2.14 or 9.14.4.

Indeed, for a general $\mathbb{E} \in \mathrm{TI}(\mathbf{C}^n)$ s.t. $\pi_+\mathbb{E}\pi_+$ is invertible, we only know the existence of a "Generalized canonical factorization" (see Section 9.15, [CG81] or [LS]) where $\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ a.e. on $i\mathbf{R}$ and the Cayley transforms of $\widehat{\mathbb{X}}, \widehat{\mathbb{Y}} \in \mathrm{H}(\mathbf{C}^+;\mathbf{C}^{n\times n})$ are invertible in H^2 over the unit disc. By Example 8.4.13, these (unique modulo a multiplicative constant) generalized factors need not be well-posed (i.e., $\widehat{\mathbb{X}}$ and $\widehat{\mathbb{Y}}$ may be unbounded at infinity) unless $n = 1$.

In and below Theorem 9.14.6, we extend the above results on generalized canonical factorization for infinite-dimensional U (assuming that $\mathbb{E}$ is the Toeplitz operator corresponding to some cost function; this is the only case for which such factorizations are needed in control theory). In this weaker result, $\widehat{\mathbb{X}}(\bar{\cdot})^*$ and $\widehat{\mathbb{X}}^{-1}$ are only known to be $\mathrm{H}^2_{\mathrm{strong}}$ over the unit disc ($\mathrm{H}^2_{\mathrm{strong}}(\mathbf{D}; \mathcal{B}(U))$).

Obviously, the Cayley transform makes spectral factorization of $\mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U))$ maps equivalent to the spectral factorization of $\mathrm{H}^\infty(\mathbf{D}; \mathcal{B}(U))$ maps. Unlike in continuous time, the generalized discrete-time canonical factors are always well-posed, by Theorem 9.14.6, but still not necessarily stable, as noted in Example 8.4.13. See also the notes on pp. 141 and 543.

Part II

Continuous-Time Control Theory

Chapter 6

Well-Posed Linear Systems (WPLS)

It must be remembered that there is nothing more difficult to plan, more doubtful of success, nor more dangerous to manage, than the creation of a new system. For the initiator has the enmity of all who would profit by the preservation of the old institutions and merely lukewarm defenders in those who would gain by the new ones.

— Niccolo Machiavelli (1469–1527)

In this chapter, we shall present a theory on most aspects WPLSs; only dynamic stabilization is left for the next chapter.

Section 6.1 treats the basic properties of WPLSs: stability, realization theory, generators and dual systems.

In Section 6.2, we list the basic facts on regularity, which means the existence of a feedthrough operator in a weak sense. This leads to generalizations of classical state-space and frequency-space formulae for the maps that comprise the system, e.g., of equations $x' = Ax + Bu,\ y = Cx + Du$ for the state and output and equation $\widehat{\mathbb{D}}(s) = D + C(s-A)^{-1}B$ for the transfer function are established in a weak sense using the Weiss extension of C.

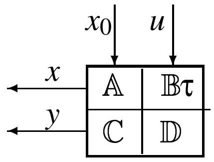
Not all WPLSs have a feedthrough operator D, but there are always ways to define a *compatible pair* (C_{ext}, D) s.t. the above formulae become valid; however this theory is less fruitful and hence less important than that of regular WPLSs. Compatible pairs are studied in Section 6.3, where we also present further results on regularity, [strong or weak] L^p impulse responses and H^p transfer functions, relations between a WPLS and its generators, and reachability and observability.

In Sections 6.4 and 6.5, we define and study coprime, spectral and lossless factorizations. The importance of these factorizations is due the fact that in many control problems, the existence of a (nonsingular) solution is equivalent to the equivalence of such a factorization; also dynamic feedback is intimately connected to coprime factorization. We also present two weak forms of coprimeness that are useful in infinite-dimensional settings, the weaker of them being invariant under (inverse) discretization and hence allowing us to reduce several results to the simpler discrete-time theory.

Sections 6.6 and 6.7 treat state feedback, output injection and static output feedback. In Section 6.8, we study systems whose semigroup is smoothing (e.g., $\mathbb{A}Bu_0 \in \mathrm{L}^p_{\text{loc}}(\mathbf{R}_+; H)$ for all $u_0 \in U$).

In Section 6.9, we show that a transfer function $\widehat{\mathbb{D}}$ has a realization with bounded B iff $\widehat{\mathbb{D}} - \widehat{\mathbb{D}}(+\infty) \in \mathrm{H}^2_{\mathrm{strong}}$ over some right half-plane. We also establish analogous results for realizations with bounded C and for Pritchard–Salamon realizations.

To get a shorter introduction to WPLSs, just read the parts of Sections 6.1, 6.2, 6.4 and 6.6 that seem interesting. Throughout this chapter, the letters H, U, W, Y, Z, H_k, U_k and Y_k ($k \in \mathbf{N}$) denote arbitrary (complex) Hilbert spaces.



$$x = \mathbb{A}x_0 + \mathbb{B}\tau u$$
$$y = \mathbb{C}x_0 + \mathbb{D}u$$

Figure 6.1: Input/state/output diagram of the system Σ

6.1 WPLS theory

Foolproof systems don't take into account the ingenuity of fools.
— Gene Brown

We start by defining WPLSs; see p. 22 for a motivation. Our formulation follows that of Olof Staffans. The correspondence to Weiss' notation is explained on pp. 158 and 166. Recall also that $\mathrm{L}^2_\omega = \mathrm{e}^{\omega\cdot}\mathrm{L}^2 = \{f \,|\, \mathrm{e}^{-\omega\cdot} f \in \mathrm{L}^2\}$.

Definition 6.1.1 (WPLS and stability) *Let* $\omega \in \mathbf{R}$. *An* ω-stable well-posed linear system on (U,H,Y) *is a quadruple*[1] $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$*, where* $\mathbb{A}$*,* $\mathbb{B}$*,* $\mathbb{C}$*, and* $\mathbb{D}$ *are bounded linear operators of the following type:*

1. $\mathbb{A}(t)\colon H \to H$ *is a strongly continuous semigroup of bounded linear operators on H satisfying* $\sup_{t\in\mathbf{R}_+} \|\mathrm{e}^{-\omega t}\mathbb{A}(t)\| < \infty$*;*

2. $\mathbb{B}\colon \mathrm{L}^2_\omega(\mathbf{R};U) \to H$ *satisfies* $\mathbb{A}(t)\mathbb{B}u = \mathbb{B}\tau(t)\pi_- u$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$ *and* $t \in \mathbf{R}_+$*;*

3. $\mathbb{C}\colon H \to \mathrm{L}^2_\omega(\mathbf{R};Y)$ *satisfies* $\mathbb{C}\mathbb{A}(t)x = \pi_+\tau(t)\mathbb{C}x$ *for all* $x \in H$ *and* $t \in \mathbf{R}_+$*;*

4. $\mathbb{D}\colon \mathrm{L}^2_\omega(\mathbf{R};U) \to \mathrm{L}^2_\omega(\mathbf{R};Y)$ *satisfies* $\tau(t)\mathbb{D}u = \mathbb{D}\tau(t)u$*,* $\pi_-\mathbb{D}\pi_+u = 0$*, and* $\pi_+\mathbb{D}\pi_-u = \mathbb{C}\mathbb{B}u$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$ *and* $t \in \mathbf{R}$.

We write $\Sigma \in \mathrm{WPLS}_\omega(U,H,Y)$ *(or* $\Sigma \in \mathrm{WPLS}$ *if we do not wish to specify stability).*

The different components of $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ *are named as follows: U is the* input space*, H the* state space*, Y the* output space*,* $\mathbb{A}$ *the* semigroup*,* $\mathbb{B}$ *the* reachability map*,* $\mathbb{C}$ *the* observability map*, and* $\mathbb{D}$ *the* I/O map *(input/output map) of* Σ*.*

We allow the right column (or the bottom row) to be empty, e.g., we call $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline & \end{array}\right]$ a WPLS iff 1. and 2. are satisfied; this is equivalent for $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline 0 & 0 \end{array}\right]$ being a WPLS (for any Y). The same applies to $\left[\begin{array}{c|c} \mathbb{A} & \\ \hline \mathbb{C} & \end{array}\right]$.

Intuitively, the reachability map $\mathbb{B}$ maps past controls into the present state, the observability map $\mathbb{C}$ maps the present state into future observations, and the I/O map $\mathbb{D}$ maps inputs into outputs in a causal way. The condition "4." imposed on $\mathbb{D}$ requires that $\mathbb{D} \in \mathrm{TIC}_\omega(U;Y)$ and that the Hankel operator induced by $\mathbb{D}$ is equal to $\mathbb{C}\mathbb{B}$. The definitions of this section are more extensively explained in [Sbook] (or in [S97b]–[S98c]).

[1] This is an ordinary matrix with operator-valued elements (cf. Appendix A). The rule lines are just for making the recognition of elements easier, especially in the case of multiblock elements.

Obviously, axioms 2. and 3. imply that $\mathbb{B}\pi_+ = 0 = \pi_-\mathbb{C}$. Since $\pi_+\mathrm{L}^2_\omega \subset \pi_+\mathrm{L}^2_{\omega'}$ and $\pi_-\mathrm{L}^2_{\omega'} \subset \pi_-\mathrm{L}^2_\omega$, continuously, for any $\omega' > \omega$, we can increase ω in Definition 6.1.1 (use also Lemma 2.1.11) when we identify $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ with their unique continuous (restricted) extensions:

Lemma 6.1.2 ($\mathbf{WPLS}_\omega \subset \mathbf{WPLS}_{\omega'}$) *Let* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega$ *for some* $\omega \in \mathbf{R}$. *Then* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_{\omega'}$ *for all* $\omega' > \omega$. □

(This is Lemma 2.4 of [S98a].)

We also note that if $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega$ and $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{H} \\ \mathbb{C} & \mathbb{G} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega$, then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{H} & \mathbb{B} \\ \mathbb{C} & \mathbb{G} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega$. However, having $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$, $\left[\begin{smallmatrix} \mathbb{A} \\ \mathbb{C} \end{smallmatrix}\right] \in \mathrm{WPLS}$ does not imply that $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}$ for any $\mathbb{D}$ (unless B or C is bounded; see Lemma 6.3.16), by Example 6.3.24.

Definition 6.1.3 (Stability) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $\omega \in \mathbf{R}$.

If axiom 1. (resp. 2., 3., 4.) of Definition 6.1.1 holds, then we say that $\mathbb{A}$ *(resp.* $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$*) is* ω*-stable and* Σ *is* internally *(resp.* input, output, I/O*)* ω-stable.

If Σ *is [internally]* ω*-stable and* $\mathrm{e}^{-\omega t}\mathbb{A}(t)x \to 0$ *strongly (resp. weakly) as* $t \to \infty$ *for all* $x \in H$*, then* Σ *is* [internally] strongly (resp. weakly) ω-stable*, and* $\mathbb{A}$ *is* strongly (resp. weakly) ω-stable*. We call* $\mathbb{B}$ strongly (resp. weakly) ω-stable *if* $\mathbb{B}$ *is* ω*-stable and* $\mathbb{B}\tau^t u \to 0$ *strongly (resp. weakly), as* $t \to +\infty$*, for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$*. For* $\mathbb{C}$ *and* $\mathbb{D}$*, "strongly stable" or "weakly stable" means "stable".*

If $\Sigma \in \mathrm{WPLS}_\omega$ *for some* $\omega < 0$*, then* Σ *is* exponentially stable.*The prefix "0-" is usually omitted (e.g.,* Σ *is strongly stable iff it is stable and* $\mathbb{A}(t)x_0 \to 0$ *as* $t \to +\infty$*, for all* $x_0 \in H$*).*

An output stable and I/O-stable WPLS is called a Stable-Output (Well-Posed Linear) System *(or SOS-stable); we denote such systems by* SOS *(or* $\mathrm{SOS}(U,H,Y)$*). Thus,* $\mathrm{WPLS}_0 \subset \mathrm{SOS} \subset \mathrm{WPLS} := \cup_{\omega\in\mathbf{R}}\mathrm{WPLS}_\omega$.

Sometimes "strongly stable" is called "asymptotically stable" and "exponentially stable" is called "uniformly stable". Obviously, "exponentially strongly" is equivalent to "exponentially".

If Σ is minimal and finite-dimensional, then it is stable iff it is exponentially stable. Whenever $\dim H < \infty$, the system is strongly (or weakly) stable iff it is exponentially stable (iff the eigenvalues of the generator of $\mathbb{A}$ have negative real parts). Therefore, in finite-dimensional control theory, "stable" usually means "exponentially stable" (and the same applies to stabilizability and detectability); this practise is common also in earlier infinite-dimensional theory, but less common in the theory of WPLSs.

The WPLS Σ is exponentially stable iff $\mathbb{A}$ is exponentially stable, by Lemma 6.1.10(a1), but the ω-stability of $\mathbb{A}$ does not imply that of Σ. If Σ is strongly stable, then so is $\mathbb{B}$ (and $\mathbb{A}$), by Lemma 6.1.13 that

SOS-stability means that the system maps any initial state $x_0 \in H$ and input $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ continuously to the output $y = \mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$. It is the weakest assumption that allows one to use the stable case methods for most control problems (this assumption was made in [WW] too).

We define dual systems as in Proposition 6.1 of [WW]:

Lemma 6.1.4 (Dual system Σ^{d}) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$, $\omega \in \mathbf{R}$. *Then its* dual system *(or (causal) adjoint system)*

$$\Sigma^{\mathrm{d}} := \left[\begin{array}{c|c}\mathbb{A}^{\mathrm{d}} & \mathbb{C}^{\mathrm{d}}\\ \hline \mathbb{B}^{\mathrm{d}} & \mathbb{D}^{\mathrm{d}}\end{array}\right] := \left[\begin{array}{c|c}\mathbb{A}^* & \mathbb{C}^*\boldsymbol{Я}\\ \hline \boldsymbol{Я}\mathbb{B}^* & \boldsymbol{Я}\mathbb{D}^*\boldsymbol{Я}\end{array}\right] \tag{6.1}$$

is in $\mathrm{WPLS}_\omega(Y,H,U)$. *Moreover,* $(\Sigma^{\mathrm{d}})^{\mathrm{d}} = \Sigma$. □

(We leave the simple verification of the lemma to the reader.) For the generators ($\left[\begin{smallmatrix}A^* & C^*\\ B^* & \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}A^* & C^*\\ B^* & D^*\end{smallmatrix}\right]$) of Σ^{d}, see Lemma 6.1.16 and Lemma 6.2.9(b). Note that our dual systems are causal unlike those in [S95]–[S98d].

Here $(\boldsymbol{Я}u)(t) := u(-t)$ (hence $\boldsymbol{Я} : \mathrm{L}^2_\omega \to \mathrm{L}^2_{-\omega}$ is an isometric isomorphism). The adjoints are taken with respect to the L^2 inner product (i.e., without a weight function); e.g., for $\mathbb{C} \in \mathcal{B}(H, \mathrm{L}^2_\omega(\mathbf{R};Y))$ we have that $\mathbb{C}^* \in \mathcal{B}(\mathrm{L}^2_{-\omega}(\mathbf{R};Y),H)$ and (cf. Lemma A.3.24)

$$\langle \mathbb{C}x, y\rangle_{\langle \mathrm{L}^2_\omega, \mathrm{L}^2_{-\omega}\rangle} := \int_{\mathbf{R}} \langle (\mathbb{C}x)(t), y(t)\rangle_Y\, dt = \langle x, \mathbb{C}^* y\rangle_H \quad (x \in H,\ y \in \mathrm{L}^2_{-\omega}(\mathbf{R};Y)). \tag{6.2}$$

(Equivalently, $\mathbb{C}^* = \mathbb{C}^{\mathrm{H}} e^{-2\omega\cdot}$, where $\mathbb{C}^{\mathrm{H}}$ is the adjoint of $\mathbb{C}$ w.r.t. $\langle\cdot,\cdot\rangle_{\mathrm{L}^2_\omega}$; note also that $\mathbb{C}^* = \mathbb{C}^*\pi_+$, $\mathbb{C}^{\mathrm{H}} = \mathbb{C}^{\mathrm{H}}\pi_+$.) This makes Σ^{d} independent of ω (cf. Lemma 6.1.2), and $\mathbb{C}^{\mathrm{d}}$ becomes α-stable iff $\mathbb{C}$ is α-stable for any $\alpha \in \mathbf{R}$; the same applies to $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{D}$.

Note that $\mathrm{L}^2_{-\omega}$ is the dual of L^2_ω with respect to this (weightless) L^2 inner product. Note also that the standard involution rules apply, e.g., $(\mathbb{D}\mathbb{C})^* = \mathbb{C}^*\mathbb{D}^*$ and $(\mathbb{C}\mathbb{B})^* = \mathbb{B}^*\mathbb{C}^*$ regardless of ω.

In control theory, one usually assumes the system to have an initial value (at $t = 0$, w.l.o.g.) and a control (on $(0,\infty)$). Sometimes the system is assumed to be controlled from $-\infty$ to ∞ (usually the latter setting is only used in proofs of results for the former setting). We formulate this in detail:

Definition 6.1.5 (State and output — $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] : \left[\begin{smallmatrix}x_0\\ u\end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]$) *In the* initial value setting *with initial time zero,* initial value $x_0 \in H$, *and* control *(or* input*)* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$, *the controlled* state $x(t) \in H$ *at time* $t \in \mathbf{R}_+$ *and the* output $y \in \mathrm{L}^2_\omega(\mathbf{R}_+,Y)$ *of* Σ *are given by (cf. Figure 6.1)*

$$\begin{bmatrix}x(t)\\ y\end{bmatrix} = \begin{bmatrix}\mathbb{A}(t) & \mathbb{B}\tau(t)\\ \mathbb{C} & \mathbb{D}\end{bmatrix}\begin{bmatrix}x_0\\ u\end{bmatrix} = \begin{bmatrix}\mathbb{A}(t)x_0 + \mathbb{B}\tau(t)u\\ \mathbb{C}x_0 + \mathbb{D}u\end{bmatrix}. \tag{6.3}$$

In the time-invariant setting, *the controlled state* $x(t) \in H$ *at time* $t \in \mathbf{R}$ *and the output* $y \in \mathrm{L}^2_\omega(\mathbf{R};Y)$ *of* Σ *with control (or input)* $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$ *are given by*

$$\begin{bmatrix}x(t)\\ y\end{bmatrix} = \begin{bmatrix}\mathbb{B}\tau(t)u\\ \mathbb{D}u\end{bmatrix}. \tag{6.4}$$

Sometimes we use the Salamon–Weiss notation ($\tau^t := \tau(t)$ and)

$$\begin{bmatrix}\mathbb{A}^t & \mathbb{B}^t\\ \mathbb{C}^t & \mathbb{D}^t\end{bmatrix} := \begin{bmatrix}\mathbb{A}(t) & \mathbb{B}\tau(t)\pi_{[0,t)}\\ \pi_{[0,t)}\mathbb{C} & \pi_{[0,t)}\mathbb{D}\pi_{[0,t)}\end{bmatrix} : \begin{bmatrix}x_0\\ u\end{bmatrix} \mapsto \begin{bmatrix}x(t)\\ \pi_{[0,t)}y\end{bmatrix}. \tag{6.5}$$

Because of (6.3), we sometimes denote Σ by $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\tau \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ (naturally, "$\Sigma : \left[\begin{smallmatrix}x_0 \\ u\end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix}x \\ y\end{smallmatrix}\right]$" refers to $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\tau \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$).

To treat both settings at once, we sometimes allow for $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$ in (6.3) (but we are interested only in cases where $x_0 = 0$ or $\pi_- u = 0$). By causality, the state and output are well defined for $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$ too (with $y \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;Y)$); cf. Definition 2.1.1.

G. Weiss et al. use symbols $\begin{bmatrix}\mathbb{T}_t & \Phi_t \\ \Sigma_t & \mathbb{F}_t\end{bmatrix} := \begin{bmatrix}\mathbb{A}^t & \mathbb{B}^t \\ \mathbb{C}^t & \mathbb{D}^t\end{bmatrix}$, and define WPLSs by requiring this quadrable to be linear and continuous $H \times \mathrm{L}^2(\mathbf{R}_+;U) \to H \times \mathrm{L}^2(\mathbf{R}_+;Y)$ (i.e., by requiring Σ to be locally continuous) and to satisfy, instead of (2.)–(4.), the algebraic conditions

$$\mathbb{B}^{s+t}(u \diamondsuit_s v) = \mathbb{A}^t \mathbb{B}^s u + \mathbb{B}^t v, \tag{6.6}$$

$$\mathbb{C}^{s+t} x_0 = \mathbb{C}^s x_0 \diamondsuit_s \mathbb{C}^t \mathbb{A}^s x_0, \tag{6.7}$$

$$\mathbb{D}^{s+t}(u \diamondsuit_s v) = \mathbb{D}^s u \diamondsuit_s (\mathbb{C}^t \mathbb{B}^s u + \mathbb{D}^t v) \qquad (u, v \in \mathrm{L}^2(\mathbf{R}_+;U),\ x_0 \in H); \tag{6.8}$$

here $u \diamondsuit_s v := \pi_{[0,s)} u + \tau^{-t} v$ ($\mathbb{A}$ is still required to be a C_0-semigroup). By discretization (see Theorems 13.4.4 and 13.4.5, or the proof of Lemma 6.1.10), it then follows from Lemma 13.3.3(b) that Σ is ω-stable for any $\omega > \omega_A$ (where ω_A is the growth rate of $\mathbb{A}$), hence the two definitions of WPLSs (and that of D. Salamon) are equivalent (see [Sbook] for details).

Of these two notations, we try to use the one (often "$\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$") that leads to a simpler formula. Obviously, (6.3) implies that, for $s, t \geq 0$, we have

$$\begin{bmatrix}\tau^s x(t) \\ \pi_+ \tau^s y\end{bmatrix} = \begin{bmatrix}x(t+s) \\ y(\cdot + s)\end{bmatrix} = \begin{bmatrix}\mathbb{A}^t & \mathbb{B}^t \\ \mathbb{C} & \mathbb{D}\end{bmatrix}\begin{bmatrix}x(s) \\ \pi_+ \tau^s u\end{bmatrix}. \tag{6.9}$$

From axiom 4. of Definition 6.1.1 we conclude that

$$\pi_{[t,\infty)} \mathbb{D} \pi_{(-\infty,t)} = \tau^{-t} \mathbb{C} \mathbb{B} \tau^t \quad (t \in \mathbf{R}). \tag{6.10}$$

"Shift semigroup systems" are often used as realizations of given I/O maps; these systems are useful for WPLSs too; indeed, every well-posed I/O map has a realization as a WPLS:

Definition 6.1.6 (Realizations) *Let* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, $\omega \in \mathbf{R}$. *If* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *for some Hilbert space H, then we call* Σ *(together with H) a* realization *of* $\mathbb{D}$.

We call the (strongly ω*-stable) system*

$$\Sigma_\omega := \left[\begin{array}{c|c}\pi_+ \tau & \pi_+ \mathbb{D} \pi_- \\ \hline I & \mathbb{D}\end{array}\right] \in \mathrm{WPLS}_\omega(U, \mathrm{L}^2_\omega(\mathbf{R}_+;Y), Y) \tag{6.11}$$

the exactly ω-observable realization *of* $\mathbb{D}$.

The system $\Sigma_\omega \in \mathrm{WPLS}_\omega(U,H,Y)$, *where* $H := \{\pi_+ \mathbb{D} \pi_- u \,|\, u \in \mathrm{L}^2(\mathbf{R};U)\}$ *and*

$$\|x_0\|_H^2 := \|x_0\|^2_{\mathrm{L}^2_\omega} + \inf_{u \in \mathrm{L}^2_\omega,\ \pi_+ \mathbb{D} \pi_- u = x_0} \|u\|^2_{\mathrm{L}^2_\omega} \qquad (x_0 \in H), \tag{6.12}$$

is the minimal ω-stable exactly ω-observable realization *of* $\mathbb{D}$.

(Note that we have identified the two systems ("Σ_ω"), although the semigroup and the output map of the latter system are actually restrictions of those of the former (and the range spaces of the semigroup and the input map are restricted analogously). The situation with Lemma 6.1.2 was somewhat analogous. Despite the name, the latter realization is not the only minimal ω-stable exactly ω-observable realization of $\mathbb{D}$.)

Analogously, the ω-stable realization $\Sigma^\omega := \left[\begin{smallmatrix}\tau\pi_- & \pi_- \\ \hline \pi_+\mathbb{D}\pi_- & \mathbb{D}\end{smallmatrix}\right]$ is exactly ω-reachable (see Definition 6.3.25).

The minimal ω-stable realizations of $\mathbb{D}$ correspond naturally one-to-one to "admissible" ω-stable factorizations of the Hankel operator $\pi_+\mathbb{D}\pi_-$, as shown in [S99] (combined with Remark 6.1.9).

Obviously, Σ_ω is a WPLS over $(U, \mathrm{L}^2_\omega(\mathbf{R}_+;Y), Y)$. It is also a WPLS over (U,H,Y):

Lemma 6.1.7 *Let* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, $\omega \in \mathbf{R}$. *The* minimal ω-stable exactly ω-observable realization *of* $\mathbb{D}$ *is an* ω*-observable, exactly* ω*-reachable (hence minimal)* ω*-stable WPLS. This realization is exactly* ω*-observable iff* $\pi_+\mathbb{D}\pi_-[\mathrm{L}^2_\omega(\mathbf{R};U)]$ *is closed in* $\mathrm{L}^2_\omega(\mathbf{R};Y)$.

Proof: Set $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] := \Sigma_\omega$. Set $\mathcal{X} := \mathrm{Ker}(\mathbb{B})^\perp \subset \mathrm{L}^2_\omega$, so that $\|\mathbb{B}u\|_H^2 := \|\mathbb{B}u\|^2_{\mathrm{L}^2_\omega} + \|u\|^2_{\mathrm{L}^2_\omega}$ for all $u \in \mathcal{X}$. Thus, $T := \mathbb{B}|_{\mathcal{X}} : \mathcal{X} \to H$ becomes coercive, hence $T \in \mathcal{GB}(\mathcal{X}, H)$. Consequently, H is complete, hence a Hilbert space. Moreover, $H \underset{c}{\subset} \mathrm{L}^2_\omega$.

We have $\pi_+\tau\mathbb{B} = \mathbb{A}^t\mathbb{B} = \pi_+\tau^t\mathbb{D}\pi_- = \mathbb{B}\tau^t\pi_-$ $(t > 0)$, hence

$$\|\mathbb{A}^t\mathbb{B}u\|_H^2 = \|\tau^t\mathbb{B}u\|_H^2 + \|\tau^t u\|^2_{\mathrm{L}^2_\omega} \le e^{2\omega t}\|\mathbb{B}u\|_H^2 \tag{6.13}$$

for $u \in \mathcal{X}$, $t \ge 0$, by (2.2), hence $\|\mathbb{A}^t\|_{\mathcal{B}(H)} \le e^{\omega t}$, i.e., $\mathbb{A}|_H$ is an ω-stable semigroup on H (it need not be strongly ω-stable, but it is weakly ω-stable), because its semigroup properties are inherited from $\mathbb{A}$ and its strong (or C_0-)continuity on H follows from the fact that $\tau^t\mathbb{B}u \to \mathbb{B}u$ and $\tau^t u \to u$ in L^2_ω, as $t \to 0+$.

Because $\mathcal{X} \subset \mathrm{L}^2_\omega$ is closed, the orthogonal projection $P : \mathrm{L}^2_\omega \to \mathcal{X}$ is continuous, hence so is $\mathbb{B} = TP \in \mathcal{B}(\mathrm{L}^2_\omega, H)$. Obviously, $\mathbb{C}$ remains continuous with this stronger topology of $H \subset \mathrm{L}^2_\omega$ and the other properties of the exactly ω-observable realization of $\mathbb{D}$ are preserved (except that Σ is exactly ω-reachable iff $\mathbb{B}$ is coercive on $\mathcal{X}$, equivalently, iff $\mathbb{B}$ has a closed-range, i.e., iff the Hankel operator $\pi_+\mathbb{D}\pi_-$ (on L^2_ω) has a closed range). It follows that $\Sigma \in \mathrm{WPLS}_\omega(U,H,Y)$. □

Example 6.1.8 The exactly observable realization of $\mathbb{D} := \tau(-1) \in \mathrm{TIC}(\mathbf{C})$ is given by

$$\Sigma := \left[\begin{array}{c|c}\mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] := \left[\begin{array}{c|c}\pi_+\tau & \tau(-1)\pi_{[-1,0)} \\ \hline I & \tau(-1)\end{array}\right] \in \mathrm{WPLS}_0(U,H,Y), \tag{6.14}$$

where $U = \mathbf{C} = Y$ and $H := \mathrm{L}^2(\mathbf{R}_+;Y)$. Thus,

$$\left[\begin{array}{c|c} \mathbb{A}^t & \mathbb{B}^t \\ \hline \mathbb{C}^t & \mathbb{D}^t \end{array}\right] = \left[\begin{array}{c|c} \pi_+\tau^t & \tau^{t-1}\pi_{[\max(0,t-1),t)} \\ \hline \pi_{[0,t)} & \tau^{-1}\pi_{[0,t-1)} \end{array}\right] : \begin{bmatrix} x_0 \\ u \end{bmatrix} \mapsto \begin{bmatrix} x(t) \\ \pi_{[0,t)}y \end{bmatrix} \tag{6.15}$$

and

$$\Sigma^{\mathrm{d}} := \left[\begin{array}{c|c} \mathbb{A}^{\mathrm{d}} & \mathbb{C}^{\mathrm{d}} \\ \hline \mathbb{B}^{\mathrm{d}} & \mathbb{D}^{\mathrm{d}} \end{array}\right] := \left[\begin{array}{c|c} \tau(-\cdot)\pi_+ & \text{Я} \\ \hline \tau^{-1}\pi_{[-1,0)}\text{Я} & \tau(-1) \end{array}\right]. \tag{6.16}$$

See Examples 6.2.14, 6.3.7, 8.3.12 and 9.8.15 for more on Σ. $\triangleleft$

We often assume that $\omega = 0$; the corresponding results for general ω can be obtained by shifting stability (we extend here Remark 2.1.6):

Remark 6.1.9 (Shifting stability) *Let* $\alpha, \omega \in \mathbf{R}$. *Let* $\mathcal{T}_\alpha$ *be the* stability shift *(or scaling operator)* $\mathbb{E} \mapsto \mathrm{e}^{\alpha\cdot}\mathbb{E}\mathrm{e}^{-\alpha\cdot}$. *Then* $\mathcal{T}_\alpha$ *is an isometric isomorphism of* TI_ω *onto* $\mathrm{TI}_{\omega+\alpha}$ *and of* TIC_ω *onto* $\mathrm{TIC}_{\omega+\alpha}$ *(because* $[\pi_+]\mathrm{L}^2_{\omega+\alpha} = \mathrm{e}^{\alpha\cdot}[\pi_+]\mathrm{L}^2_\omega$*, isometrically).*

Obviously, $\mathcal{T}_\alpha\pi_\pm = \pi_\pm\mathcal{T}_\alpha$, $\mathcal{T}_\alpha\tau(t) = \tau(t)\mathcal{T}_\alpha$, $\tau(t)\mathrm{e}^{\omega\cdot} = \mathrm{e}^{\omega t}\mathrm{e}^{\omega\cdot}\tau(t)$ *(*$t \in \mathbf{R}$*), and we have*

$$\mathcal{T}_\alpha(\mathbb{E}\mathbb{F}) = (\mathcal{T}_\alpha\mathbb{E})(\mathcal{T}_\alpha\mathbb{F}), \qquad \mathcal{T}_\alpha(\beta\mathbb{E}+\gamma\mathbb{F}) = \beta\mathcal{T}_\alpha\mathbb{E}+\gamma\mathcal{T}_\alpha\mathbb{F}, \tag{6.17}$$

$$(\mathcal{T}_\alpha\mathbb{E})^{-1} = \mathcal{T}_\alpha\mathbb{E}^{-1}, \qquad (\mathcal{T}_\alpha\mathbb{E})^* = \mathcal{T}_{-\alpha}\mathbb{E}^*, \tag{6.18}$$

$$(\mathcal{T}_\alpha\mathbb{E})^d = \mathcal{T}_\alpha\mathbb{E}^d, \qquad \widehat{\mathcal{T}_\alpha\mathbb{E}} = \tau(-\alpha)\widehat{\mathbb{E}}. \tag{6.19}$$

The operator $\mathcal{T}_\alpha$ *can be extended to a bijection of* $\mathrm{WPLS}_\omega(U,H,Y)$ *onto* $\mathrm{WPLS}_{\omega+\alpha}(U,H,Y)$*, by the rule*

$$\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \mapsto \Sigma_\alpha := \left[\begin{array}{c|c} \mathbb{A}_\alpha & \mathbb{B}_\alpha \\ \hline \mathbb{C}_\alpha & \mathbb{D}_\alpha \end{array}\right] := \left[\begin{array}{c|c} \mathrm{e}^{\alpha\cdot}\mathbb{A} & \mathbb{B}\mathrm{e}^{-\alpha\cdot} \\ \hline \mathrm{e}^{\alpha\cdot}\mathbb{C} & \mathrm{e}^{\alpha\cdot}\mathbb{D}\mathrm{e}^{-\alpha\cdot} \end{array}\right]. \tag{6.20}$$

This extended bijection preserves all properties of Σ *modulo the change in stability; e.g, the bijection does not affect the norms of* $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ *and* $\mathbb{D}$ *(remember that their domains are changed by the amount* α*), nor the regularity of* $\mathbb{D}$ *(because* $\widehat{\mathcal{T}_\alpha\mathbb{D}}(s) = \widehat{\mathbb{D}}(s-\alpha)$*), as one easily verifies (see [Sbook] for details).*

Moreover, this shift commutes with the multiplication by static operators and with the valid sums and compositions of operators, hence the shift of a system (and its admissible state or output feedback operators) corresponds to the same shift of the closed-loop system (an analogous remark applies to all closed loop systems corresponding any definition given in Summary 6.7.1). $\square$

(See Section 6.2 for regularity and Summary 6.7.1 for feedback. The formula $\widehat{\mathcal{T}_\alpha\mathbb{E}} = \widehat{\mathbb{E}}(\cdot-\alpha)$ refers to Theorem 3.1.3(a1); for $\mathbb{E} \in \mathrm{TIC}_\infty$ it also covers Theorem 6.2.1.)

In Lemma 6.2.9(c) we will show that the generators of $\mathcal{T}_\alpha \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ are $\left[\begin{array}{c|c} A+\alpha I & C \\ \hline B & * \end{array}\right]$ (or $\left[\begin{array}{c|c} A+\alpha I & C \\ \hline B & D \end{array}\right]$ if Σ is WR). Note that $\alpha > 0$ decreases stability, i.e., shifts the transfer function to the right. If $\mathbb{D}u = \mu * u$ for all $u \in \mathrm{L}^2_\omega$ for a measure μ then $\mathbb{D}_\alpha u = (\mathrm{e}^{\omega\cdot}\mu) * u$ for all $u \in \mathrm{L}^2_{\omega+\alpha}$ (see Definition 2.6.3 and Lemma D.1.12(d) for details).

A system is almost as stable as its semigroup:

Lemma 6.1.10 (Exp. stability) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $-\infty < \omega < \omega' < \infty$. *Then*

(a1) Σ *is exponentially stable iff* $\mathbb{A}$ *is exponentially stable.*

(a2) $\Sigma \in \mathrm{WPLS}_\omega$ *and* $\mathbb{B}\tau \in \mathrm{TIC}_\omega(U,H)$ *whenever* $\omega > \omega_A$.

(b1) If $\mathbb{B}$ *is* ω*-stable, then* $\mathbb{B}\tau$ *and* $\mathbb{D}$ *are* ω'*-stable.*

(b2) If $\mathbb{B}\tau$ *is* ω*-stable, then* $\mathbb{B}$ *and* $\mathbb{D}$ *are* ω*-stable.*

(b3) If $\mathbb{C}$ *is* ω*-stable, then* $\mathbb{D}$ *is* ω'*-stable,* $\mathbb{D}[\mathrm{L}^2_c] \subset \mathrm{L}^2_\omega$, *and hence Lemma 2.1.13 applies.*

(c1) If $\mathbb{A}$ *is exponentially stable, then* $\mathbb{A}, \mathbb{C} \in \mathcal{B}(H, \mathrm{L}^2)$ *and* $\mathbb{B}\tau, \mathbb{D} \in \mathrm{TIC}_{\exp}(U, *)$.

(c2) If $\mathbb{B}$ *is exponentially stable, then so are* $\mathbb{D}$ *and* $\mathbb{B}\tau$.

(c3) If $\mathbb{C}$ *is exponentially stable, then so is* $\mathbb{D}$.

See Lemma A.4.5 and Theorem 6.7.10(d) for further equivalent conditions.

When A is bounded (or $\mathbb{A}$ is compact or differentiable), exponential stabilizability is equivalent to the condition $\sigma(A) \subset \mathbf{C}^-$; for general infinite-dimensional systems the latter condition is strictly weaker, as illustrated in Example 5.1.4 of [CZ].

(The "spectrum determined growth condition" $\sup \operatorname{Re} \sigma(A) = \omega_A$ holds for any bounded A and any compact or differentiable semigroup; see [CZ] or [Sbook] for details.)

Proof: (Part (a1) was independently proved by D. Salamon and G. Weiss.) All this follows from Lemma 13.3.8 through discretization, see Theorems 13.4.4 and 13.4.5. □

If $\mathbb{C}$ is stable, then $\mathbb{D}$ is "almost stable", by (b3) above. We often use this fact combined with Lemma 2.1.13, hence we give the conclusions here explicitly:

Lemma 6.1.11 *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}$ *and let* $\mathbb{C}$ *be stable. Then* $\mathbb{D}[\mathrm{L}^2_c] \subset \mathrm{L}^2$*; in fact,* $\mathbb{D}\pi_{[-T,T)} \in \mathcal{B}(\mathrm{L}^2)$ *for all* $T > 0$. *Moreover,* $\mathbb{D} \in \mathrm{TIC}_\omega$ *for all* $\omega > 0$.

See Lemma 2.1.13 for further implications.

Proof: We have $\mathbb{D}\pi_{[0,1)} = \mathbb{D}^1 + \tau^{-1}\mathbb{C}\mathbb{B}^1 \in \mathcal{B}(\mathrm{L}^2([0,1);U), \mathrm{L}^2)$, hence the claims follow from Lemma 2.1.13. □

A map is stable iff it maps stable inputs to stable outputs:

Lemma 6.1.12 *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}$ *and* $\omega \in \mathbf{R}$. *Then* $\mathbb{C}$ *is* ω*-stable iff* $\mathbb{C}[H] \subset \mathrm{L}^2_\omega$, *and* $\mathbb{D}$ *is* ω*-stable iff* $\mathbb{D}[\pi_+\mathrm{L}^2_\omega] \subset \mathrm{L}^2_\omega$.

Proof: Let $\mathbb{C}H \subset \mathrm{L}^2_\omega$. Then $\mathbb{C} \in \mathcal{B}(H, \mathrm{L}^2_\omega)$, by Lemma A.3.6, because $\mathbb{C} \in \mathcal{B}(H, \mathrm{L}^2_\alpha)$ for $\alpha := \max\{\omega_A + 1, \omega\}$. The converse is trivial. The claim on $\mathbb{D}$ is Lemma 2.1.10(e). □

If(f) $\mathbb{B}$ is stable, then the strong stability of $\mathbb{A}$ implies that of $\mathbb{B}$:

Lemma 6.1.13 (Strongly stable $\mathbb{B}$) *Let* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}(U,H,\{0\})$. *Then the following are equivalent:*

(i) $\mathbb{B}$ *is strongly stable;*

(ii) $\mathbb{B}$ *is stable and* $\mathbb{B}\tau^t u \to 0$, *as* $t \to +\infty$, *for all* $u \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_-;U)$*;*

If $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ *is strongly stable or* $\mathbb{B}$ *is exponentially stable, then* $\mathbb{B}$ *is strongly stable.*

Proof: 1° *(i)⇔(ii):* Obviously, (i) implies (ii). Assume (ii). Then we may allow for any $u \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R};U)$ (ii) (replace t by $t+T$ for suitable $T \in \mathbf{R}$)), hence for any $u \in \mathrm{L}^2(\mathbf{R};U)$, by continuity; thus, (i) holds.

2° If $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is strongly stable, and $u \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_-;U)$, then $\mathbb{B}\tau^{T+t}u = \mathbb{A}^t\mathbb{B}\tau^T u \to 0$, as $t \to +\infty$, thus, then $\mathbb{B}$ is strongly stable, by 1°.

3° If $\mathbb{B}$ is ω-stable, $\omega < 0$, then $\|\mathbb{B}\tau^t u\|_H \le M\|\tau^t u\|_{\mathrm{L}^2_\omega} \le M\mathrm{e}^{\omega t}\|u\|_{\mathrm{L}^2_\omega} \to 0$, as $t \to +\infty$, for all $u \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+;U)$. Thus, then $\mathbb{B}$ is strongly stable, by 1°. □

Assume for a while that $\dim H < \infty$. Then $\mathbb{A}$ is stable iff $\sigma(A) \subset \overline{\mathbf{C}^-}$ and $\mathbb{A}$ is strongly (or exponentially) stable iff $\sigma(A) \subset \mathbf{C}^-$. However, for a non-strongly stable $\mathbb{A}$ (say $H = \mathbf{C}$ and $A \in i\mathbf{R}$), maps $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ are unstable unless the non-strongly stable poles of $\mathbb{A}$ are unreachable or unobservable.

The strong stability of $\mathbb{A}$ does not imply that of $\mathbb{B}$, $\mathbb{C}$ or $\mathbb{D}$, not even for bounded A, B, C and D (cf. Lemma 6.1.16):

Example 6.1.14 ($\mathbb{A}$ strongly stable $\not\Rightarrow$ $\mathbb{B}/\mathbb{C}/\mathbb{D}$ stable) (It follows from Lemma 6.3.26(f) (or (d)), that all systems below are minimal.)

(a) Let $Y := H := U := \ell^2(\mathbf{N}+1)$ (with natural base $\{e_k := \chi_{\{k\}}\}_{k=1}^\infty$). Define $A \in \mathcal{B}(H)$ by setting $Ae_k := -k^{-1}e_k$ ($k \in \mathbf{N}+1$).

Then $\|A\| \le 1$ and $\mathbb{A}^t e_k := \mathrm{e}^{At}e_k = \mathrm{e}^{-t/k}e_k$ ($k \in \mathbf{N}+1$, $t \ge 0$), hence $\|\mathbb{A}^t\| \le 1$ ($t \ge 0$). Because $\mathbb{A}$ is stable and $\mathbb{A}e_k \to 0$ for all k, the semigroup $\mathbb{A}$ is strongly stable, by Lemma A.3.4(H1). Since $A = A^*$, we have $\mathbb{A} = \mathbb{A}^*$.

By Lemma 6.3.16(a), the operators $\left[\begin{smallmatrix} A & I \\ I & 0 \end{smallmatrix}\right] \in \mathcal{B}(U \times U)$ generate a wpls $\Sigma := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,U,U)$. By Theorem 6.2.11,

$$\widehat{\mathbb{A}} = \widehat{\mathbb{B}\tau} = \widehat{\mathbb{C}} = \widehat{\mathbb{D}} = (s-A)^{-1} = \mathrm{diag}((s+k^{-1})^{-1})_{k \in \mathbf{N}+1}. \tag{6.21}$$

Because $\mathbb{A}$ is obviously not exponentially stable, we have $\widehat{\mathbb{A}} \notin \mathrm{H}^\infty$ and $\widehat{\mathbb{A}} \notin \mathrm{H}^2_{\mathrm{strong}}$, by Lemma A.4.5, hence $\mathbb{D}$, $\mathbb{C}$ and $\mathbb{B}\tau$ are unstable. Since $\mathbb{B} = \mathbb{C}^{\mathrm{d}}$, also $\mathbb{B}$ is unstable.

(b) If, instead, we take $Ce_k := k^{-1/2}e_k$, then $\widehat{\mathbb{C}}(s)e_k = \widehat{\mathbb{D}}(s)e_k = \frac{k^{1/2}}{sk+1}e_k$ ($k \in \mathbf{N}+1$), hence $\|\widehat{\mathbb{C}}x\|^2_{H^2} = \sum_k \int_{i\mathbf{R}} |x_k \frac{k^{1/2}}{sk+1}|^2 = \pi\sum_k |x_k|^2 = \pi\|x\|_2^2$ for each $x \in H$, so that then $\left[\begin{smallmatrix} \mathbb{A} \\ \hline \mathbb{C} \end{smallmatrix}\right]$ is strongly stable but still $\mathbb{D}$ is unstable (since $\|\widehat{\mathbb{D}}e_k\|_\infty = k^{1/2}$ ($k \in \mathbf{N}+1$)).

One could show that this system is exponentially stabilizable (take $K = 2$) but not detectable.

(c) Exchange C and $B = I$ in (b) to have $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ strongly stable but $\mathbb{C}$ and $\mathbb{D}$ unstable.

(d) By Example 9.13.14 (see $\mathbb{A}_{\circlearrowleft}$ and $\mathbb{A}^{\mathrm{d}}_{\circlearrowleft}$), we can have $\mathbb{A}$, $\mathbb{C}$ and $\mathbb{D}$ strongly stable but $\mathbb{B}$ and $\mathbb{B}\tau$ unstable, and $\mathbb{A}$, $\mathbb{B}$, $\mathbb{B}\tau$ and $\mathbb{D}$ strongly stable but $\mathbb{C}$ unstable.

(e) By (d), we can have Σ minimal and $\mathbb{B}\tau$ stable without Σ being exponentially stable. ◁

(This still leaves the open question whether $\mathbb{D}$ can be unstable when both $\mathbb{C}$ and $\mathbb{B}$ are stable (and Σ is well-posed; cf. Example 6.3.24). By using realization (6.11) we obtain that this is the case iff some unstable $\mathbb{D} \in \mathrm{TIC}_\infty$ has a stable Hankel operator $\pi_+\mathbb{D}\pi_-$; thus, it might be that the answer to our question is known.)

As one can easily verify, in [S97b], [S98b] and [S98c] the stability assumptions on $\mathbb{A}$ and $\mathbb{B}$ were not important:

Remark 6.1.15 (SOS and [Staffans]) *In [S97b], [S98b] and [S98c], except in [S97b, Lemma 21] and [S98a, Lemma 3.5(ii)], we may drop the assumptions on the stability of semigroups and input maps, if we do the same on conclusions.* □

In the sequel, we will refer to these articles with these weaker assumptions without any further mention.

(We could, in addition, replace joint stabilizability and detectability by r.c.-SOS-stabilizability In particular, Sections 5–7 of [S98b] are true with these two replacements and for indefinite S too, cf. [S98c, Remark 7.7]; however, we do not need this.)

Next we will present the generating operators A, B and C of a WPLS; the existence of a feedthrough operator D depends on the regularity of the system and is therefore studied in Section 6.2. As mentioned above, these lead to classical formulae $x' = Ax + Bu$, $y = Cx + Du$ and $\widehat{\mathbb{D}}(s) = D + C(s-A)^{-1}B$ and others in a weak sense.

Following the customary practice, we shall set

$$H_1 := \mathrm{Dom}(A),\ \ H_1^* := \mathrm{Dom}(A^*),\ \ H_{-1} := (H_1^*)^*,\ \ H_{-1}^* := (H_1)^*, \tag{6.22}$$

where A is the generator of $\mathbb{A}$. We shall take adjoints w.r.t. the pivot space H; e.g., $H_{-1} \times H_1^* \to \mathbf{C}$ denotes the unique continuous extension of the restriction of the inner product $H \times H \to \mathbf{C}$ to $H \times H_1^*$, see Definition A.3.23 for details.

A detailed description of this process is given in the lemma and definition below.

Lemma 6.1.16 ($\left[\begin{smallmatrix} A & B \\ C & \end{smallmatrix}\right]$, H_1, H_{-1}) *Let* $\Sigma := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$, $\omega \in \mathbf{R}$. *Let A be the generator of $\mathbb{A}$ and let $\alpha \in \sigma(A)^c$.*

We set $H_1 := \mathrm{Dom}(A)$ with $\|x\|_{H_1} := \|(\alpha - A)x\|_H$ (this is equivalent to the graph norm), and define H_{-1} to be the completion of H under the norm $\|(\alpha-A)^{-1}\cdot\|_H$ (thus $H_1 \subset H \subset H_{-1}$; H_1 and H_{-1} are independent of α modulo an equivalent norm).

The following hold:

(a) The unique extension $\mathbb{A}_{H_{-1}}$ of $\mathbb{A}$ onto H_{-1} is a semigroup isomorphic to the original $\mathbb{A}$ and the generator of $\mathbb{A}_{H_{-1}}$ an extension of A; we identify the two. The situation with $\mathbb{A}$ and $\mathbb{A}|_{H_1}$ is the same.

Thus, $\mathbb{A}(t)$ is in $\mathcal{B}(H)$, $\mathcal{B}(H_1)$ and $\mathcal{B}(H_{-1})$ for $t \geq 0$, $A \in \mathcal{B}(H_1, H)$, and $A \in \mathcal{B}(H, H_{-1})$. However, by $\mathrm{Dom}(A)$ *we always denote $H_1 = \{x_0 \in H \,\big|\, Ax_0 \in H\}$. The map $\alpha - A$ is an isometric isomorphism of H_n onto H_{n-1} ($n = 0, 1$).*

(b) There is a unique input operator $B \in \mathcal{B}(U, H_{-1})$ *s.t.*

$$\mathbb{B}\tau(t)u = (\mathbb{A}B * u)(t) = \int_0^t \mathbb{A}(t-s)Bu(s)\,ds \in H \quad (u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+; U),\ t \geq 0) \tag{6.23}$$

(above and below, the integration is carried out in H_{-1}). Consequently,

$$\mathbb{B}\tau(t)u = (\mathbb{A}B * u)(t) = \lim_{T\to\infty} \int_{-T}^t \mathbb{A}(t-s)Bu(s)\,ds \in H \quad (t \in \mathbf{R},\ u \in \mathrm{L}^2_\omega(\mathbf{R}; U)), \tag{6.24}$$

where the limit can be taken in H. Moreover, $x = \mathbb{A}x_0 + \mathbb{B}\tau u$ satisfies $x' = Ax + Bu$ in H_{-1} a.e. on $\mathbf{R}_+$ and $x(t) - x_0 = \int_0^t (Ax + Bu)\,dm$ for all $t \geq 0$, $x_0 \in H$, $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+; U)$.

(c) There is a unique output operator $C \in \mathcal{B}(H_1, Y)$ *s.t.*

$$(\mathbb{C}x_0)(t) = C\mathbb{A}(t)x_0 \quad (x_0 \in H_1,\ t \geq 0). \tag{6.25}$$

We say that Σ is generated by $\left[\begin{smallmatrix} A & B \\ C & \end{smallmatrix}\right]$, *and we call* $\left[\begin{smallmatrix} A & B \\ C & \end{smallmatrix}\right]$ *the* generators *of Σ; they are independent of α and ω (as long as $\Sigma \in$* WPLS$_\omega$*). Also the following hold:*

(d) $\left[\begin{smallmatrix} A & B \\ C & \end{smallmatrix}\right]$ determine $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \end{smallmatrix}\right]$ uniquely and $\mathbb{D}$ modulo an additive constant from $\mathcal{B}(U, Y)$ (cf. Lemma 6.2.9(a) and Lemma 6.3.10(d)).

(e) The generators of Σ^{d} are given by $\left[\begin{smallmatrix} A^* & C^* \\ B^* & \end{smallmatrix}\right]$.

(See Definition 6.2.3 and Lemma 6.2.9(a) for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, and Definition 6.1.17 for $\left[\begin{smallmatrix} A^* & C^* \\ B^* & \end{smallmatrix}\right]$.) The specific number α chosen above is irrelevant; only the topology of H_{-1} matters in application, not the particular norm of H_{-1} (the norms corresponding to different α's are equivalent).

Note from (c) that C is determined by

$$Cx_0 := (\mathbb{C}x_0)(0) \quad (x_0 \in H_1), \tag{6.26}$$

and B is determined by $B^*x_0 := (\mathbb{B}^{\mathrm{d}}x_0)(0) = (\mathbb{B}^*x_0)(0)$ ($x_0 \in H_1^* := \mathrm{Dom}(A^*)$).

We note for (b) that the (convolution) integrals converge in H_{-1}, i.e., $\mathbb{A}(t - \cdot)Bu(\cdot) \in \mathrm{L}^1((-\infty, t); H_{-1})$, for each $t \in \mathbf{R}_+$, by the Hölder Inequality (we have $\mathbb{A}B \in \mathcal{C}(\mathbf{R}_+; \mathcal{B}(U, H_{-1})) \subset \mathrm{L}^2_{\mathrm{loc}}$). However, the values of the integrals belong to H too. Recall from (B.18) that if any integral converges in H, then it converges in H_{-1}, with the same value.

In Theorem 6.2.13(a1), we shall show that x is the unique strong solution of "$x' = Ax + Bu$, $x(0) = x_0$" in the sense that $Ax + Bu$ is the distributional derivative (and derivative a.e.) of x For further details and formula $x' = Ax + Bu$ (and "$y = Cx + Du$"), see Theorem 6.2.13 and Chapter 4 of [Sbook]; for the generators of closed-loop systems, see Proposition 6.6.18.

As above, we will denote the generators with same letters as corresponding operators (as in [S95]–[S01] and [Sbook]).

Proof: (a) See Lemma A.4.6.

(b) Formula (6.23) and the existence claim for B and follow easily from Theorem 3.9 of [W89a] (note that $\pi_-\tau(t)u \in L^2_\omega$ for any $\omega \in \mathbf{R}$ and that $\mathbb{B} = \mathbb{B}\pi_-$, by Definition 6.1.1). Formula (6.24) (cf. Remark 30 of [S97b]) follows from (6.23), because $\pi_{[-T,\infty)}u \to u$ in L^2_ω, by Corollary B.3.8. The rest is from Theorem 3.9 of [W89a] (and Lemma B.7.6).

(c) This is from Theorem 3.3 of [W89b].

(d) This is obvious (use density and continuity).

(e) By Lemma A.4.2(f), the generator of $\mathbb{A}^*$ is A^*. It is easy to verify that $\langle u, \mathbb{B}^* x_0\rangle_H = \langle u(-\cdot), B^*\mathbb{A}(\cdot)^* x_0\rangle_{L^2}$ for $u \in \mathcal{C}_c(\mathbf{R}_-;U)$, $x_0 \in H_1^*$ (cf. 6.1.17). By density (note that $B^*\mathbb{A}(\cdot)^* x_0 \in \mathcal{C}(\mathbf{R}_+;U)$), $\mathbb{B}^* x_0 = B^*\mathbb{A}(\cdot)^* x_0$ for $x_0 \in H_1^*$, hence B^* corresponds to $\text{Я}\mathbb{B}^*$. By exchanging the roles of Σ and Σ^{d}, we obtain that C^* corresponds to $\mathbb{C}^*\text{Я}$. $\square$

As explained above, the spaces H_1^* and H_{-1}^* are defined as H_1 and H_{-1}, respectively, but with A^* in place of A, and $H_{-1}^* \times H_1 \to \mathbf{C}$ is the unique continuous extension of $H \times H_1 \to \mathbf{C}$. The adjoints C^* and B^* are defined w.r.t. these two sesquilinear forms (see Lemma A.3.24), so that

$$\langle Bu_0, x_0\rangle_{\langle H_{-1}, H_1^*\rangle} = \langle u_0, B^* x_0\rangle_U \qquad \text{for all } u_0 \in U,\ x_0 \in H_1^*, \tag{6.27}$$

$$\langle y_0, Cx_0\rangle_Y = \langle C^* y_0, x_0\rangle_{\langle H_{-1}^*, H_1\rangle} \qquad \text{for all } y_0 \in Y,\ x_0 \in H_1. \tag{6.28}$$

Let us state this and a bit more formally:

Definition 6.1.17 ($B^*, C^*, H_1^*, H_{-1}^*, H_B, H_C^*$) *Let the assumptions of Lemma 6.1.16 hold. We set $H_1^* := \mathrm{Dom}(A^*) \subset H \subset H_{-1}^* :=$ the completion of H under the norm $\|(\bar\alpha - A^*)^{-1}\cdot\|_H$.*

We extend $\langle h, x\rangle_{\langle H_1, H_{-1}^*\rangle} := \langle h, x\rangle_H$ for all $h \in H_1$, $x \in H$ continuously to $H_1 \times H_{-1}^*$ to get an interpretation of H_{-1}^* as the dual of H_1, and we do the same for $\langle h^*, x\rangle_{\langle H_1^*, H_{-1}\rangle} := \langle h^*, x\rangle_H$ for all $h^* \in H_1^*$, $x \in H$ (all these pairings are sesquilinear).

The adjoints $C^* \in \mathcal{B}(Y, H_{-1}^*)$ and $B^* \in \mathcal{B}(H_1^*, U)$ are taken with respect to these pairings. As above, A^* means the adjoint $A^* : \mathrm{Dom}(A^*) \to H$ of the unbounded operator A as well as its unique continuous extension $A^* \in \mathcal{B}(H, H_{-1}^*)$ (for which $\langle Az, x\rangle_H = \langle z, A^* x\rangle_{\langle H_1, H_{-1}^*\rangle}$ for all $z \in H_1$, $x \in H$). Finally, we define the Hilbert spaces (see Lemma A.3.16)

$$H_B := (\alpha - A)^{-1}[H + BU] = \{x_0 \in H \,|\, Ax_0 + Bu_0 \in H \text{ for some } u_0 \in U\} \subset H \tag{6.29}$$

with $\|z\|_{H_B} := \inf\{(\|x\|_H^2 + \|u\|_U^2)^{1/2} \,|\, (\alpha - A)^{-1}(x + Bu) = z\}$, and $H_C^* := (\bar\alpha - A^*)^{-1}[H + C^*Y] \subset H$ with $\|z\|_{H_C^*} := \inf\{\|(x,y)\|_{H\times Y} \,|\, (\bar\alpha - A^*)^{-1}(x + C^* y) = z\}$. Thus, $H_1 \subset H_B \subset H \subset H_{-1}$ and $H_1^* \subset H_C^* \subset H \subset H_{-1}^*$, continuously (see Corollary A.3.7).

We set $H_{C,K}^* := H^*_{\left[\begin{smallmatrix} C \\ K \end{smallmatrix}\right]}$ when $\left[\begin{array}{c|c} A & B \\ \hline C & \\ K & \end{array}\right]$ generate a WPLS.

All spaces in this definition are independent of $\alpha \in \sigma(A)^c$ modulo an equivalent norm, by the Resolvent Equation and Corollary A.3.7.

A mnemonic: $H_1^* := (H^*)_1 \neq (H_1)^* = (H^*)_{-1} =: H_{-1}^*$. See also Lemma 6.1.16(e). Example 6.2.14 contains examples of spaces and (extended) operators defined above, but usually suffices to remember (6.27)–(6.28). See Lemma 6.3.18 and [Sbook, Lemma 4.2.18] or [S97b, Lemma 32] for further details.

We follow the standard convention to call B *bounded* and write "$B \in \mathcal{B}(U,H)$", when $B \in \mathcal{B}(U,H_{-1})$ is such that $Bu_0 = B_0u_0$ for all $u_0 \in U$ for some $B_0 \in \mathcal{B}(U,H)$ (equivalently, $(\mathbb{A}x_0 + \mathbb{B}u)' = Ax + B_0u$ a.e. for all $x_0 \in H$ and all $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$). Cf. also part 1° of the proof of Theorem 9.9.6(a). Similarly, we call C *bounded* if "$C \in \mathcal{B}(H,Y)$". Obviously, B is bounded iff B^* is bounded.

We call B or C *unbounded* if it is not bounded. For B, this does not agree with the meaning of unboundedness in functional analysis (but it does agree for C and for B^*). In physical examples, unbounded input and output operators appear typically in connection with boundary control or boundary observation, respectively.

We have followed the WPLS formulation of O. Staffans ([S97a]–[S01]), hence we share most of his notation, but we use the part of the notation of G. Weiss that we feel more elegant or practical (cf. Definition 6.1.5).

Readers familiar with existing WPLS literature might wish to consult the following "translation table" between the notation of this book, Staffans and Weiss. Here "$H_1 \overset{\mathrm{S}}{=} W \overset{\mathrm{W}}{=} X_1$" means that we use H_1 for the W of [S95]–[S98d], which in turn equals the X_1 of [W94a], [WW] etc. We exclude the sign "$\overset{\mathrm{S}}{=}$" when the symbol of [S95]–[S98d] coincides with that of this book, e.g., by "$H \overset{\mathrm{W}}{=} X$" we mean that Staffans and we use H (for the state space) where Weiss uses X.

System theory (Chapter 6):

Our notation $\overset{\mathrm{S}}{=}$ Staffans' notations $\overset{\mathrm{W}}{=}$ Weiss' notation:

$H \overset{\mathrm{W}}{=} X$ (the state space), $U \overset{\mathrm{W}}{=} U$ (the input space), $Y \overset{\mathrm{W}}{=} Y$ (the output space) (these are complex (possibly unseparable) Hilbert spaces).

$\mathbb{A}^t := \mathbb{A}(t) \overset{\mathrm{W}}{=} \mathbb{T}_t \in \mathcal{B}(H)$ for all $t \geq 0$ (the semigroup), $A \overset{\mathrm{W}}{=} A$ (the infinitesimal generator of $\mathbb{A}$).

$\mathbb{B} \in \mathcal{B}([\pi_-]\mathrm{L}^2_\omega(\mathbf{R};U);H))$ (the reachability map) is the (unique) operator for which we have $\mathbb{B}^t := \mathbb{B}\tau(t)\pi_{[0,t)} \overset{\mathrm{W}}{=} \Phi_t$,

$\mathbb{C} \overset{\mathrm{W}}{=} \Psi_\infty \in \mathcal{B}(H,[\pi_+]\mathrm{L}^2_\omega(\mathbf{R};Y))$ (the observability map) ($\mathbb{C}^t := \pi_{[0,t)}\mathbb{C} \overset{\mathrm{W}}{=} \Psi_t$),

$\mathbb{D} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U);\mathrm{L}^2_\omega(\mathbf{R};Y))$ (the I/O map) is the (unique) causal, time-invariant operator for which $\pi_+\mathbb{D}\pi_+ \overset{\mathrm{W}}{=} \mathbb{F}_\infty$ (and hence $\mathbb{D}^t := \pi_{[0,t)}\mathbb{D}\pi_{[0,t)} \overset{\mathrm{W}}{=} \mathbb{F}_t$); see Lemma 2.1.3.

$H_1 \overset{\mathrm{S}}{=} W \overset{\mathrm{W}}{=} X_1 := \mathrm{Dom}(A)$, $H_{-1} \overset{\mathrm{S}}{=} V \overset{\mathrm{W}}{=} X_{-1} := \mathrm{cl}_{\|(\alpha-A)^{-1}\cdot\|}(H)$, hence $H_1 \underset{c}{\subset} H \underset{c}{\subset} H_{-1}$;

$H_1^* \overset{\mathrm{S}}{=} V^* \overset{\mathrm{W}}{=} Z_1 := \mathrm{Dom}(A^*)$, $H_{-1}^* \overset{\mathrm{S}}{=} W^* \overset{\mathrm{W}}{=} Z_{-1} := \mathrm{cl}_{\|(\bar\alpha-A^*)^{-1}\cdot\|}(H)$, analogously.

$B_{\mathrm{s}}^* \overset{\mathrm{S}}{=} \overline{B^*} \overset{\mathrm{W}}{=} (B^*)_\Lambda = B_\Lambda^* = \text{s-lim}_{s\to+\infty} B^*s(s-A^*)^{-1}$, hence $H_1^* \underset{c}{\subset} \mathrm{Dom}(B_{\mathrm{s}}^*) \underset{c}{\subset} H$;

$C_{\mathrm{s}} \overset{\mathrm{S}}{=} \overline{C} \overset{\mathrm{W}}{=} C_\Lambda := \text{s-lim}_{s\to+\infty} Cs(s-A)^{-1}$, $K_{\mathrm{s}} \overset{\mathrm{S}}{=} \overline{K} \overset{\mathrm{W}}{=} F_\Lambda$, (see Proposition 6.2.8).

$\widehat{\mathbb{D}} \overset{\mathrm{W}}{=} \mathbb{H} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ (the transfer function; see Theorem 6.2.1).

Optimization theory (Chapters 8–12):

$\widehat{\mathbb{D}}(s)^*J\widehat{\mathbb{D}}(s) \overset{\mathrm{W}}{=} \Pi \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ (the Popov function for stable $\mathbb{D}$),

$\mathcal{P} \overset{\text{S}}{=} \Pi \overset{\text{W}}{=} X \in \mathcal{B}(H)$ (the Riccati operator);
$\mathbb{X} \overset{\text{W}}{=} \Xi \in \mathcal{G}\mathrm{TIC}(U)$ (the spectral factor, see Definition 6.4.4)
$X \overset{\text{W}}{=} D \in \mathcal{B}(U)$ (its feedthrough operator, see Definition 6.2.3; in particular, $X^*SX \overset{\text{W}}{=} D^*D$ for $S \gg 0$; in applications we often take $X = I$, hence then $S \overset{\text{W}}{=} D^*D$ and $X \overset{\text{W}}{=} D^{-1}\Xi$);
$K \overset{\text{W}}{=} F \in \mathcal{B}(H_1, U)$ (the state feedback operator) $D^*JD \overset{\text{W}}{=} R \in \mathcal{B}(U)$ (see Remark 9.1.14).

See Remark 9.1.14 for further differences. The notation in recent works of Staffans (e.g., [Sbook]) is closer to that of Weiss and this book than that of [S95]–[S98d]. Most other existing WPLS theory seems to use notation close to that of Weiss with some exceptions in the direction of this book.

Notes

The history of WPLSs is explained in the notes to Chapter 2 of [Sbook]; see also p. 23. The early history is explained in [Helton76a]; the final rise of the theory is due to [Sal87], [Sal89], [W89a], [W89b] and [W89c], whose contributions include Lemma 6.1.16, hence the alternative name *Salamon–Weiss systems*. Ruth Curtain [Curtain89] gives a detailed account of this process and its relations to the rest of control theory.

This abstract formulation has for long been widely used also in the special case of bounded input and output operators, where also the definition of a system through generators would be possible. Naturally, in that case the theory becomes much simpler and more elegant, although very restrictive, hence it serves as a nice introduction to the general case; see [CZ] for a mature and rather extensive presentation.

This book is system theory oriented, but the literature is full of practical applications of WPLSs and its special cases, dating back to [Sal87] (see also [CZ]).

The name WPLS, part of Lemma 6.1.13, Remark 6.1.15 and most of each definition in this section are due to Olof Staffans. Lemmas 6.1.2 and 6.1.12 and Lemma 6.1.10(a) are well-known. Realization (6.11) is from Section 4 of [Sal89]. Variants of realization (6.12) and Lemma 6.1.7 were presented in [KMR] in a Pritchard–Salamon setting. The main new contribution of this section are the relations between the stabilities of different parts of a system (most of 6.1.10–6.1.14).

The reader interested in a more detailed study on WPLSs than that of this chapter must read [Sbook]. That monograph also treats the case where L^2 signals are replaced by L^p signals $(1 \leq p \leq \infty)$ and U, H and Y are allowed to be Banach spaces. Although most results and proofs on system theory and feedback theory are the same in this more general context as in our L^2 Hilbert space setting, the latter seems necessary for fruitful optimization theory and hence it is the one usually treated in the literature; this has also motivated our choice.

6.2 Regularity ($\exists\widehat{\mathbb{D}}(+\infty)$)

That is the usual method, but not mine–
My way is to begin with the beginning;
The regularity of my design
Forbids all wandering as the worst of sinning.
— Lord Byron (1788–1824), "The Bride of Abydos"

learning or genius, reader,

In this section we define certain regularity concepts for an I/O map and study the basic implications of regularity, including the classical formulae $y = Cx + Du$ and $\widehat{\mathbb{D}}(s) = D + C(s-A)^{-1}B$. This section does not contain essentially new results. Further results on regularity and the relations between a system and its generators are given in Section 6.3.

If $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is a finite-dimensional system (cf. (1.7)), then its transfer function $D + C(\cdot - A)^{-1}B$ has the limit D at infinity. In general, well-posed transfer functions having such a limit are called regular. For a mnemonic, we write formally

$$\mathbb{D} \in \mathrm{TIC}_\infty \text{ is regular } \Leftrightarrow \exists\widehat{\mathbb{D}}(+\infty); \tag{6.30}$$

see Definition 6.2.3 for exact definitions.

Most system theory can be written in terms of "integral maps" $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$, but part of the theory requires feedthrough operators. In particular, the Riccati equations used to solve optimal control problems are written in terms of the generators, including the feedthrough operator, of the system involved. Such problems can be solved in terms of factorization of the I/O map, but this approach is less useful in practical applications and therefore usually only serves as a path to Riccati equations. Fortunately, all transfer functions of practical interest seem to be regular.

The Laplace transform, defined by $\widehat{u} : s \mapsto \int_{\mathbf{R}} e^{-st} u(t)\,dt \in U$, maps function $u \in \mathrm{L}^2_\omega$ onto H^2_ω, and is an isomorphism onto, isometric times the factor $\sqrt{2\pi}$; see Appendix D for details. Well-posed I/O maps correspond to transfer functions that are bounded on some right half-plane $\mathbf{C}^+_\omega := \{s \in \mathbf{C} \mid \operatorname{Re} s > \omega\}$, i.e., that are proper; we recall this fact from Theorem 2.1.2:

Theorem 6.2.1 (Transfer functions) *Let $\omega \in \mathbf{R}$. For each $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$ there is a unique function $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$, called the* transfer function *(or symbol or Laplace transform) of $\mathbb{D}$, s.t. $\widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\hat{u}$ on $\mathbf{C}^+_\omega$ for all $u \in \mathrm{L}^2_\omega(\mathbf{R}_+; U)$. The mapping $\mathbb{D} \mapsto \widehat{\mathbb{D}}$ is an isometric isomorphism.* □

We often identify functions and corresponding multiplication operators, i.e., we consider $\widehat{\mathbb{D}}$ both as a function and as an operator on H^2_ω. Note that $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\omega$ iff $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{H}^\infty_\omega$, in particular $\widehat{\mathbb{D}^{-1}} = (\widehat{\mathbb{D}})^{-1}$ if $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\omega$.

Next we recall the last claim of Lemma 3.3.8:

Lemma 6.2.2 ($\mathbb{D}^{\mathrm{d}}$) *Let $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$. Then $\mathbb{D}^{\mathrm{d}} := \mathbf{Я}\mathbb{D}^*\mathbf{Я} \in \mathrm{TIC}_\omega(U,Y)$ and $(\mathbb{D}^{\mathrm{d}})^{\mathrm{d}} = \mathbb{D}$. Moreover, $\widehat{\mathbb{D}^{\mathrm{d}}}(s) = \widehat{\mathbb{D}}(\bar{s})^*$ for $s \in \mathbf{C}^+_\omega$.* □

George Weiss has given eight equivalent characterizations of strong regularity in [W94a, Theorem 5.8] and for weak regularity in [SW01a]; a more thorough study on these concepts is given in [Sbook]. We have chosen the simplest characterizations as the definitions: the transfer function should have a limit at infinity. However, some applications require that this limit converges in a very strong sense whereas others allow for a weaker convergence, therefore we define twelve different (combinations of) attributes of regularity:

Definition 6.2.3 (Regularity) *Let $\omega \in \mathbf{R}$, $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$, and $\mathbb{D} \in \mathcal{B}(U,Y)$.*

The map $\mathbb{D}$ is called regular (R) *with* feedthrough operator $\widehat{\mathbb{D}}(+\infty) := D \in \mathcal{B}(U,Y)$ *if $\widehat{\mathbb{D}}(s) \to D$ as $s \to +\infty$ on $(\omega,+\infty)$.*

If $\widehat{\mathbb{D}}(s) \to D$ as $\mathrm{Re}\, s \to +\infty$ on $\mathbf{C}^+_\omega$, then we call $\mathbb{D}$ line-regular (LR).

If $\mathbb{D}$ is regular and there is $\alpha > \omega$ s.t. $\widehat{\mathbb{D}}(\beta + iy) \to D$ as $y \to +\infty$, for all $\beta > \alpha$, then we call $\mathbb{D}$ vertically regular (VR).

If $\mathbb{D}$ is stable (or $\mathbb{D}[\mathrm{L}^2_c] \subset \mathrm{L}^2$) and $\widehat{\mathbb{D}}(s) \to D$ as $s \in \mathbf{C}^+$ and $|s| \to \infty$, then we call $\mathbb{D}$ half-plane regular (HPR).

In the above definitions, we modify the word "regular" with the word weakly (W), strongly (S) *or* uniformly (U) *according to the sense of convergence (of $\widehat{\mathbb{D}}(s) \to D$).*

Thus, WR *means weakly regular,* SHPR *means strongly half-plane-regular,* ULR *means uniformly line-regular etc.*

We call ($\widehat{\mathbb{D}}$ and) $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in$ WPLS *strongly regular and write "$\mathbb{D} \in$ SR" if the I/O map $\mathbb{D}$ is strongly regular, and we do analogously also for the other regularity concepts defined above.*

When Σ is WR, the generators *of Σ refer to the operators $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, we say that $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$* generate Σ *and we follow the classical convention to denote a system by its generators as in "$\Sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$" (cf. Lemma 6.1.16 and Definition 6.3.8).*

(Sometimes in the literature, the term "regular" means "strongly regular", and no other forms of regularity are defined. The operator D in "VR" refers to the one in "R", hence D is always the same for different forms of regularity.)

As above, we shall denote feedthrough operators by same letters as the corresponding WR operators ($D := \widehat{\mathbb{D}}(+\infty)$). The same also applies to the generators of the other components of WPLSs, as in Lemma 6.1.16. See Lemma 2.1.13 for more on the condition $\mathbb{D}[\mathrm{L}^2_c] \subset \mathrm{L}^2$.

We shall now make a few remarks on these different forms of regularity. Obviously, weak regularity is weaker than any other form of regularity. The definition of weak regularity means the existence of $D \in \mathcal{B}(U,Y)$ s.t.

$$\langle y_0, \widehat{\mathbb{D}}(s)u_0\rangle \to \langle y_0, Du_0\rangle \ \text{ as } s \in (\omega,+\infty) \text{ and } s \to +\infty, \text{ for all } u_0 \in U,\ y_0 \in Y. \tag{6.31}$$

It is apparent that whenever $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $\widehat{\mathbb{D}}(s)u_0$ converges weakly as $s \to +\infty$, for each $u_0 \in U$, then the limit is necessarily Du_0 for some (unique) $D \in \mathcal{B}(U,Y)$ satisfying $\|D\|_{\mathcal{B}(U,Y)} \le \|\mathbb{D}\|_{\mathrm{TIC}_\omega(U,Y)}$, hence then (and only then) $\mathbb{D}$ is WR.

Similarly, $\mathbb{D} \in \mathrm{ULR}(U,Y)$ with feedthrough operator $D \in \mathcal{B}(U,Y)$ iff $\|\widehat{\mathbb{D}}(s) - D\| \to 0$ as $\mathrm{Re}\, s \to +\infty$. Most systems treated in the literature are ULR; this includes the systems of [FLT], [LT00a] and [LT00b] (see also Lemma 6.3.16). We note that $\mathbb{D}$ is HPR iff $\widehat{\mathbb{D}} \in \mathrm{H}(\mathbf{C}^+;\mathcal{B})$ and $\widehat{\mathbb{D}} \circ \phi_{\text{Cayley}}$ has a limit at -1.

Although weak regularity suffices for most results on regularity, only uniform line-regularity makes the invertibility of a map equivalent to the invertibility of its feedthrough operator, as noted in Proposition 6.3.1(c), and hence allows for several results similar to the finite-dimensional ones. Therefore, "WR" and "ULR" are the most important ones of the twelve concepts above.

If $n := \dim Y < \infty$, then weak, strong and uniform convergence coincide (with the componentwise convergence of $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 \\ \vdots \\ \mathbb{D}_n \end{bmatrix}$).

A scalar example of an irregular transfer function is $\widehat{\mathbb{D}}(s) := \cos(\log s)$ (due to Kirsten Morris), but we do not know any physically motivated examples (crossing our fingers in hope that there were none). On the other hand, a bounded control operator B or observation operator C makes a system ULR, by Lemma 6.3.16; in particular, any Pritchard–Salamon system is ULR, by Lemma 6.9.4. Further examples of assumptions that guarantee a certain amount of regularity are given in Sections 2.6, 6.3 and 6.8.

We shall often write inclusions such as $\mathcal{B} \subset \mathrm{ULR} \subset \mathrm{TIC}_\infty$ or $\mathcal{B} \subset \mathrm{H}^\infty_{-1} \subset \mathrm{H}^\infty$ without specifying the input and output spaces (recall that $T \in \mathcal{B}$ is identified with $\mathbb{D}_T \in \mathrm{TIC}$ defined by $\mathbb{D}_T u := Tu$ for all $u \in \mathrm{L}^2$); the definition below makes this notation rigorous. (We often apply it with substitutions $\mathcal{X} := \{\text{all Hilbert spaces}\}$, $\mathcal{A}' := \mathrm{TIC}_\infty$ (or some smaller class) and $\mathcal{A}$ equal to some subclass of TIC.)

Definition 6.2.4 ($\boldsymbol{\mathcal{A} \underset{a}{\subset} \mathcal{A}'}$) *Let both $(\mathcal{A},\mathcal{X})$ and $(\mathcal{A}',\mathcal{X})$ be as in Lemma A.1.1 (or as in Remark A.1.3).*

If $\mathcal{A}(U,Y) \subset \mathcal{A}'(U,Y)$ for all $U,Y \in \mathcal{X}$, then we call $\mathcal{A}$ an algebraic subclass *of $\mathcal{A}'$ and write $\mathcal{A} \underset{a}{\subset} \mathcal{A}'$.*

Thus, $\mathrm{UHPR} \underset{a}{\subset} \mathrm{ULR} \underset{a}{\subset} \mathrm{SLR} \underset{a}{\subset} \mathrm{SR} \underset{a}{\subset} \mathrm{TIC}_\infty$ etc. Note that it follows that $\mathcal{A}(U,Y)$ is a subgroup of $\mathcal{A}'(U,Y)$ and $\mathcal{A}(U)$ is a subring of $\mathcal{A}'(U)$, for each $U,Y \in \mathcal{X}$ (where $\mathcal{X}$ is, e.g., the collection of all Hilbert spaces).

Because limits commute with most basic operations, such operations preserve regularity:

Lemma 6.2.5 (Regularity preserved) *Any form of regularity is preserved under sums and scalar multiplication, under (left or right) multiplication by static operators, and under convergence in TIC_ω ($\omega \in \mathbf{R}$).*

All strong and uniform properties are preserved under composition of maps. All weak and uniform properties are preserved under taking causal adjoints ($\mathbb{D} \mapsto \mathbb{D}^{\mathrm{d}} := \text{Я}\mathbb{D}^*\text{Я}$).

In general, uniformly $*$ $\Longrightarrow$ strongly $*$ $\Longrightarrow$ weakly $*$, when $*$ is anything suitable from Definition 6.2.3; similarly, $*$ half-plane-regular $\Longrightarrow$ $*$ line-regular $\Longrightarrow$ $*$ regular, $*$ half-plane-regular $\Longrightarrow$ $*$ vertically regular $\Longrightarrow$ $*$ regular.

Furthermore, $\mathbb{D}\mathbb{E}$ is WR with feedthrough DE (but $\mathbb{E}\mathbb{D}$ need not be WR) if $\mathbb{D}$ is WR and $\mathbb{E}$ is SR (or $\mathbb{D}^{\mathrm{d}}$ is SR and $\mathbb{E}$ is WR).

Thus, $\mathrm{TIC}_\omega(U,Y)\cap\mathrm{WR}$ is a closed subspace of TIC_ω ($\omega\in\mathbf{R}$), the same holds for SR, UR, ULR or any other regularity property in place of WR, and $\mathbb{D}\mapsto D$ is a bounded linear operation on any such subspace.

Proof: Obviously, the limits commute with sums, scalar multiplication and multiplication by static operators (e.g., for $L\in\mathcal{B}$, the map $L\mathbb{D}$ or $\mathbb{D}L$ has (at least) the same regularity properties as $\mathbb{D}$ has).

Clearly $\|D\|\le\|\widehat{\mathbb{D}}\|_{\mathrm{H}^\infty_\omega}=\|\mathbb{D}\|_{\mathrm{TIC}_\omega}$. Thus, if $\mathbb{D}_n\to\mathbb{D}$ in $\mathrm{TIC}_\omega(U,Y)$, and $\mathbb{D}_n\in\mathrm{WR}$ for all n, then $\{D_n\}$ converges in $\mathcal{B}(U,Y)$, and one easily verifies that $\mathbb{D}$ is WR and $D_n\to D$ weakly. An analogous claim holds for any other weak or uniform regularity property.

Similarly strong and uniform limits commute with composition; e.g., if $\mathbb{D},\mathbb{E}\in\mathrm{SR}$, then $\mathbb{D}\mathbb{E}\in\mathrm{SR}$ with feedthrough DE (see Lemma A.3.1(j2)).

From Lemma 6.2.2 we observe that weak and uniform limits commute with causal adjointing (e.g., if $\mathbb{D}$ is WR, then $\mathbb{D}^{\mathrm{d}}$ is WR with feedthrough D^*).

The claims on $\mathbb{D}\mathbb{E}$ and $\mathbb{E}\mathbb{D}$ follow from Lemma A.3.1(j2) and Example 6.2.6 (with $\mathbb{D}\mapsto\mathbb{D}^{\mathrm{d}}$, $\mathbb{E}\mapsto\mathbb{D}$). □

As the formulation of the above lemma hints, strong regularity is not inherited by adjoints, and weak regularity is not preserved under composition:

Example 6.2.6 ($\mathbb{D}\mathbb{D}^{\mathrm{d}}$ is not WR) Let $\mathbf{N}:=\{0,1,2,3,\ldots\}$, $U=\ell^2(\mathbf{N};\mathbf{C})$, $Y=\mathbf{C}$, $f(s):=s/(1+s)^2$, and

$$\widehat{\mathbb{D}}(s)u:=\sum_{n\in\mathbf{N}}f(10^{-n}s)u_n. \tag{6.32}$$

Then $\mathbb{D}\in\mathrm{TIC}$, $D=0$, $\mathbb{D}$ is strongly half-plane-regular, $\mathbb{D}^{\mathrm{d}}\in\mathrm{WR}\setminus\mathrm{SR}$ and $\mathbb{D}\mathbb{D}^{\mathrm{d}}\notin\mathrm{WR}$. ◁

(The above claims are explicitly or implicitly contained in the computations of Example 8.1 of [SW01b].)

It follows that weak regularity is not preserved under feedback or cascade connection (see the remarks below Proposition 6.6.18).

We have called $D:=\widehat{\mathbb{D}}(+\infty)$ the feedthrough operator of $\mathbb{D}$. This is justified, since the step response $\mathbb{D}\pi_+u_0$ is close to Du_0 near $t=0$ (in the average):

Proposition 6.2.7 ("$\mathbb{D}^t u_0\to Du_0$") *A map $\mathbb{D}\in\mathrm{TIC}_\infty(U,Y)$ is WR iff*

$$\operatorname*{w-lim}_{t\to0+}\frac{1}{t}\int_0^t\mathbb{D}\chi_{\mathbf{R}_+}u_0\,dm=:Du_0 \tag{6.33}$$

exists for all $u_0\in U$. If this is the case, then $D=\widehat{\mathbb{D}}(+\infty)$. Analogously, $\mathbb{D}$ is SR iff (6.33) converges strongly for each $u_0\in U$. □

(This is given in Theorem 4.6 of [SW01a] and Theorem 5.8 of [W94a]; it is the original definition of regularity.)

To obtain the convergence $(\mathbb{D}\chi_{\mathbf{R}_+}u_0)(t)\to Du_0$ (instead of the above convergence in the average), we seem to need a stronger assumption; e.g., for $\mathbb{D}\in\mathrm{SMTIC}_\infty$, $\mathbb{D}\chi_{\mathbf{R}_+}u_0$ becomes continuous with value Du_0 at zero, by Theorem 2.6.4(i3).

Recall from Lemma 6.1.16 and Definition 6.1.17 that $H_1 \underset{c}{\subset} H_B \underset{c}{\subset} H \underset{c}{\subset} H_{-1}$ for any $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(*,H,*)$, and that H_1 is the domain of the output operator C.

When $\mathbb{D} \in \mathrm{TIC}_\infty$ is the I/O map of a WPLS, the weak regularity of $\mathbb{D}$ is equivalent to the existence of the *(weak) Weiss extension* C_w of the output operator of the system, as shown below. This extremely important operator allows one to write the output of the system in the form $y(t) = C_{\mathrm{w}}x(t) + Du(t)$ (a.e.), and the transfer function as $\widehat{\mathbb{D}}(s) = D + C_{\mathrm{w}}(s-A)^{-1}B$, as shown later in this section.

Proposition 6.2.8 ($C_{\mathrm{w}}, C_{\mathrm{s}}, C_{\mathrm{L,s}}, C_{\mathrm{L,w}}, B^*_{\mathrm{w}}, \ldots$) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ *have generating operators* $\left[\begin{smallmatrix}A & B\\ C & *\end{smallmatrix}\right]$ *(cf. Definition 6.1.17), and let* $\omega \in \mathbf{R}$.

- *(a1)* Σ *is WR iff* $H_B \subset \mathrm{Dom}(C_{\mathrm{w}}) := \{x_0 \in H \,|\, \exists C_{\mathrm{w}}x_0 := \text{w-lim}_{s\to+\infty} Cs(s-A)^{-1}x_0\}$.
  If Σ *is WR, then* $C_{\mathrm{w}} \in \mathcal{B}(H_B,Y)$.
- *(a2)* Σ *is SR iff* $H_B \subset \mathrm{Dom}(C_{\mathrm{s}}) := \{x_0 \in H \,|\, \exists C_{\mathrm{s}}x_0 := \lim_{s\to+\infty} Cs(s-A)^{-1}x_0\}$.
  If Σ *is SR, then* $C_{\mathrm{s}} \in \mathcal{B}(H_B,Y)$.
- *(a3)* Σ *is regular in a certain sense iff* $C_{\mathrm{w}}x_0 := \lim_{s\to+\infty} Cs(s-A)^{-1}x_0$ *converges in the corresponding sense, for each* $x_0 \in (\alpha - A)^{-1}BU$ *(here* $\alpha \in \mathbf{C}^+_{\omega_A}$ *is irrelevant).*
- *(b1) We set* $\|x_0\|_{\mathrm{Dom}(C_{\mathrm{w}})} := \|x_0\|_H + \sup_{s>\omega_A+1} \|Cs(s-A)^{-1}x_0\|_Y$.

  Consequently, $\mathrm{Dom}(C_{\mathrm{w}})$ *becomes a Banach space and* $\mathrm{Dom}(C_{\mathrm{s}})$ *becomes its closed subspace. The operators* $C_{\mathrm{s}} : \mathrm{Dom}(C_{\mathrm{s}}) \to Y$ *and* $C_{\mathrm{w}} : \mathrm{Dom}(C_{\mathrm{w}}) \to Y$ *are continuous and* $H_1 \subset \mathrm{Dom}(C_{\mathrm{s}}) \subset \mathrm{Dom}(C_{\mathrm{w}}) \subset H$ *continuously.*
- *(b2) Part (b1) holds for any operator* $C \in \mathcal{B}(H_1,Y)$ *and any* C_0*-semigroup generator* A.
- *(c1)* Σ *is WR iff* $H_B \subset \mathrm{Dom}(C_{\mathrm{L,w}}) := \{x_0 \in H \,|\, \exists C_{\mathrm{L,w}}x_0 := \text{w-lim}_{t\to0+} \frac{1}{t}C\int_0^t \mathbb{A}^r x_0\,dr\}$.
- *(c2) We have* $x_0 \in \mathrm{Dom}(C_{\mathrm{L,w}})$ *iff* $\lim_{t\to0+} \frac{1}{t}\langle \mathbb{C}^*\chi_{[0,t)}y_0, x_0\rangle_H$ *exists for all* $y_0 \in H$*; if this is the case, then this limit is equal to* $\langle y_0, C_{\mathrm{L,w}}x_0\rangle_Y$.
- *(c3) We have* $x_0 \in \mathrm{Dom}(C_{\mathrm{L,w}})$ *iff* $\text{w-lim}_{t\to0+} \frac{1}{t}\int_0^t (\mathbb{C}x_0)(r)\,dr$ *exists; if this is the case, then this limit is equal to* $C_{\mathrm{L,w}}x_0$.
- *(c4) The strong forms of (c1)–(c3) also hold; the corresponding (strong Lebesgue) extension of* C *is denoted by* $C_{\mathrm{L,s}}$.
- *(c5) We set* $\|x_0\|_{\mathrm{Dom}(C_{\mathrm{L,w}})} := \|x_0\|_H + \sup_{t\in(0,1]} \|C\frac{1}{t}\int_0^t \mathbb{A}^r x_0\,dr\|_Y$.

  Consequently, $\mathrm{Dom}(C_{\mathrm{L,w}})$ *becomes a Banach space, and* $\mathrm{Dom}(C_{\mathrm{L,s}})$ *becomes its closed subspace. We have* $C_{\mathrm{L,w}} \in \mathcal{B}(\mathrm{Dom}(C_{\mathrm{L,w}}),Y)$*, and* $C_{\mathrm{L,s}} \in \mathcal{B}(\mathrm{Dom}(C_{\mathrm{L,s}}),Y)$*. Moreover,* $H_1 \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{L,s}}) \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{L,w}}) \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{w}}) \underset{c}{\subset} H$.
- *(c6)* $\mathrm{Dom}(A) \ni \frac{1}{t}\int_0^t \mathbb{A}^r x_0\,dr \to x_0$ *in* $\mathrm{Dom}(C_{\mathrm{L,s}})$*, as* $t \to 0+$*, for each* $x_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$*. In particular,* $\mathrm{Dom}(A)$ *is dense in* $\mathrm{Dom}(C_{\mathrm{L,s}})$.
- *(d1)* $C \subset C_{\mathrm{L,s}} \subset C_{\mathrm{s}} \subset C_{\mathrm{w}}$, $C \subset C_{\mathrm{L,s}} \subset C_{\mathrm{L,w}} \subset C_{\mathrm{w}}$*, hence* $B^* \subset B^*_{\mathrm{L,s}} \subset B^*_{\mathrm{s}} \subset B^*_{\mathrm{w}}$.

(d2) The domains of all operators of (d1) are dense in H.

(e) We have $B \in \mathcal{B}(U, \mathrm{Dom}(B^*_{\mathrm{L,s}})^*)$, and its adjoint is $B^*_{\mathrm{L,s}}$ (when $\mathrm{Dom}(B^*_{\mathrm{L,s}})^*$ (not H_{-1}) is considered as the range space of B and H (as always) the pivot space).

In particular, for any $u_0 \in U$, the functional $Bu_0 \in \mathrm{Dom}(B^*_{\mathrm{L,s}})^* \underset{c}{\subset} H_{-1} := \mathrm{Dom}(A^*)^*$ is the restriction of any of $(B^*_{\mathrm{L,w}})^* u_0$, $(B^*_{\mathrm{s}})^* u_0$ and $(B^*_{\mathrm{w}})^* u_0$.

(f) $A \in \mathcal{B}(H_B, \mathrm{Dom}(B^*_{\mathrm{L,s}})^*)$.

By $C \subset C_{\mathrm{s}}$ we mean that C_{s} is an extension of C (i.e., $H_1 = \mathrm{Dom}(C) \subset \mathrm{Dom}(C_{\mathrm{s}})$ and $C_{\mathrm{s}} x_0 = C x_0$ for all $x_0 \in \mathrm{Dom}(C)$). By B^*_{w} (resp. B^*_{s}, $B^*_{\mathrm{L,s}}$, $B^*_{\mathrm{L,w}}$) we always refer to $(B^*)_{\mathrm{w}}$ (resp. $(B^*)_{\mathrm{s}}$, $(B^*)_{\mathrm{L,s}}$, $(B^*)_{\mathrm{L,w}}$).

Note that, by (a1), $H^*_C \subset \mathrm{Dom}(B^*_{\mathrm{w}}) := \{x \in H \,\big|\, \exists B^*_{\mathrm{w}} x := \text{w-lim}_{s\to+\infty} B^* s(s - A^*)^{-1} x\}$ iff Σ^{d} is WR, and in that case, $B^*_{\mathrm{w}} \in \mathcal{B}(H^*_C, U)$.

Obviously, $C_{\mathrm{w}} = C_{\mathrm{s}} = C_{\mathrm{L,s}} = C_{\mathrm{L,w}} = C$ if C is bounded ($C \in \mathcal{B}(H,Y)$), and $H_B = H_1 \subset C_{\mathrm{s}}$ if B is bounded; hence Σ is SR if B or C is bounded. See Example 6.2.14 for a nontrivial example of C_{w} and B^*_{w}.

Note that we have equipped the domains of these four extensions of C with special norms instead of graph norms in order to make the domains Banach spaces (i.e., complete; it was noted in [W89b] that if $\dim Y < \infty$, then $C_{\mathrm{L,s}}$ cannot be closable unless C is bounded).

For more information on the *strong Weiss extension* C_{s} (or on $C_{\mathrm{L,s}}$), see [W89b] and [W94b, Section 5] or [Sbook]; for more information on the *weak Weiss extension* C_{w} (or on $C_{\mathrm{L,w}}$), see [SW01a], [WW, Sections 2 & 4] or [Sbook].

Proof: For the definitions of $H_B := (\alpha - A)^{-1}[H + BU]$ and H^*_C, see Definition 6.1.17.

(b) The SR counterpart of this is given on pp. 42–43 of [W94b]; the same proof applies for the WR claim mutatis mutandis, by [SW01a], and the requirements in (b2) are enough for this. (The details are in Proposition 4.3 of [W89b], also [W94a, p. 848] is relevant.) One can replace $\omega_A + 1$ by any other value $> \omega_A$ to obtain an equivalent norm, by the resolvent equation (Lemma A.4.4(a)).

(a1)&(a2) This follows from [S97b, Proposition 36], Lemma A.3.6 and (b).

(a3) Let $\mathbb{D}$ be WR. By (a1), Lemma A.4.4(a), and the linearity of C_{w}, we have

$$\begin{aligned} Cs(s-A)^{-1}(\alpha - A)^{-1} Bu_0 &= \frac{s}{s-\alpha}\big(C_{\mathrm{w}}(\alpha - A)^{-1} Bu_0 - C_{\mathrm{w}}(s-A)^{-1} Bu_0\big) \\ &\to 1 \cdot C_{\mathrm{w}}(\alpha - A)^{-1} Bu_0, \end{aligned} \tag{6.34}$$

as $s \to +\infty$, in the sense that $C_{\mathrm{w}}(s-A)^{-1} Bu_0 = \widehat{\mathbb{D}}(s) - D$ converges to zero (e.g., weakly, strongly, vertically, ..., but not independently on α), hence the claim.

(c4) The strong version of (c1) is Theorem 5.8 of [W94a]. The proofs of (c2)–(c3) apply in the strong case mutatis mutandis.

(c1) See Theorem 4.6 of [SW01a].

(c3) We have $\int_0^r \mathbb{C}x_0\,dm = C\int_0^r \mathbb{A}x_0\,dm$ for all $x_0 \in H$, because this holds on H_1 and both sides are continuous $H \to Y$. Thus, the extensions in (c1) and (c3) are equal.

(c2) We have $\langle y_0, \frac{1}{t}\int_{\mathbf{R}} \pi_{[0,t)}\mathbb{C}x_0\rangle_Y = \frac{1}{t}\int_{\mathbf{R}}\langle y_0, \pi_{[0,t)}\mathbb{C}x_0\rangle_Y = \frac{1}{t}\langle \pi_{[0,t)}y_0, \mathbb{C}x_0\rangle_{\mathrm{L}^2}$, hence (c2) follows from (c3) (use Lemma A.3.4(i3)).

(c5) It is shown in Proposition 4.3 of [W89b] that $\mathrm{Dom}(C_{\mathrm{L,s}})$ is a Banach space; the weak case is analogous (see [WW] or [Sbook]). The continuity of $C_{\mathrm{L,w}}$ and $C_{\mathrm{L,s}}$ is obvious, and so are the inclusions (use Lemma A.3.6 for continuity), except for $\mathrm{Dom}(C_{\mathrm{L,w}}) \subset \mathrm{Dom}(C_{\mathrm{w}})$, which is given in (d1).

(c6) This is given in the proof of Theorem 5.2 of [W94b]. It is an open problem whether H_1 is always dense in $\mathrm{Dom}(C_{\mathrm{s}})$.

(d1) See Proposition 4.2 of [SW01a].

(N.B. V. Katsnelson [KW] has constructed a system where $\mathrm{Dom}(C_{\mathrm{L,w}}) = \mathrm{Dom}(C_{\mathrm{L,s}}) \neq \mathrm{Dom}(C_{\mathrm{s}}) = \mathrm{Dom}(C_{\mathrm{w}})$. Example 6.2.6 shows that we may have $\mathrm{Dom}(C_{\mathrm{L,s}}) \neq \mathrm{Dom}(C_{\mathrm{L,w}})$ and $\mathrm{Dom}(C_{\mathrm{s}}) \neq \mathrm{Dom}(C_{\mathrm{w}})$. Thus, all inclusions in (d1) may be strict, as noted in Proposition 4.2 of [SW01a].)

(d2) Sets $\mathrm{Dom}(A)$ and $\mathrm{Dom}(A^*)$ are dense in H, hence so are any of their supersets.

(e) Note first that $\mathrm{Dom}(A^*) \underset{c}{\subset} \mathrm{Dom}(B^*_{\mathrm{L,s}}) \underset{c}{\subset} H \underset{c}{\subset} \mathrm{Dom}(B^*_{\mathrm{L,s}})^* \underset{c}{\subset} H_{-1}$, and that the first two (and the last) inclusions are dense, by (c6) and Lemma A.3.24 (we needed the density of H_1 in $\mathrm{Dom}(B^*_{\mathrm{L,s}})$ to obtain $\mathrm{Dom}(B^*_{\mathrm{L,s}})^* \subset H_{-1}$ (one-to-one)). For all $x_0 \in \mathrm{Dom}(A^*)$, $u_0 \in U$, we have

$$\langle x_0, Bu_0\rangle_{\langle H_1^*, H_{-1}\rangle} = \langle B^*x_0, u_0\rangle_U = \langle B^*_{\mathrm{L,s}}x_0, u_0\rangle_U = \langle x_0, (B^*_{\mathrm{L,s}})^*u_0\rangle_{\langle Z, Z^*\rangle}, \tag{6.35}$$

where $Z := \mathrm{Dom}(B_{\mathrm{L,s}})$. Thus, $Bu_0 \in H_{-1} := \mathrm{Dom}(A^*)^*$ is continuous w.r.t. to the $\|\cdot\|_{\mathrm{Dom}(B^*_{\mathrm{L,s}})}$ norm, hence and element of $\mathrm{Dom}(B^*_{\mathrm{L,s}})^*$. By (6.35) and density, this element is equal to $(B^*_{\mathrm{L,s}})^*u_0$.

Of course, we can replace $B^*_{\mathrm{L,s}}$ by any of its extensions (see (d1)) in equation $\langle x_0, Bu_0\rangle_{H_1^*, H_{-1}^*} = \langle B^*_{\mathrm{L,s}}x_0, u_0\rangle_U$ ($x_0 \in \mathrm{Dom}(A^*)$, $u_0 \in U$). (Note that this does not determine Bu_0 as an element of $\mathrm{Dom}(B^*_{\mathrm{L,w}})$ uniquely unless H_1 happens to be dense in $\mathrm{Dom}(B^*_{\mathrm{L,w}})$; situation is the same for B^*_{s} and B^*_{w}.)

(f) For any $x_0 \in H_B$, we have $z_0 := Ax_0 + Bu_0 \in H$ for some $u_0 \in U$, hence $Ax_0 = z_0 - Bu_0 \in \mathrm{Dom}(B^*_{\mathrm{L,s}})^*$, by (e). Thus, $A[H_B] \subset \mathrm{Dom}(B^*_{\mathrm{L,s}})^*$. But $A \in \mathcal{B}(H, H_{-1})$ and $H_B \underset{c}{\subset} H$, hence $A \in \mathcal{B}(H_B, H_{-1})$. Therefore, $A \in \mathcal{B}(H_B, \mathrm{Dom}(B^*_{\mathrm{L,s}})^*)$, by Lemma A.3.6 and (e). $\square$

Next we present four lemmas that explore the relation between a system and its generators:

Lemma 6.2.9 ($\left[\begin{smallmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D}\end{smallmatrix}\right]$) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U, H, Y)$ *have generators* $\left[\begin{smallmatrix} A & B \\ C & \end{smallmatrix}\right]$ *(resp.* $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$*, and* $\mathbb{D} \in \mathrm{WR}$*). Then*

(a) The operators $\begin{bmatrix} A & B \\ C & \widehat{\mathbb{D}}(s)\end{bmatrix}$ *(resp.* $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$*) determine* Σ *uniquely for any* $s \in \mathbf{C}^+_{\omega_A}$*.*

(b) The generators of Σ^{d} *are* $\left[\begin{smallmatrix} A^* & C^* \\ B^* & \end{smallmatrix}\right]$ *(resp.* $\left[\begin{smallmatrix} A^* & C^* \\ B^* & D^*\end{smallmatrix}\right]$*, and* Σ^{d} *is WR).*

(c) The generators of $\mathcal{T}_\omega\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ *are* $\left[\begin{smallmatrix}A+\omega I & C\\ B & *\end{smallmatrix}\right]$ *(resp.* $\left[\begin{smallmatrix}A+\omega I & C\\ B & D\end{smallmatrix}\right]$*, and* $\mathcal{T}_\omega\Sigma$ *is WR). Moreover,* $(C_\omega)_{\mathrm{w}} = C_{\mathrm{w}}$ *and* $(C_\omega)_{\mathrm{L,w}} = C_{\mathrm{L,w}}$.

See Proposition 6.6.18 for the generating operators of closed-loop systems.

Proof: (a) This follows from Lemma 6.1.16(d) (resp. and Theorem 6.2.13(a1)).

(b) This holds by Lemma 6.1.16(e) (resp. and Lemma 6.2.2).

(c) For generators, the simple proof is given in Example 4.7.2 of [Sbook]. Because $s(s-A_\omega)^{-1} = \frac{s}{s-\omega}(s-\omega)(s-\omega-A)^{-1}$, the C_{w} claim follows easily. The $C_{\mathrm{L,w}}$ claim as follows from the fact that (here $f := \mathbb{C}x_0 \in \mathrm{L}^2_{\mathrm{loc}} \subset \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+;Y)$):

$$t^{-1}\int_0^t (1-\mathrm{e}^{-\omega r} f(r)\,dr = t^{-1}\int_0^t \omega r \mathrm{e}^{-\omega r\xi_r} f(r)\,dr \to 0 \tag{6.36}$$

(here $\xi_r \in (0,1)$ for all $r > 0$ (use the Mean Value Theorem)), as $t \to 0+$. □

We shall soon need the following technical lemma:

Lemma 6.2.10 ($\mathbb{B}u = (s-A)^{-1}Bu_0$, $(\mathbb{D}u)(t) = \mathrm{e}^{st}\widehat{\mathbb{D}}(s)u_0$) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in$ WPLS *and let* $\mathrm{Re}\, s > \omega_A$. *Let* $u_0 \in U$ *and* $u(t) = \mathrm{e}^{st}u_0$ *(*$t \in \mathbf{R}$*), so that* $\pi_- u \in \mathrm{L}^2_{\omega_A}(\mathbf{R};U)$. *Then* $\mathbb{B}u = \mathbb{B}\pi_- u = (s-A)^{-1}Bu_0 \in H_B \subset H$, $\mathbb{B}\tau^t u = \mathrm{e}^{st}(s-A)^{-1}Bu_0$ *and* $(\mathbb{D}u)(t) = \mathrm{e}^{st}\widehat{\mathbb{D}}(s)u_0$ *for all* $t \in \mathbf{R}$. *Moreover, we have*

$$\mathbb{B}\tau^t\pi_+\mathrm{e}^{s\cdot}u_0 = (\mathrm{e}^{st} - \mathbb{A}^t)(s-A)^{-1}Bu_0 \quad (t \geq 0), \tag{6.37}$$

$$\mathbb{D}\pi_+\mathrm{e}^{s\cdot}u_0 = \pi_+\mathrm{e}^{s\cdot}\widehat{\mathbb{D}}(s)u_0 - \mathbb{C}(s-A)^{-1}Bu_0 \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U). \tag{6.38}$$

Proof: The claim about $\mathbb{D}$ is Lemma 2.1.15. Because $\widehat{\mathbb{B}\tau} = (s-A)^{-1}B$ (see Theorem 6.2.11(b1)), claims on $\mathbb{B}u$ and $\mathbb{B}\tau u$ follow from this. But $\mathbb{B}\tau^t\pi_- = \mathbb{A}^t\mathbb{B}$, hence (6.37) follows; (6.38) is obtained analogously, by using $\pi_+\mathbb{D}\pi_- = \mathbb{C}\mathbb{B}$.

(An alternative proof of Lemma 2.1.15 would be to obtain $\mathbb{B}u = (s-A)^{-1}Bu_0$ from (6.24) and Lemma A.4.4(f), and then $(\mathbb{D}u)(t) = C_{\mathrm{c}}(s-A)^{-1}B\mathrm{e}^{st}u_0 + D_{\mathrm{c}}\mathrm{e}^{st}u_0 = \mathrm{e}^{st}\widehat{\mathbb{D}}(s)u_0$ from Lemma 6.3.10(c).) □

Now we are ready to show that for any $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}_\beta(U,H,Y)$, $\beta \in \mathbf{R}$, $s \in \mathbf{C}^+_\beta$, $u \in \mathrm{L}^2_\beta(\mathbf{R}_+;U)$ and $x_0 \in H$, we have

$$\begin{cases}\widehat{x}(s) = (s-A)^{-1}x_0 + (s-A)^{-1}B\widehat{u}(s)\\ \widehat{y}(s) = C(s-A)^{-1}x_0 + C_{\mathrm{w}}(s-A)^{-1}B\widehat{u}\end{cases} \quad \text{where} \quad \begin{cases}x = \mathbb{A}x_0 + \mathbb{B}\tau u,\\ y = \mathbb{C}x_0 + \mathbb{D}u,\end{cases} \tag{6.39}$$

(the formula for $\widehat{y}$ requires that $\mathbb{D}$ is WR or $u = 0$). This and further information on the Laplace transforms of the components of a system are given below:

Theorem 6.2.11 ($\widehat{\mathbb{A}}, \widehat{\mathbb{B}}, \widehat{\mathbb{C}}, \widehat{\mathbb{D}}$) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *have generating operators* $\left[\begin{smallmatrix}A & B\\ C & *\end{smallmatrix}\right]$ *and let* $\omega := \omega_A \leq \alpha < \beta$.

(a) $\widehat{\mathbb{A}(\cdot)x_0}(s) = (s-A)^{-1}x_0$ *for all* $x_0 \in H$, $s \in \mathbf{C}^+_\omega$.

(b1) $\widehat{\mathbb{B}\tau u}(s) = (s-A)^{-1}B\widehat{u}(s)$ *for* $u \in \mathrm{L}^2_\alpha(\mathbf{R}_+;U)$ *and* $s \in \mathbf{C}^+_\alpha$.

(b2) $(s-A)^{-1}B \in \mathrm{H}^\infty(\mathbf{C}^+_\beta;\mathcal{B}(U,H)) \cap \mathrm{H}(\mathbf{C}^+_\omega;\mathcal{B}(U,H_B))$.

(b3) $\|(s-A)^{-1}B\|_{\mathcal{B}(U,H)} \le \|\mathbb{B}\|_{\mathcal{B}(\mathrm{L}^2_\gamma(\mathbf{R};U),H)}/\sqrt{2}\sqrt{\operatorname{Re}s-\gamma}$ *for* $s \in \mathbf{C}^+_\gamma$, $\gamma \in \mathbf{R}$.

(c1) $\widehat{\mathbb{C}x_0}(s) = C(s-A)^{-1}x_0$ *for* $s \in \mathbf{C}^+_\omega$, $x_0 \in H$.

(c2) $C(s-A)^{-1} \in \mathrm{H}^\infty(\mathbf{C}^+_\beta;\mathcal{B}(H,Y))$, *and* $C(s-A)^{-1}x_0 \in \mathrm{H}^2(\mathbf{C}^+_\beta;Y)$ *for* $x_0 \in H$. *Moreover,* $\|C(\cdot-A)^{-1}\|_{\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))} = \sqrt{2\pi}\|\mathbb{C}\|_{\mathcal{B}(H,\mathrm{L}^2_\omega)}$.

(c3) $\|C(s-A)^{-1}\|_{\mathcal{B}(H,Y)} \le \|\mathbb{C}\|_{\mathcal{B}(H,\mathrm{L}^2_\gamma(\mathbf{R};Y))}/\sqrt{2}\sqrt{\operatorname{Re}s-\gamma}$ *for* $s \in \mathbf{C}^+_\gamma$, $\gamma \in \mathbf{R}$.

(d1) $\widehat{\mathbb{D}}(s) = C_{\mathrm{w}}(s-A)^{-1}B + D$ *on* $\mathbf{C}^+_\omega$ *if* $\mathbb{D}$ *is WR,* [2] *hence then* $C_{\mathrm{w}}(s-A)^{-1}Bu_0 \to 0$ *weakly as* $s \to +\infty$*, for all* $u_0 \in H$*. The same holds also for* C_{s} *in place of* C_{w} *if* $\mathbb{D}$ *is SR.*

(d2) For $s, s_0 \in \mathbf{C}^+_\omega$ *we have*

$$\frac{\widehat{\mathbb{D}}(s)-\widehat{\mathbb{D}}(s_0)}{s-s_0} = -C(s-A)^{-1}(s_0-A)^{-1}B = -(\mathcal{L}(\pi_+\mathbb{D}\pi_-\mathrm{e}^{s_0\cdot}u_0))(s). \tag{6.40}$$

Proof: (a) This is Lemma A.4.4(f).

(b1) This is [W89a, Remark 3.12].

(b2) By (c2) and Lemma 6.1.4, $B^*(s-A^*)^{-1} \in \mathrm{H}^\infty(\mathbf{C}^+_{\omega'};\mathcal{B}(H,U))$, from which the "$\in \mathrm{H}^\infty$" claim follows.

Set $F(s) := (s-A)^{-1}B$. Then $F(\alpha) \in \mathcal{B}(U,H_B)$ (with $\|F(\alpha)u_0\|_{H_B} \le \|u_0\|_U$), where α is as in Definition 6.1.17, and $F(s)-F(\alpha) = (\alpha-s)(\alpha-A)^{-1}(s-A)^{-1}B \in \mathrm{H}(\mathbf{C}^+_\omega;\mathcal{B}(U,H_1))$, by Lemma A.4.4(a). But $\mathcal{B}(U,H_1) \subset \mathcal{B}(U,H_B)$ continuously, hence $F(s) \in \mathrm{H}(\mathbf{C}^+_\omega;\mathcal{B}(U,H_B))$, by Lemma D.1.2(b1).

(b3) Let $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in$ WPLS be s.t. $\mathbb{B}$ is stable, $u_0 \in U$, $\operatorname{Re}s > \omega_A$. Set $u := \pi_-\mathrm{e}^{st}u_0 \in \mathrm{L}^2(\mathbf{R}_-;U)$. By Lemma 6.2.10, we have

$$\|(s-A)^{-1}Bu_0\| = \|\mathbb{B}u\| \le \|\mathbb{B}\|_{\mathcal{B}(\mathrm{L}^2;H)}\|u\|_2 = \|\mathbb{B}\|\,\|u_0\|_H/\sqrt{2\operatorname{Re}s}. \tag{6.41}$$

By shifting, we obtain an analogous claim for an arbitrary $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in$ WPLS (note that $\|\mathbb{B}\| < \infty$ at least for $\gamma > \omega_A$).

(The condition "$\|(s-A)^{-1}Bu_0\| \le M/(\operatorname{Re}s)^{1/2}$ for all $s \in \mathbf{C}^+$" is also suffient for given $B \in \mathcal{B}(U,H_{-1})$ to generate a WPLS with A if, e.g., A is left-invertible [W91b] or A generates a contraction semigroup and U is finite-dimensional [JP]; however this is not the case in general, see [JZ00] for a counter-example.

(c1) This is (3.6) on p. 26 of [W89b].

(c2) By (c1) and the Paley–Wiener Theorem Theorem 3.3.1(b), we have $\|\widehat{\mathbb{C}}x_0\|_{\mathrm{H}^2(\mathbf{C}^+_\omega;Y)} = \sqrt{2\pi}\|\mathbb{C}x_0\|_{\mathrm{L}^2}$ for all $x_0 \in H$, hence the second and third claim hold. The first claim follows from the third claim and Lemma F.3.2(a).

[2] If $\mathbb{D} \in \mathrm{TIC}_\alpha$ for some $\alpha < \omega_A$, then, by $C_{\mathrm{w}}(s-A)^{-1}B + D$ for $s \in \mathbf{C}_\alpha \setminus \mathbf{C}_{\omega_A}$, we mean the unique analytic extension $\widehat{\mathbb{D}}(\cdot)$ of this function (otherwise the equation would not always hold for $s \in \mathbf{C}^+_\alpha \cap \sigma(A)^c \setminus \mathbf{C}^+_{\omega_A}$, see [W94a, Remark 4.8]).

(c3) This is the dual of (b3).

(d1) This is [WW, Theorem 4.4].

(d2) The first identity is given in [Sal89] (and in Theorem 4.5.9 of [Sbook]; it follows from (6.51)). For the second, we note that

$$(\mathcal{L}(\pi_+\mathbb{D}\pi_-\mathrm{e}^{s_0\cdot}u_0))(s) = (\mathcal{L}(\mathbb{C}\mathbb{B}\mathrm{e}^{s_0\cdot}u_0))(s) = (\mathcal{L}(\mathbb{C}(s_0-A)^{-1}Bu_0))(s) \tag{6.42}$$

$$= C(s-A)^{-1}(s_0-A)^{-1}Bu_0, \tag{6.43}$$

by Lemma 6.2.10 and (c1). □

We shall need several different forms of the formula "$\mathbb{C} = C\mathbb{A}$":

Lemma 6.2.12 ($\mathbb{C} = C_\mathrm{w}\mathbb{A}$) *Let $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, $\omega > \omega_A$, $x_0 \in H$, $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;H)$, $x = \mathbb{A}x_0 + \mathbb{B}\tau u$. Set $C^r_{\mathrm{L,s}}x_1 := C\frac{1}{r}\int_0^r \mathbb{A}^s x_1\,ds$ $(x_1 \in H)$. Then*

- *(a) $(\mathbb{C}x_0)(t) = C_{\mathrm{L,s}}\mathbb{A}^t x_0 = C_\mathrm{w}\mathbb{A}^t x_0$ for a.e. $t \geq 0$ (for all $t \geq 0$ if $x_0 \in \mathrm{Dom}(A)$). In particular, $\mathbb{A}^t x_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$ a.e.*
- *(b1) We have $C^r_{\mathrm{L,s}}x \to C_{\mathrm{L,s}}x$ in L^2_ω and pointwise a.e., as $r \to 0+$, (in particular, $x \in \mathrm{Dom}(C_{\mathrm{L,s}})$ a.e.) if $\mathbb{D}$ is SR.*
- *(b2) We have $C^r_{\mathrm{L,s}}x \to C_{\mathrm{L,w}}x$ weakly in L^2_ω and weakly pointwise a.e., as $r \to 0+$, (in particular, $x \in \mathrm{Dom}(C_{\mathrm{L,w}})$ a.e.) if $\mathbb{D}$ is WR.*
- *(c1) Assume that $\mathbb{D}$ is SR. Then, for any $f \in \mathrm{L}^2_{-\omega}(\mathbf{R}_+)$, we have*

$$\int_{\mathbf{R}_+} f(t)C_{\mathrm{L,s}}x(t)\,dt = C_{\mathrm{L,s}}\int_{\mathbf{R}_+} f(t)x(t)\,dt \tag{6.44}$$

  (in particular, $\int_0^\infty f(t)x(t)\,dt \in \mathrm{Dom}(C_{\mathrm{L,s}})$). Thus, then $\int_{T_1}^{T_2} f(t)C_{\mathrm{L,s}}x(t)\,dt = C_{\mathrm{L,s}}\int_{T_1}^{T_2} f(t)x(t)\,dt$ for any $T_1,T_2 \in \mathbf{R}_+$, $f \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R})$.
- *(c2) Claim (c1) also holds with replacements SR$\mapsto$WR and $C_{\mathrm{L,s}} \mapsto C_{\mathrm{L,w}}$.*
- *(c3) In (b1)–(c2), we may replace $\mathbf{R}_+$ by $[T,+\infty)$, and allow for any $u \in \mathrm{L}^2_\omega([T,+\infty);U)$ $(T \in \mathbf{R})$.*
- *(c4) In (b1)–(c2), $\mathbb{D}$ need not be regular if $u = 0$.*

Proof: (a) By [W89b, Theorem 4.5], we have $(\mathbb{C}x_0)(t) = C_{\mathrm{L,s}}\mathbb{A}^t x_0$ for any (right-)Lebesgue point t of $\mathbb{C}x_0$, hence for a.e. $t \geq 0$. Recall that $C_{\mathrm{L,s}} \subset C_\mathrm{w}$.

(b1) The L^2_ω claim is Lemma 4.4 of [W94a]; the pointwise claim follows from the fact that $x(t) \in \mathrm{Dom}(C_{\mathrm{L,s}})$ a.e., by Theorem 5.8 of [W94a].

(b2)&(c2) The proofs of (b1) and (c1) apply mutatis mutandis (which means corresponding slight changes in the proofs of [W94a]; these results were contained in a preprint of [SW01a] and implicitly used in [SW01a]).

(c1) This is essentially given in Theorem 4.6 of [W94a] (substitute $f \mapsto f\chi_{[T_1,T_2)}$ to obtain the second claim).

(Alternatively, claim (c1) follows from (b1) (the proof of (c2) is analogous), since $fC^r_{\mathrm{L,s}}x \to fC_{\mathrm{L,s}}x$ in $\mathrm{L}^1(\mathbf{R}_+;Y)$, $fx, C^r_{\mathrm{L,s}}fx, C_{\mathrm{L,s}}x \in \mathrm{L}^1$, and $C^r_{\mathrm{L,s}}$ commutes with the integral (because $C^r_{\mathrm{L,s}} \in \mathcal{B}(H,Y)$).)

(c3) W.l.o.g., we assume that $x_0 = 0$ (substract (6.44) if necessary). Just shift u (hence x too) and f, and use the original claim.

(c4) Replace B by 0 to obtain a SR (in fact, ULR) WPLS with same $\mathbb{A}$ and $\mathbb{C}$ (see Lemma 6.3.16(b)). □

Now we may extend Lemma 6.1.16 by presenting the standard formulae $x' = Ax + Bu$ and $y = Cx + Du$ with certain limitations:

Theorem 6.2.13 ($x' = Ax + Bu$, $y = Cx + Du$) *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ *and either*

(i) $x_0 \in H$, $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$ *and* $J = \mathbf{R}_+$, *or*

(ii) $x_0 = 0$, $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$ *and* $J = \mathbf{R}$.

Set

$$\begin{cases} x &= \mathbb{A}x_0 + \mathbb{B}\tau u, \\ y &= \mathbb{C}x_0 + \mathbb{D}u. \end{cases} \tag{6.45}$$

Then

(a1) $x' = Ax + Bu \in \mathrm{L}^2_{\mathrm{loc}}(J;H_{-1})$ *(a.e.) and* $x \in \mathcal{C}(J;H) \cap \mathrm{W}^{1,2}_{\mathrm{loc}}(J;H_{-1})$.

(a2) If $\mathbb{D}$ *is WR, then the following formulae hold (in particular,* $x(t) \in \mathrm{Dom}(C_{\mathrm{L,w}})$*) for almost every* $t \in J$*, including those* $t \in J$ *where* u *and* y *are right-continuous:*

$$y(t) = C_{\mathrm{L,w}}x(t) + Du(t) \tag{6.46}$$

$$= C_{\mathrm{L,w}}\int_0^t \mathbb{A}(t-s)Bu(s)\,ds + C_{\mathrm{L,w}}\mathbb{A}x_0 + Du(t) \qquad \text{(case (i) only)} \tag{6.47}$$

$$= C_{\mathrm{L,w}}\lim_{T\to-\infty}\int_{-T}^t \mathbb{A}(t-s)Bu(s)\,ds + Du(t) \qquad \text{(case (ii) only)}\,. \tag{6.48}$$

If $\mathbb{D}$ *is SR, then we can use* $C_{\mathrm{L,s}}$ *in place of* $C_{\mathrm{L,w}}$ *in (6.46)–(6.48).*

Stricten (i) and (ii) as follows:

(i) $x_0 \in H$, $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbf{R}_+;U)$, $x_0' := Ax_0 + Bu(0) \in H$ *and* $J = \mathbf{R}_+$, *or*

(ii) $x_0 = 0 = x_0'$, $u \in \mathrm{W}^{1,2}_\omega(\mathbf{R};U)$ *and* $J = \mathbf{R}$.

Then,

(b1)
$$x' = \mathbb{A}x_0' + \mathbb{B}\tau u' = Ax + Bu \in \mathcal{C}(J;H), \quad y' = \mathbb{C}x_0' + \mathbb{D}u', \tag{6.49}$$

$$e^{-\omega\cdot}x, e^{-\omega\cdot}x' \in \mathcal{C}_{\mathrm{b}}(J;H), \quad e^{-\omega\cdot}x \in \mathcal{C}_{\mathrm{b}}(J;H_B), \quad \text{and} \quad y \in \mathrm{W}^{1,2}_\omega. \tag{6.50}$$

(b2) $y \in \mathrm{W}^{1,2}_{\mathrm{loc}}$ *and* $y' = \mathbb{D}u'$. *If* $u \in \mathrm{W}^{1,2}_\omega$, *then* $y \in \mathrm{W}^{1,2}_\omega$. *Moreover, for any* $s_0 \in \mathbf{C}^+_\omega$ *we have*

$$(\mathbb{D}u)(t) = C(x(t) - (s_0 - A)^{-1}Bu(t)) + \widehat{\mathbb{D}}(s_0)u(t) \quad (t \in J). \tag{6.51}$$

(c1) If $x_0 \in H_n := \mathrm{Dom}(A^n)$, $n \in \mathbf{N}$, *then* $\mathbb{A}x_0 \in \mathcal{C}^k(\mathbf{R}_+; H_{n-k})$ $(k = 0, 1, \ldots, n)$ *and* $y := \mathbb{C}x_0 \in \mathcal{C}^{k-1}(\mathbf{R}_+; H_{n-k})$ $(k = 1, \ldots, n)$ ($y \in \mathcal{C}^k(\mathbf{R}_+; H_{n-k})$ *if* $C \in \mathcal{B}(H,Y)$).

(c2) If $u \in \mathrm{W}^{n,2}_\omega(\mathbf{R}; U)$, $n \in \mathbf{N}$, *then* $x := \mathbb{B}\tau u$, $\mathrm{e}^{-\omega\cdot} x \in \mathcal{C}^n_{\mathrm{b}}(\mathbf{R}; H) \cap \mathcal{C}^{n-1}_{\mathrm{b}}(\mathbf{R}; H_B) \cap \mathrm{W}^{n-1,2}(\mathbf{R}; H_B)$, $y := \mathbb{D}u \in \mathrm{W}^{n,2}_\omega(\mathbf{R}; Y) \subset \mathcal{C}^{n-1}$, $x^{(n)} = \mathbb{B}\tau u^{(n)}$, $y^{(n)} = \mathbb{D}u^{(n)}$.

(d) If, instead, $y \in \mathrm{L}^2_\omega(\mathbf{R}; Y)$ *is arbitrary, and* $\mathbb{D} \in \mathrm{WR}$, *then*

$$(\mathbb{D}^* y)(t) = D^* y(t) + B^*_{\mathrm{L,w}} \lim_{T\to\infty} \int_0^T \mathbb{A}^*(s) C^* y(t+s)\, ds \tag{6.52}$$

a.e. and at every $t \in \mathbf{R}$ *at which* y *and* $\mathbb{D}^* y$ *are left-continuous. (We can replace* $B^*_{\mathrm{L,w}}$ *by* $B^*_{\mathrm{L,s}}$ *if* $\mathbb{D}^{\mathrm{d}} \in \mathrm{SR}$*).*

We conclude that $\mathbb{D} = D + C\mathbb{B}\tau$ when C is bounded. Recall from Proposition 6.2.8 that $C_{\mathrm{L,s}} \subset C_{\mathrm{s}} \subset C_{\mathrm{w}}$ and $C_{\mathrm{L,w}} \subset C_{\mathrm{L,s}} \subset C_{\mathrm{w}}$.

Proof: (a)&(b) This is well-known (see, e.g., Sections 4.2 and 4.5 of [Sbook] or combine Section 4 of [WW] and Propositions 29 and 33 of [S97b] with Remark 6.1.9 and Lemma 6.1.16).

(c1) This follows from Lemma A.4.2(c5). (Recall that $\mathbb{C}x_0 = C\mathbb{A}x_0$ for $x_0 \in \mathrm{Dom}(A)$.)

(c2) (Ignore $\mathcal{C}^{n-1}$ and $\mathrm{W}^{n-1,2}_\omega$ for $n = 0$.) This follows by applying (b2) subsequently, except for the H_B claims. We have $\tau u \in \mathcal{C}^{n-1}(\mathbf{R}; \mathrm{W}^{1,2}_\omega)$, by Lemma B.7.11, and $\mathbb{B} \in \mathcal{B}(\mathrm{W}^{1,2}_\omega, H_B)$, by Lemma 6.3.19. Therefore, $\mathbb{B}\tau u \in \mathcal{C}^{n-1}(\mathbf{R}; H_B)$.

Moreover, for $u \in \mathrm{W}^{n,2}_\omega$, $n \in \mathbf{N}+1$, $x := \mathbb{B}\tau u$, we have $(\alpha - A)x = \alpha x - x' + Bu$, and $\alpha x - x' \in \mathrm{L}^2_\omega(\mathbf{R}; H)$, $u \in \mathrm{L}^2_\omega(\mathbf{R}; U)$, hence $x \in \mathrm{L}^2_\omega(\mathbf{R}; H_B)$, because $(\alpha - A)^{-1}\begin{bmatrix} I & B \end{bmatrix} \in \mathcal{B}(H \times U, H_B)$. By induction, we have $x \in \mathrm{W}^{n-1,2}(\mathbf{R}; H_B)$.

(d) We have $\mathbb{D}^* y = \boldsymbol{Я}\mathbb{D}^{\mathrm{d}}\boldsymbol{Я} y$, so that this follows from (6.48) and Lemma 6.2.9(b). □

The following standard delay line example (to which we return several times later) illustrates the symbols defined in this section:

Example 6.2.14 (C_{w} and B^*_{w}) Let $U = \mathbf{C} = Y$, $H := \mathrm{L}^2(\mathbf{R}_+; Y)$, and

$$\Sigma := \left[\begin{array}{c|c} \pi_+\tau & \pi_{[0,1)}\tau(-1) \\ \hline I & \tau(-1) \end{array}\right] \in \mathrm{WPLS}_0(U, H, Y). \tag{6.53}$$

More details of this system are given in [WZ], [S95] and [Sbook]; the reader might also wish to consult some text book on distributions (e.g., [Rud73]) for H_{-1} and H^*_{-1}, the duals of H_1 and H^*_1 w.r.t. L^2.

Now $\widehat{\mathbb{D}}(s) = \mathrm{e}^{-s} \to 0$, as $\mathrm{Re}\, s \to \infty$, hence $\mathbb{D} \in \mathrm{ULR}$ and $D = 0$ (in fact, $\mathbb{D} \in \mathrm{MTIC_d}$). By Proposition 6.2.8(a2), we have $C_{\mathrm{w}} \subset \mathcal{B}(H_B, Y)$ and $B^*_{\mathrm{w}} \subset \mathcal{B}(H^*_C, U)$. (Because $\dim U = 1 = \dim Y$, we have in fact that $C_{\mathrm{w}} = C_{\mathrm{s}}$ and $B^*_{\mathrm{w}} = B^*_{\mathrm{s}}$.)

By Proposition B.7.12, we have

$$A = \frac{d}{d\theta}, \qquad H_1 := W^{1,2}((0,\infty)) := \{f \in H \,|\, f' \in H\} \tag{6.54}$$

$$A^* = -\frac{d}{d\theta}, \qquad H_1^* = W_0^{1,2}((0,\infty)) := \{f \in W^{1,2}((0,\infty)) \,|\, f(0+) = 0\}. \tag{6.55}$$

(Here and below, $\theta \in \mathbf{R}_+$ is the argument of an element (function) of H. Recall $W^{1,2}((0,\infty)) \subset C_0(\mathbf{R}_+)$, continuously, by Theorem B.7.4 and Lemma B.7.6.)

Because $\mathbb{C} = I$, we have $x_0(t) = (\mathbb{C}x_0)(t) = C\pi_+\tau^t x_0$ for $x_0 \in H$, and $C = \delta_0^* : f \mapsto f(0)$ is the unique operator $C \in \mathcal{B}(H_1, Y)$ satisfying this. By Lemma 6.2.9(b), we have $(\text{Я}\mathbb{B}^* f)(t) = B^*\mathbb{A}^*(t)f$ for $f \in H_1^*$, hence

$$B^*\tau(-t)\pi_+ f = B^*\mathbb{A}^*(t)f = (\tau^1\pi_{[0,1)}f)(-t) = (\pi_{[0,1)}f)(1-t), \tag{6.56}$$

hence $B^* = \delta_1^* : f \mapsto f(1)$. Summarizing,

$$Bu_0 = u_0\delta_1 \in H_{-1}, \qquad C = \delta_0^* : x_0 \mapsto x_0(0) \in Y = \mathbf{C}, \quad D = 0 = D^*; \tag{6.57}$$

$$B^* = \delta_1^*, \qquad C^* y_0 = y_0\delta_0, \quad (u_0 \in U,\ x_0 \in H_1,\ y_0 \in Y). \tag{6.58}$$

By taking Laplace transforms of $\mathbb{A}x_0$, $\mathbb{A}^*x_0$ and $\mathbb{B}\tau^t\pi_+u_0$ for $x_0 \in H$ and $u_0 \in U$, we obtain

$$H_1 \ni (s-A)^{-1}x_0 : \theta \mapsto \int_\theta^\infty e^{-s(r-\theta)}x_0(r)\,dr, \tag{6.59}$$

$$H_1^* \ni (s-A^*)^{-1}x_0 : \theta \mapsto \int_0^\theta e^{-sr}x_0(\theta-r)\,dr, \tag{6.60}$$

$$H_B \ni (s-A)^{-1}Bu_0 = \pi_{[0,1)}(\cdot)e^{-s(1-\cdot)}u_0 \qquad (s \in \mathbf{C}^+). \tag{6.61}$$

Note that "$\pi_{[0,1)}$" in (6.61) (instead of "$\pi_{[0,1]}$") is just our choice — the elements of H are known just a.e.

By Definition 6.1.17, the space $H_B \subset H$ is given by

$$H_B = H_1 + \mathbf{C}\pi_{[0,1)}(\cdot)e^{-s(1-\cdot)} = \{f \in H \,|\, f' \in H + \mathbf{C}\delta_1\} \tag{6.62}$$

$$= W^{1,2}((0,1)) + W^{1,2}([1,\infty)) \tag{6.63}$$

(we used here the choice made above; note that elements of H_B are bounded and continuous from the right on $\mathbf{R}_+$). We can set, e.g., $\alpha = 1$, to make the norm on H_B equal to

$$\|(1-A)^{-1}(x_0 + Bu_0)\|_{H_B} := \|(x_0, u_0)\|_{H\times U}. \tag{6.64}$$

However, this norm serves only as an example, it is enough for us to know that the inclusions $H_1 \subset H_B \subset H$ are continuous. The operator $C_w \in \mathcal{B}(H_B, U)$ is given by

$$C_w x_0 = \operatorname*{w-lim}_{s\to+\infty} Cs(s-A)^{-1}x_0 = \operatorname*{w-lim}_{s\to+\infty} s\int_0^\infty e^{-sr}x_0(r)\,dr = x_0(0+), \tag{6.65}$$

i.e., $C_w = \delta_{0+}^* \in \mathcal{B}(H_B, U)$. From this and (6.61) we can verify the identity

$$\widehat{\mathbb{D}}(s) = C_w(s-A)^{-1}B = e^{-s} \tag{6.66}$$

as required, since $e^{-s} = \widehat{\tau(-1)}$. Furthermore, $(s-A^*)C^*y_0 = e^{-s\cdot}y_0 \in H_C^* \subset H$ for $s \in \mathbf{C}^+$, $y_0 \in \mathbf{C}$, hence $(1-A^*)C^*y_0 = y_0e^{-\cdot}$, hence $H_C^* = H_1^* + \mathbf{C}e^{-\cdot} = W^{1,2}$, by

Lemma A.3.4(I1). Finally

$$B^*s(s-A^*)x_0 = s\int_0^1 e^{-sr}x_0(1-r)\,dr \to x_0(1-), \quad \text{as } s\to+\infty, \tag{6.67}$$

for $x_0 \in H_C^*$, hence $B_{\rm w}^* = \delta_1^* : x_0 \mapsto x_0(1)$, so that $B_{\rm w}^* \in \mathcal{B}(H_C^*, U)$ as required. In fact, $B_{\rm w}^* x_0 = x_0(1-)$ for any $x_0 \in H$ that is weakly continuous to the left (or has a weak left Lebesgue value at 1, by Lemma B.5.10). $\triangleleft$

Notes

Strong and weak regularity and their core theory are due to G. Weiss [W89c], [WW]. The concept ULR is due to [Helton76a], p. 155.

Vertical and half-plane-regularity seem to be the most reasonable transfer function properties that connect the I/O map equalities to corresponding feedthrough operator equalities. Their main advantage is that they can be used to guarantee that the signature operator of an optimal control problem equals the classical one (see, e.g., Lemma 6.3.6(b) and Proposition 9.11.3(c)).

Except for Lemma 6.2.2 (in this generality) and Proposition 6.2.8(e)&(f), almost all results of this section can be found in some form in the literature, mostly due to G. Weiss. The Lemma 6.2.10, the composition part of Example 6.2.6 and some minor results are from the works of O. Staffans.

Further results on regularity are given in the next section and in [W94a], [W94b], [SW00], [SW01a] and [Sw01b] among others; see [Sbook] for an extensive treatment on the subject and for historical remarks (the notes for Chapters 4 and 5).

6.3 Further regularity and compatibility

Be regular and orderly in your life, so that you may be violent and original in your work.
— Gustave Flaubert (1821–1880)

We start this section by studying several types of regular I/O maps (regularity and invertibility, H^p transfer functions and convolution I/O maps).

Not every WPLS is regular, but the output operator C of any WPLS has an extension C_c s.t. $D_c := \widehat{\mathbb{D}} - C_c(\cdot - A)^{-1}B$ is constant; such pairs (C_c, D_c) are called *compatible output operator pairs* for the system. Also the formula $y = C_c x + D_c u$ holds if the input is smooth enough. We give a few basic results on such pairs.

Further on, we give necessary and sufficient conditions for certain operators to generate a WPLS. Then we present several auxiliary lemmas on the connection between the generators and components or signals of a system until we finish this section by a brief treatment of reachability and observability.

In connection with feedback, it is often important to know whether the inverse of an I/O map is regular and whether its feedthrough operator is invertible; we list here the basic facts on this:

Proposition 6.3.1 (Regularity of $\mathbb{X}^{-1}$) *(a) Let $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ be SR. Then*

(a1) The feedthrough operator $X := \mathbb{X}(+\infty)$ is left-invertible.
(a2) If $\mathbb{X}^{\mathrm{d}}$ is SR, then X is invertible and $\mathbb{X}^{-1}$ is SR.
(a3) $\mathbb{X}^{-1}$ is SR iff X is invertible. If $\mathbb{X}^{-1}$ is SR, then $\mathbb{X}^{-1}(+\infty) = X^{-1}$.

(b1) If $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ is UR, then $\mathbb{X}^{-1}$ is UR, $X \in \mathcal{GB}$, and $\mathbb{X}^{-1}(+\infty) = X^{-1}$.

(b2) Let $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ and $X \in \mathcal{GB}$. Then $\mathbb{X}$ is SR (resp. UR, SVR, UVR, SLR, ULR) iff $\mathbb{X}^{-1}$ is SR (resp. UR, SVR, UVR, SLR, ULR).

(b3) Let $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$. Then $\mathbb{X}$ is strongly (resp. uniformly) half-plane regular iff $\mathbb{X}^{-1}$ is.

(b4) Let $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ and $X \in \mathcal{GB}$. Then $\mathbb{D}$ and $\mathbb{X}$ are SR (resp. UR, SVR, UVR, SLR, ULR) iff $\mathbb{D}\mathbb{X}^{-1}$ and $\mathbb{X}$ are SR (resp. UR, SVR, UVR, SLR, ULR).

(c) Let $\mathbb{X} \in \mathrm{TIC}_\infty$ be ULR. *Then $\mathbb{X} \in \mathcal{G}\mathrm{ULR} \Leftrightarrow \mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow X \in \mathcal{GB}$.*

Due to (c), one can work with ULR maps in the same way as with rational maps: a map is invertible iff its feedthrough operator is invertible. Moreover, the class ULR is also closed under inverses, compositions, linear operations and causal adjoints (see Lemma 6.2.5). These properties make uniform line-regularity the most important regularity property in the optimal control theory of Part III.

Proof: (a)–(c) W.l.o.g. (see Lemma 2.2.1(c4)), we assume that $Y = U$.

(a1–3) follow from [W94b, Theorems 4.7 & 4.8] (with $\mathbf{H} := I - \widehat{\mathbb{X}}$ and $K := I$).

(b1) Now $\widehat{\mathbb{X}^{-1}}(s) = \widehat{\mathbb{X}}(s)^{-1}$ is bounded on some $\mathbf{C}^+_\omega$ and $\widehat{\mathbb{X}}(s) \to X$, hence $X \in \mathcal{GB}$, by Lemma A.3.3(A3). The rest follows as in the proof of (b2).

(b2) Let $a \in U$ be arbitrary and set $b := X^{-1}a$. Now

$$\widehat{\mathbb{X}}^{-1}(s)a - X^{-1}a = \widehat{\mathbb{X}}^{-1}(s)Xb - b = \widehat{\mathbb{X}}^{-1}(s)[Xb - \widehat{\mathbb{X}}(s)b] \to 0 \tag{6.68}$$

as $s \to \infty$ (along $\mathbf{R}_+$ or $\omega + i\mathbf{R}_+$ or $\operatorname{Re} s \to +\infty$), because $\|\widehat{\mathbb{X}}^{-1}(s)\|_{\mathcal{B}} \le \|\mathbb{X}^{-1}\|_{\mathrm{TIC}_\omega}$ on $\mathbf{C}^+_\omega$. Therefore, $\mathbb{X}^{-1}$ is SR (resp. SVR, SLR) if $\mathbb{X}$ is. The uniform properties follow by removing a and replacing b by X^{-1} in (6.68). The converses follow by exchanging the roles of $\mathbb{X}$ and $\mathbb{X}^{-1}$.

(b3) The proof of (b2) applies here too.

(b4) This follows from (b2) and Lemma 6.2.5 (these properties are preserved under compositions).

(c) This follows from (b1)–(b2). □

The obvious facts (a1)–(b) below are often needed, (c) and (d) less so:

Lemma 6.3.2 (Regular I/O maps) *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$.

(a1) If $\dim Y < \infty$*, then* $\mathbb{D}$ *is WR (resp. WLR, WVR) iff* $\mathbb{D}$ *is SR (resp. SLR, SVR).*

(a2) If $\dim U < \infty$*, then* $\mathbb{D}$ *is UR (resp. ULR, UVR) iff* $\mathbb{D}$ *is SR (resp. SLR, SVR).*

(b) $\mathbb{D}^{\mathrm{d}}$ *is WR (resp. any other weak or uniform property from Definition 6.2.3) iff* $\mathbb{D}$ *is.*

(c) If $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *is uniformly (resp. strongly, weakly) regular and* $\widehat{\mathbb{D}}$ *is holomorphic and bounded on the sector* $\{s \in \mathbf{C} \bigm| |\arg s| < \frac{\pi}{2} + \varepsilon\}$ *for some* $\varepsilon > 0$*, then* $\widehat{\mathbb{D}}$ *is uniformly (resp. strongly, weakly) half-plane-regular.*

(d) If $\mathbb{D}$ *is UR and* $D \in \mathcal{GB}$*, then there is* $R > 0$ *s.t.* $\widehat{\mathbb{D}}(s) \in \mathcal{GB}$ *for* $s > R$*.*

(e) If $\mathbb{D} \in \mathrm{TIC}_\omega \cap \mathrm{WR}$*, then* $\|D\|_{\mathcal{B}(U,Y)} \le \|\mathbb{D}\|_{\mathrm{TIC}_\omega}$*.*

Proof: (a1)&(a2) See Lemma A.3.1(k1)–(k2).

(b) This is obvious.

(c) By [HP, Theorem 3.14.3], $\|\widehat{\mathbb{D}} - D\|_{\mathcal{B}(U,Y)} \to 0$ as $|s| \to \infty$, on any closed subsector of $\{s \in \mathbf{C} \bigm| |\arg s| < \frac{\pi}{2} + \varepsilon\}$, in particular, on $\mathbf{C}^+$. (By [HP], we have the above convergence even if a disc $\{|s| \le R\}$ were excluded from the sector.)

The strong [weak] claim is proved by replacing $\widehat{\mathbb{D}}$ by $\widehat{\mathbb{D}}u_0$ [by $\langle \widehat{\mathbb{D}}u_0, y_0\rangle$] for $u_0 \in U$ [and $y_0 \in Y$].

(d) This follows from Lemma A.3.3(A2) and the continuity of $\widehat{\mathbb{D}}$ on $(\omega, +\infty]$, where ω is s.t. $\mathbb{D} \in \mathrm{TIC}_\omega$.

(N.B. There is $\mathbb{D} = \mathbb{D}^{\mathrm{d}} \in \mathrm{SR} \cap \mathrm{TIC}$ s.t. $D = I$ but $\widehat{\mathbb{D}}(n)e_n = 0$ ($n \in \mathbf{N}$): Let $U := \ell^2(\mathbf{N})$ and define $\mathbb{D}$ by $\widehat{\mathbb{D}}(s)e_n := (1 - 2e^{-s\log 2/n})e_n$ ($n \in \mathbf{N}, s \in \mathbf{C}^+$) (obviously, $\|\widehat{\mathbb{D}}\|_{\mathcal{B}(U)} \le 1$; by Lemma D.1.1(b) we have $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U))$).) □

Recall that $\mathrm{H}^p_\omega := \mathrm{H}^p(\mathbf{C}^+_\omega; *) = \tau(-\omega)\mathrm{H}^p$ $(1 \le p \le \infty,\ \omega \in \mathbf{R})$ and that $\mathrm{H}^p_\infty := \cup_{\omega \in \mathbf{R}} \mathrm{H}^p_\omega$. By Theorem 6.2.1, the set of transfer functions of WPLSs (or of TIC_∞ maps) equals H^∞_∞. If a transfer function belongs to H^p_∞ (or to weak H^p_∞, see Definition F.3.1), for any $p < \infty$, then it is necessarily uniformly line-regular and much more:

Proposition 6.3.3 ($\widehat{\mathbb{D}} \in \mathrm{H}^p_{[\text{strong}]}$) *Let $-\infty < \omega < \alpha < \infty$, $1 \le p < \infty$.*

(a) **($\mathrm{H}^p_{\text{weak},\infty} \subset \widehat{\mathrm{ULR}}$)** *Let $\widehat{\mathbb{D}} \in \mathrm{H}^p_{\text{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$. Then $\mathbb{D} \in \mathrm{TIC}_\alpha(U,Y) \cap \mathrm{ULR} \cap \mathrm{WVR}$, and $D = 0$. If $\omega < 0$, then $\mathbb{D} \in \mathrm{WHPR}$.*

The above claim also holds with replacements $\mathrm{H}^p_{\text{weak}} \mapsto \mathrm{H}^p_{\text{strong}}$ and "W"$\mapsto$"S", as well as with replacements $\mathrm{H}^p_{\text{weak}} \mapsto \mathrm{H}^p$ and "W"$\mapsto$"U".

(b1) **($\mathcal{B} + \mathrm{H}^p_{\text{strong},\infty}$ is inverse-closed in H^∞_∞)** *Let $\widehat{\mathbb{D}} \in \mathcal{B}(U,Y) + \mathrm{H}^p_{\text{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$. Then $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{H}^\infty_\infty \Leftrightarrow \widehat{\mathbb{D}} \in \mathcal{G}(\mathcal{B} + \mathrm{H}^p_{\text{strong},\infty})$. Moreover, if $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{H}^\infty_\omega$, then*

$$\|\widehat{\mathbb{D}}^{-1} - D^{-1}\|_{\mathrm{H}^p_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(Y,U))} \le \|\widehat{\mathbb{D}}^{-1}\|_{\mathrm{H}^\infty_\omega} \|\widehat{\mathbb{D}}\|_{\mathrm{H}^p_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))} \|D^{-1}\|. \tag{6.69}$$

Part (b1) also holds with "strong" removed.

(b2) If $\widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathcal{B}(U,Y) + \mathrm{H}^p_{\text{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$, then $\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{H}^\infty_\infty \Leftrightarrow \widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathcal{G}(\mathcal{B} + \mathrm{H}^p_{\text{strong},\infty})$ etc., as in (b1), by duality.

(c) If $\widehat{\mathbb{D}} \in \mathcal{B}(U,Y) + \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\alpha; \mathcal{B}(U,Y))$ and $\widehat{\mathbb{F}} \in \mathcal{B}(Y,Z) + \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(Y,Z))$, then $\widehat{\mathbb{F}}\widehat{\mathbb{D}} \in \mathcal{B}(U,Z) + \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\alpha; \mathcal{B}(U,Z))$.

(This also holds with "strong" removed.)

(d) Let $\widehat{\mathbb{D}} \in \mathrm{H}^p(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$, $p \in [2,\infty]$. Then $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \le M_{\mathbb{D},p,\varepsilon}\|u\|_{\mathrm{L}^2_{\omega-\varepsilon}}$ for any $\varepsilon > 0$, $u \in \mathrm{L}^2_{\omega-\varepsilon}(\mathbf{R}_+; U)$. In particular, $\mathbb{D}[\mathrm{L}^2_c] \subset \mathrm{L}^2_\omega$.

(e) Let $\widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(Y,U))$. Then $\mathbb{D}u \in \mathcal{C}(\mathbf{R}; U)$ and

$$\sup_{\mathbf{R}} \|e^{-\omega\cdot}\mathbb{D}u\|_Y \le \|\widehat{\mathbb{D}}(\bar{\cdot})^*\|_{\mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(Y,U))} \|u\|_{\mathrm{L}^2_\omega} \quad (u \in \mathrm{L}^2_\omega(\mathbf{R}; U)). \tag{6.70}$$

Note for (e) that $\|\widehat{\mathbb{D}}(\bar{\cdot})^*\|_{\mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(Y,U))} \le \|\widehat{\mathbb{D}}\|_{\mathrm{H}^2(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))}$ for any $\widehat{\mathbb{D}} \in \mathrm{H}^2_\omega$. By (c), $\mathcal{B} + \mathrm{H}^2_{\text{strong},\infty}$ is a subalgebra of H^∞_∞ when $U = Y$.

In Theorem 6.9.1 we shall show that $\widehat{\mathbb{D}}(\cdot - \omega) \in \mathrm{H}^2_{\text{strong}}$ for some $\omega \in \mathbf{R}$ iff $\mathbb{D}$ has a realization with a bounded B and $D = 0$ (and a dual claim holds for bounded C).

Proof: (a) 1° *$\mathbb{D} \in \mathrm{ULR} \cap \mathrm{TIC}_\alpha(U,Y)$:* By Lemma F.3.2(a), $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\alpha; \mathcal{B}(U,Y))$, hence it is the transfer function of some $\mathbb{D} \in \mathrm{TIC}_\alpha$. By Lemma F.3.2(b), $\mathbb{D}$ is ULR.

2° *Half-plane-regularity:* We give the proof for $\widehat{\mathbb{D}} \in \mathrm{H}^p(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$; (add $u_0 \in U$ [and $\Lambda \in Y^*$] for $\mathrm{H}^p_{\text{strong}}$ [or $\mathrm{H}^p_{\text{weak}}$]).

Assume that $\omega < 0$. Then $\mathbb{D}$ is uniformly half-plane-regular, by Theorem 6.4.2 of [HP] (use $\alpha := \omega/2 =: -\delta < 0$; note that $\widehat{\mathbb{D}} \in \mathrm{H}^p(\mathbf{C}_{\omega/2}; \mathcal{B})$ in the sense of [HP], i.e.,the extra assumption (iii) of Definition 6.4.1 of [HP] is satisfied for $\alpha = \omega/2$).

3° *Vertical regularity:* This follows from the half-plane-regularity of $\widehat{\mathbb{D}}(\cdot - \omega - 1)$.

(b1) This follows from Theorem 4.1.1(j). The norm estimate is obtained as in the proof of Lemma 4.1.2.

(b2) Apply (b1) to $\widehat{\mathbb{D}}(\bar{\cdot})^*$, and note that $\widehat{\mathbb{D}} \mapsto \widehat{\mathbb{D}}(\bar{\cdot})^*$ is an isometric isomorphism of H^∞_α onto H^∞_α.

(c) Set $g := \widehat{\mathbb{D}} - D$, $f := \widehat{\mathbb{F}} - F$. Obviously, we have $FD \in \mathcal{B}(U,Z)$ and $Fg, fD \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\alpha;\mathcal{B}(U,Z))$. But $f \in \mathrm{H}^\infty_\alpha$, by (a), hence $fg \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\alpha;\mathcal{B}(U,Z))$. (We can also remove "strong" from the statement and the proof of (c).)

(d) In fact, it suffices that $u \in \mathrm{L}^q_\omega(\mathbf{R}_+;U)$, where $q = (1/2+1/p)^{-1}$. Indeed, $q \in [1,2]$, hence $q' := (1/2-1/p)^{-1} \in [2,\infty]$ and $\|\widehat{u}\|_{\mathrm{H}^{q'}_\omega} \le M_q\|u\|_{\mathrm{L}^q_\omega}$, by Theorem E.1.7. On the other hand, $\|\widehat{\mathbb{D}}\widehat{u}\|_{\mathrm{H}^2_\omega} \le \|\widehat{\mathbb{D}}\|_{\mathrm{H}^p_\omega}\|\widehat{u}\|_{\mathrm{H}^{q'}_\omega}$, by Lemma B.3.13.

Thus, $\|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \le M'_p\|\widehat{\mathbb{D}}\|_{\mathrm{H}^p_\omega}\|u\|_{\mathrm{L}^q_\omega}$, where $M'_p := \sqrt{2\pi}M_q$. Because $\|u\|_{\mathrm{L}^q_\omega} \le M'_{q,\varepsilon}\|u\|_{\mathrm{L}^2_{\omega-\varepsilon}}$, by Lemma D.1.4(b4), we can set $M_{\mathbb{D},p,\varepsilon} := M'_{q,\varepsilon}M'_p\|\widehat{\mathbb{D}}\|_{\mathrm{H}^p_\omega}$.

(e) See Lemma F.3.7(b2). (N.B. although $\mathbb{D} \in \mathrm{TIC}_\alpha$ and $\mathbb{D} \in \mathcal{B}(\mathrm{L}^2_\omega, C)$ (where $\mathbb{D}$ stands for its unique continuous extension), we do not have $\mathbb{D} \in \mathcal{B}(\mathrm{L}^2_\omega,\mathrm{L}^2_\omega)$ unless $\widehat{\mathbb{D}} \in \mathrm{H}^\infty_\omega$.) □

If $F \in \mathrm{L}^1(\mathbf{R}_+;\mathcal{B}(U,Y))$, then $\mathbb{D}u := F * u$ defines a TIC map, and we have $\widehat{\mathbb{D}}\widehat{u} = \widehat{F}\widehat{u}$ (see Lemma D.1.11(c')). If merely $F \in \mathrm{L}^1_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(U,Y))$ (Definition F.1.4), i.e., $F : \mathbf{R}_+ \to \mathcal{B}(U,Y)$ is s.t. $Fu_0 \in \mathrm{L}^1(\mathbf{R}_+;Y)$ $(u_0 \in U)$, then $\mathbb{D}u := F * u$ can in general be written as an integral for finite-dimensional $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ only, but still $\widehat{F} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$, hence $\widehat{\mathbb{D}}\widehat{u} := \widehat{F}\widehat{u}$ still defines a map $\mathbb{D} \in \mathrm{TIC}(U,Y)$.

We list below further properties of convolution maps:

Proposition 6.3.4 ($\mathbb{D} \in \mathrm{L}^p_{[\mathrm{strong}]}*$) *Let $-\infty < \omega < \alpha < \infty$ and $1 \le p \le \infty$.*

(a1) Let $F \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y))$. Define $\mathbb{D}$ by $\widehat{\mathbb{D}} := \widehat{F}$.

Then $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\alpha \subset \mathrm{ULR} \cap \mathrm{UVR}$, $\mathbb{D}u := F * u$ for all $u \in \mathrm{L}^2_\alpha + \pi_+\mathrm{L}^2_{\mathrm{loc}}$, and $\widehat{\mathbb{D}^{\mathrm{d}}} = \widehat{F^*}$.

If $\omega < 0$, then $\mathbb{D}$ is UHPR. If $E \in \mathcal{B}(U,Y)$ and $E + \mathbb{D} \in \mathcal{G}\mathrm{TIC}_\alpha$, then $(E+\mathbb{D})^{-1} \in \mathcal{GB}(Y,U) + \mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha(\mathbf{R}_+;\mathcal{B}(Y,U))$. Finally, $\frac{1}{r}\mathbb{D}\chi_{[-r,0)} \to F$ in $\mathcal{B}(U,Y)$ at every Lebesgue point of F, hence a.e. (See (a2) and (a3) for further results.)

(a2) Let $\mathrm{e}^{-\omega\cdot}F \in \mathrm{L}^p_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(U,Y))$. Define $\mathbb{D}$ by $\widehat{\mathbb{D}} := \widehat{F}$. Then $\mathbb{D}^{\mathrm{d}} = \widehat{F^*}$ on $\mathbf{C}^+_\omega$, and (a3) applies.

(a3) Let $F \in \mathcal{B}(U,\mathrm{L}^p_\omega(\mathbf{R}_+;Y))$. Define $\mathbb{D}$ by $\widehat{\mathbb{D}} := \widehat{F}$.

Then $\mathbb{D} \in \mathrm{TIC}_\alpha \cap \mathrm{SLR} \cap \mathrm{SVR}$ (and $\mathbb{D} \in \mathrm{ULR}$ if $p > 1$). If $\omega < 0$, then $\mathbb{D}$ is strongly half-plane-regular. Moreover, $\mathbb{D}u = F * u$ for finite-dimensional $u \in \mathrm{L}^2_\alpha + \pi_+\mathrm{L}^2_{\mathrm{loc}}$.

If $1 \le p \le 2$ and $1/p + 1/q = 1$, then $\widehat{\mathbb{D}} \in \mathrm{H}^q_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$; if $p \ge 2$, then $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\alpha;\mathcal{B}(U,Y))$. Finally, $\frac{1}{r}\mathbb{D}\chi_{[-r,0)}u_0 \to Fu_0$ in Y at every Lebesgue point of Fu_0, hence a.e. $(u_0 \in U)$.

(a4) Let $\mathrm{e}^{-\omega\cdot}F \in \mathrm{L}^p_{\mathrm{weak}}(\mathbf{R}_+;\mathcal{B}(U,Y))$. Define $\mathbb{D}$ by $\widehat{\mathbb{D}} := \widehat{F}$. Then $\mathbb{D}^{\mathrm{d}} = \widehat{F^*}$ on $\mathbf{C}^+_\omega$, and (a5) applies.

(a5) Let $F \in \mathcal{B}(U, \mathcal{B}(Y^{\mathrm{B}}, \mathrm{L}^p_\omega(\mathbf{R}_+)))$*. Define* $\mathbb{D}$ *by* $\widehat{\mathbb{D}} := \widehat{F}$*.*

Then $\mathbb{D} \in \mathrm{TIC}_\alpha \cap \mathrm{WLR} \cap \mathrm{WVR}$ *(and* $\mathbb{D} \in \mathrm{ULR}$ *if* $p > 1$*). If* $\omega < 0$*, then* $\mathbb{D}$ *is weakly half-plane-regular. Moreover,* $\Lambda\mathbb{D}u = \Lambda F * u$ *for finite-dimensional* $u \in \mathrm{L}^2_\alpha + \pi_+\mathrm{L}^2_{\mathrm{loc}}$ *and all* $\Lambda \in Y^*$*.*

If $1 \le p \le 2$ *and* $1/p + 1/q = 1$*, then* $\widehat{\mathbb{D}} \in \mathrm{H}^q_{\mathrm{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$*; if* $p \ge 2$*, then* $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{weak}}(\mathbf{C}^+_\alpha; \mathcal{B}(U,Y))$*. Finally,* $\Lambda\frac{1}{r}\mathbb{D}\chi_{[-r,0)}u_0 \to \Lambda F u_0$ *in* $\mathbf{C}$ *at every Lebesgue point of* $\Lambda F u_0$*, hence a.e. (*$u_0 \in U$*,* $\Lambda \in Y^*$*).*

(b) For $p = 1$*, we can take* $\alpha = \omega$ *in (a1)–(a4).*

See Theorem 2.6.4 for more on the classes in (a1) and (a3). See Lemma F.2.2(a) for the "strong convolutions" appearing in (a3); they coincide with ordinary ones when $F \in \mathrm{L}^p_\omega$.

A convolution kernel ("F" above) corresponding to an I/O map of a system is often called the impulse response (since it equals "$F * \delta_0 = \mathbb{D}\delta_0$") or the weighting pattern of the system.

Proof: (a1) We have $F \in \mathrm{L}^1_\alpha$, hence $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\alpha$, by definition, and $\widehat{F * u} = \widehat{F}\widehat{u} = \widehat{\mathbb{D}}\widehat{u} = \widehat{\mathbb{D}u}$ for $u \in \mathrm{L}^2_\alpha$, by Lemma D.1.11(c'). By causality (replace u by $\pi_{[0,T)}u$, for arbitrary $T > 0$), we have $F * u = \mathbb{D}u$ also for $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+; U)$ (cf. Lemma D.1.7).

By Lemma D.1.12(d), we have $\widehat{\mathbb{D}^{\mathrm{d}}} = \widehat{F^*}$ (hence $\mathbb{D}^{\mathrm{d}} = F^* *$).

If $\omega < 0$, then we can take $\alpha = 0$ to see that $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$ (and hence $\widehat{\mathbb{D}}$ is uniformly half-plane-regular, by Theorem 2.6.4). By Lemma D.1.11(b'), we have $\mathbb{D} \in \mathrm{ULR} \cap \mathrm{UVR}$ (and $\mathbb{D}$ is uniformly half-plane-regular if $\omega < 0$ or $\omega = 0$ and $p = 1$). Apply Theorem 4.1.1(b) to $\mathcal{T}_{-\alpha}(E + F*)$ to obtain that also the inverse lies in $\mathcal{B} + (\mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha)*$ (recall that $F \in \mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha$).

At each Lebesgue point t of F, we have

$$\frac{1}{r}\mathbb{D}\chi_{[-r,0)}\cdot = \frac{1}{r}\int_{-r}^{0} F(t-s)\,ds = \frac{1}{r}\int_{0}^{r} F(t+s)\,ds \to F, \tag{6.71}$$

as $r \to 0+$. (N.B. $\frac{1}{r}\int_0^r F(t+s)\,ds$ is a continuous function of r and t, by Corollary B.3.8.)

(a2) (Note that if, instead, $\mathrm{e}^{-\omega\cdot}F^* \in \mathrm{L}^p_{\mathrm{strong}}$, then $\mathbb{D}$ is "strong* half-plane-regular" etc., by duality.)

As in (a1), one can show that $\mathbb{D} \in \mathrm{SLR} \cap \mathrm{SVR}$ and that $\mathbb{D}$ is strongly half-plane-regular if $\omega < 0$ (or $\omega = 0$ and $p = 1$).

By Lemma F.3.4(c2)&(a1), we have $\widehat{F} \in \mathrm{H}^q_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y)) \cap \mathrm{H}^\infty_\alpha$ for $p \in [1,2]$; for $p > 2$ we have $\mathrm{e}^{-\omega'\cdot}F \in \mathrm{L}^2_{\mathrm{strong}}$ for any $\omega' > \omega$. It follows that $\mathbb{D} \in \mathrm{ULR}$ if $p > 1$, by Proposition 6.3.3(a1).

For $u = \phi u_0$, $\phi \in \mathrm{L}^2_\alpha(\mathbf{R})$, $u_0 \in U$, we have $\widehat{F * u} = \widehat{Fu_0 * \phi} = \widehat{F}u_0\widehat{\phi} = \widehat{\mathbb{D}}\widehat{u}$, i.e., $F * u = \mathbb{D}u$; hence the same holds for finite-dimensional u (see the proof for $\pi_+\mathrm{L}^2_{\mathrm{loc}}$).

By Lemma 6.2.2, we have $\widehat{\mathbb{D}^{\mathrm{d}}} = \widehat{\mathbb{D}}(\bar{s})^* = \widehat{F}(\bar{s})^*$; by Lemma F.3.3(c), $\widehat{F}(\bar{s})^* = \widehat{F^*}(s)$, for $s \in \mathbf{C}^+_\omega$.

(a3) Most of this follows from Lemma F.2.2(d1)–(d3); the rest can be obtained as in (a2).

(a4) The proofs of (a1)–(a3) apply mutatis mutandis. (Recall from Remark A.3.22 that Y^{B} refers to the "linear" dual, where the scalar multiplication is defined by $(\beta\Lambda)y_0 := \beta(\Lambda y_0)$ $\alpha \in \mathbf{C}$, $\Lambda \in Y^*$, $y_0 \in Y$, as in Theorem F.2.1(f).)

(Obviously, $\mathcal{B}(U, \mathcal{B}(Y^{\mathrm{B}}, \mathrm{L}^p_\omega(\mathbf{R}_+)))$ is the space of bilinear mappings $U \times Y^{\mathrm{B}} \to \mathrm{L}^p_\omega(\mathbf{R}_+)$, i.e., the space of sesquilinear mappings $U \times Y \to \mathrm{L}^p_\omega(\mathbf{R}_+)$, with norm $\|T\| \le \sup_{\|u\|,\|y\|\le 1} \|T(u_0,y)\|_{\mathrm{L}^p_\omega}$.)

(a5) One can observe this from the proofs of (a1)–(a3). $\square$

In classical Riccati equation theory, one often needs to make conclusions such as $\mathbb{D}^*\mathbb{D} = \mathbb{X}^*\mathbb{X} \implies D^*D = X^*X$. This may fail even if $\mathbb{D}, \mathbb{X} \in \mathrm{ULR}$ (Example 6.3.7), but the following two lemmas give us important sufficient conditions:

Lemma 6.3.5 ($\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X} \implies D^*JD = X^*SX$) *Let* $\widehat{\mathbb{D}}_k \in \mathrm{H}^{p_k}_{\mathrm{strong}}(\mathbf{C}^+; \mathcal{B}(U,Y_k)) + \mathcal{B}(U,Y_k)$, $p_k \in [2,\infty)$ *($k = 1,2,3,4$) and* $Y_1 = Y_2$, $Y_3 = Y_4$. *Assume that* $\langle \mathbb{D}_1 u, \mathbb{D}_2 u\rangle \ge \langle \mathbb{D}_3 u, \mathbb{D}_4 u\rangle$ *for all* $u \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+;U)$. *Then* $D_1^*D_2 \ge D_3^*D_4$.

In particular, $\langle \mathbb{D}_1 u, \mathbb{D}_2 u\rangle = \langle \mathbb{D}_3 u, \mathbb{D}_4 u\rangle$ ($u \in \mathrm{L}^2_{\mathrm{c}}$) implies that $D_1^*D_2 = D_3^*D_4$. We may allow for any $p_k \in [1,\infty)$ under slightly stronger stability, by Lemma F.3.2(a2).

For $\mathbb{D}_k \in \mathrm{MTIC}^{\mathrm{L}^1}$ (or $\mathrm{L}^1_{\mathrm{strong}} * + \mathcal{B}$), we obtain the same from Lemma 6.3.6(b).

Proof: By Proposition 6.3.3(a)&(d), we have $\widehat{\mathbb{D}}_k \in \mathcal{B}(\mathrm{L}^2_{-\delta}, \mathrm{L}^2)$, $\mathbb{D}_k \in \mathrm{ULR}$, hence $\langle \mathbb{D}_1 u, \mathbb{D}_2 u\rangle \ge \langle \mathbb{D}_3 u, \mathbb{D}_4 u\rangle$ (use continuity and Corollary B.3.8) for all $u \in \mathrm{L}^2_{-\delta}(\mathbf{R}_+;U)$, $k = 1,2,3,4$, $\delta > 0$.

We shall assume that $\mathbb{D}_3 = 0 = \mathbb{D}_4$ to simplify the notation; the general case is analogous (just the number of terms is doubled).

Let $u_0 \in U$. Set $\alpha := \langle D_1 u_0, D_2 u_0\rangle_Y \in \mathbf{C}$, $F := \widehat{\mathbb{D}}_1 - D_1$, $G := \widehat{\mathbb{D}}_2 - D_2$. Because $Fu_0 \in \mathrm{H}^{p_1}(\mathbf{C}^+;Y_1)$ and $Gu_0 \in \mathrm{H}^{p_2}(\mathbf{C}^+;Y_1)$, they have L^{p_k} boundary functions, by Theorem 3.3.1(a2). Moreover, $g_1 := \langle Gu_0, Fu_0\rangle_{Y_1} \in \mathrm{L}^q(i\mathbf{R})$, where $q^{-1} = p_1^{-1} + p_2^{-1} \in [1,\infty)$, by Lemma B.3.13.

Given $\varepsilon > 0$, there $R_1 > 0$ s.t. $f_r := f_{1,r} \in \mathrm{L}^2_{-1/2}(\mathbf{R}_+)$ satisfies $\|f_r\|_2 = 1$ and $\int_{i\mathbf{R}} |\widehat{f}_r|^2 |g_1|\, dm < \varepsilon/3$ for all $r > R_1$, by Lemma D.1.24. Choose, analogously, numbers R_2 and R_3 for functions $g_2 := \langle Fu_0, D_2 u_0\rangle_Y \in \mathrm{L}^{p_1}(i\mathbf{R})$ and $g_3 := \langle D_1 u_0, Gu_0\rangle_Y \in \mathrm{L}^{p_2}(i\mathbf{R})$, respectively, in place of g_1. Set $r_\varepsilon := \max\{R_1,R_2,R_3\}$, $\beta_\varepsilon := \langle \mathbb{D}_1 f_r u_0, \mathbb{D}_2 f_r u_0\rangle_Y \ge 0$. Then, by (D.36), we have

$$2\pi|\beta_\varepsilon - \alpha| = \langle \widehat{\mathbb{D}}_1 f_{r_\varepsilon} u_0, \widehat{\mathbb{D}}_2 f_{r_\varepsilon} u_0\rangle_{\mathrm{L}^2(i\mathbf{R};Y_1)} - \langle D_1 f_{r_\varepsilon} u_0, D_2 f_{r_\varepsilon} u_0\rangle_{\mathrm{L}^2(i\mathbf{R};Y_1)} \tag{6.72}$$

$$\le \int_{i\mathbf{R}} |f_{r_\varepsilon}|^2 (|g_1| + |g_2| + |g_3|)\, dm < 3\varepsilon/3, \tag{6.73}$$

hence $|\beta_\varepsilon - \alpha| < \varepsilon/2\pi$. Because ε was arbitrary, we have $\inf_{\beta\in\mathbf{R}_+} |\beta - \alpha| = 0$, i.e., $\alpha \in \mathbf{R}_+$. Because u_0 was arbitrary, we have $\langle D_1 u_0, D_2 u_0\rangle_Y \ge 0$ for all $u_0 \in U$. $\square$

Since convolutions with L^1 are uniformly half-plane-regular (UHPR), we can often use the following lemma for the purpose described above:

Lemma 6.3.6 (Half-plane-regularity) *Assume that* $\mathbb{D} \in \mathrm{TIC}(U,Y)$ *and* $J = J^* \in \mathcal{B}(Y)$.

(a1) Let $\mathbb{D} \in \mathrm{TIC}(U,Y)$ *be SHPR. Then for each* $u_0 \in U$ *there is a null set* $N_{\mathbb{D},u_0} \subset \mathbf{R}$ *s.t.* $\widehat{\mathbb{D}}(ir)u_0 \to Du_0$ *as* $N^c_{\mathbb{D},u_0} \ni r \to \pm\infty$.

(a2) Let $\mathbb{D} \in \mathrm{TIC}(U,Y)$ *be UHPR, and let U be separable. Then there is a null set* $N_{\mathbb{D}} \subset \mathbf{R}$ *s.t.* $\|\widehat{\mathbb{D}}(ir) - D\| \to 0$ *as* $N^c_{\mathbb{D}} \ni r \to \pm\infty$.

(b) $(\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X} \Rightarrow D^*JD = X^*SX)$ *Let* $\mathbb{D},\mathbb{E} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{X},\mathbb{Z} \in \mathrm{TIC}(U,H)$ *be strongly half-plane-regular. Then* $\mathbb{E}^*\mathbb{D} = \mathbb{Z}^*\mathbb{X} \implies E^*D = Z^*X$*, and* $\mathbb{E}^*\mathbb{D} \geq \mathbb{Z}^*\mathbb{X} \implies E^*D \geq Z^*X$.

(c1) $(\exists(\pi_+\mathbb{D}^*J\mathbb{D}\pi_+)^{-1} \Rightarrow \exists(D^*JD)^{-1})$ *Let* $\mathbb{D},\mathbb{D}^{\mathrm{d}},\mathbb{E},\mathbb{E}^{\mathrm{d}} \in \mathrm{SHPR}\cap\mathrm{TIC}$ *and* $\pi_+\mathbb{D}^*\mathbb{E}\pi_+ \in \mathcal{GB}(L^2(\mathbf{R}_+;U))$*. Then* $D^*E \in \mathcal{GB}(U)$.

(c2) $(\mathbb{D}^*J\mathbb{D} \gg 0 \Rightarrow D^*JD \gg 0)$ *Let* $\mathbb{D},\mathbb{E} \in \mathrm{SHPR}\cap\mathrm{TIC}(U,Y)$ *and* $\mathbb{D}^*\mathbb{E} \gg 0$*. Then* $D^*E \gg 0$.

(d1) $(\exists(\pi_+\mathbb{D}^*J\mathbb{D}\pi_+)^{-1} \Rightarrow \exists(D^*JD)^{-1})$ *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *be s.t.* $\mathbb{B}\tau \in \mathrm{UHPR}\cap\mathrm{TIC}$ *and* $\mathbb{D} \in \mathrm{ULR}\cap\mathrm{TIC}$ *(or* $\mathbb{B}\tau,\mathbb{C}^{\mathrm{d}}\tau \in \mathrm{SHPR}\cap\mathrm{TIC}$ *and* $\mathbb{D},\mathbb{D}^{\mathrm{d}} \in \mathrm{SLR}\cap\mathrm{TIC}$*). If* $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \in \mathcal{GB}(L^2(\mathbf{R}_+;U))$*, then* $D^*JD \in \mathcal{GB}(U)$.

(d2) $(\mathbb{D}^*J\mathbb{D} \gg 0 \Rightarrow D^*JD \gg 0)$ *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *be s.t.* $\mathbb{B}\tau \in \mathrm{SHPR}\cap\mathrm{TIC}$ *and* $\mathbb{D} \in \mathrm{SLR}\cap\mathrm{TIC}$*. If* $\mathbb{D}^*J\mathbb{D} \geq \varepsilon I$*,* $\varepsilon > 0$*, then* $D^*JD \geq \varepsilon I$.

Parts (d1) and (d2) are mainly applied to exponentially stable (or stabilizable) systems of the type studied in Sections 6.8 and 9.2. See Lemma 9.2.17 for the unstable case.

Proof: (a1) For any $\varepsilon > 0$ choose $R_\varepsilon > 0$ s.t. $\|[\widehat{\mathbb{D}}(s) - D]u_0\| < \varepsilon$ for all $s \in \mathbf{C}^+$ with $|s| > R_\varepsilon$. Then $\|[\widehat{\mathbb{D}}(ir) - D]u_0\| = \lim_{t\to 0+}\|[\widehat{\mathbb{D}}(ir+t) - D]u_0\| \leq \varepsilon$, when $r \in \mathbf{R}\setminus N_{\mathbb{D},u_0}$ and $|r| > R_\varepsilon$, for some null set $N_{\mathbb{D},u_0}$, by Theorem 3.3.1(c1).

(a2) Modify the proof of (a1) suitably (use Theorem 3.3.1(c2)).

(b) We only prove that $\mathbb{E}^*\mathbb{D} \geq 0 \implies E^*D \geq 0$, because the latter claim follows by applying this to $\left[\mathbb{E} \ \ -\mathbb{Z}\right]^* \left[\begin{smallmatrix}\mathbb{D}\\ \mathbb{X}\end{smallmatrix}\right] \geq 0$, and the former claim follows from the latter.

We have $\langle\widehat{\mathbb{E}}u_0, \widehat{\mathbb{D}}u_0\rangle \geq 0$ a.e., by Theorem 3.1.3(e2). Letting $(N_{\mathbb{D},u_0} \cup N_{\mathbb{E},u_0})^c \ni r \to +\infty$, we get that $\langle Eu_0, Du_0\rangle_U \geq 0$, by (a). Because u_0 was arbitrary, we have $E^*D \geq 0$.

(c1) (N.B. $\mathbb{D} \in \mathrm{UHPR} \Rightarrow \mathbb{D},\mathbb{D}^{\mathrm{d}} \in \mathrm{UHPR} \subset \mathrm{SHPR}$.) Choose $\eta > 0$ s.t. $\|\pi_+\mathbb{D}^*\mathbb{E}u\|_2 \geq \eta\|u\|_2$ for all $u \in L^2(\mathbf{R}_+;U)$.

Let $\|u_0\|_U = 1$ and $\varepsilon > 0$ be arbitrary. By (a), there are $R > 0$ and a null set N s.t. $\|(\widehat{\mathbb{E}}(ir) - E)u_0\|, \|(\widehat{\mathbb{D}}^*(ir) - D^*)Eu_0\| < \varepsilon$ for $r > R$ s.t. $r \in N$. By Lemma D.1.24(a), there are $\delta > 0$ and $R' > 0$ s.t. for all $r > R'$ and $t \in (0,\delta)$ we have $\|\widehat{f}_{t,r}\|_{H^2(\mathbf{C}^+)} = \sqrt{2\pi}$ and $\|\chi_{i[-R,R]}\widehat{f}\|_{H^2}$ is arbitrarily small, hence for suitable $\delta > 0$ and $R' > 0$ we have

$$|\langle \widehat{\mathbb{D}}\widehat{v}, \widehat{\mathbb{E}}\widehat{f}_{1,r}u_0\rangle| \leq |\langle \widehat{\mathbb{D}}\widehat{v}, \widehat{\mathbb{E}}\widehat{f}_{1,r}u_0\rangle - \langle \widehat{v}, D^*E\widehat{f}_{1,r}u_0\rangle| + \sqrt{2\pi}\|\widehat{v}\|_{H^2}\|D^*Eu_0\|_U \quad (6.74)$$

$$\leq (2\pi)\varepsilon + (2\pi)\|v\|_2\|D^*Eu_0\|_U \quad (v \in \mathrm{L}^2(\mathbf{R}_+;U)). \quad (6.75)$$

But for a fixed r, there is v s.t. $\|v\|_2 = 1$ and $|\langle \widehat{\mathbb{D}}\widehat{v}, \widehat{\mathbb{E}}\widehat{f}_{1,r}u_0\rangle| > 2\pi\eta$, by Lemma A.3.1(c1)(xi), hence $\|D^*Eu_0\| > \eta - \varepsilon$. Since ε and u_0 were arbitrary, we have $\|D^*Eu_0\| \geq \eta$ whenever $\|u_0\|_U = 1$.

By exchanging the roles of $\widehat{\mathbb{E}}$ and $\widehat{\mathbb{D}}$, we obtain that $\|E^*Du_0\| \geq \eta\|u_0\|_U$ for all $u_0 \in U$. Consequently, $D^*E \in \mathcal{GB}(U)$, by Lemma A.3.1(c3)(v)&(i).

(c2) Now $\mathbb{D}^*\mathbb{E} \gg \varepsilon I$ for some $\varepsilon > 0$, hence $D^*E \gg \varepsilon I$, by (b).

(d1) By Lemma 6.3.23, we have $\mathbb{D} \in$ UHPR (or $\mathbb{D}, \mathbb{D}^{\mathrm{d}} \in$ SHPR), hence $\mathbb{D}^{\mathrm{d}}, J\mathbb{D}, \mathbb{D}^{\mathrm{d}}J \in$ UHPR $\subset$ SHPR (or $J\mathbb{D}, J\mathbb{D}^{\mathrm{d}} \in$ SHPR), hence this follows from (c1).

(d2) By Lemma 6.3.23, we have $\mathbb{D}, \mathbb{D}^{\mathrm{d}} \in$ SHPR, hence this follows from (c2). □

For non-half-plane-regular $\mathbb{D}$ and $\mathbb{X}$ (even for $\mathbb{D}, \mathbb{X} \in$ MTIC $\subset$ ULR), we really may have $D^*D \neq X^*X$ although $\mathbb{D}^*\mathbb{D} = \mathbb{X}^*\mathbb{X}$:

Example 6.3.7 ($S \neq D^*JD$) In the system of Example 6.2.14, we have $U = \mathbf{C} = Y$, $\mathbb{D} = \tau(-1)$. Take $J = I$, so that $\mathbb{D}^*J\mathbb{D} = I$ has the spectral factorization $\mathbb{X}^*S\mathbb{X}$ with $S = I = \mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ (or $\mathbb{X} = E,\ S = (EE^*)^{-1}$ with $E \in \mathcal{B}(U)$).

Clearly $D = 0$ and $X = I$, hence $D^*JD = 0 \neq I = X^*SX$. Note that $\mathbb{D}, \mathbb{X} \in$ ULR, but, of course, $\mathbb{D}$ is not (even weakly) half-plane-regular. Moreover, for $s = i\omega \in i\mathbf{R}$ we have $\mathbb{D}^*(s)J\mathbb{D}(s) = |\mathrm{e}^{-s}|^2 = 1 = \mathbb{X}^*(s)S\mathbb{X}(s)$, but $\mathbb{D}(i\omega) = \mathrm{e}^{-i\omega} \not\to D$ as $\omega \to \pm\infty$). ◁

(In Example 9.13.8 (with Σ divided by $\sqrt{2}$), we have $X^*X \neq D^*D$ even though $\mathbb{D}^*\mathbb{D} = \mathbb{X}^*\mathbb{X}$, C is bounded and $\mathbb{X} = I$.)

This unfortunate fact makes the WPLS Riccati theory more complicated than the earlier theories for smoother classes and forces us to introduce the signature operator S (in general, X^*SX) that replaces the standard term D^*JD in Riccati equations in the general case, cf. (9.3).

We shall show in Theorem 6.3.9 that the output operator of an arbitrary WPLS $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ has an extension $C_{\mathrm{c}} \in \mathcal{B}(H_B, Y)$ s.t. $\widehat{\mathbb{D}}(s) = D_{\mathrm{c}} + C_{\mathrm{c}}(s-A)^{-1}B$ for some D_{c}. Such operators are the compatible generators of Σ:

Definition 6.3.8 (Compatibility) *We call* $(C_{\mathrm{c}}, D_{\mathrm{c}})$ *a* compatible (output operator) pair *for* Σ *(or for* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \end{smallmatrix}\right]$*), and we call* $\left[\begin{smallmatrix} A & B \\ C_{\mathrm{c}} & D_{\mathrm{c}} \end{smallmatrix}\right]$ compatible generators *of* $\Sigma \in \mathrm{WPLS}_\omega(U,H,Y)$ *if* $\left[\begin{smallmatrix} A & B \\ C & \end{smallmatrix}\right]$ *are the generators of* Σ*,* $C_{\mathrm{c}} \in \mathcal{B}(W,Y)$ *is an extension of* C*, where* W *is a Banach space s.t.* $H_B \subset W \underset{c}{\subset} H$*, and* $D_{\mathrm{c}} = \widehat{\mathbb{D}}(\alpha) - C_{\mathrm{c}}(\alpha - A)^{-1}B$ *for some* $\alpha \in \mathbf{C}^+_\omega$.

As before, "generate" means "are the generators of". We consider pairs $(C_{\mathrm{c}}, D_{\mathrm{c}})$ and $(\widetilde{C}_{\mathrm{c}}, D_{\mathrm{c}})$ for $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \end{smallmatrix}\right]$ equal iff $\left[\begin{smallmatrix} \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widetilde{\mathbb{C}} & \widetilde{\mathbb{D}} \end{smallmatrix}\right]$, i.e., iff

$C_c = \widetilde{C}_c$ on H_B (the extension of C_c outside H_B is irrelevant, see below); cf. Lemma 6.3.10(d3).

The operator $D_c \in \mathcal{B}(U,Y)$ is independent of α, by (6.40) and the Resolvent Equation. By Corollary A.3.7, we have $H_B \underset{c}{\subset} W$, hence $C_c|_{H_B} \in \mathcal{B}(H_B,Y)$, so that we could always replace W by H_B (or by any other Banach space $V \underset{c}{\subset} W$ s.t. $H_B \subset V$), and so we usually do. However, sometimes we wish to choose W s.t. H_1 is dense in W (it need not be dense in H_B even if $\mathbb{D} = 0$, by Lemma 6.3.10(f)), and this can be done by taking $W := \mathrm{Dom}(C_{L,s})$ whenever $\mathbb{D}$ is SR, by Lemma 6.3.10(e) and Proposition 6.2.8(c6). Also physical considerations might sometimes make some other W more convenient.

Theorem 6.3.9 *Every WPLS has a compatible pair.*

In the proof we show that C is bounded in the H_B norm, hence it has a unique bounded extension to the closure of $\mathrm{Dom}(A)$ in H_B. We could take the zero extension of this to extend C to H_B, but this is not always the most natural choice. E.g., if C is bounded (i.e., it coincides with some $C_0 \in \mathcal{B}(H,Y)$), then $C_w = C$ is in general different from the zero extension.

Proof: Let $\Sigma := \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$. Let H_{1B} be H_1 with the topology inherited from H_B. Assume, w.l.o.g., that $\alpha > \omega_A$ (in Lemma 6.1.16).

1° *If $z_n \to 0 \in H_{1B}$, then $Cz_n \to 0$ in Y:* Assume that $\{z_n\} \subset \mathrm{Dom}(A)$ is s.t. $\|z_n\|_{H_B} \to 0$ as $n \to \infty$. By definition of $\|z_n\|_{H_B}$, there must be $\{x_n\} \subset H_1$ and $\{u_n\} \subset U$ s.t.

$$z_n = x_n + (\alpha - A)^{-1}Bu_n, \quad \|x_n\|_{H_1} \to 0, \ \ \|u_n\|_U \to 0. \tag{6.76}$$

It follows that $Cx_n \to 0$. Now $z_n, x_n \in H_1$ imply that $(\alpha - A)^{-1}Bu_n \in H_1$. Choose $\omega > \omega_A$ and set $M := \|\widehat{\mathbb{D}}\|_{\mathrm{H}^\infty_\omega} < \infty$. Then

$$\widehat{\mathbb{D}}(s)u_n = \widehat{\mathbb{D}}(\alpha)u_n - C(s-\alpha)(s-A)^{-1}(\alpha - A)^{-1}Bu_n \to \widehat{\mathbb{D}}(\alpha)u_n - C(\alpha - A)^{-1}Bu_n =: y_n, \tag{6.77}$$

and $\|y_n\|_Y \le M\|u_n\|_U$, hance $\|C(\alpha - A)^{-1}Bu_n\|_Y \le 2M\|u_n\|_U$. Consequently, $Cz_n \to 0$ as $n \to \infty$.

2° Σ *is compatible:* By density (see Lemma A.3.10), C has a continuous extension to the closure $\bar{H}_{1B}$ of H_{1B} in H_B. Because H_B is a Hilbert space, this extension has an extension $C_c \in \mathcal{B}(H_B,Y)$, by Lemma A.3.11. Thus, C_c and $D_c := \widehat{\mathbb{D}}(\alpha) - C_c(\alpha - A)^{-1}B$ form a compatible pair for Σ. □

Next we list the basic properties of compatible pairs:

Lemma 6.3.10 *Let Σ, C_c and D_c be as in Definition 6.3.8. Then*

(a) $\widehat{\mathbb{D}}(s) = D_c + C_c(s-A)^{-1}B$ *for all* $s \in \mathbf{C}^+_\omega$.

(b) If $x_0 \in H$, $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbf{R}_+;U)$, *and* $Ax_0 + Bu(0) \in H$, *then* $\mathbb{C}x_0 + \mathbb{D}u = C_c x + D_c u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\mathbf{R}_+;Y)$, *where* $x := \mathbb{A}x_0 + \mathbb{B}\tau u$.

(c) If $u \in \mathrm{W}^{1,2}_\omega(\mathbf{R};U)$, *then* $\mathbb{D}u = C_c x + D_c u \in \mathrm{W}^{1,2}_\omega(\mathbf{R};Y)$, *where* $x := \mathbb{B}\tau u$.

(d1) Σ *is uniquely determined by* $\left[\begin{smallmatrix} A & B \\ C_c & D_c \end{smallmatrix}\right]$ *(the converse holds iff* H_1 *is dense in* H_B*).*

(d2) Conversely, Σ *determines* $\left[\begin{smallmatrix} A & B \\ C & * \end{smallmatrix}\right]$ *uniquely, and* C_c *can be chosen to be any continuous extension of the closure of* C *(on the closure of* H_1 *in* W*); this choice determines* D_c *uniquely.*

(d3) Let also $(\widetilde{C_c}, \widetilde{D_c})$ *be a compatible pair for* Σ*. Then* $C = \widetilde{C}_c$ *on* H_B *iff* $D_c = \widetilde{D_c}$*.*

(e) If $\mathbb{D}$ *is WR (resp. SR), then* $(C_{L,w}, D)$ *and* (C_w, D) *(resp.* $(C_{L,s}, D)$ *and* $(C_{L,s}, D)$*) are compatible pairs for* Σ*.*

(f) C_c *and* D_c *may be nonunique even if* Σ *is very regular (e.g.,* $\mathbb{D} = 0 = \mathbb{C}$*).*

(g) $(\alpha - A)^{-1}B \in \mathcal{B}(U, H_B) \subset \mathcal{B}(U, W)$ *for any* $\alpha \in \sigma(A)^c$*, and* $H_B \underset{c}{\subset} W$*.*

By causality, in (c) it is enough that $\pi_{(-\infty,T)}u \in \mathrm{W}^{1,2}_{\omega}(\mathbf{R}; U)$ for each $T > 0$.

Proof: (a) This follows from (6.40) and the Resolvent Equation.

(b) (Note that, given just u, we can always take $x_0 := (s - A)^{-1}Bu(0)$.) Assume that $u \in \mathrm{W}^{1,2}(\mathbf{R}_+; U)$ (for the general case, choose some $T > 0$ and extend u on $(T, +\infty)$ so that $u \in \mathrm{W}^{1,2}$; the values of x and $\mathbb{C}x_0 + \mathbb{D}u$ on $[0, T]$ remain unchanged, by the causality of $\mathbb{D}$).

Because $e^{-\omega\cdot}x \in \mathcal{C}_b(\mathbf{R}_+; H_B)$, by Theorem 6.2.13(b1), the Laplace transform of $f := C_c x + D_c u$ is

$$\hat{f}(s) = C_c\widehat{x}(s) + D_c\widehat{u}(s) = C_c[(s - A)^{-1}x_0 + (s - A)^{-1}B\widehat{u}(s)] + D_c\widehat{u}(s), \quad (6.78)$$

which equals the Laplace transform of $\mathbb{C}x_0 + \mathbb{D}u$. Both functions are continuous, hence equal.

(c) See the proof of (b).

(d2) The first claim follows from Lemma 6.1.16(d). Conversely, let H' be the closure of H_1 in W, and let H'' be its orthogonal complement in W. Obviously, C_c is uniquely determined on H', and $C_c|_{H''}$ can be chosen to be any $\mathcal{B}(H'', U)$ operator; this choice determines D_c uniquely, by Definition 6.3.8.

(d1) The uniqueness follows from (a) and Lemma 6.2.9(a). The converse holds iff H_1 is dense in W, by (d2).

(d3) If $C_c = \widetilde{C}_c$ on H_B, then $D_c = \widetilde{D_c}$, by definition. Conversely, let $D_c = \widetilde{D_c}$. Then $C_c = \widetilde{C}_c$ on $(\alpha - A)^{-1}B[U]$. But $C_c = C = \widetilde{C}_c$ on H_1, hence $C_c = \widetilde{C}_c$ on $H_1 + (\alpha - A)^{-1}B[U] = H_B$, by linearity.

(e) This follows from Proposition 6.2.8.

(f) Let $\mathbb{C} = 0 = \mathbb{D}$ (so that $\mathbb{D}$ is uniformly half-plane-regular etc.), $BU \cap H = \{0\}$ and $\mathrm{Ker}(B) = \{0\}$ (so that $(\alpha - A)^{-1}B \in \mathcal{B}(U, H'')$ is an isometric isomorphism onto, in terms of the proof of (d2); here α is as in Lemma 6.1.16).

Then we can choose an arbitrary $D_c \in \mathcal{B}(U, Y)$, and define C_c on $H' = H_1$ by $C_c|_{H'} = C$, and on H'' by $C_c(\alpha - A)^{-1}B := \widehat{\mathbb{D}}(\alpha) - D_c \in \mathcal{B}(U, Y)$, so that $C_c \in \mathcal{B}(H' \times H'', Y)$, and (C_c, D_c) is a compatible output operator pair for Σ.

(g) By Corollary A.3.7, we have $H_B \underset{c}{\subset} W$, hence $\mathcal{B}(U, H_B) \subset \mathcal{B}(U, W)$. Trivially, $\|(alpha - A)^{-1}Bu_0\|_{H_B} \leq \|u_0\|_U$ for all $u_0 \in U$ if α is as in Definition 6.1.17. Since a different α leads to an equivalent norm on H_B, as noted at the end of Definition 6.1.17, we have $(\alpha - A)^{-1}B \in \mathcal{B}(U, H_B)$ for any α. □

If B is bounded, then $H_B = H_1$, so that $C_c = C$ is essentially the only possible choice of C_c. If B is unbouded and $U = \mathbf{C}$ (even if C were bounded), then D_c can be chosen arbitrarily:

Lemma 6.3.11 *Let $\Sigma \in \mathrm{WPLS}(\mathbf{C}, H, Y)$ have B unbounded, i.e., $B1 \notin H$. Choose $r > \omega_A$ and set $x_B := (r - A)^{-1}B1$. Then $\|x_1 + \alpha x_B\|_{H_B'} := |\alpha|^2 + \|x_1\|_{H_1}^2$ is an equivalent norm for $H_B = \{x_1 + \alpha x_B \,|\, x_1 \in H_1,\ \alpha \in \mathbf{C}\}$.*

Let $y_0 \in Y$ be arbitrary. Define, $C_c(x_1 + \alpha x_B) := Cx_1 + \alpha y_0$, so that $C_c \in \mathcal{B}(H_B, Y)$ becomes an extension of $C \in \mathcal{B}(H_1, Y)$ and hence (C_c, Dc) are compatible generators of Σ, where

$$D_c := \widehat{\mathbb{D}}(r) - C_c(r - A)^{-1}B = \widehat{\mathbb{D}}(r) - C_c x_B = \widehat{\mathbb{D}}(r) - \alpha y_0 \in \mathcal{B}(\mathbf{C}, Y) = Y. \quad (6.79)$$

Thus, $D_c \in \mathcal{B}(\mathbf{C}, Y)$ can be chosen arbitrarily. □

(All this is straightforward.)

In particular, the "compatible feedthrough operator" $D_c \in \mathcal{B}(\mathbf{C})$ can be chosen arbitrarily in Example 6.2.14. If we take $D_c \neq 0$, then it becomes compensated in C_c. If we replace C by 0 in Example 6.2.14, then $D_c \neq 0$ implies that $C_c \neq 0$ — thus, the compatible generators are not unique even for bounded C.

How to recognize a suitable pair (C_c, D_c)? Quite easily:

Lemma 6.3.12 *Let $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_\omega(U, H, Y)$. If (1.) or (2.) holds, then (C_c, D_c) is a compatible pair for Σ.*

(1.) $C_c : H_B \to Y$ is linear, $D_c \in \mathcal{B}(U, Y)$, $C \subset C_c$ and $\widehat{\mathbb{D}}(s) = D_c + C_c(s - A)^{-1}B$ for some $s \in \mathbf{C}^+_\omega$;

(2.) $C_c \in \mathcal{B}(W, Y)$, $D_c \in \mathcal{B}(U, Y)$, $C \subset C_c$, and $(\mathbb{D}u)(0) = D_c u(0) + C_c x(0)$ whenever $u = \phi u_0$, $u_0 \in U$ and $\phi \in C_c^\infty(\mathbf{R})$.

Recall that "$C \subset C_c$" means that C_c is an extension of C.

Proof:

(1.) Now $C_c(\alpha - A)^{-1}(x_0 + Bu_0) = C(\alpha - A)^{-1}x_0 + [\widehat{\mathbb{D}}(\alpha) - D_c]u_0 \leq M[\|x_0\| + \|u_0\|]$ for all $x_0 \in H$, $u_0 \in U$, where $M := \max\{\|C(\alpha - A)^{-1}\|, \|\widehat{\mathbb{D}}(\alpha) - D\|\}$, hence $C_c \in \mathcal{B}(H_B, Y)$, hence (C_c, D_c) is a compatible pair for Σ.

(2.) Because $C_c^\infty(\mathbf{R})$ is dense in $\mathrm{W}^{1,2}_\omega(\mathbf{R})$, by Theorem B.7.3 and Lemma B.7.10, we may choose any $\phi \in \mathrm{W}^{1,2}_\omega(\mathbf{R})$, by continuity (indeed, by Lemma 6.3.19, we have $\mathbb{B} \in \mathcal{B}(\mathrm{W}^{1,2}_\omega; H_B)$; by Theorem B.7.4 and Lemma B.7.10, we also have $(\phi u_0 \mapsto \phi(0)u_0) \in \mathcal{B}(\mathrm{W}^{1,2}_\omega, U)$; by Theorem 6.2.13(b1), $y \in \mathrm{W}^{1,2}_\omega$, hence also $y \mapsto y(0)$ is continuous).

By causality, we may take $\phi = e^{st}$ for any $s \in \mathbf{C}^+_\omega$ (since $\pi_{(-\infty,0]}\phi \in \mathrm{W}^{1,2}_\omega$). From Lemma 6.2.10 we obtain that

$$\widehat{\mathbb{D}}(s)u_0 = D_c u_0 + C_c \mathbb{B}u = (D_c + C_c(s - A)^{-1}B)u_0. \quad (6.80)$$

Since $u_0 \in U$ was arbitrary, we have $D_c = \widehat{\mathbb{D}}(s) - C_c(s - A)^{-1}B$. □

Next few lemmas will give necessary and sufficient conditions for given operators to be the generators of a WPLS:

Lemma 6.3.13 (Generating a WPLS) *Operators* $\left[\begin{array}{c|c} A & B \\ \hline C_c & D_c \end{array}\right]$ *are compatible generators of a WPLS* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *iff the following conditions hold:*

(1.) A is the generator of a C_0-semigroup $\mathbb{A}$ on H.

(2.) $B \in \mathcal{B}(U,H_{-1})$, and there is $T > 0$ s.t. for all $u \in \mathrm{L}^2([-T,0);U)$ we have

$$\mathbb{B}u := \int_{-T}^{0} \mathbb{A}(-s)Bu(s)\,ds \in H. \tag{6.81}$$

(3.) $C_c \in \mathcal{B}(W,Y)$, $D_c \in \mathcal{B}(U,Y)$ and $H_B \subset W \underset{c}{\subset} H$.

(4.) The map $\mathbb{C} : H_1 \to \mathcal{C}(\mathbf{R}_+;Y)$ defined by

$$(\mathbb{C}x)(t) := C_c\mathbb{A}(t)x, \quad (x \in H_1,\ t \geq 0) \tag{6.82}$$

can be extended to a continuous map $H \mapsto \mathrm{L}^2([0,T);Y)$ for some $T > 0$.

(5.) For some $\omega > \omega_A$ and $T > 0$, the map $\mathbb{D} : \mathcal{C}_c^\infty((0,T);U) \to \mathcal{C}_b((0,T);Y)$ defined by

$$(\mathbb{D}u)(t) := C_c\mathbb{B}\tau(t)u + D_cu(t) \quad (t \in \mathbf{R}), \tag{6.83}$$

can be extended to a continuous map $\mathrm{L}^2_\omega([0,T);U) \to \mathrm{L}^2_\omega([0,T);Y)$ for some $T > 0$.

If this is the case, then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ *for any $\omega > \omega_A$; consequently, $\widehat{\mathbb{D}}(s) = D_c + C_c(s-A)^{-1}B$ for $s \in \mathbf{C}^+_{\omega_A}$. (Here $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ are the unique continuous extensions of operators defined in (6.81)–(6.83).)*

By the last claim, $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ is WR iff $C_c(s-A)^{-1}B$ converges weakly as $s \to +\infty$ (the limit is then $D - D_c$, which need not be zero, cf. Lemma 6.3.10(e)).

If we know that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}$, then (1.) and (2.) are redundant; if $\left[\begin{smallmatrix} \mathbb{A} \\ \mathbb{C} \end{smallmatrix}\right] \in \mathrm{WPLS}$, then (1.) and (4.) are redundant.

Proof: 1° *"Only if":* This follows from Lemma 6.1.16 and Lemma 6.3.10(c).

2° *"If":* Let $\omega_A < \alpha < \omega$, and choose M s.t. $\|\mathbb{A}(t)\| \leq Me^{\alpha t}$ $(t \geq 0)$. Note first that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}_\omega(U,H_{-1},\{0\})$, which follows from the α-boundedness of $\mathbb{A}$ and the extended $\mathbb{B}$ defined by $\mathbb{B}u := \int_{\mathbf{R}_-} \mathbb{A}(-\cdot)Bu$. But the range of $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2([-T,0);U),H_{-1})$ is in H, hence $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2([-T,0);U),H)$, by Lemma A.3.6.

Then $\left(\begin{array}{c|c} \mathbb{A}^T & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right)$ generate a e^ω-stable wpls, say $\Delta^S\widetilde{\Sigma}$ (see Section 13.4); let $\left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \widetilde{\mathbb{D}} \end{array}\right] := \Delta^{S^{-1}}(\Delta^S\widetilde{\Sigma})$ be the corresponding "WPLS". Then the quadruple $\left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \widetilde{\mathbb{D}} \end{array}\right]$ is "ω-stable", by Lemma 13.3.8.

One easily verifies that this quadruple is a WPLS and has generators $\left[\begin{smallmatrix} A & B \\ C & * \end{smallmatrix}\right]$ (hence compatible generators $\left[\begin{smallmatrix} A & B \\ C_c & D_c \end{smallmatrix}\right]$).

3° The last two claims follow from Lemma 6.1.10 and Theorem 6.2.11(d1). □

Corollary 6.3.14 ($\left[\begin{smallmatrix} \mathbb{A} \\ \mathbb{C} \end{smallmatrix}\right]$) *Assume that $\mathbb{A}$ is a C_0-semigroup and that $C \in \mathcal{B}(H_1, Y)$. Then $\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]$ generate a WPLS iff there are $T > 0$, $M < \infty$ s.t. $\|C\mathbb{A}x_0\|_{\mathrm{L}^2([0,T);Y)} \le M\|x_0\|_H$ for all $x_0 \in H_1$.* □

(Apply Lemma 6.3.13 with $B = 0$, $D = 0$ and $W = H_1$.)

We sometimes need the following frequency-domain variant of Lemma 6.3.13:

Lemma 6.3.15 (Generating a $\widehat{\textbf{WPLS}}$) *Operators $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H_1 \times U, H \times Y)$ generate a WPLS iff (1.) A generates a C_0-semigroup on H (see Theorem A.4.3), and there are $\varepsilon > 0$ and $\omega > \omega_A$ and $s_0 \in \mathbf{C}^+_{\omega_A}$ s.t. for each $x_0 \in H$ we have*

(2.) $B^*(\cdot - A^*)^{-1}x_0 \in \mathrm{H}^2(\mathbf{C}^+_\omega; U)$;

(4.) $C(\cdot - A)^{-1}x_0 \in \mathrm{H}^2(\mathbf{C}^+_\omega; Y)$;

(5.) $(\cdot - s_0)C(\cdot - A)^{-1}(s_0 - A)^{-1}B \in \mathrm{H}^\infty_\infty$

For $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to generate a WR WPLS, we may replace (5.) by the assumption that $D + C_{\mathrm{w}}(\cdot - A)^{-1}B \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$, but we must require that $H_B \subset \mathrm{Dom}(C_{\mathrm{w}})$, i.e., that $Cr(r-A)^{-1}(\omega - A)^{-1}Bu_0$ converges weakly for all $u_0 \in U$.

One may replace (5.) by the assumption that the map (6.51) extends to a continuous operator $\mathrm{L}^2([0,T);U) \to \mathrm{L}^2([0,T);Y)$. Naturally, the WPLS is ω-stable for any $\omega > \omega_A$.

Proof: (Analogous results are given in Chapter 9 of [Sbook], so we only sketch a proof.) Necessity follows from Theorem 6.2.11. For the converse, assume (1.)–(5.). The operator $\mathbb{C}$ of (6.82) satisfies $\widehat{\mathbb{C}x_0}(s) = C(s-A)^{-1}x_0$ for all $x_0 \in H_1$, hence (1.), (4.) and Lemma 6.3.13 apply that $\left[\begin{smallmatrix} \mathbb{A} \\ \mathbb{C} \end{smallmatrix}\right] \in$ WPLS. Analogously, $\left[\begin{smallmatrix} \mathbb{A}^{\mathrm{d}} \\ \mathbb{B}^{\mathrm{d}} \end{smallmatrix}\right] \in$ WPLS, hence $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in$ WPLS. By (d2) (see also (d1)) of Theorem 6.2.11, Lemma D.1.26 and Lemma 6.2.10, we have that $\pi_+\mathbb{D}\pi_- = \mathbb{C}\mathbb{B}$, hence $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in$ WPLS. □

A bounded operator will always do as a generator:

Lemma 6.3.16 (Bounded *B* or *C*)

(a) Let $B \in \mathcal{B}(U,H)$, $C \in \mathcal{B}(H,Y)$ and $D \in \mathcal{B}(U,Y)$, and let A generate a C_0-semigroup on H (e.g., $A \in \mathcal{B}(H)$). Then $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ generate a WPLS that is ULR.

(b) **(Bounded B)** *Let* $\left[\frac{A}{C}\right]$ *generate* $\left[\frac{\mathbb{A}}{\mathbb{C}}\right] \in \mathrm{WPLS}(\{0\}, H, Y)$*, i.e., let A generate a C_0-semigroup $\mathbb{A}$, and let $C \in \mathcal{B}(\mathrm{Dom}(A), Y)$ be s.t. the map $\mathbb{C} : \mathrm{Dom}(A) \to C(\mathbf{R}_+; Y)$ defined by*

$$(\mathbb{C}x)(t) := C\mathbb{A}(t)x, \quad (x \in \mathrm{Dom}(A),\ t \geq 0) \tag{6.84}$$

can be extended to a continuous operator $H \mapsto \mathrm{L}^2([0, \varepsilon); Y)$ for some $\varepsilon > 0$.

Assume that $B \in \mathcal{B}(U, H)$ and $D \in \mathcal{B}(U, Y)$. Then $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *generate a WPLS, $\mathbb{D}$ is ULR, and $H_B = \mathrm{Dom}(A)$ with equivalent norms.*

If $\omega_A < 0$, then $\mathbb{D}$ is strongly half-plane-regular. If $\mathbb{C}$ is ω-stable, then $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U, Y))$.

(c) **(Bounded C)** *Let* $\left[\begin{array}{c|c} A & B \end{array}\right]$ *generate* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}(U, H, \{0\})$*, i.e., let (1.)–(2.) of Lemma 6.3.13 hold.*

Assume that $C \in \mathcal{B}(H, Y)$ and $D \in \mathcal{B}(U, Y)$. Then $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *generate a WPLS Σ, and $\mathbb{D}$ is ULR.*

(d) In (a)–(c), Σ and Σ^{d} are ULR and ω-stable for any $\omega > \omega_A$, and $\widehat{\mathbb{D}}(s) = D + C(s - A)^{-1}B$ ($s \in \mathbf{C}^+_{\omega_A}$). If $\omega_A < 0$, then $\mathbb{D}$ and $\mathbb{D}^{\mathrm{d}}$ are weakly half-plane-regular.

Thus, any WPLS with a bounded B or C is ULR; in particular, $\mathbb{B}\tau$ is always ULR (with zero feedthrough!), since it has the realization $\left(\begin{array}{c|c} A & B \\ \hline I & 0 \end{array}\right)$. Since any bounded A ($\in \mathcal{B}(H)$) generates the (uniformly continuous) C_0-semigroup e^{At} with $H_1 = H = H_{-1}$, the operators B, C and D are necessarily bounded if A is bounded.

Proof: (a) It follows from, e.g., (c), that $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ generate a WPLS. By Lemma A.4.4(c3), this WPLS is ULR.

(b) Apply (c) to Σ^{d} (see Lemma 6.1.4) to see that $\Sigma^{\mathrm{d}} \in \mathrm{WPLS}$ and Σ^{d} is ULR (hence so is Σ). If $\omega_A < r < 0$, then $C(s - A)^{-1}B \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_r; \mathcal{B}(U, Y))$, by Theorem 6.2.11(c2), hence $\mathbb{D} - D$ is strongly half-plane-regular (hence so is $\mathbb{D}$), by Proposition 6.3.3(a).

Obviously, $H_B = \{x_0 \in H \,|\, Ax_0 \in H\} = \mathrm{Dom}(A)$. Because $H_B \subset H$ and $\mathrm{Dom}(A) \subset H$, continuously, their topologies coincide, by Corollary A.3.7, i.e., they have equivalent norms.

If $\mathbb{C}$ is ω-stable, then $C(s - A)^{-1} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(H, Y))$, by Theorem 6.2.11(c2), hence also the last claim holds.

(c) 1° *WPLS:* Let $\omega > \omega_A$ so that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}_\omega$, by Lemma 6.1.10. Set $\mathbb{C}x_0 := \pi_+ C\mathbb{A}(\cdot)x_0$ for $x_0 \in H$. Obviously, $\mathbb{C} \in \mathcal{B}(H, \mathrm{L}^2_\omega)$, hence (1.)–(4.) of Lemma 6.3.13 hold.

Set $(\mathbb{D}u)(t) := D + C\mathbb{B}\tau(t)u$ (i.e., "$y = Cx + Du$") for $u \in \mathrm{L}^2_\omega$, $t \in \mathbf{R}$. By Theorem 6.2.11(a)&(b1), we have $\widehat{\mathbb{D}u} = D\widehat{u} + C(s - A)^{-1}B\widehat{u}$ for $u \in \mathrm{L}^2_\omega(\mathbf{R}_+; U)$. But $C(\cdot - A)^{-1}B \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U, Y))$, by Theorem 6.2.11(b2), hence $D + C(s - A)^{-1}B$ defines some $\mathbb{D}' \in \mathrm{TIC}_\omega(U, Y)$. By the obvious time-invariance of $\mathbb{D}$, we have $\mathbb{D}u = \mathbb{D}'u$ for $u \in \mathrm{L}^2_\omega(\mathbf{R}_+; U)$, hence also (5.) of Lemma 6.3.13 is satisfied.

2° *ULR:* By Theorem 6.2.11(b3), we have $\|C(s - A)^{-1}B\|_{\mathcal{B}(U,Y)} \leq M/\sqrt{\mathrm{Re}\, s - \gamma}$ for $\gamma > \omega_A$, $s \in \mathbf{C}^+_\gamma$, $M := \|\mathbb{B}\|\|C\|/\sqrt{2}$, hence $\mathbb{D} \in \mathrm{ULR}$.

(d) See the proofs of (b) and (c) (and use duality). □

The state feedback operator corresponding to a Riccati operator is of the following form (except that T is not always bounded):

Lemma 6.3.17 *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, $K = SC + T$, $S \in \mathcal{B}(Y,Z)$, $T \in \mathcal{B}(H,Z)$.

Then $\left[\begin{array}{c|c} A & B \\ \hline K & \end{array}\right]$ *generate* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{K} & \mathbb{F}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Z)$. *Moreover,* $\mathbb{F} = S\mathbb{D} + T\mathbb{B}\tau + E$, *where* $E \in \mathcal{B}(U,Z)$ *is arbitrary, and* $H^*_{C,K} = H^*_C$.

If $\mathbb{D}$ *is WR (resp. SR, UR, WLR, SLR, ULR, WVR), then so is* $\mathbb{F}$, *and* $F = SD + E$, *i.e.,* $\mathbb{F} = S(\mathbb{D} - D) + T\mathbb{B}\tau + F$. *In particular, if* $\mathbb{D} \in \mathrm{ULR}$, *then* $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow I - F \in \mathcal{GB}(U)$,

Thus, if $\mathbb{D} \in \mathrm{ULR}$ and $Z = U$, then K is an ULR admissible state-feedback operator (see Definition 6.6.10).

Proof: 1° $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{K} & \mathbb{F}\end{smallmatrix}\right] \in \mathrm{WPLS}$*:* Now $K\mathbb{A}x_0 = S\mathbb{C}x_0 + T\mathbb{A}x_0$ for $x_0 \in H_1$, hence $\left[\begin{smallmatrix}\mathbb{A}\\ \mathbb{K}\end{smallmatrix}\right] \in \mathrm{WPLS}$. But if we set $\mathbb{F} = S\mathbb{D} + T\mathbb{B}\tau \in \mathrm{TIC}_\infty(U,Z)$, then

$$\mathbb{K}\mathbb{B} = \pi_+(S\mathbb{D} + T\mathbb{A}\mathbb{B})\pi_- = \pi_+(S\mathbb{D} + T\mathbb{B}\tau)\pi_- = \pi_+\mathbb{F}\pi_-, \tag{6.85}$$

hence $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{K} & \mathbb{F}\end{smallmatrix}\right]$ is a WPLS. By Lemma 6.1.16(d), $\mathbb{F}$ is unique modulo a constant $E \in \mathcal{B}(U,Z)$.

2° Now $\mathrm{Ran}(K^*) = \mathrm{Ran}(C^*S^* + T^*) \subset H + \mathrm{Ran}(C^*)$, hence $(\bar{\alpha} - A^*)^{-1}K^* \subset H^*_C$ (see Definition 6.1.17).

3° Because $\mathbb{B}\tau \in \mathrm{ULR} \cap \mathrm{WVR}$ with feedthrough zero, by Lemma 6.3.16(c)&(d), $\mathbb{F}$ inherits the regularity properties of $\mathbb{D}$ up to ULR and WVR, and $F = SD + E$. The last claim follows from Proposition 6.3.1(c). □

We still need several more auxiliary technical lemmas on generators for the needs of later chapters. The following equivalent norm on H_B makes their proofs simpler:

Lemma 6.3.18 ($\|x_0\|_{H_B}$) *The norm* $\|\cdot\|'_{H_B}$, *defined by*

$$\|x_0\|'_{H_B} := \|x_0\|_H + \inf_{u_0 \in U}\left(\|Ax_0 + Bu_0\|_H + \|u_0\|_U\right), \tag{6.86}$$

is equivalent to $\|\cdot\|_{H_B}$.

In particular, if $u_n \to u_\infty$ *in* U, $x_n \to x_\infty$ *in* H *and* $Ax_n + Bu_n \to Ax_\infty + Bu_\infty$ *in* H, *as* $n \to \infty$, *then* $x_n \to x_\infty$ *in* H_B.

Even $\|x_0\|''_{H_B} := \|x_0\|_H + \inf_{u_0 \in U}\left(\|Ax_0 + Bu_0 + sx_0\|_H + \|u_0\|_U\right)$, where $s \in \mathbf{C}$ is fixed, is equivalent to $\|x_0\|_{H_B}$, as one can see from the proof below.

Proof: (Recall that $\|x_1\|_H := \infty$ for $x_1 \notin H$ etc.) Let α be as in Definition 6.1.17.

1° *The equivalence:* One can rewrite the definition of $\|x_0\|_{H_B}$ by

$$\|x_0\|_{H_B} := \inf\{(\|Ax_0 + Bu_0 - \alpha x_0\|_H^2 + \|u_0\|_U^2)^{1/2} \mid u_0 \in U,\ Ax_0 + Bu_0 \in U\}. \tag{6.87}$$

Thus, $\|x_0\|_{H_B} \le \max\{|\alpha|,1\}\|x_0\|'_{H_B}$. It follows that $\|x_0\|'_{H_B} = 0 \Rightarrow x_0 = 0$. Now the reader can easily verify that $\|\cdot\|'_{H_B}$ is a norm.

Choose M s.t. $\|\cdot\|_H \le M\|\cdot\|_{H_B}$. Assume that $\|x_0\|_{H_B} < 1$, so that $\|Ax_0 + Bu_0 - \alpha x_0\|_H < 1$ and $\|u_0\|_U < 1$ for some $u_0 \in U$, by (6.87). It follows that $\|x_0\|'_{H_B} < M + (1 + \alpha M + 1)$. Thus, $\|\cdot\|'_{H_B} \le (2 + (|\alpha| + 1)M)\|\cdot\|_{H_B}$. Consequently, the two norms are equivalent.

2° *"$x_n \to x_\infty$":* Obviously, $\|x_n - x_\infty\|'_{H_B} \le \|A(x_n - x_\infty) + B(u_n - u_\infty)\|_H + \gamma\|x_n - x_\infty\|_H + \|u_n - u_\infty\|_U \to 0$ as $n \to \infty$. □

By definition, an input map $\mathbb{B}$ maps $\mathrm{L}^2_\omega \to H$ continuously. Smoother inputs give "smoother states", as noted in Theorem 6.2.13(c1)&(c2) and below:

Lemma 6.3.19 ($\mathbb{B} : \mathrm{W}^{1,2}_\omega \to H_B$) *Let* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}_\omega(U,H,\{0\})$. *Then there is* $M < \infty$ *s.t.* $\|\mathbb{B}\tau^t u\|_{H_B} \le M e^{\omega t}\|u\|_{\mathrm{W}^{1,2}_\omega((-\infty,t);U)}$ *and* $\|(\mathbb{B}\tau u)^{(n)}(t)\|_H \le M e^{\omega t}\|u\|_{\mathrm{W}^{n+1,2}}$ *for all* $u \in \mathrm{W}^{n,2}_\omega((-\infty,t);U)$, $n \in \mathbf{N}$. *Moreover,* $\|(\alpha - A)^{-1}\begin{bmatrix} I & B \end{bmatrix}\|_{\mathcal{B}(H\times U,H_B)} \le 1$.

Proof: The last claim follows from the definition of H_B. For the others, we take $t = 0$ w.l.o.g. (note that $\|\tau^t u\|_{\mathrm{W}^{n+1,2}_\omega((-\infty,0);U)} = e^{\omega t}\|u\|_{\mathrm{W}^{n+1,2}_\omega((-\infty,t);U)}$). Let $x := \mathbb{B}\tau u$, so that on $(-\infty,t)$ we have $x' = Ax + Bu$, i.e.,

$$x = (\alpha - A)^{-1}(\alpha x - x' + Bu), \tag{6.88}$$

(let α be the number in Definition 6.1.17). Therefore,

$$\|x(t)\|_{H_B} \le \|u(t)\|_U + |\alpha|\|x(t)\|_H + \|x'(t)\|_H. \tag{6.89}$$

In particular, $\|x(0)\|_{H_B} \le M_1\|u\|_{\mathrm{W}^{1,2}_\omega} + |\alpha|\|\mathbb{B}\|\,\|u\|_{\mathrm{L}^2_\omega} + \|\mathbb{B}\|\,\|u'\|_{\mathrm{L}^2_\omega}$, by Theorem B.7.4 (recall that $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2_\omega, H)$ and that $x' = \mathbb{B}\tau u'$). Use induction for the nth derivative. □

If A is not exponentially stable, then $(s - A)^{-1}$ might not exist for $s \in i\mathbf{R}$. Nevertheless, equations $\widehat{x} = (\cdot - A)^{-1}B\widehat{u}$ and $\widehat{y} = C\widehat{x} + D\widehat{u}$ can be partially recovered on $i\mathbf{R}$ when u, x and y are stable (take $\omega = 0$):

Lemma 6.3.20 *Let* (C_c, D_c) *be a compatible pair for* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$. *Let* $\omega \in \mathbf{R}$, $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$ *and* $x := \mathbb{B}\tau u \in \mathrm{L}^2_\omega(\mathbf{R}_+;H)$. *Then* $(s - A)\widehat{x}(s) = B\widehat{u}(s) \in H_{-1}$ *(in particular,* $\widehat{x}(s) \in H_B$*) for a.e.* $s \in \omega + i\mathbf{R}$.

Assume, in addition, that $y := \mathbb{D}u \in \mathrm{L}^2_\omega$. *Then* $\widehat{y} = C_c\widehat{x} + D_c\widehat{u} \in Y$ *a.e. on* $\omega + i\mathbf{R}$. *In particular, for* $\omega = 0$ *and* $J \in \mathcal{B}(Y)$ *we have*

$$\langle \mathbb{D}u, J\mathbb{D}u\rangle_{\mathrm{L}^2(\mathbf{R};Y)} = (2\pi)^{-1}\left\langle \begin{bmatrix} \widehat{x} \\ \widehat{u} \end{bmatrix}, \kappa \begin{bmatrix} \widehat{x} \\ \widehat{u} \end{bmatrix} \right\rangle_{\mathrm{L}^2(i\mathbf{R};Y)}, \tag{6.90}$$

where $\kappa := \begin{bmatrix} C_c & D_c \end{bmatrix}^* J \begin{bmatrix} C_c & D_c \end{bmatrix}$.

Note that we have $\langle \mathbb{D}u, J\mathbb{D}u\rangle = \langle \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right], \kappa\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\rangle$ whenever $u \in \mathrm{W}^{1,2}(\mathbf{R}_+;U)$, $u(0) = 0$, $\mathbb{D}u \in \mathrm{L}^2$, by Lemma 6.3.10(b) (for general $u \in \mathrm{L}^2$ having $\mathbb{D} \in \mathrm{L}^2$, an analogous result holds when $\mathbb{D}$ is WR). Here the integrands are continuous, whereas those of (6.90) are defined only a.e. In Lemma 6.7.8, we shall show that the assumption that $y \in L^2_\omega$ is redundant.

Proof: 1° *"$(\cdot - A)\widehat{x} = B\widehat{u}$ a.e. on $\omega + i\mathbf{R}$":* Recall that $A \in \mathcal{B}(H, H_{-1})$ and $B \in \mathcal{B}(U, H_{-1})$. We have $(\cdot - A)\widehat{x} = B\widehat{u}$ on $\mathbf{C}^+_{\omega_A}$, by Theorem 6.2.11(b1), hence on $\mathbf{C}^+_\omega$, by Lemma D.1.2(e) (both functions are holomorphic $\mathbf{C}^+_\omega \to H_{-1}$). Because both $\widehat{x}$ and $\widehat{u}$ converge a.e. to their boundary functions, by Theorem 3.3.1(a), also the boundary functions must satisfy $(\cdot - A)\widehat{x} = B\widehat{u}$ a.e.

2° *"$\widehat{y} = C_c\widehat{x} + D_c\widehat{u} \in Y$ a.e. on $\omega + i\mathbf{R}$":* Choose a null set $N \subset \mathbf{R}$ s.t. $\widehat{x}(\omega + ir + t) \to \widehat{x}(\omega + ir)$, $\widehat{y}(\omega + ir + t) \to \widehat{y}(\omega + ir)$ and $\widehat{u}(\omega + ir + t) \to \widehat{u}(\omega + ir)$, as $t \to 0+$, for all $r \in \mathbf{R} \setminus N$. Let $r \in \mathbf{R} \setminus N$. Then

$$A\widehat{x}(\omega + ir + t) + B\widehat{u}(\omega + ir + t) = (\omega + ir + t)\widehat{x}(\omega + ir + t) \tag{6.91}$$

$$\to (\omega + ir)\widehat{x}(\omega + ir) = A\widehat{x}(\omega + ir) + B\widehat{u}(\omega + ir), \tag{6.92}$$

as $t \to 0+$. Therefore, $\widehat{x}(\omega + ir + t) \to \widehat{x}(\omega + ir)$ in H_B, by Lemma 6.3.18. Analogously, we see that $\widehat{x} \in \mathcal{C}(\mathbf{C}^+_\omega; H_B)$.

But $\widehat{y} = C_c\widehat{x} + D_c\widehat{u}$ on $\mathbf{C}^+_{\omega_A}$, by Lemma 6.3.10(a), hence $\widehat{y} = C_c\widehat{x} + D_c\widehat{u}$ on $\mathbf{C}^+_\omega$, by continuity. Consequently, $\widehat{y} = C_c\widehat{x} + D_c\widehat{u}$ on $\omega + i(\mathbf{R} \setminus N)$.

3° We get (6.90) from (3.34). □

In the setting of the above lemma, the following estimate is often useful:

Lemma 6.3.21 ($\|C_c x_0\| \le M(\|x_0\| + \|u_0\|)$) *Let (C_c, D_c) be a compatible pair for $\Sigma \in$ WPLS(U, H, Y). For each $\omega \in \mathbf{R}$, there is $M_\omega = M_{\omega,\Sigma,D_c} < \infty$ s.t. for all $x_0 \in H$, $u_0 \in U$, $s \in \mathbf{C}^+_\omega$ we have*

$$\|C_c x_0\|_Y \le M_\omega(\|x_0\|_H + \|u_0\|_U + \|(s - A)x_0 - Bu_0\|_H), \quad \textit{in particular}, \tag{6.93}$$

$$r \in \mathbf{R} \ \& \ (ir - A)x_0 = Bu_0 \implies \|C_c x_0\|_Y \le M_0(\|x_0\|_H + \|u_0\|_U). \tag{6.94}$$

Proof: Choose $\alpha > \omega_A$. Choose $z \in \mathbf{C}^+_{\alpha-\omega}$, so that $s + z \in \mathbf{C}^+_\alpha$. Set $x_1 := sx_0 - Ax_0 - Bu_0$.

Then $(s + z - A)x_0 = zx_0 + x_1 + Bu_0$, i.e., $x_0 = (s + z - A)^{-1}(zx_0 + x_1) + (s + z - A)^{-1}Bu_0$. Thus, by Lemma 6.3.10(a),

$$C_c x_0 = C(s + z - A)^{-1}(zx_0 + x_1) + (\widehat{\mathbb{D}}(s + z) - D_c)u_0. \tag{6.95}$$

Consequently, we can take $M_\omega := \|\widehat{\mathbb{C}}\|_{H^\infty(\mathbf{C}^+_\alpha; \mathcal{B}(H,Y))}(|z| + 1) + \|\mathbb{D}\|_{\mathrm{TIC}_\alpha} + \|D_c\|_{\mathcal{B}(U,Y)} < \infty$, by Theorem 6.2.11(c2).

(Note that $M_\omega = M_{\Sigma,D_c,z}$, where we can fix, e.g., $z := \omega_A + 1 - \omega$ to obtain $M_\omega = M_{\Sigma,D_c,\omega}$.) □

If $\mathbb{D}$ is ULR (or SLR), then we can improve the above estimate:

Lemma 6.3.22 ($\|C_w x_0\| \le M\|x_0\| + \varepsilon\|u_0\|$) *Let $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in$ WPLS(U, H, Y). If $\mathbb{D}$ is ULR, then, for each $\omega \in \mathbf{R}$ and $\varepsilon > 0$, there is $M_{\omega,\varepsilon} < \infty$ s.t.*

$$\|C_w x_0\|_Y \le \varepsilon\|u_0\|_U + M_{\omega,\varepsilon}(\|x_0\|_H + \|(s - A)x_0 - Bu_0\|_H), \tag{6.96}$$

for all $x_0 \in H$, $u_0 \in U$ *and* $s \in \mathbf{C}^+_\omega$. *If* $\mathbb{D}$ *is merely SLR, then we can still choose* $M_{u_0,\omega,\varepsilon} < \infty$ *satisfying (6.96) for any (fixed)* $u_0 \in U$, $\omega \in \mathbf{R}$ *and* $\varepsilon > 0$.

Proof: Assume that $\mathbb{D}$ is ULR. Given $\varepsilon > 0$, choose $\alpha > \omega_A$ s.t. $\|\widehat{\mathbb{D}}(z) - D\|_{\mathcal{B}} < \varepsilon$ for all $z \in \mathbf{C}^+_\alpha$. Work otherwise as in the proof of Lemma 6.3.21 (with $D_c := D$, $C_c := C_w$). Then the norm of the last term in (6.95) is at most $\varepsilon\|u_0\|$, hence we obtain $M_{\omega,\varepsilon}$ as above.

If $\mathbb{D}$ is SLR, then the above proof applies except that we have to choose $R > \omega$ s.t. $\|\widehat{\mathbb{D}}(s)u_0 - Du_0\|_{\mathcal{B}} < \varepsilon$ for all $s \in \mathbf{C}^+_R$. □

From the above estimate, we obtain the following implications:

Lemma 6.3.23 ($\mathbb{B}\tau \in$ SHPR&$\mathbb{D} \in$ SLR $\Rightarrow \mathbb{D} \in$ SHPR) *If* $\mathbb{B}\tau$ *is UHPR and* $\mathbb{D}$ *is ULR, then* $\mathbb{D}$ *is UHPR. If* $\mathbb{B}\tau$ *is UVR and* $\mathbb{D}$ *is ULR, then* $\mathbb{D}$ *is UVR. This lemma also holds with S in place of U.*

Proof: 1° $\mathbb{B}\tau \in$ UHPR*:* Let $\mathbb{B}\tau$ be UHPR. Then $\mathbb{B}\tau \in \mathrm{TIC}_\omega(U,H)$ for all $\omega > 0$, hence $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$ for all $\omega > 0$, by Lemma 6.1.10(b2). Given $\varepsilon > 0$, choose $M := M_{0,\varepsilon/2}$ as in Lemma 6.3.22. Choose $R > 0$ s.t. $\|(s-A)^{-1}B\| < \varepsilon/2M$ for $s \in \mathbf{C}^+$ s.t. $|s| > R$.

Let $u_0 \in U$ and $\|u_0\|_U = 1$. Set $u := u_0\chi_{\mathbf{R}_+}$, $x := \mathbb{B}\tau u$, $y := \mathbb{D}u$, so that $u,x,y \in \mathrm{L}^2_\omega$ for all $\omega > 0$. Obviously, $\widehat{u}(s) = u_0/s$ ($s \in \mathbf{C}^+$).

By Lemma 6.3.20, we have $(s-A)\widehat{x}(s) = B\widehat{u}(s)$ (hence $(s-A)s\widehat{x}(s) = Bu_0$) and $\widehat{y} = C_w\widehat{x} + D\widehat{u}$ on $\mathbf{C}^+$ (even if $(s-A)^{-1}$ does not exist for all $s \in \mathbf{C}^+$). Therefore, $\widehat{\mathbb{D}}(s)u_0 = s\widehat{y}(s) = C_w s\widehat{x}(s) + Bu_0$ for $s \in \mathbf{C}^+$, hence

$$\|(\widehat{\mathbb{D}}(s) - D)u_0\|_Y = \|C_w s\widehat{x}(s)\| \leq \frac{\varepsilon}{2}\|u_0\| + M\|s\widehat{x}(s)\| \quad (s \in \mathbf{C}^+). \tag{6.97}$$

But $s\widehat{x}(s) = \widehat{\mathbb{B}\tau}(s)u_0$ (since both sides are holomorphic on $\mathbf{C}^+$ and equal to $(s-A)^{-1}Bu_0$ on $\mathbf{C}^+_{\max\{\omega_A,0\}}$), hence $\|(\widehat{\mathbb{D}}(s) - D)u_0\| \leq \varepsilon/2 + M\varepsilon/2M = \varepsilon$ when $|s| > R$.

2° $\mathbb{B}\tau \in$ UVR*:* This goes as in 1°, except that now we choose $R > 0$ for given $\beta > \alpha_{\mathbb{D}} := \max\{\alpha_{\mathbb{B}\tau}, \omega_A\}$ (cf. Definition 6.2.3).

3° *Cases SHPR and SVR:* Work as in 1° or 2° but choose R and M for a fixed $u_0 \in U$ (alternatively, replace $\mathbb{B}$ by $\mathbb{B}P$ and $\mathbb{D}$ by $\mathbb{D}P$, where $P\alpha := \alpha u_0$ for all $\alpha \in \mathbf{C}$, so that $\Sigma \in \mathrm{WPLS}(\mathbf{C},H,Y)$ (cf. Lemma 6.7.17) and strong becomes equivalent to uniform, so that we can apply 1° or 2°). □

Curtain and Weiss [CW89] have shown that even if both B and C "fit to A", they need not "fit" simultaneously (by Lemma 6.3.16, this cannot happen if B or C is bounded):

Example 6.3.24 **($\left[\begin{smallmatrix} A & B \\ C & \end{smallmatrix}\right] \notin$ WPLS)** Let $H := \ell^2(\mathbf{N}$ and $U := Y := \mathbf{C}$. Then the system defined by

$$A := \begin{bmatrix} -1 & & & \\ & -2 & & \\ & & -3 & \\ & & & \ddots \end{bmatrix}, \quad B := \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} C \qquad := \begin{bmatrix} 1 & 1 & 1 & \dots \end{bmatrix} \tag{6.98}$$

(and any $D \in \mathcal{B}(C)$) is not well posed, but if we replace C by

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & \dots \end{bmatrix}, \tag{6.99}$$

then the system becomes well-posed (and UR). (See Example 6.1 of [CW89] for proofs.) $\triangleleft$

Given any WPLSs $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ and $\left[\begin{smallmatrix} \mathbb{A} \\ \hline \mathbb{C} \end{smallmatrix}\right]$, the operators $\left[\begin{array}{c|c} A & B \\ \hline C & \end{array}\right]$ are the generators of a WPLS iff

$$(\alpha - \cdot)C(\cdot - A)^{-1}(\alpha - A)^{-1}B \in \mathrm{H}^\infty_\infty \tag{6.100}$$

for some (hence all) $\alpha \in \mathbf{C}^+_{\omega_A}$ (see, e.g., Theorem 9.4.6(iv) of [Sbook]; in this case, the transfer function of the system is (6.100) plus an arbitrary constant in $\mathcal{B}$).

Reachability means that we can control any initial state approximately to zero and observability means that any initial state can be observed from the output:

Definition 6.3.25 (Reachability and observability) *The* reachability subspace $H_{\mathbb{B}}$ *and* observability subpace $H_{\mathbb{C}}$ *of* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$ *are defined by*

$$H_{\mathbb{B}} := \overline{\mathbb{B}[\mathrm{L}^2_{\mathrm{c}}(\mathbf{R};U)]} \subset H, \quad H_{\mathbb{C}} := \mathrm{Ker}(\mathbb{C})^\perp \subset H. \tag{6.101}$$

We call Σ (approximately) reachable *if* $H_{\mathbb{B}} = H$*, and* (approximately) observable *if* $H_{\mathbb{C}} = H$*. We call* Σ exactly ω-reachable *(in infinite time) if* $\mathbb{B}$ *is* ω*-stable and* $\mathbb{B}[\mathrm{L}^2_\omega(\mathbf{R};U)] = H$*; we call* Σ exactly ω-observable *(in infinite time) if* $\mathbb{C}$ *is* ω*-stable and* $\mathbb{C} \in \mathcal{B}(H, \mathrm{L}^2_\omega)$ *is coercive (we may drop* ω *for* $\omega = 0$*). If* Σ *is both reachable and observable, then* Σ *is called* minimal.

By exact reachability one sometimes means exact reachability in finite time (i.e., that $\mathrm{Ran}(\mathbb{B}^T) = H$ for some $T > 0$, see Definition 4.6 of [WR00]).

By (d) below, reachability and observability are extensions of the corresponding classical (finite-dimensional) concepts:

Lemma 6.3.26 *Let* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$*,* $\alpha \in \mathbf{R}$*. Then the following hold:*

(a1) Σ *is [exactly* α*-]reachable iff* Σ^{d} *is [exactly* α*-]observable.*

(a2) $H_{\mathbb{B}} = H_{\mathbb{B}^{\mathrm{d}}} = \overline{\mathrm{Ran}(\mathbb{B})}$*, and* $H_{\mathbb{C}} = H_{\mathbb{C}^{\mathrm{d}}} = \overline{\mathrm{Ran}(\mathbb{C}^*)}$*.*

(b1) $H_{\mathbb{B}}$ *is the closure in* H *of any of* $\mathbb{B}[\mathrm{L}^2_\omega(\mathbf{R};U)]$, $\mathbb{B}[C^\infty_{\mathrm{c}}(\mathbf{R}_-;U)]$, $\cup_{t>0}\mathrm{Ran}(\mathbb{B}^t)$*.*

(b2) $H_{\mathbb{C}}^\perp = \cap_{t>0}\mathrm{Ker}(\mathbb{C}^t)$*.*

(b3) Let $\Sigma' \in \mathrm{WPLS}$*. If* $\mathbb{B}' = \mathbb{B}\mathbb{M}$ *for some* $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$*, then* $H_{\mathbb{B}'} = H_{\mathbb{B}}$*. If* $\mathbb{C}' = \mathbb{M}\mathbb{C}$ *for some* $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$*, then* $H_{\mathbb{C}'} = H_{\mathbb{C}}$*.*

(c1) Let Σ be stable. Then Σ is [exactly] reachable if $\mathbb{B}\mathbb{B}^* > 0$ [$\mathbb{B}\mathbb{B}^* \gg 0$] on L^2; Σ is [exactly] observable if $\mathbb{C}^*\mathbb{C} > 0$ [$\mathbb{C}^*\mathbb{C} \gg 0$] on L^2.

(c2) If Σ is exactly ω-observable (resp. exactly ω-reachable), then Σ is observable (resp. reachable).

(d) Let $A \in \mathcal{B}(H)$. Then, for any $t > 0$, we have

$$H_{\mathbb{B}} = \overline{\mathrm{Ran}(\mathbb{B}^t)} = \overline{\mathrm{span}(\cup_{n\in\mathbf{N}}\mathrm{Ran}(A^nB))}, \tag{6.102}$$

$$\mathrm{Ker}(\mathbb{C}) = \mathrm{Ker}(\mathbb{C}^t) = \cap_{n\in\mathbf{N}}\mathrm{Ker}(CA^n). \tag{6.103}$$

(e) If we replace H by $H_{\mathbb{B}}$ (with the topology inherited from H), we get a reachable realization

$$\left[\begin{array}{c|c} P\mathbb{A}P^* & P\mathbb{B} \\ \hline \mathbb{C}P^* & \mathbb{D}\end{array}\right] \in \mathrm{WPLS}(U, H_{\mathbb{B}}, Y) \tag{6.104}$$

of $\mathbb{D}$, where $P \in \mathcal{B}(H, H_{\mathbb{B}})$ is the orthogonal projection $H \to H_{\mathbb{B}}$. If $B \in \mathcal{B}(U,H)$, then the input operator of (6.104) is $PB \in \mathcal{B}(U, H_{\mathbb{B}})$.

By taking, instead, P to be the orthogonal projection $H \to H_{\mathbb{C}}$, we get an observable realization of $\mathbb{D}$; if C is bounded, then $CP^* \in \mathcal{B}(H_{\mathbb{C}}, Y)$ is the (bounded) output operator of (6.104).

(f) If C is bounded and injective, then Σ is observable.

(g) $\mathbb{A}^t[H_{\mathbb{B}}] \subset H_{\mathbb{B}}$ and $(\mathbb{A}^*)^t[H_{\mathbb{C}}] \subset H_{\mathbb{C}}$ for all $t \geq 0$.

By (b3), state feedback preserves reachability and output injection preserves observability; see Section 6.6 for details. Relations to stabilizability and detectability are explained on p. 241.

Part (d) does not hold for unbounded A in general, e.g., if Σ is the exactly reachable realization of $\mathbb{D} = \tau^{-2}$ (see p. 159), then $\chi_{[-1,0)}u_0 \in \mathrm{Ker}(\pi_{[0,1)}\mathbb{C}) \setminus \mathrm{Ker}(\mathbb{C})$, (and $CA^n\chi_{[-1,0)}u_0$ is not defined).

Proof: (b1) By definition, $H_{\mathbb{B}}$ is the closure of

$$\cup_{t>0}\mathrm{Ran}(\mathbb{B}^t) = \cup_{t<0}\mathrm{Ran}(\mathbb{B}\pi_{[-t,0)}) = \mathrm{Ran}(\mathbb{B}\cup_{t<0}\mathrm{L}^2([-t,0);U)). \tag{6.105}$$

But $\cup_{t<0}\mathrm{L}^2([-t,0);U)) = \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_-;U)$ and $\mathcal{C}^\infty_{\mathrm{c}}(\mathbf{R}_-;U)$ are dense in $\mathrm{L}^2_\omega(\mathbf{R}_-;U)$, by Theorem B.3.11, hence (b1) holds (note that $\mathbb{B}[\mathrm{L}^2_\omega(\mathbf{R};U)] = \mathbb{B}[\mathrm{L}^2_\omega(\mathbf{R}_-;U)]$).

(b2) Trivially, $\mathrm{Ker}(\mathbb{C}) \subset \mathrm{Ker}(\mathbb{C}^t)$ for any $t > 0$. Conversely, if $\mathbb{C}x \neq 0$, then $\mathbb{C}^t x \neq 0$ for some $t > 0$.

(a2) Now $\mathrm{Ran}(\mathbb{C}^{\mathrm{d}}) = \mathrm{Ran}(\mathbb{C}^*) = \mathrm{Ran}(\mathbb{C}^{\mathrm{H}})$ (see (6.2)), and $\mathrm{Ran}(\mathbb{C}^{\mathrm{H}})^\perp = \mathrm{Ker}(\mathbb{C})$, by Lemma A.3.1(c7). Therefore, $H_{\mathbb{C}} = \overline{\mathrm{Ran}(\mathbb{C}^{\mathrm{d}})} = H_{\mathbb{C}^{\mathrm{d}}}$, by (b1). Exchange Σ^{d} and $\Sigma = (\Sigma^{\mathrm{d}})^{\mathrm{d}}$ to obtain that $H_{\mathbb{B}^{\mathrm{d}}} = H_{\mathbb{B}}$.

(a1) The reachability claim follows from (a2). The exact reachability claim follows from the identity $\mathrm{Ran}(\mathbb{C}^{\mathrm{d}}) = \mathrm{Ran}(\mathbb{C}^{\mathrm{H}})$ (from (a2)).

(b3) We have $\mathbb{M}^t := \pi_{[0,t)}\mathbb{M}\pi_{[0,t)} \in \mathcal{GB}(\pi_{[0,t)}\mathrm{L}^2)$, by Lemma 2.2.8, and $(\mathbb{B}')^t := \mathbb{B}'\tau^t\pi_+ = \mathbb{B}^t\mathbb{M}^t$, hence $H_{\mathbb{B}'} = H_{\mathbb{B}}$. Apply this for $(\Sigma')^{\mathrm{d}}$ to obtain the $H_{\mathbb{C}'} = H_{\mathbb{C}}$ claim.

(c1) This follows from (c1) and (c9) of Lemma A.3.1 and (a2) above.

(c2) This follows from (b1) and (a1).

(d) Set $X := \cap_{n\in\mathbf{N}} \mathrm{Ker}(CA^n)$. Now $\mathbb{A}^t = e^{At} := \sum_{n\in\mathbf{N}} A^n t^n/n!$, hence $x \in X \Rightarrow (\mathbb{C}x)(t) = C\mathbb{A}^t x = 0$ for all $t \geq 0$, hence $X \subset \mathrm{Ker}(\mathbb{C})$. Let $t > 0$. By (b2), we have $\mathrm{Ker}(\mathbb{C}) \subset \mathrm{Ker}(\mathbb{C}^t)$. Let $x \in \mathrm{Ker}(\mathbb{C}^t)$, so that $Ce^{As}x = 0$ for $s \in [0,t)$. Differentiate this n times and set $s = 0$ to obtain that $CA^n x = 0$; because $n \in \mathbf{N}$ was arbitrary, we have $x \in X$. Thus, $\mathrm{Ker}(\mathbb{C}^t) \subset X$, and hence (6.103) holds.

Therefore, $H_{\mathbb{B}}^{\perp} = H_{\mathbb{B}^{\mathrm{d}}}^{\perp} = \mathrm{Ker}(\mathbb{B}^{\mathrm{d}}) = \cap_{n\in\mathbf{N}} \mathrm{Ker}(B^*(A^*)^n)$, hence

$$H_{\mathbb{B}} = (\cap_{n\in\mathbf{N}} \mathrm{Ran}(A^n B)^{\perp})^{\perp} = ((\cup_{n\in\mathbf{N}} \mathrm{Ran}(A^n B))^{\perp})^{\perp} = \overline{\mathrm{span}(\cup_{n\in\mathbf{N}} \mathrm{Ran}(A^n B))}. \tag{6.106}$$

Set $\widetilde{\mathbb{C}} := \mathbb{B}^{\mathrm{d}}$. Then $\mathrm{Ran}(\mathbb{B}^t)^{\perp} = \mathrm{Ker}((\mathbb{B}^t)^*) = \mathrm{Ker}(\pi_- \tau^{-t}\mathbb{B}) = \mathrm{Ker}(\widetilde{\mathbb{C}}^t) = \mathrm{Ker}(\widetilde{\mathbb{C}}) = \mathrm{Ran}(\mathbb{B})^{\perp} = H_{\mathbb{B}}^{\perp}$, hence also (6.102) holds.

(e) By (g), we have $P^*P\mathbb{A}P^*P = \mathbb{A}P^*P$ (note that $P^* \in \mathcal{B}(H_{\mathbb{B}}, H)$ is the embedding $H_{\mathbb{B}} \to H$, $P^*P \in \mathcal{B}(H)$ is the orthogonal projection $H \to H_{\mathbb{B}}$ with range space H, and $PP^* = I_{H_{\mathbb{B}}}$).

Because $P^*P\mathbb{B} = \mathbb{B}$, also the new system is an ω-stable WPLS, by Lemma 6.7.17.

An analogous claim holds for $\mathrm{H}_{\mathbb{C}}$. If $C \in \mathcal{B}(H,Y)$, then $Cx_0 = (\mathbb{C}x_0)(0) = 0$ for all $x_0 \in H_{\mathbb{C}}^{\perp}$, hence then (here $P \in \mathcal{B}(H, H_{\mathbb{C}})$ is the orthogonal projection) $CP\mathbb{A}P^* = C\mathbb{A}P^* = \mathbb{C}P^*$, i.e., C is the (unique) generator of $\mathbb{C}P^*$. By duality, we get the claim on $B \in \mathcal{B}(U,H)$.

(f) Obviously, now $\mathrm{Ker}(C\mathbb{A}) = \{0\}$.

(g) Let $t \geq 0$. We have $\mathbb{A}^t x_0 \in H_{\mathbb{B}}$ for all $x_0 \in \mathbb{B}\mathrm{L}^2_{\omega}$, by "2." of Definition 6.1.1, hence $\mathbb{A}^t x_0 \in H_{\mathbb{B}}$ for all $x_0 \in H_{\mathbb{B}}$, by continuity. Thus, $\mathbb{A}^t[H_{\mathbb{B}}] \subset H_{\mathbb{B}}$.

Apply this to Σ^{d} to observe that $(\mathbb{A}^{\mathrm{d}})^t[H_{\mathbb{C}^{\mathrm{d}}}] \subset H_{\mathbb{C}^{\mathrm{d}}}$; which by (a2) means that $(\mathbb{A}^*)^t[H_{\mathbb{C}}] \subset H_{\mathbb{C}}$. □

Notes

Parts of Proposition 6.3.1 and Lemma 6.3.2 are found in the literature, as stated in the proofs; the rest is rather obvious.

Theorem 6.3.9 is due to G. Weiss [SW01a] (with a somewhat different proof) and Lemma 6.3.13 is from [Sbook], which also contains methods similar to those in Lemma 6.3.11. Corollary 6.3.14 is well known.

An implicit form of Lemma 6.3.22 for bounded B and $x_0 \in H_1$ is contained in [Keu], p. 96. Example 6.3.24 is a simple conclusion of [PW]. Definition 6.3.25 is essentially from [Sbook], which contains further results on most subjects of this section (its final version probably overlaps more than explained above).

The regularity theory is usually more fruitful than compatibility theory, but the latter covers all WPLSs. In some applications, another approach that also covers all WPLSs, namely the use of a combined "*C&D*" operator, might be more practical, see [AN] and [Sbook] for details. Next we shall motivate the concept of a compatible pair and explain the history of this notion.

The Riccati equation theory of [WW] and of several articles by O. Staffans is based on the assumption that the system is regular and that the spectral factor of an optimation problem is SR and has an invertible feedthrough, so that also the corresponding (optimal) closed-loop system is regular (this corresponds to Proposition 6.6.18(d4)).

If the spectral factor is merely WR with invertible feedthrough, then the optimal closed-loop system may be irregular (i.e., not even WR). Nevertheless, we found that one can still find closed-loop "generators" that produce the output pointwise from the input and the state ("$y = C_c x + D_c u$"), and that by such methods one can extend most of Riccati equation theory for arbitrary control problems (regardless of regularity, we only need compatible output operators) as long as some "compatible feedthrough operator" of the spectral factor is invertible. It also appeared that by using these methods, the complete Riccati equation theory for optimal control can be extended to the case of a WR system and WR spectral factor with invertible feedthrough.

This fact lead us to define two weaker "regularity properties" for a WPLS in [Mik97a], the more general of which ("infraregular output operators") is equivalent to compatible output operators. We then developed a brief compatibility theory (including early versions of Lemmas 6.3.10 and 6.3.12 and Proposition 6.6.18) and used it to derive this extended Riccati equation theory. (We used the theory in the manuscript of [Mik97b] for the WR case, but in the final version of [Mik97b] the theory is used only implicitly, for briefness.)

After finding that William Helton used a similar concept for ULR systems in [Helton76a], O. Staffans developed the theory to a rather mature state in [Sbook]. Staffans and Weiss also presented this theory in [SWcompatible] and [SW01a], the former of which treats the relations between the three approaches mentioned above (compatibility, regularity and "*C&D*"; note that at that time not all WPLSs were known to have compatible pairs). We refer the reader to these works for further information on compatibility.

6.4 Spectral and coprime factorizations ($\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$)

Science is spectral analysis. Art is light synthesis.

— Karl Kraus (1874–1936)

In this section, we shall define spectral factorization, right, left and doubly coprime factorization, weak forms of coprimeness (quasi-, pseudo-), and inner and lossless factorization. We then explain the basic properties of these concepts to the degree required by the next two sections and Chapter 7.

The readers who wish to have a deeper understanding of the subject, may wish to read also Section 6.5, which is a further study on coprimeness, or Chapter 5 on spectral factorization. Other readers may skip the next section and visit it only when pointed by a reference.

The importance of the factorizations mentioned above is due to several reasons. For example, dynamic stabilization is intimately connected with coprime factorizations of the I/O map of the plant (see Chapter 7), and so is joint stabilizability and detectability (Theorem 6.6.28).

Stable control problems can be solved by using a spectral factorization of the corresponding Popov operator ($\mathbb{D}^* J \mathbb{D}$, where $\langle y, Jy \rangle_{\mathrm{L}^2(\mathbf{R}_+;Y)}$ is the cost function of the problem), and unstable problems by using a coprime, inner-right/left or lossless factorization (depending on the problem) of the I/O map ($\mathbb{D}$) of the system; this will be explained in Part III.

We shall start this section by defining three forms of coprimeness. In number theory, the word "coprime" means having no common divisors (except units). Thus, numbers $n, m \in \mathbf{Z}$ are coprime iff $n = n_0 k,\ m = m_0 k,\ n_0, m_0, k \in \mathbf{Z}$ implies that k is a unit (i.e., invertible, hence $k = \pm 1$). It is well known that an equivalent condition is that $xm + yn = 1$ for some $x, y \in \mathbf{Z}$, i.e., that $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$ has a left inverse (e.g., $\begin{bmatrix} x & y \end{bmatrix} \in \mathbf{Z} \times \mathbf{Z}$).

If $\widehat{\mathbb{N}}, \widehat{\mathbb{M}} \in \mathcal{R} := \{\text{rational bounded scalar functions } \mathbf{C}^+ \to \mathbf{C}\}$, then $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{M}}$ have no common divisor (except units, i.e., elements of $\mathcal{G}\mathcal{R}$) iff $\begin{bmatrix} \widehat{\mathbb{N}} \\ \widehat{\mathbb{M}} \end{bmatrix}$ has no left inverse in $\mathcal{R}$ (equivalently, in $\mathrm{H}^\infty(\mathbf{C}^+)$), by pp. 70 and 386 of [Vid] (an analogous claim holds for any other principal ideal domain in place of $\mathcal{R}$).

For $\mathrm{H}^\infty(\mathbf{C}^+)$ in place of $\mathcal{R}$, the latter condition (traditionally called coprimeness) becomes strictly stronger than the former (which is sometimes called weak coprimeness; we shall not need it). For matrix- or operator-valued functions, we must distinguish between right and left coprimeness (which imply having no common right or left divisors, respectively; see the comments below Lemma 6.5.2 for further information). Therefore, we shall use the traditional definition of coprimeness; we supplement it by two weaker concepts:

Definition 6.4.1 (Coprime)

(a) The operators $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{M} \in \mathrm{TIC}(U)$ *are* right coprime (r.c.)*, if* $\mathbb{N}$ *and* $\mathbb{M}$ *together with some* $\widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}} \in \mathrm{TIC}$ *satisfy the (right) Bezout identity*

$$\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I_U. \tag{6.107}$$

(b) The operators $\widetilde{\mathbb{N}} \in \mathrm{TIC}(U,Y)$ *and* $\widetilde{\mathbb{M}} \in \mathrm{TIC}(Y)$ *are* left coprime (l.c.)*, if* $\widetilde{\mathbb{N}}$ *and* $\widetilde{\mathbb{M}}$ *together with some* $\mathbb{Y}, \mathbb{X} \in \mathrm{TIC}$ *satisfy the (left) Bezout identity*

$$\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} = I_Y. \tag{6.108}$$

(c) The operators $\mathbb{N}, \mathbb{M}, \widetilde{\mathbb{N}}, \mathbb{M} \in \mathrm{TIC}$ *are* doubly coprime (d.c.)*, if they together with some* $\mathbb{Y}, \mathbb{X}, \widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}} \in \mathrm{TIC}$ *satisfy the double Bezout identity*

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = \begin{bmatrix} I_U & 0 \\ 0 & I_Y \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix}. \tag{6.109}$$

*In (a)–(c), we add the words "*over $\mathcal{A}$*", if* $\mathcal{A} \subset \mathrm{TIC}$ *and the requirements are met with* $\mathcal{A}$ *in place of* TIC*. The word* exponential*, e.g., in "exponentially d.c." will refer to "over* $\mathrm{TIC}_{\mathrm{exp}}$*".*

(d) The operators $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{M} \in \mathrm{TIC}(U)$ *are* pseudo–right coprime (p.r.c.) *if* $\widehat{\mathbb{N}}^* \widehat{\mathbb{N}} + \widehat{\mathbb{M}}^* \widehat{\mathbb{M}} \geq \varepsilon I$ *on* $\mathbf{C}^+$ *for some* $\varepsilon > 0$*.*

We call $\mathbb{N}^{\mathrm{d}}, \mathbb{M}^{\mathrm{d}}$ *pseudo–left coprime (p.l.c.) if* $\mathbb{N}, \mathbb{M}$ *are p.r.c. (i.e., iff* $\widehat{\mathbb{N}}\widehat{\mathbb{N}}^* + \widehat{\mathbb{M}}\widehat{\mathbb{M}}^* \geq \varepsilon I$ *on* $\mathbf{C}^+$ *for some* $\varepsilon > 0$*).*

(e) The operators $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{M} \in \mathrm{TIC}(U)$ *are* quasi–right coprime (q.r.c.) *if* $\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right] u \notin \mathrm{L}^2$ *whenever* $u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U) \setminus \mathrm{L}^2$*.*

We call $\mathbb{N}^{\mathrm{d}}, \mathbb{M}^{\mathrm{d}}$ *quasi–left coprime (q.l.c.) if* $\mathbb{N}, \mathbb{M}$ *are q.r.c.*

(f) By the coprimeness of $\widehat{\mathbb{N}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$ *and* $\widehat{\mathbb{M}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U))$ *we refer to the coprimeness of* $\mathbb{N}$ *and* $\mathbb{M}$ *(in any of the above senses). An analogous comment applies to Definition 6.4.4.*

(Recall that $\mathrm{L}^2_\infty := \cup_{\omega \in \mathbf{R}} \mathrm{L}^2_\omega$; see Theorem 6.2.1 for (f).) Before motivating the above definitions, we observe some basic facts:

Lemma 6.4.2 *D.c. implies r.c. and l.c., r.c. implies p.r.c., and p.r.c. implies q.r.c.*

The maps $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{M} \in \mathrm{TIC}(U)$ *are [p.]r.c. iff* $\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right]$ *is [pseudo-]left-invertible in* $\mathrm{TIC}(U)$*, or equivalently, iff* $\mathbb{N}^{\mathrm{d}}$ *and* $\mathbb{M}^{\mathrm{d}}$ *are [p.]l.c. The maps* $\mathbb{N}$ *and* $\mathbb{M}$ *are q.r.c. iff* $\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right]$ *is quasi-left-invertible in* $\mathrm{TIC}(U)$*.*

(See pp. 128 and 131 for pseudo/quasi-left-invertibility.)

The above facts will be used in the sequel without further mention; the same applies to the obvious fact that if $\mathbb{M} \in \mathcal{G}\mathrm{TIC}(U)$ and $\mathbb{N} \in \mathrm{TIC}(U,*)$ (resp. $\mathbb{N} \in \mathrm{TIC}(*,U)$), then $\mathbb{M}$ and $\mathbb{N}$ are r.c. (resp. l.c.). (Analogous claims hold for pseudo- or quasi-left-invertible $\mathbb{M} \in \mathrm{TIC}(U)$; e.g., τ^{-1} and $\mathbb{N}$ are q.r.c. (resp. q.l.c.).)

The use of any “left” results is minimal in this monograph, we prefer using “right”[3] results and the duality stated in the lemma. Explicit forms of many such “left” results can be found in [Sbook].

Proof of Lemma 6.4.2: Trivially, d.c. implies r.c. and l.c.; the other two implications follows from Lemma 6.5.2(ii)&(b1). The equivalence follows directly from the definitions. □

It is instructive to observe the meaning of coprimeness in the case of scalar transfer functions:

Lemma 6.4.3 *Let* $\mathbb{N}, \mathbb{M} \in \mathrm{TIC}(\mathbf{C})$*. Then the following are equivalent:*

(i) $\mathbb{N}$ *and* $\mathbb{M}$ *are [p.]r.c.*

(ii) $\mathbb{N}$ *and* $\mathbb{M}$ *are [p.]l.c.*

(iii) $|\widehat{\mathbb{N}}| + |\widehat{\mathbb{M}}| \geq \varepsilon$ *on* $\mathbf{C}^+$ *for some* $\varepsilon > 0$*.*

If $\widehat{\mathbb{N}}$ *and* $\widehat{\mathbb{M}}$ *are continuous on* $\overline{\mathbf{C}^+} \cup \{\infty\}$ *(e.g., they are rational or in* $\widehat{\mathrm{MTIC}^{\mathrm{L}^1}}$*), then (iii) holds iff* $\widehat{\mathbb{N}}$ *and* $\widehat{\mathbb{M}}$ *have no common zeros on* $\overline{\mathbf{C}^+} \cup \{\infty\}$*.* □

(This follows from Lemma 6.5.3(a)&(c) and the compactness of $\overline{\mathbf{C}^+} \cup \{\infty\}$.)

Thus, if $\mathbb{N}$ and $\mathbb{M}$ are scalar and coprime, and $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$, then “$\mathbb{N}$ cancels no poles of $\mathbb{M}^{-1}$”, i.e., “$\mathbb{M}^{-1}$ and $\mathbb{N}\mathbb{M}^{-1}$ have the same poles”. See Lemma 6.5.4 for the general case.

Classical coprimeness has its advantages, especially in dynamic feedback (see Chapter 7) including the H^∞ four-block problem. However, the most useful properties of coprimeness are the ones given in (b1) and (c1) of Lemma 6.5.1, hence for most results using coprimeness, also quasi-coprimeness is a sufficient assumption.

Furthermore, quasi-coprimeness has two important advantages to coprimeness: it can often be more easily verified and it is preserved in inverse discretization (see Theorem 13.4.4(e1); we do not know if this is the case for pseudo-coprimeness), thus allowing us to prove several important results in discrete time. Indeed, the verification of pseudo-coprimeness is a simple, nonconstructive process, and “p.r.c.” implies “q.r.c.”. Moreover, the I/O map of an exponentially stabilizable and detectable system has a q.r.c. factorization, by Theorem 6.7.15(c2) (see Corollary 6.7.16 for similar implications).

For the above reasons, we usually use coprimeness in connection with dynamic stabilization and quasi-coprimeness for other occasions, including state feedback. Pseudo-coprimeness seldom implies anything useful that quasi-coprimeness would not imply, hence we mostly neglect it.

Next we define several notions that are used in connection with feedback and optimal control:

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[3]This is not an ideological statement.

Definition 6.4.4 (Factorizations) *Let $J = J^* \in \mathcal{B}(Y)$, $S \in \mathcal{GB}(U)$, and $\mathcal{A} \subset$* TIC.

(a) An operator $\mathbb{N} \in \text{TIC}(U,Y)$ *is* (J,S)-inner, *if* $\mathbb{N}^* J \mathbb{N} = S$.

(b) An operator $\mathbb{N} \in \text{TIC}(U,Y)$ *is* (J,S)-lossless, *if* $\mathbb{N}^* J \mathbb{N} = S$ *and* $\mathbb{N}^* \pi_- J \mathbb{N} \leq \pi_- S$.

(c) **(SpF)** *A factorization* $\mathbb{E} = \mathbb{X}^* S \mathbb{X}$ *is a* spectral factorization [over $\mathcal{A}$] *of* $\mathbb{E} = \mathbb{E}^* \in \text{TI}(U)$, *if* $\mathbb{X} \in \mathcal{G}\text{TIC}(U)$ *[*$\mathbb{X} \in \mathcal{GA}(U)$*]. In this case we call* $\mathbb{X}$ *an* S-spectral factor *of* $\mathbb{E} = \mathbb{X}^* S \mathbb{X}$.

(d1) **(r.c.f.)** *A factorization* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a* [quasi–]right coprime factorization ([q.]r.c.f.) *of* $\mathbb{D} \in \text{TIC}_\infty(U,Y)$, *if* $\mathbb{N}, \mathbb{M} \in$ TIC *are [q.]r.c. and* $\mathbb{M} \in \mathcal{G}\text{TIC}_\infty(U)$.

If, in addition, $\mathbb{N}$ *is* (J,S)*-inner (resp.* (J,S)*-lossless), then* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a* (J,S)-inner [q.]r.c.f. *(resp.* (J,S)-lossless [q.]r.c.f.*).*

(d2) A factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a* right factorization *of* $\mathbb{D} \in \text{TIC}_\infty(U,Y)$, *if* $\mathbb{N}, \mathbb{M} \in$ TIC *and* $\mathbb{M} \in \mathcal{G}\text{TIC}_\infty(U)$.

If, in addition, $\mathbb{N}^* J \mathbb{N} = S$*, then* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a* (J,S)-inner-right factorization

(e) A factorization $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ *is a* [quasi–]left coprime factorization ([q.]l.c.f.) *of* $\mathbb{D} \in \text{TIC}_\infty(U,Y)$, *if* $\widetilde{\mathbb{N}}, \widetilde{\mathbb{M}} \in$ TIC *are [q.]l.c. and* $\widetilde{\mathbb{M}} \in \mathcal{G}\text{TIC}_\infty(Y)$.

(f) **(d.c.f.)** *A factorization* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ *is a* doubly coprime factorization (d.c.f.) [over $\mathcal{A}$] *of* $\mathbb{D} \in \text{TIC}_\infty(U,Y)$, *if* $\mathbb{N}, \mathbb{M}, \widetilde{\mathbb{N}}, \widetilde{\mathbb{M}}$ *are d.c. [over* $\mathcal{A}$*] and* $\mathbb{M}, \widetilde{\mathbb{M}} \in \mathcal{G}\text{TIC}_\infty$.

If (6.109) is the corresponding double Bezout identity, we say that $\mathbb{D}$ *and* $\mathbb{Y}\mathbb{X}^{-1}$ *(equivalently,* $\mathbb{D}$ *and* $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$*) have a* joint d.c.f. [over $\mathcal{A}$].[4]

*As in (f), we add the words "*over $\mathcal{A}$*" in (d) and (e) too, if the factors are coprime over* $\mathcal{A}$ *instead of* TIC. *For example, the factorization* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a* r.c.f. over $\mathcal{A}$, *if* $\mathbb{M} \in \mathcal{G}\text{TIC}_\infty$ *and* $\mathbb{M}, \mathbb{N}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$ *satisfy (6.107).*

We also use the above definitions with "pseudo" in place of "quasi" and "p." in place of "q.".

The words exponentially stable*, e.g., in "an exponentially stable d.c.f.", will refer to "over* TIC_{exp}*".*

In parts (a)–(d2), we call S a signature operator *(note that necessarily* $S = S^*$*). By "*$(J,*)$*" we mean "*(J,S) *for some* $S \in \mathcal{GB}$*".*

Thus, a d.c.f. contains a r.c.f. and a l.c.f.; the converse is given in Lemma 6.5.8; moreover, a r.c.f. is a right factorization. These factorizations are unique up to a unit, see Lemmas 6.4.5 and 6.5.9. In (d)–(f), the operators $\mathbb{N}$ and $\widetilde{\mathbb{N}}$ are called *numerators* and $\mathbb{M}$ and $\widetilde{\mathbb{M}}$ *denominators*.

Inner-right factorizations are often called inner-outer factorizations if the right factor has a stable inverse. Losslessness is further treated in Section 2.5.

[4] If $\mathbb{X}$ is not invertible in a reasonable sense (e.g., in TIC_∞), this will be considered merely as a formal definition until Section 7.3. Note that $\widetilde{\mathbb{X}}\mathbb{Y} = \widetilde{\mathbb{Y}}\mathbb{X}$.

Note that we have required $\mathbb{X}$ and S to be invertible unlike some authors do. This way (c) coincides with Definition 5.1.1, by Lemma 5.2.1(d), and these strong definitions are more useful for the results presented in this monograph.

We remark that, by Lemma 6.5.9(a2), any operators satisfying (6.109) constitute a d.c.f. of $\mathbb{N}\mathbb{M}^{-1}$ if(f) $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$. In the finite-dimensional case, on often constructs a d.c.f. of a map by choosing a minimal, hence (exponentially) jointly stabilizable and detectable realization. The same approach can be used for the I/O map for any WPLS having SR jointly stabilizing state feedback and output injection operators, as noted below Theorem 6.6.28.

When $\dim U < \infty$, one sometimes does not require that $\mathbb{M}^{-1} \in \mathrm{TIC}_\infty$, just that $\det \widehat{\mathbb{M}} \not\equiv 0$, and thus (d1) becomes the definition of a r.c. "$\mathrm{H}^\infty/\mathrm{H}^\infty$ factorization". By Lemma 6.5.4(d2), this is equivalent to our definition when $\mathbb{D} \in \mathrm{TIC}_\infty$.

Observe from Lemma 6.4.3 that a scalar right factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is coprime iff $\mathbb{N}$ does not cancel any poles of $\mathbb{M}^{-1}$, i.e., iff $\mathbb{M}^{-1}$ and $\mathbb{D}$ have the same poles. In a sense, the same holds also in the infinite-dimensional case, by the comments below Lemma 6.5.4.

Coprime and spectral factorizations are unique up to unit:

Lemma 6.4.5 (Uniqueness of factorizations)

(a) Let $\mathbb{X}_0^* S_0 \mathbb{X}_0$ be a spectral factorization of $\mathbb{E} = \mathbb{E}^* \in \mathrm{TI}(U)$. Then all spectral factorizations of $\mathbb{E}$ are given by $\mathbb{X} = E\mathbb{X}_0$, $S = (E^*)^{-1} S_0 E^{-1}$ with $E \in \mathcal{GB}(U)$.

(b) Let $(\mathbb{N}_0, \mathbb{M}_0)$ be a right factorization of $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Then all right (not necessarily coprime) factorizations of $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ with $\mathbb{N}, \mathbb{M} \in \mathrm{TIC}$ are given by $\mathbb{N} = \mathbb{N}_0\mathbb{U}$, $\mathbb{M} = \mathbb{M}_0\mathbb{U}$ with $\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ ($\mathbb{U} \in \mathrm{TIC}(U) \cap \mathcal{G}\mathrm{TIC}_\infty(U)$ if $\mathbb{D} = \mathbb{N}_0\mathbb{M}_0^{-1}$ is a q.r.c.f.).

(c) Let $(\mathbb{N}_0, \mathbb{M}_0)$ be a q.r.c.f. of $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Then all q.r.c.f.'s of $\mathbb{D}$ are given by $\mathbb{N} = \mathbb{N}_0\mathbb{U}$, $\mathbb{M} = \mathbb{M}_0\mathbb{U}$ with $\mathbb{U} \in \mathcal{G}\mathrm{TIC}(U)$.

Moreover, if one of them is a [p.]r.c.f., then all of them are [p.]r.c.f.'s.

(d) Let $(\widetilde{\mathbb{N}}_0, \widetilde{\mathbb{M}}_0)$ be a [q.]l.c.f. of $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Then all [q.]l.c.f.'s of $\mathbb{D}$ are given by $\widetilde{\mathbb{N}} = \mathbb{U}\widetilde{\mathbb{N}}_0$, $\widetilde{\mathbb{M}} = \mathbb{U}\widetilde{\mathbb{M}}_0$ with $\mathbb{U} \in \mathcal{G}\mathrm{TIC}(U)$.

(e) Let $(\mathbb{N}_0, \mathbb{M}_0)$ be a (J, S_0)-inner [q.]r.c.f. of $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Then all $(J,*)$-inner [q.]r.c.f.'s of $\mathbb{D}$ are given by $\mathbb{N} = \mathbb{N}_0 E$, $\mathbb{M} = \mathbb{M}_0 E$ (and $S = E^* S_0 E$) with $E \in \mathcal{GB}(U)$.

Moreover, all $(J,*)$-inner-right factorizations of $\mathbb{D}$ are [q.]r.c.f.'s, hence of the above form. (See also (c).)

(f) Parts (a) and (e) hold even if the signature operators (S and S_0) are required to be merely one-to-one (not necessarily invertible).

Note that for $\mathbb{E} \gg 0$ one can take $E := \sqrt{S_0}$ to get $\mathbb{E}^* = \mathbb{X}^*\mathbb{X}$ in (i), as is often done (e.g., in [WW]). For WR $\mathbb{X}_0$ with an invertible feedthrough operator $X_0 := \widehat{\mathbb{X}}_0(+\infty)$, another common normalization is to take $E := X_0^{-1}$ (i.e., $X = I$;

this corresponds to zero feedthrough when $\mathbb{F} := I - \mathbb{X}$ is used to construct a state feedback pair, as in Theorem 9.9.10(g1) and Corollary 9.9.11).

Proof: (a) This follows from Lemma 5.2.1(d)&(f).

(b) Set $\mathbb{U} := \mathbb{M}_0^{-1}\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$ to obtain $\mathbb{N} = \mathbb{N}_0\mathbb{U}$, $\mathbb{M} = \mathbb{M}_0\mathbb{U}$. If $\mathbb{N}_0$ and $\mathbb{M}_0$ are q.r.c., then $\mathbb{U} \in \mathrm{TIC}$, by Lemma 6.5.1(c1).

(c) The parametrization follows from (b) (interchange the roles of $(\mathbb{N}, \mathbb{M})$ and $(\mathbb{N}_0, \mathbb{M}_0)$ to see that $\mathbb{U}^{-1} \in \mathrm{TIC}$).

If $\mathbb{N}_0, \mathbb{M}_0$ are r.c., i.e., $\widetilde{\mathbb{X}}\mathbb{N}_0 - \widetilde{\mathbb{Y}}\mathbb{M}_0 = I$ for some $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$, then $\widetilde{\mathbb{U}}^{-1}\widetilde{\mathbb{X}}\mathbb{N} - \widetilde{\mathbb{U}}^{-1}\widetilde{\mathbb{Y}}\mathbb{M} = I$, hence then $\mathbb{N}, \mathbb{M}$ are r.c. The p.r.c. claim follows analogously from, e.g., Lemma 6.5.2(ii).

(d) Apply (c) to $\mathbb{D}^{\mathrm{d}}$. (Of course, the other claims in (c) also have their duals for (d).)

(e) It is obvious that $\mathbb{D} = (\mathbb{N}_0 E)(\mathbb{M}_0 E)^{-1}$ is a $(J, E^* S_0 E)$-inner [q.]r.c.f. for each $E \in \mathcal{G}\mathcal{B}(U)$ (see (c)), hence we only have to study the (extended) converse.

Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a (J,S)-inner-right factorization of $\mathbb{D}$. Set $E := \mathbb{M}_0^{-1}\mathbb{M} \in \mathrm{TIC}(U) \cap \mathcal{G}\mathrm{TIC}_\infty(U)$ as in (b), so that $\mathbb{N} = \mathbb{N}_0 E$. Then

$$E^* S_0 E = E^* \mathbb{N}_0^* J \mathbb{N}_0 E = \mathbb{N}^* J \mathbb{N} = S, \tag{6.110}$$

hence $E \in \mathcal{G}\mathcal{B}$, by Lemma 6.5.5(a); in particular, $\mathbb{N}\mathbb{M}^{-1}$ is a [q.]r.c.f., by (c).

(f) For (a) we note that $\mathbb{X}^* S \mathbb{X} = \mathbb{X}_0^* S_0 \mathbb{X}_0$ implies that $E^* S E = S_0$, where $E = \mathbb{X}\mathbb{X}_0^{-1} \in \mathcal{G}\mathrm{TIC}$, hence $E \in \mathcal{G}\mathcal{B}$, by Lemma 6.5.5(a). The converse is trivial.

The proof of (e) above applies for noninvertible S too. □

Next we recall two important facts from Lemma 2.2.2(a1)&(d):

Lemma 6.4.6 *Let $\mathbb{E} \in \mathrm{TI}(U)$. We have $\mathbb{E} \gg 0$ iff $\pi_+\mathbb{E}\pi_+ \gg 0$. Moreover, if $\pi_+\mathbb{E}\pi_+$ is invertible (on $\pi_+\mathrm{L}^2$), then so is $\mathbb{E}$ (on L^2, i.e., in $\mathrm{TI}(U)$).* □

The converse to the latter claim is not true (e.g., take $\mathbb{E} = \tau(1)$).

We repeat here some results from Lemma 5.2.1:

Lemma 6.4.7 (SpF) *Let $\mathbb{E} = \mathbb{E}^* \in \mathrm{TI}(U)$. Then we have the following:*

(a) $\mathbb{E} \gg 0$ iff $\mathbb{E}$ has the spectral factorization $\mathbb{E} = \mathbb{X}^*\mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$.

Assume now that $\mathbb{E} \in \mathrm{TI}(U)$ has a spectral factorization $\mathbb{E} = \mathbb{X}^* S \mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$, $S = S^* \in \mathcal{G}\mathcal{B}(U)$. Then we have the following:

(b) The Toeplitz operator $\pi_+\mathbb{E}\pi_+$ is invertible on $\pi_+\mathrm{L}^2$, and $\pi_+\mathbb{X}^{-1}\pi_+ S^{-1}\mathbb{X}^{-*}\pi_+$ is its inverse.

(See Theorem 8.4.12 for the converse for $\mathbb{E} \in \mathrm{MTI}$.)

(c) If, in addition, $\mathbb{E} \in \mathrm{TI}_\omega(U)$ for some $\omega \neq 0$, then $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_{\exp}(U)$.

(d) The map $\mathbb{E}^{\mathrm{d}} := \boldsymbol{Я}\mathbb{E}\boldsymbol{Я} \in \mathrm{TI}(U)$ has the co-spectral factorization $\mathbb{E}^{\mathrm{d}} = \mathbb{X}^{\mathrm{d}} S (\mathbb{X}^{\mathrm{d}})^*$ *(where $\mathbb{X}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}(U)$).* □

(This is a direct consequence of Lemma 5.2.1.) Clearly $\mathbb{E}^{\rm d}$ has a co-spectral factorization iff $\mathbb{E}$ has a spectral factorization, i.e., the converse to (d) is also true.

Section 8.4 contains a study on the equivalence between spectral or coprime factorizations and the coercivity of the cost function of a control problem; see especially Theorem 8.4.12 for MTIC classes. In Section 9.1, we establish a third equivalent condition in terms of Riccati equations. For very regular systems, we give more neat results in Theorem 9.2.14 and Corollary 9.2.15. See also Chapter 5 on spectral factorization.

We finish this section by noting that the search for a (J,S)-inner r.c.f. can be reduced to a spectral factorization problem:

Lemma 6.4.8 ((J,S)-inner r.c.f. vs. SpF) *The following hold:*

(a) Let $\mathbb{D} \in \mathrm{TIC}(U,Y)$ and $J = J^* \in \mathcal{B}(Y)$. If $\mathbb{X}^* S\mathbb{X}$ is a spectral factorization of $\mathbb{D}^* J\mathbb{D}$, then $(\mathbb{D}\mathbb{X}^{-1}, \mathbb{X}^{-1})$ is a (J,S)-inner r.c.f. of $\mathbb{D}$. Conversely, if $(\mathbb{N}, \mathbb{M})$ is a (J,S)-inner q.r.c.f. of $\mathbb{D}$, then $\mathbb{M}^{-1}S\mathbb{M}^{-1}$ is a spectral factorization of $\mathbb{D}^* J\mathbb{D}$.

(b) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a [q.]r.c.f., let $J = J^* \in \mathcal{B}(Y)$, and let $S = S^* \in \mathcal{GB}(U)$. Then $\mathbb{D}$ has a (J,S)-inner [q.]r.c.f. iff $\mathbb{N}^* J\mathbb{N}$ has an S-spectral factor.

(c) Let $\mathbb{E} \in \mathcal{G}\mathrm{TI}(U \times W)$. Then $\mathbb{E}^* J_1\mathbb{E} = J_1 \Leftrightarrow \mathbb{E}J_1\mathbb{E}^* = J_1 \Leftrightarrow \mathbb{E}^{-*}J_1\mathbb{E}^{-1} = J_1$.

Proof: (a) The first claim follows from (denote $\mathbb{M} := \mathbb{X}^{-1}$) $\mathbb{X}^* S\mathbb{X} = \mathbb{D}^* J\mathbb{D} \Leftrightarrow S = (\mathbb{D}\mathbb{M})^* J(\mathbb{D}\mathbb{M})$ and from $0 + \mathbb{X}\mathbb{M} = I$. The converse follows from the fact that $\mathbb{X} := \mathbb{M}^{-1} \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.5.6(b).

(b) If $\mathbb{N}'\mathbb{M}'^{-1}$ is a (J,S)-inner [q.]r.c.f., then $\mathbb{N}' = \mathbb{N}\mathbb{X}^{-1}$, $\mathbb{M}' = \mathbb{M}\mathbb{X}^{-1}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.4.5(c), and $(\mathbb{N}\mathbb{X}^{-1})^* J\mathbb{N}\mathbb{X}^{-1} = S$ implies that $\mathbb{X}S\mathbb{X} = \mathbb{N}^* J\mathbb{N}$. By going backwards we get the converse implication.

(c) The first equivalence follows from Lemma A.1.1(h1); the second is obvious (recall that $J_1 := \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$). □

Notes

Definition 6.4.1(a)–(c), Lemma 6.4.6, Lemma 6.4.7(a)&(b), and most of Definition 6.4.4 and Lemmas 6.4.5 and 6.4.8 are from [S98a] and [S98c] (see also [Sbook]); most of these are classical results/definitions (except possibly for signs). Also Lemma 6.4.2 and 6.4.3 are well known.

Naturally, from each result of this section on coprimeness, one can obtain a corresponding result on "ω-coprimeness" or on exponential coprimeness, by shifting (see Remark 6.1.9); an analogous claim applies to most other results concerning stability.

The class $\mathcal{F} := \{\widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1} \mid \widehat{\mathbb{N}}, \widehat{\mathbb{M}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathbf{C}^{*\times m}),\ \det\widehat{\mathbb{M}} \not\equiv 0\}$ of matrix-valued "$\mathrm{H}^\infty/\mathrm{H}^\infty$ transfer functions"has been studied extensively in the literature. This class is not contained in, nor does it contain the class of matrix-valued well-posed transfer functions. We can apply our theory to any $\mathbb{D} \in \mathrm{TIC}(\mathbf{C}^n, \mathbf{C}^m)$ having a right factorization (such a factorization is implied by any kind of stabilizability), due to Lemma 6.5.4(d2). Conversely, most proofs in this and the next section also apply (mutatis mutandis) to the $\mathcal{F}$ setting or to its generalization to infinite-dimensional input and output spaces with $\widehat{\mathbb{M}}$ required to be invertible at least at one point of $\mathbf{C}^+$.

The monograph [Vid] is an excellent reference to coprime factorization. Its emphasis is on rational matrix-valued transfer functions (equivalently, on the I/O maps having a finite-dimensional realization), but it also contains some results on several more general classes. Also [Logemann93] treats coprime factorizations in various subclasses of $\mathcal{F}$ (cf. Remark 6.5.11), and [Smith] is instructive in the general $\mathcal{F}$ setting.

During the preparation of this book, we have received several articles on coprime factorization and dynamic feedback in a WPLS setting, these include [CWW96], [WC] and [CWW01]. Also [Sbook] contains further results.

The notions of quasi- and pseudo-coprimeness may be new. Ruth Curtain [Curtain02] has lately studied the latter using so called *reci-procal systems*; one of her results is stated below Lemma 6.5.10.

6.5 Further coprimeness and factorizations

I tell them to turn to the study of mathematics, for it is only there that they might escape the lusts of the flesh.
— Thomas Mann (1875–1955), "The Magic Mountain"

In this section we give deeper explanations of the nature of classical, pseudo- and quasi-coprimeness and their relations to each other. We also give certain further results on coprime factorizations and inner maps.

We first deduce the basic facts on quasi–right coprime maps from those of quasi-left-invertible ones:

Lemma 6.5.1 (q.r.c.) *Assume that* $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{M} \in \mathrm{TIC}(U)$ *are q.r.c. Then, for any* $\omega \geq 0$*, we have*

(a) $\mathbb{N}^*\mathbb{N} + \mathbb{M}^*\mathbb{M} \gg 0$.

(b1) $\mathbb{N}u, \mathbb{M}u \in \mathrm{L}^2 \Leftrightarrow u \in \mathrm{L}^2$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.

(b3) There is $\varepsilon > 0$ *s.t. s.t.* $\varepsilon\|u\|_{\mathrm{L}^2} \leq \|\mathbb{N}u\|_{\mathrm{L}^2} + \|\mathbb{M}u\|_{\mathrm{L}^2} \leq \varepsilon^{-1}\|u\|_{\mathrm{L}^2}$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U) + \mathrm{L}^2_\infty(\mathbf{R}_+;U)$.

(c1) We have $\mathbb{N}\mathbb{D}, \mathbb{M}\mathbb{D} \in \mathrm{TIC} \Leftrightarrow \mathbb{D} \in \mathrm{TIC}$ *(and* $\|\mathbb{D}\|_{\mathrm{TIC}} \leq \varepsilon^{-1}(\|\mathbb{N}\mathbb{D}\|_{\mathrm{TIC}} + \|\mathbb{M}\mathbb{D}\|_{\mathrm{TIC}}$*) when* $\mathbb{D} \in \mathrm{TIC}_\infty(H,U)$.

(c2) We have $\mathbb{N}\mathbb{C}, \mathbb{M}\mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2) \Leftrightarrow \mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2)$ *(and* $\|\mathbb{C}\| \leq \varepsilon^{-1}(\|\mathbb{N}\mathbb{C}\| + \|\mathbb{M}\mathbb{C}\|)$*) when* X *is a normed space and* $\mathbb{C} \in \mathcal{B}(X,\mathrm{L}^2_\omega(\mathbf{R}_+;U))$.

(d) If $\mathbb{N}$ *and* $\mathbb{M}$ *are [q.]r.c. and* $\begin{bmatrix}\mathbb{D}_1\\ \mathbb{D}_2\end{bmatrix} \in \mathrm{TIC}(Y \times U)$ *is [quasi–]left invertible (over* TIC*), then* $\mathbb{D}_1\begin{bmatrix}\mathbb{N}\\ \mathbb{M}\end{bmatrix}$ *and* $\mathbb{D}_2\begin{bmatrix}\mathbb{N}\\ \mathbb{M}\end{bmatrix}$ *are [q.]r.c.; in particular,* $\mathbb{N}$ *and* $\mathbb{M} + \mathbb{U}\mathbb{N}$ *are [q.]r.c. for any* $\mathbb{U} \in \mathrm{TIC}$.

(e) For each $u_0 \in U \setminus \{0\}$*, we have* $\|\widehat{\mathbb{N}}u_0\|_Y + \|\widehat{\mathbb{M}}u_0\|_U \neq 0$ *on* $\mathbf{C}^+$ *and* $\|\widehat{\mathbb{N}}(ir)u_0\|_Y + \|\widehat{\mathbb{M}}(ir)u_0\|_U \geq \varepsilon\|u_0\|_U$ *for a.e.* $r \in \mathbf{R}$ *and some* $\varepsilon > 0$.

(f) If $\begin{bmatrix}\mathbb{N}\\ \mathbb{M}\end{bmatrix} = \begin{bmatrix}\mathbb{N}_0\\ \mathbb{M}_0\end{bmatrix}\mathbb{U}$*, and* $\mathbb{U}, \mathbb{N}_0, \mathbb{M}_0 \in \mathrm{TIC}$*, then* $\mathbb{U}$ *is quasi–left-invertible (left-invertible over* TIC *if* $\mathbb{N}, \mathbb{M}$ *are r.c.).*

Proof: This follows from corresponding claims in Lemma 4.1.8. (Set $\mathbb{D} = \begin{bmatrix} I & 0\\ \mathbb{U} & I\end{bmatrix}$ in (d). The "non-quasi" claims in (d) and (f) are obvious.) □

We go on to list some properties of p.r.c. maps (hence of r.c. maps too):

Lemma 6.5.2 (p.r.c.) *Let* $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *and* $\mathbb{M} \in \mathrm{TIC}(U)$*. The following are equivalent:*

(i) $\mathbb{N}$ *and* $\mathbb{M}$ *are p.r.c.;*

(ii) $\widehat{\widetilde{\mathbb{X}}}\widehat{\mathbb{M}} - \widehat{\widetilde{\mathbb{Y}}}\widehat{\mathbb{N}} \equiv I$ *on* $\mathbf{C}^+$ *for some bounded* $\widehat{\widetilde{\mathbb{X}}}, \widehat{\widetilde{\mathbb{Y}}} : \mathbf{C}^+ \to \mathcal{B}$*;*

(iii) There is $M < \infty$ *s.t. for each* $\omega \geq 0$ *there are* $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TI}_\omega(Y,U)$ *satisfying* $\|\widetilde{\mathbb{X}}\|_{\mathrm{TI}_\omega} \leq M$*,* $\|\widetilde{\mathbb{Y}}\|_{\mathrm{TI}_\omega} \leq M$*, and* $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I_{\mathrm{TI}_\omega}$*;*

(iv) There is $\varepsilon > 0$ *s.t.* $\|\mathbb{M}u\|_{\mathrm{L}^2_\omega} + \|\mathbb{N}u\|_{\mathrm{L}^2_\omega} \geq \varepsilon\|u\|_{\mathrm{L}^2_\omega}$ *(*$u \in \mathrm{L}^2_\omega(\mathbf{R};U)$*) for all* $\omega \geq 0$.

Moreover, if $\mathbb{N}$ *and* $\mathbb{M}$ *are p.r.c.,* $\alpha \geq \omega \geq 0$*, and* $T, \beta \in \mathbf{R}$*, then*

(a) Condition (ii) holds with some $\widehat{\widehat{\mathbb{X}}}, \widehat{\widehat{\mathbb{Y}}} \in \mathcal{C}_{\mathrm{b}}(\mathbf{C}^+;\mathcal{B})$.

(b1) $\mathbb{N}u, \mathbb{M}u \in \mathrm{L}^2_\omega \Leftrightarrow u \in \mathrm{L}^2_\omega$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R};U)$.

(b2) $\mathbb{N}u, \mathbb{M}u \in \pi_{[T,\infty)}\mathrm{L}^2_\omega \Leftrightarrow u \in \pi_{[T,\infty)}\mathrm{L}^2_\omega$ *for all* $u \in \mathrm{L}^2_\alpha(\mathbf{R};U)$.

(b3) There is $\varepsilon > 0$ *s.t.*

$$\varepsilon\|u\|_{\mathrm{L}^2_\omega} \leq \|\mathbb{M}u\|_{\mathrm{L}^2_\omega} + \|\mathbb{N}u\|_{\mathrm{L}^2_\omega} \leq \varepsilon^{-1}\|u\|_{\mathrm{L}^2_\omega}, \quad (u \in \mathrm{L}^2_\alpha(\mathbf{R};U)). \tag{6.111}$$

(c1) Let $\mathbb{D} \in \mathrm{TIC}_\infty(H,U)$*. Then* $\mathbb{ND}, \mathbb{MD} \in \mathrm{TIC}_\omega \Leftrightarrow \mathbb{D} \in \mathrm{TIC}_\omega$.

(c2) Let $\mathbb{C} \in \mathcal{B}(H,\mathrm{L}^2_\beta(\mathbf{R}_+;U))$*. Then* $\mathbb{NC}, \mathbb{MC} \in \mathcal{B}(H,\mathrm{L}^2_\omega) \Leftrightarrow \mathbb{C} \in \mathcal{B}(H,\mathrm{L}^2_\omega)$.

(d) If $\mathbb{N}$ *and* $\mathbb{M}$ *are [p.]r.c., then so are* $\mathbb{N}$ *and* $\mathbb{M} + \mathbb{UN}$ *for any* $\mathbb{U} \in \mathrm{TIC}$.

(e) We have $\mathbb{N}^*\mathbb{N} + \mathbb{M}^*\mathbb{M} \gg 0$.

(f) If $\left[\begin{smallmatrix}\mathbb{N}\\ \mathbb{M}\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mathbb{N}_0\\ \mathbb{M}_0\end{smallmatrix}\right]\mathbb{U}$*, and* $\mathbb{U}, \mathbb{N}_0, \mathbb{M}_0 \in \mathrm{TIC}$*, then* $\mathbb{U}$ *is pseudo–left-invertible on* TIC *(left-invertible if* $\mathbb{N}, \mathbb{M}$ *are r.c.).*

Note that if $\dim U < \infty$, then $\mathbb{U} \in \mathcal{G}\mathrm{TIC}(U)$ in (f), by Proposition 2.2.5(3). Thus, when $\dim U < \infty$, all common right divisors of [p.]r.c. maps are invertible. We conclude that “p.r.c.”, is a stronger property than “weakly r.c.”; in fact, it is strictly stronger (e.g., take $f(s) = se^{-s}/(s+1)$, $g = 1/(s+1)$, so that $f, g \in \mathrm{H}^\infty(\mathbf{C}^+;\mathbf{C})$ are “weakly r.c.” but not [p.]r.c., because $f(+\infty) = 0 = g(+\infty)$; this is Example of [Smith]).

On the other hand, “weakly r.c.” is not implied by “q.r.c.”: τ^{-1} and τ^{-1} are q.r.c. but not “weakly r.c.”, because τ^{-1} (or τ^{-r} for any $r \in (0,1]$) is their common divisor and not a unit (in TIC). We note that also the definition of “weakly r.c.” used in part III of [Smith] is weaker than r.c. (=p.r.c., by Lemma 6.5.3), by Lemma 4 of [Smith]. However, we shall not use the concept “weakly r.c.”.

Proof of Lemma 6.5.2: The equivalence and (a)–(c2) follow by setting $\mathbb{D} := \left[\begin{smallmatrix}\mathbb{M}\\ \mathbb{N}\end{smallmatrix}\right]$, $[\widehat{\widehat{\mathbb{X}}}\ -\widehat{\widehat{\mathbb{Y}}}] := \widehat{\mathbb{V}}$ in Proposition 4.1.7 (for (iv) and (b3) we use $(|r| + |s|)/2\|\left[\begin{smallmatrix}r\\ s\end{smallmatrix}\right]\| \leq (|r| + |s|)$) which also gives additional equivalent conditions.

(d) If $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$, then $\widetilde{\mathbb{X}}(\mathbb{M} + \mathbb{UN}) - (\widetilde{\mathbb{X}}\mathbb{U} + \widetilde{\mathbb{Y}})\mathbb{N} = I$; use (ii) for the p.r.c. case.

(e) By (v) (with $\omega = 0$), we have $\mathbb{M}^*\mathbb{M} + \mathbb{N}^*\mathbb{N} \geq \varepsilon I$.

(f) Now $(\widehat{\widehat{\mathbb{X}}}\widehat{\mathbb{M}}_0 - \widehat{\widehat{\mathbb{Y}}}\widehat{\mathbb{N}}_0)\mathbb{U} = I$, by (ii). □

Due to (a) and (c) below, one does not meet the notions “p.r.c.” and “q.r.c.” in the finite-dimensional theory:

Lemma 6.5.3 *Let* $\mathbb{M} \in \mathrm{TIC}(U)$ *and* $\mathbb{N} \in \mathrm{TIC}(U,Y)$*. Then*

(a) If $\dim U < \infty$*, then* $\mathbb{M}$ *and* $\mathbb{N}$ *are p.r.c. iff they are r.c.*

(b) If $\dim U = \infty$*, then* $\mathbb{M}$ *and* $\mathbb{N}$ *may be p.r.c. even if they are not r.c.*

(c) If $\dim U < \infty$ *and* $\widehat{\mathbb{M}}, \widehat{\mathbb{N}} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,*))$ *are rational and* $M \in \mathcal{G}\mathcal{B}(U)$*, then* $\widehat{\mathbb{M}}$*,* $\widehat{\mathbb{N}}$ *are q.r.c. iff they are r.c. (iff* $\|\widehat{\mathbb{M}}u_0\|_U + \|\widehat{\mathbb{N}}u_0\|_Y \neq 0$ *on* $\overline{\mathbf{C}^+}$ *for all* $u_0 \in U \setminus \{0\}$*).*

(d) A rational q.r.c.f. is a r.c.f.

See (the Corona) Theorem 4.1.6 for further results in case $\dim U < \infty$.

Even when $U = \mathbf{C} = Y$, we may have $\widehat{\mathbb{M}} = e^{-s}$, $\widehat{\mathbb{N}} = 0$ so that $\mathbb{M}$ and $\mathbb{N}$ are q.r.c. but not p.r.c. We do not know whether $\mathbb{M}$ and $\mathbb{N}$ can be q.r.c. without being p.r.c. if we require that $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

Proof: (a) This holds by Theorem 4.1.6(b).

(b) Let $\mathbb{M}$ be the map $\mathbb{D}$ of Lemma 4.1.10, and set $\mathbb{N} = 0$. Then $\widehat{\mathbb{M}}^*\widehat{\mathbb{M}} + 0^*0 \geq I$ on $\mathbf{C}^+$, hence $\mathbb{M}$ and $\mathbb{N}$ are p.r.c., but $\left[\begin{smallmatrix}\mathbb{M}\\ \mathbb{N}\end{smallmatrix}\right]$ is not left-invertible on TIC, i.e., $\mathbb{M}$ and $\mathbb{N}$ are not r.c.

(c) (In fact, $\widehat{\mathbb{M}}, \widehat{\mathbb{N}}$ need not be rational, it suffices that $\dim U < \infty$, $\widehat{\mathbb{M}}, \widehat{\mathbb{N}} \in \mathcal{C}(\overline{\mathbf{C}^+} \cup \{\infty\}; \mathcal{B}(U,*)) \cap \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U,*))$ and $M \in \mathcal{GB}(U)$.)

Set $\varepsilon := \min_{\|u_0\|_U = 1,\ s \in \overline{\mathbf{C}^+} \cup \{\infty\}} f(s, u_0)$ (this exists, since $\overline{\mathbf{C}^+} \cup \{\infty\}$ is compact), where $f(s, u_0) := \|\widehat{\mathbb{M}}(s)u_0\|^2 + \|\widehat{\mathbb{N}}(s)u_0\|^2$. If $\varepsilon > 0$, then $\mathbb{M}$ and $\mathbb{N}$ are p.r.c., hence r.c. (by (a)). If $\mathbb{M}$ and $\mathbb{N}$ are r.c., then they are q.r.c. It remains to show that $\varepsilon = 0$ implies that $\mathbb{N}$ and $\mathbb{M}$ are not q.r.c.

Assume that $\varepsilon = 0$. Then there are $\{s_n\}$, $\{u_n\}$ as above s.t. $f(s_n, u_n) \to 0$ as $n \to +\infty$. Replaces the above sequences by subsequences so that $s_n \to s$ and $u_n \to u$ for some s and u (since $M \in \mathcal{GB}(U)$ we have $s \neq \infty$). Use the uniform continuity of f on $\overline{\mathbf{C}^+} \cup \{\infty\} \times \{\|v\|_U = 1\}$ to obtain that $f(s_n, u) \to 0$, hence $f(s, u) = 0$, hence $\mathbb{M}$ and $\mathbb{N}$ are not q.r.c., by Lemma 6.5.1(e).

(d) This follows from (c) (since $M := \widehat{\mathbb{M}}(+\infty) \in \mathcal{GB}(U)$, by Proposition 6.3.1(c)). □

By Lemma 6.4.3, a scalar right factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is coprime iff $\mathbb{N}$ does not cancel any poles of $\mathbb{M}^{-1}$, i.e., iff $\mathbb{M}^{-1}$ and $\mathbb{D}$ have the same poles. Claim (c) above is an extension of this, below we give similar claims in the general case:

Lemma 6.5.4 (R.c. maps do not have common zeros) *Let* $\mathbb{M} \in \mathrm{TIC}(U)$ *and* $\mathbb{N} \in \mathrm{TIC}(U,Y)$ *be p.r.c. Let* $\Omega := \{s \in \mathbf{C}^+ \,|\, \widehat{\mathbb{M}}(s) \in \mathcal{GB}(U)\} \neq \emptyset$ *(this set is open, by Lemma A.3.3(A2)). Let* $\Omega' \subset \mathbf{C}^+$ *be open and connected, let* $\Omega_1 \subset \Omega$ *be bounded, and let* $\Omega_2 \subset \Omega$ *satisfy* $0 \notin \overline{\Omega_2}$*. Then the following hold:*

(a) $\widehat{\mathbb{M}}^{-1} \in \mathrm{H}(\Omega; \mathcal{B}(U))$.

(b) Let s_0 *be a boundary point of* Ω *in* $\mathbf{C}^+$*. Then* $\|(\widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1})(s)\| \to \infty$*, as* $s \to s_0$ *and* $s \in \Omega$.

(c1) Let $s_0 \in \overline{\Omega}$ *and* $N \in \mathbf{N}$*. Then* $(s - s_0)^N(\widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1})(s)$ *is bounded on* Ω_1 *iff* $(s - s_0)^N\widehat{\mathbb{M}}^{-1}(s)$ *is bounded on* Ω_1.

(c2) Let $N \in \mathbf{N}$*. Then* $s^{-N}(\widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1})(s)$ *is bounded on* Ω_2 *iff* $s^{-N}\widehat{\mathbb{M}}^{-1}(s)$ *is bounded on* Ω_2.

(c3) Let $\Omega_3 \subset \Omega$*. Then* $\widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ *is bounded on* Ω_3 *iff* $\widehat{\mathbb{M}}^{-1}$ *is bounded on* Ω_3.

(d1) If $\widehat{\mathbb{D}} \in \mathrm{H}(\Omega'; \mathcal{B}(U,Y))$ *(resp.* $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\Omega'; \mathcal{B}(U,Y))$*) and* $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ *on some open, nonempty subset of* $\Omega \cap \Omega'$*, then* $\Omega' \subset \Omega$*,* $\widehat{\mathbb{M}}^{-1} \in \mathrm{H}(\Omega'; \mathcal{B}(U))$ *(resp.* $\widehat{\mathbb{M}}^{-1} \in \mathrm{H}^\infty(\Omega'; \mathcal{B}(U))$*) and* $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ *on* Ω'.

(d2) If some $\mathbb{D} \in \mathrm{TIC}_\omega$ satisfies $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ on some open subset of $\mathbf{C}^+_\omega$, then $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\omega$ and $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$.

In the finite-dimensional setting, a r.c.f. is often defined so that $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ is required only on the set where $\widehat{\mathbb{M}}^{-1}$ exists. By (d2), this is equivalent to our definition whenever $\mathbb{D}$ is well-posed (i.e., in TIC_∞).

If a map $\mathbb{D} \in \mathrm{TIC}_\infty$ can be written as $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, where $\mathbb{N}$ and $\mathbb{M}$ are p.r.c. and $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$, then the poles of $\mathbb{M}^{-1}$ on $\overline{\mathbf{C}^+} \cup \{\infty\}$ are exactly those of $\mathbb{D}$, and these poles have same multiplicities, by (c1) and (c2) (the latter treats the point ∞).

In a sense, also the converse holds (if $\mathbb{N}$ and $\mathbb{M}$ are not p.r.c., then there are $\{s_n\} \subset \mathbf{C}^+$, $s_\infty \in \overline{\mathbf{C}^+} \cup \{\infty\}$, $\{u_n\} \subset U$ s.t. $\|u_n\| = 1$ $(n \in \mathbf{N})$, $s_n \to s_\infty$, and $\widehat{\mathbb{N}}(s_n)u_n, \widehat{\mathbb{M}}(s_n)u_n \to 0$, so that $\widehat{\mathbb{N}}$ has a zero and $\widehat{\mathbb{M}}^{-1}$ has a nonremovable singularity at s_∞.

Note that, by Lemma 3.3.9, for a map $\mathbb{M} \in \mathrm{TIC}(U)$ with U infinite-dimensional, the set of singularities ("poles") of $\widehat{\mathbb{M}}^{-1}$ can be any closed subset of $\mathbf{C}^+$ (excluding $\mathbf{C}^+_\omega$ if we wish that $\mathbb{M} \in \mathrm{TIC} \cap \mathcal{G}\mathrm{TIC}_\omega$; take, e.g., $\mathbb{N} = I$ to obtain a r.c.f.).

Proof: (a) This follows from Lemma D.1.2(b2).

(b) This follows from Lemma A.3.3(A3)) and the proof of (c1) with $N = 0$.

(c1) Let $\widehat{\widehat{\mathbb{X}}}\widehat{\mathbb{M}} - \widehat{\widehat{\mathbb{Y}}}\widehat{\mathbb{N}} = I$ on $\mathbf{C}^+$ and $M := \max(\|\widehat{\widehat{\mathbb{X}}}\|, \|\widehat{\widehat{\mathbb{Y}}}\|, d(\Omega_1)) < \infty$, as in Lemma 6.5.2(ii). Assume that $(s-s_0)^N\widehat{\mathbb{M}}^{-1}(s)$ is unbounded on Ω_1, i.e., that there is $\{s_n\} \subset \Omega$ s.t. $\|\widehat{\mathbb{M}}(s_n)^{-1}\| > n$ $(n \in \mathbf{N})$. Choose $\{u_n\} \subset U$ s.t. $\|v_n\| > n$ and $\|u_n\| < 1$ for $n \in \mathbf{N}$, where $v_n := (s_n - s_0)^N\widehat{\mathbb{M}}(s_n)^{-1}u_n$. Note that $\|\widehat{\mathbb{M}}(s_n)v_n\| < |s_n - s_0|^N \le M^N$. Now

$$M\|(s_n-s_0)^N\widehat{\mathbb{N}}(s_n)\widehat{\mathbb{M}}(s_n)^{-1}u_n\| = M\|\widehat{\mathbb{N}}(s_n)v_n\| \ge \|\widehat{\widehat{\mathbb{Y}}}(s_n)\widehat{\mathbb{N}}(s_n)v_n\| \tag{6.112}$$

$$\ge \|Iv_n\| - \|\widehat{\widehat{\mathbb{X}}}(s_n)\widehat{\mathbb{M}}(s_n)v_n\| > n - M|s_n - s_0|^N \to \infty, \tag{6.113}$$

as $n \to \infty$, as desired.

(c2) Replace $(s-s_0)$ by $1/s$ in the proof of (c1) (and let $\infty > M \ge \sup_{s\in\Omega_2} |1/s|$).

(c3) This follows from the proof of (c1) with $N = 0$, $|s_n - s_0|^N = 1$.

(d1) By Lemma D.1.2(e), we have $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ on $\Omega \cap \Omega'$, If there were some $s_0 \in \Omega' \cap \partial\Omega$, then we would have $\|\widehat{\mathbb{D}}(s)\| \to \infty$ as $s \to s_0$ in $\Omega' \cap \Omega$, by (a), hence $\Omega' = \Omega \cup (\Omega' \cap \overline{\Omega}^c)$. Because Ω' is connected, this implies that $\Omega' \cap \overline{\Omega}^c = \emptyset$, hence $\Omega' \subset \Omega$; consequently, $\widehat{\mathbb{M}}^{-1} \in \mathrm{H}(\Omega'; \mathcal{B}(U))$. If $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ is bounded on Ω', then so is $\widehat{\mathbb{M}}^{-1}$, by (c3).

(d2) If some $\mathbb{D} \in \mathrm{TIC}_\omega$ satisfies $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ on some open subset of $\mathbf{C}^+_\omega$, then $\widehat{\mathbb{M}} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U))$, by (d1), hence $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\omega$ and $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$. Conversely, if $\mathbb{M}^{-1} \in \mathrm{TIC}_\omega$, then $\mathbb{N}\mathbb{M}^{-1} \in \mathrm{TIC}_\omega$. □

We take now a look at the approach of Frank Callier, Charles Desoer and others to obtain a d.c.f. Assume that $\mathbb{N} \in \mathrm{TIC}_{-\varepsilon}(\mathbf{C}^m, Y)$ and $\mathbb{M} \in \mathrm{TIC}_{-\varepsilon}(\mathbf{C}^m) \cap \mathcal{G}\mathrm{TIC}_\infty$ for some $m \in \mathbf{N}+1$, $\varepsilon > 0$. Then $\widehat{\mathbb{D}} := \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ is meromorphic on $\mathbf{C}^+_{-\varepsilon}$.

By Lemma D.1.2(e), the set $Z := \{s \in \mathbf{C}^+_{-\varepsilon} \mid \det\widehat{\mathbb{M}}(s) = 0\}$ has no limit points on $\mathbf{C}^+_{-\varepsilon}$. Assume now that $|\det\widehat{\mathbb{M}}| \ge \varepsilon$ when $s \in \mathbf{C}^+$ and $|s| > R$, for some $R, \varepsilon > 0$ (e.g., $\mathbb{M} \in$ UHPR). Then ∞ is not a limit point of $Z \cap \overline{\mathbf{C}^+}$ (see also Lemma 6.3.6(a2)), hence then $Z \cap \overline{\mathbf{C}^+}$ is finite.

It has been shown for several subclasses of such transfer functions that from these zeros one can construct a rational (exponentially stable) denominator $\widehat{\mathbb{M}}_0 \in \mathrm{H}^\infty(\mathbf{C}^+;\mathbf{C}^n)$ s.t. $\mathbb{D} = \mathbb{N}_0\mathbb{M}_0^{-1}$ is a r.c.f., where $\mathbb{N}_0 := \mathbb{D}\mathbb{M}_0 \in \mathrm{TIC}_{\exp}$, and that $\mathbb{D} = \mathbb{D}_1 + \mathbb{D}_2$, where $\mathbb{D}_1 \in \mathrm{TIC}_{\exp}$ and $\widehat{\mathbb{D}}_2 \in \mathrm{H}^\infty_\infty$ is rational. See, e.g., [CD78], [CD80], [Logemann93] or [Logemann87] for further details and information.

The following is a generalization of a standard result:

Lemma 6.5.5 (Inner–outer is constant)

(a) If $J, S \in \mathcal{B}$, S is one-to-one, $\mathbb{N}^*J\mathbb{N} = S$ and $\mathbb{N} \in \mathcal{G}\mathrm{TIC}$, then $\mathbb{N} \in \mathcal{G}\mathcal{B}$.

(b) Let $J = J^* \in \mathcal{B}$ and $S = S^* \in \mathcal{G}\mathcal{B}$. If $\mathbb{N} \in$ is (J,S)-inner and $\mathbb{N}$ is outer *(i.e., $\mathbb{N}\pi_+\mathrm{L}^2$ is dense in $\pi_+\mathrm{L}^2$), then $\mathbb{N} \in \mathcal{G}\mathcal{B}$.*

Proof: (a) Now $\mathrm{TIC} \ni S\mathbb{N}^{-1} = \mathbb{N}^*J \in \mathrm{TIC}^*$, hence $L := S\mathbb{N}^{-1} \in \mathcal{B}$, by Lemma 2.1.7, Because $S\widehat{\mathbb{N}}^{-1} \equiv L$ on $\mathbf{C}^+$ (see Theorem 6.2.1), $\widehat{\mathbb{N}}^{-1}$ is a constant$\in \mathcal{B}$, hence so is $\mathbb{N}$.

(b) Because $\pi_+(S^{-1}\mathbb{N}^*J)\pi_+\mathbb{N}\pi_+ = \pi_+$ (recall that $\pi_+\mathbb{N}\pi_+ = \mathbb{N}\pi_+$), the range $\mathbb{N}\pi_+\mathrm{L}^2$ is closed in $\pi_+\mathrm{L}^2$, by Lemma A.3.1(v)&(iv), hence $\pi_+\mathbb{N}\pi_+\mathrm{L}^2 = \pi_+\mathrm{L}^2$. Being coercive and onto on $\pi_+\mathrm{L}^2$, $\mathbb{N}$ is invertible on $\pi_+\mathrm{L}^2$, by Lemma A.3.1(c3)(ii)&(i), hence $\mathbb{N} \in \mathcal{G}\mathrm{TIC}$, by Lemma 2.2.3. Consequently, $\mathbb{N} \in \mathcal{G}\mathcal{B}$, by (a). □

As noted above, "$\mathbb{D}$ is as stable as $\mathbb{M}^{-1}$" if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a [p.]r.c.f. This implies several useful facts:

Lemma 6.5.6 ($\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $\alpha \ge \omega \ge 0$. Then the following hold:*

(a1) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a p.r.c.f. and $u \in \mathrm{L}^2_\alpha$. Then $u, \mathbb{D}u \in \mathrm{L}^2_\omega \Leftrightarrow \mathbb{M}^{-1}u \in \mathrm{L}^2_\omega$.

(a2) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a p.r.c.f. Then there is $\varepsilon > 0$ s.t.

$$\varepsilon\|\mathbb{M}^{-1}u\|_{\mathrm{L}^2_\omega} \le \|u\|_{\mathrm{L}^2_\omega} + \|\mathbb{D}u\|_{\mathrm{L}^2_\omega} \le \varepsilon^{-1}\|\mathbb{M}^{-1}u\|_{\mathrm{L}^2_\omega} \quad (u \in \mathrm{L}^2_\alpha(\mathbf{R};U)). \tag{6.114}$$

(b) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a [p.]r.c.f. and $\mathbb{T} \in \mathrm{TIC}(H,U)$. Then $\mathbb{D} \in \mathrm{TIC}_\omega \Leftrightarrow \mathbb{M}^{-1} \in \mathrm{TIC}_\omega$, and $\mathbb{D}\mathbb{T} \in \mathrm{TIC}_\omega \Leftrightarrow \mathbb{M}^{-1}\mathbb{T} \in \mathrm{TIC}_\omega$.

(c) $\mathbb{N}\mathbb{M}^{-1}$ is a [p.]r.c.f. of $\mathbb{D}$ iff $\mathbb{M}^{-\mathrm{d}}\mathbb{N}^{\mathrm{d}}$ is a [p.]l.c.f. of $\mathbb{D}^{\mathrm{d}}$.

(d) If $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a [p.]r.c.f., $U = U_1 \times U_2$, and $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}(U_2)$, then $\mathbb{D}$ has a [p.]r.c.f. of the form

$$\mathbb{D} = \mathbb{N}' \begin{bmatrix} \mathbb{M}'_{11} & \mathbb{M}'_{12} \\ 0 & I \end{bmatrix}^{-1}. \tag{6.115}$$

If $\mathcal{A} \subset \mathrm{TIC}$ and $\mathbb{N}, \mathbb{M}, \mathbb{M}_{22}^{-1} \in \mathcal{A}$, then we can take $\mathbb{N}', \mathbb{M}' \in \mathcal{A}$; if $\mathbb{M}_{22}^{-1} \in \mathcal{A}$ and $\mathbb{N}\mathbb{M}^{-1}$ is a [p.]r.c.f. over $\mathcal{A}$, then also $\mathbb{N}'(\mathbb{M}')^{-1}$ is a [p.]r.c.f. over $\mathcal{A}$,.

(e) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a [p.]r.c.f. Then $\mathbb{D}$ has a (normalized) [p.]r.c.f. $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1}$ s.t. $\mathbb{N}'^*\mathbb{N}' + \mathbb{M}'^*\mathbb{M}' = I$.

(f) For $\omega = 0$, we can replace "p." by "q." in (a1)–(e).

Proof: (a1)&(a2) Apply Lemma 6.5.2(b1)&(b3) to $u_{\circlearrowleft} := \mathbb{M}^{-1}u$.

(b) Apply Lemma 6.5.2(c) to $\mathbb{M}^{-1}\mathbb{T} \in \mathrm{TIC}_\infty$.

(c) This is trivial (recall that $\mathbb{D}^{\mathrm{d}} := \text{Я}\mathbb{D}^*\text{Я} \in \mathrm{TIC}_\infty$).

(d) Now $\mathbb{U} := \begin{bmatrix} I & 0 \\ -\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{M}_{22}^{-1} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}(U_1 \times U_2)$, by Lemma A.1.1(b1), hence $\mathbb{D} = (\mathbb{N}\mathbb{U})(\mathbb{M}\mathbb{U})^{-1}$ is obviously an [p.]r.c.f. too. But $\mathbb{M}\mathbb{U} = \begin{bmatrix} \mathbb{M}_{11} - \mathbb{M}_{12}\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{M}_{12}\mathbb{M}_{22}^{-1} \\ 0 & I \end{bmatrix}$, as desired.

The $\mathcal{A}$ case is obvious (if $\mathbb{U}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$, then so are $\mathbb{U}^{-1}\widetilde{\mathbb{X}}$ and $\mathbb{U}^{-1}\widetilde{\mathbb{Y}}$).

(e) By Lemma 6.5.2(e) and Lemma 6.4.7(a), we have $\mathbb{X}^*\mathbb{X} = \mathbb{N}^*\mathbb{N} + \mathbb{M}^*\mathbb{M}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$. Set $\mathbb{N}' := \mathbb{N}\mathbb{X}^{-1}$, $\mathbb{M}' := \mathbb{M}\mathbb{X}^{-1}$.

(f) Use Lemma 6.5.1 instead of Lemma 6.5.2 in the proofs of (a1)–(e). □

See Theorem 4.1.6(d) on the connection between right and left coprime factorizations.

A coprime factorization of a perturbed map is obtained as follows:

Lemma 6.5.7 (D.c.f.+TIC) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ and $\widetilde{\mathbb{D}} \in \mathrm{TIC}(U,Y)$. Then the following hold:*

(a) If $\mathbb{D}$ has the d.c.f. (6.109), then

$$\begin{bmatrix} I & 0 \\ \widetilde{\mathbb{D}} & I \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} = \left(\begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\widetilde{\mathbb{D}} & I \end{bmatrix} \right)^{-1} \tag{6.116}$$

is a d.c.f. of $\mathbb{D} + \widetilde{\mathbb{D}}$.

(b) Similarly, $\widetilde{\mathbb{D}} + \mathbb{N}\mathbb{M}^{-1} = (\mathbb{N} + \widetilde{\mathbb{D}}\mathbb{M})\mathbb{M}^{-1}$ is a [q.]r.c.f., whenever $\mathbb{N}\mathbb{M}^{-1}$ is.

(c) $\left[\begin{smallmatrix} \mathbb{D} \\ I \end{smallmatrix}\right] = \widetilde{\mathbb{N}}\mathbb{M}^{-1}$ is a [q.]r.c.f. iff $\widetilde{\mathbb{N}} = \left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right]$ for a [q.]r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$.

Thus, when factorizing of a regular transfer function $\widehat{\mathbb{D}}$, we may assume that $D := \widehat{\mathbb{D}}(+\infty) = 0$.

Proof: (a) This is obvious.

(b) This follows from Lemma 6.5.1(e).

(c) Let $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty(U)$. We have $\left[\begin{smallmatrix} \mathbb{D} \\ I \end{smallmatrix}\right] = \widetilde{\mathbb{N}}\mathbb{M}^{-1}$ iff $\widetilde{\mathbb{N}} = \left[\begin{smallmatrix} \mathbb{D} \\ I \end{smallmatrix}\right]\mathbb{M} =: \left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right]$. Obviously, $\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right]$ and $\mathbb{M}$ are [q.]r.c. iff $\mathbb{N}$ and $\mathbb{M}$ are [q.]r.c. □

A map has a d.c.f. iff it has a r.c.f. and a l.c.f.:

Lemma 6.5.8 (D.c.f.⇔r.c.f. & l.c.f.) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ have the r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ [over $\mathcal{A}$] and the l.c.f. $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ [over $\mathcal{A}$]. Then these factorizations can be extended to a d.c.f.*

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = I = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix}, \tag{6.117}$$

[over $\mathcal{A}$] of $\mathbb{D}$. Moreover, if $\mathbb{X}, \mathbb{Y} \in$ TIC satisfying (6.108) are given, there are $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in$ TIC satisfying (6.117) (the dual claim with left and right interchanged holds as well).

See Theorem 6.6.28 for the equivalence of joint stability and detectability and the existence of a d.c.f.

By Lemma 6.5.6(e) (and its dual), we can above require the normalizations $\mathbb{N}^*\mathbb{N} + \mathbb{M}^*\mathbb{M} = I$ and $\widetilde{\mathbb{M}}\widetilde{\mathbb{M}}^* + \widetilde{\mathbb{N}}\widetilde{\mathbb{N}}^* = I$.

Proof of Lemma 6.5.8 Lemma A.1.1(e1) shows that if $\mathbb{X}_0, \mathbb{Y}_0, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in$ TIC satisfying (6.108) and (6.107), respectively, are given, then $\mathbb{X} := \mathbb{X}_0 + \mathbb{N}(\widetilde{\mathbb{Y}}\mathbb{X}_0 - \widetilde{\mathbb{X}}\mathbb{Y}_0)$ and $\mathbb{Y} := \mathbb{Y}_0 + \mathbb{M}(\widetilde{\mathbb{Y}}\mathbb{X}_0 - \widetilde{\mathbb{X}}\mathbb{Y}_0)$ satisfy the second equality in (6.117) [note that also $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$]. The equation $\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = I$ holds in TIC_∞, by Lemma A.1.1(e5), hence in TIC (by analytic extension on $\mathbf{C}^+$). By taking (causal) adjoints, one gets the dual claim. □

(The invertibility of $\mathbb{M}$ was used in the above proof; the corresponding assumption in Lemma A.1.1(e5) is not superfluous.)

Given a d.c.f. of a TIC_∞ map, all d.c.f.'s of that map are obtained from (d) below:

Lemma 6.5.9 (All d.c.f.'s) *Let*

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = I = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \tag{6.118}$$

in $\text{TIC}(U \times Y)$. *Then we have the following:*

(a1) $\mathbb{M} \in \mathcal{G}\text{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{M}} \in \mathcal{G}\text{TIC}_\infty$.

(a2) If $\mathbb{M} \in \mathcal{G}\text{TIC}_\infty$*, then (6.118) is a d.c.f. of* $\mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$.

(b) All possible choices $\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in$ TIC *in (6.118) (for fixed* $\mathbb{M}, \mathbb{N}, \widetilde{\mathbb{M}}, \widetilde{\mathbb{N}}$*) are parametrized by*

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} + \mathbb{M}\mathbb{U} \\ \mathbb{N} & \mathbb{X} + \mathbb{N}\mathbb{U} \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{\mathbb{X}} + \mathbb{U}\widetilde{\mathbb{N}} & -(\widetilde{\mathbb{Y}} + \mathbb{U}\widetilde{\mathbb{M}}) \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \quad (\mathbb{U} \in \text{TIC}). \tag{6.119}$$

(c) All completions of $\begin{bmatrix} \mathbb{M} \\ \mathbb{N} \end{bmatrix}$ *to an invertible operator in* $\text{TIC}(U \times Y)$ *are parametrized by*

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \begin{bmatrix} I & \mathbb{U} \\ 0 & \mathbb{V} \end{bmatrix} = \begin{bmatrix} \mathbb{M} & \mathbb{Y}\mathbb{V} + \mathbb{M}\mathbb{U} \\ \mathbb{N} & \mathbb{X}\mathbb{V} + \mathbb{N}\mathbb{U} \end{bmatrix} \quad (\mathbb{U} \in \text{TIC},\ \mathbb{V} \in \mathcal{G}\text{TIC}). \tag{6.120}$$

(d) If (6.118) is a d.c.f. (i.e., if $\mathbb{M} \in \mathcal{G}\text{TIC}_\infty$*), then all d.c.f.'s of* $\mathbb{N}\mathbb{M}^{-1}$ *are given by*

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \begin{bmatrix} \mathbb{W} & \mathbb{U} \\ 0 & \mathbb{V} \end{bmatrix} = \left(\begin{bmatrix} \mathbb{W} & \mathbb{U} \\ 0 & \mathbb{V} \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \right)^{-1} \quad (\mathbb{U} \in \text{TIC},\ \mathbb{V}, \mathbb{W} \in \mathcal{G}\text{TIC}). \tag{6.121}$$

(e) Let (6.118) be a d.c.f., and let $\mathbb{M} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, $\widetilde{\mathbb{M}} = \left[\begin{smallmatrix} I & * \\ 0 & * \end{smallmatrix}\right]$, $\mathbb{X} = \left[\begin{smallmatrix} I & * \\ 0 & * \end{smallmatrix}\right]$, $\widetilde{\mathbb{X}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$. *Then all d.c.f.'s of* $\mathbb{N}\mathbb{M}^{-1}$ *with such* $\mathbb{M}, \widetilde{\mathbb{M}}, \mathbb{X}, \widetilde{\mathbb{X}}$ *are given by (6.121) with extra requirements* $\mathbb{W} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, $\mathbb{U} = \left[\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right]$, $\mathbb{V} = \left[\begin{smallmatrix} I & * \\ 0 & * \end{smallmatrix}\right]$.

Proof: (a) Part (a1) follows from Lemma A.1.1(c1) and (a2) is trivial.

(b) This follows from (c), because fixing $\widetilde{\mathbb{M}}$ and $\widetilde{\mathbb{N}}$ forces $\mathbb{V}$ to be I.

(c) Clearly $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right]\left[\begin{smallmatrix} I & \mathbb{U} \\ 0 & \mathbb{V} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ whenever $\mathbb{U} \in \mathrm{TIC}$ and $\mathbb{V} \in \mathcal{G}\mathrm{TIC}$. For the converse, assume that $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}(U \times Y)$. Set

$$\begin{bmatrix} I & \mathbb{U} \\ 0 & \mathbb{V} \end{bmatrix} := \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}.$$

We must have $\mathbb{V} \in \mathcal{G}\mathrm{TIC}$, hence we get (6.120) by multiplying the above equation by $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right]$ to the left.

(d) Clearly $\left[\begin{smallmatrix} \mathbb{M}\mathbb{W} & \mathbb{Y} \\ \mathbb{N}\mathbb{W} & \mathbb{X} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbb{W} & 0 \\ 0 & I \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$. By (b), all d.c.f.'s corresponding to the pair $(\mathbb{N}\mathbb{W}, \mathbb{M}\mathbb{W})$ are given by $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbb{W} & 0 \\ 0 & I \end{smallmatrix}\right]\left[\begin{smallmatrix} I & \mathbb{U} \\ 0 & \mathbb{V} \end{smallmatrix}\right]$, and these are of the form (6.121) (with $\mathbb{W}\mathbb{U}$ in place of $\mathbb{U}$). On the other hand, all r.c.f.'s of $\mathbb{N}\mathbb{M}^{-1}$ are of the form $(\mathbb{N}\mathbb{W}, \mathbb{M}\mathbb{W})$ for some $\mathbb{W} \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.4.5.

(e) This follows by writing (6.121) out (the equation $\mathbb{U}_{11} = 0$ is obtained from the right-hand-side and the equation $\mathbb{W}\left[\begin{smallmatrix} 0 & * \\ 0 & * \end{smallmatrix}\right]\mathbb{V} = \left[\begin{smallmatrix} 0 & * \\ 0 & * \end{smallmatrix}\right]$, or, alternatively, from the fact that $\mathbb{M}_{11}$ is injective). □

An operator not having a d.c.f. is somewhat pathological, as shown below:

Lemma 6.5.10 *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U, Y)$. *The existence of a d.c.f. is guaranteed in the following cases:*

(a) If $\mathbb{D}$ *is stable, then it has the d.c.f.* $\left[\begin{smallmatrix} I & 0 \\ \mathbb{D} & I \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} I & 0 \\ -\mathbb{D} & I \end{smallmatrix}\right]$.

(b1) If $\widehat{\mathbb{D}}$ *is rational, i.e.,* $\mathbb{D}$ *has a finite-dimensional realization, then* $\mathbb{D}$ *has a d.c.f.*

(c) If $\widehat{\mathbb{D}}$ *belongs to the Callier–Desoer class "*$\mathbb{B}(0)^{n\times m}$*", then* $\mathbb{D}$ *has a d.c.f. with* $\widehat{\mathbb{M}}$ *and* $\widehat{\widetilde{\mathbb{M}}}$ *rational.*

(d) If $\mathbb{D}$ *has a jointly stabilizable and detectable realization, then it has a d.c.f.*

(e1) Assume that $\mathbb{D}$ *is DF-stabilizable (stabilizable by dynamic (output) feedback; see Section 7.1). If* $\dim U, \dim Y < \infty$ *or some DF-stabilizing controller of* $\mathbb{D}$ *has a d.c.f., then* $\mathbb{D}$ *has a d.c.f.*

(e2) If $\mathbb{D}$ *DF-stabilizable by a stable controller, then* $\mathbb{D}$ *has a d.c.f. In particular, this is the case if* $\mathbb{D}$ *is stabilizable by static output feedback.*

(e3) If $\mathbb{D}$ *has an exponentially DF-stabilizable realization with bounded input and output operators (or as a WPLS of the form of Lemma 6.8.5), then* $\mathbb{D}$ *has an exponential d.c.f. in* $\mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}$.

By (e1), the maps not having a d.c.f. are not interesting from the point of view of DF-stabilization and optimization, at least not in the case with finite-dimensional input and output spaces.

In cases (b) and (c), the d.c.f. can, in fact, be chosen to be exponentially coprime (and with rational $\mathbb{M}$ and $\widetilde{\mathbb{M}}$).

According to [Curtain02], a d.c.f. also exists if Σ and Σ^d are SOS-stabilizable, or Σ and Σ^d are state-and-output-stabilizable, provided that $\dim U, \dim Y < \infty$ (or that we accept pseudo-coprimeness) and that $\sigma(A)$ satisfies certain assumptions.

Proof: (a) This is clear.

(b) By [Vid, p. 387], the set $\mathbf{C}[s]$ of (complex) rational functions over the field $\mathbf{C}$ (or any other field) is a proper Euclidean domain, hence a Bezout domain [Vid, Fact A.4.6], so every $\widehat{\mathbb{D}} \in \mathbf{C}[s]^{n\times m}$ has a r.c.f. and a l.c.f., by [Vid, Corollary 8.1.8]. (Take a minimal realization (see, e.g., Section 6.4 of [LR]) and use Theorem 6.6.28 to obtain an alternative proof.)

(c) This is contained in Theorem 2.1 of [CD80]. The class "$\mathbb{B}(0)^{n\times m}$" refers to $\mathbf{C}^{n\times m}$-valued (matrix) functions with elements of form $\widehat{f}/\widehat{g}$ s.t. $f, g \in \mathrm{MTIC}^{\mathrm{L}^1}_{\exp}(\mathbf{C})$ and $g \in \mathcal{G}\mathrm{TIC}_\infty(\mathbf{C})$.

(d) This is proved in Theorem 6.6.28.

(e1) The first claim is Lemma 7.1.4 and the second one is Proposition 7.1.6(d) and contains (e2) as a special case, by (a).

(e3) This follows from Theorem 7.2.4(a) and Corollary 9.2.13(c). □

The factorization results over TIC (as well as most results of Chapter 7 among others) could as well have been stated over MTIC or over any other structures where $\mathrm{TIC}(U)$ is replaced with a ring with identity etc. (cf. Lemma A.1.1) we state this notion in a form that will be applied in Section 7.1:

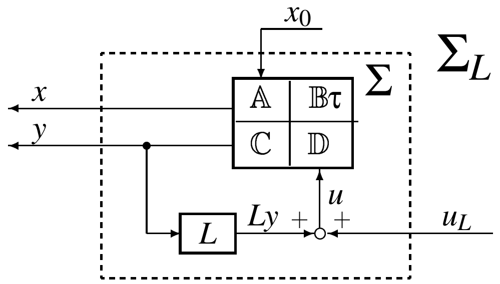
Remark 6.5.11 *Let $\mathcal{X}$, $\mathcal{A}'$ and $\mathcal{A} \subset \mathcal{A}'$ be as in Definition 6.2.4. Define r.c.f.'s, l.c.f.'s and d.c.f.'s as above, with $\mathcal{A}$ in place of* TIC *and $\mathcal{A}'$ in place of* TIC_∞ *(i.e., consider $\mathcal{A}'$ as the class of all admissible I/O maps and $\mathcal{A}$ as the class of "stable" I/O maps).*

Then Lemma 6.4.5(b)–(d), Lemmas 6.5.9 and 6.5.7, and most of Lemmas 6.5.8 and 6.5.6 (and almost all I/O results of Section 6.6 and most of Chapter 7; cf. Remark 7.0.1) hold with $\mathcal{A}$ in place of TIC*, $\mathcal{A}'$ in place of* TIC_∞ *and "q." and "p." removed.*

This is particularly useful when we let $\mathcal{A}$ be MTIC *or some of its subclasses, and take $\mathcal{X} := \{$all Hilbert spaces$\}$ and $\mathcal{A}' :=$* TIC_∞*, or when $\mathcal{A} =$* tic *and* $\mathcal{A}' = \mathrm{tic}_\infty$.

Notes

Lemma 6.5.8 is essentially from [S98a]. Lemmas 6.5.5, 6.5.7 and 6.5.9 are known at least to some extent. Probably also many of the other results are known at least in the case where "pseudo-" or "quasi-" is dropped from the assumptions and $\dim U, \dim Y < \infty$. See also the references in the text and the notes to Section 6.4.



$$x = \mathbb{A}x_0 + \mathbb{B}\tau u$$
$$y = \mathbb{C}x_0 + \mathbb{D}u$$
$$u = Ly + u_L$$

Figure 6.2: Static output feedback

6.6 Feedback and Stabilization (Σ_L, $\Sigma_\flat$, $\Sigma_\sharp$)

The Universe is populated by stable things.

— Richard Dawkins

One often wants to stabilize and possibly also regulate or optimize a system by feeding its state or output back into the system through some kind of controller, as in, e.g., Figure 6.2. All control problems of Chapters 8–12 are of this form.

In this section, we shall introduce static output feedback (Figure 6.2), state feedback (Figure 6.3) and output injection (Figure 6.5), hence also the concepts "stabilizability" and "detectability" in their various forms.

In finite-dimensional control theory, optimization is usually restricted by the requirement that the (controlled) closed-loop system should be exponentially stable. However, sometimes strong stability or some other form of stability has been allowed, and for infinite-dimensional systems this has become increasingly popular during recent years. Therefore, we have decided to study all forms of feedback w.r.t. all forms of stability in this section.

Most of our definitions and some of the results follow [S97b], [S98a] and [Sbook]; in particular, we reduce all forms of feedback to static output feedback. We note that by shifting (see Remark 6.1.9) any result on stabilization, one obtains a result on exponential stabilization, but the converse is not true, and results on (nonexponential) stabilization are often weaker or harder to prove.

In Section 6.7, we shall present further results and related concepts. Different forms of dynamic [partial] output feedback are the subject of Chapter 7. We have collected the definition of all forms of feedback to Summary 6.7.1.

Now we shall introduce static output feedback, where we feed a part Ly of the output y of Σ back into the input, as Figure 6.2 shows; here $L \in \mathcal{B}(Y,U)$ and $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$. Under an external output $u_L : \mathbf{R}_+ \to U$ (external input, control, disturbance or analogous), the effective input u becomes equal to $Ly + u_L$. In the initial value setting (6.3) with initial value $x_0 \in H$, the signals in the resulting dashed *closed-loop system* Σ_L of Figure 6.2 clearly satisfy the following equations:

$$x(t) = \mathbb{A}(t)x_0 + \mathbb{B}\tau(t)u \qquad (t \geq 0), \tag{6.122}$$

$$y = \mathbb{C}x_0 + \mathbb{D}u, \tag{6.123}$$

$$u = Ly + u_L. \tag{6.124}$$

Obviously, this can be uniquely solved in terms of x_0 and u_L iff $(I - L\mathbb{D})$ is invertible (the corresponding solution is given by (6.125)). We call such an L admissible:

Definition 6.6.1 (Admissible static output feedback) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$. *An operator* $L \in \mathcal{B}(Y,U)$ *is called an* admissible (static) output feedback operator *for* Σ *(or for* $\mathbb{D}$*) if* $I - L\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty(U)$*, or equivalently, if* $I - \mathbb{D}L \in \mathcal{G}\mathrm{TIC}_\infty(Y)$.

This is the case iff $I - \mathbb{D}L$ is locally invertible, by Lemma 2.2.8. For $\mathbb{D} \in \mathrm{ULR}$, this is equivalent to $I - DL \in \mathcal{GB}(Y)$, by Proposition 6.3.1(c). A more thorough motivation of admissibility is given in [W94b, Proposition 3.6].

As shown in [W94b, Section 6], the signals x and y in (6.122)–(6.124) can be interpreted as the state and output of another well-posed linear system:

Proposition 6.6.2 **(Σ_L)** *Let* $L \in \mathcal{B}(Y,U)$ *be an admissible output feedback operator for* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$.

Then $\Sigma_L \in \mathrm{WPLS}(U,H,Y)$*, where*

$$\Sigma_L := \left[\begin{array}{c|c} \mathbb{A}_L & \mathbb{B}_L \\ \hline \mathbb{C}_L & \mathbb{D}_L \end{array}\right] := \left[\begin{array}{c|c} \mathbb{A} + \mathbb{B}\tau L(I - \mathbb{D}L)^{-1}\mathbb{C} & \mathbb{B}(I - L\mathbb{D})^{-1} \\ \hline (I - \mathbb{D}L)^{-1}\mathbb{C} & \mathbb{D}(I - L\mathbb{D})^{-1} \end{array}\right] \tag{6.125}$$

$$= \Sigma \begin{bmatrix} I & 0 \\ -L\mathbb{C} & I - L\mathbb{D} \end{bmatrix}^{-1} = \Sigma \begin{bmatrix} I & 0 \\ (I - L\mathbb{D})^{-1}L\mathbb{C} & (I - L\mathbb{D})^{-1} \end{bmatrix} : \begin{bmatrix} x_0 \\ u_L \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}. \tag{6.126}$$

We call Σ_L *the* closed-loop system *with output feedback operator* L*. In the initial value setting (6.3) with initial value* x_0 *and control* u_L*, the controlled state* $x(t)$ *at time* t *and the output* y *of* Σ_L *form the unique solution of equations (6.122)–(6.124).* □

Note that (6.126) follows easily from

$$\begin{bmatrix} x_0 \\ u_L \end{bmatrix} = \begin{bmatrix} I & 0 \\ -L\mathbb{C} & I - L\mathbb{D} \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} x_0 \\ u \end{bmatrix} = \begin{bmatrix} I & 0 \\ (I - L\mathbb{D})^{-1}L\mathbb{C} & (I - L\mathbb{D})^{-1} \end{bmatrix} \begin{bmatrix} x_0 \\ u_L \end{bmatrix} \tag{6.127}$$

(here $u, u_L : \mathbf{R}_+ \to U$ and Σ refers to $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B}\tau \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$; cf. (6.123)–(6.124)).

Repeated feedback behaves in the expected way:

Lemma 6.6.3 ($\boldsymbol{\Sigma_{L+K} = (\Sigma_L)_K}$) *Let* L *be admissible for* Σ *as above. Then* $K \in \mathcal{B}(Y,U)$ *is admissible for* Σ_L *iff* $K + L$ *is admissible for* Σ*; and in that case* $\Sigma_{L+K} = (\Sigma_L)_K$ *(in particular,* $(\Sigma_L)_{-L} = \Sigma$*).* □

(See [W94b, Remark 6.5] for the proof.)

Definition 6.6.4 (Stabilizing L) *Let $L \in \mathcal{B}(Y,U)$ be an admissible output feedback operator for $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$. The operator L is* stabilizing *for Σ if the corresponding closed loop system Σ_L is stable.*

The same applies to all stability concepts (prefices for "stabilizing") defined in Definition 6.1.3. E.g., L is strongly internally ω-stabilizing if Σ_L is strongly internally ω-stable, i.e., if $\mathbb{A}_L$ is strongly ω-stable.

Analogously, we call L $\mathbb{B}$-stabilizing *if $\mathbb{B}_L$ is stable; the same applies to other components of Σ_L.*

We call L [exponentially] stabilizing for $\mathbb{D}$ *if L is admissible and makes $\mathbb{D}_L = \mathbb{D}(I - L\mathbb{D})^{-1}$ [exponentially] stable.*

Stabilizes *means* is stabilizing for.

Thus, an admissible output feedback operator L I/O-stabilizes $\Sigma \in \mathrm{WPLS}(U,H,Y)$ iff $y \in \mathrm{L}^2$ for all $x_0 \in H$ and $u_L \in \mathrm{L}^2(\mathbf{R}_+;U)$, and L stabilizes Σ [strongly] iff x is bounded [and $x(t) \to 0$ as $t \to \infty$] and $y \in \mathrm{L}^2(\mathbf{R}_+;Y)$ for all $x_0 \in H$ and $u_L \in \mathrm{L}^2(\mathbf{R}_+;U)$. The operator L is exponentially stabilizing for Σ iff $x \in \mathrm{L}^2(\mathbf{R}_+;H)$ for all $x_0 \in H$ (and $u_L = 0$); or equivalently, iff $x, y \in \mathrm{L}^2$ for all $x_0 \in H$ and $u_L \in \mathrm{L}^2(\mathbf{R}_+;U)$, by Lemma 6.1.10(a1) and Lemma A.4.5. (Here $x := \mathbb{A}_L x_0 + \mathbb{B}_L \tau u_L$ and $y := \mathbb{C}_L x_0 + \mathbb{D}_L u_L$ are the state and output of the closed-loop system Σ_L with initial state x_0 and input u_L.)

Usually we shall only need the definitions of two first paragraphs of Definition 6.6.4. Terms like "$\mathbb{B}$-stabilizing" are useful only when referring to a part of a component of Σ (above we could say "input-stabilizing"). The last definition will be needed in connection with dynamic output feedback (when only I/O maps are specified).

Note that L is admissible for $\mathbb{D}$ iff L is admissible for Σ, but L may stabilize $\mathbb{D}$ even if L does not stabilize Σ. The same, of course, holds for the admissibility and stability concepts derived from this later.

The maps $u_L, y_L \mapsto u, y$ in a stabilizing feedback induce a d.c.f.:

Lemma 6.6.5 *The operator $L \in \mathcal{B}(Y,U)$ is a stabilizing output feedback operator for $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ iff*

$$\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix} := \begin{bmatrix}(I - L\mathbb{D})^{-1} & L\\ \mathbb{D}_L & I\end{bmatrix} = \begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}^{-1} := \begin{bmatrix}I & -L\\ -\mathbb{D}_L & (I - \mathbb{D}L)^{-1}\end{bmatrix}^{-1} \tag{6.128}$$

defines a d.c.f. of $\mathbb{D}$ (still $\mathbb{D}_L := \mathbb{D}(I - L\mathbb{D})^{-1}$), i.e., iff the operators in (6.128) are (well-posed and) stable.

If we input an extra signal y_L to the output y (as y is output from Σ in Figure 6.2), then the closed loop map $\left[\begin{smallmatrix}u_L\\ y_L\end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix}u\\ y\end{smallmatrix}\right]$ is the stable mapping

$$\begin{bmatrix}u\\ y\end{bmatrix} = \begin{bmatrix}\mathbb{M} & \widetilde{\mathbb{N}}\\ \mathbb{N} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}u_L\\ y_L\end{bmatrix}. \tag{6.129}$$

This is based on the useful formula $(I-L\mathbb{D})^{-1} = I + L\mathbb{D}_L$.

Proof: Obviously, (6.128) is sufficient for L to stabilize $\mathbb{D}$ (because then $\mathbb{D}_L$ is well-posed and stable), so we will assume that L stabilizes $\mathbb{D}$ and prove (6.128).

We have $\mathbb{M} := (I-L\mathbb{D})^{-1} = I + L\mathbb{D}(I-L\mathbb{D})^{-1} = I + L\mathbb{D}_L \in \mathrm{TIC}$, and $\widetilde{\mathbb{M}} := (I-\mathbb{D}L)^{-1} = I + (I-\mathbb{D}L)^{-1}\mathbb{D}L = I + \mathbb{D}_L L \in \mathrm{TIC}$ (because $\mathbb{D}_L := \mathbb{D}(I-L\mathbb{D})^{-1} = (I-\mathbb{D}L)^{-1}\mathbb{D}$, by Lemma A.1.1(f6)); clearly (6.128) follows from this.

Equation (6.129) is obviously the solution of equations

$$y = y_L + \mathbb{D}u \tag{6.130}$$

$$u = u_L + Ly. \tag{6.131}$$

□

If $\mathbb{D}$ has a r.c.f., then it is enough to stabilize $u_L \mapsto u$:

Lemma 6.6.6 *Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a r.c.f. of $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Then $L \in \mathcal{B}(U,Y)$ is an admissible output feedback operator for $\mathbb{D}$ iff $\mathbb{M} - L\mathbb{N} \in \mathcal{G}\mathrm{TIC}_\infty(U)$.*

An admissible L is $\mathbb{D}$-stabilizing (i.e., $\mathbb{D}_L = \mathbb{N}(\mathbb{M}-L\mathbb{N})^{-1}$ is stable) iff $(\mathbb{M}-L\mathbb{N})^{-1} \in \mathrm{TIC}$ is stable. Note that then $(I-L\mathbb{D})^{-1} = \mathbb{M}(\mathbb{M}-L\mathbb{N})^{-1} : u_L \mapsto u$.

Proof: Now $I - L\mathbb{D} = (\mathbb{M}-L\mathbb{N})\mathbb{M}^{-1}$, so admissibility is equivalent to $\mathbb{M} - L\mathbb{N} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, and we have $(I-L\mathbb{D})^{-1} = \mathbb{M}(\mathbb{M}-L\mathbb{N})^{-1} =: \mathbb{M}_L$ and $\mathbb{D}_L := \mathbb{D}(I-L\mathbb{D})^{-1} = \mathbb{N}(\mathbb{M}-L\mathbb{N})^{-1}$.

If L is stabilizing, then $\mathbb{M}_L$ and $\mathbb{D}_L$ are (stable and) r.c., by Lemma 6.6.5, and this in turn implies that $(\mathbb{M}-L\mathbb{N})^{-1} = \mathbb{X}\mathbb{M}_L - \mathbb{Y}\mathbb{D}_L$ is stable, if $\mathbb{X}, \mathbb{Y} \in \mathrm{TIC}$ are s.t. $\mathbb{X}\mathbb{M} - \mathbb{Y}\mathbb{N} = I$. On the other hand, if $(\mathbb{M}-L\mathbb{N})^{-1}$ is stable, then so is $\mathbb{D}_L = \mathbb{N}(\mathbb{M}-L\mathbb{N})^{-1}$. □

Lemma 6.6.7 ($\mathbb{A} + \mathbb{B}\tau\widetilde{\mathbb{C}}$ [strongly] stable) *Assume that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}(U,H,\{0\})$ and $\widetilde{\mathbb{C}} \in \mathcal{B}(H, \mathrm{L}^2(\mathbf{R};U))$, and that $\mathbb{A} + \mathbb{B}\tau\widetilde{\mathbb{C}}$ is a C_0-semigroup.*

If $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is [[exponentially] strongly] stable, then $\mathbb{A} + \mathbb{B}\tau\widetilde{\mathbb{C}}$ is [[exponentially] strongly] stable.

Proof: Since $\mathbb{B}\tau\widetilde{\mathbb{C}}$ is bounded, the stable case is obvious. If $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is strongly stable, then so is $\mathbb{B}$, by Lemma 6.1.13, hence then $\mathbb{A} + \mathbb{B}\tau\widetilde{\mathbb{C}}$ is strongly stable.

If $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is exponentially stable, then $\mathbb{B}\tau$ is stable, by Lemma 6.1.10(a2), hence then $(\mathbb{A} + \mathbb{B}\tau\widetilde{\mathbb{C}})x_0 \in \mathrm{L}^2$ for all $x_0 \in H$, so that $\mathbb{A} + \mathbb{B}\tau\widetilde{\mathbb{C}}$ is exponentially stable, by Lemma A.4.5. □

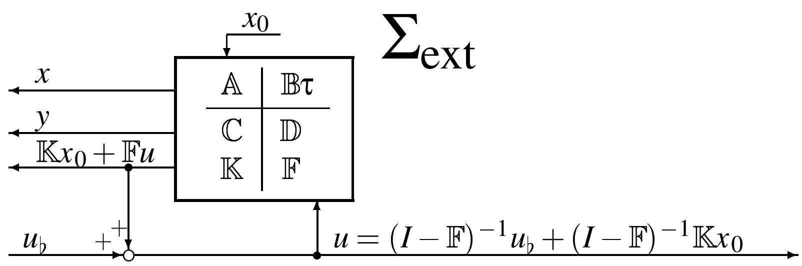


Figure 6.3: State feedback connection

From the above lemma we conclude the following:

Lemma 6.6.8 ($\mathbb{A}$ iff $\mathbb{A}_L$ stable) *Let L, Σ, and Σ_L be as in Proposition 6.6.2.*

(a) Assume that $\mathbb{B}$ and $L\mathbb{C}_L$ are stable. If $\mathbb{A}$ is [strongly] stable, then $\mathbb{A}_L$ is [strongly] stable.

(b) Assume that $\mathbb{B}_L L$ and $\mathbb{C}$ are stable. Then $\mathbb{A}$ is stable iff $\mathbb{A}_L$ is stable.

(c) Assume that $\mathbb{B}_L L$ or $L\mathbb{C}_L$ is stable. If $\mathbb{A}$ is exponentially stable, then $\mathbb{A}_L$ is exponentially stable.

Proof: (a) By (6.125), $\mathbb{A}_L - \mathbb{A} = \mathbb{B}\tau L\mathbb{C}_{\circlearrowleft}$, hence this follows from Lemma 6.6.7.

(b) By (6.125), $\mathbb{A}_L - \mathbb{A} = \mathbb{B}\tau(I - L\mathbb{D})^{-1}L\mathbb{C} = \mathbb{B}_L L\tau\mathbb{C}$, which is bounded, hence (b) holds.

(c) If $L\mathbb{C}_L$ is stable, then this follows from Lemma 6.6.7 as in (a). If $\mathbb{B}_L L$ is stable, then we apply the same for Σ^{d} and L^{d}. □

When Σ and Σ_L are I/O-stable, their stabilities are equivalent:

Corollary 6.6.9 (Σ iff Σ_L stable) *Let the assumptions of Proposition 6.6.2 hold. Let $I - L\mathbb{D} \in \mathcal{G}$TIC. Then Σ is [SOS-/strongly/exponentially] stable iff Σ_L is.*

Note that $\mathbb{D}, \mathbb{D}_L \in \mathrm{TIC} \Rightarrow I - L\mathbb{D} \in \mathcal{G}\mathrm{TIC}$, because $I + L\mathbb{D}_L = (I - L\mathbb{D})^{-1}$.

Proof: "Only if": The stable and SOS-stable case follow from (6.125); the strongly [exponentially] stable case follows from the stable case and Lemma 6.6.8(a)[(c)]. "If": Exchange the roles of Σ and Σ_L. □

Next we shall formulate and study stabilizability and detectability. A reader more familiar with matrix presentations of systems than (abstract) integral operators might wish to still think the systems as groups of generating operators ($\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$), which can be done in the regular case (and the formulae look the same, just "the font is changed", from $\mathbb{A}$ to A etc., as in Proposition 6.6.18).

A state feedback can be reduced to an output feedback as follows. The appropriate connection has been drawn in Figure 6.3.

Definition 6.6.10 (Admissible state feedback) *1. A pair* $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *is called an* admissible state feedback pair *for* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *if the extended system*

$$\Sigma_{\mathrm{ext}} := \left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\tau\\ \hline \mathbb{C} & \mathbb{D}\\ \mathbb{K} & \mathbb{F}\end{array}\right] \tag{6.132}$$

is a WPLS *and* $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty(U)$*, i.e.,* $L := \begin{bmatrix}0 & I\end{bmatrix}$ *is an admissible output feedback operator for* Σ_{ext}.

We often denote the corresponding closed loop system by (here $\mathbb{M} := (I - \mathbb{F})^{-1}$*)*

$$\Sigma_\flat = \left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat\tau\\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\\ \mathbb{K}_\flat & \mathbb{F}_\flat\end{array}\right] = \left[\begin{array}{c|c}\mathbb{A}+\mathbb{B}\tau\mathbb{M}\mathbb{K} & \mathbb{B}\mathbb{M}\tau\\ \hline \mathbb{C}+\mathbb{D}\mathbb{M}\mathbb{K} & \mathbb{D}\mathbb{M}\\ \mathbb{M}\mathbb{K} & \mathbb{M}-I\end{array}\right] \tag{6.133}$$

$$= \Sigma_{\mathrm{ext}}\begin{bmatrix}I & 0\\ -\mathbb{K} & I-\mathbb{F}\end{bmatrix}^{-1} = \Sigma_{\mathrm{ext}}\begin{bmatrix}I & 0\\ \mathbb{M}\mathbb{K} & \mathbb{M}\end{bmatrix} : \begin{bmatrix}x_0\\ u_\flat\end{bmatrix} \mapsto \begin{bmatrix}x\\ y\\ u-u_\flat\end{bmatrix}. \tag{6.134}$$

2. The pair $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *is a* stabilizing state feedback pair *if, in addition, L is a stabilizing output feedback operator for* Σ_{ext}*; we use prefices as in Definition 6.6.4 above, and add, in addition, the prefix "*[q.]r.c.*" (resp. suffix "*in $\mathcal{A}$*"), if* $\mathbb{N} := \mathbb{D}\mathbb{M}$ *and* $\mathbb{M}$ *are [q.]r.c. (resp.* $\mathbb{N}, \mathbb{M} \in \mathcal{A}$*; cf. Definition 6.2.4). The pair* $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *is* exponentially [q.]r.c.-stabilizing *for* Σ *if* $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *is exponentially stabilizing and* $\mathbb{N}$ *and* $\mathbb{M}$ *are exponentially [q.]r.c.; equivalently, if there is* $\omega > 0$ *s.t.* $\mathcal{T}_\omega\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *is [q.]r.c.-stabilizing for* $\mathcal{T}_\omega\Sigma$ *(see Remark 6.1.9).*

3. The system $\Sigma \in$ WPLS *is* stabilizable *if it has a stabilizing feedback pair; we use here prefices and suffices as above (e.g.,* strongly r.c.-stabilizable*).*

4. We call $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *stable iff* $\mathbb{K}$ *and* $\mathbb{F}$ *are stable. We call* $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *WR (resp. SR, UR, ...) iff* $\mathbb{F}$ *is WR (resp. SR, UR, ...).*

5. We call K_{c} *an* admissible compatible state feedback operator *for* Σ *if* $(K_{\mathrm{c}}, 0)$ *is a compatible pair for* $\left[\begin{smallmatrix}A & B\\ K & F\end{smallmatrix}\right]$*. We call* K_{w} *an* admissible WR state feedback operator *for* Σ *if* $\mathbb{F}$ *is WR and* $\left[\begin{smallmatrix}A & B\\ K_{\mathrm{w}} & 0\end{smallmatrix}\right]$ *generate* $\left[\begin{smallmatrix}A & B\\ K & F\end{smallmatrix}\right]$*. We use here prefices and suffices as above (see 2. and 4.).*

Thus, e.g., a stabilizing $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is *stabilizing in* MTIC for Σ iff $\mathbb{M}, \mathbb{D}\mathbb{M} \in$ MTIC, equivalently, iff $\left[\begin{smallmatrix}\mathbb{D}_\flat\\ \mathbb{F}_\flat\end{smallmatrix}\right] \in$ MTIC; an admissible $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is $\left[\begin{array}{c|c}\mathbb{C} & \mathbb{D}\end{array}\right]$-stabilizing for Σ iff $\mathbb{C}_\flat$ and $\mathbb{D}_\flat$ are stable. Of course, $\Sigma_{\mathrm{ext}} \in$ WPLS iff $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{K} & \mathbb{F}\end{smallmatrix}\right] \in$ WPLS. Thus, $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is admissible for Σ iff $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is admissible for $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\end{array}\right]$.

In the literature, notion "K is admissible for A (resp. Σ)" often means that $\left[\begin{smallmatrix}A\\ K\end{smallmatrix}\right]$ (resp. $\left[\begin{smallmatrix}A & B\\ C & *\\ K & *\end{smallmatrix}\right]$) generate a WPLS, i.e., that K "fits" to A (resp. to Σ). To avoid misinterpretation, we never use this convention, hence admissibility always contains the requirement that the corresponding feedback is well-posed, e.g., $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is admissible for Σ iff it "fits" to Σ *and* $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty$. Thus, we follow the convention of [S95]–[S98d].

If $\dim H < \infty$, then a reachable system is exponentially stabilizable, by p. 91 of [LR]. This is not the case in general: if $\mathbb{D} \in \mathrm{TIC}(U,Y)$, then the dual of (6.11) is stable and exactly 0-stabilizable (in infinite time) but not even optimizable (see Definition 6.7.3), hence not exponentially stabilizable.

Any bounded K is an admissible ULR state feedback operator:

Lemma 6.6.11 (Bounded *K*) *Let $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ and $\begin{bmatrix}K & F\end{bmatrix} \in \mathcal{B}(H \times U, U)$.*

Then $\begin{bmatrix}K & F\end{bmatrix}$ generate an admissible state feedback pair for Σ iff $I - F \in \mathcal{GB}(U)$. If $I - F \in \mathcal{GB}(U)$, then $(I-F)^{-1}K \in \mathcal{B}(H,U)$ is an ULR admissible state feedback operator for Σ, and corresponding closed-loop systems are identical modulo $E := (I-F)^{-1}$ (see (6.136)).

By Proposition 6.6.18(d), for $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\end{array}\right] \in \mathrm{WPLS}(U,H,\{0\})$, the closed-loop system corresponding to $K \in \mathcal{B}(H,U)$ is generated by $\left[\begin{smallmatrix}A+BK & B\\ K & 0\end{smallmatrix}\right]$.

Proof: By Lemma 6.3.16(c), $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{K} & \mathbb{F}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,U)$ and $\mathbb{F}$ is ULR. By Proposition 6.3.1(c), $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is admissible iff $I - F \in \mathcal{GB}(U)$.

Assume that $I - F \in \mathcal{GB}(U)$. Set $K' := EK$, where $E := (I-F)^{-1}$. Then K' is admissible and ULR, by the above. If $\left[\begin{array}{c|c}qK' & \mathbb{F}'\end{array}\right]$ is the corresponding state feedback pair, then

$$\mathbb{F}' = K'\mathbb{B}\tau = E(K\mathbb{B}\tau + F) - EF = E\mathbb{F} + I - E, \tag{6.135}$$

because $EF = E - I$, hence then Lemma 6.6.12 applies. □

A coordinate transform in the feedback signal does not essentially affect the system:

Lemma 6.6.12 (Making *F* zero) *Let $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$, Σ and $\Sigma_\flat$ be as in Definition 6.6.10. Let $E \in \mathcal{GB}$. Then also $\left[\begin{array}{c|c}E\mathbb{K} & E\mathbb{F} + I - E\end{array}\right]$ is an admissible state feedback pair for Σ (having same prefices and suffices as $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ does), and the corresponding closed-loop system is given by*

$$\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat E^{-1}\tau\\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat E^{-1}\\ \mathbb{K}_\flat & \mathbb{F}_\flat E^{-1} + E^{-1} - I\end{array}\right] \tag{6.136}$$

□

(We leave the easy verification of this and the following two lemmas to the reader.)

Thus, if $\mathbb{F}$ is WR and $E := I - F \in \mathcal{GB}(U)$ (this is necessary if $\mathbb{F}$ is UR or $\mathbb{F}, \mathbb{F}^* \in \mathrm{SR}$, by Proposition 6.3.1(a1)&(b1)), then we can take $F = 0$ w.l.o.g. In particular, any UR state feedback is equivalent (in the sense of the lemma) to a UR state feedback operator. We shall often use this fact.

The following is obvious:

Lemma 6.6.13 (Stable is stabilizable) *The pair $\left[\begin{array}{c|c}0 & 0\end{array}\right]$ is [IO-/SOS-/strongly/exponentially] r.c.-stabilizing for Σ iff Σ is [IO-/SOS-/strongly/exponentially] stable.*

Thus, an [IO-/SOS-/strongly/exponentially] stable system is [IO-/SOS-/strongly/exponentially] r.c.-stabilizable (but not vice versa). □

Naturally, analogous claims also hold for detectability and for joint stabilizability and detectability (Definition 6.6.21).

We can always use negative feedback to recover the original system:

Lemma 6.6.14 ($\left[\begin{array}{c|c} -\mathbb{K}_\flat & -\mathbb{F}_\flat \end{array}\right]$ for $\Sigma_\flat$) *Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, Σ and $\Sigma_\flat$ be as in Definition 6.6.10. Then $\left[\begin{array}{c|c} -\mathbb{K}_\flat & -\mathbb{F}_\flat \end{array}\right] = \left[\begin{array}{c|c} -\mathbb{M}\mathbb{K} & I-\mathbb{M} \end{array}\right]$ is an admissible state feedback pair for $\Sigma_\flat$, and the corresponding closed-loop system is Σ_{ext} with the added row $\left[\begin{array}{c|c} -\mathbb{K} & -\mathbb{F} \end{array}\right]$.* □

Remark 6.6.15 (Historical definitions of exponential stabilizability) *In most infinite-dimensional classes, one traditionally defines only exponential stabilizability (not stabilizability, strong stabilizability etc.) and uses a stronger definition: one requires the existence of a* bounded *($K \in \mathcal{B}(H,U)$) exponentially stabilizing state-feedback operator (see, e.g., [CZ]).*

For Pritchard–Salamon-systems (see Section 6.9), the definition is even stronger: $\Sigma_\flat$ must be exponentially stable also in the larger state space "$\mathcal{V}$", not merely in $H := \mathcal{W}$.

Our definition was adopted from [S98a]. However, in some articles on regular WPLSs (e.g., in [WR00]), one requires the existence of a SR exponentially stabilizing state-feedback operator; also this definition is stronger than our definition of exponential stabilizability (we do not know whether it is strictly stronger).

In the works of I. Lasiecka, R. Triggiani and others (e.g., in [FLT]), sometimes also "non-admissible" feedback is accepted, i.e., the map $u \mapsto u_L$ need not be well-posed; this is a special case of the setting of Definition 8.3.15 and Section 9.7 (see also (8.34)), Therefore, under that convention one usually treats only "the left column of $\Sigma_\flat$", which is well posed; any external input ("$u_\flat$"; e.g., disturbance or modeling error) might "explode" the system. This convention makes optimizability equivalent to exponential stabilizability. (Optimizability is the weakest reasonable extension of the classical concept "exponential stabilizability"; see Definition 6.7.3.)

By Theorem 9.2.12, all these definitions of exponential stability coincide for systems having B bounded (or $\mathbb{A}Bu_0 \in \mathrm{L}^1_{\text{loc}}(\mathbf{R}_+;H)$ for each $u_0 \in U$). Note also that by "stabilizability" one often means exponential stabilizability. □

All this applies also to (exponential) detectability (Definition 6.6.21) and estimatability, mutatis mutandis.

A state feedback pair induces a factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ of $\mathbb{D}$. However, most such factorizations do not correspond to state feedback.

It is not enough to find a right factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ to obtain an I/O-stabilizing feedback pair, there must also exist a suitable $\mathbb{K}$ for $\mathbb{F} := I - \mathbb{M}^{-1}$. Therefore, $\widehat{\mathbb{F}}$ must be analytic outside the spectrum of A (this is most obvious when $\widehat{\mathbb{F}} = K(s-A)^{-1}B + F$ is rational; in the general case one should use (6.40) with $\widehat{\mathbb{D}} \mapsto \widehat{\mathbb{F}}$ and $C \mapsto K$ on some right half-plane and (possibly) analytic extension elsewhere). Hence, in the finite-dimensional case, the McMillan degree of $\mathbb{F}$ must not exceed the dimension of the state space H. Thus, only a small fraction of all right factorizations of $\mathbb{D}$ correspond to I/O-stabilizing state feedback pairs.

The concept "[q.]r.c.-stabilizability" is equivalent to stabilizability by such a pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ that the right factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ mentioned above is a [q.]r.c.f. Since this concept is new, we shall look at it more closely below.

In Chapters 8.3–12.3, the importance of this concept will become obvious; e.g., the Popov operator of a sufficiently regular strongly stable system has a spectral factorization iff the corresponding Riccati equation has a q.r.c.-stabilizing solution, and under suitable assumptions such solutions lead to solutions of several control problems also in the unstable case. Also Remark 9.9.9 explains the advantages of q.r.c.-stabilizability in optimal control; some sufficient conditions for q.r.c.-stabilizability are given in Corollary 6.7.16.

By definition, [q.]r.c.-stabilization means such a stabilization that the map $\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right] : u_\flat \mapsto \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ (see (6.133)–(6.134) or Figure 6.3) becomes [quasi–]left invertible over TIC. In particular, for such a pair the map $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \mapsto u_\flat$ is well-defined and stable.

Therefore, to construct a system that is stabilizable but not q.r.c.-stabilizable, we let $\widehat{\mathbb{B}}$ have a pole (at 1) that is not shared by $\widehat{\mathbb{D}}$, so that we must have $\widehat{\mathbb{M}}(1) = 0$ to make $\mathbb{B}_\flat := \mathbb{B}\mathbb{M}$ stable, in which case $\widehat{\mathbb{N}} := \widehat{\mathbb{D}}\widehat{\mathbb{M}}$ also has a zero at 1, i.e., $\mathbb{N}$ and $\mathbb{M}$ are not q.r.c.:

Example 6.6.16 (Exponentially stabilizable but not q.r.c.-stabilizable) For the system $\Sigma := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ generated by $\left[\begin{array}{c|c} 1 & 1 \\ \hline 0 & 1 \end{array}\right]$, a state feedback operator $K \in \mathbf{C}$ results in $\widehat{\mathbb{M}}(s) = (s-1)/(s-1-K) = \widehat{\mathbb{N}}(s)$, $A + BK = 1 + K$, hence only $K = 0$ is q.r.c.-I/O-stabilizing (even q.r.c.-SOS-stabilizing), whereas $K < -1$ is exponentially stabilizing, $K = -1$ is $\left[\begin{smallmatrix} \mathbb{A} \\ \mathbb{C} \end{smallmatrix}\right]$-stabilizing (but not I/O-stabilizing), and $K > -1, K \neq 0$ is only output-stabilizing. In particular, this system is not q.r.c.-stabilizable (see Example 8.4.4 of [Sbook] for a less trivial example). This follows from the fact that $\mathbb{A}$ has an unobservable pole. $\triangleleft$

As above, lack of q.r.c.-stabilizability typically corresponds to situations where the semigroup has unobservable poles. Conversely, one can often make a stabilizing pair r.c.-stabilizing by canceling the common zeros of $\widehat{\mathbb{N}}$ and $\widehat{\mathbb{M}}$. Indeed, from Lemma 6.5.4 one observes that in the finite-dimensional case, $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a (q.)r.c.f. iff the zeros of $\widehat{\mathbb{M}}$ coincide with the poles of $\widehat{\mathbb{D}}$, up to the multiplicity. Thus, then (q.)r.c.-I/O-stabilization means I/O-stabilization with a minimal number of zeros. (See also Lemma 6.5.3(d).) The infinite-dimensional situation is similar.

For an exponentially stabilizable and detectable system (e.g., a minimal finite-dimensional system), any I/O-stabilizing state feedback pair is exponentially q.r.c.-stabilizing, by Theorem 6.7.15(c1). This is one of the reasons why [q.]r.c.-stabilizability is not as important in the finite-dimensional case.

We do not know whether there is a SOS-stabilizable system that is not q.r.c.-SOS-stabilizable (or a (well-posed) right factorization that cannot be reduced to a q.r.c.f.).

For a stable system, any stable, stabilizing state feedback pair is [q.]r.c.-stabilizing (and conversely, see (c)). This and related results are given below:

Lemma 6.6.17 (Stable case: $\mathbb{F} \in$ TIC $\Leftrightarrow$ r.c.-stab.) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in$ WPLS.

(a) Let $\mathbb{D}$ *be stable, and let* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *I/O-stabilize* Σ*. Then the following are equivalent:*

(i) $\mathbb{F}$ *is stable;*
(ii) $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is r.c.-I/O-stabilizing for* Σ*;*
(iii) $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is q.r.c.-I/O-stabilizing for* Σ*;*
(iv) $I - \mathbb{F} \in \mathcal{G}$TIC.

Moreover, $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is [q.]r.c.-SOS-stabilizing iff it is stable (i.e., iff* $\mathbb{K}$ *and* $\mathbb{F}$ *are stable) and* $\Sigma \in$ SOS.

(b) If $\Sigma \in$ SOS*, then a SOS-stabilizing feedback pair for* Σ *is stable iff it is [q.]r.c.-SOS-stabilizing.*

(c) Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *be a stable I/O-stabilizing feedback pair for* Σ*, and let* $\mathbb{D}$ *be stable.*

Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is r.c.-I/O-stabilizing. Moreover, it is r.c.-SOS-stabilizing iff* $\Sigma \in$ SOS*, and it is [strongly/exponentially] r.c.-stabilizing iff* Σ *is [strongly/exponentially] stable.*

(d) Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *be admissible for* $\Sigma \in$ SOS*. Then* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is r.c.-SOS-stabilizing iff it is stable and* $(I-\mathbb{F})^{-1} \in$ TIC.

Proof: (a) If $\mathbb{F} \in$ TIC, then $\mathbb{M} := (I-\mathbb{F})^{-1} \in \mathcal{G}$TIC, which makes $\mathbb{N} := \mathbb{D}(I-\mathbb{F})^{-1} \in$ TIC and $\mathbb{M} := (I-\mathbb{F})^{-1} \in$ TIC r.c. (see Definition 6.6.10). Conversely, if $\mathbb{M}$ and $\mathbb{N}$ are q.r.c., then $\mathbb{M}^{-1} \in$ TIC, by Lemma 6.5.6(b), hence then $\mathbb{F} \in$ TIC.

Let $\mathbb{F}$ be stable. The pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is [q.]r.c.-SOS-stabilizing iff $\mathbb{K}_\flat := \mathbb{M}\mathbb{K}$ and $\mathbb{C}_\flat = \mathbb{C} + \mathbb{N}\mathbb{K}$ are stable. But $\mathbb{M}\mathbb{K}$ is stable iff $\mathbb{K} := \mathbb{M}^{-1}\mathbb{M}\mathbb{K}$ is (because $\mathbb{M} \in \mathcal{G}$TIC). If $\mathbb{K}$ is stable, then $\mathbb{C}$ is stable iff $\mathbb{C}_\flat$ is.

(b) This follows from (a).

(c) The first two claims are from (a), the last one follows from Corollary 6.6.9.

(d) This follows from (a) and (b). □

If B and C are bounded, then it easy to verify that the generators of (6.125) are given by $\left[\begin{array}{c|c} A+BL(I-DL)^{-1}C & B(I-LD)^{-1} \\ (I-DL)^{-1}C & D(I-LD)^{-1} \end{array}\right]$, i.e., formally we just (remove τ and) replace each operator by its feedthrough operator; an analogous comment applies to other forms of feedback too. For unbounded B and C, the situation is similar but much trickier, as obvious from the following important results:

Proposition 6.6.18 (Generators $\left[\begin{array}{c|c} A_L & B_L \\ C_L & D_L \end{array}\right]$ of Σ_L) *Make the assumptions and use the notation of Proposition 6.6.2. Then (a1)–(c4) hold:*

(a1) $\mathrm{Dom}(A_L) \subset H_{B_L} = H_B$ *and* $H^*_{C_L} = H^*_C$.

(a2) For $\mathrm{Re}\, s > \max\{\omega_A, \omega_{A_L}\}$ *we have*

$$(s-A_L)^{-1} - (s-A)^{-1} = (s-A)^{-1}BLC_L(s-A_L)^{-1} \tag{6.137}$$

$$= (s-A_L)^{-1}B_LLC(s-A)^{-1} \in \mathcal{B}(H, H_B). \tag{6.138}$$

(a3) The generators A and A_L are given by

$$A_L = A + BLC_L : \mathrm{Dom}(A_L) \to H, \tag{6.139}$$

$$A = A_L - B_LLC : \mathrm{Dom}(A) \to H. \tag{6.140}$$

(b1) **(Σ_L)** *Let Σ have compatible output operator pair (C_c, D) s.t. $I - LD$ or $I - DL$ is left-invertible.*

Then both $I - LD$ and $I - DL$ are left-invertible, Σ_L has compatible generators (here $(I-DL)^{-1}_{\mathrm{left}}$ may be any left inverse of $I - DL$)

$$\left[\begin{array}{c|c} A_L & B_L \\ \hline (C_L)_c & (D_L)_c \end{array}\right] = \left[\begin{array}{c|c} A + BLC_L & B_L \\ \hline (I-DL)^{-1}_{\mathrm{left}}C_c & (I-DL)^{-1}_{\mathrm{left}}D \end{array}\right] \text{ on } \mathrm{Dom}(A_L) \times U, \tag{6.141}$$

$A = A_L - B_LLC_c \in \mathcal{B}(H_B, V)$ $(C_L)_c = (I-DL)^{-1}_{\mathrm{left}}C_c \in \mathcal{B}(H_B, Y)$ and $B = B_L(I - LD) \in \mathcal{B}(U, V)$ (see the proof for the identification of $B_LU \subset H^L_{-1}$ and $BU \subset H_{-1}$; here V is a Banach space s.t. $H \underset{c}{\subset} V \underset{c}{\subset} H_{-1}$ and $V \underset{c}{\subset} H^L_{-1}$).

If $I - LD \in \mathcal{GB}$, then $B_L = B(I-LD)^{-1} \in \mathcal{B}(U, V)$ and $A_L = A + BL(C_L)_c \in \mathcal{B}(H_B, V)$.

(b2) Let Σ be WR and let $I - LD$ or $I - DL$ be left-invertible. Then (b1) applies with $(C_c, D) = (C_w, D)$.

If, in addition, $I - LD \in \mathcal{GB}$, then $B^*_L = (I - D^*L^*)^{-1}B^*_w$ on $\mathrm{Dom}(A_L)$.

(b3) Let Σ be SR. Then $I - LD$ and $I - DL$ are left-invertible, and (b1) and (b2) apply with $(C_c, D) = (C_s, D)$. Moreover, then Σ_L is SR *iff $I - LD \in \mathcal{GB}$.*

If $I - LD \in \mathcal{GB}$, then

$$\left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right] = \left[\begin{array}{c|c} A + BLC_L & B(I-LD)^{-1} \\ \hline (I-DL)^{-1}C_s & (I-DL)^{-1}D \end{array}\right] \in \mathcal{B}(H_B \times U, V \times Y). \tag{6.142}$$

(see the proof for the identification of $B_L[U]$ and $B[U]$), $A_L = A + B(I - LD)^{-1}LC_s \in \mathcal{B}(H_B, V)$, $B, B_L \in \mathcal{B}(U, V)$, where V is a Banach space s.t. $H \underset{c}{\subset} V$ densely, $V \underset{c}{\subset} H_{-1}$, $V \underset{c}{\subset} H^L_{-1}$. See also (c4).

(b4) Let Σ be UR. Then $I - LD \in \mathcal{GB}$, Σ_L is UR and (6.142) holds.

(c1) Let $\mathbb{D}$ be SR. Then $\mathbb{D}_L$ is SR iff $I - DL \in \mathcal{GB}(Y)$.

(c2) Let $\mathbb{D}L$ *be WR and let* $I-DL$ *have a left inverse* $(I-DL)^{-1}_{\text{left}}$*. Then* $(C_L)_{\text{s}} \subset (I-DL)^{-1}_{\text{left}} C_{\text{w}}$*.*

(c3) Let $\mathbb{D}L$ *be SR. Then* $I-DL$ *has a left inverse* $(I-DL)^{-1}_{\text{left}}$*, and* $(C_L)_{\text{s}} \subset (I-DL)^{-1}_{\text{left}} C_{\text{s}}$*.*

(c4) Let $\mathbb{D}$ *be SR and* $I-DL \in \mathcal{GB}(Y)$*. Then* $(C_L)_{\text{s}} = (I-DL)^{-1}C_{\text{s}}$*, and* $(B_L^*)_{\text{s}} = (I-D^*L^*)^{-1}B_{\text{s}}^*$*.*

In particular, $\mathrm{Dom}((C_L)_{\text{s}}) = \mathrm{Dom}(C_{\text{s}})$ *and* $\mathrm{Dom}((B_L^*)_{\text{s}}) = \mathrm{Dom}(B_{\text{s}}^*)$ *(with equivalent norms).*

(c5) Let $(\mathbb{D}L)^{\text{d}}$ *be SR and let* $I-DL$ *have a left inverse. Then* $I-DL \in \mathcal{GB}(Y)$ *and* $(C_L)_{\text{w}} = (I-DL)^{-1}C_{\text{w}}$*.*

(c6) Let $\mathbb{D}$ *be SR. Assume that* $\mathbb{D}L \in \mathrm{MTIC}_\infty$*. Then* $I-DL \in \mathcal{GB}(Y)$*,* $(C_L)_{\text{L,s}} = (I-DL)^{-1}C_{\text{L,s}}$*, and* $(B_L^*)_{\text{L,s}} = (I-D^*L^*)^{-1}B_{\text{L,s}}^*$*.*

In particular, $\mathrm{Dom}((C_L)_{\text{L,s}}) = \mathrm{Dom}(C_{\text{L,s}})$ *and* $\mathrm{Dom}((B_L^*)_{\text{L,s}}) = \mathrm{Dom}(B_{\text{L,s}}^*)$ *(with equivalent norms).*

Make the assumptions and use the notation of Definition 6.6.10. Then

(d1) $(\boldsymbol{\Sigma_\flat})$ *Let* $(\begin{bmatrix} C_{\text{c}} \\ K_{\text{c}} \end{bmatrix}, \begin{bmatrix} D \\ F \end{bmatrix})$ *be a compatible pair for* Σ_{ext}*. Assume that* $X := I-F$ *has a left inverse* M*. Then* $\Sigma_\flat$ *is compatible with*

$$\left[\begin{array}{c|c} A_\flat & B_\flat \\ \hline C_\flat & D_\flat \\ K_\flat & F_\flat \end{array}\right] = \left[\begin{array}{c|c} A+B_\flat K_{\text{c}} & B_\flat \\ \hline C_{\text{c}}+DMK_{\text{c}} & DM \\ MK_{\text{c}} & M-I \end{array}\right] \text{ on } H_B \times U, \tag{6.143}$$

$A_\flat = A+BMK_{\text{c}}$ *on* $\mathrm{Dom}(A)$*, and* $B = B_\flat X$ *on* U *(the remarks of (b1) apply). By (a),* $\mathrm{Dom}(A_\flat) \subset H_{B_\flat} = H_B$*.*

Moreover, $\widehat{\mathbb{X}}(s) = X - K_{\text{c}}(s-A)^{-1}B$*, and* $\widehat{\mathbb{M}}(s) = M + MK_{\text{c}}(s-A_\flat)^{-1}B_\flat$*. If* $\mathbb{X}$ *is WR, then* $\mathbb{K}_{\circlearrowleft} x_0 = MK_{\text{w}}\mathbb{A}_{\circlearrowleft}(\cdot)x_0$ *a.e.; if* $\mathbb{X}$ *is SR, then* $\mathbb{K}_{\circlearrowleft} x_0 = MK_{\text{s}}\mathbb{A}_{\circlearrowleft}(\cdot)x_0$ *a.e., for any* $x_0 \in H$*.*

If $X \in \mathcal{GB}$*, then* $B_\flat = BM$ *on* U*, and* $A_\flat = A+BMK_{\text{c}}$ *on* H_B*.*

(d2) $((\boldsymbol{K_{\text{c}}}, \mathbf{0}))$ *Let* $(\begin{bmatrix} C_{\text{c}} \\ K_{\text{c}} \end{bmatrix}, \begin{bmatrix} D \\ 0 \end{bmatrix})$ *be a compatible pair for* Σ_{ext}*. Then* $\Sigma_\flat$ *is compatible with*

$$\left[\begin{array}{c|c} A_\flat & B_\flat \\ \hline C_\flat & D_\flat \\ K_\flat & F_\flat \end{array}\right] = \left[\begin{array}{c|c} A+BK_{\text{c}} & B \\ \hline C_{\text{c}}+DK_{\text{c}} & D \\ K_{\text{c}} & 0 \end{array}\right] \text{ on } H_B \times U, \tag{6.144}$$

(the remarks of (b1) apply). Moreover, (d1) applies (with $M = I = X$*).*

In particular, $-K_{\text{c}}$ *is admissible for* $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$*, and the resulting closed-loop system is* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ -\mathbb{K} & -\mathbb{F} \end{array}\right]$*.*

$$\begin{tikzcd} & \Sigma \arrow[dl, "K"'] \arrow[dr, "K' = K + K_{\natural_{\mathrm{w}}}"] & \\ \Sigma_\flat \arrow[rr, "K_\natural"'] & & \Sigma_\circlearrowleft, \left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}'_\circlearrowleft & \mathbb{F}'_\circlearrowleft \end{array}\right] \end{tikzcd}$$

Figure 6.4: The setting of Proposition 6.6.18(f)

(d3) Assume that $\mathbb{D}, \mathbb{F} \in \mathrm{WR}$ *and* $F = 0$. *Then* $\Sigma_\flat$ *has compatible generators*

$$\left[\begin{array}{c|c} A_\flat & B_\flat \\ \hline C_\flat & D_\flat \\ K_\flat & F_\flat \end{array}\right] = \left[\begin{array}{c|c} A + BK_{\mathrm{w}} & B \\ \hline C_{\mathrm{w}} + DK_{\mathrm{w}} & D \\ K_{\mathrm{w}} & 0 \end{array}\right] \text{ on } H_B \times U. \tag{6.145}$$

Moreover, then $(C_\flat)_{\mathrm{s}} \subset C_{\mathrm{w}} + DK_{\mathrm{w}}$, $(K_\flat)_{\mathrm{s}} \subset K_{\mathrm{w}}$, $\widehat{\mathbb{X}}(s) = I - K_{\mathrm{w}}(s-A)^{-1}B$, $\widehat{\mathbb{M}}(s) = I + K_{\mathrm{w}}(s - A - BK_{\mathrm{w}})^{-1}B$, *and* $\mathbb{K}_\circlearrowleft x_0 = K_{\mathrm{w}} \mathbb{A}_\circlearrowleft(\cdot) x_0$ *a.e.*

(d4) $((K_{\mathrm{s}}, 0))$ *Assume that* $\mathbb{D} \in \mathrm{WR}$, $\mathbb{F} \in \mathrm{SR}$ *and* $F = 0$. *Then* $\mathbb{F}_\flat$ *is SR,* $\Sigma_\flat$ *is WR with generators (6.145) (which holds also with* K_{s} *in place of* K_{w}*). Moreover, then* $(C_\flat)_{\mathrm{w}} = C_{\mathrm{w}} + DK_{\mathrm{s}}$ *on* H_B, *and* $(K_\flat)_{\mathrm{s}} = K_{\mathrm{s}}$.

If $D = 0$, *then* $(C_\flat)_{\mathrm{w}} = C_{\mathrm{w}}$. *If* $\mathbb{D}$ *is SR, then* $\Sigma_\flat$ *is SR and* $(C_\flat)_{\mathrm{s}} = C_{\mathrm{s}} + DK_{\mathrm{s}}$.

(e) Assume that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is SR and that* $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{S} & \mathbb{T} \end{array}\right] \in \mathrm{WPLS}(U, H, Z)$. *If* $\mathbb{T}$ *is WR (resp. SR), then the weak (resp. strong) Weiss extensions of* $S : \mathrm{Dom}(A_\flat) \to Z$ *and* $S_{\mathrm{w}}|_{\mathrm{Dom}(A)} : \mathrm{Dom}(A) \to Z$ *are identical on* H_B.

(f) Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *be SR and* $F = 0$. *Let* $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ *be WR and admissible for* $\Sigma_\flat$ *with closed-loop system* $\Sigma_\circlearrowleft$ *and* $F_\natural = 0$.

Then $K' := K + K_{\natural_{\mathrm{w}}}|_{\mathrm{Dom}(A)}$ *is WR and admissible for* Σ, *and* $\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \end{array}\right]$ *forms the top two rows of the corresponding closed-loop system (and* $K'_{\mathrm{w}} = K_{\mathrm{s}} + K_{\natural_{\mathrm{w}}}$ *on* H_B*, by (e)). The pair generated by* K' *and corresponding closed-loop maps are given by*

$$\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}_\natural + \mathbb{X}_\natural \mathbb{K} & \mathbb{F} + \mathbb{F}_\natural - \mathbb{F}_\natural \mathbb{F} \end{array}\right], \tag{6.146}$$

$$\left[\begin{array}{c|c} \mathbb{K}'_\circlearrowleft & \mathbb{F}'_\circlearrowleft \end{array}\right] = \left[\begin{array}{c|c} \mathbb{M}\mathbb{K}_\circlearrowleft + \mathbb{K}_\flat & \mathbb{M}\mathbb{M}_\natural - I \end{array}\right]. \tag{6.147}$$

Thus, if K *and* $K_\natural$ *have some strong or uniform regularity property, then so do* K' *and* $K'_\circlearrowleft$.

(g) We have $K' = K''$ *(on* H_B*)* $\Leftrightarrow \mathbb{K}'_\flat = \mathbb{K}''_\flat \Leftrightarrow \left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}'' & \mathbb{F}'' \end{array}\right]$ *for any compatible admissible state feedback operators* K' *and* K''.

Recall that $(C_L)_{\mathrm{s}} \subset (I - DL)^{-1}_{\mathrm{left}} C_{\mathrm{w}}$ means that the latter is an extension of the former, i.e., that $\mathrm{Dom}((C_L)_{\mathrm{s}}) \subset \mathrm{Dom}((I - DL)^{-1}_{\mathrm{left}} C_{\mathrm{w}})$ and $(C_L)_{\mathrm{s}} x_0 = (I - DL)^{-1}_{\mathrm{left}} C_{\mathrm{w}} x_0$ for all $x_0 \in \mathrm{Dom}((C_L)_{\mathrm{s}})$. In (c2)–(c5), "$DL$" denotes the feedthrough operator of $\mathbb{D}L \in \mathrm{WR}$; obviously, "$DL$"$= D \cdot L$ whenever $\mathbb{D} \in \mathrm{WR}$ (so that "D" exists).

By (c4), we have $\mathrm{Dom}((C_L)_{\mathrm{s}}) = (I - DL)^{-1}\,\mathrm{Dom}(C_{\mathrm{s}})$ despite the fact that the strong extensions are computed w.r.t. different semigroup generators (A_L and A, respectively); therefore, $(K_\flat)_{\mathrm{s}} = K_{\mathrm{s}}$ in (d4) (see the proofs of (c3) and (c2) for details; also (e) relates to this).

It is not known, whether, e.g., (c4) holds with $C_{\mathrm{L,s}}$ in place of C_{s} etc. (with the exception of (c6); do not mix the generator C_L of $\mathbb{C}_L$ with the strong Lebesgue extension $C_{\mathrm{L,s}}$ of C). However, when $\mathbb{D}$ is SR, we have $C_{\mathrm{s}} = C_{\mathrm{L,s}}$ on H_B, by Proposition 6.2.8(c1)&(c4)&(d1). Analogously, most of the above proposition can be written with Lebesgue extensions.

Recall also that compatible output operators (e.g., $(C_L)_{\mathrm{c}}$ and $(D_L)_{\mathrm{c}}$) are not unique in general and may differ from $(C_L)_{\mathrm{w}}$ and $(D_L)_{\mathrm{w}}$ even if Σ_L is WR (but $(C_L)_{\mathrm{w}}$ and $(D_L)_{\mathrm{w}}$ are unique).

The closed-loop system Σ_L need not be WR in (b2):

Example 6.6.19 If we set $\widetilde{\mathbb{D}} := \begin{bmatrix} 0 & \mathbb{D} \\ \mathbb{D}^{\mathrm{d}} & 0 \end{bmatrix} \in \mathrm{TIC} \cap \mathrm{WHPR}$ and $L := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$, where $\mathbb{D} \in \mathrm{TIC} \cap \mathrm{SHPR}$ is as in Example 6.2.6, then $\widetilde{\mathbb{D}}_L = \begin{bmatrix} \mathbb{D}\mathbb{D}^{\mathrm{d}} & \mathbb{D} \\ \mathbb{D}^{\mathrm{d}} & 0 \end{bmatrix} \notin \mathrm{WR}$, although $I - L\widetilde{D} = I \in \mathcal{GB}$. Thus, if Σ is any realization of $\widetilde{\mathbb{D}}$, then (b1) and (b2) apply but Σ_L is not regular. $\triangleleft$

This fact and the need for closed-loop generators for Riccati equation theory was our main motivation to introduce compatible generators in [Mik97a]. Cf. also Lemma 6.3.11.

Proof of Proposition 6.6.18: (a1) By [S97b, Proposition 37], we have $H_B = H_{B_L} := (s - A_L)^{-1}[H + BU] \supset \mathrm{Dom}(A_L)$, and $H^*_{C_L} = H^*_C$ (for the stable case; the proofs apply in the general case too; alternatively, use shifting (see Remark 6.1.9) or see [Sbook] for the general case).

(a2) This is Proposition 6.6 of [W94b] (just take the Laplace transform of the identity $\mathbb{A}_L - \mathbb{A} = \mathbb{B}\tau L\mathbb{C}_L = \mathbb{B}_L\tau L\mathbb{C}$, which follows from (6.125)).

(b1) The (left-)invertibility of $I - DL$ is equivalent to that of $I - LD$, by Lemma A.1.1(f6). The other claims are given in Theorem 6.5.1 of [Sbook] (originally mostly from [Mik97a] except for B_L). We sketch the proof below.

1° *A_L, $(C_L)_{\mathrm{c}}$ and $(D_L)_{\mathrm{c}}$:* The formula for A was given in (a3). Set $\omega := \max\{\omega_A, \omega_{A_L}\}$. By Theorem 6.2.11(c1), Lemma 6.3.10(a) and (6.125), we have

$$\widehat{\mathbb{C}}(s) = C(s - A),\ \ \widehat{\mathbb{C}}_L(s) = C_L(s - A_L)^{-1}, \tag{6.148}$$

$$\widehat{\mathbb{D}}(s) = D + C_{\mathrm{c}}(s - A^{-1})B \ \text{ and } \ (I - \widehat{\mathbb{D}}L)\widehat{\mathbb{C}}_L = \widehat{\mathbb{C}} \ \ (s \in \mathbf{C}^+_\omega). \tag{6.149}$$

Thus, for any $s \in \mathbf{C}^+_\omega$, we have

$$(I-DL)C_L(s-A_L)^{-1} = C_c\left((s-A)^{-1} + (s-A)^{-1}BLC_L(s-A_L)^{-1}\right) \quad (6.150)$$

$$= C_c(s-A_L)^{-1} \in \mathcal{B}(H,Y), \quad (6.151)$$

by (a3), hence $C_L = (I-DL)^{-1}_{\text{left}}C_c$ on $\text{Ran}(s-A_L) = \text{Dom}(A_L)$. Since $(s-A_L)^{-1}B_L = (s-A)^{-1}B(I-L\widehat{\mathbb{D}}(s))^{-1}$, by (6.125) and Theorem 6.2.11, we obtain that

$$D + C_c(s-A_L)^{-1}B_L(I-L\widehat{\mathbb{D}}(s)) = (I-DL)\widehat{\mathbb{D}}(s), \quad \text{hence} \quad (6.152)$$

$$(I-DL)^{-1}(D+C_c(s-A_L)^{-1}B_L) = \widehat{\mathbb{D}}(s)(I-L\widehat{\mathbb{D}}(s))^{-1} = \widehat{\mathbb{D}}_L(s), \quad (6.153)$$

by (6.125), so that $(I-DL)^{-1}(C_c, D_c)$ is a compatible pair for Σ_L (we can use for $(C_L)_c := (I-DL)^{-1}C_c$ the same W as for C_c), by Definition 6.3.10. (An alternative proof would have used Lemma 6.3.12.)

2° B_L: Choose $\alpha \in \sigma(A)^c \cap \sigma(A_L)^c$, so that $(\alpha - A) \in \mathcal{GB}(H, H_{-1})$ and $(\alpha - A_L)\mathcal{GB}(H, H^L_{-1})$, where $H_{-1} := \text{Dom}(A^*)^*$, $H^L_{-1} := \text{Dom}(A_L^*)^*$, by Lemma A.4.6. Set

$$V := (\alpha - A)W, \quad \|(\alpha - A)x_0\|_V := \|x_0\|_W, \quad (6.154)$$

$$V_L := (\alpha - A_L)W, \quad \|(\alpha - A_L)x_0\|_{V_L} := \|x_0\|_W, \ \ (x_0 \in W); \quad (6.155)$$

$$J := (\alpha - A_L + B_L L C_c)(\alpha - A)^{-1} \in \mathcal{B}(V, V_L) \quad (6.156)$$

$$J_L := (\alpha - A - BL(C_L)_c)(\alpha - A_L)^{-1} \in \mathcal{B}(V_L, V). \quad (6.157)$$

It follows that $H \underset{c}{\subset} V \underset{c}{\subset} H_{-1}$, $H \underset{c}{\subset} V_L \underset{c}{\subset} H^L_{-1}$, and that V and V_L are independent of α (obviously, $H = (\alpha - A)H_1 \subset V$ and $V \underset{c}{\subset} H_{-1}$; by the resolvent equation, V is independent of α as a set; by Lemma A.3.6, also the topology of V is independent of α, by Lemma A.4.6).

Assuming that $(I-DL)$ is left-invertible, a direct computation (see [Sbook]) shows that $J_L J = I \in \mathcal{B}(V)$, i.e., that $(\alpha - A)^{-1}J_L J(\alpha - A) = I \in \mathcal{B}(W)$. Since J is boundedly left-invertible, we have $J \in \mathcal{GB}(V, V')$, were $V' := J[V] \subset V_L$. Thus, J defines an embedding $V \to V_L$ with range V'. By (a3), $J_{|H} = I = J_{L|H}$, hence we can and will identify V to V' (note that this may alter the norm of V from the one defined above, but the new norm is equivalent, i.e., the topology of V is unchanged).

Then J becomes the identity on V, hence $\alpha - A = \alpha - (A_L - B_L L C_c) \in \mathcal{B}(W, V)$ (use the definition of J). By Lemma 6.3.10(g), it follows that $B \in$

$\mathcal{B}(U,V)$. From equation

$$(s-A_L)^{-1}B_L = (s-A)^{-1}B(I-L\mathbb{D}(\alpha))^{-1} \in \mathcal{B}(U,W), \tag{6.158}$$

one can compute that $B_L(I-LD) = B$ (and $B_L \in \mathcal{B}(U,V_L)$, by Lemma 6.3.10(g)).

If $I-LD \in \mathcal{GB}(U)$, then we get the final two claims by exchanging the roles of Σ_L and Σ (note that then $JJ_L = I_{V_L}$, hence then $V' = V_L$, i.e., $V = V_L$).

3° *Remark:* We do not know whether necessarily $(C_L)_{\mathrm{w}} = (C_L)_{\mathrm{c}} := (I-DL)^{-1}_{\mathrm{left}}C_{\mathrm{c}}$, or equivalently (by Lemma 6.3.10(d3)), $D_L = (D_L)_{\mathrm{c}}$, when Σ and Σ_L are WR and $I-DL \in \mathcal{GB}(Y)$. Of course, this is not the case in (6.142), where "WR" is replaced by "SR".

(b2) This follows from (b1), except for $B_L^* = (I-D^*L^*)^{-1}B_{\mathrm{w}}^*$, which is from (c2) applied to $\Sigma^{\mathrm{d}} \in \mathrm{WR}(Y,H,U)$ and L^*, giving $(B_L^*)_{\mathrm{s}} = (I-D^*L^*)^{-1}B_{\mathrm{w}}^*$ on $\mathrm{Dom}((B_L^*)_{\mathrm{s}})$.

(As noted below the proposition, Σ_L need not be WR even if Σ is WR and $I-LD \in \mathcal{GB}(U)$.)

(b3) (Note that $\mathrm{Dom}(A) =: H_1$ and $\mathrm{Dom}(A_L)$ are in general proper subsets of H_B; the notion "$A \in \mathcal{B}(H_B,V)$" in (b3) refers to $A|_{H_B}$, where A stands for, e.g., its extension to H (see Lemma 6.1.16(a)). The same applies for A_L.)

We prove the case $I-DL \in \mathcal{GB}(Y)$ below; the rest follows from (b2) and Proposition 6.3.1(a1).

1° *The proof:* We fix $\alpha \in \sigma(A)^c \cap \sigma(A_L)^c$, so that $(\alpha - A) \in \mathcal{GB}(H_k, H_{k-1})$ ($k \in \mathbf{Z}$). Let W be the closure of H_1 in $\mathrm{Dom}(C_{\mathrm{s}})$ (equivalently, in $\mathrm{Dom}((C_L)_{\mathrm{s}})$, by (c4)). Since H_1 is dense in $\mathrm{Dom}(C_{\mathrm{L,s}})$, and $\mathrm{Dom}(C_{\mathrm{L,s}}) \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{s}})$, we have

$$H_1 \underset{c}{\subset} H_B \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{L,s}}) \underset{c}{\subset} W \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{s}}) \underset{c}{\subset} H \underset{c}{\subset} V := (\alpha - A)W \underset{c}{\subset} H_{-1}. \tag{6.159}$$

Analogously, we see that W is the closure of $\mathrm{Dom}(A)$ in $\mathrm{Dom}((C_L)_{\mathrm{s}}) = \mathrm{Dom}(C_{\mathrm{s}})$. Now H is dense in V, because H_1 is dense in W. Therefore, J in (b1) becomes the unique continuous extension of $I : H \to H$ to $V \to V_L$.

This way, we have $C_{\mathrm{c}} := C_{\mathrm{s}}|_W$, $(I-DL)^{-1}C_{\mathrm{c}} = (C_L)_{\mathrm{s}}|_W$, $(I-DL)^{-1}D$ becomes the feedthrough operator of $\mathbb{D}_L$, and we obtain (b3) from the formulae of (b1) and (c4) (note from the proof of (b1) that now $V = V_L$ and $B, B_L \in \mathcal{B}(U,V)$).

2° *An equivalent construction from [W94b], pp. 52–56:* G. Weiss shows that, for each $x_0 \in V$, the element $Jx_0 := V_L\text{-}\lim_{r\to+\infty} r(r-A)^{-1}x_0 \in V_L$ exists. Conversely $J^L x_0 := V\text{-}\lim_{r\to+\infty} r(r-A_L)^{-1}x_0 \in V$ exists for each $x_0 \in V_L$ and $J = (J^L)^{-1} \in \mathcal{GB}(V,V_L)$. He then shows that J and J^L are (unique) continuous extensions of $I : H \to H$. By density, they must be equal to the extensions of 1° (i.e., of (b1)). It follows that

$$x_0 = \lim_{r\to+\infty} r(r-A)^{-1}x_0 = \lim_{r\to+\infty} r(r-A_L)^{-1}x_0 \quad (x_0 \in V = V_L) \tag{6.160}$$

(both limits converge in both H_{-1} and H_{-1}^L).

By Proposition 6.2.8(e), we also have $B \in \mathcal{B}(U, \mathrm{Dom}(B_{\mathrm{L,s}}^*))$. We do not know whether $\mathrm{Dom}(B_{\mathrm{L,s}}^*)^*$ (and $\mathrm{Dom}((B_L^*)_{\mathrm{L,s}})^*$) is always contained in V.

(b4) By Proposition 6.3.1(b1), the admissibility of L implies that Σ_L is uniformly regular and $I - DL \in \mathcal{GB}$; the rest follows from (b3).

(c1) This follows from Proposition 6.3.1(a3) (and Lemma 6.2.5).

(c2) (Here "DL" denotes the feedthrough operator of $\mathbb{D}L$; if $\mathbb{D} \in$ WR, then, obviously, $DL = D \cdot L$).

We use the argument of Proposition 7.1 of [W94b] (which contains (c3) and (c4)):

For $x_0 \in \mathrm{Dom}((C_L)_\mathrm{s})$, by Lemma 6.2.12, the limit

$$C_\mathrm{w} x_0 = \text{w-lim} Cs(s-A)^{-1}x_0 = \text{w-lim} s\widehat{\mathbb{C}x_0}(s) \tag{6.161}$$

$$= \text{w-lim} s\widehat{(I-\mathbb{D}L)\mathbb{C}_L x_0}(s) = \text{w-lim} s(I - \widehat{\mathbb{D}}(s)L)\widehat{\mathbb{C}_L x_0}(s) \tag{6.162}$$

$$= \text{w-lim}(I - \widehat{\mathbb{D}}(s)L)C_L s(s-A_L)^{-1}x_0 = (I-DL)(C_L)_\mathrm{s} x_0 \tag{6.163}$$

exists as $s \to +\infty$ (because $C_L s(s - A_\circlearrowleft)^{-1}x_0 \to (C_L)_\mathrm{s} x_0$ strongly and $(I - \widehat{\mathbb{D}}(s)L) \to I - DL$ weakly; see also Lemma A.3.1(j2)), hence C_w is an extension of $(I-DL)(C_L)_\mathrm{s}$. Because $I - DL$ is left-invertible, we obtain (c1).

(c3) By Proposition 6.3.1(a1), $I - DL$ has a left inverse. The second claim is obtained exactly as in (c2) (with strong limits).

(c4) The $(C_L)_\mathrm{s}$ formula follows from (c3) and (c1) (and Lemma 6.6.3); in particular, $\mathrm{Dom}((C_L)_\mathrm{s}) = \mathrm{Dom}(C_\mathrm{s})$. By Lemma A.3.6, the domains must have equal norms (since both $\underset{c}{\subset} H$). The $(B_L^*)_\mathrm{s}$ formula follows from this (as in the proof of (b2)).

(c5) By Proposition 6.3.1(a1), $(I-DL)^*$ has a left inverse, i.e., $I - DL$ has a right inverse, hence $I - DL \in \mathcal{GB}(Y)$. Now the proof of (c2) shows that $C_\mathrm{w} x_0 = (I-DL)(C_L)_\mathrm{w} x_0$ for all $x_0 \in \mathrm{Dom}((C_L)_\mathrm{w})$, hence $(C_L)_\mathrm{w} \subset C_\mathrm{w}$.

But $((I-\mathbb{D}L)^{-1})^\mathrm{d} = ((I-\mathbb{D}L)^\mathrm{d})^{-1} \in$ SR, by Proposition 6.3.1(a3), hence one can analogously verify that $C_\mathrm{w} \subset (C_L)_\mathrm{w}$.

(c6) By Theorem 2.6.4(c2) (shifted), $\mathbb{D}L \in \mathrm{MTIC}_\infty \Leftrightarrow \mathbb{E} := (I-\mathbb{D}L)^{-1} \in \mathrm{MTIC}_\infty$; assume that this holds and that $\mathbb{D}$ is SR.

It follows that $\mathbb{E} \in \mathcal{G}$ULR, $E \in \mathcal{GB}(Y)$ and $I - \mathbb{D}L = \mathbb{E}^{-1} \in$ ULR, by Proposition 6.3.1(c)&(a3), so that $I - DL = E^{-1} \in \mathcal{GB}(Y)$ and $\mathbb{D}_L \in$ SR. Set $g* := \mathbb{E} - E$. Note that $\mathbb{C}_L = \mathbb{E}\mathbb{C} = E\mathbb{C} + g * \mathbb{C}$.

Choose $x_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$. Then $f := \mathbb{C}x_0 \in \mathrm{L}^2_\infty$ and $\frac{1}{t}\int_0^t f\,dm =: y_0 + \varepsilon(t) \to y_0$ as $t \to 0+$, where $y_0 := EC_{\mathrm{L,s}} x_0 \in Y$, by Proposition 6.2.8(c4)&(c3). If $g \in \mathrm{L}^1_\infty$, then, by the Fubini Theorem, we have

$$\frac{1}{t}\int_0^t g * \mathbb{C}x_0\,dm = \frac{1}{t}\int_0^t \int_0^s g(r) f(s-r)\,dr\,ds = \frac{1}{t}\int_0^t g(r)\int_r^t f(s-r)\,ds\,dr \tag{6.164}$$

$$= \int_0^t g(r)\frac{t-r}{t}(y_0 + \varepsilon(t-r))\,dr \to 0, \tag{6.165}$$

as $t \to 0+$. Analogously, if $\sum_k \|E_k\|_{\mathcal{B}(Y)} < \infty$, and $T_k > 0$ for all $k \in \mathbf{N}$, then

$$\|\frac{1}{t}\int_0^t \sum_k E_k \tau^{-T_k} f\|_Y \leq \sum_k \|E_k\| \|\frac{t-T_k}{t}(y_0 + \varepsilon(t-T_k))\|_Y \to 0, \tag{6.166}$$

as $t \to 0+$, where the sum runs over $\{k \in \mathbf{N} \mid T_k < t\}$. From (6.164) and (6.166) we obtain that $\frac{1}{t}\int_0^t (\mathbb{E} - E) f \, dm \to 0$, as $t \to 0+$. Because $\frac{1}{t}\int_0^t E f \, dm \to E y_0$, as $t \to 0+$, this implies that $(C_L)_{\mathrm{L,s}} x_0$ exists and is equal to $y_0 = E C_{\mathrm{L,s}} x_0$, by Proposition 6.2.8(c4)&(c3).

Since $x_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$ was arbitrary, we have $C_{\mathrm{L,s}} \subset E(C_L)_{\mathrm{L,s}}$. Exchange the roles of Σ and Σ_L to obtain that $(C_L)_{\mathrm{L,s}} \subset E^{-1} C_{\mathrm{L,s}}$. The claim on B_L^* follows by duality.

(d1) This follows from (b1), except for the $\widehat{\mathbb{X}}$, $\widehat{\mathbb{M}}$ and $\mathbb{K}_{\circlearrowleft}$ formulae, which are from Lemma 6.3.10(a) and Lemma 6.2.12 (we have $(K_\flat)_{\mathrm{s}} \subset K_{\mathrm{w}}$ $[(K_\flat)_{\mathrm{s}} = K_{\mathrm{w}}]$, by (c2) [(c4)]). Note that the "$(I - DL)^{-1}_{\text{left}}$" of (b1) is given by $(I - \left[\begin{smallmatrix} D \\ F \end{smallmatrix}\right] \left[\begin{smallmatrix} 0 & I \end{smallmatrix}\right])^{-1}_{\text{left}} = \left[\begin{smallmatrix} I & DM \\ 0 & M \end{smallmatrix}\right]$.

(d2) This follows from (d1).

(d3) If $\mathbb{X} \in \mathrm{SR}$, then $\mathbb{M} \in \mathrm{SR}$, by Proposition 6.3.1(a3), hence then $\mathbb{D}_\flat = \mathbb{D}\mathbb{M} \in \mathrm{WR}$. The rest follows from (b1), (c2) and (d1). (Note that (6.145)$\in \mathcal{B}(H_B \times U, V \times Y \times U)$, by (b1).)

(d4) Now $Ks(s-A)^{-1} = s\widehat{\mathbb{X}}(s) K_\flat (s - A_\flat)^{-1}$ and $C_\flat s(s-A_\flat)^{-1} = Cs(s-A)^{-1} + \widehat{\mathbb{D}}(s)\widehat{\mathbb{M}}(s) Ks(s-A)^{-1}$ (cf. (c2)); the claims follow from these equations and Proposition 6.3.1(a3) (because $H_B \subset \mathrm{Dom}(K_{\mathrm{s}})$ etc.).

(e) (In fact, without any regularity assumptions on $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, we have the equality on $\mathrm{Dom}((K_\flat)_{\mathrm{s}})$; i.e., for $x_0 \in \mathrm{Dom}((K_\flat)_{\mathrm{s}})$ we have $x_0 \in \mathrm{Dom}(S_{\mathrm{s}}) \Leftrightarrow x_0 \in \mathrm{Dom}(S_{\mathrm{s}}^A)$ with equal values on domains, as one easily observes from the proof.)

Set $S_{\mathrm{w}} := \text{w-lim}_{s \to +\infty} Ss(s-A_\flat)^{-1}$, $S_{\mathrm{w}}^A := \text{w-lim}_{s \to +\infty} S_{\mathrm{w}} s(s-A)^{-1}$.

Let $x_0 \in H_B$. Then, by (a2), we have

$$S_{\mathrm{w}} s(s-A)^{-1} x_0 = Ss(s-A_\flat)^{-1} x_0 + S_{\mathrm{w}}(s-A)^{-1} B K_\flat s(s-A_\flat)^{-1} x_0 \rightharpoonup S_{\mathrm{w}} x_0 + 0, \tag{6.167}$$

because $S_{\mathrm{w}}(s-A)^{-1}B \rightharpoonup 0$ and $K_\flat s(s-A_\flat)^{-1} x_0 \to (K_\flat)_{\mathrm{s}} x_0$ (see also Lemma A.3.1(j2)). Therefore, $x_0 \in \mathrm{Dom}(S_{\mathrm{w}}^A)$ and $S_{\mathrm{w}}^A x_0 = S_{\mathrm{w}} x_0$. The SR case is analogous.

(f) Recall from (a1) that "H_B" is the same for all four systems. From Lemma 6.7.12 we obtain (6.146) and (6.147).

Because $s\widehat{\mathbb{K}}'(s) x_0 \rightharpoonup (K_\natural)_{\mathrm{w}} x_0 + IK_{\mathrm{s}} x_0)$, as $s \to +\infty$, for all $x_0 \in H_B$ (even for all $x_0 \in \mathrm{Dom}(K_{\mathrm{s}}) \cap \mathrm{Dom}((K_\natural)_{\mathrm{w}})$), the operator $\mathbb{F}'$ is WR and $K_{\mathrm{w}}' = K_{\mathrm{s}} + (K_\natural)_{\mathrm{w}}$ on H_B (even on $\mathrm{Dom}(K_{\mathrm{s}}) \cap \mathrm{Dom}((K_\natural)_{\mathrm{w}})$). Thus, $K' = (K_{\mathrm{w}}')|_{\mathrm{Dom}(A)} = K + (K_\natural)_{\mathrm{w}}|_{\mathrm{Dom}(A)}$.

By (6.146) and Lemma 6.2.5, we have $F' = F + F_\natural - F_\natural F = 0$ (since $\mathbb{F}$ is SR). Recall from Lemma 6.2.5 that strong and uniform regularity properties are preserved under composition and vector operations.

(g) By Lemma 6.3.10(d3), $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}'' & \mathbb{F}'' \end{array}\right]$ iff $K' = K''$ on H_B. Obviously, either implies that $\mathbb{K}'_\flat = \mathbb{K}''_\flat$.

Assume then that $\mathbb{K}'_\flat = \mathbb{K}''_\flat$. Then $\mathbb{A}'_\flat = \mathbb{A} + \mathbb{B}\tau\mathbb{K}'_\flat = \mathbb{A}''_\flat$, and $\mathbb{C}'_\flat = \mathbb{C} + \mathbb{D}\tau\mathbb{K}'_\flat = \mathbb{C}''_\flat$, hence $A'_\flat = A''_\flat$, $C'_\flat = C''_\flat$ and $K'_\flat = K''_\flat$. By (6.144), we have $\Sigma'_\flat = \Sigma''_\flat$. Since $-K'_\flat = -K''_\flat$ is admissible for $\Sigma'_\flat$, we obtain that $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}'' & \mathbb{F}'' \end{array}\right]$, by Lemma 6.6.14. $\square$

The formulae between different operators in Proposition 6.6.18 are straightforward as long as we stay in H (which obviously includes the domains and extended domains of C, K and B^*). However, when write formulae for B and B_L, or for A and A_L with input space such as H_B, larger than the domain of the original operator, we face the fact that $H_{-1} \neq H^L_{-1}$ in general.

Thus, these formulae depend on how we identify a part of H_{-1} (containing $H + B[U]$) to a part of H^L_{-1} (containing $H + B_L[U]$). We make the following remark on this:

Remark 6.6.20 ($\boldsymbol{B = B_\flat}$) *(1.) The identification between B and $B_\flat$ in Proposition 6.6.18(d2) (the compatible case without feedthrough, due to O. Staffans [Sbook] (or [SW00])) is based on an isometric isomorphism $J : (\alpha - A)W \to (\alpha - A_\flat)W$ (see the proof of (b1)), which is a continuous extension of $I : H \to H$; here $W \supset H_B$ is the domain of $\begin{bmatrix} C_c \\ K_c \end{bmatrix}$.*

If H_1 is dense in W, then H is dense in $(\alpha - A)W$ and $(\alpha - A_\flat)W$, and thus this extension is unique. If this is not the case, then Σ has several compatible pairs for the same W, and the identification depends on the pair chosen. Thus, when applying the formula $B = B_\flat$, one should write out the spaces $(\alpha - A)W$ and $(\alpha - A_\flat)W$ for the system under study; in several applications it is completely natural to identify them.

If $I - F$ is merely left-invertible, then we can still embed $(\alpha - A)W$ into $(\alpha - A_\flat)W$, still so that the embedding is a continuous extension of $I : H \to H$, and we obtain a weaker connection between the operators, as shown in (d1) (or (b1)).

(2.) The identification in the SR case (e.g., Proposition 6.6.18(d4)), due to G. Weiss [W94b], is based on a rather natural identification; the space W has been chosen so that H_1 is dense in W. Of course, instead of V, one could choose to close H w.r.t. a different norm than that of V and try to obtain a space containing $H + BU$ and embeddable into H_{-1}.

Still, we recommend the reader to always check the meaning of V before applying the formula $B = B_\flat$ (or the formula for $A_\flat$ outside $\mathrm{Dom}(A_\flat)$*).*

Naturally, the case for static output feedback is analogous. See the proofs of (b1) and (b3) for details on the two identifications mentioned above.

As observed in the proof of (b3), the identification of Weiss is a special case of the identification of Staffans. $\square$

If Σ can be stabilized by output injection (i.e., by feeding the output back to the system through an extra input, as in Figure 6.5, then we call Σ detectable (we use duality to avoid a longer treatment analogous to that for state feedback stabilizability):

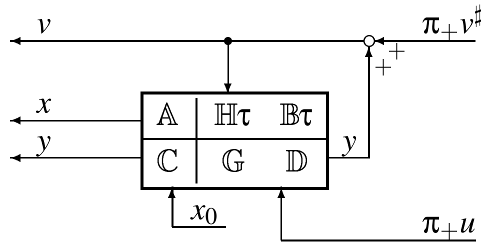


Figure 6.5: Output injection connection

Definition 6.6.21 (Detectability) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$. *A pair* $\left[\begin{smallmatrix}\mathbb{H}\\ \mathbb{G}\end{smallmatrix}\right]$ *is an* admissible output injection pair *for* Σ *if the extended system* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{H} & \mathbb{B}\\ \mathbb{C} & \mathbb{G} & \mathbb{D}\end{smallmatrix}\right]$ *is a* WPLS *and* $I - \mathbb{G} \in \mathcal{G}\mathrm{TIC}_\infty$ *(i.e.,* $L = \left[\begin{smallmatrix}I\\ 0\end{smallmatrix}\right]$ *is admissible for* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{H} & \mathbb{B}\\ \mathbb{C} & \mathbb{G} & \mathbb{D}\end{smallmatrix}\right]$*).*

Such a pair $\left[\begin{smallmatrix}\mathbb{H}\\ \mathbb{G}\end{smallmatrix}\right]$ *is* stabilizing*if the resulting closed loop system*

$$\Sigma_\sharp := \left[\begin{array}{c|cc} \mathbb{A} + \mathbb{H}\tau\widetilde{\mathbb{M}}\mathbb{C} & \mathbb{H}\widetilde{\mathbb{M}} & \mathbb{B} + \mathbb{H}\widetilde{\mathbb{M}}\mathbb{D} \\ \hline \widetilde{\mathbb{M}}\mathbb{C} & \widetilde{\mathbb{M}} - I & \widetilde{\mathbb{M}}\mathbb{D} \end{array}\right] \tag{6.168}$$

(here $\widetilde{\mathbb{M}} := (I - \mathbb{G})^{-1}$*) is stable, i.e., if* L *is stabilizing; in this case, we call* Σ detectable.

If $\left[\begin{smallmatrix}A & \mathtt{H}\\ C & 0\end{smallmatrix}\right]$ *generate a WR WPLS* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{H}\\ \mathbb{C} & \mathbb{G}\end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix}\mathbb{H}\\ \mathbb{G}\end{smallmatrix}\right]$ *is an admissible (resp. stabilizing) output injection pair for* Σ*, then* $\mathtt{H}$ *is called a* WR admissible (resp. stabilizing) output injection operator *for* Σ*.*

Admissible output injection and state feedback pairs $\left[\begin{smallmatrix}\mathbb{H}\\ \mathbb{G}\end{smallmatrix}\right]$ *and* $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *are called* jointly admissible *for* Σ *if they are parts of a single* WPLS

$$\Sigma_{\mathrm{Total}} := \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{H} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{G} & \mathbb{D} \\ \mathbb{K} & \mathbb{E} & \mathbb{F} \end{array}\right] \in \mathrm{WPLS}(Y \times U, H, Y \times U) \tag{6.169}$$

(with some interaction operator $\mathbb{E} \in \mathrm{TIC}_\infty(Y,U)$*). If the closed-loop systems of* Σ_{Total} *corresponding to* $L = \left[\begin{smallmatrix}0 & 0\\ 0 & I\end{smallmatrix}\right]$ *and* $\widetilde{L} = \left[\begin{smallmatrix}I & 0\\ 0 & 0\end{smallmatrix}\right]$ *are stable, then the pairs are* jointly stabilizing *and* Σ *is* jointly stabilizable and detectable.

We use prefices and suffices as in Definitions 6.6.10 and 6.6.4 above (e.g., strongly l.c.-detectable *means having a admissible pair* $\left[\begin{smallmatrix}\mathbb{H}\\ \mathbb{G}\end{smallmatrix}\right]$ *s.t.* $\Sigma_\sharp$ *is strongly stable and* $\widetilde{\mathbb{M}}$ *and* $\widetilde{\mathbb{M}}\mathbb{D}$ *are l.c.). Prefices preceding the word "jointly" apply to both closed-loop systems (hence to both pairs).*

The two closed-loop systems corresponding to jointly admissible pairs are

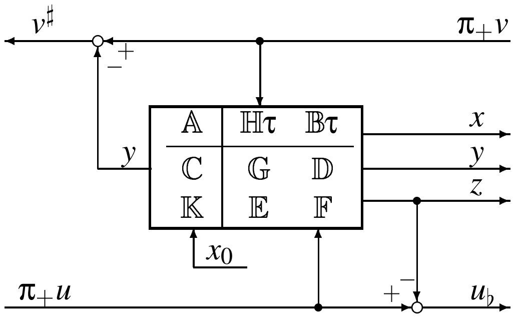


Figure 6.6: The extended system Σ_{Total}

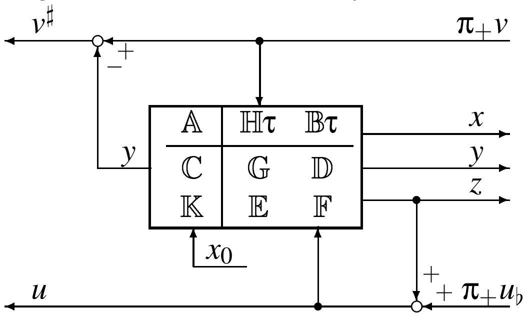


Figure 6.7: The closed-loop system $(\Sigma_{\text{Total}})_L$

easily seen to be given by

$$(\Sigma_{\text{Total}})_L = \left[\begin{array}{c|cc} \mathbb{A}+\mathbb{B}\tau(I-\mathbb{F})^{-1}\mathbb{K} & \mathbb{H}+\mathbb{B}(I-\mathbb{F})^{-1}\mathbb{E} & \mathbb{B}(I-\mathbb{F})^{-1} \\ \hline \mathbb{C}+\mathbb{D}(I-\mathbb{F})^{-1}\mathbb{K} & \mathbb{G}+\mathbb{D}(I-\mathbb{F})^{-1}\mathbb{E} & \mathbb{D}(I-\mathbb{F})^{-1} \\ (I-\mathbb{F})^{-1}\mathbb{K} & (I-\mathbb{F})^{-1}\mathbb{E} & (I-\mathbb{F})^{-1}-I \end{array}\right], \tag{6.170}$$

$$(\Sigma_{\text{Total}})_{\widetilde{L}} = \left[\begin{array}{c|cc} \mathbb{A}+\mathbb{H}\tau(I-\mathbb{G})^{-1}\mathbb{C} & \mathbb{H}(I-\mathbb{G})^{-1} & \mathbb{B}+\mathbb{H}(I-\mathbb{G})^{-1}\mathbb{D} \\ \hline (I-\mathbb{G})^{-1}\mathbb{C} & (I-\mathbb{G})^{-1}-I & (I-\mathbb{G})^{-1}\mathbb{D} \\ \mathbb{K}+\mathbb{E}(I-\mathbb{G})^{-1}\mathbb{C} & \mathbb{E}(I-\mathbb{G})^{-1} & \mathbb{F}+\mathbb{E}(I-\mathbb{G})^{-1}\mathbb{D} \end{array}\right]. \tag{6.171}$$

Any [strongly/exponentially] stable Σ is [strongly/exponentially] jointly r.c.-stabilizable and l.c.-detectable (take $\mathbb{F} = 0 = \mathbb{G} = \mathbb{K} = \mathbb{H} = \mathbb{E}$). Because of the duality explained in Lemma 6.7.2, we omit most left definitions and left results.

Detectability means roughly that the unstable parts of the system can be detected in the output. Observability means that the whole state space can be observed in the output. The latter condition sounds stronger, and, indeed, an observable system is exponentially detectable, by p. 91 of [LR], if $\dim H < \infty$. In general this is not the case (recall the word "roughly" above): for any $\mathbb{D} \in \text{TIC}(U,Y)$, the (infinite-dimensional) system (6.11) is stable and exactly 0-observable (in infinite time) but not even estimatable (see Definition 6.7.3).

Detectability does not imply observability even for $\dim H < \infty$ (just take any exponentially stable $\mathbb{A}$ with $\mathbb{C} = 0$). By duality, the relations between reachability and stabilizability are analogous.

Next we shall list a few facts on joint stabilizability and detectability. We start with a rather obvious remark:

Lemma 6.6.22 (Jointly stabilizing ⇒ separately) *If* $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ *and* $\left[\ \mathbb{K} \mid \mathbb{F}\ \right]$ *are jointly admissible [stabilizing] for some WPLS* Σ*, then* $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ *and* $\left[\ \mathbb{K} \mid \mathbb{F}\ \right]$ *are admissible [stabilizing] for* Σ*; all prefices and suffices apply.*

Proof: 1° *Admissibility:* Joint admissibility is obviously equivalent to the invertibility of $I-\mathbb{F}$ and $I-\mathbb{G}$ in TIC_∞, hence the admissibility claim holds.

2° *Stabilization:* Compare (6.170) to (6.133) and (6.171) to (6.168) to obtain the claim in brackets and the claim on prefices and suffices (note that by "r.c.-" we refer to $\mathbb{M} := (I-\mathbb{F})^{-1}$ and $\mathbb{N} := \mathbb{D}\mathbb{M}$ and by "l.c.-" to $\widetilde{\mathbb{M}} := (I-\mathbb{G})^{-1}$ and $\widetilde{\mathbb{N}} := \widetilde{\mathbb{M}}\mathbb{D}$, is in Definitions 6.6.10 and 6.6.21). □

The converse is not true:

Example 6.6.23 (Stabilizing but not jointly stabilizing) If A, B and C are as in Example 6.3.24, then the output injection pair (right column) in $\left(\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right)$ and the state feedback pair (lower row) in $\left(\begin{array}{c|c} A & 0 \\ \hline C & 0 \end{array}\right)$ are admissible for $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right)$, by Lemma 6.3.16 and Proposition 6.3.1(c), but they are not jointly admissible, by Example 6.3.24 (there is no suitable interaction operator $\mathbb{E}$).

By replacing A by $A+\omega$ for a suitable $\omega \in \mathbf{R}$ (see Remark 6.1.9), we can make the two pairs stabilizing but yet not jointly stabilizing (not even jointly admissible). ◁

Nevertheless, we do not know whether there is a system that is [exponentially] stabilizable and detectable but not [exponentially] jointly stabilizable and detectable. [Neither do we know whether optimizability and estimatability (see Definition 6.7.3) is strictly weaker than the above two conditions. All these conditions are equivalent if the system is sufficiently regular, by Theorem 7.2.4(c).]

Lemma 6.6.24 *We have "exponentially jointly stabilizable and detectable" = "jointly exponentially stabilizable and detectable" and "strongly jointly stabilizable and detectable" = "jointly strongly stabilizable and detectable".*

Proof: Let $\left[\ \mathbb{K} \mid \mathbb{F}\ \right]$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ be jointly stabilizing, so that $I-(\widetilde{L}-L)\mathbb{D}_{\mathrm{Total}} = \left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ in terms of (6.172). By Lemma 6.6.3, $((\Sigma_{\mathrm{Total}})_L)_{\widetilde{L}-L} = (\Sigma_{\mathrm{Total}})_{\widetilde{L}}$, hence $(\Sigma_{\mathrm{Total}})_L$ is [exponentially] strongly stable iff $(\Sigma_{\mathrm{Total}})_{\widetilde{L}}$ is [exponentially] strongly stable, by Corollary 6.6.9. □

If C is boundedly left-invertible, then one can detect everything about the state from the output; in particular:

Lemma 6.6.25 *Let* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ *be s.t.* $TC = I$ *for some* $T \in \mathcal{B}(Y,H)$*. Then* Σ *is exponentially detectable.*

In standard stabilization problems we have $C = \left[\begin{smallmatrix} I \\ * \end{smallmatrix}\right]$, so that one can choose $T := \begin{bmatrix} I & 0 \end{bmatrix}$.

Proof: Take $\mathtt{H} := -rT \in \mathcal{B}(Y,H)$, $G = 0$, $r > \omega$. Then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{H} \\ \hline \mathbb{C} & \mathbb{G} \end{smallmatrix}\right]$ is well-posed, by Lemma 6.3.16(b), hence so is $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{H} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{G} & \mathbb{D} \end{smallmatrix}\right]$, and $\mathbb{G}$ is ULR. Moreover, the generator of $\mathbb{A}_\sharp$ is $A + \mathtt{H}C_s = A - rI$, by Proposition 6.6.18(b3), hence $\mathbb{A}_\sharp$ is exponentially stable, by Lemma A.4.2(g2). □

In the classical case, any admissible state feedback and output-injection pairs are jointly admissible:

Lemma 6.6.26 (K and $\mathtt{H}$ jointly) *Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ be admissible (resp. exponentially stabilizing) pairs for Σ.*

If K or $\mathtt{H}$ is bounded, then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ are jointly admissible (resp. jointly exponentially r.c.- and l.c.-stabilizing).

Proof: Let K be bounded (the other case is analogous). Then $\left[\begin{array}{c|cc} K & 0 & F \end{array}\right]$ extends $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{H} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{G} & \mathbb{D} \end{smallmatrix}\right]$ to a WPLS, by Lemma 6.3.16(c), hence $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ are jointly admissible (resp. and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially r.c.-stabilizing and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ is exponentially l.c.-stabilizing, by Theorem 6.6.28 (and Remark 6.1.9). □

In the uniformly regular case, jointly stabilizing pairs can be replaced by jointly stabilizing K and $\mathtt{H}$:

Lemma 6.6.27 *If $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ are jointly stabilizing for $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in$ WPLS(U,H,Y) with interaction operator $\mathbb{E}$, then so are $\left[\begin{array}{c|c} R\mathbb{K} & I - R(I-\mathbb{F}) \end{array}\right]$ and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ with $R\mathbb{E}$, and so are also $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and $\left[\begin{array}{c|c} \mathbb{H}S & I-(I-\mathbb{G})S \end{array}\right]$ with $\mathbb{E}S$, for any $R \in \mathcal{GB}(U)$, $S \in \mathcal{GB}(Y)$. All prefices and suffices apply.*

In particular, if Σ_{Total} is UR, then we can have $F = 0$, $G = 0$ and $E = 0$.

Proof: 1° We observe from equations (6.170)–(6.171) that corresponding closed-loop systems are equal except that $\mathbb{B}_L \mapsto \mathbb{B}_L R^{-1}$, $\mathbb{D}_L \mapsto \mathbb{D}_L R^{-1}$ and $\mathbb{M}_L \mapsto \mathbb{M}_L R^{-1}$ (hence $\mathbb{F}_L \mapsto I-(I-\mathbb{F}_L)R^{-1} = \mathbb{F}_L R^{-1} + I - R^{-1}$), and $(\Sigma_{\text{Total}})_{\widetilde{L}}$ is affected analogously.

2° If Σ_{Total} is UR, then $I-F, I-G \in \mathcal{GB}$, by Proposition 6.3.1(b1), hence we can have $F=0$ and $G=0$ (take $R=(I-F)^{-1}$, $S=(I-G)^{-1}$ in 1°). Moreover, if we replace $\mathbb{E}$ by $\mathbb{E}-E$, then we add the elements of the third column of (6.170) plus $\left[\begin{smallmatrix} 0 \mid 0 \mid I \end{smallmatrix}\right]^{\mathrm{T}}$, times $-E$, to the second column of (6.170), hence the stability of (6.170) is not affected. An analogous claim holds for (6.171). Thus, the new pairs (those with $F=0$, $G=0$) with $\mathbb{E}-E$ are stabilizing exactly in the same sense as the old ones. □

We recall the main result of [S98a]:

Theorem 6.6.28 (d.c.f. ⇔ jointly) *Let $\mathbb{D} \in \text{TIC}_\infty(U,Y)$. Then the following are equivalent:*

(i) $\mathbb{D}$ has a d.c.f.;

(ii) $\mathbb{D}$ *has a jointly I/O-stabilizable and I/O-detectable realization;*

(iii) $\mathbb{D}$ *has a strongly jointly r.c.-stabilizable and l.c.-detectable realization.*

Moreover, any jointly I/O-stabilizing pairs (if any) for any realization of $\mathbb{D}$ *are jointly r.c.- and l.c.-I/O-stabilizing.*

Recall from the definition that (iii) means that both (6.170) and (6.171) become strongly stable and that $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f. and $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ is a l.c.f., where $\widetilde{\mathbb{N}} := \widetilde{\mathbb{M}}\mathbb{D}$, $\widetilde{\mathbb{M}} := (I - \mathbb{G})^{-1}$.

If Σ is the corresponding realization and $\begin{bmatrix} \mathbb{K} & \big| & \mathbb{F} \end{bmatrix}$ corresponds to a stabilizing SR $K \in \mathcal{B}(H_1, U)$, then $\widehat{\mathbb{M}}(s) = I + K_s(s-A)^{-1}B$ and $\widehat{\mathbb{N}}(s) = D + (C_s + DK_s)(s - A)^{-1}B$ are SR, by Proposition 6.6.18(b3). An analogous claim obviously holds for the whole d.c.f. when also $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ corresponds to a SR operator. (If K and H are merely WR, then the same formulae can be applied but the factorization need not be WR.) See Section 5 of [WC] and Section 3 of [CWW96] for analogous results (in the SR exponentially stabilizing case, which can be obtained by shifting the above result).

Proof: This follows from the proof of Theorem 4.4 of [S98a]; we sketch the proof below.

1° *"(iii)⇒(ii)":* This is trivial.

2° *"(ii)⇒(i)":* Assume (ii) and denote the I/O maps of (6.170) and (6.170) by $\begin{bmatrix} \mathbb{G}_L & \mathbb{D}_L \\ \mathbb{E}_L & \mathbb{F}_L \end{bmatrix}$ and $\begin{bmatrix} \mathbb{G}_{\widetilde{L}} & \mathbb{D}_{\widetilde{L}} \\ \mathbb{E}_{\widetilde{L}} & \mathbb{F}_{\widetilde{L}} \end{bmatrix}$, respectively. Then the $\mathrm{TIC}(U \times Y)$ maps

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} = \begin{bmatrix} I + \mathbb{F}_L & -\mathbb{E}_L \\ \mathbb{D}_L & I - \mathbb{G}_L \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = \begin{bmatrix} I - \mathbb{F}_{\widetilde{L}} & \mathbb{E}_{\widetilde{L}} \\ -\mathbb{D}_{\widetilde{L}} & I + \mathbb{G}_{\widetilde{L}} \end{bmatrix} \tag{6.172}$$

are inverses of each other; here $\Sigma_\sharp$ is the system (6.168) (see [S98a] for details (or use direct computation); note that there $\mathbb{Y} =$"$\mathbb{E}_\flat$"$= -\mathbb{E}_L$ and $\mathbb{Y} =$"$\mathbb{E}_\sharp$"$= -\mathbb{E}_{\widetilde{L}}$ because of different signs in the Bezout equations of [S98a]). Thus, if the pairs are jointly I/O-stabilizing, then (6.172) defines a d.c.f. of $\mathbb{D}$ (and $\begin{bmatrix} \mathbb{K} & \big| & \mathbb{F} \end{bmatrix}$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ are jointly r.c.- and l.c.-I/O-stabilizing). Consequently, we have (i) holds.

2° *"(i)⇒(iii)":* We work as in Lemma 6.6.29: Let (6.109) be a d.c.f. of $\mathbb{D}$. Start with, e.g., a strongly stable realization (say, Σ_L) of

$$\begin{bmatrix} \mathbb{G}_L & \mathbb{D}_L \\ \mathbb{E}_L & \mathbb{F}_L \end{bmatrix} := \begin{bmatrix} I - \mathbb{X} & \mathbb{N} \\ -\mathbb{Y} & \mathbb{M} - I \end{bmatrix} \tag{6.173}$$

(cf. Definition 6.1.6), where Close it with the output feedback operator $L' := \begin{bmatrix} 0 & 0 \\ 0 & -I \end{bmatrix}$ (which is admissible since $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$) to obtain a WPLS $\Sigma_{\mathrm{Total}} := (\Sigma_L)_{L'}$; drop the middle column and bottom line of Σ_{Total} to obtain $\Sigma \in \mathrm{WPLS}(U, H, Y)$.

The additional operators in Σ_{Total} constitute strongly jointly stabilizing and detecting pairs and an interaction operator as in Definition 6.6.21: indeed, $(\Sigma_{\mathrm{Total}})_{-L'} = \Sigma_L$, and, by Lemma 6.6.3, (6.171) is strongly stable iff

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I - \mathbb{X} & \mathbb{N} \\ -\mathbb{Y} & \mathbb{M} - I \end{bmatrix} = \begin{bmatrix} \mathbb{X} & -\mathbb{N} \\ -\mathbb{Y} & \mathbb{M} \end{bmatrix} \tag{6.174}$$

is in $\mathcal{G}$TIC, by Corollary 6.6.9, and this is the case, by assumption (see (6.109)). $\square$

By same methods, we obtain the following:

Lemma 6.6.29 *If(f) $\mathbb{D} \in \mathrm{TIC}_\infty$ has a right factorization [[q.]r.c.f.] $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, then $\mathbb{D}$ has a strongly [[q.]r.c.-]stabilizable realization.*

The converse is also true, by Definition 6.6.10. If $\mathbb{M} \in \mathrm{UR}$ or $\mathbb{M}, \mathbb{M}^{-1} \in \mathrm{SR}$, we can take $M = I$ above, by Proposition 6.3.1(a3)&(b1) and Lemma 6.4.5, i.e., then the feedback pair $\left[\begin{array}{c|c} \mathbb{K} & I - \mathbb{M}^{-1} \end{array}\right]$ in the proof below has no feedthrough: $\left[\begin{array}{c|c} K & I - M^{-1} \end{array}\right] = \left[\begin{array}{c|c} K & 0 \end{array}\right]$.

Proof: Take a strongly stable realization $\Sigma_\flat$ of $\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M}-I \end{smallmatrix}\right]$ (e.g. $\Sigma_\flat := \left[\begin{array}{c|c} \pi_+\tau & \pi_+\left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M} \end{smallmatrix}\right]\pi_- \\ \hline I & \left[\begin{smallmatrix} \mathbb{N} \\ \mathbb{M}-I \end{smallmatrix}\right] \end{array}\right] \in \mathrm{WPLS}_0(U, \mathrm{L}^2_\omega(\mathbf{R}_+;Y), Y))$, and close it with $L := \begin{bmatrix} 0 & -I \end{bmatrix}$ to get a realization

$$\Sigma_{\mathrm{ext}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & I - \mathbb{M}^{-1} \end{array}\right] := (\Sigma_\flat)_L \in \mathrm{WPLS}. \tag{6.175}$$

By Lemma 6.6.3, $\left[\begin{array}{c|c} \mathbb{K} & I - \mathbb{M}^{-1} \end{array}\right]$ is strongly [[q.]r.c.-]stabilizing for $\Sigma := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$. $\square$

Notes

All definitions are basically from [S98a]; the suffices, some prefices (such as r.c.-stabilization) and the concept "compatible state feedback (or output injection) operator" are new. The definition of stabilizability in [WC] and [CWW96] is equivalent to the existence of a SR exponentially stabilizing state feedback operator; a dual remark applies to detectability.

Proposition 6.6.2 is [S98a, Proposition 3.2], which is a variant of [W94b, Proposition 3.6] (the semigroup part of this was contained already in [Sal87]).

Lemma 6.6.3 is [W94b, Remark 6.5]. Part of Lemmas 6.6.7 and 6.6.8 and Corollary 6.6.9 are based on Lemma 21 of [S97b] and on its proof.

Most of parts (a1)–(a3), (b3), (c1)–(c5) and (d4) of Proposition 6.6.18 are based on [W94b] or on the methods used in its proofs. Most of (b1) and (d1) are from [Mik97a]; the claims on B_L (including V and the corresponding identification) were added in [Sbook].

Example 6.6.19 is based on [SW01b]. Theorem 6.6.28 is essentially Theorem 4.4 of [S98a] (Theorems 3.2 and 3.4 of [CWW96] present an independent variant for regular WPLSs); the proof of Lemma 6.6.29 is analogous.

A classical WPLS reference for output feedback is [W94b], which contains the rudiments of static feedback and state feedback; the most complete reference at present is [Sbook, Chapters 7&8], whose results are partially contained in [S98a]. Most existing literature treats exponential stability rather than stability; however, the results on the latter always imply analogous results on the former, but the converse does not hold.

6.7 Further feedback results

Stability itself is nothing else than a more sluggish motion.

In this section, we present further results on feedback; especially on stabilizability. We also define and study optimizability and estimatability, which are weak forms of exponential stabilizability and exponential detectability, respectively.

For the reader to distinguish between the several stability concepts introduced above and in the next chapter, we give here a summary of all such concepts:

Summary 6.7.1 (Stabizability concepts) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $\omega \in \mathbf{R}$.

(a) [6.6.4] An operator $L \in \mathcal{B}(Y,U)$ *is called a* stabilizing (static) output feedback operator *for* Σ *if* L *is admissible* ($I - L\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty(U)$) *and the resulting closed-loop system* $\Sigma_L = \left[\begin{smallmatrix} \mathbb{A}_L & \mathbb{B}_L \\ \mathbb{C}_L & \mathbb{D}_L \end{smallmatrix}\right]$ *is stable.*

If such an L *exists, we call* Σ stabilizable by static output feedback.

Whenever L *is s.t.* $\mathbb{B}_L$ *is stable,* L *is called* $\mathbb{B}$-stabilizing *for* Σ*; the applies also to the other components of* Σ.

Even without reference to Σ*, we say that* L *is* stabilizing *for* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *when* L *is admissible and* $\mathbb{D} := \mathbb{D}(I - L\mathbb{D})^{-1}$ *is stable.*

(b) [6.6.10] A pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is called a* stabilizing state feedback pair *for* Σ *if the extended system* Σ_{ext} *is a* WPLS *and* $L := \begin{bmatrix} 0 & I \end{bmatrix}$ *stabilizes* Σ_{ext}.

If such a pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *exists, we call* Σ stabilizable.

When we add the prefix "[q.]r.c.-" *(resp. suffix* "in $\mathcal{A}$"*) (e.g.,* "$\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is [q.]r.c.-stabilizing"), we mean that* $\mathbb{N}$ *and* $\mathbb{M}$ *are [q.]r.c. (resp.* $\mathbb{N}, \mathbb{M} \in \mathcal{A}$*), where* $\mathbb{M} := (I - \mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$.

The pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is* exponentially [q.]r.c.-stabilizing *for* Σ *if* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is exponentially stabilizing and* $\mathbb{N}$ *and* $\mathbb{M}$ *are exponentially [q.]r.c.; equivalently, if* $\mathcal{T}_\omega \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is [q.]r.c.-stabilizing for* $\mathcal{T}_\omega \Sigma$ *for some* $\omega > 0$ *(see Remark 6.1.9).*

(c) [6.6.21] A pair $\left[\begin{smallmatrix} \mathbb{H} \\ \mathbb{G} \end{smallmatrix}\right]$ *is an* stabilizing output injection pair *for* Σ *if* $\left[\begin{array}{c|c} \mathbb{H}^{\mathrm{d}} & \mathbb{G}^{\mathrm{d}} \end{array}\right]$ *is a stabilizing state feedback pair for* Σ.

If such a pair $\left[\begin{smallmatrix} \mathbb{H} \\ \mathbb{G} \end{smallmatrix}\right]$ *exists, we call* Σ detectable. *We use prefices (e.g., "[q.]l.c.") as in (b).*

(d) [6.6.21] The output injection and state feedback pairs $\left[\begin{smallmatrix} \mathbb{H} \\ \mathbb{G} \end{smallmatrix}\right]$ *and* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *are called* jointly stabilizing *for* Σ *if they are part of a single* WPLS *(6.169) and both* $L = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ *and* $L = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ *stabilize this* WPLS.

If such pairs exist, we call Σ jointly stabilizable and detectable.

By dynamic feedback (DF) we mean output feedback similar to that defined in (a) but with a dynamic controller L*, i.e.,* $L \in \mathrm{TIC}_\infty(Y,U)$ *need not be static. In dynamic partial feedback (DPF, aka. measurement feedback) we mean the* *situation where the input and output of the controller are connected only to a part of the output and input of the system to be controlled. Maps with internal loop are a generalized concept of* TIC$_\infty$ *maps. This concept makes the algebraic stabilization theory more complete, and it has also a reasonable physical interpretation.*

These concepts are studied in Chapter 7, but we include them in this summary for easy comparison; here also Ξ *is an arbitrary Hilbert space:*

- *(e)* **(DF)** *[7.1.1] A map* $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ *is a* stabilizing (DF-)controller *for* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *if* $L = I$ *is stabilizing for* $\begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix}$.

  If such a $\mathbb{Q}$ *exists, we say that* $\mathbb{D}$ *is* DF-stabilizable *and that* $\mathbb{Q}$ DF-stabilizes $\mathbb{D}$.
- *(e')* **(DF-IL)** *[7.2.1] A map* $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ *is a* stabilizing (DF-)controller for with internal loop $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *if* $L = I$ *is stabilizing for (7.20).*

  If such an $\mathbb{O}$ *exists, we say that* $\mathbb{D}$ *is* DF-stabilizable with internal loop *and that* $\mathbb{O}$ DF-stabilizes $\mathbb{D}$ with internal loop.
- *(f)* **(DPF)** *[7.3.1] A map* $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ *is a* stabilizing DPF-controller *for* $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ *if* $\begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix}$ *is a stabilizing DF-controller for* $\mathbb{D}$.
- *(f')* **(DPF-IL)** *[7.3.1] A map* $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ *is a* stabilizing (DPF-)controller with internal loop for $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ *if (7.58) is an admissible [stabilizing] (DF-)controller for* $\mathbb{D}$ *with internal loop.*

  If such an $\mathbb{O}$ *exists, we say that* $\mathbb{D}$ *is* DPF-stabilizable with internal loop *and that* $\mathbb{O}$ DPF-stabilizes $\mathbb{D}$ with internal loop.

In (e)–(f'), one can make analogous definitions with $\mathbb{D}$ *replaced by its realization [and* $\mathbb{Q}$ *or* $\mathbb{O}$ *replaced by its realization], see Definitions 7.1.1, 7.2.1 and 7.3.1.*

When we say that L *is* strongly internally ω-stabilizing *for* Σ*, we mean that we do not require the corresponding closed loop system* Σ_L*, to be stable but strongly internally* ω*-stable; the same applies to all other stability concepts (prefices) of Definition 6.1.3.*

Same prefices are used for stabilizability by static output feedback (e.g., "Σ *is strongly internally* ω*-stabilizable by static output feedback"), and these are inherited by definitions (b)–(f'), where also further prefices and suffices can be used (see 2. and 4. of Definition 6.6.10). In (d), prefices preceding the word "jointly" apply to both pairs.*

In (a)–(f'), we use the word admissible *instead of stabilizing if* L *is (merely) admissible. The word* stabilizes *means "is stabilizing for". See Remark 6.7.19 for* ω*-stabilization (*$\omega \in \mathbf{R}$*).*

See Definition 7.2.11 for maps with a *coprime* (that is, a d.c., r.c., or l.c.) internal loop. Remark 6.7.19 explains ω-stabilization further. We usually say "stabilizing" instead of "admissible stabilizing".

The concepts of Summary 6.7.1 are invariant under duality:

Lemma 6.7.2 (Duality) *Make the assumptions of Summary 6.7.1.*

Then the properties (a)–(f') of the summary are invariant to the following extent:

(a) L is admissible for Σ iff L^{d} is admissible for Σ^{d}.

If L is admissible, then $(\Sigma^{\mathrm{d}})_{L^{\mathrm{d}}} = (\Sigma_L)^{\mathrm{d}}$. Thus, e.g., L is [exponentially] stabilizing for Σ iff L is [exponentially] stabilizing for Σ^{d}, etc.

(b)&(c) A pair $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ is an admissible output injection pair for Σ iff $\left[\ \mathbb{H}^{\mathrm{d}}\ \middle|\ \mathbb{G}^{\mathrm{d}}\ \right]$ is an admissible state feedback pair for Σ^{d}.

Moreover, the corresponding closed-loop systems (see (6.168) and (6.133)) are duals of each other, hence $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ is [exponentially / exponentially [q.]l.c.- / [q.]l.c.-]stabilizing for Σ iff $\left[\ \mathbb{H}^{\mathrm{d}}\ \middle|\ \mathbb{G}^{\mathrm{d}}\ \right]$ is [exponentially / exponentially [q.]r.c.- / [q.]r.c.-]stabilizing for Σ^{d}.

Thus, Σ is stabilizable iff Σ^{d} is detectable, etc.

(d) The pairs $\left[\ \mathbb{K}\ \middle|\ \mathbb{F}\ \right]$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ are jointly admissible (resp. [exponentially] jointly stabilizing) for Σ iff $\left[\ \mathbb{H}^{\mathrm{d}}\ \middle|\ \mathbb{G}^{\mathrm{d}}\ \right]$ and $\left[\frac{\mathbb{K}^{\mathrm{d}}}{\mathbb{F}^{\mathrm{d}}}\right]$ are jointly admissible (resp. [exponentially] jointly stabilizing) for Σ^{d}. The corresponding closed-loop systems are duals of each other modulo permutations of the two I/O rows and columns.

Thus, Σ is jointly stabilizable and detectable iff Σ^{d} is jointly stabilizable and detectable, etc.

(e)–(f') Analogously, $\mathbb{Q}$ is admissible for $\mathbb{D}$ in the sense of (e) (resp. (e'), (f), (f')) iff $\mathbb{Q}^{\mathrm{d}}$ is admissible for $\mathbb{D}_{\mathrm{d}}$ in the sense of (e) (resp. (e'), (f), (f')).

Moreover, the corresponding closed-loop systems are duals of each other modulo permutations of I/O rows and columns. Thus, $\mathbb{Q}$ is [exponentially] stabilizing for $\mathbb{D}$ in the sense of (e) (resp. (e'), (f), (f')) iff $\mathbb{Q}^{\mathrm{d}}$ is [exponentially] stabilizing for $\mathbb{D}_{\mathrm{d}}$ in the sense of (e) (resp. (e'), (f), (f')).

Analogous remarks apply to realizations of $\mathbb{D}$ (and $\mathbb{Q}$).

By $\mathbb{D}_{\mathrm{d}}$ we mean $\mathbb{D}^{\mathrm{d}}$ in case of (e) or (e') and $\begin{bmatrix} \mathbb{D}_{22}^{\mathrm{d}} & \mathbb{D}_{12}^{\mathrm{d}} \\ \mathbb{D}_{21}^{\mathrm{d}} & \mathbb{D}_{11}^{\mathrm{d}} \end{bmatrix}$ in case of (f) or (f') (note that these correspond to $\mathbb{O}^{\mathrm{d}}$, not to $\mathbb{O}_{\mathrm{d}}$, in Definitions 7.2.1 and 7.3.1). Cf. also Proposition 7.2.5(d), Lemma 7.2.6 and Proposition 7.3.4(d).

Because of the duality between the closed-loop systems, also most prefices and suffices (e.g., "internally", "I/O-", "weakly", "ω-", and "in $\mathcal{A}$" if $\mathcal{A} = \mathcal{A}^{\mathrm{d}}$) are preserved (although "output" becomes "input", "[q.]r.c." becomes "[q.]l.c." etc.). However, the "strong" properties (e.g., "strongly detectable") are exceptions to this duality, because the adjoint of a strongly stable system need not be strongly stable. In the following we will apply these facts without further mention.

Proof: (a) $(I - \mathbb{D}L)^{\mathrm{d}} = I - L^{\mathrm{d}}\mathbb{D}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}_\infty$ iff $I - \mathbb{D}L \in \mathcal{G}\mathrm{TIC}_\infty$ (because $(\cdot)^{\mathrm{d}} \in \mathcal{GB}(\mathrm{TIC}_\omega)$ for any $\omega \in \mathbf{R}$) but also iff $I - \mathbb{D}^{\mathrm{d}}L^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}_\infty$. Thus, L is admissible iff L^{d} is. Obviously, $(\Sigma^{\mathrm{d}})_{L^{\mathrm{d}}} = (\Sigma_L)^{\mathrm{d}}$. The remaining claims follow from this.

(b)–(d) These follow from (a), because the extended systems are duals of each other modulo the permutation of I/O rows and/or columns (rows for (b), columns for (c) and both for (d)).

(e)–(f') (Here $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ or $\mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$.)

These follow from (a), because the dual of (7.21) is equal to its counterpart for Σ^{d} and $\widetilde{\Sigma}^{\mathrm{d}}$ modulo the permutation of its two first I/O rows and columns. The well-posed case (i.e., the case without internal loop) is a special case of this (alternatively, use the same proof with (7.4 in place of (7.21)).

(This applies to "$\mathbb{Q}$ stabilizes $\mathbb{D}$", "$\mathbb{Q}$ stabilizes Σ" and "$\widetilde{\Sigma}$ stabilizes Σ", in the sense of any of Definitions 7.1.1, 7.2.1 and 7.3.1, with or without internal loop.)

(See Proposition 7.2.5(d) and Proposition 7.3.4(d) for alternative partial proofs.) □

Optimizability is the weakest reasonable extension of the finite-dimensional concept exponential stabilizability. In the infinite-dimensional theory, the former concept often takes the place of exponential stabilizability (as formulated in Definition 6.6.10), hence we shall study this concept briefly:

Definition 6.7.3 (Optimizability and estimatability) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$.

If for each $x_0 \in H$ there is $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ s.t. $x := \mathbb{A}x_0 + \mathbb{B}\tau u$ is in L^2, then we call Σ optimizable. *We call Σ* estimatable *if Σ^{d} is optimizable.*

Obviously, Σ is optimizable iff $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable (one often says that (A,B) is optimizable).

It follows from the definition that Σ is optimizable iff the cost $\|x\|_2^2 + \|u\|_2^2$ is finite for each $x_0 \in H$ there is $u \in \mathrm{L}^2(\mathbf{R}_+;U)$. Therefore, the assumption that a system is optimizable is often called the finite cost condition. The concept "$\mathcal{U}_{\mathrm{exp}}^{\Sigma}(x_0) \neq \emptyset$ for all $x_0 \in H$" of Section 8.3 is also equivalent to optimizability.

Exponential stabilizability (by state feedback) implies optimizability. In fact, even exponential stabilizability by (static or dynamic, even partial and/or with internal loop) output feedback implies optimizability, by Lemma 6.7.6, Theorem 7.2.3(c1) and Theorem 7.3.11(c1). Therefore, optimizability is a necessary condition for the solvability of any standard control problem over exponentially stabilizing state or output feedback controllers.

If, e.g., B is bounded (see Theorem 9.2.12 for weaker sufficient conditions), then optimizability is equivalent to exponential stabilizability (this is also the case for any discrete-time systems, by Proposition 13.3.14). However, it is not known whether this equivalence holds for general WPLSs (or equivalently, by Remark 6.9.5, for general Pritchard–Salamon systems). The situation is analogous with the dual properties, estimatability and exponential detectability.

The following is obvious:

Lemma 6.7.4 *If* $\left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \end{array}\right] \in \mathrm{WPLS}$ *is s.t.* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \end{array}\right]$ *is optimizable, then so is* $\left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \end{array}\right]$. □

By duality, if $\begin{bmatrix} \mathbb{A} \\ \hline \mathbb{C}_1 \\ \mathbb{C}_2 \end{bmatrix} \in$ WPLS is s.t. $\begin{bmatrix} \mathbb{A} \\ \hline \mathbb{C}_1 \end{bmatrix}$ is estimatable, then $\begin{bmatrix} \mathbb{A} \\ \hline \mathbb{C}_1 \\ \mathbb{C}_2 \end{bmatrix}$ estimatable. Next we prove the two lemmas mentioned above.

Lemma 6.7.5 *Any exponentially stabilizable system is optimizable.*

Hence any exponentially detectable system is estimatable, by duality.

Proof: Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be exponentially stabilizing for Σ. By Lemma 6.1.10(c1), $\mathrm{L}^2 \ni \mathbb{A}_\sharp x_0 = \mathbb{A}x_0 + \mathbb{B}\tau\mathbb{K}_\flat x_0$ and $\mathbb{K}_\flat x_0 \in L^2$ for all $x_0 \in H$ (see (6.133)), hence Σ is optimizable. (A second proof: apply Lemma 6.7.6 to Σ_{ext}.) $\square$

Lemma 6.7.6 (Σ_L exp. stable $\Longrightarrow$ Σ is opt. & est.) *If there is an exponentially stabilizing output feedback operator for $\Sigma \in$ WPLS, then Σ is optimizable and estimatable.*

We extend this (and partially the converse) for dynamic output feedback in Theorem 7.2.3(c1) and Theorem 7.3.12.

Proof: Now (6.125) is exponentially stable for some L, hence, for any $x_0 \in H$, the function $u := L\mathbb{C}_L x_0 \in \mathrm{L}^2$ satisfies $\mathbb{A}x_0 + \mathbb{B}\tau u = \mathbb{A}_L \in \mathrm{L}^2$. Thus, Σ is optimizable. By duality (see Lemma 6.7.2(a)), Σ is also estimatable. $\square$

We have "$u, y \in \mathrm{L}^2 \Rightarrow x \in \mathrm{L}^2$" for estimatable systems:

Theorem 6.7.7 ($u, y \in \mathbf{L}^2 \Rightarrow x \in \mathbf{L}^2$) *Let $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$ be estimatable. Then there is $M < \infty$ s.t. if $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ and $x_0 \in H$ are s.t. $y := \mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$, then $x := \mathbb{A}x_0 + \mathbb{B}\tau u \in \mathrm{L}^2 \cap \mathcal{C}_0$ and $\|x\|_2 \le M(\|x_0\|_H + \|u\|_2 + \|y\|_2)$.*

Proof: The proof will be given in Section 8.3, see Lemma 8.3.20. (Except for the $\mathcal{C}_0$ property, this would follow from Theorem 13.3.15 and Theorem 13.4.4(a3)&(e3)). $\square$

We need two more implications between the signals:

Lemma 6.7.8 ($u, x \in \mathbf{L}^2 \Rightarrow y \in \mathbf{L}^2$) *Let $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$. If $u, x \in \pi_+\mathrm{L}^2$, then $y \in \pi_+\mathrm{L}^2$, where $x_0 \in H$ is arbitrary and $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}\tau \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]\left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right]$. In fact, there is $M = M_\Sigma < \infty$ s.t. $\|y\|_2 \le M(\|u\|_2 + \|x\|_2 + \|x_0\|_H)$.*

Conversely, if $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$, $\omega \in \mathbf{R}$, $x, y \in \mathrm{L}^2$ and $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$, then $u \in \mathrm{L}^2$.

By combining the above lemma with Theorem 6.7.7, we see that "$u, x \in \mathrm{L}^2 \Rightarrow y \in \mathrm{L}^2$" holds for arbitrary WPLSs, "$x, y \in \mathrm{L}^2 \Rightarrow u \in \mathrm{L}^2$" for WPLSs with $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$, and "$u, y \in \mathrm{L}^2 \Rightarrow x \in \mathrm{L}^2$" for estimatable WPLSs.

Proof: By the second inequality of Theorem 13.4.4(a3), we have

$$\|x\|_{\ell^2_{r'}(\mathbf{N};H)} + \|\Delta^{\mathrm{S}} u\|_{\ell^2_{r'}(\mathbf{N};U)} \le M'(\|x_0\|_H + \|x\|_{\mathrm{L}^2_{\omega'}(J;H)} + \|u\|_{\mathrm{L}^2_{\omega'}(J;U)}) \qquad (6.176)$$

for some $M' = M'_\Sigma < \infty$. Combine this with Lemma 13.3.18 to obtain the first claim (thus, our M depends on Σ only).

The converse for $\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$ follows from the fact that x and u are the state and output of the closed-loop system $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}-I\end{array}\right]_{-I}$ (the "flow-inverted system") with input y and initial state x_0 (it is straightforward to verify this, see Example 6.2.4 of [Sbook]). □

For an exponentially detectable system, output-stabilization is equivalent to exponential stabilization:

Lemma 6.7.9 *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *be estimatable.*

If a state feedback pair $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *or an output injection pair* $\left[\begin{smallmatrix}\mathbb{H}\\ \mathbb{G}\end{smallmatrix}\right]$ *or a static output feedback operator L output-stabilizes Σ, then it stabilizes Σ exponentially.*

Proof: 1° *Output feedback:* Let L output-stabilize Σ. Let $x_0 \in H$. Set $u := L\mathbb{C}_L x_0 \in \mathrm{L}^2$, $x := \mathbb{A}_L x_0$, $\left[\begin{smallmatrix}y_1\\ y_2\end{smallmatrix}\right] := y := \mathbb{C}_L x_0 \in \mathrm{L}^2$, so that

$$x = \mathbb{A}x_0 + \mathbb{B}\tau u, \quad y = \mathbb{C}x_0 + \mathbb{D}u, \quad y_1 = \mathbb{C}_1 x_0 + \mathbb{D}_1 u, \tag{6.177}$$

by (6.126). Consequently, $x \in \mathrm{L}^2$, by Theorem 6.7.7. Because x_0 was arbitrary, $\mathbb{A}_L$ is exponentially stable, by Lemma A.4.5.

2° *State feedback or output injection:* Apply 1° with the extended system (which is also estimatable, by Lemma 6.7.4) in place of Σ. □

Next we explore the connection between stability and stabilizability (see Definitions 6.1.3, 6.6.10 and 6.6.21):

Theorem 6.7.10 *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}$.

(a) **(Stability)** *The following are equivalent:*

(i) Σ is stable (i.e., $\Sigma \in \mathrm{WPLS}_0$);
(ii) $\mathbb{D}$ is stable and Σ is detectable and output-stabilizable;
(iii) $\mathbb{D}$ is stable and Σ is q.r.c.-stabilizable;
(iv) $\mathbb{D}$ is stable and Σ is q.l.c.-detectable.

(b) **(SOS-Stability)** *The following are equivalent:*

(i) Σ is SOS-stable (i.e., $\Sigma \in \mathrm{SOS}$);
(ii) $\mathbb{D}$ is stable and Σ is output-stabilizable;
(iii) $\mathbb{D}$ is stable and Σ is q.l.c.-output-detectable.

(c) **(Strong stability)** *The following are equivalent:*

(i) Σ is strongly stable;
(ii) Σ is stable and strongly detectable;
(iii) $\mathbb{D}$ is stable, Σ is output-stabilizable, and strongly detectable;
(iv) $\mathbb{D}$ is stable and Σ is strongly q.r.c.-stabilizable;
(v) $\mathbb{D}$ is stable and Σ is strongly q.l.c.-detectable.

(d) **(Exponential stability)** *The following are equivalent:*

(i) Σ *is exponentially stable (i.e.,* $\Sigma \in \mathrm{WPLS}_\omega$ *for some* $\omega < 0$*);*
(ii) $\mathbb{B}$ *is stable (or* $(s-A)^{-1}B \in \mathrm{H}^\infty$*) and* Σ *is optimizable;*
(iii) $\mathbb{C}$ *is stable (or* $C(s-A)^{-1} \in \mathrm{H}^\infty$*) and* Σ *is estimatable;*
(iv) $\mathbb{D}$ *is stable, and* Σ *is optimizable and input-detectable;*
(v) $\mathbb{D}$ *is stable and* Σ *is output-stabilizable and estimatable;*
(vi) $\mathbb{D}$ *is stable and* Σ *is optimizable and q.r.c.-stabilizable;*
(vii) $\mathbb{D}$ *is stable and* Σ *is estimatable and q.l.c.-detectable;*
(viii) $\mathbb{D}$ *is stable and* Σ *is optimizable and estimatable.*

Proof: In (a)–(d), obviously (i) always implies all the other conditions (use, e.g., $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \end{array}\right]$), hence we only prove the converse claims.

We give more detailed proofs for part (b); the other proofs are given briefly (and they are more or less analogous to those in (b)).

(b) "(ii)⇒(i)": Assume (ii), i.e., that $\mathbb{D}$, $\mathbb{C}_\flat$ and $\mathbb{K}_\flat$ are stable in (6.133) (for some $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$). Then also $\mathbb{C} = \mathbb{C}_\flat - \mathbb{D}\mathbb{K}_\flat$ is stable, hence (i) holds.

"(iii)⇒(i)": Assume (iii), i.e., that $\mathbb{D}$ is stable and there is an admissible output injection pair $\left[\begin{smallmatrix} \mathbb{H} \\ \mathbb{G} \end{smallmatrix}\right]$ for Σ s.t. $\widetilde{\mathbb{M}}\mathbb{C}$ is stable, where $\widetilde{\mathbb{M}} := (I - \mathbb{G})^{-1}$, and $\widetilde{\mathbb{N}} := \widetilde{\mathbb{M}}\mathbb{D}$ and $\widetilde{\mathbb{M}}$ are q.l.c. Then also $\mathbb{D} = I^{-1}(\mathbb{D})$ is a q.l.c.f. of $\mathbb{D}$, hence $\widetilde{\mathbb{M}} = \widetilde{\mathbb{M}}I \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.4.5(d). Consequently, $\mathbb{C} = \widetilde{\mathbb{M}}^{-1}(\widetilde{\mathbb{M}}\mathbb{C})$ is stable, hence (i) holds.

(a) "(ii)$\Longrightarrow$(i)": Assume (ii). Then $\mathbb{C}$ is stable, by (b). From (6.168) we see that also $\mathbb{B} = \mathbb{B}_\sharp - \mathbb{H}_\sharp\mathbb{D}$ and $\mathbb{A} = \mathbb{A}_\sharp - \mathbb{H}_\sharp\tau\mathbb{C}$ are stable.

"(iii)⇒(i)": This follows from Lemma 6.6.17(a)&(c). "(iv)⇒(i)" is the dual result of this.

(c) "(ii)⇒(i)": This follows from Lemma 6.6.8 (where "Σ":=(6.168) and $L := \begin{bmatrix} -I & 0 \end{bmatrix}$). (Note that an analogous stabilizability result would require $\mathbb{K}$ to be stable.) "(iii)⇒(ii)" & "(v)⇒(ii)": These hold by (a). "(iv)⇒(i)": This follows from Lemma 6.6.17(a)&(c).

(d) "(iii)⇒(i)": If Σ is estimatable and $C(s-A)^{-1} \in \mathrm{H}^\infty$, then Σ is exponentially stable, by Proposition 6.2 of [WR00] (use discretization and Theorem 13.3.13 for an alternative proof; this also applies to "(viii)⇔(i)").

Assume then that $\mathbb{C}$ is stable and Σ is estimatable. By Theorem 6.7.7, $\mathbb{A}x_0 \in \mathrm{L}^2$ for all $x_0 \in H$ (take $u = 0$ and note that $y := \mathbb{C}x_0 \in \mathrm{L}^2$), hence $\mathbb{A}$ is exponentially stable, by Lemma A.4.5. An alternative proof (for the dual claim (ii)⇒(i)) is obtained by slightly modifying the proof of Proposition 6.1 of [WR00] (combined to the dual of Theorem 6.2.11(c2)).

"(v)⇒(iii)": Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be output-stabilizing for Σ. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially stabilizing, by Lemma 6.7.9, hence Σ is stable, by (a)(ii).

"(ii)⇒(i)"&"(iv)⇒(ii)": These are the duals of "(iii)⇒(i)" and "(v)⇒(iii)" (note also that $(s-A)^{-1}B \in \mathrm{H}^\infty \Leftrightarrow \mathbb{B}\tau \in \mathrm{TIC}$).

"(vi)⇒(ii)": ["(vii)⇒(iii)"]: This follows from implication (iii)[(iv)]⇒(i) of (a).

"(viii)⇔(i)": This is Theorem 6.3 of [WR00]. □

Any admissible state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ for a system Σ preserves stabilizability in the following way: if $\Sigma_\flat$ is the corresponding closed-loop system and $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ is stabilizable in some sense, then the components $\left[\begin{smallmatrix}\mathbb{A}_\flat & \mathbb{B}_\flat\\ \mathbb{C}_\flat & \mathbb{D}_\flat\end{smallmatrix}\right]$ of $\Sigma_\flat$ can be stabilized exactly in the same sense (see (a) below). This only applies two the top two rows of the closed-loop systems; we need coprimeness or exponential stabilization in order to guarantee that also the third row of the closed-loop system The situation with output injection is analogous (see (b)), whereas static output feedback (see (c)) preserves anything:

Lemma 6.7.11 (Stabilizability preserved) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$.

(a) $(\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right])$ *Optimizability, exponential stabilizability, and all* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$*-stabilizability properties (see (a')) are invariant under admissible state feedback. Moreover, if* Σ *is estimatable, then so is* $\Sigma_\flat$ *(of (6.178)).*

(a') The stabilizability of the "$\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ *part" of a system is invariant under admissible state feedback:*

Let $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *and* $\left[\begin{array}{c|c}\mathbb{K}^2 & \mathbb{F}^2\end{array}\right]$ *be admissible state feedback pairs for* Σ*, and let*

$$\Sigma_\flat := \left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat\\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\\ \mathbb{K}_\flat & \mathbb{F}_\flat\end{array}\right] \text{ and } \Sigma_{\flat'} := \left[\begin{array}{c|c}\mathbb{A}_{\flat'} & \mathbb{B}_{\flat'}\\ \hline \mathbb{C}_{\flat'} & \mathbb{D}_{\flat'}\\ \mathbb{K}^2_{\flat'} & \mathbb{F}^2_{\flat'}\end{array}\right] \tag{6.178}$$

be the corresponding closed-loop systems, respectively.

Then the state feedback pair $\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$ *defined by (6.180) is admissible for* $\Sigma_\flat$*, and the two top rows of the corresponding closed-loop system are the two top rows* $\left[\begin{smallmatrix}\mathbb{A}_{\flat'} & \mathbb{B}_{\flat'}\\ \mathbb{C}_{\flat'} & \mathbb{D}_{\flat'}\end{smallmatrix}\right]$ *of* $\Sigma_{\flat'}$*.*

Moreover, with additional assumptions, this pair $\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$ *stabilizes even more (here* $\Sigma^1_\flat := \left[\begin{smallmatrix}\mathbb{A}_\flat & \mathbb{B}_\flat\\ \mathbb{C}_\flat & \mathbb{D}_\flat\end{smallmatrix}\right]$*):*

(a1) Let $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *be q.r.c.-I/O-stabilizing for* Σ*. Then* $\left[\begin{array}{c|c}\mathbb{K}^2 & \mathbb{F}^2\end{array}\right]$ *is [q.r.c.-]I/O-stabilizing for* Σ *iff* $\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$ *is [[q]r.c.-]I/O-stabilizing for* $\Sigma^1_\flat$ *[(iff* $I - \mathbb{F}_\natural \in \mathcal{G}\mathrm{TIC}$*)].*

(a2) Let $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *be q.r.c.-SOS-stabilizing for* Σ*. Then* $\left[\begin{array}{c|c}\mathbb{K}^2 & \mathbb{F}^2\end{array}\right]$ *is [q.r.c.-]SOS-stabilizing for* Σ *iff* $\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$ *is [stable and r.c.-]SOS-stabilizing for* $\Sigma^1_\flat$*.*

Thus, $\left[\begin{array}{c|c}\mathbb{K}^2 & \mathbb{F}^2\end{array}\right]$ *[q.r.c.-]stabilizes* Σ *(resp. weakly, strongly) iff* $\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$ *[[q.]r.c.-]stabilizes* $\Sigma^1_\flat$ *(resp. weakly, strongly).*

(a3) The pair $\left[\begin{array}{c|c}\mathbb{K}^2 & \mathbb{F}^2\end{array}\right]$ *is exponentially stabilizing for* Σ *iff* $\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$ *is exponentially stabilizing for* $\Sigma^1_\flat$*.*

(a4) Let $\left[\begin{array}{c|c}\mathbb{K}^2 & \mathbb{F}^2\end{array}\right]$ *be exponentially [q.]r.c.-stabilizing for* Σ*, and let* $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *be I/O-stabilizing for* Σ*.*

Then $\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$ *is exponentially r.c.-stabilizing for* $\Sigma^1_\flat$*, and* $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *is exponentially [q.]r.c.-stabilizing for* Σ*.*

$$\begin{tikzcd} & \Sigma \arrow[dl, "{\left[\, \mathbb{K} \mid \mathbb{F} \,\right]}"'] \arrow[dr, "{\left[\, \mathbb{K}^2 \mid \mathbb{F}^2 \,\right]}"] & \\ \Sigma_\flat \arrow[rr, "{\left[\, \mathbb{K}_\natural \mid \mathbb{F}_\natural \,\right]}"'] & & \Sigma_{\flat'}, \Sigma_{\circlearrowleft} \end{tikzcd}$$

Figure 6.8: The setting of Lemma 6.7.11(a')

(a4') Let $\left[\, \mathbb{K} \mid \mathbb{F} \,\right]$ *be exponentially [[q.]r.c.-]stabilizing for* Σ. *Then* $\left[\, \mathbb{K}^2 \mid \mathbb{F}^2 \,\right]$ *is exponentially [[q.]r.c.-]stabilizing for* Σ *iff* $\left[\, \mathbb{K}_\natural \mid \mathbb{F}_\natural \,\right]$ *is I/O-stabilizing (or input-stabilizing) for* $\Sigma_\flat^1$ *(in which case* $\left[\, \mathbb{K}_\natural \mid \mathbb{F}_\natural \,\right]$ *is exponentially stable and exponentially r.c.-stabilizing for* $\Sigma_\flat^1$*).*

(a5) Let $\left[\, \mathbb{K}^2 \mid \mathbb{F}^2 \,\right]$ *be exponentially stabilizing and [q.]r.c.-stabilizing for* Σ*, and let* $\left[\, \mathbb{K} \mid \mathbb{F} \,\right]$ *be I/O-stabilizing for* Σ.

Then $\left[\, \mathbb{K}_\natural \mid \mathbb{F}_\natural \,\right]$ *is exponentially r.c.-stabilizing for* $\Sigma_\flat^1$*, and* $\left[\, \mathbb{K} \mid \mathbb{F} \,\right]$ *is exponentially stabilizing and [q.]r.c.-stabilizing for* Σ.

(a6) Claims (a1)–(a5) also hold with $\Sigma_\flat$ *in place of* $\Sigma_\flat^1$.

(b) $\left(\left[\begin{smallmatrix}\mathbb{H}\\ \hline \mathbb{G}\end{smallmatrix}\right]\right)$ *Estimatability, exponential detectability, and all* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$*-detectability properties (cf. (a)) are invariant under admissible output injection. Moreover, if* Σ *is optimizable, then so is* $\Sigma_\sharp$ *(of (6.168).*

(c) **(Σ_L)** *All different versions of stabilizability and detectability listed in Summary 6.7.1(a)–(d) (including prefices and suffices except those corresponding to 4. of Definition 6.6.10), as well as optimizability and estimatability are preserved under admissible static output feedback, i.e., the systems* Σ *and* Σ_L *of Proposition 6.6.2 are output feedback stabilizable, stabilizable, detectable or jointly stabilizable and detectable exactly in the same sense.*

Proof: (a) All $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$-stabilizability properties of Σ and $\Sigma_\flat$ are equal, by the first claim in (a'). By (a3), Σ is (equivalently, $\left[\, \mathbb{A} \mid \mathbb{B} \,\right]$ is) exponentially stabilizable iff $\Sigma_\flat$ is (interchange their roles for the converse).

In discrete time, optimizability is equivalent to exponential stabilizability, by Proposition 13.3.14, hence invariant under state feedback. By Theorem 13.4.4(e3)&(e1), optimizability and state feedback are invariant under discretization, hence optimizability is invariant under state feedback in continuous time too.

Finally, for estimatability (of $\Sigma_\flat$, not $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$), we can use discretization as above (since exponential detectability is preserved in discrete time: $A + \mathtt{H}C = A + BMK + \begin{bmatrix}\mathtt{H} & \widetilde{H}\end{bmatrix}\begin{bmatrix}C+DMK\\ MK\end{bmatrix}$, where $\widetilde{H} := -\mathtt{H}D - B$). Alternatively, we can establish the dual claim by noting that if $u \in \mathrm{L}^2$ is s.t. $x := \mathbb{A}x_0 + \mathbb{B}\tau u \in \mathrm{L}^2$, then $y := \mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$, by Lemma 6.7.8. But $\mathbb{A}_\sharp x_0 + \mathbb{B}_\sharp \tau u + \mathbb{H}_\sharp \tau(-y) = x \in \mathrm{L}^2$. Consequently, if Σ is estimatable, then so is $\Sigma_\sharp$, for any output injection pair $\left[\begin{smallmatrix}\mathbb{H}\\ \hline \mathbb{G}\end{smallmatrix}\right]$.

(a') Extend Σ to Σ_{Ext2} with these two state feedback pairs, and let $\Sigma_{\flat 2}$ be the closed-loop system of Σ_{Ext2} corresponding to $L := \begin{bmatrix} 0 & I & 0 \end{bmatrix}$, i.e.,

$$\Sigma_{\mathrm{Ext2}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F} \\ \mathbb{K}^2 & \mathbb{F}^2 \end{array}\right], \quad \Sigma_{\flat 2} := \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \\ \mathbb{K}_\flat & \mathbb{F}_\flat \\ \mathbb{K}^2_\flat & \mathbb{F}^2_\flat \end{array}\right] = \left[\begin{array}{c|c} \mathbb{A} + \mathbb{B}\tau\mathbb{M}\mathbb{K} & \mathbb{B}\mathbb{M} \\ \hline \mathbb{C} + \mathbb{D}\mathbb{M}\mathbb{K} & \mathbb{D}\mathbb{M} \\ \mathbb{M}\mathbb{K} & \mathbb{M} - I \\ \mathbb{K}^2 + \mathbb{F}^2\mathbb{M}\mathbb{K} & \mathbb{F}^2\mathbb{M} \end{array}\right], \tag{6.179}$$

where $\mathbb{M} := \mathbb{X}^{-1} := (I - \mathbb{F})^{-1} \in \mathcal{G}\mathrm{TIC}_\infty$. Because $L' := \begin{bmatrix} 0 & 0 & I \end{bmatrix}$ makes the first, second and fourth rows of Σ_{Ext2} equal to $\Sigma_{\flat'}$, it follows from Proposition 6.6.3 that $L' - L$ does the same for $\Sigma_{\flat 2}$. Therefore, the state feedback pair (here $\mathbb{M}^2 := (I - \mathbb{F}^2)^{-1}$)

$$\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right] := \left[\begin{array}{c|c} \mathbb{K}^2_\flat - \mathbb{K}_\flat & \mathbb{F}^2_\flat - \mathbb{F}_\flat \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}^2 - \mathbb{X}^2\mathbb{M}\mathbb{K} & I - \mathbb{X}^2\mathbb{M} \end{array}\right] \tag{6.180}$$

also does the same for $\Sigma_{\flat 2}$, in particular, it is admissible for $\Sigma_\flat$, and the corresponding closed-loop system

$$\Sigma_{\circlearrowleft} := \left[\begin{array}{c|c} \mathbb{A}_{\flat'} & \mathbb{B}_{\flat'} \\ \hline \mathbb{C}_{\flat'} & \mathbb{D}_{\flat'} \\ \mathbb{K}_{\flat\circlearrowleft} & \mathbb{F}_{\flat\circlearrowleft} \\ \mathbb{K}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft} \end{array}\right] := \left[\begin{array}{c|c} \mathbb{A}_\flat + \mathbb{B}_\flat\tau\mathbb{K}_{\circlearrowleft} & \mathbb{B}_\flat\mathbb{M}_\natural \\ \hline \mathbb{C}_\flat + \mathbb{D}_\flat\mathbb{K}_{\circlearrowleft} & \mathbb{D}_\flat\mathbb{M}_\natural \\ \mathbb{K}_\flat + \mathbb{F}_\flat\mathbb{K}_{\circlearrowleft} & \mathbb{F}_\flat\mathbb{M}_\natural \\ \mathbb{M}_\natural\mathbb{K}_\natural & \mathbb{M}_\natural - I \end{array}\right], \tag{6.181}$$

where $\mathbb{M}_\natural := (I - \mathbb{F}_\natural)^{-1}$. In particular, the two top rows of $\Sigma_{\circlearrowleft}$ are equal to the two top rows of $(\Sigma_{\flat 2})_{(L'-L)}$, i.e., to those of $\Sigma_{\flat'}$.

Indeed, the $\left[\begin{smallmatrix} x_0 \\ u_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right]$ map "$\left[\begin{smallmatrix} I & 0 \\ -L\mathbb{C} & I-L\mathbb{D} \end{smallmatrix}\right]^{-1}$" (cf. (6.127)) of this state feedback connection is equal to that corresponding to $L' - L$ with $\Sigma_\flat$, i.e., to $\left[\begin{smallmatrix} I & 0 \\ \mathbb{M}_\natural\mathbb{K}_\natural & \mathbb{M}_\natural \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ \mathbb{K}_{\circlearrowleft} & \mathbb{M}_\natural \end{smallmatrix}\right]$ see also formula (6.134).

The fourth row of $\Sigma_{\circlearrowleft}$ is given by

$$\left[\begin{array}{c|c} \mathbb{K}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft} \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right] \begin{bmatrix} I & 0 \\ \mathbb{M}_\natural\mathbb{K}_\natural & \mathbb{M}_\natural \end{bmatrix} = \left[\begin{array}{c|c} \mathbb{M}_\natural\mathbb{K}^2 - \mathbb{K} & \mathbb{M}_\natural - I \end{array}\right] \tag{6.182}$$

$$= \left[\begin{array}{c|c} \mathbb{X}\mathbb{M}^2\mathbb{K}^2 - \mathbb{K} & \mathbb{X}\mathbb{M}^2 \end{array}\right], \tag{6.183}$$

since $\mathbb{M}_\natural = (I - \mathbb{F}_\natural)^{-1} = (\mathbb{X}^2\mathbb{M})^{-1} = \mathbb{X}\mathbb{M}^2$.

(a1) Now $\mathbb{D} = \mathbb{D}_\flat\mathbb{M}^{-1}$ is a q.r.c.f. The maps $\mathbb{D}_{\flat'}$ and $\mathbb{M}^2$ are [q.r.c. and] stable, iff $\mathbb{M}_\natural = \mathbb{M}^{-1}\mathbb{M}^2 \in \mathrm{TIC}(U) \cap \mathcal{G}\mathrm{TIC}_\infty(U)$ $[\cap\mathcal{G}\mathrm{TIC}(U)]$, by Lemma 6.4.5(b)[(c)]. But $\mathbb{F}_{\circlearrowleft} + I = \mathbb{M}_\natural \in \mathrm{TIC}(U)[\cap\mathcal{G}\mathrm{TIC}(U)]$ iff $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is [r.c.]-I/O-stabilizing [(equivalently, [q.r.c.]-I/O-stabilizing), by Lemma 6.6.17(a)].

(Actually $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ will then [r.c.-]I/O-stabilize the whole $\Sigma_{\flat 2}$, because also the additional row $\left[\begin{array}{c|c} \mathbb{K}^2_\flat & \mathbb{F}^2_\flat \end{array}\right] \left[\begin{smallmatrix} I & 0 \\ \mathbb{M}_\natural\mathbb{K}_\natural & \mathbb{M}_\natural \end{smallmatrix}\right]$ is equal to $\left[\begin{array}{c|c} \mathbb{K}^2_{\flat'} & \mathbb{F}^2_{\flat'} \end{array}\right]$, hence I/O-stable, by assumption. A similar comment applies to (a2).)

(a2) Because the second row of $\Sigma_{\circlearrowleft}$ (i.e., of $\Sigma_{\flat'}$) is now stable, and $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is [r.c.-]I/O-stabilizing for $\Sigma_\flat$ under either assumption, by (a1), we only have to show that $\mathbb{K}_{\circlearrowleft}$ is stable iff $\mathbb{K}_{\flat'}$ is stable (since $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is q.r.c.-SOS-stabilizing iff it is stable and r.c.-SOS-stabilizing, by Lemma 6.6.17(b)).

We have

$$\mathbb{N}\mathbb{K}_{\circlearrowleft} = (\mathbb{D}\mathbb{M})(\mathbb{M}_{\natural}\mathbb{K}^2 - \mathbb{K}) = \mathbb{D}\mathbb{M}^2\mathbb{K}^2 - \mathbb{D}\mathbb{M}\mathbb{K} = \mathbb{C}_{\flat'} - \mathbb{C}_{\flat}, \quad \text{and} \tag{6.184}$$

$$\mathbb{M}\mathbb{K}_{\circlearrowleft} = \mathbb{M}\mathbb{M}_{\natural}\mathbb{K}_{\natural} = \mathbb{M}\mathbb{M}_{\natural}(\mathbb{K}^2 - (\mathbb{M}^2)^{-1}\mathbb{M}\mathbb{K}) = \mathbb{M}^2\mathbb{K}^2 - \mathbb{M}\mathbb{K} = \mathbb{K}^2_{\flat'} - \mathbb{K}_{\flat}. \tag{6.185}$$

Since $\mathbb{C}_{\flat'}$, $\mathbb{C}_{\flat}$ and $\mathbb{K}_{\flat}$ are stable under either assumption, it follows that $\mathbb{M}\mathbb{K}_{\circlearrowleft}$ and $\mathbb{N}\mathbb{K}_{\circlearrowleft}$ are stable iff $\mathbb{K}^2_{\flat'}$ is stable. But this holds iff $\mathbb{K}_{\circlearrowleft}$ is stable, by Lemma 6.5.2(c2), as required.

(The "Thus" comment follows from the fact that the first row $\left[\begin{array}{c|c} \mathbb{A}_{\flat'} & \mathbb{B}_{\flat'} \end{array}\right]$ is the same for $\Sigma_{\circlearrowleft}$ and $\Sigma_{\flat'}$.)

(a3) Now $\Sigma_{\flat'}$ is exponentially stable, hence so are $\mathbb{A}_{\flat'} = \mathbb{A}_{\circlearrowleft}$ and $\Sigma_{\circlearrowleft}$, by Lemma 6.1.10.

(a4) 1° *"Only if":* By (a3), $\mathbb{M}_{\natural} = \mathbb{F}_{\circlearrowleft} + I$, $\Sigma_{\flat'}$ and $\Sigma_{\circlearrowleft}$ are exponentially stable. But $\mathbb{M}_{\natural}^{-1} = (\mathbb{M}^2)^{-1}\mathbb{M}$ is stable, by Lemma 6.4.5(b) (with $\mathbb{U} := \mathbb{M}_{\natural}^{-1}$). Therefore, by Lemma 2.2.7, there is $\varepsilon > 0$ s.t. $\mathbb{M}_{\natural} \in \mathcal{G}\mathrm{TIC}_{-\varepsilon}(U)$ and $\Sigma_{\circlearrowleft} \in \mathrm{WPLS}_{-\varepsilon}$.

It follows that $\mathbb{M} = \mathbb{M}^2\mathbb{M}_{\natural}^{-1}$ and $\mathbb{D}_{\flat} = \mathbb{D}_{\flat'}\mathbb{M}_{\natural}^{-1}$ are exponentially [q.]r.c., by Lemma 6.4.5(c) (shifted by $-\varepsilon$; cf. Remark 2.1.6).

Moreover, $\mathbb{M}_{\natural}$ is exponentially r.c. with any $\mathrm{TIC}_{-\varepsilon}(U,*)$ operator (in particular, with the other I/O components of $\Sigma_{\circlearrowleft}$), hence $\left[\begin{array}{c|c} \mathbb{K}_{\natural} & \mathbb{F}_{\natural} \end{array}\right]$ is exponentially r.c.-stabilizing for $\Sigma_{\flat}$.

By Theorem 6.7.10(d)(vi), $\Sigma_{\flat}$ is exponentially stable, hence $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially [q.]r.c.-stabilizing for Σ.

2° *"If":* Since $\left[\begin{array}{c|c} \mathbb{K}_{\natural} & \mathbb{F}_{\natural} \end{array}\right]$ is exponentially stable (since $\mathbb{A}_{\flat}$ is), it follows that $\mathbb{M}_{\natural} \in \mathcal{G}\mathrm{TIC}$ (even exponentially, as in 1°). The rest follows as in 1°.

(a4') Since $\mathbb{A}_{\flat'} = \mathbb{A}_{\circlearrowleft}$ is common for $\Sigma_{\flat'}$ and $\Sigma_{\circlearrowleft}$, one of them is exponentially stabilizing iff the other is. By Theorem 6.7.15(b1), $\left[\begin{array}{c|c} \mathbb{K}_{\natural} & \mathbb{F}_{\natural} \end{array}\right]$ is exponentially r.c.-stabilizing iff it is I/O-stabilizing (or input-stabilizing) (since $\Sigma_{\flat}$ is exponentially stable).

Assume that this is the case. Then $\mathbb{M}_{\natural} \in \mathcal{G}\mathrm{TIC}_{-\varepsilon}(U)$ for some $\varepsilon > 0$, so that $\left[\begin{array}{c|c} \mathbb{K}^2 & \mathbb{F}^2 \end{array}\right]$ is exponentially [q.]r.c.-stabilizing iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is, by Lemma 6.4.5(c).

(a5) The proof of (a4) applies, except that now we have to apply the original (unshifted) version of Lemma 6.4.5(c).

(a6) Equation (6.181) shows that also $\left[\begin{array}{c|c} \mathbb{K}_{\flat\circlearrowleft} & \mathbb{F}_{\flat\circlearrowleft} \end{array}\right]$ of (6.181) is stable (resp. I/O-stable) whenever $\left[\begin{array}{c|c} \mathbb{K}_{\flat} & \mathbb{F}_{\flat} \end{array}\right]$ and $\left[\begin{array}{c|c} \mathbb{K}_{\circlearrowleft} & \mathbb{M}_{\natural} \end{array}\right]$ are stable (resp. I/O-stable).

Therefore, (a3)–(a5) and the "only if" parts of (a1)–(a2) hold for $\Sigma_{\flat}$ in place of $\Sigma^1_{\flat}$ (for the [q.]r.c. claims this is trivial since in those cases we always have (exponentially in (a4)) $\mathbb{M}_{\natural} \in \mathcal{G}\mathrm{TIC}$, as shown in the proof of (a1), so that in those cases $\left[\begin{array}{c|c} \mathbb{K}_{\natural} & \mathbb{F}_{\natural} \end{array}\right]$ is r.c.-I/O-stabilizing whenever it is I/O-stabilizing (exponentially r.c.-stabilizing in (a4))). For the "if" parts of (a1)–(a2), we note that

$$\begin{bmatrix} \mathbb{D}_{\flat'} \\ \mathbb{F}_{\flat'\circlearrowleft} \\ \mathbb{M}_\natural \end{bmatrix} = \begin{bmatrix} \mathbb{D}_{\flat'} \\ (\mathbb{M}-I)\mathbb{M}_\natural \\ \mathbb{M}_\natural \end{bmatrix} \tag{6.186}$$

is [quasi-]left-invertible over TIC iff $\begin{bmatrix} \mathbb{D}_{\flat'} \\ \mathbb{M}_\natural \end{bmatrix}$ is [quasi-]left-invertible over TIC.

(b) This follows from (a) by duality, i.e., by taking causal adjoints (note that "strongly stable" maps to "strongly* stable").

(c) The preservation of optimizability and estimatability is shown in Theorem 7.3 of [WR00]; for the rest we deduce as follows:

Let L be an admissible static output feedback operator for Σ.

1° For (a) of Summary 6.7.1, the claim follows from Lemma 6.6.3 (what K is for Σ, that $K-L$ is for Σ_L).

2° Part (b) of Summary 6.7.1: Use the notation of Proposition 6.6.2 and Definition 6.6.10 and L and let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be admissible for Σ. Set

$$\left[\begin{array}{c|c} \mathbb{K}_L & \mathbb{F}_L \end{array}\right] := \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] \left[\begin{smallmatrix} I & 0 \\ L\mathbb{C}_L & (I-L\mathbb{D})^{-1} \end{smallmatrix}\right] \tag{6.187}$$

(cf. (6.126); thus this is the bottom line of $(\Sigma_{\mathrm{ext}})_{[L\ 0]}$). Then $\left[\begin{array}{c|c} \mathbb{K}_L & \mathbb{F}_L \end{array}\right]$ is admissible for Σ_L. Now (we leave the details to the reader)

$$\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right] := \left[\begin{array}{c|c} \mathbb{K}_L - L\mathbb{C}_L & \mathbb{F}_L - L\mathbb{D}_L \end{array}\right] \tag{6.188}$$

is admissible for Σ_L, because

$$(I-\mathbb{F}_\natural)^{-1} = (\mathbb{X}(I+L\mathbb{D}_L))^{-1} = (I-L\mathbb{D})\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty(U). \tag{6.189}$$

Therefore, $\left[\begin{smallmatrix} I & 0 \\ -\mathbb{K}_\natural & I-\mathbb{F}_\natural \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} I & 0 \\ (I-L\mathbb{D})\mathbb{M}\mathbb{K} & (I-L\mathbb{D})\mathbb{M} \end{smallmatrix}\right]$. Consequently, the combined $\left[\begin{smallmatrix} x_0 \\ u_\flat \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} x_0 \\ u_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right]$ map is given by

$$\begin{bmatrix} I & 0 \\ L\mathbb{C}_L & (I-L\mathbb{D})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ (I-L\mathbb{D})\mathbb{M}\mathbb{K} & (I-L\mathbb{D})\mathbb{M} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathbb{M}\mathbb{K} & \mathbb{K} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -\mathbb{K} & I-\mathbb{F} \end{bmatrix}. \tag{6.190}$$

Consequently, the closed-loop system of Σ_L with the state feedback pair $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is given by

$$\left[\begin{array}{c|c} \mathbb{A}_L & \mathbb{B}_L \\ \hline \mathbb{C}_L & \mathbb{D}_L \\ \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]_{[L\ 0]} = \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \\ \mathbb{K}_\flat - L\mathbb{C}_\flat & \mathbb{F}_\flat - L\mathbb{D}_\flat \end{array}\right]. \tag{6.191}$$

Thus, if $\Sigma_\flat$ is stable in some sense, then so is (6.191). Moreover, $\mathbb{D}_\flat$ and $(\mathbb{F}_\flat + I) - L\mathbb{D}_\flat$ are [q.]r.c. iff $\mathbb{N} := \mathbb{D}_\flat$ and $\mathbb{M} := (\mathbb{F}_\flat + I)$ are [q.]r.c., by Lemma 6.5.1(d). Obviously, $\mathbb{D}_\flat, \mathbb{F}_\flat + I \in \widetilde{\mathcal{A}}$ iff $\mathbb{D}_\flat, (\mathbb{F}_\flat + I) - L\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$. Thus, all prefices and suffices are preserved except those of 4. of Definition 6.6.10.

(Remark: the prefices in 4. of Definition 6.6.10 need not preserve, i.e., $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ might be, e.g., SR or exponentially stable even if $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ were not. Our claim concerns only the stabilizability of Σ and Σ_L, including prefices and suffices others than those concerning the direct properties of $\mathbb{K}$ and $\mathbb{F}$.)

3° Part (c) of Summary 6.7.1: this is analogous to 2° (use $\left[\begin{smallmatrix} \mathbb{H}_L - \mathbb{B}_L L \\ \hline \mathbb{G}_L - \mathbb{D}_L L \end{smallmatrix}\right]$).

4° Part (d) of Summary 6.7.1: Let Σ and (6.169) be as in Definition 6.6.21, let $L \in \mathcal{B}(Y,U)$ be admissible for Σ and let $\left[\begin{array}{c|cc} \mathbb{A}_L & \mathbb{H}_L & \mathbb{B}_L \\ \hline \mathbb{C}_L & \mathbb{G}_L & \mathbb{D}_L \\ \mathbb{K}_L & \mathbb{E}_L & \mathbb{F}_L \end{array}\right]$ be the closed-loop system of (6.169) induced by the output feedback operator $\left[\begin{smallmatrix} L & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then one can easily verify that $\left[\begin{array}{c|c} \mathbb{K}_L - L\mathbb{C}_L & \mathbb{F}_L - L\mathbb{D}_L \end{array}\right]$ and $\left[\begin{smallmatrix} \mathbb{H}_L - \mathbb{B}_L L \\ \hline \mathbb{G}_L - \mathbb{D}_L L \end{smallmatrix}\right]$ are jointly admissible (with the interaction operator $\mathbb{E}_L' := \mathbb{E}_L - L\mathbb{G}_L + \mathbb{F}_L L - L\mathbb{D}_L L$) for $\Sigma_L := \left[\begin{array}{c|c} \mathbb{A}_L & \mathbb{B}_L \\ \hline \mathbb{C}_L & \mathbb{D}_L \end{array}\right]$ (see the proof of [Sbook, Lemma 8.2.7] for a heuristic derivation for the formulae).

Straightforward computations show that the closed-loop system corresponding to output feedback operator $\left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ is

$$\left[\begin{array}{c|cc} \mathbb{A}_L & \mathbb{H}_L - \mathbb{B}_L L & \mathbb{B}_L \\ \hline \mathbb{C}_L & \mathbb{G}_L - \mathbb{D}_L L & \mathbb{D}_L \\ \mathbb{K}_L - L\mathbb{C}_L & \mathbb{E}_L - L\mathbb{F}_L + \mathbb{G}_L L & \mathbb{F}_L \end{array}\right]_{\left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]} = \left[\begin{array}{c|cc} \mathbb{A}_\flat & \mathbb{H}_\flat - \mathbb{B}_\flat L & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{G}_\flat - \mathbb{D}_\flat L & \mathbb{D}_\flat \\ \mathbb{K}_\flat - L\mathbb{C}_\flat & \mathbb{E}_\flat - L\mathbb{F}_\flat - \mathbb{G}_\flat L & \mathbb{F}_\flat \end{array}\right]. \tag{6.192}$$

Thus, if $\left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ is stabilizing for (6.169), then it is stabilizing for Σ_{ext}^L; the same applies all prefices and suffices (except for the ones concerning the stability and regularity of $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$; cf. 2° above). A similar computation shows that the same applies the output feedback operator $\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Therefore, the system Σ_L inherits the joint stabilizability properties of Σ.

□

We often need to apply part (a) of the above lemma with Σ and $\Sigma_\flat$ interchanged. Let us make this explicit:

Lemma 6.7.12 *Let $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ be an admissible state feedback pair for Σ with closed-loop system $\Sigma_\flat$, and let $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ be an admissible state feedback pair for $\Sigma_\flat$ with closed-loop system $\Sigma_\circlearrowleft$. Then*

$$\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}_\natural + \mathbb{X}_\natural \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} + \mathbb{F}_\natural - \mathbb{F}_\natural \widetilde{\mathbb{F}} \end{array}\right] = \left[\begin{array}{c|c} \mathbb{X}'\mathbb{K}_\flat + \mathbb{K}_\natural & \mathbb{F}' \end{array}\right] \tag{6.193}$$

is an admissible state feedback pair for Σ with closed-loop system

$$\Sigma_\circlearrowleft' := \left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}_\circlearrowleft' & \mathbb{F}_\circlearrowleft' \end{array}\right] = \left[\begin{array}{c|c} \mathbb{A}_\flat + \mathbb{B}\mathbb{M}'\tau\mathbb{K}_\natural & \mathbb{B}\mathbb{M}' \\ \hline \mathbb{C}_\flat + \mathbb{N}'\mathbb{K}_\natural & \mathbb{N}' \\ \mathbb{K}_\flat + \mathbb{M}'\mathbb{K}_\natural & \mathbb{M}' - I \end{array}\right], \tag{6.194}$$

where $\mathbb{X}' := I - \mathbb{F}'$, $\mathbb{M}' := (\mathbb{X}')^{-1}$, $\mathbb{N}' := \mathbb{D}\mathbb{M}' = \mathbb{D}_\flat \mathbb{X}_\natural^{-1}$, $\mathbb{X}_\natural := I - \mathbb{F}_\natural$, $\widetilde{\mathbb{X}} := I - \widetilde{\mathbb{F}}$. Moreover, $\mathbb{X}' = \mathbb{X}_\natural \widetilde{\mathbb{X}}$, $\mathbb{K}_\circlearrowleft' = \mathbb{K}_\flat + \widetilde{\mathbb{M}}\mathbb{K}_\circlearrowleft$ and $\mathbb{M}' = \widetilde{\mathbb{M}}\mathbb{M}_\natural$.

In particular, $\Sigma_\circlearrowleft'$ is equal to $\Sigma_\circlearrowleft$ except for $\mathbb{K}_\circlearrowleft'$ and $\mathbb{F}_\circlearrowleft'$. Also Lemma 9.12.3(a)–(c) apply (with same proofs); see also Proposition 6.6.18(f).

Proof: Apply Lemma 6.7.11(a) with substitutions $\Sigma \mapsto \Sigma_\flat$, $\Sigma_\flat \mapsto \Sigma$, $\Sigma_{\flat'} \mapsto \Sigma_\circlearrowleft$, so that "$\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$"$= \left[\begin{array}{c|c} -\mathbb{K}_\flat & -\mathbb{F}_\flat \end{array}\right]$, by Lemma 6.6.14,

$$\begin{tikzcd}
 & \Sigma \arrow[dl, "{\left[\begin{array}{c|c}\widetilde{\mathbb{K}} & \widetilde{\mathbb{F}}\end{array}\right]}"'] \arrow[dr, "{\left[\begin{array}{c|c}\mathbb{K}' & \mathbb{F}'\end{array}\right]}"] & \\
\Sigma_\flat \arrow[rr, "{\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]}"'] & & \Sigma_\circlearrowleft, \Sigma'_\circlearrowleft
\end{tikzcd}$$

Figure 6.9: The setting of Lemma 6.7.12

"$\left[\begin{array}{c|c}\mathbb{K}^2 & \mathbb{F}^2\end{array}\right]$"$= \left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$ and "$\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$"$= \left[\begin{array}{c|c}\mathbb{K}' & \mathbb{F}'\end{array}\right]$. $\square$

In optimization problems, one often first stabilizes a system exponentially and then optimizes the exponentially stable closed-loop system with the aid of a spectral factorization.

Since optimization should be independent on preliminary stabilization, one would expect the same for the existence of a spectral factorization; this is indeed the case:

Lemma 6.7.13 (Exp. stabilized SpF) *Let $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ and $\left[\begin{array}{c|c}\mathbb{K}^2 & \mathbb{F}^2\end{array}\right]$ be exponentially stabilizing state feedback pairs for $\Sigma \in \mathrm{WPLS}(U,H,Y)$ with closed-loop systems $\Sigma_\flat$ and $\Sigma_\circlearrowleft$, respectively.*

If $\mathbb{D}_\circlearrowleft^* J\mathbb{D}_\circlearrowleft = \widetilde{\mathbb{X}}^* S\widetilde{\mathbb{X}}$ for some $J = J^* \in \mathcal{B}(Y)$, $S \in \mathcal{GB}(U)$ and $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}(U)$, then $\mathbb{D}_\flat^* J\mathbb{D}_\flat = (\mathbb{X}')^* J\mathbb{X}'$, where $\mathbb{X}' := \widetilde{\mathbb{X}}(I-\mathbb{F}^2)(I-\mathbb{F})^{-1} \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}(U)$.

Proof: By Lemma 6.7.11(a'), $\left[\begin{array}{c|c}\mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$:=(6.180) is exponentially stable and exponentially stabilizing for $\Sigma_\flat$ (since $\mathbb{A}_\flat$ and $\mathbb{A}_\circlearrowleft$ are exponentially stable; see Lemma 6.1.10), and it leads to closed-loop system with same top two rows. In particular, $\mathbb{X}_\natural := I - \mathbb{F}_\natural = (I-\mathbb{F}^2)(I-\mathbb{F})^{-1} \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}(U)$, and $\mathbb{D}_\circlearrowleft = \mathbb{D}_\flat \mathbb{X}_\natural^{-1}$.

Consequently, $\mathbb{D}_\flat^* J\mathbb{D}_\flat = \mathbb{X}_\natural^* \mathbb{D}_\circlearrowleft^* J\mathbb{D}_\circlearrowleft \mathbb{X}_\natural = (\mathbb{X}')^* J\mathbb{X}'$, where $\mathbb{X}' := \widetilde{\mathbb{X}}\mathbb{X}_\natural$. By Lemma 6.4.7(c), $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$, hence $\mathbb{X}' \in \mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$. $\square$

By combining Lemma 6.7.11 and Theorem 6.7.10 we obtain the following result:

Proposition 6.7.14 (An I/O-stabilizing L is stabilizing) *Let $\Sigma \in \mathrm{WPLS}$.*

(a) **(SOS)** *If Σ is SOS-stabilizable, then any I/O-stabilizing static output feedback operator L for Σ is SOS-stabilizing.*

(b) **([Strong] stability)** *Suppose that any of the following conditions holds:*

(1.) Σ is [[exponentially] strongly] q.r.c.-stabilizable.

(2.) Σ is [[exponentially] strongly] q.l.c.-detectable.

(3.) Σ is SOS-stabilizable and [[exponentially] strongly] detectable.

(4.) Σ is detectable and [exponentially] stabilizable.

Then any I/O-stabilizing static output feedback operator L for Σ is [[exponentially] strongly] stabilizing.

(c) **(Exponential stability)** *Suppose that any of the following conditions holds:*

(1.) Σ is optimizable and estimatable;
(2.) Σ is optimizable and input-detectable;
(3.) Σ is estimatable and output-stabilizable;
(4.) Σ is optimizable and q.r.c.-stabilizable;
(5.) Σ is estimatable and q.l.c.-detectable.

Then any I/O-stabilizing static output feedback operator L for Σ is exponentially stabilizing. Conversely, if some I/O-stabilizing output feedback operator for Σ is exponentially stabilizing, then (1.) holds.

Note the "missing strong stabilizability result" in (4.) corresponding to the analogous "results" missing in Theorems 6.7.10(c), 7.2.3 and 7.3.11 (we do not even know whether such "results" are true).

Of course, one can analogously deduce from Theorem 6.7.10, that, e.g., if Σ is estimatable, then an output-stabilizing L is exponentially stabilizing, but the above results will be applied to the I/O-stabilization theory of Chapter 7, hence we are only interested of consequences of I/O-stabilization only.

Proof: (a)&(b)(1.) By Lemma 6.7.11(c), the resulting closed-loop system Σ_L is also [SOS-/strongly/exponentially] q.r.c.-stabilizable. Because it is I/O-stable, by the assumption, it is [SOS-/strongly/exponentially] stable, by Theorem 6.7.10 (note that "q.r.c." is not needed in the SOS case).

The proofs of (b) assuming (2.), (3.) or (4.) are analogous. (We do not know whether the (mere) "strong" version of (4.) holds; cf. Theorem 6.7.10(c)&(d)(iv).)

(c) The proof of (c) is analogous to that of (b), except for the necessity of (1.): If $L \in \mathcal{B}(Y,U)$ is exponentially stabilizing for Σ, then $u := L(I-\mathbb{D}L)^{-1}\mathbb{C}x_0$ is in L^2 and satisfies $\mathbb{A}x_0 + \mathbb{B}u \in \mathrm{L}^2$, by (6.125). By duality (see Lemma 6.7.2(a)), also Σ^{d} is optimizable, i.e., Σ is estimatable. □

We now list similar (often weaker) results on state feedback stabilization:

Theorem 6.7.15 (An I/O-stabilizing $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is stabilizing) *Let $\Sigma \in$ WPLS, let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be an admissible state feedback pair for Σ.*

(a1) Let Σ be [q.]r.c.-stabilizable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is [q.]r.c.-stabilizing for Σ iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is q.r.c.-SOS-stabilizing for Σ.

(a2) Let Σ be strongly [q.]r.c.-stabilizable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is strongly [q.]r.c.-stabilizing for Σ iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is q.r.c.-SOS-stabilizing for Σ.

(b1) Let Σ be exponentially [q.]r.c.-stabilizable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially [q.]r.c.-stabilizing for Σ iff it is I/O-stabilizing or input-stabilizing.

(b2) Let Σ have a exponentially stabilizing and [q.]r.c.-stabilizing state feedback pair. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially stabilizing and [q.]r.c.-stabilizing for Σ iff it is I/O-stabilizing or input-stabilizing.

(b3) Let Σ be optimizable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially stabilizing for Σ iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is input-stabilizing for Σ.

(c1) Let Σ be output stabilizable (or optimizable) and estimatable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially q.r.c.-stabilizing for Σ iff it is I/O-stabilizing, output-stabilizing or input-stabilizing.

(c2) Let Σ be estimatable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is q.r.c.-I/O-stabilizing iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is I/O-stabilizing.

Moreover, $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially q.r.c.-stabilizing iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is output-stabilizing.

(c3) Let Σ be input-detectable. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially q.r.c.-stabilizing iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially stabilizing.

(d) Let $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ be [strongly] stable. If $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is output-stabilizing, then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ [strongly] stabilizes $\mathbb{A}$.

(e) If Σ is exponentially stable, then the following are equivalent:

(i) $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is I/O-stabilizing
(ii) $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is input-stabilizing
(iii) $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is output-stabilizing
(iv) $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially r.c.-stabilizing;
(v) $(I-\mathbb{F})^{-1} \in \mathrm{TIC}$.

We leave the dual results (concerning output injection) to the reader (see [Sbook, Theorem 8.4.11]). Note that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is I/O-stabilizing iff $\mathbb{N}, \mathbb{M} \in \mathrm{TIC}$, where $\mathbb{M} = (I-\mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$ (hence $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$).

The conclusion of part (b1) is quite strong: any I/O-stabilizing feedback pair for Σ is exponentially q.r.c.-stabilizing (note that the q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ mentioned above is exponentially q.r.c.); this conclusion is based on Lemma 6.1.10(a1).

Recall from Lemma 6.6.13 that an exponentially stable system is obviously exponentially r.c.-stabilizable and estimatable, and a strongly stable system is strongly r.c.-detectable.

The results in (a) are not as strong as their counterparts for static output feedback. The reason is that we had to assume that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is q.r.c.-SOS-stabilizing, because an I/O-stabilizing pair would only guarantee that some pair would stabilize $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$, by Lemma 6.7.11(a), and that is not enough for Theorem 6.7.10.

Proof of Theorem 6.7.15: The "only if" parts are trivial, so we only prove the "if" parts, with the notation of Definition 6.6.10.

(a1)[&(a2)] By Lemma 6.7.11(a2), some stable pair $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ r.c.-stabilizes $\Sigma_\flat$. By Lemma 6.7.10(a)(iii)[(c)(iv)], $\Sigma_\flat$ is [strongly] stable.

If Σ has a r.c.f., then the q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f., by Lemma 6.4.5(c).

(b1)&(b2) These follow from Lemma 6.7.11(a4)&(a5).

(b3) If Σ is exponentially stabilizable, then, by Lemma 6.7.11(a3), $\Sigma_\flat$ is exponentially stabilizable, hence it is exponentially stable, by Theorem 6.7.10(d)(ii).

Assume then that Σ is optimizable. Then $\Delta^{S}\Sigma_\flat$ is exponentially stable, by the discrete-time version of this claim, hence $\Sigma_\flat$ is exponentially stable.

(c1) By (c2), Σ is exponentially q.r.c.-stabilizable. Therefore, the claims on input- and I/O-stabilization follow from (b1); the claim on output-stabilization follows from (c2).

(If Σ is optimizable and estimatable, then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially q.r.c.-stabilizing by discretization of Σ, $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and this result (see Proposition 13.3.14).)

(c2) 1° Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be output-stabilizing. By Lemma 13.3.17(b), $\Delta^{S}\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially r.c.-stabilizing for $\Delta^{S}\Sigma$, hence $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially q.r.c.-stabilizing for Σ, by Theorem 13.4.4(e1).

2° Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be I/O-stabilizing. By Lemma 13.3.17(c), $\Delta^{S}\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially r.c.-stabilizing for $\Delta^{S}\Sigma$, hence $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially q.r.c.-I/O-stabilizing for Σ, by Theorem 13.4.4(e1).

(c3) By the dual of (c1), any input-stabilizable $\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right]$ is exponentially q.l.c.-stabilizing, hence Σ is exponentially detectable. Therefore, (c3) follows from (c1).

(d) This follows from Lemma 6.6.8. (Note that if $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is SOS-stabilizing, then it is [strongly] stabilizing.)

(e) This follows from (c1) and the fact that $\mathbb{D}$ and $\mathbb{F}$ are necessarily (exponentially) stable (by Lemma 6.1.10(a1)). □

Since SOS-q.r.c.-stabilizability is rather important for optimization theory (unless we require the closed-loop system to be exponentially stable), we summarize below three cases in which this property can be deduced from other stabilizability and detectability properties:

Corollary 6.7.16 (q.r.c.-stabilizable) *If Σ is [strongly/SOS-]stable, or Σ is jointly [strongly/SOS-]stabilizable and I/O-detectable, or Σ is output-stabilizable and estimatable, then Σ is [strongly/SOS-]q.r.c.-stabilizable.* □

(This follows from Lemma 6.6.13 (r.c.), Theorem 6.6.28 (r.c.) and Theorem 6.7.15(c1) (exponentially q.r.c.).)

We note that certain kind of similarity transforms do not affect the properties of a system:

Lemma 6.7.17 (Permutations) *Let $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$. Let also U', H' and Y' be Hilbert spaces, $F \in \mathcal{GB}(U',U)$, $E \in \mathcal{GB}(H',H)$, and $G \in \mathcal{GB}(Y',Y)$. Then the following systems are also WPLSs and have the exactly the same stability, stabilizability and detectability properties (those listed in Summary 6.7.1, including prefices and suffices and optimizability and estimatability) as Σ has:*

$$\left[\begin{array}{c|c} E^{-1}\mathbb{A}E & E^{-1}\mathbb{B} \\ \hline \mathbb{C}E & \mathbb{D} \end{array}\right], \quad \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}F \\ \hline \mathbb{C} & \mathbb{D}F \end{array}\right], \quad \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline G\mathbb{C} & G\mathbb{D} \end{array}\right]. \tag{6.195}$$

If, instead, we do not require F and G to be invertible, then the above systems are still WPLSs and have same stability properties and Σ (but the stabilizability and detectability properties may be weaker or stronger in general).

The same holds for

$$\left[\begin{array}{c|c} L\mathbb{A}R & L\mathbb{B} \\ \hline \mathbb{C}R & \mathbb{D} \end{array}\right] \tag{6.196}$$

if $R \in \mathcal{B}(H', H)$ and $L \in \mathcal{B}(H, H')$ are s.t. $LR = I_{H'}$ and $\mathbb{C}P\mathbb{A}P = \mathbb{C}\mathbb{A}P$, $P\mathbb{A}P\mathbb{B} = P\mathbb{A}\mathbb{B}$ and $P\mathbb{A}P\mathbb{A}P = P\mathbb{A}P$, where $P := RL \in \mathcal{B}(H)$.

Obviously, invertible E, F and G do not essentially affect the other properties of a system either. In particular, permutations (of the above form) of rows and columns do not affect the properties of a system (matrix).

Proof: (We will use this lemma only for properties (a)–(d) of Summary 6.7.1, so the reader need not worry about that (e)–(f') have not been studied this far.)

We only prove the "if" claims; the converses will follow from this by applying E^{-1} (resp. F^{-1}, G^{-1}) in place of E (resp. F, G).

It is obvious that the three systems of (6.195) are in WPLS_ω if Σ is (for any $\omega \in \mathbf{R}$).

For E, the operators stabilizing/detecting Σ will stabilize/detect the first system in (6.195) in any case.

For F, we may use $F^{-1}L$ in (a) to obtain $\left[\begin{smallmatrix} \mathbb{A}_L & \mathbb{B}_L F \\ \mathbb{C}_L & \mathbb{D}_L F \end{smallmatrix}\right]$ as the closed-loop system; similar remarks prove the other cases too (with the notations of the corresponding definitions, use $\left[\begin{array}{c|c} F^{-1}\mathbb{K} & F^{-1}\mathbb{F}F \end{array}\right]$ in (b), $F^{-1}\mathbb{Q}$ in (c), multiply (7.20) by $H := \left[\begin{smallmatrix} F & 0 \\ 0 & I_{Y\times\Xi} \end{smallmatrix}\right]$ to the right and by its inverse to the left (so that $L = I = H^{-1}IH$ is stabilizing for the resulting system, by (a)) to prove (e'), and so on.

One easily verifies the claim on (6.196) by using Lemma A.4.2(h2). □

In the dynamic output feedback theory of Chapter 7, we will often combine a plant $\Sigma_1 \in \mathrm{WPLS}(U, H, Y)$ and its controller $\Sigma_2 \in \mathrm{WPLS}(Y, H_2, U)$ to form a larger system in the following way (with the third and fourth rows interchanged for convenience):

Lemma 6.7.18 *Let $\Sigma_i = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U_i, H_i, Y_i)$ for $i = 1, 2$. If Σ_1 and Σ_2 have some property (with some allowable prefices and suffices) listed in (a)–(d) of Summary 6.7.1, then so does their* parallel connection

$$\Sigma := \left[\begin{array}{cc|cc} \mathbb{A}_1 & 0 & \mathbb{B}_1 & 0 \\ 0 & \mathbb{A}_2 & 0 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & 0 & \mathbb{D}_1 & 0 \\ 0 & \mathbb{C}_2 & 0 & \mathbb{D}_2 \end{array}\right]. \tag{6.197}$$

Moreover, Σ_1 and Σ_2 are optimizable (resp. estimatable) iff Σ is optimizable (resp. estimatable).

Proof: This is obvious, because the required stabilizing/detecting operators for Σ can be combined from those for Σ_i ($i = 1, 2$); e.g., we use $L := \left[\begin{smallmatrix} L_1 & 0 \\ 0 & L_2 \end{smallmatrix}\right]$

and $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right] := \left[\begin{array}{cc|cc}\mathbb{K}_1 & 0 & \mathbb{F}_1 & 0\\ 0 & \mathbb{K}_2 & 0 & \mathbb{F}_2\end{array}\right]$ (the symbols correspond to those in Summary 6.7.1). □

The concept "ω-stabilization" is sometimes used in the literature. All our stabilization results can be shifted to ω-stabilization results:

Remark 6.7.19 (ω-stabilization) *It follows from Remark 6.1.9 that any feedback or injection X (see Summary 6.7.1) is ω-stabilizing for $\Sigma \in$ WPLS iff $\mathcal{T}_{-\omega}X$ is stabilizing for $\mathcal{T}_{-\omega}\Sigma$ (all prefices and suffices apply).*

In particular, one gets directly a corollary about ω-stabilization of any of the results in this section (or in those to follow).

For example, if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f. over TIC_ω, then L ω-stabilizes $\mathbb{D}$ iff $(\mathbb{M} - L\mathbb{N})^{-1} \in \mathrm{TIC}_\omega$, by Lemma 6.6.6. These corollaries can be used to deduce results on exponential stabilization. □

Recall that "exponentially" means "for some $\omega < 0$" and that hence "exponentially strongly" is equivalent to "exponentially".

Notes

Most of Lemma 6.7.2 is well known. Optimizability has become popular since (or before) [FLT], estimatability is from [WR00]. Lemmas 6.7.5 and 6.7.6 are from [WR00] (in fact, from earlier conference versions of [WR00]). While writing these notes, we found an independent copy of Lemma 6.7.8 in [WR00]. The implication $u, y \in \mathrm{L}^2 \Rightarrow x \in \mathrm{L}^2$ of Theorem 6.7.7 was given in Lemma 14.1 of [ZDG] for finite-dimensional exponentially detectable systems.

In the spring of 1999, the manuscript of this monograph contained (a) and the q.r.c. and q.l.c. parts of Proposition 6.7.14; parts of Theorem 6.7.10 and Lemma 6.7.11(c) were implicitly contained in its proof. O. Staffans adopted these into [Sbook] and expanded them to early variants of Theorem 6.7.10 (including (c)(ii)&(iii) and weaker variants of (a)(ii) and (d)(ii)&(iii)) and of Lemma 6.7.11. We then adopted these and expanded them to Theorem 6.7.10 by adding further results and (d)(iii) from Theorem 7.4 of [WR00] (which extended Corollary 1.8 of [Rebarber]), and the H^∞ parts of (d)(ii)&(iii) from Propositions 6.1&6.2 of [WR00].

The claims on optimizability and estimatability in Lemma 6.7.11(c) are from Theorem 7.3 of [WR00]; most of the rest is from Lemma 8.2.7 of [Sbook]. Much of Lemma 6.7.11(a)&(b) and Lemma 6.7.12 is based on Lemma 4.5 of [S98a].

The r.c. parts of (a1) and (a2) and the (r.c.) I/O part of (b1) of Theorem 6.7.15 are from Theorem 8.4.11 of [Sbook].

Several of the above results can be found in [Sbook] (probably even more in its final version) with a more detailed proof; see [Sbook] also for further results. The article [WR00] includes further results on optimizability and estimatability.

6.8 Systems with $\mathbb{A}Bu_0 \in \mathrm{L}^p([0,1];H)$

I must Create a System, or be enslav'd by another Man's;
I will not Reason and Compare: my business is to Create.
— William Blake (1757–1827), "The Words of Los"

In this section we shall study the properties of systems whose semigroup is smoothing in the sense described below. In Section 9.2, we shall establish a rather complete Riccati equation theory for such systems.

If B is bounded ($B \in \mathcal{B}(U,H)$) or A is smoothing, then we may have $\mathbb{A}(t)Bu_0 \in H$ for a.e. $t > 0$ whenever $u_0 \in U$ (this is typical for parabolic-type systems), in which case actually $\mathbb{A}B \in \mathcal{C}((0,\infty);\mathcal{B}(U,H))$, as shown in (b1) below. However, unless $Bu_0 \in H$, we have $\|\mathbb{A}(t)Bu_0\|_H \to \infty$ as $t \to 0+$. Nevertheless, one often has $\mathbb{A}Bu_0 \in \mathrm{L}^p((0,\varepsilon);H)$ for some (hence all) $\varepsilon > 0$; in that case we actually have $\mathbb{A}Bu_0 \in \mathrm{L}^p_\omega(\mathbf{R}_+;H)$ for any $\omega > \omega_A$. If the above condition is satisfied for all $u_0 \in U$, then $\mathbb{A}B \in \mathcal{B}(U,\mathrm{L}^p_\omega(\mathbf{R}_+;H))$ for all $\omega > \omega_A$ and we have a number of additional tools and regularity properties at hands, as one observes from the results of this section. Naturally, an analogous claim applies to the dual property $\mathbb{A}^*C^*y_0 \in \mathrm{L}^p((0,\varepsilon);H)$ ($y_0 \in Y$).

As above, by "$\mathbb{A}Bu_0 \in \mathrm{L}^p([0,T);*)$" we mean just that $\pi_{[0,T)}\mathbb{A}Bu_0 \in \mathrm{L}^p([0,T);*)$ (i.e., this expression does not say anything about $\pi_{[T,\infty)}\mathbb{A}Bu_0$). The reader might wish to recall Proposition 6.3.4 before going on.

Lemma 6.8.1 ($\mathbb{A}B \in \mathcal{B}(U,\mathrm{L}^p)$) *Let $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, $\omega > \omega_A$, $T > 0$ and $p \in [1,\infty]$.*

(a) We have $\mathbb{A}B \in \mathcal{B}(U,\mathrm{L}^p_\omega(\mathbf{R}_+;H))$ iff $\mathbb{A}Bu_0 \in \mathrm{L}^p([0,T);H)$ for all $u_0 \in U$.

This holds for $p = 2$ iff $(s-A)^{-1}B \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,H))$.

(b1) For any $t > 0$, the following are equivalent:

(i) $\mathbb{A}^t B[U] \subset H$;
(ii) $\mathbb{A}^{t\,*}[H] \subset \mathrm{Dom}(B^*_{\mathrm{w}})$;
(iii) $\mathbb{A}^{t\,*}[H] \subset \mathrm{Dom}(B^*_{\mathrm{L,s}})$;
(iv) $B^*\mathbb{A}^{t\,*}$ extends to $\mathcal{B}(H,U)$.

If (i) holds, then $\mathbb{A}^s B \in \mathcal{B}(U,H)$, $\mathbb{A}^{s*} \in \mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}}))$ and $(\mathbb{A}^s B)^* = B^*_{\mathrm{L,s}}\mathbb{A}^{s*}$ for all $s \geq t$, $\mathbb{A}B \in \mathcal{C}([t,\infty);\mathcal{B}(U,H))$ and $\mathbb{A}^* \in \mathcal{C}([t,\infty);\mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}})))$.

(b2) $\mathbb{A}^t Bu_0 \in H$ for a.e. $t \in [0,T)$, for all $u_0 \in U$ iff $B^*\mathbb{A}^{t\,*}$ extends to $\mathcal{B}(H,U)$ for a.e. $t \in [0,T)$.

Assume, in addition, that $\mathbb{A}Bu_0 \in H$ a.e. on $[0,T)$ for all $u_0 \in U$, and that $q \in [1,2]$, $\alpha \in \mathbf{R}$.

(c) $\mathbb{A}B \in \mathcal{C}((0,\infty);\mathcal{B}(U,H))$, $\mathbb{A}^* \in \mathcal{C}((0,\infty);\mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}})))$ and $C_{\mathrm{L,s}}\mathbb{A}Bu_0 \in \mathrm{L}^q_{\mathrm{loc}}((0,\infty);Y) \cap \mathrm{L}^q_\omega([T,\infty);Y)$ (in particular, $\mathbb{A}Bu_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$ a.e. on $\mathbf{R}_+$) for all $u_0 \in U$.

(d1) We have $C_{\mathrm{L,s}}\mathbb{A}B \in \mathcal{B}(U, \mathrm{L}^q_\omega(\mathbf{R}_+;Y))$ *iff* $C_{\mathrm{L,s}}\mathbb{A}Bu_0 \in \mathrm{L}^q([0,T);Y)$ *for all* $u_0 \in U$.

This holds for $q = 2$ *iff* $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$.

(d2) We have $C_{\mathrm{L,s}}\mathbb{A}B \in \mathcal{B}(U, \mathcal{B}(Y^{\mathrm{B}}, \mathrm{L}^q_\omega(\mathbf{R}_+)))$ *iff* $\langle C_{\mathrm{L,s}}\mathbb{A}Bu_0, y_0\rangle_Y \in \mathrm{L}^q([0,T))$ *for all* $u_0 \in U$, $y_0 \in Y$.

This holds for $q = 2$ *iff* $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$.

(e1) If $Fu_0 := C_{\mathrm{L,s}}\mathbb{A}Bu_0 \in \mathrm{L}^q([0,T);Y)$ *for all* $u_0 \in U$, *then* $\mathbb{D} \in \mathrm{SLR} \cap \mathrm{SVR}$ *(and* $\mathbb{D} \in \mathrm{ULR}$ *if* $q > 1$*),* $\widehat{\mathbb{D}} = \widehat{F} + D$ *and* $\mathbb{D}u = F * u + Du$ *for all finite-dimensional* $u \in \mathrm{L}^2_\omega(\mathbf{R};U) + \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$ *(for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U) + \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$ *if* $F \in \mathrm{L}^p(\mathbf{R}_+;\mathcal{B}(U,Y))$*).*

(e2) Conversely, if $\widehat{\mathbb{D}} = \widehat{F} + D$ *for some* $F \in \mathcal{B}(U, \mathrm{L}^p_\alpha(\mathbf{R}_+;Y))$*, then* $F = C_{\mathrm{L,s}}\mathbb{A}B$.

If, in addition, $F \in \mathrm{L}(\mathbf{R}_+;\mathcal{B}(U,Y))$*, then* $\mathbb{A}B \in \mathcal{B}(U, \mathrm{Dom}(C_{\mathrm{L,s}}))$ *a.e. and* $C_{\mathrm{L,s}}\mathbb{A}B = F$ *a.e., hence then* $C_{\mathrm{L,s}}\mathbb{A}B \in e^{\alpha\cdot}\mathrm{L}^p_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(U,Y)) \cap \mathrm{L}(\mathbf{R}_+;\mathcal{B}(U,Y))$.

(e3) Claims (e1)–(e2) also hold with replacements $\mathcal{B}(U, \mathrm{L}^p_\alpha(\mathbf{R}_+;Y)) \mapsto \mathcal{B}(U, \mathcal{B}(Y^{\mathrm{B}}, \mathrm{L}^p_\alpha(\mathbf{R}_+)))$, $\mathrm{L}^p_\alpha \mapsto \mathrm{L}^p_{\mathrm{weak}}$, $\mathrm{L}^q \mapsto \mathrm{L}^p_{\mathrm{weak}}$, $\mathrm{L}^p_{\mathrm{strong}} \mapsto \mathrm{L}^p_{\mathrm{weak}}$, $\mathrm{SLR} \mapsto \mathrm{WLR}$, $\mathrm{SVR} \mapsto \mathrm{WVR}$ *(and* $C_{\mathrm{L,s}} \mapsto C_{\mathrm{L,w}}$ *in the "in addition" paragraph).*

(f) We have $\pi_+\mathbb{D}\pi_- u = \pi_+(C_{\mathrm{L,s}}\mathbb{A}B * \pi_- u) \in \mathcal{C}((0,\infty);Y)$ *for any finite-dimensional* $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$.

Actually, in (a) we have $\mathbb{A}B \in \mathrm{L}^p_{\mathrm{strong},\omega}(\mathbf{R}_+;\mathcal{B}(U,H))$, by (c). Note also that $\mathrm{L}^p([0,T);H) \subset \mathrm{L}^{p'}([0,T);H)$ for $p' \in [1,p]$, hence the case $p = 1$ is the weakest one (this applies to Lemma 6.8.3(a) too).

We rephrase the most important results of (d1), (e1) and (e2) as follows:

Corollary 6.8.2 ($\mathbb{D} = D + C_{\mathrm{L,s}}\mathbb{A}B*$) *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *be WR and s.t.* $\mathbb{A}Bu_0 \in H$ *a.e. for all* $u_0 \in U$. *Then* $\mathbb{D} - D$ *is a strong convolution iff* $C_{\mathrm{w}}\mathbb{A}Bu_0 \in \mathrm{L}^q([0,T);Y)$ *for some* $q \in [1,2]$ *and all* $u_0 \in U$.

If this is the case, then $C_{\mathrm{w}}\mathbb{A}B = C_{\mathrm{L,s}}\mathbb{A}B \in \mathcal{B}(U, \mathrm{L}^q_\omega(\mathbf{R}_+;Y))$*, and* $(\mathbb{D} - D)u = C_{\mathrm{w}}\mathbb{A}B * u = C_{\mathrm{L,s}}\mathbb{A}B * u$ *for each* $\omega > \omega_A$ *and each finite-dimensional* $u \in \mathrm{L}^2_\omega$. □

(Note that "$C_{\mathrm{w}}\mathbb{A}Bu_0 \in \mathrm{L}^q([0,T);Y)$" includes the assumption that $\mathbb{A}^t Bu_0 \in \mathrm{Dom}(C_{\mathrm{w}})$ for a.e. $t \in [0,T)$. Since $\mathrm{L}^q([0,T);Y) \subset \mathrm{L}^2([0,T);Y)$ for $q \in (2,\infty]$, we could allow for any $q \in [1,\infty]$ in the above equivalence (but possibly not in the latter paragraph).)

In deriving (b)–(f), we take advantage of the fact that $\mathbb{C}$ is "almost $\mathrm{L}^2_{\mathrm{strong},\omega}(\mathbf{R}_+;\mathcal{B}(U,H))$", i.e., $\mathbb{C} \in \mathcal{B}(U, \mathrm{L}^2_\omega(\mathbf{R}_+;H))$. Since $\mathbb{C}$ need not satisfy corresponding "uniform" condition, we cannot present complete "uniform" analogies of (b)–(f) in Lemma 6.8.3.

The reader might wish to consult Lemmas F.2.2–F.2.4 (resp. Lemma D.1.7) for convolutions corresponding to (a) (resp. to Lemma 6.8.3(a); the former ("strong") convolutions coincide with these standard ("uniform") convolutions for L^p_ω functions).

Proof of Lemma 6.8.1: (The logical order of the proof goes as follows: (a)–(b2), (c), (f), (d1)&(d2)1°, (e1)–(e3), (d1)&(d2)2°.)

(a) 1° *The first equivalence:* "Only if" is obvious, so assume that $\mathbb{A}Bu_0 \in \mathrm{L}^p([0,T);H)$ for all $u_0 \in U$.

Let $u_0 \in U$ and choose $t \in (0,T)$ s.t. $x_t := \mathbb{A}^t Bu_0 \in H$ (recall that $\mathbb{A}Bu_0 \in \mathcal{C}(\mathbf{R}_+;H_{-1})$, so that $\mathbb{A}^t Bu_0$ is well-defined for each $t \geq 0$).

Then $\mathbb{A}^{t+\cdot}Bu_0 = \mathbb{A}x_t \in \mathrm{L}^p_\omega(\mathbf{R}_+;H) \cap \mathcal{C}(\mathbf{R}_+;H)$, by Lemma A.4.5. Consequently, $\mathbb{A}^{t+\cdot}Bu_0 \in \mathrm{L}^p_\omega(\mathbf{R}_+;H)$. Because $u_0 \in U$ was arbitrary, we have $\mathbb{A}B[U] \subset \mathrm{L}^p_\omega(\mathbf{R}_+;H)$. But, $\mathbb{A}B \in \mathcal{B}(U,\mathrm{L}^p_\omega(\mathbf{R}_+;H_{-1}))$, hence $\mathbb{A}B \in \mathcal{B}(U,\mathrm{L}^p_\omega(\mathbf{R}_+;H))$, by Lemma A.3.6.

2° *The claim on* $\mathrm{H}^2_{\text{strong}}$*:* We have $\|(s-A)^{-1}Bu_0\|_{\mathrm{H}^2(\mathbf{C}^+_\omega;H)} = \sqrt{2\pi}\|\mathbb{A}Bu_0\|_{\mathrm{L}^2_\omega}$, by (D.36).

(b1) Note first that $\mathbb{A}^t B \in \mathcal{B}(U,H)$ follows from (i) and $\mathbb{A}^{t\,*} \in \mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}}))$ follows from (iii), by Lemma A.3.6. Because $\mathbb{A}^{\cdot - t} \in \mathcal{C}([t,\infty);\mathcal{B}(H))$, these imply that $\mathbb{A}B \in \mathcal{C}([t,\infty);\mathcal{B}(U,H))$ and $\mathbb{A}^* \in \mathcal{C}([t,\infty);\mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}}))$, and we have $\mathbb{A}^s B \in \mathcal{B}(U,H)$, $\mathbb{A}^{s\,*} \in \mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}}))$ for all $s \geq t$.

1° *(i)⇔(ii),* $(\mathbb{A}^t B)^* = B^*_{\mathrm{L,s}}\mathbb{A}^{t\,*}$*:* For each $x_0 \in H_1$, we have

$$\langle x_0, \mathbb{A}^t Bu_0\rangle_{\langle H_1,H_{-1}\rangle} = \lim_{r\to 0+}\langle r(r-A^*)^{-1}x_0, \mathbb{A}^t Bu_0\rangle \tag{6.198}$$

$$= \lim_{r\to 0+}\langle B^* r(r-A^*)^{-1}\mathbb{A}^{t\,*}x_0, u_0\rangle_U = \langle B^*_{\mathrm{w}}\mathbb{A}^{t\,*}x_0, u_0\rangle_U. \tag{6.199}$$

If (i) holds, then the above limit exists for all $x_0 \in H$, hence then (ii) holds and $(\mathbb{A}^t B)^* = B^*_{\mathrm{L,s}}\mathbb{A}^{t\,*}$.

Conversely, if (ii) holds, then (6.198) is bounded w.r.t. $\|x_0\|_H$ for all $u_0 \in H$, i.e., $\mathbb{A}^t Bu_0 \in H$ for all $u_0 \in H$ (see Definition A.3.23).

2° *The rest:* Because $\mathrm{Dom}(B^*_{\mathrm{L,s}}) \subset \mathrm{Dom}(B^*_{\mathrm{w}})$, we have (iii)⇒(ii). Obviously, (ii)⇒(iv). Finally, assume (iv), i.e., that $B^*\mathbb{A}^{t\,*} \in \mathcal{B}(H_1^*,U)$ has an extension $R \in \mathcal{B}(H,U)$. Then

$$B^*\frac{1}{r}\int_0^r \mathbb{A}^{s\,*}\mathbb{A}^{t\,*}x_0\,ds = R\mathbb{A}^{t\,*}\frac{1}{r}\int_0^r \mathbb{A}^{s\,*}x_0\,ds \to R\mathbb{A}^{t\,*}x_0, \tag{6.200}$$

by continuity, hence $B^*_{\mathrm{L,s}}\mathbb{A}^t x_0 = R\mathbb{A}^t$ exists, for any $x_0 \in H$, i.e., (iii) holds.

(b2) "Only if" follows from (b1). Assume then that $\mathbb{A}^t Bu_0 \in H$ for a.e. $t \in [0,T)$, for all $u_0 \in U$. Let $u_0 \in U$ be arbitrary. Then $\mathbb{A}^t Bu_0 \in H$ for arbitrarily small $t > 0$, and for such t we have $\mathbb{A}^{t+\cdot}Bu_0 = \mathbb{A}^{\cdot}\mathbb{A}^t Bu_0 \in \mathcal{C}(\mathbf{R}_+;H)$. Thus, $\mathbb{A}^s Bu_0 \in H$ for all $s > 0$.

Because $u_0 \in U$ was arbitrary, we have $\mathbb{A}^s B[U] \subset H$ for all $s > 0$, hence $B^*\mathbb{A}^{s\,*}$ extends to $\mathcal{B}(H,Y)$ for all $s > 0$, by (b1)(i)&(iv).

(c) Let $u_0 \in U$. For arbitrarily small $t > 0$, we have $\mathbb{A}^t Bu_0 \in H$, hence $\mathbb{A}^{t+\cdot}Bu_0 = \mathbb{A}^{\cdot}\mathbb{A}^t Bu_0 \in \mathcal{C}(\mathbf{R}_+;H)$, i.e., $\pi_{[t,\infty)}\mathbb{A}Bu_0 \in \mathcal{C}([t,+\infty);H)$, $\mathbb{A}^{t+r}Bu_0 \in \mathrm{Dom}(C_{\mathrm{L,s}})$ for a.e. $r > 0$, and $C_{\mathrm{L,s}}\mathbb{A}^{t+\cdot}Bu_0 = \mathbb{C}(\mathbb{A}^t Bu_0)(\cdot) \in \mathrm{L}^2_\omega(\mathbf{R}_+;Y)$, by Lemma 6.2.12(a).

In particular, $\mathbb{A}^t B[U] \subset H$ for any $t > 0$, hence $\mathbb{A}B \in \mathcal{C}((0,+\infty);\mathcal{B}(U,H))$, and $\mathbb{A}^* \in \mathcal{C}((0,\infty);\mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}})))$, by (b1).

Now we have established (c) for $q = 2$. If $q < 2$, replace ω by some $\alpha \in (\omega_A, \omega)$ and recall that $L^2_\alpha([T,\infty);Y) \subset L^q_\omega([T,\infty);Y)$.

(d1) 1° The first claim follows from (c).

2° We have $\widehat{\mathbb{D}} - D \in H^2_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ iff $\widehat{\mathbb{D}} = \widehat{F}$ for some $F \in \mathcal{B}(U, L^2_\omega(\mathbf{R}^+;Y))$, by Lemma F.3.4(d). Thus, "only if" follows from (e1) and 1°, and "if" from (e2).

(d2) The proof of (d1) applies mutatis mutandis.

(e1) We have $F \in \mathcal{B}(U, L^q_\alpha(\mathbf{R}_+;Y))$ for any $\alpha > \omega_A$, by (d1). Thus, $\widehat{\widetilde{\mathbb{D}}} := \widehat{F}$ defines an operator $\widetilde{\mathbb{D}} \in \text{TIC}_\omega(U,Y)$ with the properties claimed in (e1), by Proposition 6.3.4(a3)(&(a1)). By (f), density and continuity, we have $\pi_+\mathbb{D}\pi_- = \pi_+\widetilde{\mathbb{D}}\pi_-$, hence $\mathbb{D} = \widetilde{\mathbb{D}} + D$, by Corollary 2.1.8.

(e2) 1° By Proposition 6.3.4(a3), we have $(\mathbb{D} - D)u = F * u$ for finite-dimensional $u \in L^2_\omega(\mathbf{R};U)$. Let $u_0 \in U$. Substitute $f := \frac{1}{r}\chi_{[-r,0)}$ to (6.204) to observe that

$$\frac{1}{r}(\mathbb{D} - D)\chi_{[-r,0)}u_0 \to C_{\text{L,s}}\mathbb{A}^t Bu_0 \tag{6.201}$$

for each t s.t. $\mathbb{A}^t Bu_0 \in \text{Dom}(C_{\text{L,s}})$, hence a.e. By combining this with Proposition 6.3.4(a3), we obtain that $Fu_0 = C_{\text{L,s}}\mathbb{A}^t Bu_0$ a.e. Because $u_0 \in U$ was arbitrary, we have $F = C_{\text{L,s}}\mathbb{A}^t B$ as elements of $\mathcal{B}(U, L^q_\alpha)$.

2° Assume that, in addition, $F \in L(R_+;\mathcal{B}(U,Y))$. Then the limit $C_{\text{L,s}}\mathbb{A}^t Bu_0 = F(t)u_0$ exists for all $u_0 \in U$ at each Lebesgue point t of F, by the computations in 1°. Therefore, $\mathbb{A}^t Bu_0 \in \text{Dom}(C_{\text{L,s}})$ (hence $C_{\text{L,s}}\mathbb{A}^t B \in \mathcal{B}(U,Y)$) and $C_{\text{L,s}}\mathbb{A}^t B = F(t)$ for such t, hence for a.e. $t \in \mathbf{R}_+$.

(e3) The proofs of (e1)–(e2) apply mutatis mutandis: add $\Lambda \in Y^{\text{B}}$ to the left of suitable terms (i.e., use $\not\!\!\int$ instead of $\int$ etc.).

By (b1) (applied to Σ^{d}), we may use $C_{\text{L,s}}$ instead of $C_{\text{L,w}}$ everywhere except possibly in the "in addition" claim of (e2) (we only know that $\mathbb{A}B \in \mathcal{B}(U, \text{Dom}(C_{\text{L,w}}))$ a.e., hence $C_{\text{L,w}}\mathbb{A}B \in \mathcal{B}(U,Y)$ a.e.; we do not know whether $C_{\text{L,s}}\mathbb{A}^t B$ is defined for all $u_0 \in U$ at any $t \in \mathbf{R}_+$).

(f) Let $u = fu_0$, $f \in L^2_\omega(\mathbf{R}_-)$, $u_0 \in U$ (the general case follows by linearity). For a.e. $t > 0$, we have (use 2.&4. of Definition 6.1.1, (6.24) and Lemma 6.2.12(c1)&(c4), and note that $x_0 := \mathbb{A}^t Bu_0 \in H$))

$$(\mathbb{D}\pi_- u)(t) = (\mathbb{C}\mathbb{B}u)(t) = C_{\text{L,s}}(\mathbb{A}^t\mathbb{B}u) = C_{\text{L,s}}(\mathbb{B}\tau^t u) = C_{\text{L,s}}(\mathbb{A}B * u) \tag{6.202}$$

$$= C_{\text{L,s}}\int_{-\infty}^0 \mathbb{A}^{t-s}Bu(s)\,ds = C_{\text{L,s}}\int_0^\infty \mathbb{A}^s\mathbb{A}^t Bu_0 f(-s)\,ds \tag{6.203}$$

$$= \int_0^\infty C_{\text{L,s}}\mathbb{A}^s\mathbb{A}^t Bu_0 f(-s)\,ds = (C_{\text{L,s}}\mathbb{A}B * u)(t). \tag{6.204}$$

(We did not have to write $\lim_{T\to+\infty}\int_{-T}^0$ in (6.203), since we had $\mathbb{A}^{-\cdot}x_0 f(\cdot) \in L^1(\mathbf{R}_-;H)$, because $\mathbb{A}^{-\cdot}x_0 \in \text{Я}L^2_\omega = L^2_{-\omega}$.)

Since $\mathbb{A}^t Bu_0 \in \mathcal{C}((0,\infty);H)$, by (c), $C_{\text{L,s}}\mathbb{A}^{\cdot} = \mathbb{C} \in \mathcal{B}(H, L^2_\omega)$ and $f(-\cdot) \in L^2_{-\omega}$, we have $C_{\text{L,s}}\mathbb{A}^{\cdot}\mathbb{A}^t Bu_0 f(-\cdot) \in \mathcal{C}((0,\infty);L^1(\mathbf{R}_+;Y))$, hence (6.204)$\in \mathcal{C}((0,\infty);Y)$. $\square$

Next we present "uniform" counterparts of the "strong" claims presented in the above lemma:

Lemma 6.8.3 ($\mathbb{A}B \in \mathrm{L}^p$) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$*,* $\omega > \omega_A$*,* $T > 0$ *and* $p \in [1,\infty]$.

(a) The following are equivalent:

(i) $\mathbb{A}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$*;*
(ii) $\mathbb{A}B \in \mathrm{L}^p([0,T);\mathcal{B}(U,H))$*;*
(iii) $\mathbb{A}Bu_0 = Fu_0$ *a.e. on* $[0,T)$ *for all* $u_0 \in U$ *and some* $F \in \mathrm{L}^p([0,T);\mathcal{B}(U,H))$*;*
(iv) $\mathbb{B}\tau^T\phi u_0 = (Fu_0 * \phi)(T)$ *for all* $u_0 \in U$ *and* $\phi \in C_c^\infty((0,T))$*, and some* $F \in \mathrm{L}^p([0,T);\mathcal{B}(U,H)) + \mathrm{L}^1_\infty(\mathbf{R}_+;\mathcal{B}(U,H_{-1}))$.
(v) $B_{\mathrm{w}}^*\mathbb{A}^* \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(H,U))$*;*
(vi) $B_{\mathrm{w}}^*\mathbb{A}^* \in \mathrm{L}^p([0,T);\mathcal{B}(H,U))$*;*
(vii) $B^*\mathbb{A}^*x_0 = Fx_0$ *a.e. on* $[0,T)$ *for all* $x_0 \in H_1^*$ *and some* $F \in \mathrm{L}^p([0,T);\mathcal{B}(H,U))$*;*

If (i) holds, then $\mathbb{A}^* \in C((0,\infty);\mathcal{B}(H,\mathrm{Dom}(B_{\mathrm{L,s}}^*)))$*, hence then we may above replace* B_{w}^* *by* $B_{\mathrm{L,s}}^*$*,* $B_{\mathrm{L,w}}^*$ *or* B_{s}^*.

(b) If $\mathbb{C}x_0 = Fx_0$ *a.e. on* $[0,T)$ *for all* $x_0 \in H_1$ *and some* $F \in \mathrm{L}^p([0,T);\mathcal{B}(U,H))$*, then* $\mathbb{A} \in C((0,\infty);\mathcal{B}(H,\mathrm{Dom}(C_{\mathrm{L,s}})))$ *and* $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$.

(c) If $C_{\mathrm{w}}\mathbb{A} \in \mathcal{B}(H,Y)$ *and* $\mathbb{A}B \in \mathcal{B}(U,H)$ *a.e. on* $[0,T)$*, and* $C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^p([0,T);\mathcal{B}(U,Y))$*, then* $C_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y)) \cap C((0,\infty);\mathcal{B}(U,Y))$ *and* $\mathbb{D} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}(U,Y)$.

Naturally, if we apply (a) to Σ^{d}, then (v)–(vii) turn to results on C and $\mathbb{A}$. Thus, we may use $C_{\mathrm{L,w}}$, C_{s} or C_{w} instead of $C_{\mathrm{L,s}}$ in (d) and (e). Sometimes one may also wish to use the fact that $B_{\mathrm{L,s}}^*$ is the dual of B (see Proposition 6.2.8(e)).

Note that the assumptions in (a)–(c) are satisfied by parabolic systems of the type described in Hypothesis 9.5.1.

Proof: (a) 1° *(i)⇒(iv):* This follows from (6.23).

2° *(ii)⇔(i):* Fix $t > 0$ s.t. $\mathbb{A}^tB \in \mathcal{B}(U,H)$ and work as in the proof of Lemma 6.8.1(a).

3° *(iii)⇒(ii):* For any $t > 0$, we have $\mathbb{A}^tB[U] \in H$, by Lemma 6.8.1(b), hence $\mathbb{A}B \in C((0,\infty);\mathcal{B}(U,H))$, by (b). But $F(t)u_0 = \mathbb{A}^tBu_0$ for all u_0 at every Lebesgue point t of F, hence $\mathbb{A}B = F$ a.e. on $[0,T)$, hence $\pi_{[0,T)}\mathbb{A}B \in \mathrm{L}^p([0,T);\mathcal{B})$.

4° *(iv)⇒(iii):* By (6.23), we have $\int_0^T(\mathbb{A}Bu_0 - Fu_0)\phi\,dm = 0$ (the integral is taken in H_{-1}) for all ϕ, hence $\mathbb{A}Bu_0 = Fu_0$ as elements of $\mathrm{L}^1([0,T);H_{-1})$, by Theorem B.4.12(d), hence a.e. on $[0,T)$, i.e., (iii) holds.

5° *The rest:* By Lemma 6.8.1(b)&(c), any of (i)–(ii) and (v)–(vii) implies that $B^*\mathbb{A}^{t*}$ extends to $\mathcal{B}(H,U)$ a.e., $\mathbb{A}^* \in C((0,\infty);\mathcal{B}(H,\mathrm{Dom}(B_{\mathrm{L,s}})))$, and $(B_{\mathrm{L,s}}^*\mathbb{A}^*)^* = \mathbb{A}B \in C((0,\infty);\mathcal{B}(U,H))$.

Therefore, B_{w}^* and $B_{\mathrm{L,s}}^*$ are interchangeable everywhere in (a), and we have the equivalencies "(i)⇔(v)", "(ii)⇔(vi)", and "(vi)⇔(vii)" (because the unique extension $F(t)$ of $B^*\mathbb{A}^{t\,*}$ must be $B^*_{\mathrm{w}}\mathbb{A}^{t\,*}$ wherever the equality holds in (vii), hence a.e.).

(b) Because $C\mathbb{A} \in \mathcal{C}(\mathbf{R}_+;\mathcal{B}(H_1,Y))$, we have $C\mathbb{A}^t x_0 = F(t)x_0$ for all $x_0 \in H_1$ at each Lebesgue point t of F. Thus, $C\mathbb{A}^t$ extends to $F(t) \in \mathcal{B}(H,Y)$ at those points, so that $\mathbb{A} \in \mathcal{C}((0,\infty);\mathcal{B}(H,\mathrm{Dom}(C_{\mathrm{L,s}})))$, by Lemma 6.8.1(b), and $C\mathbb{A}^t = F(t)$ at those points; in particular, $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^p([0,T);\mathcal{B}(U,H))$.

Let $\alpha \in (\omega_A,\omega)$. Set $M := \|\mathrm{e}^{-\omega\cdot}C_{\mathrm{L,s}}\mathbb{A}\|_{\mathrm{L}^p([0,T);\mathcal{B})}$, $M_\alpha := \|\mathrm{e}^{-\alpha\cdot}\mathbb{A}\|_\infty$. If $p = \infty$, then

$$\|\mathrm{e}^{-\omega t}C_{\mathrm{L,s}}\mathbb{A}^t\|_{\mathcal{B}} \le \|\mathrm{e}^{-\omega T}C_{\mathrm{L,s}}\mathbb{A}^T\|\|\mathrm{e}^{-\omega(t-T)}\mathbb{A}^{t-T}\|, \tag{6.205}$$

which is bounded for $t > T$. Thus, $\|\mathrm{e}^{-\omega\cdot}C_{\mathrm{L,s}}\mathbb{A}\|_{\mathcal{B}}$ is then bounded. Assume then that $p < \infty$. Then

$$\int_0^\infty \|\mathrm{e}^{-\omega t}C_{\mathrm{L,s}}\mathbb{A}^t\|^p_{\mathcal{B}}\,dt = \sum_{n\in\mathbf{N}}\int_0^T \|\mathrm{e}^{-\omega t}C_{\mathrm{L,s}}\mathbb{A}^t\|^p_{\mathcal{B}}\,dt\|\mathrm{e}^{-\omega Tn}\mathbb{A}^{Tn}\|^p_{\mathcal{B}} \tag{6.206}$$

$$\le M^p\sum_{n\in\mathbf{N}} M^p_\alpha \mathrm{e}^{-p(\omega-\alpha)Tn} < \infty. \tag{6.207}$$

(c) By Lemma 6.8.1(b1) (applied to Σ and Σ^{d}), we have $\mathbb{A}B \in \mathcal{C}((0,\infty);\mathcal{B}(U,H))$ and $C_{\mathrm{L,s}}\mathbb{A} \in \mathcal{C}((0,\infty);\mathcal{B}(H,Y))$. In particular, $C_{\mathrm{L,s}}\mathbb{A}B \in \mathcal{C}((0,\infty);\mathcal{B}(U,Y))$, $C_{\mathrm{L,s}}\mathbb{A}^{T/2} \in \mathcal{B}(H,Y)$ and $\mathbb{A}^{T/2}B \in \mathcal{B}(U,H)$. Therefore, $\|\mathrm{e}^{-\alpha t}C_{\mathrm{L,s}}\mathbb{A}^{T/2}\mathbb{A}^t\mathbb{A}^{T/2}B\|_{\mathcal{B}(U,Y)}$ is bounded for each $\alpha > \omega_A$. Consequently,

$$C_{\mathrm{L,s}}\mathbb{A}^{T/2}\mathbb{A}^{\cdot}\mathbb{A}^{T/2}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y)), \tag{6.208}$$

hence $C_{\mathrm{L,s}}\mathbb{A}^{\cdot}B \in \mathrm{L}^p([0,T);\mathcal{B}(U,Y)) \cap \tau^{-T}\mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y)) = \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y))$. Consequently, $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U,Y)$, by Lemma 6.8.1(e1) and density (see Theorem B.3.11). □

The $\mathrm{L}^2_{\mathrm{strong}}$ properties and all L^p properties described above are unaffected by bounded state feedback operators:

Lemma 6.8.4 (Bounded *K*) *Assume that* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $K \in \mathcal{B}(H,U)$*, and let* $\Sigma_\flat$ *be the corresponding closed-loop system, so that* $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat\end{array}\right]$ *is generated by* $\left[\begin{array}{c|c}A+BK & B\end{array}\right]$*. Let* $p \in [1,\infty]$*.*

(a1) If $\mathbb{A}B \in \mathrm{L}^p([0,1];\mathcal{B}(U,H))$*, then* $\mathbb{A}_\flat B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$ *for all* $\omega > \omega_{A_\flat}$*.*

(a2) If $\mathbb{A}B \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$ *and* $\mathbb{M} \in \mathrm{TIC}_\omega(U)$ *for some* $\omega \in \mathbf{R}$*, then* $\mathbb{A}_\flat B \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$ *and* $\mathbb{M} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1}_\omega(U)$*.*

(b) If $\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1];H)$ *for all* $u_0 \in U$*, then* $\mathbb{A}_\flat Bu_0 \in \mathrm{L}^2_\omega(\mathbf{R}_+;H)$ *for all* $\omega > \omega_{A_\flat}$ *and* $u_0 \in U$*.*

(c1) If $\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$*,* $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^1([0,1);\mathcal{B}(H,Y))$ *and* $C_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,Y))$*, then* $\mathbb{D}_\flat \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U,Y)$ *for any* $\omega > \omega_{\mathbb{A}_\flat}$*.*

(c2) If $\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$ *and* $C_{\mathrm{L,s}}\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1);Y)$ *for all* $u_0 \in U$*, then* $\mathbb{D}_\flat - D \in \mathcal{B}(U,\mathrm{L}^1_\omega(\mathbf{R}_+;Y))*$ *(i.e.,* $(C_\flat)_{\mathrm{L,s}}\mathbb{A}_\flat Bu_0 \in \mathrm{L}^1_\omega(\mathbf{R}_+;Y)$ *for all* $u_0 \in U$*) for all* $\omega > \omega_{A_\flat}$*.*

(c3) If $\mathbb{A}Bu_0, C_{\mathrm{L,s}}\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1];*)$ *for all* $u_0 \in U$*, then* $\widehat{\mathbb{D}_\flat} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ *for all* $\omega > \omega_{A_\flat}$*.*

(d) Assume that $\omega_{A_\flat} < 0$*. Then* $\mathbb{B}_\flat\tau, \mathbb{M} \in \mathrm{ULR} \cap \mathrm{SHPR}$ *in (a1)–(c3); in (c1)–(c3) we also have* $\mathbb{D}_\flat \in \mathrm{SHPR}$ *(and* $\mathbb{D}_\flat \in \mathrm{ULR}$ *in (c1) and (c3)).*

Note that the assumption in (c2) holds iff $\mathbb{B}\tau \in \mathrm{MTIC}^{\mathrm{L}^1}_\infty$ and $\mathbb{D} - D \in \mathcal{B}(U, \mathrm{L}^1_\alpha(\mathbf{R}_+;Y))*$ for some $\alpha \in \mathbf{R}$. As usual, $\mathbb{M} := (I - \mathbb{F})^{-1}$, where $\left[\ \mathbb{K}\ \middle|\ \mathbb{F}\ \right]$ is generated by K.

Proof: (As in Definition 6.6.10, we have set $\mathbb{M} := I - \mathbb{F}$, where $\left[\ \mathbb{K}\ \middle|\ \mathbb{F}\ \right]$ is the state feedback pair corresponding to K. See Proposition 6.6.18(d3) for the generators of $\Sigma_\flat$.)

(a2) By Lemma 6.1.16(b), we have $\mathbb{B}\tau u = \mathbb{A}B * u$; by (6.46), we have $\mathbb{F} = K\mathbb{B}\tau = K\mathbb{A}B*$. But $K\mathbb{A}B \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$, hence $\mathbb{F} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U) \subset \mathrm{TIC}_\omega(U)$.

Thus, $\mathbb{X} := I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\omega(U) \cap \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U)$, hence $\mathbb{M}^{-1} = \mathbb{X} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1}_\omega(U)$, by Theorem 4.1.1(b)(i)&(ii) and Remark 6.1.9.

But then $\mathbb{B}_\flat\tau = \mathbb{B}\tau\mathbb{M} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U,H)$ (by Lemma D.1.7), i.e., $\mathbb{A}_\flat B_\flat = \mathbb{A}_\flat B \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$ (see Lemma 6.8.3(a)(iv)&(i)).

(a1) Obviously, we may replace L^1 by $\mathrm{L}^1 \cap \mathrm{L}^p$ in (a2) and its proof.

Choose some $\alpha > \max\{\omega, \omega_A\}$. By Lemma 6.8.3(a), we have $\mathbb{A}B \in \mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha(\mathbf{R}_+;\mathcal{B}(U,H))$. But $\Sigma_\flat$ is ω-stable, hence $\mathbb{M} \in \mathrm{TIC}_\omega$. Consequently, $\mathbb{A}_\flat B \in \mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha(\mathbf{R}_+;\mathcal{B}(U,H))$, by (a2) (modified, as noted above). By Lemma 6.8.3(a), $\mathbb{A}_\flat B \in \mathrm{L}^1_\alpha \cap \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$.

(b) By Lemma 6.3.3(b1), we have $\widehat{\mathbb{M}} \in \mathcal{B} + \mathrm{H}^2_{\mathrm{strong},\infty}$, hence $\widehat{\mathbb{B}_\flat\tau} = \widehat{\mathbb{B}\tau}\widehat{\mathbb{M}} \in \mathcal{B} + \mathrm{H}^2_{\mathrm{strong},\infty}$, by Lemma 6.3.3(b1). But $\widehat{\mathbb{B}_\flat\tau}(+\infty) = 0$, hence $\widehat{\mathbb{B}_\flat\tau} \in \mathrm{H}^2_{\mathrm{strong},\infty}$. By Lemma 6.8.1(a), we have $\mathbb{A}_\flat Bu_0 \in \mathrm{L}^2_\omega(\mathbf{R}_+;H)$ for all $\omega > \omega_{A_\flat}$ and $u_0 \in U$.

(c1) (As one observes from the proof, we have $C_{\mathrm{L,s}}\mathbb{A}_\flat B \in \mathrm{L}^p_\omega$ if $\mathbb{A}B, C_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^p$ and $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^1$ on $[0,1)$.)

We have

$$\mathbb{C}_\flat = \mathbb{C} + \mathbb{D}K\mathbb{A}_\flat \in \mathrm{L}^1 + \mathrm{L}^p * \mathrm{L}^\infty \subset \mathrm{L}^1 \tag{6.209}$$

on $[0,1)$ (because $\pi_{[0,1)}\mathrm{L}^1 * \pi_{[0,1)}\mathrm{L}^1 \subset \pi_{[0,1)}\mathrm{L}^1$, by Lemma D.1.7, and $K\mathbb{A}_\flat \in \mathcal{C} \subset \mathrm{L}^\infty_{\mathrm{loc}}$). Therefore, $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^1_\infty$, by Lemma 6.8.3(b). But $\mathbb{A}_\flat B \in \mathrm{L}^1_\infty$ and $\mathbb{M} \in \mathrm{MTIC}^{\mathrm{L}^1}_\infty(U,Y)$, by (a1), hence $\mathbb{D}_\flat = \mathbb{D}\mathbb{M} \in \mathrm{MTIC}^{\mathrm{L}^1}_\infty(U,Y)$. Consequently, $\mathbb{D}_\flat \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U,Y)$, by Lemma 6.8.3(c).

(c2) By (a1), we have $K\mathbb{A}_\flat B \in \mathrm{L}^1([0,1];\mathcal{B}(U))$, hence $\mathbb{M} \in \mathrm{MTIC}^{\mathrm{L}^1}{}_\infty(U)$. Since $\mathbb{D} \in \mathrm{SMTIC}^{\mathrm{L}^1}{}_\infty$, by the assumption, hence $\mathbb{D}_\flat := \mathbb{D}\mathbb{M} \in \mathrm{SMTIC}^{\mathrm{L}^1}{}_\infty$ and $D_\flat = D$, by Theorem 2.6.4(a1)&(h1)&(d). Thus, we obtain (c2) from this and Lemma 6.8.1(e2)&(d1)

(c3) This follows from (b) and Lemma 6.8.1(d1).

(d) Choose some $\omega \in (\omega_{A_\flat}, 0)$. In (a1) and (a2), we have $\mathbb{A}_\flat B, K\mathbb{A}_\flat B \in \mathrm{L}^1_\omega$, hence $\mathbb{B}_\flat\tau, \mathbb{M} \in \mathrm{MTIC}^{\mathrm{L}^1} \subset \mathrm{ULR} \cap \mathrm{UHPR}$. In (b), we have $\mathbb{A}_\flat B, K\mathbb{A}_\flat B \in \mathrm{L}^2_\omega$, hence $\mathbb{B}_\flat\tau, \mathbb{M} \in \mathrm{ULR} \cap \mathrm{SHPR}$, by Proposition 6.3.4(a3).

The claim on $\mathbb{D}_\flat$ follows analogously from (c1)–(c3): use Proposition 6.3.4(a1)&(a3) for (c1)&(c2), respectively, and Proposition 6.3.3(a) for (c3). □

The systems described in the following lemma are in certain sense the most general class of systems to which we can extend the full connection between optimal control and exponentially stabilizing solutions Riccati equations without any a priori factorization or stability assumptions (see, e.g., Theorem 9.2.18):

Lemma 6.8.5 ($\mathbb{A}B, C_{\mathrm{w}}\mathbb{A}, C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1_{\mathrm{loc}}$) *Assume that* $\Sigma := \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, $p,q \in [1,\infty]$, $\mathbb{A}B \in \mathrm{L}^p([0,t);\mathcal{B}(U,H))$, $C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^q([0,t);\mathcal{B}(H,Y))$, *and* $C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^p([0,t);\mathcal{B}(U,Y))$ *for some* $t>0$.

(a) Then $\mathbb{A}B \in \mathrm{L}^p_\omega$, $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^q_\omega$ *and* $C_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^p_\omega$ *on* $\mathbf{R}_+$ *(in particular,* $\mathbb{B}\tau, \mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega(U,*) \subset \mathrm{ULR}\cap\mathrm{UVR}$*) for any* $\omega > \omega_A$.

(b) If $L \in \mathcal{B}(Y,U)$ *is an admissible output feedback operator for* Σ*, then* $\mathbb{A}_L B_L \in \mathrm{L}^p_\omega$, $(C_L)_{\mathrm{L,s}}\mathbb{A}_L \in \mathrm{L}^q_\omega$ *and* $(C_L)_{\mathrm{L,s}}\mathbb{A}_L B_L \in \mathrm{L}^p_\omega$ *on* $\mathbf{R}_+$ *for any* $\omega > \omega_{A_L}$.

(c) Let $K = SC + T$, $S \in \mathcal{B}(Y,U)$, $T \in \mathcal{B}(H,U)$*. Then* K *is a ULR admissible state feedback operator for* Σ*, and also the corresponding extended system* Σ_{ext} *and hence the closed-loop system* $\Sigma_\flat$ *satisfy the assumptions of this lemma.*

(d) If $p=q$*, then also* Σ^{d} *satisfies the assumptions of this lemma.*

(Naturally, we can replace the exponents p and q in the conclusion parts by smaller ones.)

We conclude that the above type of systems are closed w.r.t. static output feedback and w.r.t. state feedback of kind described in (c) (which often appears in connection with Riccati equations and optimal control). We observe from Lemma 9.5.4 and Proposition 6.6.18(b3) that an analogous claim holds for systems of the (parabolic) type of Hypothesis 9.5.1 as well as for those of the type of Hypothesis 9.5.7(3.).

Proof: (We shall use the facts that $\mathrm{L}^p([0,t);*) \subset \mathrm{L}^1([0,t);*)$ and that $\mathbb{A} \in \mathcal{C}\cap\mathrm{L}^s_\omega(\mathbf{R}_+;\mathcal{B}(H))$ for any $s\in[1,\infty]$ and $\omega>\omega_A$. See Lemma D.1.7 for convolutions. We observe from the proof that we could use a third exponent $r\in[1,\infty]$ (instead of p) for $C_{\mathrm{w}}\mathbb{A}B$ and $C_{\mathrm{L,s}}\mathbb{A}B$, but then (b) and (c) would no longer hold.)

(a) Let $\omega>\omega_A$. By Lemma 6.8.3(a) (applied to Σ^{d} and Σ), we can replace C_{w} by $C_{\mathrm{L,s}}$ and we have $\mathbb{A}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$ and $C_{\mathrm{L,s}}\mathbb{A} \in \mathrm{L}^q_\omega(\mathbf{R}_+;\mathcal{B}(H,Y))$. By Lemma 6.8.3(c), it follows that $C_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y))$ (in particular, $\mathbb{A}(t)B \in \mathcal{B}(U,\mathrm{Dom}(C_{\mathrm{L,s}}))$ for a.e. $t\in\mathbf{R}_+$). We conclude that $\mathbb{B}\tau,\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega$, by Corollary 6.8.2. Recall from Proposition 6.3.4(a1) that $\mathrm{MTIC}^{\mathrm{L}^1}_\infty \subset \mathrm{ULR}\cap\mathrm{UVR}$.

(b) Let $\alpha > \max\{\omega_A,\omega_{A_\flat}\}$. Set $\mathbb{X} := I - L\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty(U)$. Then $\mathbb{D} \in \mathcal{B}(U,Y) + (\mathrm{L}^1_\alpha\cap\mathrm{L}^p_\alpha)*$, by (a), hence $\mathbb{X} \in \mathcal{B}(U) + (\mathrm{L}^1_\alpha\cap\mathrm{L}^p_\alpha)*$.

We have $\mathbb{D},\mathbb{X},\mathbb{D}_L \in \mathrm{TIC}_\alpha$. Since $\mathbb{M} := \mathbb{X}^{-1} = I + L\mathbb{D}_L \in \mathrm{TIC}_\alpha$, we obtain from Proposition 6.3.4(a1) that $\mathbb{M} \in \mathcal{B}(U) + (\mathrm{L}^1_\alpha\cap\mathrm{L}^p_\alpha)*$. Analogously, we conclude that $(I-\mathbb{D}L)^{\pm 1} \in \mathcal{B}(Y) + (\mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha)*$. Consequently, (see Proposition 6.6.18)

$$\mathbb{B}_L\tau = \mathbb{B}_L\mathbb{M}\tau = \mathbb{B}\tau\mathbb{M} \in \mathrm{L}^p_\alpha * (I + \mathrm{L}^p_\alpha *) = \mathrm{L}^p_\alpha, \tag{6.210}$$

$$\mathbb{D}_L = \mathbb{D}\mathbb{M} \in (D + (\mathrm{L}^1_\alpha \cap \mathrm{L}^p_\alpha)*)(M + \mathrm{L}^p_\alpha *) \subset DM + \mathrm{L}^p_\alpha *, \quad \text{and} \tag{6.211}$$

$$\mathbb{C}_L = (I-\mathbb{D}L)^{-1}\mathbb{C} \in (\mathcal{B} + \mathrm{L}^1_\alpha *)\mathrm{L}^p_\alpha = \mathrm{L}^p_\alpha. \tag{6.212}$$

We conclude from Corollary 6.8.2 that $\mathbb{A}_L B_L, (C_L)_{\mathrm{L,s}}\mathbb{A}_L B_L \in \mathrm{L}^p_\alpha(\mathbf{R}_+;\mathcal{B}(U,*))$. Apply (a) to Σ_L to obtain the rest of (b).

(c) By Lemma 6.3.17, K is admissible and ULR for Σ. Obviously, $SC_{\mathrm{L,s}} + T \subset K_{\mathrm{L,s}}$ (in particular, $\mathrm{Dom}(C_{\mathrm{L,s}}) \subset \mathrm{Dom}(K_{\mathrm{L,s}})$), hence $K_{\mathrm{L,s}}\mathbb{A} = (SC_{\mathrm{L,s}} + T)\mathbb{A} \in \mathrm{L}^p([0,t);\mathcal{B}(H,U))$ (where the equality holds a.e.). Analogously, $K_{\mathrm{L,s}}\mathbb{A}B = (SC_{\mathrm{L,s}} + T)\mathbb{A}B \in \mathrm{L}^s([0,t);\mathcal{B}(U))$ (since $SC_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^r([0,t);\mathcal{B}(U))$ and $T\mathbb{A}B \in \mathrm{L}^p([0,t);\mathcal{B}(U))$), where $s := \min\{p,r\}$. Thus, Σ_{ext} (see (6.132)) satisfies the assumptions of this lemma with s in place of r, hence so does $\Sigma_\flat$, by (b).

(d) By Lemma 6.8.3(a)(ii)&(vi), we have $B^*_{\mathrm{w}}\mathbb{A}^* \in \mathrm{L}^p([0,t);\mathcal{B}(H,U))$ and $\mathbb{A}^*C^* \in \mathrm{L}^p([0,t);\mathcal{B}(Y,H))$. By Lemma 6.8.3(c), $F := C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y))$, hence $F^* \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(Y,U))$, by Lemma B.3.6.

By Corollary 6.8.2, $\mathbb{D} - D = F*$, hence $\mathbb{D}^{\mathrm{d}} - D^* = (\mathbb{D} - D)^{\mathrm{d}} = F^* *$, by Proposition 6.3.4(a1), hence $F^* = B^*_{\mathrm{w}}\mathbb{A}^*C^*$, by Corollary 6.8.2. Thus, $B^*_{\mathrm{w}}\mathbb{A}^*C^* \in \mathrm{L}^p([0,t);\mathcal{B}(Y,U)$. □

Notes

Pritchard–Salamon (PS) systems often satisfy the assumptions of Lemma 6.8.5 for $p = 2 = q$ (cf. Lemma 9.5.2), and in a sense the lemma allows for twice as much unboundedness as the axioms of PS-systems, but in general a PS-system might violate the assumptions of the lemma (but not those of Hypothesis 9.2.2, hence our complete "smooth Riccati equation theory", which uses the properties established in this section, covers also PS-systems).

In the control theory of optimal control of partial differential equations, one often makes similar assumptions on $C\mathbb{A}B$ with C bounded. E.g., in Section 8 of [LT00b], I. Lasiecka and R. Triggiany do not pose the assumption on $\mathbb{A}Bu_0 \in \mathrm{L}^p_{\mathrm{loc}}$ ($u_0 \in U$) and compensated this by a stronger assumption on C. They require the original system to be a WPLS but do not study the well-posedness of closed-loop systems.

The well-posedness of a closed-loop system means that under any error, disturbance or other external input to the feedback loop (the signal u_L of (6.124), the state, effective control and output of the system remain well defined and their dependence on this external input is continuous (from $\mathrm{L}^2_{\mathrm{loc}}$ to H, $\mathrm{L}^2_{\mathrm{loc}}$ and $\mathrm{L}^2_{\mathrm{loc}}$, respectively). In particular, finite input energy cannot lead to infinite output energy (or to undefined state and output) under a finite period of time. If, in addition, the closed-loop system is strongly stable, then the effect of any external L^2 signal vanishes asymptotically with time.

6.9 Bounded B, bounded C, PS-systems

'And death', said Thingol, 'thou shouldst taste,
had I not sworn an oath in haste
that blade nor chain thy flesh should mar.
Yet captive bound by never a bar
unchained, unfettered shalt thou be
in lightless labyrinths, endlessly.
— J.R.R. Tolkien (1892–1973), "The Lay of Leithian"

In this section we shall show that a transfer function $\widehat{\mathbb{D}}$ has a realization with a bounded input operator B iff $\widehat{\mathbb{D}} - \widehat{\mathbb{D}}(+\infty) \in \mathrm{H}^2_{\mathrm{strong}}$ over some right half-plane. We also establish analogous results for realizations with a bounded C and for Pritchard–Salamon realizations. In addition to U, H and Y, also $\mathcal{V}$ and $\mathcal{W}$ denote (arbitrary) Hilbert spaces in this section.

By Proposition 6.3.3(a), any $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$ is the transfer function of some $\mathbb{D} \in \mathrm{TIC}_{\omega+\varepsilon}(U,Y) \cap \mathrm{ULR}$ with $D = 0$ (for any $\varepsilon > 0$). By strengthening the assumption slightly, we get a necessary and sufficient condition for $\widehat{\mathbb{D}}$ being the transfer function of a WPLS with a bounded B or C:

Theorem 6.9.1 ($\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}} \Leftrightarrow B$ bounded) *Let $\omega \in \mathbf{R}$ and $\widehat{\mathbb{D}} \in \mathrm{H}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$.*

(a) $\widehat{\mathbb{D}}$ is the transfer function of the I/O map of some $\Sigma \in \mathrm{WPLS}_\omega$ with a bounded B iff $\widehat{\mathbb{D}} - D \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y)) \cap \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$.

(b) $\widehat{\mathbb{D}}$ is the transfer function of the I/O map of some $\Sigma \in \mathrm{WPLS}_\omega$ with a bounded C iff $\widehat{\mathbb{D}}(\bar{\cdot})^* - D^* \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(Y,U)) \cap \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(Y,U))$.

(c) The corresponding realizations can be chosen so that they are minimal and they satisfy $\|B\|_{\mathcal{B}(U,H)} \le \|\widehat{\mathbb{D}}\|_{\mathrm{H}^2_{\mathrm{strong}}}$ in (a), (or $\|C\|_{\mathcal{B}(H,Y)} \le \|\widehat{\mathbb{D}}(\bar{\cdot})\|_{\mathrm{H}^2_{\mathrm{strong}}}$ in (b)), where H is the state space of the corresponding realization.

(d1) If we drop the assumption $\widehat{\mathbb{D}} - D \in \mathrm{H}^\infty_\omega$, then (a)–(c) still hold except that $\mathbb{B}$ and $\mathbb{D}$ in (a) (or $\mathbb{C}$ and $\mathbb{D}$ in (b)) are only known to be ω'-stable for any $\omega' > \omega$.

(d2) We can replace "$\Sigma \in \mathrm{WPLS}_\omega$" by "$\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}$ s.t. $\mathbb{C}$ and $\mathbb{D}$ are ω-stable" in (a).

Analogously, in (b) it suffices to require $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}$ be s.t. $\mathbb{B}$ and $\mathbb{D}$ are ω-stable. We may require Σ to be strongly ω-stable in (b).

Thus, $\mathbb{D}$ has a realization with a bounded B and $D = 0$ iff $\widehat{\mathbb{D}}(\cdot - \omega) \in \mathrm{H}^2_{\mathrm{strong}}$ for some $\omega \in \mathbf{R}$ (see Lemma F.3.2(a)); a dual claim holds for C.

Proof: (a) 1° *"Only if":* Let $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ and $B \in \mathcal{B}(U,H)$ (in fact, $\Sigma \in \mathrm{SOS}_\omega$ is enough). W.l.o.g., we assume that $D = 0$ (see also Lemma 6.3.16(b)). We have $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$, by Theorem 6.2.1. Moreover, $\widehat{\mathbb{C}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(H,Y))$, by Theorem 6.2.11(c2), hence $\widehat{\mathbb{D}} = C(\cdot - A)^{-1}B \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$ (and $\|\widehat{\mathbb{D}}\|_{\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))} \le \sqrt{2\pi}\|\mathbb{C}\|\,\|B\| < \infty$).

2° *"If":* Apply "2°" of (b) to $\mathbb{D}^{\mathrm{d}}$. In fact, by applying Lemma 6.7.17 with $E = \mathbf{Я}$ for the resulting realization of $\mathbb{D}$, we see that the input operator of the strongly ω-stable WPLS

$$\Sigma := \left[\begin{array}{c|c} \pi_+\tau & \pi_+\mathbb{D}\pi_- \\ \hline I & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_\omega(U, \mathrm{L}^2_\omega(\mathbf{R}_+;Y), Y) \tag{6.213}$$

is bounded.

(b) 1° *"Only if":* Apply "1°" of (a) to $\mathbb{D}^{\mathrm{d}}$.

2° *"If":* W.l.o.g. we assume that $\omega = 0$. It suffices to show the stable (and exactly reachable) realization

$$\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] := \left[\begin{array}{c|c} \tau\pi_- & \pi_- \\ \hline \pi_+\mathbb{D}\pi_- & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_0(U, H, Y) \tag{6.214}$$

on $H := \mathrm{L}^2(\mathbf{R}_-;U)$ has a bounded output operator C. By Lemma F.3.7(b2), the map

$$\bar{C}x_0 := (\mathbb{C}x_0)(0) = (\pi_+\mathbb{D}\pi_-x_0)(0) = (\mathbb{D}x_0)(0) \tag{6.215}$$

satisfies $\bar{C} \in \mathcal{B}(H,Y)$, $\|\bar{C}\| \leq \|\widehat{\mathbb{D}}(\cdot)\|_{\mathrm{H}^2_{\mathrm{strong}}}$ (recall from (6.26) that $\bar{C}$ is an extension of the output operator $C \in \mathcal{B}(H_1,Y)$ of (6.214), and that C is called "bounded" iff it has an extension to $\mathcal{B}(H,Y)$ (which is necessarily unique, by density), and that C is identified with this extension (i.e., we write $C = \bar{C} \in \mathcal{B}(H,Y)$)).

(c) We prove this for (a); use duality for (b): Let $H_{\mathbb{B}}$ be the closure of $\pi_+\mathbb{D}\pi_-[\mathrm{L}^2_{\mathrm{c}}]$ in $\mathrm{L}^2_\omega(\mathbf{R}_+;Y)$ (i.e., the reachability subspace). Let P be the orthogonal projection $\mathrm{L}^2_\omega(\mathbf{R}_+;Y) \to H_{\mathbb{B}}$. By Lemma 6.3.26(e), $\Sigma' := \left[\begin{array}{c|c} \pi_+\tau & \pi_+\mathbb{D}\pi_- \\ \hline I & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U, H_{\mathbb{B}}, Y)$ (note that $\pi_+\tau[H_{\mathbb{B}}] \subset H_{\mathbb{B}}$) is reachable and also Σ' has bounded input operator. Because Σ' is (exactly ω-)observable, it is minimal.

(d1) We prove this for (b); use duality for (a):

Observe from the proof that C is bounded whenever $\widehat{\mathbb{D}}(\cdot) \in \mathrm{H}^2_{\mathrm{strong}}$, but if we do not assume that $\widehat{\mathbb{D}} \in \mathrm{H}^\infty$, then we only know that $\mathbb{C}$ (by Lemma F.3.7(b2)) and $\mathbb{D}$ (by Lemma F.3.2(a)) are ω-bounded for any $\omega > 0$.

(d2) One observes from part 1° of the proof of (a) that Σ need not be ω-stable, it suffices that $\mathbb{C}$ and $\mathbb{D}$ are ω-stable. Part 2° of the proof of (a) shows that Σ can be required to be strongly ω-stable in (a) (and strongly* ω-stable in (b)). □

We shall soon show that a transfer function can be realized as a PS-system iff it has a (WPLS) realization with bounded B and one with bounded C. Before this we must define PS-systems:

Definition 6.9.2 (PS-systems) *A system* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U, \mathcal{V}, Y)$ *is a* Pritchard–Salamon system (PS-system) *iff* $B \in \mathcal{B}(U, \mathcal{V})$ *and there is a Hilbert space* $\mathcal{W} \subset \mathcal{V}$ *s.t. 1. the embedding* $\mathcal{W} \to \mathcal{V}$ *is dense and continuous, 2.* $\mathbb{A}|_{\mathcal{W}}$

is an ω*-stable* C_0*-semigroup on* $\mathcal{W}$*, 3.* $\mathbb{B}^t \in \mathcal{B}(\mathrm{L}^2([0,t);U),\mathcal{W})$ *for some (hence all)* $t>0$*, and 4. some* $C' \in \mathcal{B}(\mathcal{W},Y)$ *satisfies* $C'\mathbb{A} = \mathbb{C}$ *on* $\mathcal{W}$*.*

If, in addition, $\omega \in \mathbf{R}$ *is s.t.* $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{V},Y)\cap\mathrm{WPLS}_\omega(U,\mathcal{W},Y)$*, then* Σ *is called* an ω-stable PS-system. *If* $\mathrm{Dom}(A) \subset \mathcal{W}$*, then* Σ *is called a* smooth PS-system.

(We also say that Σ is a *PS-system w.r.t.* U*,* $\mathcal{W}$*,* $\mathcal{V}$ *and* Y*.*)

Remark 6.9.3 *Definition 6.9.2 is equivalent to the standard definition of PS-systems (see, e.g., [KMR] or Definition 2.3 of [Keu]).*

(Note that it is not a limitation that [Keu] assumes the Hilbert spaces to be real — any complex Hilbert space is also a real Hilbert space.)

The above definition of ω-stability makes also "exponential stability" (i.e., being ω-stable for some $\omega<0$) equivalent to the standard one, by Lemma 6.9.4.

Proof: Sufficiency is rather obvious. Conversely, if Σ is a PS-system, then the axioms of the above definition are satisfied as follows: The maps in (2.4) and (2.5) of [Keu] define maps $\mathbb{B}$ and $\mathbb{C}$. By (2.6) of [Keu], we have $\left[\begin{smallmatrix}\mathbb{A}\\ \mathbb{C}\end{smallmatrix}\right] \in$ WPLS; let $\left[\begin{smallmatrix}A\\ C''\end{smallmatrix}\right]$ be its generators. Then $\left[\begin{smallmatrix}A & B\\ C'' & D\end{smallmatrix}\right]$ generate a WPLS $\Sigma \in \mathrm{WPLS}(U,\mathcal{V},Y)$, by Lemma 6.3.13. Obviously, Σ is the original (PS-)system. □

Thus, given a PS-system and a minimization problem, we "have a bounded B" (w.r.t. $\mathcal{V}$, i.e., $B \in \mathcal{B}(U,\mathcal{V})$). However, we do not necessarily "have a bounded C": if, for example, we are given a function such as $\mathcal{J}(x_0,u) := \|x\|^2_{\mathrm{L}^2(\mathbf{R}_+;\mathcal{V})} + \|u\|_2^2$ to be minimized, we would more be interested in $C \in \mathcal{B}(\mathcal{V},Y)$ rather than in $C \in \mathcal{B}(\mathcal{W},Y)$ in order to use the tools corresponding to a "bounded C", and C need not belong to $\mathcal{B}(\mathcal{V},Y)$. (To minimize, instead, $\|x\|^2_{\mathrm{L}^2(\mathbf{R}_+;\mathcal{W})} + \|u\|_2^2$, one could use the system $\Sigma \in \mathrm{WPLS}(U,\mathcal{W},Y)$ which does have a "bounded C".)

Sometimes the following characterization of PS-systems is more useful:

Lemma 6.9.4 (PS-systems) *A system* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,\mathcal{V},Y)$ *is an* ω*-stable PS-system iff 1.* $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{V},Y)\cap\mathrm{WPLS}_\omega(U,\mathcal{W},Y)$*, where* $\mathcal{W} \subset \mathcal{V}$ *densely and continuously, 2.* $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{V},Y)$ *has a bounded input operator, and 3.* $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{W},Y)$ *has a bounded output operator.*

Any PS-system is ULR and ω*-stable for some* $\omega \in \mathbf{R}$*; in particular, it is exponentially stable (i.e.,* ω*-stable for some* $\omega<0$*) iff* $\mathbb{A}$ *is exponentially stable on* $\mathcal{W}$ *and on* $\mathcal{V}$*.*

The system in 3. is obtained from that in 2. by just replacing $\mathcal{V}$ by $\mathcal{W}$ (as the domain and/or range space of $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$).

Proof: The first claim is obviously true (the operator $C' \in \mathcal{B}(\mathcal{W},Y)$ is the output operator of $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{W},Y)$). By Lemma 6.3.16(b) (or (c)), the I/O map of a PS-system ULR.

Assume that Σ is a PS-system. Choose $\omega \in \mathbf{R}$ s.t. $\mathbb{A}$ is ω'-stable on $\mathcal{W}$ and $\mathcal{V}$ for some $\omega'<\omega$. One easily verifies (cf. the reasoning on p. 158) that $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2_\omega,\mathcal{W})$; it follows that $\Sigma \in \mathrm{WPLS}(U,\mathcal{W},Y)$. By Lemma 6.1.10, both systems are ω-stable WPLSs, hence Σ is an ω-stable PS-system w.r.t. U, $\mathcal{W}$, $\mathcal{V}$ and Y. □

PS-systems are a strict subset of WPLSs; e.g., any H^∞_∞ transfer function has a realization as a WPLS (see Definition 6.1.6), but the same does not hold for PS-systems, by Theorem 6.9.6. For a PS-system, the unboundedness of C severely limits the possible unboundedness of B, and vice versa:

Remark 6.9.5 (*B* and *C* of PS-systems may be as unbounded as those of WPLSs)

If $\begin{bmatrix} \mathbb{A} & \big| & \mathbb{B} \end{bmatrix} \in \mathrm{WPLS}(U,H,\{0\})$, *then* $\left(\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right)$ *(or* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *for any* $\begin{bmatrix} C & D \end{bmatrix} \in \mathcal{B}(H_{-1} \times U, Y)$*) is a PS-system w.r.t.* $\mathcal{V} := H_{-1}$ *and* $\mathcal{W} := H$*, by Lemmas 6.9.4 and 6.3.16. An analogous claim holds for any* $\left[\begin{smallmatrix} \mathbb{A} \\ \mathbb{C} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,\{0\})$.

Thus, the input and output operators in a PS-system may be exactly as unbounded as in a WPLS, but not simultaneously*, since B must be "compatible" with* $\mathcal{W}$ *and C with* $\mathcal{V}$. □

For example, any parabolic system of Hypothesis 9.5.1 with "unboundedness distance of C and B" $\gamma - \beta < 1$ is a WPLS, but we must require that $\gamma - \beta < 1/2$ in order to make sure that it is a PS-system. Note also the while B can be as unbounded w.r.t. $\mathcal{W}$ as the input operator of any WPLS, the operator B must be bounded w.r.t. $\mathcal{V}$.

From Theorem 6.9.1 we observe that $\widehat{\mathbb{D}}$ and $\widehat{\mathbb{D}}^{\mathrm{d}}$ must be $\mathrm{H}^2_{\mathrm{strong}}$ over some right half-plane for $\mathbb{D}$ to have a PS-realization. This condition is also sufficient:

Theorem 6.9.6 ($\widehat{\mathbb{D}}, \widehat{\mathbb{D}}^{\mathrm{d}} \in \mathrm{H}^2_{\mathrm{strong}} \Leftrightarrow$ PS-realization) *Let* $\omega \in \mathbf{R}$ *and* $\widehat{\mathbb{D}} : \mathbf{C}^+_\omega \to \mathcal{B}(U,Y)$*. Then* $\widehat{\mathbb{D}}$ *is the transfer function of the I/O map of some* ω*-stable PS-system with* $D = 0$ *iff* $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(U,Y)) \cap \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$ *and* $\widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(Y,U))$.

We note two sufficient conditions:

1. if $\widehat{\mathbb{D}}(\cdot - \omega) \in \mathrm{H}^2(\mathbf{C}^+; \mathcal{B}(U,Y))$ for some $\omega \in \mathbf{R}$, then $\widehat{\mathbb{D}}$ has a ($\omega + \varepsilon$-stable) PS-realization (with $D = 0$).

2. By Lemma F.3.3(c2), $\mathcal{L}\mathrm{L}^2_{\mathrm{strong}} \subset \mathrm{H}^2_{\mathrm{strong}}$, hence $\mathbb{D}$ has a PS-realization (with $D = 0$) whenever $\mathbb{D}u = F * u$ (for all u), where $e^{-\omega\cdot} F, e^{-\omega\cdot} F^* \in \mathrm{L}^2_{\mathrm{strong}}$ for some $\omega \in \mathbf{R}$ (this has already been shown in [KMR]).

Most PS theory (e.g., [Keu]) cover only smooth PS-systems, for which the above theorem gives only necessary conditions. In this monograph, a PS-system satisfies the regularity assumptions of almost any result (see Theorem 8.4.9(γ)), hence the above conditions are more than sufficient for our results.

The requirement $D = 0$ simplifies the theorem but does not restrict generality: given an arbitrary $\widehat{\mathbb{D}}$, set $D := \widehat{\mathbb{D}}(+\infty)$ (regularity is a necessary condition) and apply Theorem 6.9.6 to $\widehat{\mathbb{D}} - D$.

Proof of Theorem 6.9.6:

Part I — "only if": This follows from Lemma 6.9.4 and Theorem 6.9.1.

Part II — "if": We shall show that conditions 1.–3. of Lemma 6.9.4 are satisfied. Set $H := \mathrm{L}^2_\omega(\mathbf{R}_+; Y)$, and let $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ be the strongly ω-stable realization (6.11). Let $\mathcal{W}$ be the space "H":= $\mathbb{B}[\mathcal{X}]$ of Definition 6.1.6.

1° $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{W},Y)$*:* This is shown in Lemma 6.1.7.

2° *There is* $C' \in \mathcal{B}(\mathcal{W},Y)$ *s.t.* $C'\mathbb{A} = \mathbb{C}$*:* As noted in the proof of Lemma 6.1.7, $T := \mathbb{B}|_{\mathcal{X}} : \mathcal{X} \to \mathcal{W}$ satisfies $T \in \mathcal{GB}(\mathcal{X},\mathcal{W})$, where $\mathcal{X} := \mathrm{Ran}(\mathbb{B})^\perp$. Consequently, $C' \in \mathrm{Hom}(\mathcal{W},Y)$ becomes well-defined by setting $C'\mathbb{B}u := (\mathbb{D}u)(0)$ $(u \in \mathcal{X})$, because then

$$\|C'\mathbb{B}u\|_Y \le \|\widehat{\mathbb{D}}(\bar{\cdot})^*\|_{\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))}\|u\|_{\mathrm{L}^2_\omega} \quad (u \in \mathcal{X}), \tag{6.216}$$

by Lemma F.3.7(b2), so that actually $C' \in \mathcal{B}(\mathcal{W},Y)$ (note that $C' = (\mathbb{D}T^{-1}\cdot)(0)$). Now, for $u \in \mathcal{X}$, we have

$$C\mathbb{A}^t\mathbb{B}u = C\mathbb{B}\tau^t u = (\mathbb{D}\tau^t u)(0) = (\mathbb{D}u)(t) = (\pi_+\mathbb{D}\pi_- u)(t) = (\mathbb{C}\mathbb{B}u)(t) \tag{6.217}$$

for $t \ge 0$, i.e., $C'\mathbb{A}x_0 = \mathbb{C}x_0$ $(= x_0 \in \mathcal{W})$ for all $x_0 \in \mathbb{B}[\mathcal{X}] = \mathcal{W}$,

(Now we have established all claims of Lemma 6.9.4 that do not include $\mathcal{V}$, and we can complete the proof by establishing also the part concerning $\mathcal{V}$.)

3° $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{V},Y)$ *(with a bounded input operator):* Let $\mathcal{V}$ be the closure of $\mathcal{W}$ in H (note also that $\mathcal{W} \subset H$ continuously, hence $\mathcal{W} \subset V$ continuously). By 1°, $\mathbb{A}^t x_0 \in \mathcal{W}$ for all $x_0 \in \mathcal{W}$; since $\mathbb{A}^t \in \mathcal{B}(H)$, it follows that $\mathbb{A}^t[\overline{\mathcal{W}}] \subset \overline{\mathcal{W}}$, i.e., that $\mathbb{A}^t x_0 \in \mathcal{V}$ for all $x_0 \in \mathcal{V}$, for any $t \ge 0$. Therefore, $\mathbb{A}$ is a (strongly) ω-stable C_0-semigroup on $\mathcal{V}$ (since $\mathbb{A}$ is a (strongly) ω-stable C_0-semigroup on H, as noted in Definition 6.1.6).

Moreover, $\mathrm{Ran}(\mathbb{B}) = \mathcal{W} \subset \mathcal{V}$. Therefore, $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{V},Y)$ (the properties 1.–4. of Definition 6.1.1 are inherited from $\Sigma \in \mathrm{WPLS}_\omega(U,H,Y)$).

By (the proof of) Theorem 6.9.1(a), the input operator B of $\Sigma \in \mathrm{WPLS}_\omega(U,H,Y)$ is bounded ($B \in \mathcal{B}(U,H)$); by uniqueness (see Lemma 6.1.16(b)) the input operator of $\Sigma \in \mathrm{WPLS}_\omega(U,\mathcal{V},Y)$ is again B $(\in \mathcal{B}(U,\mathcal{V})$, i.e., we restrict its range space to $\mathcal{V}$). □

Thus, if $\dim Y < \infty$, then a system with a bounded C has a realization with a bounded B:

Corollary 6.9.7 *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$*. If* C *is bounded and* $\dim Y < \infty$*, or* B *is bounded and* $\dim U < \infty$*, then* $\mathbb{D}$ *has an ω-stable PS-realization.*

In fact, it suffices that C *is bounded,* $\mathbb{B}$ *and* $\mathbb{D}$ *are ω-stable and* $\dim Y < \infty$*.*

Proof: As the proof of Theorem 6.9.1 shows, we only need the ω-stability of $\mathbb{C}$ and $\mathbb{D}$ for bounded B, and the ω-stability of $\mathbb{B}$ and $\mathbb{D}$ for bounded C.

By Theorem 6.9.1, we have $\widehat{\mathbb{D}}(\cdot - \omega) - D \in \mathrm{H}^\infty \cap \mathrm{H}^2$, hence $\mathbb{D}$ has an ω-stable PS-realization, by Theorem 6.9.6. □

Finally, we present the simplest case:

Corollary 6.9.8 *Let* $\dim U, \dim Y < \infty$*,* $\omega \in \mathbf{R}$ *and* $\mathbb{D} \in \mathrm{TIC}_\omega(U,Y)$*. Then the following are equivalent:*

(i) There is $f \in \mathrm{L}^2_\omega(\mathbf{R}_+;\mathcal{B}(U,Y))$ *s.t.* $\mathbb{D}u = f * u + Du$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$*;*

(ii) $\mathbb{D}$ *has an* ω*-stable realization with bounded B;*

(iii) $\mathbb{D}$ *has an* ω*-stable realization with bounded C;*

(iv) $\mathbb{D}$ *has an* ω*-stable PS-realization;*

(v) $\widehat{\mathbb{D}} - D \in \mathrm{H}^2(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$.

(Without the assumption $\mathbb{D} \in \mathrm{TIC}_\omega$ we would lose an ε of stability in (i) and (iv). E.g., for an arbitrary $f \in \mathrm{L}^2_\omega(\mathbf{R}_+; \mathcal{B}(U,Y))$ we would only know that $u \mapsto f * u$ has an ω'-stable PS-realization for any $\omega' > \omega$.)

By analogous arguments one observes that if $m := \dim U < \infty$ (so that $\mathcal{B}(U,Y) \cong Y^m$), then we have (i)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv); if $\dim Y < \infty$, then (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iv).

The condition "$\mathbb{D}u = f * u + Du$ for some $f : \mathbf{R}_+ \to \mathcal{B}(U,Y)$ s.t. $fu_0, f^* y_0 \in \mathrm{L}^2_\omega$ for all $u_0 \in U$ and $y_0 \in Y$" is not necessary for general U and Y (see the notes below).

Proof of Corollary 6.9.8: (Naturally, in (i) it would suffice to assume $\mathbb{D}u = f * u$ for all $u \in \mathcal{C}_c([0,T]; U)$ for some $T > 0$.) We take $D = 0$ w.l.o.g.

By Corollary 6.9.7 and its proof, (ii)–(iv) are equivalent. Set $m := \dim U$, $n := \dim Y$. By Theorem 6.9.1, condition (ii) holds iff (v) holds, i.e., iff $\widehat{\mathbb{D}} \in \mathrm{H}^2(\mathbf{C}^+_\omega; \mathbf{C}^{n\times m})$. But the latter holds iff $\widehat{\mathbb{D}} = \widehat{f}$ for some $f \in \mathrm{L}^2_\omega(\mathbf{R}_+; \mathbf{C}^{n\times m})$, by Theorem 3.3.1(b). On the other hand, $\widehat{f}\widehat{u} = \widehat{\mathbb{D}}\widehat{u}$ on $\mathbf{C}^+_{\omega+1}$ (hence on $\overline{\mathbf{C}^+_\omega}$) iff $\mathbb{D}u = f * u$, by Lemma D.1.11(c') (because $f \in \mathrm{L}^1_{\omega+1}$). □

Notes

The equivalence of (i)–(iii) of Corollary 6.9.8 is essentially contained in Theorem 5.2 of [Sal89], and his proof covered the cases where B was bounded and $\dim Y < \infty$ or C was bounded and $\dim U < \infty$, as noted in [WW].

In [KMR] it was stated and "proved" that $\mathbb{D}$ has a PS-realization iff $\mathbb{D}u = f * u$ ($u \in \mathrm{L}^2$), where $f, f^* \in \mathrm{L}^2_{\mathrm{strong}}$, the convolution existing as a weak (i.e., Pettis) integral. However, that condition is only sufficient but not necessary.

The fault was in the claim that the term $f := \mathbb{C}B$ would be well defined (a.e.) as a function $\mathbf{R}_+ \to \mathcal{B}(U,Y)$ (we do have $f \in \mathcal{B}(U, \mathrm{L}^2_{\mathrm{loc}})$ since B is bounded). Indeed, let $U = \ell^2(\mathbf{N})$ and $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+; \mathcal{B}(U))$ be as in Example F.3.6, so that $\widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+; \mathcal{B}(U))$. By Theorem 6.9.6, there is an ω-stable PS-realization $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ of $\mathbb{D}$ for any $\omega > 0$. Moreover, $f := \mathbb{C}B \in \mathcal{B}(U, \mathrm{L}^2_\omega)$ satisfies $\widehat{f} = \widehat{\mathbb{D}}$, by (e1) (or (e2)) of Lemma 6.8.1, hence f equals the operator F of Example F.1.10, hence f does not have a representation of form $\mathbf{R}_+ \to \mathcal{B}(U,Y)$ (as shown in the example, this would lead to the contradiction $\|f(t)\|_{\mathcal{B}(U,Y)} = \infty$ for a.e. $t \geq 0$).

Nevertheless, the paper [KMR] is an elegant introduction to PS-systems, and it inspired us to write Theorem 6.9.6. Moreover, the definition of $\mathcal{W}$ in the proof Theorem 6.9.6 is from [KMR] (the rest of our techniques are different and keep an exact track on stability).

The existence parts of the results of [Sal89], [KMR] and ours are based on the shift semigroup system (see Definition 6.1.6). We do not know corresponding conditions for smooth PS-realizations (this refers to the additional condition that $\mathrm{Dom}(A_{\mathcal{V}}) \subset \mathcal{W}$).

Chapter 7

Dynamic Stabilization

And when winds are at war with the ocean,
As the breasts I believed in with me,
If their billows excite an emotion,
It is that they bear me from thee.
— Lord Byron (1788–1824), "Stanzas to Augusta"

In this chapter we shall study different forms of dynamic stabilization, extend standard classical results (see, e.g., pp. 15–17 and 26–47 of [Francis]) for WPLSs and supplement them with new ones.

We assume that we are given a fixed *plant*, e.g., an I/O map $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ (alternatively, a WPLS) that we wish to control. In the case of *dynamic output feedback* (cf. pp. 36–42 of [Francis]) the output (y) of the plant is fed back to the input (u) through a *Dynamic Output Feedback Controller (DF-controller)* in order to stabilize and control the plant, as in Figure 7.1. Here u_L is the actual input and y as the final output; y_L can be considered as the disturbance in the feedback loop and u as the controller output. (In the literature, one sometimes uses the word "compensator" or "regulator" in place of "controller".)

By *DF-stabilization* of $\mathbb{D} \in \mathrm{TIC}_\infty(Y,U)$ we mean that we choose $\mathbb{Q} \in \mathrm{TIC}_\infty(U,Y)$ so that the map $\left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ (equivalently, $(I_{U\times W} - \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right])^{-1} : \left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$; cf. Figure 7.1) becomes stable; one often also wishes to minimize the norm $\|u_L \mapsto y\|_{\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U),\mathrm{L}^2(\mathbf{R}_+;Y))}$.

In Section 7.1 we shall extend several classical finite-dimensional results on DF-stabilization to general WPLSs; these results include the Youla parametrization of all stabilizing controllers (Corollary 7.1.8) based on a doubly coprime factorization (d.c.f.) of $\mathbb{D}$. However, it is not known whether each DF-stabilizable map has a d.c.f. (unless $\dim U, \dim Y < \infty$, see Lemma 7.1.4), hence we also present a theory for general TIC_∞ maps. (This applies to Sections 7.2 and 7.3 too.)

In DF-stabilization, we require that the controller ($\mathbb{Q}$) is well-posed (or proper, i.e., $\mathbb{Q} \in \mathrm{TIC}_\infty$). In finite-dimensional theory, one sometimes allows for improper controllers ("$\widehat{\mathbb{Q}} \in \mathrm{H}^\infty/\mathrm{H}^\infty$", i.e., $\mathbb{Q}$ is allowed to have a pole at infinity) while the closed-loop map ($(I_{U\times W} - \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right])^{-1}$) is always required to be well-posed. The infinite-dimensional counterpart of this concept, *a DF-controller with* *internal loop*, was introduced in [WC], by George Weiss and Ruth Curtain. This generalization of the concept of DF-controllers will be treated in Section 7.2.1.

In the H^∞ *Four-Block Problem (*H^∞ *4BP)* of Chapter 12, the controller may use only a part (y) of the of the output ($\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$) as its input and it can control only a part (u) of the input ($\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$) of the plant, as in Figure 7.8. Such a controller is called a *Dynamic Partial Output Feedback Controller (DPF-controller)* (cf. pp. 26–36 and 42–47 of [Francis]). We develop the theory for DPF-controllers (with or without internal loop) in Section 7.3. However, if $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$, then any stabilizing DPF-controller for $\mathbb{D}$ is a stabilizing DF-controller for $\mathbb{D}_{21}$, and under reasonable assumptions also the converse holds, by Lemmas 7.3.5 and 7.3.6. Therefore, much of this theory is obtained as a corollary of Section 7.2.

We have above treated only the I/O theory, while one is often more interested in a system stabilizing another system (also internally); cf. Figures 7.2 and 7.9. However, if Σ is a realization of the plant ($\mathbb{D}$) and $\widetilde{\Sigma}$ is a realization of the controller ($\mathbb{Q}$), then $\widetilde{\Sigma}$ stabilizes Σ exponentially iff $\mathbb{Q}$ stabilizes $\mathbb{D}$ and Σ and $\widetilde{\Sigma}$ are optimizable and estimatable (recall from Definition 6.7.3 and Corollary 9.2.13 that at least if Σ is sufficiently regular, then this is equivalent to exponential stabilizability and detectability), by Theorems 7.2.3 and 7.3.11. We also present some further results on "$\widetilde{\Sigma}$ stabilizing Σ".

We give most of our results for (non-exponential) stabilization, because the exponential analogies of such results can be obtained through shifting, as in Remarks 7.2.19 and 7.3.24 (but the converse is not true). However, there are some results that seem to hold for exponential stabilization only; such results are given explicitly.

Remark 7.0.1 *Almost all I/O results in this chapter are purely algebraic (and do not assume commutativity, neither a matrix structure over some commutative ring), hence they are valid when we replace* TIC_∞ *by* $\mathcal{A}'$ *and* TIC *by* $\mathcal{A}$*, where* $\mathcal{A}$ *and* $\mathcal{A}'$ *(and* X*) are as in Remark 6.5.11.*

Thus, one can have a given plant $\mathbb{D} \in \mathcal{A}'(U,Y)$ $(U,Y \in X)$ *and seek for a* $\mathbb{Q} \in \mathcal{A}'(Y,U)$ *that makes* $\mathbb{D}_l^o \in \mathcal{A}$*, i.e., "stable"; see Definitions 7.1.1, 7.2.1, and 7.3.1 for details.*

This holds for the results concerning the I/O maps only, i.e., the frequency-domain results; the generalization of state-space results requires, of course, further assumptions (which are often simple, cf. Chapter 13 for a discrete-time application).

7.1 Dynamic feedback (DF) stabilization

A fail-safe circuit will destroy others.
— Klipstein

As explained above, in this section we generalize several classical dynamic output feedback (DF) results (cf. [Francis, Section 4]) to the infinite-dimensional case (see, e.g., Theorem 7.1.7); most others are generalized in Section 7.2.

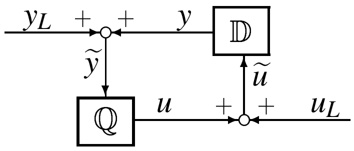


Figure 7.1: DF-controller $\mathbb{Q}$ for $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$

Every DF-stabilizable rational transfer function has a d.c.f., and classical DF-stabilization theory is based on d.c.f.'s. We believe that not every DF-stabilizable $\mathbb{D} \in \mathrm{TIC}_\infty$ has a d.c.f. (cf. Lemma 7.1.4), therefore we also develop a DF-stabilization theory for general TIC_∞ maps (and for general WPLSs).

In Figure 7.1, we have

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix} \begin{bmatrix} u+u_L \\ y+y_L \end{bmatrix}, \tag{7.1}$$

or, by setting $\widetilde{u} := u + u_L$, $\widetilde{y} := y + y_L$,

$$\begin{bmatrix} \widetilde{u} \\ \widetilde{y} \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix} \begin{bmatrix} \widetilde{u} \\ \widetilde{y} \end{bmatrix} + \begin{bmatrix} u_L \\ y_L \end{bmatrix}; \text{ or equivalently,} \tag{7.2}$$

$$\begin{bmatrix} \widetilde{u} \\ \widetilde{y} \end{bmatrix} = \left(I - \begin{bmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} u_L \\ y_L \end{bmatrix} \tag{7.3}$$

provided that $\mathbb{Q}$ is an admissible DF-controller for $\mathbb{D}$, i.e., that $I - \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}_\infty$, which is equivalent to $I - \mathbb{D}\mathbb{Q} \in \mathcal{G}\mathrm{TIC}_\infty$ (by Lemma A.1.1(d1)).

Note that this corresponds to $L = I$ in the setting of Definition 6.6.1 (applied with substitutions $\mathbb{D} \mapsto \mathbb{D}^o := \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right]$, $y \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$, $u_L \mapsto \left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right]$; compare (7.1) and (6.123)–(6.124) with $x_0 = 0$), hence the solvability (in TIC_∞) of the above equations is, indeed, equivalent to the admissibility of L, i.e., to condition $I - \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}_\infty$. We conclude that the corresponding closed-loop map is given by $\mathbb{D}^o_I$. Analogously, for the setting of Figure 7.4, the corresponding closed-loop system is given by the system $\Sigma^o_I : \begin{bmatrix} x_0 & \widetilde{x}_0 & u_L & y_L \end{bmatrix}^{\mathrm{T}} \mapsto \begin{bmatrix} x & \widetilde{x} & u & y \end{bmatrix}^{\mathrm{T}}$ defined below.

Therefore, we define:

Definition 7.1.1 (DF-stabilization) *We call $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ an* admissible [stabilizing] (DF-)controller *for $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ if $L = I$ is admissible [stabilizing] for $\mathbb{D}^o := \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right]$.*

We call $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{Q} \end{array}\right] \in \mathrm{WPLS}(Y,\widetilde{H},U)$ an admissible [stabilizing] (DF-)controller for $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$ *[and we say that $\widetilde{\Sigma}$* (DF-)stabilizes *Σ] if $L = I$ is admissible [stabilizing] for the (permutated) parallel connection*

$$\Sigma^o := \left[\begin{array}{cc|cc} \mathbb{A} & 0 & \mathbb{B} & 0 \\ 0 & \widetilde{\mathbb{A}} & 0 & \widetilde{\mathbb{B}} \\ \hline 0 & \widetilde{\mathbb{C}} & 0 & \mathbb{Q} \\ \mathbb{C} & 0 & \mathbb{D} & 0 \end{array}\right] \in \mathrm{WPLS}(U \times Y, H \times \widetilde{H}, U \times Y) \tag{7.4}$$

(we use prefices as in Definition 6.6.4).

We call $\mathbb{D}$ (resp. Σ) DF-stabilizable *if it has a stabilizing controller $\mathbb{Q}$ (resp.*

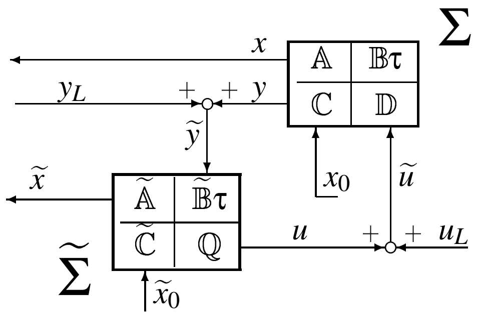


Figure 7.2: DF-controller $\widetilde{\Sigma}$ for $\Sigma \in \mathrm{WPLS}(U,H,Y)$

$\widetilde{\Sigma}$), and we use prefices as in Definition 6.6.4.

We usually say "stabilizing" instead of "admissible stabilizing" (for any meaning of these two words used in this monograph). Note that by "controller for $\mathbb{D}$" we refer to a I/O map ($\mathbb{O}$) and by "controller for Σ" we refer to system ($\widetilde{\Sigma}$). In classical theory one often does not make any difference for these two concepts (but we do).

Thus, $\mathbb{Q}$ is admissible [stabilizing] for $\mathbb{D}$ iff the closed-loop system in Figure 7.1 is well-posed [and stable, i.e., $u,y \in \mathrm{L}^2$ for all $u_L, y_L \in \mathrm{L}^2$]. Analogously, $\widetilde{\Sigma}$ is admissible [stabilizing] for Σ iff the closed-loop system in Figure 7.2 is well-posed [and stable, i.e., $u,y \in \mathrm{L}^2$ and x and $\widetilde{x}$ are bounded for all $u_L, y_L \in \mathrm{L}^2(\mathbf{R}_+;*)$, $x_0 \in H$ and $\widetilde{x}_0 \in \widetilde{H}$]. By Lemma A.4.5 and Lemma 6.1.10(a1), $\widetilde{\Sigma}$ is exponentially stabilizing for Σ iff $x, \widetilde{x} \in \mathrm{L}^2$ (and hence $u,y \in \mathrm{L}^2$) for all $u_L, y_L \in \mathrm{L}^2$, $x_0 \in H$ and $\widetilde{x}_0 \in \widetilde{H}$.

Obviously, $\mathbb{D}^o$ and hence $\mathbb{D}^o_I$ are the same in both settings (i.e., in the setting of Figure 7.1 and in that of Figure 7.2). Thus, $\widetilde{\Sigma}$ is I/O-stabilizing for Σ iff $\mathbb{Q}$ is stabilizing for $\mathbb{D}$. An analogous comment applies to Definitions 7.2.1 and 7.3.1 too.

Recall from Definition 6.6.10 that we follow the standard convention to use the word "stabilization" for state-feedback stabilization. Therefore, we have chosen the term "DF-stabilization" for dynamic output feedback, but we drop "DF-" when there is no danger of misinterpretation.

In some classical texts, one loosely speaks of "$\mathbb{Q}$ stabilizing Σ", but one then usually means the concept "$\widetilde{\Sigma}$ stabilizing Σ" for a suitable realization of $\mathbb{Q}$. However, we pay some attention to this "concept" in Remark 12.5.8.

From the above definition and Definition 6.6.1 we observe that $\mathbb{Q}$ is admissible for $\mathbb{D}$ iff $\widetilde{\Sigma}$ is admissible for Σ. We list here several additional equivalent conditions:

Lemma 7.1.2 (DF-admissibility) *A map $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ is admissible [stabilizing] for $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ iff $\begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(U \times Y)$ [and $\begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix}^{-1} \in \mathrm{TIC}(U \times Y)$]; or equivalently, if the closed-loop I/O map $\mathbb{D}^o_I : \left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \to \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$, given by*

(cf. Figure 7.1)

$$\begin{aligned}\mathbb{D}_I^o &= \begin{bmatrix}0 & \mathbb{Q}\\ \mathbb{D} & 0\end{bmatrix}\left(I-\begin{bmatrix}0 & \mathbb{Q}\\ \mathbb{D} & 0\end{bmatrix}\right)^{-1} = \left(I-\begin{bmatrix}0 & \mathbb{Q}\\ \mathbb{D} & 0\end{bmatrix}\right)^{-1} - I\\ &= \begin{bmatrix}I & -\mathbb{Q}\\ -\mathbb{D} & I\end{bmatrix}^{-1} - I = \begin{bmatrix}(I-\mathbb{Q}\mathbb{D})^{-1}-I & (I-\mathbb{Q}\mathbb{D})^{-1}\mathbb{Q}\\ \mathbb{D}(I-\mathbb{Q}\mathbb{D})^{-1} & (I-\mathbb{D}\mathbb{Q})^{-1}-I\end{bmatrix}\end{aligned} \tag{7.5}$$

is well-posed [and stable (i.e., $\mathbb{D}_I^o \in \mathrm{TIC}$)].

Moreover, $\mathbb{Q}$ is admissible for $\mathbb{D}$ (equivalently, $\widetilde{\Sigma}$ is admissible for Σ) iff $I-\mathbb{Q}\mathbb{D}\in \mathcal{G}\mathrm{TIC}_\infty(U)$ (equivalently, $I-\mathbb{D}\mathbb{Q}\in\mathcal{G}\mathrm{TIC}_\infty(Y)$). For $\mathbb{D},\mathbb{Q}\in\mathrm{ULR}$ this is equivalent to $I-DQ\in\mathcal{G}\mathcal{B}(Y)$.

Proof: We have $I-L\mathbb{D}^o = \left[\begin{smallmatrix}I & -\mathbb{Q}\\ -\mathbb{D} & I\end{smallmatrix}\right]$, hence the first paragraph follows from Definition 6.6.1 and Proposition 6.6.2. Use Lemma A.1.1(d1) (note also (A.12)) and Proposition 6.3.1(c) for the second paragraph. □

The roles of $\mathbb{D}$ and $\mathbb{Q}$ (resp. Σ and $\widetilde{\Sigma}$), are identical; e.g., $\mathbb{Q}$ stabilizes $\mathbb{D}$ iff $\mathbb{D}$ stabilizes $\mathbb{Q}$. This will not be the case in the dynamic partial (output) feedback, in Section 7.3, where the input of $\mathbb{Q}$ is only a part of the output of $\mathbb{D}$, and the input of $\mathbb{D}$ consists only partially of the output of $\mathbb{Q}$.

Unlike for admissibility, $\widetilde{\Sigma}$ being stabilizing for Σ is a stronger condition than $\mathbb{Q}$ being stabilizing for $\mathbb{D}$, since in Figure 7.2 there are more signals (or maps between signals) to be stabilized (by the choice of $\widetilde{\Sigma}$) than in Figure 7.1.

Indeed, $\widetilde{\Sigma}$ I/O-stabilizes Σ iff $\mathbb{Q}$ stabilizes $\mathbb{D}$. In this chapter, we will concentrate on I/O-stabilization, because for optimizable and estimatable Σ and $\widetilde{\Sigma}$, I/O-stabilization is equivalent to exponential stabilization, by Theorem 7.2.3(d)&(c1). (See the theorem for several analogous results.)

If $\mathbb{D}$ has a d.c.f. and $\mathbb{Q}$ stabilizes $\mathbb{D}$, then $\mathbb{D}$ and $\mathbb{Q}$ have jointly [strongly] stabilizable and detectable realizations, by Theorem 6.6.28 and Proposition 7.1.6. If Σ and $\widetilde{\Sigma}$ are such realizations and we connect their inputs and outputs (as in Figure 7.2 and Definition 7.1.1), then the resulting combined closed-loop system becomes [strongly] stable, by Theorem 7.2.3. (If $\mathbb{D}$ has an exponential d.c.f. and $\mathbb{Q}$ stabilizes $\mathbb{D}$ exponentially, then we can choose Σ and $\widetilde{\Sigma}$ so that the closed-loop system becomes exponentially stable.)

Note that we have assumed $\mathbb{Q}$ to be well-posed, that is, in TIC_∞ (i.e., $\widehat{\mathbb{Q}}\in \mathrm{H}^\infty_\infty$). See Section 7.2 for non-well-posed controllers.

A stable map (or system) is stabilized by any sufficiently small stable perturbation:

Lemma 7.1.3 (Small Gain Theorem) *Let $\|\mathbb{D}\|_{\mathrm{TIC}(U,Y)}\|\mathbb{Q}\|_{\mathrm{TIC}(Y,U)}<1$. Then $\mathbb{Q}$ stabilizes $\mathbb{D}$.*

If Σ and $\widetilde{\Sigma}$ are [SOS-/strongly/exponentially] stable realizations of $\mathbb{D}$ and $\mathbb{Q}$, respectively, then $\widetilde{\Sigma}$ [SOS-/strongly/exponentially] stabilizes Σ.

Proof: 1° *$\mathbb{Q}$ stabilizes $\mathbb{D}$:* Now $I-\mathbb{D}\mathbb{Q}\in\mathcal{G}\mathrm{TIC}$, by Lemma A.3.3(A0), hence (7.5) is stable.

2° *Σ stabilizes $\widetilde{\Sigma}$:* This follows from Theorem 7.2.3(d)&(a)&(b). □

We will often assume that $\mathbb{D}$ has a d.c.f. If U and Y are finite-dimensional, then this does not reduce generality (we suspect that this is not the case in general):

Lemma 7.1.4 (D.c.f.) *If $\mathbb{D} \in \mathrm{TIC}_\infty(\mathbf{C}^n, \mathbf{C}^m)$ is DF-stabilizable, then $\mathbb{D}$ has a d.c.f.*

However, not all distributed scalar systems (functions f/g, where $f, g \in \mathrm{H}^\infty(\mathbf{C}^+)$ and $g \not\equiv 0$) have coprime factorizations, because H^∞ is not a Bezout domain (see [Vid]), although we do not know if this applies also to well-posed scalar transfer functions (those with f/g bounded on some right half-plane).

If $\mathbb{D}$ has a r.c.f., then it has at least a DF-stabilizing controller with internal loop, by Corollary 7.2.13; see also Proposition 7.1.6(b1).

If $\mathbb{D} \in \mathrm{TIC}_\infty(\mathbf{C})$ has a r.c.f., then there is a *stable* stabilizing DF-controller for $\mathbb{D}$. Indeed, if $\widehat{\mathbb{N}}, \widehat{\mathbb{M}} \in \mathrm{H}^\infty(\mathbf{C}^+)$ are coprime, then $\widehat{\mathbb{M}} - \widehat{\mathbb{Q}}\widehat{\mathbb{N}} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{C}^+)$, for some $\widehat{\mathbb{Q}} \in \mathrm{H}^\infty(\mathbf{C}^+)$, by [Treil92], hence then $\mathbb{Q} \in \mathrm{TIC}(\mathbf{C})$ is stabilizing for $\mathbb{D}$, by Proposition 7.1.6(b1). (This would not be the case if the scalar field was real, see [S92].)

Naturally, possible extensions of this "stable (Bass) rank" result by Serge Treil for multi- or infinite-dimensional Hilbert spaces would extend the above conclusion correspondingly.

Proof of Lemma 7.1.4: Now $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, where $\mathbb{N} := \mathbb{D}(I - \mathbb{Q}\mathbb{D})^{-1}$ and $\mathbb{M} := (I - \mathbb{Q}\mathbb{D})^{-1}$ are stable, by (7.5). Thus, $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1} \in \mathrm{H}^\infty/\mathrm{H}^\infty$ (and $\widehat{\mathbb{D}}$ is DF-stabilizable), so by [Smith, Theorem 1], $\widehat{\mathbb{D}}$ has a generalized r.c.f. and a generalized l.c.f. in the sense that $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}_1\widehat{\mathbb{M}}_1^{-1}$ and $\widehat{\mathbb{D}} = \widehat{\widetilde{\mathbb{M}}}_1^{-1}\widehat{\widetilde{\mathbb{N}}}_1$ for some $\mathbb{N}_1, \mathbb{M}_1, \widetilde{\mathbb{N}}_1, \widetilde{\mathbb{M}}_1 \in \mathrm{TIC}$ with $\mathbb{N}_1, \mathbb{M}_1$ r.c. and $\widetilde{\mathbb{N}}_1, \widetilde{\mathbb{M}}_1$ l.c.

By Lemma 6.5.4(d2), $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$ and $\mathbb{D} = \mathbb{N}_1\mathbb{M}_1^{-1}$ — this is a r.c.f. Similarly, $\mathbb{D}^{\mathrm{d}} = \widetilde{\mathbb{N}}_1^{\mathrm{d}}(\widetilde{\mathbb{M}}_1^{\mathrm{d}})^{-1}$ is a r.c.f., i.e., $\mathbb{D} = \widetilde{\mathbb{M}}_1^{-1}\widetilde{\mathbb{N}}_1$ is a l.c.f.. Thus, they can be completed to a d.c.f., by Lemma 6.5.8. □

Lemma 7.1.5 *Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ and $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ be r.c.f.'s. Then $\left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & \mathbb{Y} \\ \mathbb{N} & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbb{M} & 0 \\ 0 & \mathbb{X} \end{smallmatrix}\right]^{-1}$ is a r.c.f. (of $\left[\begin{smallmatrix} 0 & \mathbb{Q} \\ \mathbb{D} & 0 \end{smallmatrix}\right]$). Moreover, we have the following:*

(a) The DF-controller $\mathbb{Q}$ is admissible for $\mathbb{D}$ iff $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}_\infty$; if this is the case, then

$$\mathbb{D}_I^o := \begin{bmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{bmatrix}^{-1} - I = \begin{bmatrix} 0 & \mathbb{Y} \\ \mathbb{N} & 0 \end{bmatrix}\begin{bmatrix} \mathbb{M} & -\mathbb{Y} \\ -\mathbb{N} & \mathbb{X} \end{bmatrix}^{-1} \tag{7.6}$$

(b) The DF-controller $\mathbb{Q}$ stabilizes $\mathbb{D}$ iff $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$; if this is the case, and we set

$$\begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} := \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix}^{-1}, \tag{7.7}$$

then $\widetilde{\mathbb{M}}, \widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$, $\widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ is a l.c.f. of $\mathbb{D}$, and $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ is a l.c.f. of $\mathbb{Q}$.

The obvious dual results for l.c.f.'s are true as well.

So if $\mathbb{Q}$ stabilizes $\mathbb{D}$ and these maps have coprime factorizations from same side, then we actually have the d.c.f. (7.7); cf. Proposition 7.1.6(a).

Proof: Clearly $\left[\begin{smallmatrix}0 & \mathbb{Q}\\ \mathbb{D} & 0\end{smallmatrix}\right]=\left[\begin{smallmatrix}0 & \mathbb{Y}\\ \mathbb{N} & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathbb{M} & 0\\ 0 & \mathbb{X}\end{smallmatrix}\right]^{-1}$ is a r.c.f., so (a) and the equivalence in (b) hold by Lemma 6.6.6 (and Lemma A.1.1(c3)) (recall that $L=I$ in Definition 7.1.1).

Assume now that $\mathbb{Q}$ DF-stabilizes $\mathbb{D}$ Then $\widetilde{\mathbb{M}},\widetilde{\mathbb{X}}\in \mathcal{G}\mathrm{TIC}_\infty$, by Lemma A.1.1(c1) (because $\mathbb{M},\mathbb{X}$ do), and (7.7) shows that $\widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}=\mathbb{N}\mathbb{M}^{-1}$, $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}=\mathbb{Y}\mathbb{X}^{-1}$, and that these factorizations are coprime.

By taking (causal) adjoints, one gets the dual results. □

Proposition 7.1.6 *Let* $\mathbb{D}\in \mathrm{TIC}_\infty(U,Y)$.

(a) Any stabilizing DF-controller of $\mathbb{D}$ *has a l.c.f. (resp. r.c.f.) iff* $\mathbb{D}$ *has a r.c.f. (resp. l.c.f.).*

(b) If $\mathbb{D}$ *has a r.c.f.* $\mathbb{D}=\mathbb{N}\mathbb{M}^{-1}$, *then (b1)–(b3) hold.*

(b1) A map $\mathbb{Q}\in \mathrm{TIC}_\infty$ *DF-stabilizes* $\mathbb{D}$ *iff* $\mathbb{Q}$ *has a l.c.f.* $\mathbb{Q}=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *s.t.* $\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N}=I$. *If* $\widetilde{\mathbb{X}}$ *and* $\widetilde{\mathbb{Y}}$ *are such, then*

$$\widetilde{\mathbb{X}}=(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1},\quad \widetilde{\mathbb{Y}}=(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}\mathbb{Q},\quad \textit{and} \tag{7.8}$$

$$\mathbb{D}_I^o=\begin{bmatrix}\mathbb{M}\widetilde{\mathbb{X}}-I & \mathbb{M}\widetilde{\mathbb{Y}}\\ \mathbb{N}\widetilde{\mathbb{X}} & \mathbb{N}\widetilde{\mathbb{Y}}\end{bmatrix}:\begin{bmatrix}u_L\\ y_L\end{bmatrix}\to\begin{bmatrix}u\\ y\end{bmatrix}. \tag{7.9}$$

(b2) Let $\mathbb{Q}=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *be a l.c.f. Then* $\mathbb{Q}$ *DF-stabilizes* $\mathbb{D}$ *iff* $\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N}\in \mathcal{G}\mathrm{TIC}$.

(b3) Let $\mathbb{Q}=\mathbb{Y}\mathbb{X}^{-1}$ *be a r.c.f. Then* $\mathbb{Q}$ *DF-stabilizes* $\mathbb{D}$ *iff* $\left[\begin{smallmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{smallmatrix}\right]\in \mathcal{G}\mathrm{TIC}$.

(c) If $\mathbb{D}$ *has a l.c.f.* $\mathbb{D}=\widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$, *then (c1)–(c3) hold.*

(c1) A map $\mathbb{Q}\in \mathrm{TIC}_\infty$ *DF-stabilizes* $\mathbb{D}$ *iff* $\mathbb{Q}$ *has a r.c.f.* $\mathbb{Q}=\mathbb{Y}\mathbb{X}^{-1}$ *s.t.* $\widetilde{\mathbb{M}}\mathbb{X}-\widetilde{\mathbb{N}}\mathbb{Y}=I$. *If* $\mathbb{X}$ *and* $\mathbb{Y}$ *are such, then*

$$\mathbb{X}=(\widetilde{\mathbb{M}}-\widetilde{\mathbb{N}}\mathbb{Q})^{-1},\quad \mathbb{Y}=(\widetilde{\mathbb{M}}-\widetilde{\mathbb{N}}\mathbb{Q})^{-1}\mathbb{Q}. \tag{7.10}$$

(c2) Let $\mathbb{Q}=\mathbb{Y}\mathbb{X}^{-1}$ *be a r.c.f. Then* $\mathbb{Q}$ *DF-stabilizes* $\mathbb{D}$ *iff* $\widetilde{\mathbb{M}}\mathbb{X}-\widetilde{\mathbb{N}}\mathbb{Y}\in \mathcal{G}\mathrm{TIC}$.

(c3) Let $\mathbb{Q}=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *be a l.c.f. Then* $\mathbb{Q}$ *DF-stabilizes* $\mathbb{D}$ *iff* $\left[\begin{smallmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{smallmatrix}\right]\in \mathcal{G}\mathrm{TIC}$.

(d) Let $\mathbb{Q}$ *DF-stabilize* $\mathbb{D}$. *Then* $\mathbb{D}$ *has a d.c.f. iff* $\mathbb{Q}$ *has a d.c.f.*

Note that $\left[\begin{smallmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{smallmatrix}\right]\in \mathcal{G}\mathrm{TIC}\Leftrightarrow\left[\begin{smallmatrix}\widetilde{\mathbb{X}} & \widetilde{\mathbb{Y}}\\ \widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{smallmatrix}\right]\in \mathcal{G}\mathrm{TIC}$, by Lemma A.1.1(c3).

In (b1), clearly $\left[\begin{smallmatrix}I & -\mathbb{Q}\\ -\mathbb{D} & I\end{smallmatrix}\right]=\left[\begin{smallmatrix}\mathbb{M}\widetilde{\mathbb{X}} & \mathbb{M}\widetilde{\mathbb{Y}}\\ \mathbb{N}\widetilde{\mathbb{X}} & \mathbb{M}\widetilde{\mathbb{X}}\end{smallmatrix}\right]:\left[\begin{smallmatrix}u_L\\ y_L\end{smallmatrix}\right]\to\left[\begin{smallmatrix}\tilde{u}\\ \tilde{y}\end{smallmatrix}\right]$.

Proof: (a) This follows from (b1) and (c1), because we can interchange the roles of $\mathbb{D}$ and $\mathbb{Q}$.

(b1) 1° Let $\mathbb{D}$ have a r.c.f. $(\mathbb{N},\mathbb{M})$ and $\widetilde{\mathbb{S}}\mathbb{M}-\widetilde{\mathbb{T}}\mathbb{N}=I$, $\widetilde{\mathbb{T}},\widetilde{\mathbb{S}}\in \mathrm{TIC}$. Let $\mathbb{Q}$ stabilize $\mathbb{D}$, so that $I-\mathbb{Q}\mathbb{D}=I-\mathbb{Q}\mathbb{N}\mathbb{M}^{-1}\in \mathcal{G}\mathrm{TIC}_\infty$ and $\mathbb{D}_I^o\in \mathrm{TIC}$, in particular, $\mathbb{M}-\mathbb{Q}\mathbb{N}\in \mathcal{G}\mathrm{TIC}_\infty$.

The stability of $\mathbb{D}[\mathbb{M}(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}]=\mathbb{D}(I-\mathbb{Q}\mathbb{D})^{-1}$ and $\mathbb{D}[\mathbb{M}(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}\mathbb{Q}]=\mathbb{D}(I-\mathbb{Q}\mathbb{D})^{-1}\mathbb{Q}=(I-\mathbb{D}\mathbb{Q})^{-1}-I$, from (7.5) (and Lemma A.1.1(f6)), implies that of $\widetilde{\mathbb{X}}:=(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}=\mathbb{M}^{-1}[\mathbb{M}(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}]$ and $\widetilde{\mathbb{Y}}:=(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}\mathbb{Q}$, by Lemma 6.5.6(b). Clearly $\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N}=(\mathbb{M}-\mathbb{Q}\mathbb{N})^{-1}(\mathbb{M}-\mathbb{Q}\mathbb{N})=I$, so $\widetilde{\mathbb{X}}\in \mathcal{G}\mathrm{TIC}_\infty$ and $\widetilde{\mathbb{Y}}$ are l.c.

2° Conversely, if $\mathbb{Q}=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ is a l.c.f. and $\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N}=I$, then $(I-\mathbb{Q}\mathbb{D})^{-1}=[\widetilde{\mathbb{X}}^{-1}(\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N})\mathbb{M}^{-1}]^{-1}=\mathbb{M}\widetilde{\mathbb{X}}$ etc., hence (7.9) holds, so $\mathbb{D}_I^o\in\mathrm{TIC}$, i.e., $\mathbb{Q}$ is stabilizing.

3° By Lemma 6.4.5(d), the $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ constructed in 1° are uniquely determined by $\mathbb{Q}$.

(b2) By Lemma 6.4.5, all l.c.f.'s of $\mathbb{Q}$ are given by $(\mathbb{U}\widetilde{\mathbb{Y}},\mathbb{U}\widetilde{\mathbb{X}})$ with $\mathbb{U}\in \mathcal{G}\mathrm{TIC}$, Therefore, $\mathbb{Q}$ has a l.c.f. of the form described in (a) iff $\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N}\in \mathcal{G}\mathrm{TIC}$.

(b3) This follows from Lemma 7.1.5.

(c) This is proved analogously (or by taking (causal) adjoints in (b)). Of course, we could write a dual formula for $\mathbb{D}_I^o$ too.

(d) This follows from (a) and from the fact that a well-posed map has a d.c.f. iff it has a r.c.f. and a l.c.f. [Lemma 6.5.8].

□

In most control theory one studies proper rational transfer functions (i.e., those with a (WPLS) realization with $\dim U, \dim H, \dim Y < \infty$); they always have a d.c.f. If $\dim U, \dim Y < \infty$, then $\mathbb{D}\in \mathrm{TIC}_\infty(U,Y)$ must have a d.c.f. in order to be DF-stabilizable, by Lemma 7.1.4. See Lemma 6.5.10 for further sufficient conditions for the existence of a d.c.f.

For these reasons, we shall often assume the existence of a d.c.f. This assumption enables us to generalize the Youla parameterization of all stabilizing controllers:

Theorem 7.1.7 (Stabilizing DF-controllers) *Let $\mathbb{D}\in\mathrm{TIC}_\infty(U,Y)$ have a r.c.f. and a l.c.f. $\mathbb{D}=\mathbb{N}\mathbb{M}^{-1}=\widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$. Then $\mathbb{D}$ has the d.c.f.*

$$\begin{bmatrix}\mathbb{M} & \mathbb{T}\\ \mathbb{N} & \mathbb{S}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}=I=\begin{bmatrix}\widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}\mathbb{M} & \mathbb{T}\\ \mathbb{N} & \mathbb{S}\end{bmatrix} \tag{7.11}$$

for some $\mathbb{T},\mathbb{S},\widetilde{\mathbb{T}},\widetilde{\mathbb{S}}\in\mathrm{TIC}$, and the following are equivalent:

(i) $\mathbb{Q}$ DF-stabilizes $\mathbb{D}$.

(ii) $\left[\begin{smallmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{smallmatrix}\right]\in\mathcal{G}\mathrm{TIC}$, $\mathbb{X}\in\mathcal{G}\mathrm{TIC}_\infty$ and $\mathbb{Q}=\mathbb{Y}\mathbb{X}^{-1}$.

(iii) $\left[\begin{smallmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{smallmatrix}\right]\in\mathcal{G}\mathrm{TIC}$, $\widetilde{\mathbb{X}}\in\mathcal{G}\mathrm{TIC}_\infty$ and $\mathbb{Q}=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$.

(iv) $\mathbb{Q}$ *has a d.c.f.* $\mathbb{Q}=\mathbb{Y}\mathbb{X}^{-1}=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *s.t.*

$$\begin{bmatrix}\mathbb{M}&\mathbb{Y}\\ \mathbb{N}&\mathbb{X}\end{bmatrix}^{-1}=\begin{bmatrix}\widetilde{\mathbb{X}}&-\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}}&\widetilde{\mathbb{M}}\end{bmatrix}. \tag{7.12}$$

(v) [Youla] $\mathbb{Q}=(\mathbb{T}+\mathbb{M}\mathbb{U})(\mathbb{S}+\mathbb{N}\mathbb{U})^{-1}$ *for some* $\mathbb{U}\in\mathrm{TIC}$ *s.t.* $\mathbb{S}+\mathbb{N}\mathbb{U}\in\mathcal{G}\mathrm{TIC}_\infty$.

(vi) [Youla] $\mathbb{Q}=(\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}})^{-1}(\widetilde{\mathbb{T}}+\mathbb{U}\widetilde{\mathbb{M}})$ *for some* $\mathbb{U}\in\mathrm{TIC}$ *s.t.* $\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}}\in\mathcal{G}\mathrm{TIC}_\infty$.

(vii) $\mathbb{Q}=\mathbb{Y}\mathbb{X}^{-1}$*, where* $\left[\begin{smallmatrix}\mathbb{Y}\\ \mathbb{X}\end{smallmatrix}\right]=\left[\begin{smallmatrix}\mathbb{M}&\mathbb{T}\\ \mathbb{N}&\mathbb{S}\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right]$ *and* $\mathbb{U}\in\mathrm{TIC}$ *is s.t.* $\mathbb{X}=\mathbb{S}+\mathbb{N}\mathbb{U}\in\mathcal{G}\mathrm{TIC}_\infty$.

(viii) $\mathbb{Q}=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$*, where* $\left[\begin{smallmatrix}\widetilde{\mathbb{X}}&\widetilde{\mathbb{Y}}\end{smallmatrix}\right]=\left[\begin{smallmatrix}I&\mathbb{U}\end{smallmatrix}\right]\left[\begin{smallmatrix}\widetilde{\mathbb{S}}&\widetilde{\mathbb{T}}\\ \widetilde{\mathbb{N}}&\widetilde{\mathbb{M}}\end{smallmatrix}\right]$ *and* $\mathbb{U}\in\mathrm{TIC}$ *is s.t.* $\widetilde{\mathbb{X}}=\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}}\in\mathcal{G}\mathrm{TIC}_\infty$.

Moreover, for $\mathbb{U}\in\mathrm{TIC}$ *we have* $\mathbb{S}+\mathbb{N}\mathbb{U}\in\mathcal{G}\mathrm{TIC}_\infty\Leftrightarrow\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}}\in\mathcal{G}\mathrm{TIC}_\infty$*, and if either is true, then*

$$(\mathbb{T}+\mathbb{M}\mathbb{U})(\mathbb{S}+\mathbb{N}\mathbb{U})^{-1}=(\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}})^{-1}(\widetilde{\mathbb{T}}+\mathbb{U}\widetilde{\mathbb{M}}). \tag{7.13}$$

Thus, any $\mathbb{D}\in\mathrm{TIC}_\infty$ having a d.c.f. (denoted by (7.11)) is DF-stabilizable iff $\mathbb{S}+\mathbb{N}\mathbb{U}\in\mathcal{G}\mathrm{TIC}_\infty$ for some $\mathbb{U}\in\mathrm{TIC}$, or equivalently, iff the $\mathbb{S}$ in (7.11) can be chosen so that $\mathbb{S}\in\mathcal{G}\mathrm{TIC}_\infty\cap\mathrm{TIC}$. Those factorizations (7.11), in which $\mathbb{S}\notin\mathcal{G}\mathrm{TIC}_\infty$, can be thought as defining non-well-posed (improper) DF-controllers; see Theorem 7.2.14 for a generalization containing also such controllers.

One faces the same problem in the finite-dimensional theory (i.e., the theory for rational transfer functions with $\dim U,\dim Y<\infty$): unless $\widehat{\mathbb{S}}+\widehat{\mathbb{N}}\widehat{\mathbb{U}}\in\mathcal{G}\mathrm{H}^\infty_\infty$, the controller $\mathbb{Q}$ is ill-posed (i.e., not proper, that is, unbounded in any right half-plane). If $\det(\widehat{\mathbb{S}}+\widehat{\mathbb{N}}\widehat{\mathbb{U}})\equiv 0$, then $\widehat{\mathbb{Q}}$ is not defined anywhere. However, regardless of $\det(\widehat{\mathbb{S}}+\widehat{\mathbb{N}}\widehat{\mathbb{U}})$, the combined closed-loop condition (in Figure 7.1) is well-posed. This kind of non-well-posed controllers ("controllers with internal loop") are treated in Section 7.2.

Note that all factorizations of $\mathbb{Q}$ (and $\mathbb{D}$) in the theorem are obviously coprime.

We recall from Lemma A.1.1(c3) that $\left[\begin{smallmatrix}\widetilde{\mathbb{X}}&-\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}}&\widetilde{\mathbb{M}}\end{smallmatrix}\right]\in\mathcal{G}\mathrm{TIC}$ iff $\left[\begin{smallmatrix}\widetilde{\mathbb{X}}&\widetilde{\mathbb{Y}}\\ \widetilde{\mathbb{N}}&\widetilde{\mathbb{M}}\end{smallmatrix}\right]\in\mathcal{G}\mathrm{TIC}$.

Proof: The d.c.f. (7.11) exists, by Lemma 6.5.8. Thus, any stabilizing controller of $\mathbb{D}$ has a d.c.f., by Proposition 7.1.6(d).

"(i)⇔(ii)": This follows from Lemma 7.1.5(b). Note that $\left[\begin{smallmatrix}\mathbb{M}&\mathbb{Y}\\ \mathbb{N}&\mathbb{X}\end{smallmatrix}\right]\in\mathcal{G}\mathrm{TIC}$ implies that $\mathbb{Y}$ and $\mathbb{X}$ are r.c.

"(ii)⇔(iii)": These are adjoints of each other.

"(ii)⇔(vii)⇔(iv)": By Lemma 6.5.9(b), all completions $\left[\begin{smallmatrix}\mathbb{Y}\\ \mathbb{X}\end{smallmatrix}\right]$ such as in (ii) are given by $\left[\begin{smallmatrix}\mathbb{T}\mathbb{V}+\mathbb{M}\widetilde{\mathbb{U}}\\ \mathbb{S}\mathbb{V}+\mathbb{N}\widetilde{\mathbb{U}}\end{smallmatrix}\right]$ with $\widetilde{\mathbb{U}}\in\mathrm{TIC}$ and $\mathbb{V}\in\mathcal{G}\mathrm{TIC}$. The stabilizing controllers are, by (ii), the ones corresponding to $\mathbb{S}\mathbb{V}+\mathbb{N}\widetilde{\mathbb{U}}\in\mathcal{G}\mathrm{TIC}_\infty$, so, by Lemma 6.4.5, we may take $\mathbb{V}=I$ (and $\mathbb{U}:=\widetilde{\mathbb{U}}\mathbb{V}^{-1}\in\mathrm{TIC}$ arbitrary) without altering $\mathbb{Q}$, and thus we obtain the equivalent parametrization (vii). Moreover, in this case

$$\begin{bmatrix}\mathbb{M}&\mathbb{T}+\mathbb{M}\mathbb{U}\\ \mathbb{N}&\mathbb{S}+\mathbb{N}\mathbb{U}\end{bmatrix}^{-1}=\begin{bmatrix}\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}}&-(\widetilde{\mathbb{T}}+\mathbb{U}\widetilde{\mathbb{M}})\\ -\widetilde{\mathbb{N}}&\widetilde{\mathbb{M}}\end{bmatrix}, \tag{7.14}$$

by Lemma 6.5.9(c).

Claim (v) is a reformulation of (vii); claims (viii) and (vi) are the duals of (vii) and (v), respectively.

To prove the final claim about Youla parametrization, we note that, by (7.14) and Lemma A.1.1(c1), $\mathbb{S}+\mathbb{N}\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}} \in \mathcal{G}\mathrm{TIC}_\infty$. Moreover, (7.14) implies (7.13) if $\mathbb{S}+\mathbb{N}\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty$. □

Directly from the theorem we get:

Corollary 7.1.8 (Youla-parametrization) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ have a r.c.f. and a l.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$. Then $\mathbb{D}$ has the d.c.f.*

$$\begin{bmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = I = \begin{bmatrix} \widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{bmatrix}. \tag{7.15}$$

for some $\mathbb{T},\mathbb{S},\widetilde{\mathbb{T}},\widetilde{\mathbb{S}} \in \mathrm{TIC}$.

Moreover, the following are equivalent:

(i) $\mathbb{D}$ is DF-stabilizable.

(ii) $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ *for some* $\mathbb{X},\mathbb{Y}$ *s.t.* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$.

(iii) $\left[\begin{smallmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ *for some* $\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}}$ *s.t.* $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$.

(vi) [Youla] $\mathbb{S}+\mathbb{N}\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty$ *for some* $\mathbb{U} \in \mathrm{TIC}$.

(vii) [Youla] $\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}} \in \mathcal{G}\mathrm{TIC}_\infty$ *for some* $\mathbb{U} \in \mathrm{TIC}$.

Finally, if these conditions are satisfied, then all DF-stabilizing controllers of $\mathbb{D}$ are parametrized by

$$\mathbb{Q} = (\mathbb{T}+\mathbb{M}\mathbb{U})(\mathbb{S}+\mathbb{N}\mathbb{U})^{-1} = (\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}})^{-1}(\widetilde{\mathbb{T}}+\mathbb{U}\widetilde{\mathbb{M}}). \tag{7.16}$$

where $\mathbb{U}$ ranges over those $\mathbb{U} \in \mathrm{TIC}$ *for which* $\mathbb{S}+\mathbb{N}\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty$ *(equivalently,* $\widetilde{\mathbb{S}}+\mathbb{U}\widetilde{\mathbb{N}} \in \mathcal{G}\mathrm{TIC}_\infty$*).*

An alternative parametrization is $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ s.t. $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ *and* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$*; a third one is $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ s.t.* $\left[\begin{smallmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ *and* $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$. □

Given certain regularity, we can make the controller corresponding to a r.c.f. well-posed:

Corollary 7.1.9 *Let $\mathbb{D}$ have a r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$ s.t.* $\widetilde{\mathbb{X}},\mathbb{M} \in \mathrm{ULR}$. *Then $\mathbb{D}$ is DF-stabilizable.*

Proof: Define

$$\widetilde{\mathbb{S}} := M^{-1} + \widetilde{\mathbb{X}} - M^{-1}\mathbb{M}\widetilde{\mathbb{X}}, \quad \widetilde{\mathbb{T}} := \widetilde{\mathbb{Y}} - M^{-1}\mathbb{M}\widetilde{\mathbb{Y}} \tag{7.17}$$

to obtain that $\widetilde{\mathbb{S}}\mathbb{M} - \widetilde{\mathbb{T}}\mathbb{Y} = I$, $\widetilde{\mathbb{S}} \in \mathrm{ULR}$ and $\widetilde{S} = M^{-1} + \widetilde{X} - \widetilde{X} = M^{-1} \in \mathcal{G}\mathcal{B}$ (by Proposition 6.3.1(c)), hence $\widetilde{\mathbb{S}} \in \mathcal{G}\mathrm{TIC}_\infty$. Thus, $\widetilde{\mathbb{S}}^{-1}\widetilde{\mathbb{T}}$ DF-stabilizes $\mathbb{D}$, by Proposition 7.1.6(b1). □

Naturally, Youla parametrization can be applied also when one one wishes to work in a subclass of TIC:

Proposition 7.1.10 *Assume that* $\mathcal{B} \underset{a}{\subset} \mathcal{A} \underset{a}{\subset} \mathrm{ULR} \cap \mathrm{TIC}$ *(e.g.,* $\mathcal{A} = \mathrm{MTIC}$ *or* $\mathcal{A} = \mathrm{ULR} \cap \mathrm{TIC}$*, see Definition 6.2.4). Assume that* $\mathbb{D}$ *has a d.c.f. over* $\mathcal{A}$*, i.e., (7.11) holds with* $\mathbb{M}, \mathbb{N}, \mathbb{S}, \mathbb{T}, \widetilde{\mathbb{M}}, \widetilde{\mathbb{N}}, \widetilde{\mathbb{S}}, \widetilde{\mathbb{T}} \in \mathcal{A}$.

Then all stabilizing DF-controllers of $\mathbb{D}$ *are parametrized in Theorem 7.1.7, and the ones that have a d.c.f. over* $\mathcal{A}$ *are exactly those whose parameter(s) are in* $\mathcal{A}$*, i.e., which satisfy any (hence all) of the following equivalent conditions:*

(ii) $\mathbb{Q}$ *has a r.c.f.* $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ *s.t.* $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$*;*

(iii) $\mathbb{Q}$ *has a l.c.f.* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *s.t.* $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$*;*

(v) $\mathbb{U} \in \mathcal{A}$ *in (v), (vi), (vii) or (viii) of Theorem 7.1.7.*

If $\mathcal{A} \underset{a}{\subset} \mathrm{ULR}$*, then* $\mathbb{D}$ *has stabilizing DF-controllers.*

Note that $\mathcal{A} = \mathrm{MTIC}$ and $\mathcal{A} = \mathrm{ULR} \cap \mathrm{TIC}$ satisfy all above assumptions (cf. Definition 6.2.4). See also (the Corona) Theorem 4.1.6(d) for such d.c.f.'s.

Proof: Theorem 7.1.7 parametrizes all DF-stabilizing controllers of $\mathbb{D}$, in particular, by (vi'), (vi") and (7.13) of Theorem 7.1.7, they satisfy

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} := \begin{bmatrix} \mathbb{M} & \mathbb{T} \\ \mathbb{N} & \mathbb{S} \end{bmatrix} \begin{bmatrix} I & \mathbb{U} \\ 0 & I \end{bmatrix} \text{ and } \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} := \begin{bmatrix} I & -\mathbb{U} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \tag{7.18}$$

for some $\mathbb{U} \in \mathrm{TIC}$. If $\mathbb{U} \in \mathcal{A}$, then clearly $\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$. Conversely, if $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$, then $\left[\begin{smallmatrix} I & \mathbb{U} \\ 0 & I \end{smallmatrix}\right] = \left[\begin{smallmatrix} \widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{A}$ (analogously, $\mathbb{U} \in \mathcal{A}$ iff $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$).

If $\mathcal{A} \underset{a}{\subset} \mathrm{ULR}$, then the existence of a stabilizing controller is guaranteed, by Corollary 7.1.9. (Alternatively, we can take $\mathbb{U} := -M^{-1}T$, because then $(\widetilde{\mathbb{S}} + \mathbb{U}\widetilde{\mathbb{N}})(+\infty) = M^{-1}(M\widetilde{S} - T\widetilde{N}) = M^{-1} \in \mathcal{GB}(U)$, by $(7.11)(+\infty)_{1,1}$ and Proposition 6.3.1(c).) □

Notes

The connection between coprime factorization and dynamic stabilization (in Theorem 7.1.7 and Corollary 7.1.8) is well-known; see, e.g., pp. 36–42 of [Francis] or Chapter 12 of [ZDG] for classical presentations and [CWW96] and [CWW01] for results for WPLSs. O. Staffans has recently included some further results in [Sbook].

The class of matrix-valued "$\mathrm{H}^\infty/\mathrm{H}^\infty$ transfer functions" is not contained in, nor does it contain the class matrix-valued well-posed transfer functions. (Remark 7.2.20 sketches an infinite-dimensional theory that covers both classes.)

The dynamic I/O-stabilization theory based on fractional representations was first introduced in [DLMS] for rational functions. Also more general cases have been studied extensively; see, e.g., [GS] for the general case of matrix-valued "$\mathrm{H}^\infty/\mathrm{H}^\infty$ transfer functions", [CZ] for the special case of a Callier–Desoer class (from [CD78]), and [Logemann93] for certain other special cases (with applications to PS-systems). An excellent classical reference is [Vid], which covers all these classes to some extent. See the notes to Chapters 7 and 9 of [CZ] for further historical notes (these also cover the results postponed to Section 7.2).

Above we have presented here only the core results of the theory and those results that require the controller to be well-posed. In the rest of this chapter we shall present further results on DF-stabilization under more general assumptions.

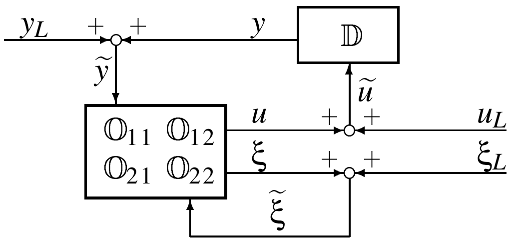


Figure 7.3: DF-controller $\mathbb{O}$ with internal loop for $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$

7.2 DF-stabilization with internal loop ($\begin{bmatrix} 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{D} & 0 & 0 \\ 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix}$)

It's not an optical illusion, it just looks like one.
— Phil White

The restriction $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ (or $\mathbb{S} + \mathbb{N}\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty$) in the Youla parametrization of Theorem 7.1.7 might feel somewhat artificial: it is only needed in order to have the open-loop map $\mathbb{Q} : \widetilde{y} \mapsto u$ of the controller well-posed (or proper, i.e., $\mathbb{Q} \in \mathrm{TIC}_\infty$), but even without that condition, all closed-loop maps are well-posed (once we connect the controller to the plant).

Therefore, in finite-dimensional theory, one sometimes allows for improper (or non-well-posed) controllers. To cover such controllers in addition to the proper ones, G. Weiss and R. Curtain introduced *DF-controllers with internal loop* in [WC].

This concept allows us to have mathematically more beautiful formulae and offers a solution to certain problems that cannot be solved by well-posed controllers (see the example at the beginning of [CWW01]). Nevertheless, in our most important results, we also point out when such a controller can be replaced by a well-posed controller.

Well-posed controllers, i.e., those of Section 7.1, are a subset of controllers with internal loop (and so are all $\mathrm{H}^\infty/\mathrm{H}^\infty$ fractional controllers, see Remark 7.2.8), hence many results concerning them were omitted in the previous section and are presented here under wider generality.

On the other hand, the proofs of most results for controllers with internal loop could be reduced to the well-posed case, by Lemma 7.2.6.

A map $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$, where Ξ is an arbitrary Hilbert space, becomes a DF-controller with internal loop when we connect its second output to its second input, as in Figure 7.3. This resulting controller need not be well-posed, i.e., closing the internal (ξ) loop only need not be an admissible operation (when $\mathbb{O}$ is uncoupled from $\mathbb{D}$); it is enough that the combined closed-loop system of Figure 7.3 becomes well-posed.

As above, a DF-controller with internal loop has an internal signal $\xi \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};\Xi)$, where Ξ is some Hilbert space. Note that whereas a given plant fixes the signal spaces U and Y of any of its controllers, the space Ξ may be different for different controllers.

In Figure 7.3, we obviously have

$$\begin{bmatrix} u \\ y \\ \xi \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{D} & 0 & 0 \\ 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \begin{bmatrix} u+u_L \\ y+y_L \\ \xi+\xi_L \end{bmatrix} =: \mathbb{D}^o \begin{bmatrix} u+u_L \\ y+y_L \\ \xi+\xi_L \end{bmatrix}. \tag{7.19}$$

As under (7.1), we observe that the corresponding closed-loop map is given by $\mathbb{D}^o_I = (I-\mathbb{D}^o)^{-1}\mathbb{D}^o$, and that the corresponding closed-loop system is given by Σ^o_I given below. Therefore, we make the following definitions:

Definition 7.2.1 (DF-stabilization with internal loop) *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. *A map* $\mathbb{O} = \left[\begin{smallmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ *(where also* Ξ *is a Hilbert space) is an* admissible [stabilizing] (DF-)controller with internal loop for $\mathbb{D}$ *if the output feedback operator* $L = I$ *is admissible [stabilizing] for*

$$\mathbb{D}^o := \begin{bmatrix} 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{D} & 0 & 0 \\ 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(U \times Y \times \Xi). \tag{7.20}$$

We call $\widetilde{\Sigma} = \left[\begin{smallmatrix} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \widetilde{\mathbb{C}} & \mathbb{O} \end{smallmatrix}\right] \in \mathrm{WPLS}(Y \times \Xi, \widetilde{H}, U \times \Xi)$ *an* admissible [stabilizing] (DF-)controller with internal loop for $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *if* $L = I$ *is admissible [stabilizing] for the (permutated) parallel connection*

$$\Sigma^o := \left[\begin{array}{cc|ccc} \mathbb{A} & 0 & \mathbb{B} & 0 & 0 \\ 0 & \widetilde{\mathbb{A}} & 0 & \widetilde{\mathbb{B}}_1 & \widetilde{\mathbb{B}}_2 \\ \hline 0 & \widetilde{\mathbb{C}}_1 & 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{C} & 0 & \mathbb{D} & 0 & 0 \\ 0 & \widetilde{\mathbb{C}}_2 & 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{array}\right] \in \mathrm{WPLS}(U \times Y \times \Xi, H \times \widetilde{H}, U \times Y \times \Xi). \tag{7.21}$$

We use prefices as in Definition 6.6.4 with Σ^o_I *in place of* Σ_L.

We call $\mathbb{D}$ *(resp.* Σ*)* DF-stabilizable with internal loop *if there is a stabilizing controller with internal loop for* $\mathbb{D}$ *(resp. for* Σ*), and we use prefices as above.*

We call two admissible DF-controllers for $\mathbb{D}$ *(resp. for* Σ*) with internal loop* equivalent for $\mathbb{D}$ *(resp. for* Σ*) if the corresponding* $(1\text{–}2, 1\text{–}2)$*-blocks of* $\mathbb{D}^o_I := (I - \mathbb{D}^o)^{-1} - I$ *are equal, i.e., if they determine same maps from* u_L, y_L *to* u, y.

If $\mathbb{O} = \left[\begin{smallmatrix} \mathbb{O}_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$*, then we may remove the words "with internal loop" everywhere in this definition and identify* $\mathbb{O}$ *with* $\mathbb{O}_{11} \in \mathrm{TIC}_\infty(Y,U)$ *(cf. Lemma 7.2.7).*

Naturally, "*[DF-]stabilizes*" means "is stabilizing for", in any of the above settings.

Note that $\mathbb{D}^o_I$ maps $(u_L, y_L, \xi_L) \mapsto (u, y, \xi)$. See also Figures 7.3 and 7.4 and the comments below Definition 7.1.1 and Summary 6.7.1.

Lemma 7.2.2 (DF-Admissibility and equivalence) *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. *A map* $\mathbb{O} = \left[\begin{smallmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ *is an admissible [stabilizing] controller with internal loop for* $\mathbb{D}$ *iff the connection if Figure 7.3 is well-posed [and stable, i.e.,*

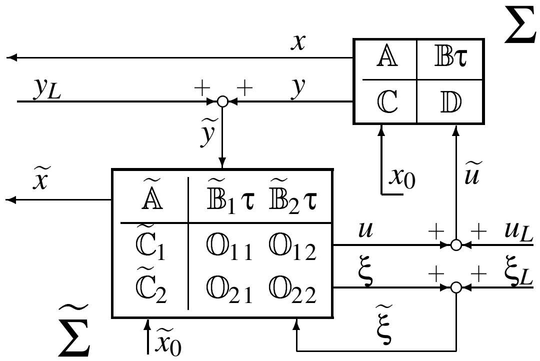


Figure 7.4: DF-controller $\widetilde{\Sigma}$ with internal loop for $\Sigma \in \mathrm{WPLS}(U,H,Y)$

$u,y,\xi \in \mathrm{L}^2$ for all $u_L,y_L,\xi_L \in \mathrm{L}^2$], equivalently, iff $I-\mathbb{D}^o \in \mathcal{G}\mathrm{TIC}_\infty(U\times Y\times \Xi)$ [and $(I-\mathbb{D}^o)^{-1} \in \mathrm{TIC}$].

Moreover, $\widetilde{\Sigma} = \left[\begin{smallmatrix} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \widetilde{\mathbb{C}} & \mathbb{O}\end{smallmatrix}\right]$ is admissible with internal loop for $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ iff $\mathbb{O}$ is admissible with internal loop for $\mathbb{D}$.

Finally, if two admissible controllers for Σ are equivalent for Σ (i.e., their I/O maps are equivalent for $\mathbb{D}$), then the maps from x_0,u_L,y_L to x,u,y are equal for the two closed-loop systems.

Analogously, $\widetilde{\Sigma}$ is admissible [stabilizing] for Σ iff the closed-loop system in Figure 7.4 is well-posed [and stable, i.e., $u,y,\xi \in \mathrm{L}^2$ and x and $\widetilde{x}$ are bounded for all $u_L,y_L,\xi_L \in \mathrm{L}^2(\mathbf{R}_+;*)$, $x_0 \in H$ and $\widetilde{x}_0 \in \widetilde{H}$]. (We note that exponential stability is equivalent to $x,\widetilde{x} \in \mathrm{L}^2$ (and hence $u,y,\xi \in \mathrm{L}^2$) for all $u_L,y_L,\xi_L \in \mathrm{L}^2$, $x_0 \in H$ and $\widetilde{x} \in \widetilde{H}$, by Lemma A.4.5 and Lemma 6.1.10(a1).)

We observe that only the maps concerning $\widetilde{x}$, $\widetilde{x}_0$, ξ and ξ_L may differ for equivalent controllers for Σ; thus, there is no difference from the part of $\widetilde{\Sigma}$ visible for Σ.

Proof: The claim on $I-\mathbb{D}^o$ and the "moreover claim" follow from Definitions 7.2.1 and 6.6.1. The reference to Figure 7.3 is obvious (cf. (6.122)–(6.124) and (6.127)).

The final claim (which could be observed from Figure 7.4) follows by computing Σ_I^o from (6.125) and observing that its first, third and fourth rows and columns depend only on Σ and $\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0\end{smallmatrix}\right]\mathbb{D}_I^o\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0\end{smallmatrix}\right]^{\mathrm{T}}$ (use the fact that $\mathbb{D}_I^o := \mathbb{D}^o(I-\mathbb{D}^o)^{-1} = (I-\mathbb{D}^o)^{-1} - I$). □

The identification of $\left[\begin{smallmatrix} \mathbb{O}_{11} & 0 \\ 0 & 0\end{smallmatrix}\right]$ and $\mathbb{O}_{11}$ is natural: all open-loop and closed-loop signals in Figures 7.3 and 7.1 become equal (except that in Figure 7.3, we have the additional, negligible signals $\xi = 0$ and $\widetilde{\xi} = \xi_L$). Thus, a (well-posed) stabilizing controller (in the sense of Definition 7.1.1) is a special case of a stabilizing controller with internal loop (see also Lemma 7.2.7). The situation with systems is the same (cf. Figures 7.4 and 7.2).

We stress that we mention the words "internal loop" explicitly whenever we speak of such controllers; all other maps are assumed to be well-posed, i.e., $\in \mathrm{TIC}_\infty$ (which is also stated explicitly in theorems and definitions), so no confusion should arise. The same applies to maps with coprime internal loop

(Definition 7.2.11) and also elsewhere is this chapter.

In connection with the H^∞ Four-Block Problem, however, the theory for controllers with internal loop becomes more natural and beautiful than the part restricted to well-posed controllers. Therefore, in Chapter 12, contrary to the practice of this chapter, a "[stabilizing] controller" is allowed to possess an internal loop, and "well-posed" is always written explicitly, never tacitly.

Trivially, $\widetilde{\Sigma}$ I/O-DF-stabilizes Σ iff $\mathbb{O}$ DF-stabilizes $\mathbb{D}$ (i.e., iff $\mathbb{D}_I^o$ becomes stable). Under standard assumptions, this is also equivalent to the stronger condition that $\widetilde{\Sigma}$ DF-stabilizes Σ (i.e., that the whole Σ_I^o becomes stable):

Theorem 7.2.3 ($\widetilde{\Sigma}$ stabilizes $\Sigma \Leftrightarrow \mathbb{O}$ stabilizes $\mathbb{D}$) *Let* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{O} \end{array}\right] \in \mathrm{WPLS}(Y \times \Xi, \widetilde{H}, U \times \Xi)$.

(a) Suppose that Σ and $\widetilde{\Sigma}$ are SOS-stabilizable. Then $\widetilde{\Sigma}$ SOS-stabilizes Σ with internal loop iff $\widetilde{\Sigma}$ I/O-stabilizes Σ with internal loop.

(b) **([Strong] stability)** *Suppose that any of the following conditions holds:*

(1.) both Σ and $\widetilde{\Sigma}$ are [[exponentially] strongly] q.r.c.-stabilizable;

(2.) both Σ and $\widetilde{\Sigma}$ are [[exponentially] strongly] q.l.c.-detectable;

(3.) both Σ and $\widetilde{\Sigma}$ are SOS-stabilizable and [[exponentially] strongly] detectable;

(4.) both Σ and $\widetilde{\Sigma}$ are detectable and [exponentially] stabilizable.

Then $\widetilde{\Sigma}$ [[exponentially] strongly] stabilizes Σ with internal loop iff $\widetilde{\Sigma}$ I/O-stabilizes Σ with internal loop.

(c1) **(Exponential stability)** *The system $\widetilde{\Sigma}$ stabilizes Σ exponentially with internal loop iff $\widetilde{\Sigma}$ I/O-stabilizes Σ with internal loop and Σ and $\widetilde{\Sigma}$ are optimizable and estimatable.*

(c2) Suppose that any of the following conditions holds:

(1.) both Σ and $\widetilde{\Sigma}$ are optimizable and estimatable;

(2.) both Σ and $\widetilde{\Sigma}$ are optimizable and input-detectable;

(3.) both Σ and $\widetilde{\Sigma}$ are estimatable and output-stabilizable;

(4.) both Σ and $\widetilde{\Sigma}$ are optimizable and q.r.c.-stabilizable;

(5.) both Σ and $\widetilde{\Sigma}$ are estimatable and q.l.c.-detectable.

Then $\widetilde{\Sigma}$ stabilizes Σ exponentially with internal loop iff $\widetilde{\Sigma}$ I/O-stabilizes Σ with internal loop.

(d) **(Well-posed controllers)** *Suppose that, instead,* $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{Q} \end{array}\right] \in \mathrm{WPLS}(Y, \widetilde{H}, U)$. *Then (a)–(c2) hold if we delete the words "with internal loop" everywhere in this theorem.*

Thus, all maps between signals (i.e., $\Sigma_I^o : x_0, \widetilde{x}_0, u_L, y_L, \xi_L \mapsto x, \widetilde{x}, u, y, \xi$ (and $\mapsto \widetilde{u}, \widetilde{y}, \widetilde{\xi}$)) in Figure 7.4 are (SOS-/strongly/exponentially/...) stable iff the maps from u_L, y_L and ξ_L to u, y and ξ are stable and Σ and $\widetilde{\Sigma}$ have the corresponding stabilizability listed above.

Therefore, we can often concentrate on the I/O theory. E.g., if Σ is jointly stabilizable and detectable, and we find a stabilizing controller for $\mathbb{D}$, then any of its q.r.c.-stabilizable realizations (cf. Theorems 7.2.14 and 6.6.28) stabilizes Σ. Analogous claims hold under other assumptions for Σ.

Proof of Theorem 7.2.3: (a)&(b)&(c2) By Lemma 6.7.18 (and Lemma 6.7.17 with $G = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$), Σ^o inherits the stabilizability and detectability properties of Σ and $\widetilde{\Sigma}$. Therefore the above assumptions imply, by Proposition 6.7.14, that $L = I$ is (SOS-/strongly/exponentially, depending on the assumptions) stabilizing for Σ^o iff it is I/O-stabilizing.

(c1) This is Theorem 7.4 of [WR00] (alternatively, "if" follows from Theorem 6.7.10(d)(viii), and the converse from (6.126) for optimizability (note that Σ^o is obviously optimizable iff Σ and $\widetilde{\Sigma}$ are optimizable) and by duality for estimatability (see Lemma 6.7.2(e'))).

(d) The above proofs still apply (use $\mathbb{O} = \left[\begin{smallmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ etc.). □

We can now extend the standard result (cf. p. 303 of [ZDG] and Theorem 7.3.12) to infinite-dimensional systems (although the converse in (b) is incomplete and those in (c) and (d) do not cover all WPLSs):

Theorem 7.2.4 (Exp. DF-stabilizable ⇔ opt. & est.) *Let* $\Sigma := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$.

(a) If Σ is exponentially DF-stabilizable [with internal loop], then Σ is optimizable and estimatable.

(b) Conversely, if Σ is jointly [[exponentially] strongly] stabilizable and detectable, then Σ is [[exponentially] strongly] DF-stabilizable with internal loop.

(c) Assume that $\mathbb{A}Bu_0, \mathbb{A}^*C^*y_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+;H)$ for all $u_0 \in U$ and $y_0 \in Y$, and that $\mathbb{D}$ is ULR. Then the following are equivalent:

(i) Σ is exponentially DF-stabilizable;

(ii) Σ is exponentially DF-stabilizable with internal loop;

(iii) Σ is optimizable and estimatable;

(iv) Σ is exponentially jointly stabilizable and detectable;

(v) Σ is exponentially jointly stabilizable and detectable by some bounded K and H.

Moreover, if (v) holds, then (d) applies with the same K and H (hence (6.169) and (7.22) become ULR).

(d) If K and H are [[exponentially] strongly] jointly stabilizing with (6.169) being SR, and $I - \mathbb{G}_L \in \mathcal{G}\mathrm{TIC}_\infty(Y)$ (this holds if (6.169) is ULR), then

$$\left(\begin{array}{c|c} A + BK_\mathrm{s} + HC_\mathrm{s} + HDK_\mathrm{s} & -H \\ \hline K & 0 \end{array}\right) \in \mathrm{WPLS}(Y,H,U) \tag{7.22}$$

is a [[exponentially] strongly] DF-stabilizing controller for Σ. Moreover, (7.22) is SR and [[exponentially] strongly] jointly stabilizable and detectable.

Note that the assumptions of (c) hold if B and C are bounded (or if $\mathbb{A}$ is somewhat smoothing, e.g., if Hypothesis 9.5.1 holds), hence always in discrete time.

A weaker form of the exponential part of the theorem is well-known for Pritchard–Salamon systems; e.g., Theorem 2.30 of [Keu] is a special case of (d) (since PS-systems are ULR and stabilizability (and detectability) are defined in a very strong sense for PS-systems; see Remark 6.6.15). However, the result (c) seems to be new in this generality

If we drop the requirement "$I - \mathbb{G}_L \notin \mathcal{G}\mathrm{TIC}_\infty$" from (d), then "(7.22)" can still be formulated as a controller with internal loop; see Proposition 5.3 of [WC] for an exponential version of this claim (or modify our proof slightly).

Proof: (a) This follows from Theorem 7.2.3(c1).

(b) By the proof of Theorem 6.6.28, we have the d.c.f. (6.172), and Σ is [strongly] r.c.-stabilizable. Consequently,

$$\left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{O} \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A}_L & 0 & \mathbb{H}_L \\ \hline -\mathbb{K}_L & 0 & -\mathbb{E}_L \\ \mathbb{C}_L & I & \mathbb{G}_L \end{array}\right] \tag{7.23}$$

is an I/O-stabilizing controller with internal loop for Σ (i.e., $\mathbb{O}$ DF-stabilizes $\mathbb{D}$ with internal loop), by, e.g., Theorem 7.2.14(i) (or the dual of Lemma 7.2.10(a); we could use (7.13) instead if one would assume that $I - \mathbb{G}_L \in \mathcal{G}\mathrm{TIC}_\infty(Y)$, i.e., that $\mathbb{Q}$ were well-posed).

But (7.23) is [[exponentially] strongly] stable, hence [strongly] r.c.-stabilizable [[and optimizable and estimatable]], hence it DF-stabilizes Σ [[exponentially] strongly] with internal loop, by Theorem 7.2.3(b)[(c1)].

(c) By Corollary 9.2.13, (iii)–(v) are equivalent and the "moreover"-claim holds (with the ULR-property from Lemma 6.3.16(d)); in particular, (v) implies (i), by (d). Implication "(i)⇒(ii)" is trivial, and "(ii)⇒(iii)" was given in (a).

(d) (Note that we have adopted the notation of Definition 6.6.21. Naturally, the signs of K and H can be interchanged.)

If (6.169) is ULR, UR, SLR, UVR or SVR, then so are all systems and maps appearing below (including (7.22)), by Proposition 6.3.1(b2); for the same reason, all of them are always SR. (See also Lemma 6.6.27.)

1° *When (6.169) is ULR:* If (6.169) is ULR, then $I - \mathbb{G}_L$ is invertible, by Proposition 6.3.1(c), since the I/O map of (6.169) (and hence of (6.170) and (6.171)) corresponding to K and H is given by $\left[\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right]$; in particular, $G_L = 0 (= G)$.

2° *DF-stabilizing $\mathbb{Q}$:* By the proof of Theorem 6.6.28, we have the d.c.f. (6.172). By (6.172) and Lemma 6.5.9(a1), the invertibility of $I - \mathbb{G}_L$ implies that of $I - \mathbb{F}_{\widetilde{L}}$. By (7.13) (with $\mathbb{U} = 0$), the map

$$\mathbb{Q} := -\mathbb{E}_L (I - \mathbb{G}_L)^{-1} = -(I - \mathbb{F}_{\widetilde{L}})^{-1} \mathbb{E}_{\widetilde{L}} \in \mathrm{TIC}_\infty(Y, U) \tag{7.24}$$

is a DF-stabilizing controller for $\mathbb{D}$ and (7.24) is a [[exponential]] d.c.f.

3° *(7.22) is a SR WPLS with I/O map* $\mathbb{Q}$*:* By Definition 6.6.21, $L := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ is admissible for $\Sigma_{\rm Total}$. Assumption $I - \mathbb{G}_L \in \mathcal{G}{\rm TIC}_\infty$ says that $\widetilde{L} := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ is admissible for $(\Sigma_{\rm Total})_L$; from (6.125) we observe that the corresponding system $(\Sigma_{\rm Total})_{L+\widetilde{L}} = (\Sigma_{\rm Total})_I$ has $\begin{bmatrix} * & * \\ -\mathbb{Q} & * \end{bmatrix}$ as its I/O map.

Apply (6.142) (and Proposition 6.6.18(a1) and Proposition 6.3.1(a3)) twice to observe that the generators of $(\Sigma_{\rm Total})_I$ are given by

$$\left[\begin{array}{c|cc} A+BK_{\rm s}+HC_{\rm s}+HDK_{\rm s} & H & B+HD \\ \hline C_{\rm s}+DK_{\rm s} & 0 & D \\ K_{\rm s} & 0 & 0 \end{array}\right]. \tag{7.25}$$

We conclude that (7.22) is a SR WPLS with I/O map $\mathbb{Q}$.

4° *The rest:*

Let Σ' be the system generated by

$$\left[\begin{array}{c|cc} A+BK_{\rm s}+HC_{\rm s}+HDK_{\rm s} & -H & -(B+HD) \\ \hline C_{\rm s}+DK_{\rm s} & 0 & D \\ K_{\rm s} & 0 & 0 \end{array}\right]. \tag{7.26}$$

Then Σ' with its second and third columns multiplied by -1 equals $(\Sigma_{\rm Total})_I$. From (6.126 we observe that $(\Sigma')_L$ with its second and third columns multiplied by -1 equals $((\Sigma_{\rm Total})_I)_{-L}$ $(= (\Sigma_{\rm Total})_{\widetilde{L}}$, by Lemma 6.6.3), and that $(\Sigma')_{\widetilde{L}}$ with its second and third columns multiplied by -1 equals $((\Sigma_{\rm Total})_I)_{-\widetilde{L}} = (\Sigma_{\rm Total})_L$.

From this and (7.26) we observe that $C_{\rm s} + DK_{\rm s}$ and $-(B+HD)$ are [[exponentially] strongly] jointly stabilizing for (7.22) (with "$E = 0$"). In particular, (7.22) is [[exponentially] strongly] r.c.-stabilizable, by Theorem 6.6.28 (and Lemma 6.6.22). By this, 2° and Theorem 7.2.3(b)(1.), (7.22) [[exponentially] strongly] DF-stabilizes Σ. □

Formally, a controller $\mathbb{O}$ with internal loop maps $y \mapsto u = (\mathbb{O}_{11} + \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21})y$. (If $I - \mathbb{O}_{22} \in \mathcal{G}{\rm TIC}_\infty(\Xi)$, then this formula is not merely formal, by Lemma 7.2.7.)

Thus, also the controllers of form "right coprime $\mathrm{H}^\infty/\mathrm{H}^\infty$" (of form $\mathbb{Y}\mathbb{X}^{-1}$ with $\mathbb{X}, \mathbb{Y} \in$ TIC being r.c.) can be written as controllers with internal loop, by taking $\mathbb{O} = \begin{bmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X} \end{bmatrix}$. We will call such controllers maps with r.c. internal loop (they are the canonical controllers of [CWW01]); see Definition 7.2.11 for details. Here we require neither $\mathbb{X}$ nor $\widetilde{\mathbb{X}}$ to be invertible, it is enough that the system $\mathbb{D}_I^o$ produced by closing the two loops simultaneously is well-posed.

The surprising fact is that all stabilizing controllers are of this form (modulo equivalence), whenever $\mathbb{D}$ has a l.c.f. This fact is the main theorem of [CWW01], but we give here (part (b) below) a shorter proof instead of the original seven pages long one. We also give a necessary and sufficient condition ((a) or its dual (a')) in the general case:

Proposition 7.2.5 (I/O-DF-stabilizing controller with IL) *Let* $\mathbb{D} \in {\rm TIC}_\infty(U,Y)$ *and* $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in {\rm TIC}_\infty(Y \times \Xi, U \times \Xi)$.

(a) $\mathbb{O}$ *is admissible with internal loop for* $\mathbb{D}$ *iff* $\mathbb{H} := \begin{bmatrix} I-\mathbb{D}\mathbb{O}_{11} & -\mathbb{D}\mathbb{O}_{12} \\ -\mathbb{O}_{21} & I-\mathbb{O}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(Y \times \Xi)$. *Moreover,* $\mathbb{O}$ *is [exponentially] stabilizing with internal loop iff the corresponding closed-loop map*

$$(I-\mathbb{D}^o)^{-1} = \begin{bmatrix} I_U + \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{H}^{-1}\begin{bmatrix}\mathbb{D}\\0\end{bmatrix} & \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{H}^{-1} \\ \mathbb{H}^{-1}\begin{bmatrix}\mathbb{D}\\0\end{bmatrix} & \mathbb{H}^{-1} \end{bmatrix} : \begin{bmatrix}u_L\\y_L\\\xi_L\end{bmatrix} \mapsto \begin{bmatrix}u+u_L\\y+y_L\\\xi+\xi_L\end{bmatrix} \tag{7.27}$$

is [exponentially] stable.

(a') $\mathbb{O}$ *is admissible with internal loop for* $\mathbb{D}$ *iff* $\mathbb{R} := \begin{bmatrix} I-\mathbb{O}_{11}\mathbb{D} & -\mathbb{O}_{12} \\ -\mathbb{O}_{21}\mathbb{D} & I-\mathbb{O}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(U \times \Xi)$. *Moreover,* $\mathbb{O}$ *is [exponentially] stabilizing with internal loop iff the corresponding closed-loop map*

$$\begin{bmatrix} I_Y + \begin{bmatrix}\mathbb{D} & 0\end{bmatrix}\mathbb{R}^{-1}\begin{bmatrix}\mathbb{O}_{11}\\\mathbb{O}_{21}\end{bmatrix} & \begin{bmatrix}\mathbb{D} & 0\end{bmatrix}\mathbb{R}^{-1} \\ \mathbb{R}^{-1}\begin{bmatrix}\mathbb{O}_{11}\\\mathbb{O}_{21}\end{bmatrix} & \mathbb{R}^{-1} \end{bmatrix} : \begin{bmatrix}y_L\\u_L\\\xi_L\end{bmatrix} \mapsto \begin{bmatrix}y+y_L\\u+u_L\\\xi+\xi_L\end{bmatrix} \tag{7.28}$$

is [exponentially] stable.

(b) Let $\mathbb{D}$ *have a l.c.f.* $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$. *Then* $\mathbb{O}$ *is admissible with internal loop for* $\mathbb{D}$ *iff* $\mathbb{F} := \begin{bmatrix} \widetilde{\mathbb{M}}-\widetilde{\mathbb{N}}\mathbb{O}_{11} & -\widetilde{\mathbb{N}}\mathbb{O}_{12} \\ -\mathbb{O}_{21} & I-\mathbb{O}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(Y \times \Xi)$, *and* $\mathbb{O}$ *is stabilizing with internal loop iff* $\mathbb{F}^{-1}, \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{F}^{-1} \in \mathrm{TIC}$.

Moreover, if $\mathbb{O}$ *is stabilizing with internal loop for* $\mathbb{D}$ *and we set*

$$\begin{bmatrix}\mathbb{Y}\\\mathbb{X}\end{bmatrix} := \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\\ I_Y & 0\end{bmatrix}\mathbb{F}^{-1}\begin{bmatrix}I_Y\\0\end{bmatrix}, \tag{7.29}$$

then $\mathbb{O}' := \begin{bmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X}\end{bmatrix} \in \mathrm{TIC}(Y\times Y, U\times Y)$, $\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} = I_Y$, $\mathbb{O}'$ *is an equivalent (to* $\mathbb{O}$*) stabilizing controller with internal loop for* $\mathbb{D}$, *and* $\mathbb{O} = \mathbb{Y}_o\mathbb{X}_o^{-1}$ *is a r.c.f., where* $\mathbb{X}_o := \mathbb{F}^{-1}$ *and* $\mathbb{Y}_o := \mathbb{O}\mathbb{F}^{-1}$ *(in particular,* $\mathbb{O}\mathbb{F}^{-1} \in \mathrm{TIC}$*).*

Of course, the corresponding dual result holds for $\mathbb{D}$ *having a r.c.f.*

(c) $\mathbb{O}$ *is admissible (resp. [exponentially] stabilizing) with internal loop for* $\mathbb{D}$ *iff* $\mathbb{O}$ *is admissible (resp. [exponentially] stabilizing) for* $\begin{bmatrix}\mathbb{D} & 0\\ 0 & I_\Xi\end{bmatrix}$.

(d) $\mathbb{O}$ *is admissible (resp. [exponentially] stabilizing) with internal loop for* $\mathbb{D}$ *iff* $\mathbb{O}^{\mathrm{d}}$ *is admissible (resp. [exponentially] stabilizing) for* $\mathbb{D}^{\mathrm{d}}$ *with internal loop.*

(e) $\mathbb{O}$ *and* $\widetilde{\mathbb{O}} \in \mathrm{TIC}_\infty(Y\times\Xi', U\times\Xi')$ *are equivalent for* $\mathbb{D}$ *iff* $(\mathbb{H}^{-1})_{11} = (\widetilde{\mathbb{H}}^{-1})_{11}$ *and* $(\begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{H}^{-1})_1 = (\begin{bmatrix}\widetilde{\mathbb{O}}_{11} & \widetilde{\mathbb{O}}_{12}\end{bmatrix}\widetilde{\mathbb{H}}^{-1})_1$, *where* $\widetilde{\mathbb{H}}$ *corresponds to* $\widetilde{\mathbb{O}}$ *as in (a).*

If $\mathbb{O}$ is merely admissible in (b), then one observes from the proof that the conclusions of (b) still hold except that $\mathbb{O}' \in \mathrm{TIC}_\infty$ (instead of TIC_0) and that $\mathbb{O}'$ need not be stabilizing (but it is admissible, because it is equivalent to $\mathbb{O}$).

Proof: (a) This follows by applying Lemma A.1.1(d1) to $A := I - \mathbb{D}^o$ (so that $\mathbb{H} = A_{22} - A_{21}A_{11}^{-1}A_{12}$).

(N.B. two admissible controllers with internal loop are equivalent for $\mathbb{D}$ iff they produce same maps $(\mathbb{H}^{-1})_{11}$ and $(\begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{H}^{-1})_1$, since then the $(1\text{–}2, 1\text{–}2)$-blocks of (7.27) are the same.)

(a') Now we set $A := T(I - \mathbb{D}^o)T$, where $T := \begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$, and apply Lemma A.1.1(d1) as in (a) (note that $\mathbb{R} = A_{22} - A_{21}A_{11}^{-1}A_{12}$).

(b) 1° Clearly $\mathbb{H} = \begin{bmatrix} \widetilde{\mathbb{M}} & 0 \\ 0 & I \end{bmatrix}^{-1} \mathbb{F}$, so $\mathbb{H} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty$. The corresponding closed-loop map is

$$(I - \mathbb{D}^o)^{-1} = \begin{bmatrix} I + \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{F}^{-1}\begin{bmatrix}\widetilde{\mathbb{N}} \\ 0\end{bmatrix} & \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{F}^{-1}\begin{bmatrix}\widetilde{\mathbb{M}} & 0 \\ 0 & I\end{bmatrix} \\ \mathbb{F}^{-1}\begin{bmatrix}\widetilde{\mathbb{N}} \\ 0\end{bmatrix} & \mathbb{F}^{-1}\begin{bmatrix}\widetilde{\mathbb{M}} & 0 \\ 0 & I\end{bmatrix} \end{bmatrix}, \tag{7.30}$$

so $\mathbb{F}^{-1}, \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{F}^{-1} \in \mathrm{TIC}$ is clearly sufficient for $(I - \mathbb{D}^o)^{-1} \in \mathrm{TIC}$.

2° For the converse, note that (here $\widetilde{\mathbb{M}}\mathbb{S} - \widetilde{\mathbb{N}}\mathbb{T} = I$, $\mathbb{S}, \mathbb{T} \in \mathrm{TIC}$)

$$\begin{bmatrix}\widetilde{\mathbb{M}} & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix}\mathbb{S} & 0 \\ 0 & I\end{bmatrix} - \begin{bmatrix}\widetilde{\mathbb{N}} \\ 0\end{bmatrix}\begin{bmatrix}\mathbb{T} & 0\end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I\end{bmatrix}, \tag{7.31}$$

so the stability of (7.30) implies that of $\mathbb{F}^{-1}$ and $\begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{F}^{-1}$. Therefore, also $\mathbb{O}'$ is stable in this case.

3° For the rest of the proof, we will assume that $\mathbb{O}$ stabilizes $\mathbb{D}$. Now the $(2,2)$-block of $I = (I - \mathbb{D}^o)(I - \mathbb{D}^o)^{-1}$ gives $I = -\mathbb{D}\mathbb{Y}\widetilde{\mathbb{M}} + I\mathbb{X}\widetilde{\mathbb{M}} + 0$, i.e., $I = \widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y}$. Using Lemma A.1.1(d1), one obtains that $\mathbb{O}'$ is admissible and (we set $\Delta := \mathbb{X} - \mathbb{D}\mathbb{Y} = \widetilde{\mathbb{M}}^{-1}$)

$$(I - \mathbb{D}^o_{\mathbb{O}'})^{-1} = \begin{bmatrix} I + \mathbb{Y}\Delta^{-1}\mathbb{D} & \mathbb{Y}\Delta^{-1} & \mathbb{Y}\Delta^{-1} \\ (I + \mathbb{D}\mathbb{Y}\Delta^{-1})\mathbb{D} & I + \mathbb{D}\mathbb{Y}\Delta^{-1} & \mathbb{D}\mathbb{Y}\Delta^{-1} \\ \Delta^{-1}\mathbb{D} & \Delta^{-1} & \Delta^{-1} \end{bmatrix}, \tag{7.32}$$

where $\mathbb{D}^o_{\mathbb{O}'}$ is as $\mathbb{D}^o$, except that $\mathbb{O}$ is replaced by $\mathbb{O}'$. This shows that $\mathbb{O}'$ is also stabilizing.

4° $\mathbb{O}'$ is equivalent to $\mathbb{O}$, because the $(1\text{–}2, 1\text{–}2)$-block of (7.32) equals that of (7.30):

$$\begin{aligned} &I + \mathbb{Y}\Delta^{-1}\mathbb{D} = I + \mathbb{Y}\widetilde{\mathbb{N}}, && \mathbb{Y}\Delta^{-1} = \mathbb{Y}\widetilde{\mathbb{M}}; \\ &(I + \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}\mathbb{Y}\widetilde{\mathbb{M}})\mathbb{D} = \widetilde{\mathbb{M}}^{-1}(I + \widetilde{\mathbb{N}}\mathbb{Y})\widetilde{\mathbb{N}} = \mathbb{X}\widetilde{\mathbb{N}}, && I + \mathbb{D}\mathbb{Y}\mathbb{M} = \mathbb{X}\mathbb{M}. \end{aligned} \tag{7.33}$$

5° Set $\widetilde{\mathbb{M}}_o := \begin{bmatrix}\widetilde{\mathbb{M}} & 0 \\ 0 & I\end{bmatrix}$, $\widetilde{\mathbb{N}}_o := \begin{bmatrix}\widetilde{\mathbb{N}} & 0 \\ 0 & I\end{bmatrix}$ to obtain $\mathbb{F} = \widetilde{\mathbb{M}}_o - \widetilde{\mathbb{N}}_o\mathbb{O}$, so that

$$\widetilde{\mathbb{M}}_o\mathbb{F}^{-1} - \widetilde{\mathbb{N}}_o\mathbb{O}\mathbb{F}^{-1} = I, \tag{7.34}$$

i.e., $\mathbb{X}_o$ and $\mathbb{Y}_o$ are r.c. (because (7.34) implies that the lower row of $\mathbb{Y}_o$ is stable, and the upper row was proved stable in 2°).

The dual result can be proved analogously from (a') (alternatively, use (d)).

(c) 0° First proof: One way to prove the rest is to interchange the second and fourth columns and the second and third rows of "$I-\mathbb{D}^o$" corresponding to $\left[\begin{smallmatrix}\mathbb{D} & 0\\ 0 & I\end{smallmatrix}\right]$, and then apply (A.11) (with the rows and columns of A and A^{-1} interchanged) to the resulting matrix

$$\begin{bmatrix}\begin{bmatrix} I & -\mathbb{O}_{12} & -\mathbb{O}_{11}\\ -\mathbb{D} & 0 & I\\ 0 & -\mathbb{O}_{22} & -\mathbb{O}_{21}\end{bmatrix} & \begin{bmatrix}0\\0\\I\end{bmatrix}\\ \begin{bmatrix}0 & I & 0\end{bmatrix} & -I\end{bmatrix} \tag{7.35}$$

to obtain that the invertibility of "$I-\mathbb{D}^o$" (i.e., the admissibility of $\mathbb{O}$ for $\left[\begin{smallmatrix}\mathbb{D} & 0\\ 0 & I\end{smallmatrix}\right]$) is equivalent to the invertibility of $I-\mathbb{D}^o$, and that both inverses are stable iff either is (since "$(I-\mathbb{D}^o)$" consists of $I-\mathbb{D}^o$ and some copies of its elements).

However, for Lemma 7.2.6 we need the alternative proof given in 1°–3°:

1° The admissibility claim follows from (a), because $\mathbb{H}=I-\underline{\mathbb{D}}\mathbb{O}$, where $\underline{\mathbb{D}}:=\left[\begin{smallmatrix}\mathbb{D} & 0\\ 0 & I\end{smallmatrix}\right]$.

2° Assume that (7.27) is [exponentially] stable, so that also (7.28) is [exponentially] stable. Then, by Lemma A.1.1(f6),

$$\begin{bmatrix}\mathbb{D} & 0\\ 0 & I\end{bmatrix}(I-\mathbb{O}\underline{\mathbb{D}})^{-1}\mathbb{O}=\mathbb{H}^{-1}-I, \tag{7.36}$$

which is [exponentially] stable, hence $\begin{bmatrix}0 & I\end{bmatrix}(I-\mathbb{O}\underline{\mathbb{D}})^{-1}\mathbb{O}=\begin{bmatrix}0 & I\end{bmatrix}\mathbb{O}(I-\underline{\mathbb{D}}\mathbb{O})^{-1}=\begin{bmatrix}\mathbb{O}_{21} & \mathbb{O}_{22}\end{bmatrix}\mathbb{H}^{-1}$ is [exponentially] stable. Combine this with the right top corner of (7.27) to observe that $\mathbb{O}\mathbb{H}^{-1}$ is [exponentially] stable. By (7.27), so is also $\mathbb{H}^{-1}\underline{\mathbb{D}}$ (because so are $\mathbb{H}^{-1}\left[\begin{smallmatrix}\mathbb{D}\\ 0\end{smallmatrix}\right]$ and $\mathbb{H}^{-1}$).

Since $\mathbb{R}=I-\mathbb{O}\underline{\mathbb{D}}$, we analogously observe from (7.28) that $(I-\mathbb{O}\underline{\mathbb{D}})^{-1}$ is [exponentially] stable, hence so is (7.5) (with substitutions $\mathbb{Q}\mapsto\mathbb{O}$, $\mathbb{D}\mapsto\underline{\mathbb{D}}$), equivalently, the map

$$\begin{bmatrix} I & -\mathbb{O}\\ -\underline{\mathbb{D}} & I\end{bmatrix}^{-1}=\begin{bmatrix}\mathbb{R}^{-1} & \mathbb{O}\mathbb{H}^{-1}\\ \mathbb{H}^{-1}\underline{\mathbb{D}} & \mathbb{H}^{-1}\end{bmatrix}=\begin{bmatrix}\mathbb{T}_{11} & \mathbb{T}_{13} & \mathbb{T}_{12} & \mathbb{T}_{13}\\ \mathbb{T}_{31} & \mathbb{T}_{33} & \mathbb{T}_{32} & \mathbb{T}_{33}-I\\ \mathbb{T}_{21} & \mathbb{T}_{23} & \mathbb{T}_{22} & \mathbb{T}_{23}\\ \mathbb{T}_{31} & \mathbb{T}_{33} & \mathbb{T}_{32} & \mathbb{T}_{33}\end{bmatrix} \tag{7.37}$$

(here $\mathbb{T}:=(I-\mathbb{D}^o)^{-1}$; we have used (7.27) and (7.28) above).

3° Conversely, if (7.37) is [exponentially] stable, then so is $\mathbb{T}=(I-\mathbb{D}^o)^{-1}$, hence so is (7.27).

(d) This follows from (a) and (a'). (Note that (d) is contained in Lemma 6.7.2(e'), but this second proof will be useful later.)

(e) We observe from (7.27) that $\left[\begin{smallmatrix}I & 0 & 0\\ 0 & I & 0\end{smallmatrix}\right]\mathbb{D}^o_I\left[\begin{smallmatrix}I & 0 & 0\\ 0 & I & 0\end{smallmatrix}\right]^{\mathrm{T}}$ (equivalently, the first and second rows and columns of (7.27)$=\mathbb{D}^o_I+I$, because $\mathbb{D}^o_I=(I-\mathbb{D}^o)^{-1}-I$) depends on $(\mathbb{H}^{-1})_{11}$ and $(\begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\mathbb{H}^{-1})_1$. □

Part (a) also shows that $\left[\begin{smallmatrix}\mathbb{D}\\ 0\end{smallmatrix}\right]=\widetilde{\mathbb{S}}^{-1}\widetilde{\mathbb{T}}$ (i.e., "H$^\infty$/H$^\infty$") with $\widetilde{\mathbb{S}},\widetilde{\mathbb{T}}\in$ TIC. If U and Y are finite-dimensional, we can write also $\mathbb{D}$ in "H$^\infty$/H$^\infty$" form (i.e., as the inverse of a stable, (TIC$_\infty$-)invertible determinant times a stable matrix), but we do not know whether these factors can be chosen to be coprime, as they are in the case of well-posed controllers, by Lemma 7.1.4.

We give here the equivalents of (c) and (d) for systems:

Lemma 7.2.6 ($\widetilde{\boldsymbol{\Sigma}}$: DF-IL vs. DF) *Let* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $\widetilde{\Sigma} = \left[\begin{smallmatrix}\widetilde{\mathbb{A}} & \widetilde{\mathbb{B}}\\ \widetilde{\mathbb{C}} & \mathbb{O}\end{smallmatrix}\right] \in \mathrm{WPLS}(Y\times\Xi,\widetilde{H},U\times\Xi)$.

Then $\widetilde{\Sigma}$ *is admissible (resp. [exponentially] stabilizing) with internal loop for* Σ *iff* $\widetilde{\Sigma}$ *is admissible (resp. [exponentially] stabilizing) for* $\underline{\Sigma} := \left[\begin{smallmatrix}\Sigma & 0\\ 0 & I_\Xi\end{smallmatrix}\right] \in$ $\mathrm{WPLS}(U\times\Xi,H,Y\times\Xi)$. *All prefices apply.*

Moreover, $\widetilde{\Sigma}$ *is admissible (resp. [exponentially] stabilizing) with internal loop for* Σ
iff $\widetilde{\Sigma}^{\mathrm{d}}$ *is admissible (resp. [exponentially] stabilizing) for* Σ^{d} *with internal loop.*

Thus, dynamic feedback with internal loop can be reduced to (proper) dynamic feedback (this could also be observed directly from Figures 7.4 and 7.3 or from equations 7.19: as u goes through $\mathbb{D}$ back to $\mathbb{O}$, we let ξ go through I back to $\mathbb{O}$).

As noted below Lemma 6.7.2, the prefix "strongly" does not apply to the duality claim but several others do.

Proof: 1° $\underline{\Sigma}$*:* The admissibility claim is contained in Proposition 7.2.5(c), whose proof shows that $\underline{\mathbb{D}}_I^o + I = \begin{bmatrix} I & -\mathbb{O}\\ -\underline{\mathbb{D}} & I\end{bmatrix}^{-1} - I$ contains $\mathbb{D}_I^o$ plus some copies of parts of it (plus one identity operator). We shall show below that the same holds for $\underline{\mathbb{A}}_I^o$, $\underline{\mathbb{B}}_I^o$ and $\underline{\mathbb{C}}_I^o$; this proves the claim.

From (7.37) we observe that

$$\underline{\mathbb{C}}_I^o = \begin{bmatrix} I & 0 & 0\\ 0 & 0 & I\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}\mathbb{C}_I^o \quad\text{and}\quad \underline{\mathbb{B}}_I^o = \mathbb{B}_I^o\begin{bmatrix} I & 0 & 0 & 0\\ 0 & 0 & I & 0\\ 0 & I & 0 & 0\end{bmatrix}. \tag{7.38}$$

It follows that $\underline{\mathbb{A}}_I^o = \underline{\mathbb{A}} + \underline{\mathbb{B}}^o\tau\underline{\mathbb{C}}_I^o = \mathbb{A} + \mathbb{B}^o I\tau\mathbb{C}_I^o = \mathbb{A}_I^o$.

2° *Duality:* This is contained in Lemma 6.7.2(e') (see the last claim of the lemma — or its proof). □

The closed-loop map $u_L, y_L \mapsto u, y$ corresponds to that of a well-posed controller iff $I - \mathbb{O}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$:

Lemma 7.2.7 (Well-posed $\mathbb{Q} = \mathbb{O}_{11} + \mathbb{O}_{12}(I-\mathbb{O}_{22})^{-1}\mathbb{O}_{21}$) *Let* $\mathbb{O} = \left[\begin{smallmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\\ \mathbb{O}_{21} & \mathbb{O}_{22}\end{smallmatrix}\right] \in$ $\mathrm{TIC}_\infty(Y\times\Xi,U\times\Xi)$ *be admissible with internal loop for* $\mathbb{D}\in\mathrm{TIC}_\infty(U,Y)$.

Then $\mathbb{O}$ *is equivalent to a well-posed controller iff* $I-\mathbb{O}_{22}\in\mathcal{G}\mathrm{TIC}_\infty$*; if this is the case, then that well-posed controller is given by* $\mathbb{O}_{11}+\mathbb{O}_{12}(I-\mathbb{O}_{22})^{-1}\mathbb{O}_{21}$ *(in particular, it is unique).*

This is exactly what one would have expected: the internal loop can be opened iff $L := \left[\begin{smallmatrix}0 & 0\\ 0 & I\end{smallmatrix}\right]$ is admissible for $\mathbb{O}$, and in that case, $\mathbb{O}$ is equivalent to $(\mathbb{O}_L)_{11} = \mathbb{O}_{11}+\mathbb{O}_{12}(I-\mathbb{O}_{22})^{-1}\mathbb{O}_{21}$ (see (6.125)).

Two different well-posed controllers induce different closed loop maps $\mathbb{D}_L :=$ $\begin{bmatrix} I & -\mathbb{Q}\\ -\mathbb{D} & I\end{bmatrix}^{-1} - I \in \mathrm{TIC}_\infty$, because inverses are unique.

Proof: 1° For any $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$, the maps $\mathbb{D}_L, \left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \end{smallmatrix}\right](I - \mathbb{D}^o)^{-1}\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \end{smallmatrix}\right]^{\mathrm{T}} - I : \left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ exist and are equal iff the maps $\mathbb{D}_L + I = \left[\begin{smallmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{smallmatrix}\right]^{-1}$ and $\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \end{smallmatrix}\right](I - \mathbb{D}^o)^{-1}\left[\begin{smallmatrix} I & 0 & 0 \\ 0 & I & 0 \end{smallmatrix}\right]^{\mathrm{T}}$ exist and are equal.

If this is the case, then $\left[\begin{smallmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{smallmatrix}\right]^{-1}$ equals the $(1\text{–}2, 1\text{–}2)$-block of

$$(I - \mathbb{D}^o)^{-1} = \begin{bmatrix} I & -\mathbb{O}_{11} & -\mathbb{O}_{12} \\ -\mathbb{D} & I & 0 \\ 0 & -\mathbb{O}_{21} & I - \mathbb{O}_{22} \end{bmatrix}^{-1} : \begin{bmatrix} u_L \\ y_L \\ \xi_L \end{bmatrix} \mapsto \begin{bmatrix} u + u_L \\ y + y_L \\ \xi_L \end{bmatrix}. \tag{7.39}$$

Therefore, by Lemma A.1.1(c1) (with $A := I - \mathbb{D}^o$ so that $B_{11} = \left[\begin{smallmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{smallmatrix}\right]^{-1}$), from $\left[\begin{smallmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{smallmatrix}\right]^{-1} \in \mathcal{G}\mathrm{TIC}_\infty$ we obtain that $(A_{22} =) I - \mathbb{O}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$, and $(B_{11} =)\left[\begin{smallmatrix} I & -\mathbb{Q} \\ -\mathbb{D} & I \end{smallmatrix}\right]^{-1}$ is equal to the inverse of

$$\begin{bmatrix} I & -\mathbb{O}_{11} \\ -\mathbb{D} & I \end{bmatrix} - \begin{bmatrix} -\mathbb{O}_{12} \\ 0 \end{bmatrix}(I - \mathbb{O}_{22})^{-1}\begin{bmatrix} 0 & -\mathbb{O}_{21} \end{bmatrix} = \begin{bmatrix} I & -\mathbb{O}_{11} - \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21} \\ -\mathbb{D} & I \end{bmatrix}, \tag{7.40}$$

hence $-\mathbb{Q} = -\mathbb{O}_{11} - \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21}$ (this also shows that $\mathbb{Q}$ is unique).

2° Conversely, if $I - \mathbb{O}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$ and one defines $\mathbb{Q} := \mathbb{O}_{11} + \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21}$, then $\mathbb{Q}$ and $\mathbb{O}$ determine same closed-loop maps $\left[\begin{smallmatrix} u_L \\ y_L \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u + u_L \\ y + y_L \end{smallmatrix}\right]$ (i.e., they are equivalent), as one can see by reversing the above calculations. □

Remark 7.2.8 ("$\mathbb{Q} \in \mathrm{H}^\infty/\mathrm{H}^\infty$") *As one easily observes from the proof, Lemma 7.2.7 actually covers a more general class of systems and controllers: If there are $\mathbb{O} \in \mathrm{TIC}_\infty$ and a holomorphic function $\widehat{\mathbb{Q}} \in \mathrm{H}(\Omega; \mathcal{B}(Y,U))$ with $\Omega \subset \mathbf{C}$ open, s.t. $\widehat{\mathbb{Q}}$, $\left[\begin{smallmatrix} I & -\widehat{\mathbb{Q}} \\ -\widehat{\mathbb{D}} & I \end{smallmatrix}\right]$ and $(I - \widehat{\mathbb{D}^o})$ are invertible at some $s_0 \in \mathbf{C}$ and $\left[\begin{smallmatrix} I & -\widehat{\mathbb{Q}} \\ -\widehat{\mathbb{D}} & I \end{smallmatrix}\right]^{-1}$ equals the $(1\text{–}2, 1\text{–}2)$-block of $(I - \widehat{\mathbb{D}^o})^{-1}$ on a neighborhood of s_0, then $\mathbb{Q} = \mathbb{O}_{11} + \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21}$. Obviously the converse holds too.*

Thus, if $I - \widehat{\mathbb{O}}_{22}$ and $(I - \widehat{\mathbb{D}^o})^{-1}$ are invertible at any $s_0 \in \mathbf{C}$, then the transfer function of the controller defined by $\mathbb{O}$ is $\widehat{\mathbb{O}}_{11} + \widehat{\mathbb{O}}_{12}(I - \widehat{\mathbb{O}}_{22})^{-1}\widehat{\mathbb{O}}_{21}$ (on the open subset of $\mathbf{C}$ where these inverses exist).

Therefore, controllers with internal loop cover (but are not covered by) all controllers whose transfer functions are of the form $\widehat{\mathbb{O}}_{11} + \widehat{\mathbb{O}}_{12}(I - \widehat{\mathbb{O}}_{22})^{-1}\widehat{\mathbb{O}}_{21}$ (and well-defined at least at one point $s_0 \in \mathbf{C}$), where $\mathbb{O} \in \mathrm{TIC}_\infty$; in particular, all "$\mathrm{H}^\infty/\mathrm{H}^\infty$" transfer functions are covered.

We now show by a simple example that the transfer function of the internal loop of a controller need not be invertible anywhere:

Example 7.2.9 ($\nexists\widehat{\mathbb{Q}}$) Take $\widehat{\mathbb{D}}(s) := (s-1)/(s+1)$, $\widehat{\mathbb{O}} = \left[\begin{smallmatrix} \widehat{\mathbb{D}}^{-1} & -1 \\ -1 & 1 \end{smallmatrix}\right]$ (exponentially stable), so that

$$(I - \widehat{\mathbb{D}^o})^{-1} = \begin{bmatrix} 1 & -\widehat{\mathbb{D}}^{-1} & 1 \\ \widehat{\mathbb{D}} & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -\widehat{\mathbb{D}}^{-1} & \widehat{\mathbb{D}}^{-1} \\ 0 & 0 & 1 \\ 1 & \widehat{\mathbb{D}}^{-1} & 0 \end{bmatrix} \tag{7.41}$$

(cf. Proposition 7.2.5(a)) is stable, because $\widehat{\mathbb{D}}^{-1} \in \mathrm{H}^\infty$.

However, $1 - \widehat{\mathbb{O}}_{22} \equiv 0$ is nowhere invertible, so one cannot close the internal loop in the controller if one does not connect the controller to the plant $\mathbb{D}$ to be controlled; neither can one close the upper loop only (admissibly), because $\mathbb{O}_{11} = I$ is not admissible for $\mathbb{D}$ ($I - \mathbb{D}\mathbb{O}_{11} = 0$); the setting becomes well-posed only when both loops are closed.

By Proposition 7.2.5(b), the corresponding map with coprime internal loop is $\mathbb{Y}\mathbb{X}^{-1} = -1(0)^{-1}$, i.e., $\mathbb{O}' = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right]$ is equivalent to $\mathbb{O}$; this 0^{-1} shows that the controller has something resembling a short circuit. In fact, in Example 2.3 of [CWW01] exactly this map with coprime internal loop (in its adjoint form) is used as a short circuit regulating an electrical circuit (whose transfer function $2/(1+\mathrm{e}^{-2s})$ has infinitely many poles on the imaginary axis).

By Lemma 7.2.7, neither $\mathbb{O}$ nor $\mathbb{O}'$ is equivalent to any well-posed controller. This is not really surprising, because we have $\widetilde{y} \equiv 0$ ($\widetilde{y} = \xi_L$, i.e., $y = -y_L + \xi_L$ if there is an external input ξ_L into the internal loop), and this poses the requirement "$(I - \mathbb{D}\mathbb{Q})^{-1} = 0$" (by formula (7.5), which has an extra $-I$), which is impossible for a well-posed controller and even for controllers of the form "$\mathrm{H}^\infty/\mathrm{H}^\infty$" (even for those with $(I - \widehat{\mathbb{D}}\widehat{\mathbb{Q}})^{-1}$ well-defined on any open subset of the complex plane, cf. Remark 7.2.8). Note that also the outputs cancel the corresponding inputs completely (i.e., the diagonal of (7.41) is zero), which could not be achieved by an admissible well-posed controller either. $\triangleleft$

For physically motivated examples, see, e.g., Example 2.3 of [CWW01]. Example 4.8 of [CWW01] illustrates a problem that can only be solved by using a non-well-posed controller.

By Proposition 7.2.5(b), it is enough to study the controllers of the following form (if we exclude maps that do not have coprime factorizations):

Lemma 7.2.10 *Let* $\mathbb{O} = \left[\begin{smallmatrix} 0 & I \\ \widetilde{\mathbb{Y}} & I-\widetilde{\mathbb{X}} \end{smallmatrix}\right] \in \mathrm{TIC}(Y \times U, U \times U)$ *and* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$.

(a) The map $\mathbb{O} = \left[\begin{smallmatrix} 0 & I \\ \widetilde{\mathbb{Y}} & I-\widetilde{\mathbb{X}} \end{smallmatrix}\right] \in \mathrm{TIC}$ *is admissible with internal loop for* $\mathbb{D}$ *iff* $\widetilde{\Delta} := \widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$*; it is stabilizing with internal loop for* $\mathbb{D}$ *iff* $\widetilde{\Delta}^{-1}, \mathbb{D}\widetilde{\Delta}^{-1} \in \mathrm{TIC}$.

(b) Let $\mathbb{O}$ *be stabilizing. Then* $\widetilde{\mathbb{X}}$ *and* $\widetilde{\mathbb{Y}}$ *are l.c., i.e., "*$\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *is a stabilizing DF-controller for* $\mathbb{D}$ *with l.c. internal loop". Moreover then, with* $\mathbb{M} := \widetilde{\Delta}^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$*, the factorization* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a r.c.f. of* $\mathbb{D}$*, it satisfies* $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$*, and the* $(1\text{–}2, 1\text{–}2)$ *blocks of the closed-loop map* $\mathbb{D}^o_I := \mathbb{D}^o(I - \mathbb{D}^o)^{-1}$ *are given by*

$$= \begin{bmatrix} \mathbb{M}\widetilde{\mathbb{X}} - I & \mathbb{M}\widetilde{\mathbb{Y}} \\ \mathbb{N}\widetilde{\mathbb{X}} & \mathbb{N}\widetilde{\mathbb{Y}} \end{bmatrix} : \begin{bmatrix} u_L \\ y_L \end{bmatrix} \mapsto \begin{bmatrix} u \\ y \end{bmatrix}, \tag{7.42}$$

as in the well-posed case, i.e., in (7.9) (cf. (7.5)).

(c) Conversely, if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is a r.c.f. with* $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$*, then* $\mathbb{O}$ *stabilizes* $\mathbb{D}$ *with internal loop.*

(d) If $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$, then $\mathbb{O}$ is admissible for $\mathbb{D}$ iff $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ is an admissible (well-posed) DF-controller for $\mathbb{D}$. If $\mathbb{O}$ is admissible and $\mathbb{Q}$ is an admissible (well-posed) DF-controller for $\mathbb{D}$, then the closed-loop maps $u_L, y_L \mapsto u, y$ determined by $\mathbb{O}$ and $\mathbb{Q}$ are identical iff $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$ and $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\mathbb{Y}$.

The corresponding dual claims (with $\Delta := \mathbb{X} - \mathbb{D}\mathbb{Y}$) hold as well.

Proof: (a)&(b)&(c) 1° With the notation of Proposition 7.2.5, we have $\mathbb{H} = \begin{bmatrix} I & -\mathbb{D} \\ -\widetilde{\mathbb{Y}} & \widetilde{\mathbb{X}} \end{bmatrix}$. By Lemma A.1.1(d1), we have $\mathbb{H} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\Delta} \in \mathcal{G}\mathrm{TIC}_\infty$, so the admissibility claim follows from Proposition 7.2.5(a) as well as the formula

$$(I - \mathbb{D}^o)^{-1} = \begin{bmatrix} I + \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}}\mathbb{D} & \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}} & \widetilde{\Delta}^{-1} \\ \mathbb{D}(I + \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}}\mathbb{D}) & I + \mathbb{D}\widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}} & \mathbb{D}\widetilde{\Delta}^{-1} \\ \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}}\mathbb{D} & \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}} & \widetilde{\Delta}^{-1} \end{bmatrix} \tag{7.43}$$

$$= \begin{bmatrix} \mathbb{M}\widetilde{\mathbb{X}} & \mathbb{M}\widetilde{\mathbb{Y}} & \mathbb{M} \\ \mathbb{N}\widetilde{\mathbb{X}} & \mathbb{N}\widetilde{\mathbb{Y}} + I & \mathbb{N} \\ \mathbb{M}\widetilde{\mathbb{X}} - I & \mathbb{M}\widetilde{\mathbb{Y}} & \mathbb{M} \end{bmatrix} : \begin{bmatrix} u_L \\ y_L \\ \xi_L \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{u} \\ \widetilde{y} \\ \widetilde{\xi} \end{bmatrix} \tag{7.44}$$

(for the second identity, we have set $\mathbb{M} := \widetilde{\Delta}^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$ and used (7.45)).

2° If (7.43) is stable, then $\mathbb{M} := \widetilde{\Delta}^{-1}$ and $\mathbb{N} := \mathbb{D}\mathbb{M}$ are also stable. Moreover, $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f., because $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = \widetilde{\mathbb{X}}(\widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D})^{-1} - \widetilde{\mathbb{Y}}\mathbb{D}(\widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D})^{-1} = I$. Formula (7.42) follows from this.

3° Conversely, if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f. and $\mathbb{O}$ is s.t. $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$ (as above), then (7.43) is stable, because then

$$I + \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}}\mathbb{D} = \widetilde{\Delta}^{-1}(\widetilde{\Delta} + \widetilde{\mathbb{Y}}\mathbb{D}) = \widetilde{\Delta}^{-1}\widetilde{\mathbb{X}} \in \mathrm{TIC}, \tag{7.45}$$

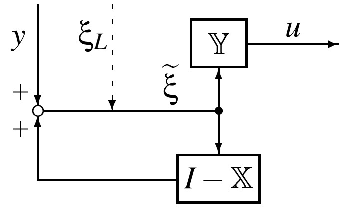
and $\mathbb{D}(I + \widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}}\mathbb{D}) = \mathbb{D}\widetilde{\Delta}^{-1}\widetilde{\mathbb{X}} \in \mathrm{TIC}$.

(d) This follows from Lemma 7.2.7. □

If a plant has a (right or left) coprime factorization, then all of its stabilizing controllers are equivalent to some of the form studied in Lemma 7.2.10, by Proposition 7.2.5(b) (or its dual). Therefore, the latter ones were called "canonical controllers" in [CWW01]. To be able to extend the Youla parametrization (Theorem 7.2.14) and related results to cover also the non-well-posed case, we shall define the concept *map with coprime internal loop* below as the equivalence class of a "canonical controller" modulo "being equal".

It follows that, for a plant having a coprime factorization, each stabilizing controllers with internal loop is equivalent to one and only one map with a coprime internal loop, by Lemma 7.2.12(c).

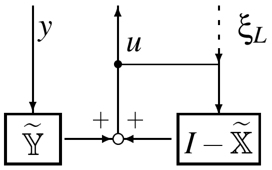
Definition 7.2.11 (Maps with coprime internal loop) *Let $(\mathbb{Y}, \mathbb{X})$ be r.c. and $(\widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}})$ be l.c. We call the (equivalence class (modulo equality; see below) of the) map $\begin{bmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X} \end{bmatrix}$ (resp. $\begin{bmatrix} 0 & I \\ \widetilde{\mathbb{Y}} & I-\widetilde{\mathbb{X}} \end{bmatrix}$) a* map with r.c. internal loop *(resp. a* map with l.c. internal loop*) and denote it by $\mathbb{Y}\mathbb{X}^{-1}$ (resp. by $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$).*



$y = \mathbb{X}\widetilde{\xi}$

$u = \mathbb{Y}\widetilde{\xi}$

Figure 7.5: Controller $\mathbb{Y}\mathbb{X}^{-1}$ with r.c. internal loop



$\widetilde{\mathbb{X}}u = \widetilde{\mathbb{Y}}y$

Figure 7.6: Controller $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ with l.c. internal loop

If, in addition, $\mathbb{X}$, $\mathbb{Y}$, $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ can be extended to satisfy the doubly coprime product

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} = I = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \tag{7.46}$$

in $\mathrm{TIC}(U \times Y)$ *for some* $\mathbb{M}, \mathbb{N}, \widetilde{\mathbb{M}}, \widetilde{\mathbb{N}}$, *then we consider* $\mathbb{Y}\mathbb{X}^{-1}$ *and* $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *equal and call* $\mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *a* map with d.c. internal loop. *We add the words "*over $\mathcal{A}$*", if* $\mathcal{A} \subset \mathrm{TIC}$ *and the elements of (7.46) can be chosen from* $\mathcal{A}$.

We consider the maps $\mathbb{Y}\mathbb{X}^{-1}$ *and* $\mathbb{Y}_0\mathbb{X}_0^{-1}$ *with r.c. internal loop* equal *if* $(\mathbb{Y}_0, \mathbb{X}_0) = (\mathbb{Y}\mathbb{U}, \mathbb{X}\mathbb{U})$ *for some* $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$. *We consider the maps* $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *and* $\widetilde{\mathbb{X}}_0^{-1}\widetilde{\mathbb{Y}}_0$ *with l.c. internal loop* equal *if* $(\widetilde{\mathbb{Y}}_0, \widetilde{\mathbb{X}}_0) = (\mathbb{U}\widetilde{\mathbb{Y}}, \mathbb{U}\widetilde{\mathbb{X}})$ *for some* $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$.

If $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$, *then we identify* $\mathbb{Y}\mathbb{X}^{-1}$ *in the usual sense (in* TIC_∞*) and* $\mathbb{Y}\mathbb{X}^{-1}$ *as a map with r.c. internal loop; we do the analogous identification for maps with l.c. internal loop too.*

A map with coprime internal loop *means a map with r.c. or l.c. internal loop. A controller with internal loop for* $\Sigma \in \mathrm{WPLS}$ *is called a* controller with coprime internal loop *if its I/O map is a representative of a map with coprime internal loop.*

Let $\mathbb{Q}$ *be a map with coprime internal loop. Then we say that* $\mathbb{Q}$ is admissible [stabilizing] for $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *[or that* $\mathbb{Q}$ stabilizes $\mathbb{D}$*] if some (hence any, by Lemma 7.2.12(c)) of its representatives is admissible [stabilizing] for* $\mathbb{D}$ *with internal loop. We use prefices as in Definition 7.2.1.*

At this stage the serious reader has several serious questions about this definition and its justification. Lemma 7.2.12 below answers these questions in the expected way.

Thus, given r.c. maps $\mathbb{Y}$ and $\mathbb{X}$ s.t. $\left[\begin{smallmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X} \end{smallmatrix}\right] \in \mathrm{TIC}(Y \times U, U \times U)$, the equivalence class of $\left[\begin{smallmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X} \end{smallmatrix}\right]$ (modulo equality, in the collection of all maps of the same form) is given by $\{\left[\begin{smallmatrix} 0 & \mathbb{Y}\mathbb{U} \\ I & I-\mathbb{X}\mathbb{U} \end{smallmatrix}\right] \mid \mathbb{U} \in \mathcal{G}\mathrm{TIC}(U)\}$ (cf. Lemma 7.2.12(a1)). Analogous claims hold for maps with l.c. or d.c. internal loop.

Recall from Definition 6.4.4(f) that (7.46) is called a *joint d.c.f. of* $\mathbb{D}$ *and* $\mathbb{Y}\mathbb{X}^{-1}$ (or of $\mathbb{D}$ and $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$) if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ (equivalently, $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$).

We warn the reader that if the left equation from (7.46) were removed, then "equality" would not be an equivalence relation. Even if both $\mathbb{Y}\mathbb{X}^{-1}$ and $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ were maps with d.c. internal loop, and $\widetilde{\mathbb{X}}\mathbb{Y} = \widetilde{\mathbb{Y}}\mathbb{X}$, these two maps need not be equal; a necessary and sufficient condition can be seen from Lemma A.1.1(e4) (although that is not needed here). From (7.46) one can also note that a pair $\mathbb{Y},\mathbb{X}$ defines a map with d.c. internal loop iff it can be extended to a invertible pair $\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$; Lemma A.1.1(e) gives some (necessary and) sufficient conditions for this.

The last identification above corresponds to the equivalence of $\begin{bmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X} \end{bmatrix}$ and $\begin{bmatrix} \mathbb{Y}\mathbb{X}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ noted in Lemma 7.2.7 (and is hence justified). This identification makes maps with coprime internal loop a natural extension of well-posed maps having a r.c.f. or a l.c.f. However, one can show by a simple example, that if $\mathbb{X}$ were not assumed to be in $\mathcal{G}\mathrm{TIC}_\infty$, then $\mathbb{Q}\mathbb{X} = \mathbb{Y}$ (for general r.c. $(\mathbb{Y},\mathbb{X})$ and some $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$) would not guarantee that $\mathbb{Y}\mathbb{X}^{-1}$ and $\mathbb{Q}$ were equivalent for all $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$; in fact, with those assumptions $\widehat{\mathbb{X}}$ might be nowhere invertible (although $\mathbb{X}$ is necessarily left-invertible on TIC) and $\mathbb{Y}\mathbb{X}$ might stabilize different plants than $\mathbb{Q}$.

By Lemma 7.2.10, a well-posed $\mathbb{D}$ has a r.c.f. (resp. a l.c.f.) iff it can be stabilized by a map with l.c. (resp. r.c.) internal loop.

From this on, we shall often use the word "map" of both members of a map (equivalence class) and of the class itself when there should be no risk of ambiguity.

Lemma 7.2.12 (Equal; well-posed) *Let $\mathbb{Y}\mathbb{X}^{-1}$, $\mathbb{Y}_0\mathbb{X}_0^{-1}$ and $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ be maps with coprime internal loop and let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. We have the following:*

- *(a1) Being equal is an equivalence relation.*
- *(a2) Two well-posed maps with coprime internal loop are equal iff they are equal in* TIC_∞.
- *(a3) A well-posed map is a map with coprime internal loop iff it has a l.c.f. or a r.c.f.*
- *(a4) If a well-posed map and a map with coprime internal loop are equivalent controllers for $\mathbb{D}$, then they are equal. (See (c) for the converse.)*
- *(b) If $\mathbb{Y}\mathbb{X}^{-1} = \mathbb{Y}_0\mathbb{X}_0^{-1}$, then $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \mathbb{X}_0 \in \mathcal{G}\mathrm{TIC}_\infty$. If $\widetilde{\mathbb{Y}}\widetilde{\mathbb{X}}^{-1} = \widetilde{\mathbb{Y}}_0\widetilde{\mathbb{X}}_0^{-1}$, then $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{X}}_0 \in \mathcal{G}\mathrm{TIC}_\infty$. If $\mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$, then $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$. In particular, $\mathbb{Y}\mathbb{X}^{-1}$ is well-posed iff $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$.*
- *(c) Let $\mathbb{Q}$ and $\mathbb{Q}'$ be maps with coprime internal loop.*
  - *(c1) If $\mathbb{Q}$ and $\mathbb{Q}'$ are equal, then either both are admissible for $\mathbb{D}$ or neither is admissible for $\mathbb{D}$.*
  - *(c2) If $\mathbb{Q}$ and $\mathbb{Q}'$ are admissible for $\mathbb{D}$, then they are equal iff they are equivalent, that is, iff they determine the same map $u_L, y_L \mapsto u, y$. In particular, $\mathbb{Q}$ is stabilizing for $\mathbb{D}$ iff $\mathbb{Q}'$ is.*

Thus, Definition 7.2.11 is justified (its last identification was justified in Lemma 7.2.7).

By (c), equivalence of maps with coprime internal loop does not depend on the plant $\mathbb{D}$ (except that equivalence is not defined for non-admissible maps). By (b), a map is well-posed iff any (hence all) of its representatives is well-posed.

Proof: (a1) The only nonobvious requirement is transitivity (of being equal), so we take a look at it:

1° If $\mathbb{Y}\mathbb{X}^{-1}$ is a map with d.c. internal loop and equal to $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$, then

$$\begin{bmatrix}\mathbb{U}\widetilde{\mathbb{X}} & -\mathbb{U}\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix} = \begin{bmatrix}\mathbb{U} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix} \in \mathcal{G}\mathrm{TIC} \tag{7.47}$$

for any $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$, hence any map with l.c. internal loop equal to $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ is equal to $\mathbb{Y}\mathbb{X}^{-1}$ (insert $\left[\begin{smallmatrix}\mathbb{U} & 0\\ 0 & I\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}\mathbb{U}^{-1} & 0\\ 0 & I\end{smallmatrix}\right]$ into (7.46)). (Thus the concept "map with d.c. internal loop" is well defined: if a map is such, then so is any equal map.)

Conversely, the (b) (and (d)) of Lemma 6.5.9 (with the columns and rows interchanged) shows that all map with r.c. internal loops equal to $\mathbb{Y}\mathbb{X}^{-1}$ are equal to $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ (in particular, they have d.c. internal loops).

(Thus, (6.121) gives all doubly coprime products for any left and right maps equal to $\mathbb{N}\mathbb{M}^{-1}$.)

2° If $\mathbb{Y}\mathbb{X}^{-1}$ does not have a d.c. internal loop, then neither does any equal map with a coprime internal loop by 1°, and transitivity is obvious (i.e., $\left[\begin{smallmatrix}\mathbb{S}\\ \mathbb{R}\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mathbb{X}\\ \mathbb{Y}\end{smallmatrix}\right]\mathbb{U}$, $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$, and $\left[\begin{smallmatrix}\mathbb{P}\\ \mathbb{O}\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mathbb{S}\\ \mathbb{R}\end{smallmatrix}\right]\mathbb{V}$, $\mathbb{V} \in \mathcal{G}\mathrm{TIC}$ imply that $\left[\begin{smallmatrix}\mathbb{P}\\ \mathbb{O}\end{smallmatrix}\right] = \left[\begin{smallmatrix}\mathbb{X}\\ \mathbb{Y}\end{smallmatrix}\right]\mathbb{W}$ for some $\mathbb{W} \in \mathcal{G}\mathrm{TIC}$ (namely for $\mathbb{W} = \mathbb{U}\mathbb{V}$)). The dual claim is analogous.

(a2) This follows from (b) combined with Lemma 6.4.5 in the left or right case and with Lemma 6.5.8 in the left-right case.

(a3) This is a restatement of the last identification in Definition 7.2.11.

(a4) Let the two maps be $\mathbb{Q} \in \mathrm{TIC}_\infty$ and $\mathbb{Y}\mathbb{X}^{-1}$, respectively (the case for $\widetilde{\mathbb{X}}^{-1}\mathbb{Y}$ is analogous). By Lemma 7.2.7, $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ and $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$.

(b) The first two claims follows from $\mathbb{X} = \mathbb{X}_0\mathbb{U} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \mathbb{X}_0 \in \mathcal{G}\mathrm{TIC}_\infty$. so $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$, The third one follows from Lemma A.1.1(c1). Thus, $\mathbb{Y}\mathbb{X}^{-1}$ is equal to a well-posed map iff $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$.

(c1) If $\mathbb{Q}$ and $\mathbb{Q}'$ have l.c. (resp. r.c.) internal loops, then this is obvious (because the admissibility is equivalent to $\widetilde{\Delta} := \widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$ (resp. $\Delta := \mathbb{X} - \mathbb{D}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}_\infty$), by Lemma 7.2.10(a)). Thus, we assume (7.46). Then

$$\begin{bmatrix} I & 0\\ -\mathbb{D} & I\end{bmatrix}\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix} = \begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} - \mathbb{D}\mathbb{M} & \mathbb{X} - \mathbb{D}\mathbb{Y}\end{bmatrix} = \left(\begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix} I & 0\\ \mathbb{D} & I\end{bmatrix}\right)^{-1}$$

in TIC_∞, so $\mathbb{X} - \mathbb{D}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$, by Lemma A.1.1(c1).

(c2) 1° We start from the case of two maps with l.c. internal loop. The formula (7.43) shows that maps $u_L, y_L \mapsto u, y$ are equal for $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ and $\mathbb{Q}' = \widetilde{\mathbb{X}}_0^{-1}\widetilde{\mathbb{Y}}_0$ iff the corresponding terms $\widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}}$ and $\widetilde{\Delta}_0^{-1}\widetilde{\mathbb{Y}}_0$ are equal, i.e., $\widetilde{\mathbb{Y}} = \mathbb{U}\widetilde{\mathbb{Y}}_0$, where $\mathbb{U} := \widetilde{\Delta}\widetilde{\Delta}_0^{-1} \in \mathcal{G}\mathrm{TIC}_\infty$. But then $\widetilde{\Delta}^{-1}\widetilde{\mathbb{X}} = I + \widetilde{\Delta}^{-1}\mathbb{Y}\mathbb{D} = \widetilde{\Delta}_0^{-1}\widetilde{\mathbb{X}}_0$, i.e., $\widetilde{\mathbb{X}} = \mathbb{U}\widetilde{\mathbb{X}}_0$. So if the maps $u_L, y_L \mapsto u, y$ are equal, then $\mathbb{U} \in \mathrm{TIC}$ (by the dual of Lemma 6.5.1(c1)), because $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ are l.c., and $\mathbb{U}^{-1} \in \mathrm{TIC}$, because $\widetilde{\mathbb{X}}_0$ and $\widetilde{\mathbb{Y}}_0$ are l.c.; thus, then $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}} = \widetilde{\mathbb{X}}_0^{-1}\widetilde{\mathbb{Y}}_0$.

Conversely, if $\begin{bmatrix}\widetilde{\mathbb{Y}} & \widetilde{\mathbb{X}}\end{bmatrix} = \mathbb{U}\begin{bmatrix}\widetilde{\mathbb{Y}}_0 & \widetilde{\mathbb{X}}_0\end{bmatrix}$ with $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$, then $\widetilde{\Delta} = \mathbb{U}\widetilde{\Delta}_0$ and hence $\widetilde{\Delta}^{-1}\widetilde{\mathbb{Y}} = \widetilde{\Delta}_0^{-1}\widetilde{\mathbb{Y}}_0$, so the maps $u_L, y_L \mapsto u, y$ are equal, as noted above.

2° From (7.32) one gets the corresponding right result analogously.

3° Similarly, from (7.32) and (7.43) one notices that $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ and $\mathbb{Q}' = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ determine the same $u_L, y_L \mapsto u, y$ iff $\mathbb{Y}(\mathbb{X} - \mathbb{D}\mathbb{Y})^{-1} = (\widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D})^{-1}\widetilde{\mathbb{Y}}$, i.e., iff $\widetilde{\mathbb{X}}\mathbb{Y} = \widetilde{\mathbb{Y}}\mathbb{X}$.

Thus, equality implies equivalence, so we assume equivalence and prove that $\mathbb{Q}$ and $\mathbb{Q}'$ are equal. Because $\widetilde{\mathbb{X}}\mathbb{Y} = \widetilde{\mathbb{Y}}\mathbb{X}$, as noted above, we may choose $\mathbb{M}, \mathbb{N}, \widetilde{\mathbb{M}}, \widetilde{\mathbb{N}}$ as in Lemma A.1.1(e1) (interchange the rows and columns) to obtain

$$I = \begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix} = \begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}I & 0\\ \mathbb{D} & I\end{bmatrix}\begin{bmatrix}I & 0\\ -\mathbb{D} & I\end{bmatrix}\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix}. \tag{7.48}$$

By Lemma A.1.1(e5) and the assumed invertibility of Δ and $\widetilde{\Delta}$, we have

$$I = \begin{bmatrix}I & 0\\ -\mathbb{D} & I\end{bmatrix}\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}I & 0\\ -\mathbb{D} & I\end{bmatrix}$$

in TIC_∞, hence we have (7.46) in TIC_∞, so it must hold in TIC too, by the density of $\mathcal{C}_c^\infty$. □

We now parametrize all stabilizing controllers by combining Proposition 7.2.5(b) and Lemma 7.2.10(a):

Corollary 7.2.13 *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U, Y)$. Then the following claims and their duals hold:*

(a1) If $\mathbb{D}$ has a r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, then each stabilizing controller with internal loop for $\mathbb{D}$ is equivalent to a unique map with l.c. internal loop

$$\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}} \quad \textit{such that} \quad \widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I \tag{7.49}$$

(in particular, a different pair $(\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}})$ defines a different stabilizing map $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$). The dual result for l.c.f.'s $\mathbb{D} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ holds as well.

(a2) The map with l.c. internal loop $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ is admissible (resp. stabilizing) for $\mathbb{D}$ iff $\widetilde{\Delta} := \widetilde{\mathbb{X}} - \widetilde{\mathbb{Y}}\mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ (resp. $\widetilde{\Delta}^{-1}, \mathbb{D}\widetilde{\Delta}^{-1} \in \mathrm{TIC}$).

(a2') The map with r.c. internal loop $\mathbb{Y}\mathbb{X}^{-1}$ is admissible (resp. stabilizing) for $\mathbb{D}$ iff $\Delta := \mathbb{X} - \mathbb{D}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ (resp. $\Delta^{-1}, \Delta^{-1}\mathbb{D} \in \mathrm{TIC}$).

(a3) If $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f., then the map $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ with l.c. internal loop is admissible (resp. stabilizing) for $\mathbb{D}$ iff $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ (resp. $\in \mathcal{G}\mathrm{TIC}(U)$).

(b) The following are equivalent:

(i) $\mathbb{D}$ has a r.c.f. (resp. a l.c.f., a d.c.f.);

(ii) $\mathbb{D}$ is stabilizable by a map with l.c. (resp. r.c., d.c.) internal loop;

Moreover, if (i) holds, then each stabilizing controller for $\mathbb{D}$ with internal loop is equivalent to one with l.c. (resp. r.c., d.c.) internal loop.

Unfortunately, we do not know, whether any $\mathbb{D}$ that is stabilizable with internal loop has a r.c.f. (or a l.c.f.), so it may be that some pathological plants (having no stabilizing controllers with coprime internal loop) might not meet the above requirements.

Proof: (We obtain the dual claims by taking the adjoints of (a1)–(b); this is explicitly illustrated in (a2').)

(a1) This follows from Proposition 7.2.5(b). The definition of equality [Definition 7.2.11] shows that $\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N}=I$ determines $(\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}})$ uniquely.

(a2) This is (most of) Lemma 7.2.10(a) and (a2') is its dual.

(a3) Now $\widetilde{\Delta}\mathbb{M}=\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N}$, so the stability of its inverse $\mathbb{M}^{-1}\widetilde{\Delta}^{-1}$ is equivalent to that of $\mathbb{D}\widetilde{\Delta}^{-1}$, by Lemma 6.5.6(b), and clearly implies the stability of Δ^{-1}, so we get (a3) from (a2).

(b) "(iii)$\Rightarrow$(ii)": Any map with l.c. [r.c.] internal loop that stabilizes $\mathbb{D}$ defines a r.c.f. [l.c.f.] of $\mathbb{D}$, by Lemma 7.2.10(b).

"(i)$\Rightarrow$(ii)": This follows from (a1) (and the definition of maps with d.c. internal loop: just take the factors $\mathbb{X},\mathbb{Y},\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}}\in\mathrm{TIC}$ of any d.c.f. of $\mathbb{D}$); and so does the "moreover" claim. □

Now we can present five equivalent parametrizations for all (modulo being equivalent) stabilizing controllers with internal loop for any fixed $\mathbb{D}\in\mathrm{TIC}_\infty$ having a d.c.f.:

Theorem 7.2.14 (All stabilizing controllers) *Let $\mathbb{D}\in\mathrm{TIC}_\infty(U,Y)$ have the d.c.f.*

$$\begin{bmatrix}\mathbb{M} & \mathbb{T}\\ \mathbb{N} & \mathbb{S}\end{bmatrix}\begin{bmatrix}\widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}=I=\begin{bmatrix}\widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix}\begin{bmatrix}\mathbb{M} & \mathbb{T}\\ \mathbb{N} & \mathbb{S}\end{bmatrix}. \tag{7.50}$$

Then each controller that stabilizes $\mathbb{D}$ with internal loop is equivalent to a unique map with d.c. internal loop (in the sense that both controllers determine the same closed-loop map $u_L,y_L\mapsto u,y$).

The following parametrizations are alternative (equivalent) parametrizations of all controllers $\mathbb{Q}$ with d.c. internal loop that stabilize $\mathbb{D}$, and each parameter *($(\mathbb{X},\mathbb{Y})$ in (i) and (iii), $(\widetilde{\mathbb{Y}},\widetilde{\mathbb{X}})$ in (i'), and $\mathbb{U}$ in (ii) and (ii'); these all are required to be stable) determines a different (nonequal) map $\mathbb{Q}$ with d.c. internal loop.*

(i) $\mathbb{Q}=\mathbb{Y}\mathbb{X}^{-1}$ such that $\widetilde{\mathbb{M}}\mathbb{X}-\widetilde{\mathbb{N}}\mathbb{Y}=I$.

(i') $\mathbb{Q}=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ such that $\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N}=I$.

(ii) **(Youla)** *$\mathbb{Q}=(\mathbb{T}+\mathbb{M}\mathbb{U})(\mathbb{S}+\mathbb{N}\mathbb{U})^{-1}$ (i.e., $\left[\begin{smallmatrix}\mathbb{Y}\\ \mathbb{X}\end{smallmatrix}\right]=\left[\begin{smallmatrix}\mathbb{M} & \mathbb{T}\\ \mathbb{N} & \mathbb{S}\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right]$), where $\mathbb{U}\in$ $\mathrm{TIC}(U)$.*

(ii') $\mathbb{Q}=(\widetilde{\mathbb{S}}+\widetilde{\mathbb{N}}\mathbb{U})^{-1}(\widetilde{\mathbb{T}}+\widetilde{\mathbb{M}}\mathbb{U})$ (i.e., $\left[\begin{smallmatrix}\widetilde{\mathbb{X}} & \widetilde{\mathbb{Y}}\end{smallmatrix}\right]=\left[\begin{smallmatrix}I & \mathbb{U}\end{smallmatrix}\right]\left[\begin{smallmatrix}\widetilde{\mathbb{S}} & \widetilde{\mathbb{T}}\\ \widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{smallmatrix}\right]$), where $\mathbb{U}\in\mathrm{TIC}(U)$.

(iii) $\mathbb{Q}=\mathbb{Y}\mathbb{X}^{-1}$ ($=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$), where $\left[\begin{smallmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{smallmatrix}\right]^{-1}=\left[\begin{smallmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{smallmatrix}\right]\in\mathcal{G}\mathrm{TIC}(U\times Y)$.

The well-posed ones (if any) are exactly those for which the "denominator" in is $\mathcal{G}\mathrm{TIC}_\infty$ (cf. Theorem 7.1.7).

Moreover, for any $\mathbb{U} \in \mathrm{TIC}$ we have (identity as equal maps with coprime internal loop)

$$(\mathbb{T}+\mathbb{M}\mathbb{U})(\mathbb{S}+\mathbb{N}\mathbb{U})^{-1} = (\widetilde{\mathbb{S}}+\widetilde{\mathbb{N}}\widetilde{\mathbb{U}})^{-1}(\widetilde{\mathbb{T}}+\widetilde{\mathbb{M}}\widetilde{\mathbb{U}}). \tag{7.51}$$

Finally, if (i) and (i') hold, then the $(1\text{–}2,1\text{–}2)$-block of the closed-loop map $\mathbb{D}_I^o := \mathbb{D}^o(I-\mathbb{D}^o)^{-1}$ is given by

$$\begin{bmatrix} \mathbb{Y}\widetilde{\mathbb{N}} & \mathbb{Y}\widetilde{\mathbb{M}} \\ \mathbb{X}\widetilde{\mathbb{N}} & \mathbb{X}\widetilde{\mathbb{M}}-I \end{bmatrix} = \begin{bmatrix} \mathbb{M}\widetilde{\mathbb{X}}-I & \mathbb{M}\widetilde{\mathbb{Y}} \\ \mathbb{N}\widetilde{\mathbb{X}} & \mathbb{N}\widetilde{\mathbb{Y}} \end{bmatrix} : \begin{bmatrix} u_L \\ y_L \end{bmatrix} \mapsto \begin{bmatrix} u \\ y \end{bmatrix}. \tag{7.52}$$

Recall from Lemma 7.2.12(c2), that the maps (7.52) depend (of course) on $\mathbb{D}$ and $\mathbb{Q}$ only, not of the particular coprime factors ($\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}}, \mathbb{N}, \mathbb{M}, \widetilde{\mathbb{N}}, \widetilde{\mathbb{M}}$) of $\mathbb{Q}$ and $\mathbb{D}$ (except that (i) and (i') are required to hold).

Proof: The first claim is from Proposition 7.2.5(b) (and its dual). The parametrizations (i) and (i') are Corollary 7.2.13(a1).

For any stable pair $(\mathbb{Y},\mathbb{X})$ there are $\widetilde{\mathbb{Y}}$ and $\widetilde{\mathbb{X}}$ satisfying (iii) iff $(\mathbb{Y},\mathbb{X})$ satisfies (i), by Lemma 6.5.8. Now the parametrizations (ii) and (ii') and equation (7.51) follow from (iii) and Lemma 6.5.9(c).

The well-posedness claim is Lemma 7.2.12(b), and (7.52) is from (7.42) (alternatively, directly from $\mathbb{D}_I^o = (I-\mathbb{D}^o)^{-1} - I$). □

To check whether a given controller with coprime internal loop stabilizes $\mathbb{D}$, one can use the following corollary:

Corollary 7.2.15 *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ have the d.c.f. (7.50). Let $\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$. Then $\mathbb{Y}\mathbb{X}^{-1}$ (resp. $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$) is a map with d.c. internal loop and stabilizes $\mathbb{D}$ iff any (hence all) of (i)–(iii) (resp. (i')–(iii')) holds:*

(i) $\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}(Y)$;

(ii) $\left[\begin{smallmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}(U\times Y)$;

(iii) There is a r.c.f. $\mathbb{D} = \mathbb{N}_0\mathbb{M}_0^{-1}$ *s.t.* $\left[\begin{smallmatrix} \mathbb{M}_0 & \mathbb{Y} \\ \mathbb{N}_0 & \mathbb{X} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$.

(i') $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *for some* $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$.

(ii') $\left[\begin{smallmatrix} \widetilde{\mathbb{X}} & \widetilde{\mathbb{Y}} \\ \widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ *and* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *for some* $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$.

(iii') There is a l.c.f. $\mathbb{D} = \widetilde{\mathbb{M}}_0^{-1}\widetilde{\mathbb{N}}_0$ *s.t.* $\left[\begin{smallmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_0 & \widetilde{\mathbb{M}}_0 \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$.

Moreover, the map $\mathbb{Y}\mathbb{X}^{-1}$ (resp. $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$) is well-posed iff $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ (resp. $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$).

Proof: Any of the conditions shows that $\mathbb{Y}\mathbb{X}^{-1}$ (resp. $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$) are coprime. If it stabilizes $\mathbb{D}$, then it is d.c., by Theorem 7.2.14.

The parametrizations (i) and (i') are from Corollary 7.2.13(a3) and its dual. Part (ii) defines a d.c.f. of $\mathbb{D}$, hence it is sufficient (take $\mathbb{U}=0$ in Theorem 7.2.14(ii)). Conversely, if $\mathbb{Y}\mathbb{X}^{-1}$ is stabilizing, then

$$\begin{bmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{bmatrix}\begin{bmatrix}I & 0\\ 0 & \mathbb{V}\end{bmatrix}=\begin{bmatrix}\mathbb{M} & \mathbb{Y}\mathbb{V}\\ \mathbb{N} & \mathbb{X}\mathbb{V}\end{bmatrix}\in\mathcal{G}\mathrm{TIC}$$

for some $\mathbb{V}\in\mathcal{G}\mathrm{TIC}$, by Theorem 7.2.14(iii), hence then $\left[\begin{smallmatrix}\mathbb{M} & \mathbb{Y}\\ \mathbb{N} & \mathbb{X}\end{smallmatrix}\right]\in\mathcal{G}\mathrm{TIC}$, i.e., (ii) holds. Condition (ii') is the dual of (ii).

(iii) The condition (iii) is sufficient, by Theorem 7.2.14(iii). Conversely, if (i) holds (and hence (i') too; thus situation is as in the "furthermore" claim) and we set

$$\begin{bmatrix}\widetilde{\mathbb{N}}_0 & \widetilde{\mathbb{M}}_0\end{bmatrix}:=(\widetilde{\mathbb{M}}\mathbb{X}-\widetilde{\mathbb{N}}\mathbb{Y})^{-1}\begin{bmatrix}\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{bmatrix},\quad \begin{bmatrix}\mathbb{M}_0\\ \mathbb{N}_0\end{bmatrix}:=\begin{bmatrix}\mathbb{M}\\ \mathbb{N}\end{bmatrix}(\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N})^{-1},\tag{7.53}$$

then, obviously, $\left[\begin{smallmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}}_0 & \widetilde{\mathbb{M}}_0\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathbb{M}_0 & \mathbb{Y}\\ \mathbb{N}_0 & \mathbb{X}\end{smallmatrix}\right]=I$, and the dual equation $\left[\begin{smallmatrix}\mathbb{M}_0 & \mathbb{Y}\\ \mathbb{N}_0 & \mathbb{X}\end{smallmatrix}\right]\left[\begin{smallmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}}_0 & \widetilde{\mathbb{M}}_0\end{smallmatrix}\right]=I$ follows from Lemma A.1.1(e5).

The well-posedness claim is Lemma 7.2.12(b). □

Next we given two lemmas that are useful when one wants to work in a subalgebra of TIC (e.g., in MTIC; cf. Theorem 4.1.1):

Lemma 7.2.16 (Predetermining the joint d.c.f. of $\mathbb{D}$ and $\mathbb{Q}$) *Let $\mathbb{D}=\mathbb{N}\mathbb{M}^{-1}=\widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ be a d.c.f., and let $\mathcal{A}\underset{a}{\subset}\mathrm{TIC}$ be inverse closed.*

(a) If $\mathbb{N},\mathbb{M},\widetilde{\mathbb{M}},\widetilde{\mathbb{N}},\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}}\in\mathcal{A}$ and $\mathbb{Q}:=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ stabilizes $\mathbb{D}$, then the d.c.f. $\mathbb{D}=\mathbb{N}\mathbb{M}^{-1}=\widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ is over $\mathcal{A}$, even joint with $\mathbb{Q}$.

(b) If $\mathbb{Q}$ stabilizes $\mathbb{D}$, then for any r.c.f. $\mathbb{Q}=\mathbb{Y}\mathbb{X}^{-1}$ and l.c.f. $\mathbb{Q}=\widetilde{\mathbb{X}}^{-1}\mathbb{Y}$, there is a joint d.c.f. $\left[\begin{smallmatrix}\mathbb{M}_0 & \mathbb{Y}\\ \mathbb{N}_0 & \mathbb{X}\end{smallmatrix}\right]^{-1}=\left[\begin{smallmatrix}\widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}}\\ -\widetilde{\mathbb{N}}_0 & \widetilde{\mathbb{M}}_0\end{smallmatrix}\right]\in\mathcal{G}\mathrm{TIC}$ of $\mathbb{D}$ and $\mathbb{Y}\mathbb{X}^{-1}$; if, in addition, $\mathbb{M},\mathbb{N},\mathbb{X},\mathbb{Y},\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}}\in\mathcal{A}$, then we can take $\mathbb{M}_0,\mathbb{N}_0,\widetilde{\mathbb{M}}_0,\widetilde{\mathbb{N}}_0\in\mathcal{A}$.

Let $\mathbb{D}=\mathbb{N}\mathbb{M}^{-1}=\widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}\in\mathrm{TIC}_\infty(U,Y)$. Assume that $\mathbb{Q}=\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ stabilizes $\mathbb{D}$. If $\mathbb{N},\mathbb{M},\widetilde{\mathbb{M}},\widetilde{\mathbb{N}},\widetilde{\mathbb{X}},\widetilde{\mathbb{Y}}\in\mathcal{A}$ and $\mathcal{A}\subset\mathrm{TIC}$ is inverse closed, then the d.c.f. $\mathbb{D}=\mathbb{N}\mathbb{M}^{-1}=\widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ is over $\mathcal{A}$.

Proof: (a) Set $\begin{bmatrix}\widetilde{\mathbb{X}}' & \widetilde{\mathbb{Y}}'\end{bmatrix}:=(\widetilde{\mathbb{X}}\mathbb{M}-\widetilde{\mathbb{Y}}\mathbb{N})^{-1}\begin{bmatrix}\widetilde{\mathbb{X}} & \widetilde{\mathbb{Y}}\end{bmatrix}\in\mathcal{A}$. Then equation $\left[\begin{smallmatrix}\mathbb{M} & \mathbb{Y}'\\ \mathbb{N} & \mathbb{X}'\end{smallmatrix}\right]^{-1}=\left[\begin{smallmatrix}\widetilde{\mathbb{X}}' & -\widetilde{\mathbb{Y}}'\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{smallmatrix}\right]$ is a joint d.c.f. of $\mathbb{D}$ and $\mathbb{Q}$ for some $\mathbb{X}',\mathbb{Y}'\in\mathrm{TIC}$, by Lemma 6.5.8. Because $\mathcal{A}$ is inverse closed, we have $\left[\begin{smallmatrix}\widetilde{\mathbb{X}}' & -\widetilde{\mathbb{Y}}'\\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}}\end{smallmatrix}\right]^{\pm1}\in\mathcal{A}$ too.

(b) The proof is similar to that of (a) and hence omitted. □

Proposition 7.2.17 ($\mathcal{A}$ case) *Let $\mathbb{D}$ have a d.c.f. over $\mathcal{A}$, where $\mathcal{A}\underset{a}{\subset}\mathrm{TIC}$ (cf. Proposition 7.1.10).*

If the elements of (7.50) are chosen from $\mathcal{A}$, then all stabilizing controllers of $\mathbb{D}$ with a (d.c.) internal loop are the ones parametrized in Theorem 7.2.14, and the

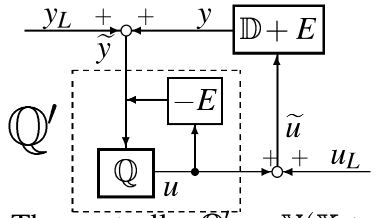


Figure 7.7: The controller $\mathbb{Q}' := \mathbb{Y}(\mathbb{X}+E\mathbb{Y})^{-1} : \widetilde{y} \mapsto u$ for $\mathbb{D}+E$

ones with d.c. internal loop over $\mathcal{A}$ are exactly those with $\mathbb{U} \in \mathcal{A}$. If, in addition, $\mathcal{B} \underset{a}{\subset} \mathcal{A} \underset{a}{\subset}$ ULR$\cap$TIC, then the one with $\mathbb{U} = -M^{-1}T$ is well-posed. □

(The proof is virtually a subset of the proof of Proposition 7.1.10 and hence omitted.)

Recall from Theorem 4.1.6(d), that if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a r.c.f. and $\mathbb{N}, \mathbb{M} \in \mathrm{MTIC}_{T\mathbf{Z}}(\mathbf{C}^n, Y)$, then $\mathbb{D}$ has a d.c.f. over $\mathrm{MTIC}_{T\mathbf{Z}}$, hence then $\mathbb{D}$ has a well-posed stabilizing controller having a d.c.f. over $\mathrm{MTIC}_{T\mathbf{Z}}$, by the above proposition.

Lemma 7.2.18 ($D = 0$ w.l.o.g.) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. Let $E \in \mathcal{B}(U,Y)$.*

Then $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ is a stabilizing DF-controller with d.c. internal loop for $\mathbb{D}$ iff $\mathbb{Q}' = \mathbb{Y}(\mathbb{X}+E\mathbb{Y})^{-1} = (\widetilde{\mathbb{X}}+\widetilde{\mathbb{Y}}E)^{-1}\widetilde{\mathbb{Y}}$ is a stabilizing DF-controller with d.c. internal loop for $\mathbb{D}+E$. The corresponding closed-loop maps $y_L \mapsto u$ (see (7.52)) are identical.

The controller $\mathbb{Q}'$ can be realized by adding to $\mathbb{Q}$ an output feedback through $-E$, as in Figure 7.7.

If one replaces $\mathbb{D}+E$ by a parallel connection of $\mathbb{D}$ and E in Figure 7.7, then it becomes obvious that E and $-E$ cancel each other and we are left with the original connection of $\mathbb{Q}$ and $\mathbb{D}$; this allows one to write down the correspondence between the original and perturbed settings. See also Lemma 7.3.23.

(Note also that one should draw some external inputs "z_L and y_L'" to Figure 7.7 (just before $-E$ and just before $\mathbb{Q}$) and the internal loop (the signals ξ, ξ_L and $\widetilde{\xi}$) of $\mathbb{Q}$ if $\mathbb{Q}$ is non-well-posed.)

Naturally, one of $\mathbb{Q}$ and $\mathbb{Q}'$ may be non-well-posed even if the other is well-posed (but the closed-loop systems are both well-posed if one is).

If $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ and $I+E\mathbb{Q} \in \mathcal{G}\mathrm{TIC}_\infty(Y)$, then one more formula for $\mathbb{Q}'$ is obviously given by $\mathbb{Q}' = (I+\mathbb{Q}E)^{-1}\mathbb{Q}$ $(= \mathbb{Q}(I+E\mathbb{Q})^{-1})$.

Proof: 1° Given any joint d.c.f. (7.46) of $\mathbb{D}$ of $\mathbb{Q}$, a joint d.c.f. of $\mathbb{D}+E$ and $\mathbb{Q}'$ is obviously given by

$$\begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N}+E\mathbb{M} & \mathbb{X}+E\mathbb{Y} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{X}}+\widetilde{\mathbb{Y}}E & -\widetilde{\mathbb{Y}} \\ -(\widetilde{\mathbb{N}}+\widetilde{\mathbb{M}}E) & \widetilde{\mathbb{M}} \end{bmatrix}^{-1} \in \mathcal{G}\mathrm{TIC}(U \times Y). \tag{7.54}$$

By exchanging $\mathbb{D}$ and $\mathbb{D}+E$, we obtain from Theorem 7.2.14(iii) that the stabilizing DPF-controllers for $\mathbb{D}$ and $\mathbb{D}+E$ correspond to each other as in the statement of the lemma.

Given $\mathbb{Q}$, the map $\mathbb{Y}\widetilde{\mathbb{M}} = \mathbb{M}\widetilde{\mathbb{Y}} : y_L \mapsto u$ is common for both closed-loop systems, by (7.52) (since $\mathbb{M}, \widetilde{\mathbb{M}}, \mathbb{Y}, \widetilde{\mathbb{Y}}$ are unaffected). (N.B. if we fix some representation $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ of $\mathbb{Q}$, then $\begin{bmatrix} \mathbb{M}\widetilde{\mathbb{Y}}-I & \mathbb{M} \\ \mathbb{M}\widetilde{\mathbb{Y}} & \mathbb{M}-I \end{bmatrix} : \begin{bmatrix} y_L \\ \xi_L \end{bmatrix} \mapsto \begin{bmatrix} u \\ \xi \end{bmatrix}$ is unaffected, by (7.44).)

2° Thus, the "rigorous" part of the proof is complete, and we only have to show that $\mathbb{Q}'$ is a model for the map in Figure 7.7, i.e., that the maps $u_L, y_L \mapsto u, y$ for $\mathbb{Q}'$ and $\mathbb{D}+E$ become equal to those obtained by solving the equations modeled in the figure.

By writing the equations for $\widetilde{u}$, $\widetilde{y}$ and $\widetilde{\xi}$, one obtains

$$(I-\mathbb{D}^{o\prime})\begin{bmatrix} u+u_L \\ y+y_L \\ \xi+\xi_L \end{bmatrix} = \begin{bmatrix} \widetilde{u}_L \\ \widetilde{y}_L \\ \widetilde{\xi}_L+\widetilde{\mathbb{Y}}(y_L'+Ez_L) \end{bmatrix}, \tag{7.55}$$

(at the moment we are not interested in the additional inputs z_L and y_L') where

$$\mathbb{D}^{o\prime} := \begin{bmatrix} 0 & 0 & I \\ \mathbb{D} & 0 & 0 \\ 0 & \widetilde{\mathbb{Y}} & I-(\widetilde{\mathbb{X}}+\widetilde{\mathbb{Y}}E) \end{bmatrix}, \tag{7.56}$$

i.e., $\mathbb{D}^{o\prime}$ has $\mathbb{D}+E$ in place of $\mathbb{D}$ and $\mathbb{O}' = \begin{bmatrix} 0 & I \\ \widetilde{\mathbb{Y}} & I-(\widetilde{\mathbb{X}}+\widetilde{\mathbb{Y}}E) \end{bmatrix}$ (a representative of $\mathbb{Q}'$) in place of $\mathbb{O} = \begin{bmatrix} 0 & I \\ \widetilde{\mathbb{Y}} & I-\widetilde{\mathbb{X}} \end{bmatrix}$ (a representative of $\mathbb{Q}$).

But once we let the additional inputs y_L' and z_L be zero, equation (7.55) becomes $(I-\mathbb{D}^{o\prime})\begin{bmatrix} u \\ y \\ \xi \end{bmatrix} = \mathbb{D}^{o\prime}\begin{bmatrix} u_L \\ y_L \\ \xi_L \end{bmatrix}$, i.e., the equation (7.19) for $\mathbb{D}+E$ and $\mathbb{O}'$. Thus, we can consider $\mathbb{O}'$ as a model for the controller (in the dashed square) in Figure 7.7. Summarizing, the map $\mathbb{Q} \mapsto \mathbb{Q}'$ corresponds to an output feedback through $-E$. □

From Remark 6.7.19 we deduce that if $\mathbb{D}$ is replaced by $\mathcal{T}_\omega\mathbb{D}$ and $\mathbb{Q}$ by $\mathcal{T}_\omega\mathbb{Q}$ for some $\omega \in \mathbf{R}$, then $\mathbb{D}_I^o$ becomes replaced by $\mathcal{T}_\omega\mathbb{D}_I^o$. From this we conclude the following:

Remark 7.2.19 (Exponential stabilization) *By Remark 6.7.19, from any claims in this section (and others), we can deduce the corresponding results about ω-stabilization for some $\omega \in \mathbf{R}$ (instead of the (0-)stabilization treated in most above results), hence also for exponential stabilization.*

For example, assume that $\mathbb{D} \in \mathrm{TIC}_\infty$ has an exponentially stable d.c.f., say (7.50) (i.e., the maps in (7.50) belong to $\mathrm{TIC}_{\mathrm{exp}}$). Then the maps DF-stabilize $\mathbb{D}$ exponentially with internal loop are exactly the maps with (exponentially) d.c. internal loop parametrized in Theorem 7.2.14 (where we must require the parameters to be exponentially stable).

Remark 7.2.20 (Plants with internal loop) *We could, of course, study more general plants, those with internal loop. One easily (though sometimes with tedious applications of Lemma A.1.1) generalizes most results of this section to the case where both $\mathbb{D}$ and $\mathbb{Q}$ have internal loops, e.g., if $\mathbb{D} = \widetilde{\mathbb{N}}\widetilde{\mathbb{M}}^{-1}$ is a map with l.c. internal loop, then $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ stabilizes $\mathbb{D}$ iff $\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}$. This way one could cover all "$\mathrm{H}^\infty/\mathrm{H}^\infty$" transfer functions (the quotient field of H^∞) and more.*

Notes

Controllers with internal loop were first introduced in [WC], which covers also some corresponding state-space theory for regular WPLSs. This notion was further developed in the frequency-domain article [CWW01]. Our theory was built on an early form of [CWW01], which we were given in late 1996. The actual article will be published late 2001.

Part (c1) of Theorem 7.2.3 is Theorem 7.4 of [WR00]. Lemma 7.2.7 is at least partially contained in Section 6 of [WC]. Proposition 7.2.5(b), Lemma 7.2.10(a)&(b), Corollary 7.2.13 and Corollary 7.2.15 are at least implicitly contained in [CWW01] (some of them with different proofs). Part (d) of Theorem 7.2.4 was written as a generalization of the corresponding classical result (see, e.g., Lemma 12.1 of [ZDG]). Proposition 5.3 of [WC] seems to be its analogy for exponential DF-stabilization with internal loop.

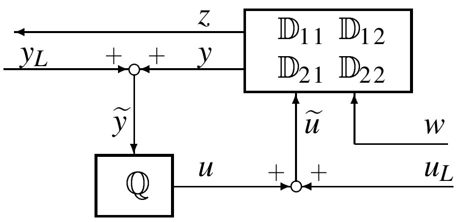


Figure 7.8: DPF-controller $\mathbb{Q}$ for $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$

7.3 DPF-stabilization ($\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})$)

Sir, it's very possible this asteroid is not stable.
— C3P0

In Section 7.1, we studied dynamic output-feedback stabilization by a well-posed controller and, in Section 7.2, that by a controller with internal loop (such controllers contain all well-posed controllers).

In this section we shall use those results to obtain a theory for dynamic partial feedback (with internal loop; we also give some further results for the special case of a well-posed controller).

A general DPF-controller differs from the special case of a DF-controller in the sense that the former has only a partial access to the inputs and outputs of the plant, as illustrated in Figure 7.8 (cf. to Figure 7.1).

In the important H$^\infty$ *Four-Block Problem (*H$^\infty$ *4BP)* (or the standard H$^\infty$ problem) of Chapter 12, one tries to find a DPF-controller that stabilizes the plant and makes the norm $\|w \mapsto z\|$ in Figure 7.8 less than a given constant $\gamma > 0$. This problem is the main motivator of the theory of this chapter. The signal y can be considered as a measure accessible for the controller and u as the controller output, whereas w often represents the disturbances in a system and z stands for the actual (objective) output.

Our choice to have u before w is contrary to the standard practice in DPF-stabilization and the H$^\infty$ 4BP theory (this corresponds to $\begin{bmatrix} \mathbb{D}_{12} & \mathbb{D}_{11} \\ \mathbb{D}_{22} & \mathbb{D}_{21} \end{bmatrix}$ in place of $\mathbb{D}$), which is better suited for DPF duality results.

However, our choice is the standard practice in the H$^\infty$ FICP theory (see Chapter 11), being more natural for that theory (e.g., it allows us to have I's on the diagonal in several FICP and 4BP formulae).

Therefore, when comparing the formulae to most studies on DPF-stabilization (e.g., [Francis], [Keu], [Green] or [ZDG]), one has to interchange the (second) indices corresponding u and w, whereas the FICP results (e.g., [S98d], [Green], [CG97], [LT00a]) can directly be compared.

If we delete the rest of $\mathbb{D}$ except $\mathbb{D}_{21}$ in Figure 7.10, we end up with Figure 7.3. Therefore, the maps $u_L, y_L \mapsto u, y$ become the same as in the DF-stabilization of $\mathbb{D}_{21}$, and the map of u_L, w, y_L to z, y, u is obtained from this and the equation

$$\begin{bmatrix} z \\ y \end{bmatrix} = \mathbb{D} \begin{bmatrix} u + u_L \\ w \end{bmatrix}. \tag{7.57}$$

In particular, the controller is admissible for $\mathbb{D}$ iff it is admissible for $\mathbb{D}_{21}$.

(Note that we use w instead of $w_L = w$, because there is no feedback to the disturbance signal w. Models should contain additional inputs representing the disturbances in each loop, but since there is no feedback (loop) for w, such an additional input would be redundant. The situation with z is similar.)

However, it is easiest to identify any DPF-controller $\widetilde{\mathbb{Q}} \in \mathrm{TIC}_\infty(Y,U)$ with the DF-controller $\mathbb{Q} : \left[\begin{smallmatrix} \widetilde{z} \\ \widetilde{y} \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ of form $\mathbb{Q} = \left[\begin{smallmatrix} 0 & \widetilde{\mathbb{Q}} \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(Z \times Y, U \times W)$ (so that $\widetilde{\mathbb{Q}} \in \mathrm{TIC}_\infty(Y,U)$ maps $\widetilde{y} \mapsto u$, and $\widetilde{z}$ and w are uncoupled from the controller). Obviously, this definition is equivalent to the one above. Its rigorous form is contained in the following definition (the case $\mathbb{O} = \left[\begin{smallmatrix} \widetilde{\mathbb{Q}} & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$), which also covers the case with internal loop:

Definition 7.3.1 (DPF-stabilization [with internal loop], $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})$) *Let* $\mathbb{D} = \left[\begin{smallmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$. *We call* $\mathbb{O} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ *(where also* Ξ *is a Hilbert space) an* admissible (resp. stabilizing) DPF-controller with internal loop *for* $\mathbb{D}$ *if*

$$\mathbb{O}_{\mathrm{DF}} := \begin{bmatrix} 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ 0 & 0 & 0 \\ 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty((Z \times Y) \times \Xi, (U \times W) \times \Xi) \tag{7.58}$$

is an admissible (resp. stabilizing) DF-controller with internal loop for $\mathbb{D}$.

We call $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{O} \end{array}\right] \in \mathrm{WPLS}(Y \times \Xi, \widetilde{H}, U \times \Xi)$ *an* admissible (resp. stabilizing) DPF-controller with internal loop for $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Z \times Y)$ *if*

$$\widetilde{\Sigma}_{\mathrm{DF}} := \left[\begin{array}{c|ccc} \widetilde{\mathbb{A}} & 0 & \widetilde{\mathbb{B}}_1 & \widetilde{\mathbb{B}}_2 \\ \hline \widetilde{\mathbb{C}}_1 & 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ 0 & 0 & 0 & 0 \\ \widetilde{\mathbb{C}}_2 & 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{array}\right] \in \mathrm{WPLS}(Z \times Y \times \Xi, \widetilde{H}, U \times W \times \Xi) \tag{7.59}$$

is an admissible (resp. stabilizing) DF-controller with internal loop for Σ.

In either case, by $\mathcal{F}_\ell(\mathbb{D},\mathbb{O})$ *we denote the map* $w \mapsto z$ *of* $\mathbb{D}_I^o$ *(see (7.64), (7.68) and (7.98)).*

We call two admissible DPF-controllers with internal loop for $\mathbb{D}$ *(resp. for* Σ*)* equivalent for $\mathbb{D}$ *(resp. for* Σ*) if they determine same maps from* u_L, y_L *to* u, y.

We call $\mathbb{D}$ *(resp.* Σ*)* DPF-stabilizable with internal loop *if there is a stabilizing DPF-controller with internal loop for* $\mathbb{D}$ *(resp. for* Σ*). and we use prefices as above. (We use prefices as in Definition 7.2.1.)*

If $\mathbb{O}_{\mathrm{DF}}$ *is a well-posed DF-controller (equivalently,* $\mathbb{O} = \left[\begin{smallmatrix} \mathbb{O}_{11} & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$*), then we may remove the words "with internal loop" everywhere above in this definition and identify* $\mathbb{O}$ *with* $\mathbb{O}_{11} \in \mathrm{TIC}_\infty(Y,U)$.

If $\mathbb{Q}$ *is a map with coprime internal loop, then we call* $\mathbb{Q}$ *an* admissible (resp. stabilizing) DPF-controller with coprime internal loop for $\mathbb{D}$ *if* $\left[\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right]$ *is an admissible (resp. stabilizing) DF-controller with coprime internal loop for* $\mathbb{D}$.

As before, "*[DPF-]stabilizes*" means "is [DPF-]stabilizing for", in any of the above settings. (We use the prefix "DPF-" whenever there is a risk of confusion.)

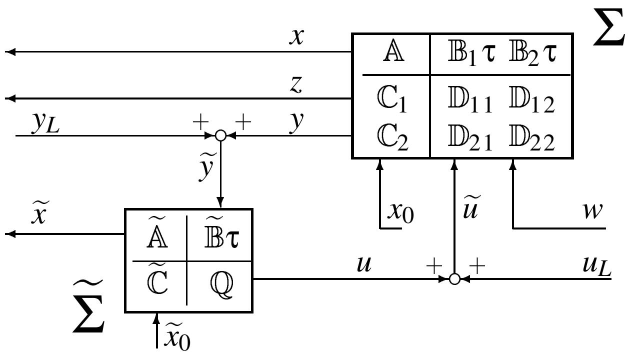


Figure 7.9: DPF-controller $\widetilde{\Sigma}$ for $\Sigma \in \mathrm{WPLS}(U \times W, H, Z \times Y)$

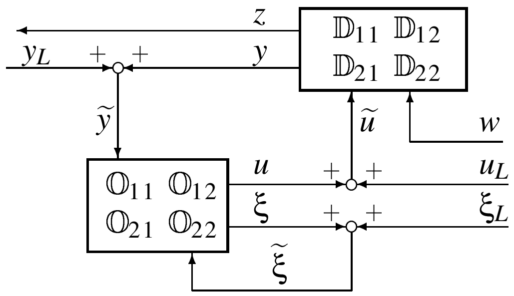


Figure 7.10: DPF-controller $\mathbb{O}$ with internal loop for $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$

Lemma 7.3.10 shows that also the coprime part of Definition 7.3.1 is justified.

Note that $\mathbb{D}^o_I$ maps $(u_L, w, z_L, y_L, \xi_L) \mapsto (u, w, z, y, \xi)$ (cf. (7.63) and recall that $\mathbb{D}^o_I = (I - \mathbb{D}^o)^{-1} - I$). Therefore, $\mathbb{O}$ stabilizes $\mathbb{D}$ iff $u, (w,) z, y, \xi \in \mathrm{L}^2$ for all $u_L, w, z_L, y_L, \xi_L \in \mathrm{L}^2$. See also Figures 7.10 and 7.11 and the comments below Definition 7.1.1 and Summary 6.7.1.

The combined open-loop system of (7.21) corresponding to the DF-controller $\widetilde{\Sigma}_{\mathrm{DF}}$ with internal loop for Σ (i.e., the DPF-controller $\widetilde{\Sigma}$ with internal loop for Σ), is obviously given by

$$\Sigma^o := \left[\begin{array}{cc|ccccc} \mathbb{A} & 0 & \mathbb{B}_1 & \mathbb{B}_2 & 0 & 0 & 0 \\ 0 & \widetilde{\mathbb{A}} & 0 & 0 & 0 & \widetilde{\mathbb{B}}_1 & \widetilde{\mathbb{B}}_2 \\ \hline 0 & \widetilde{\mathbb{C}}_1 & 0 & 0 & 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbb{C}_1 & 0 & \mathbb{D}_{11} & \mathbb{D}_{12} & 0 & 0 & 0 \\ \mathbb{C}_2 & 0 & \mathbb{D}_{21} & \mathbb{D}_{22} & 0 & 0 & 0 \\ 0 & \widetilde{\mathbb{C}}_2 & 0 & 0 & 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{array}\right] \tag{7.60}$$

$\in \mathrm{WPLS}(U \times W \times Z \times Y \times \Xi, H \times \widetilde{H}, U \times W \times Z \times Y \times \Xi)$. Thus, $\widetilde{\Sigma}$ is an admissible [stabilizing] DPF-controller with internal loop for Σ iff Σ^o_I is well-posed [and stable] (cf. Definition 6.6.4); see (6.125) for the closed-loop system Σ^o_I.

If $\mathbb{O} = \left[\begin{smallmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{smallmatrix}\right]$, then we can simplify the above definition as follows:

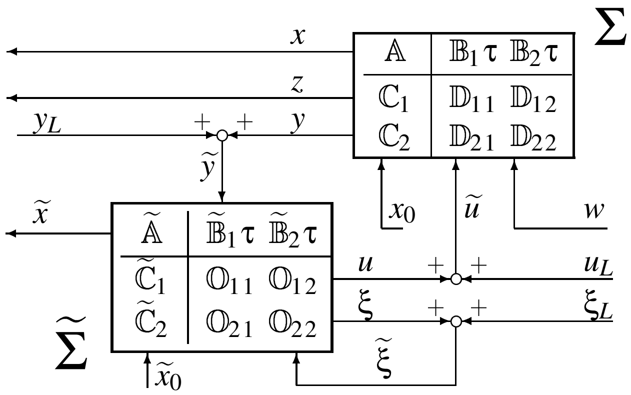


Figure 7.11: DPF-controller $\widetilde{\Sigma}$ with internal loop for $\Sigma \in \mathrm{WPLS}(U \times W, H, Z \times Y)$

Lemma 7.3.2 (Well-posed DPF-controllers) *A (well-posed) DPF-controller $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ is admissible [stabilizing] for $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ iff $L := I$ is admissible [stabilizing] for $\begin{bmatrix} 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 \\ \mathbb{D} & 0 & 0 \end{bmatrix} \mathrm{TIC}_\infty(U \times W \times Y, U \times Z \times Y)$; all prefices apply.*

A (well-posed) DPF-controller $\widetilde{\Sigma} \in \mathrm{WPLS}(Y, \widetilde{H}, U)$ is admissible [stabilizing] for $\Sigma \in \mathrm{WPLS}(U \times W, H, Z \times Y)$ iff $L := I$ is admissible [stabilizing] for

$$\Sigma^o := \left[\begin{array}{cc|ccc} \mathbb{A} & 0 & \mathbb{B} & 0 & 0 \\ 0 & \widetilde{\mathbb{A}} & 0 & 0 & \widetilde{\mathbb{B}} \\ \hline 0 & \widetilde{\mathbb{C}} & 0 & 0 & \mathbb{Q} \\ 0 & 0 & 0 & 0 & 0 \\ \mathbb{C} & 0 & \mathbb{D} & 0 & 0 \end{array}\right] \in \mathrm{WPLS}(U \times W \times Z \times Y, H \times \widetilde{H}, U \times W \times Z \times Y); \tag{7.61}$$

all prefices apply. In either setting, admissibility is equivalent to condition $I - \mathbb{Q}\mathbb{D}_{21} \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

The last condition is equivalent to "$I - QD_{21} \in \mathcal{G}\mathcal{B}(U)$" if $\mathbb{Q}, \mathbb{D}_{21} \in \mathrm{ULR}$, by Proposition 6.3.1(c).

Cf. again Figures 7.8 and 7.9 to Figures 7.10 and 7.11, respectively. Note also that (7.61) equals (7.21) for Σ and

$$\widetilde{\Sigma}' := \left[\begin{array}{c|cc} \widetilde{\mathbb{A}} & 0 & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & 0 & \mathbb{Q} \\ 0 & 0 & 0 \end{array}\right] \in \mathrm{WPLS}(Z \times Y, \widetilde{H}, U \times W) \tag{7.62}$$

i.e., it corresponds to the DF-controller $\widetilde{\Sigma}'$ for Σ.

Proof: Denote (7.60) by $\Sigma^{o\prime}$. Obviously, $\Sigma^{o\prime} = \begin{bmatrix} \Sigma^o & 0 \\ 0 & 0 \end{bmatrix}$. One easily verifies that $L = I := I_{U \times W \times Z \times Y}$ is admissible [stabilizing] for Σ^o iff $\begin{bmatrix} I & 0 \\ 0 & I_\Xi \end{bmatrix}$ is admissible [stabilizing] for $\Sigma^{o\prime}$ (because $\Sigma^{o\prime}_I = \begin{bmatrix} \Sigma^o_I & 0 \\ 0 & 0 \end{bmatrix}$, by (6.125); from this we also observe that all prefices of Definition 6.6.4 apply).

Condition $I - \mathbb{Q}\mathbb{D}_{21} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ can be obtained from Lemmas 7.3.5 and 7.1.2 (or from a direct computation). □

Lemma 7.3.3 (DPF-controllers with IL) *A map* $\mathbb{O} = \left[\begin{smallmatrix} \mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22} \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ *is an admissible [stabilizing] DPF-controller with internal loop for* $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ *iff the connection if Figure 7.10 is well-posed [and stable], equivalently, iff*

$$\mathbb{D}^o := \begin{bmatrix} 0 & 0 & 0 & \mathbb{O}_{11} & \mathbb{O}_{12} \\ 0 & 0 & 0 & 0 & 0 \\ \mathbb{D}_{11} & \mathbb{D}_{12} & 0 & 0 & 0 \\ \mathbb{D}_{21} & \mathbb{D}_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{O}_{21} & \mathbb{O}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W \times Z \times Y \times \Xi) \tag{7.63}$$

satisfies $I - \mathbb{D}^o \in \mathcal{G}\mathrm{TIC}_\infty$ *[and* $(I - \mathbb{D}^o)^{-1} \in \mathrm{TIC}$*].* □

(This follows from Lemma 7.2.2.)

Analogously, $\widetilde{\Sigma}$ is admissible [stabilizing] for Σ iff the closed-loop system (Σ_I^o; cf. (7.60)) in Figure 7.11 is well-posed [and stable, i.e., $u, y, z, \xi \in \mathrm{L}^2$ and x and $\widetilde{x}$ are bounded for all $u_L, w, y_L, \xi_L \in \mathrm{L}^2(\mathbf{R}_+; *)$, $x_0 \in H$ and $\widetilde{x}_0 \in \widetilde{H}$]. (We note that exponential stability is equivalent to $x, \widetilde{x} \in \mathrm{L}^2$ (and hence $u, y, z, \xi \in \mathrm{L}^2$) for all $u_L, w, y_L, \xi_L \in \mathrm{L}^2$, $x_0 \in H$ and $\widetilde{x} \in \widetilde{H}$, by Lemma A.4.5 and Lemma 6.1.10(a1).)

As before, we identify a well-posed controller $\mathbb{Q} \in \mathrm{TIC}_\infty(Y, U)$ with the controller $\left[\begin{smallmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ with internal loop. From (7.58) we observe that $\mathbb{Q} \in \mathrm{TIC}_\infty$ is an admissible [stabilizing] DPF-controller for $\mathbb{D}$ iff $\left[\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(Z \times Y, U \times W)$ is an admissible [stabilizing] (DF-)controller for $\mathbb{D}$ (obviously, (7.58) is a well-posed DF-controller iff $\mathbb{O}$ is a well-posed DPF-controller; see the end of Definition 7.2.1). This can be compared to Figure 7.8, where $\left[\begin{smallmatrix} \widetilde{u} \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} z \\ y + y_L \end{smallmatrix}\right] + \left[\begin{smallmatrix} u_L \\ w \end{smallmatrix}\right]$, whereas $\widetilde{u} = \mathbb{Q}(y + y_L) + u_L$ in Figure 7.1; the differences are explained by the facts that we need no z_L and that $w = w_L = \widetilde{w}$, due to lack of feedback in these loops.

Obviously, a (well-posed) map $\mathbb{Q} \in \mathrm{TIC}_\infty$ is admissible iff $(I - \mathbb{D}_{21}\mathbb{Q}) \in \mathcal{G}\mathrm{TIC}_\infty$. The closed loop map $w \mapsto z$ (from the second input to the first output) is given by the standard linear fractional transformation formula

$$\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q}) := \mathbb{D}_{12} + \mathbb{D}_{11}\mathbb{Q}(I - \mathbb{D}_{21}\mathbb{Q})\mathbb{D}_{22} = \mathcal{F}_\ell(\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]\mathbb{D}^{\mathrm{d}}\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right], \mathbb{Q}^{\mathrm{d}})^{\mathrm{d}}. \tag{7.64}$$

The full map $u_L, w, y_L \mapsto u, z, y$ is given by

$$\begin{bmatrix} (I - \mathbb{Q}\mathbb{D}_{21})^{-1} & (I - \mathbb{Q}\mathbb{D}_{21})^{-1}\mathbb{Q}\mathbb{D}_{22} & (I - \mathbb{Q}\mathbb{D}_{21})^{-1}\mathbb{Q} \\ \mathbb{D}_{11}(I - \mathbb{Q}\mathbb{D}_{21})^{-1} & \mathbb{D}_{12} + \mathbb{D}_{11}(I - \mathbb{Q}\mathbb{D}_{21})^{-1}\mathbb{Q}\mathbb{D}_{22} & \mathbb{D}_{11}(I - \mathbb{Q}\mathbb{D}_{21})^{-1}\mathbb{Q} \\ \mathbb{D}_{21}(I - \mathbb{Q}\mathbb{D}_{21})^{-1} & \mathbb{D}_{22} + (I - \mathbb{D}_{21}\mathbb{Q})^{-1}\mathbb{D}_{22} & (I - \mathbb{D}_{21}\mathbb{Q})^{-1} \end{bmatrix}. \tag{7.65}$$

Thus, all admissibility results of Sections 7.2 and 7.1 are valid (for DPF) with $\mathbb{D}_{21}$ in place of $\mathbb{D}$, but for the stabilizability, we must add the requirement that the maps to z and the maps from w also become stable.

We usually study only DPF-controllers with coprime internal loop, because the standard stabilizability and detectability assumptions for the H^∞ 4BP imply that no other controllers stabilize the plant (assuming sufficient regularity or a discrete-time setting; cf. Section 12.5, Lemmas 12.6.6 and 12.5.3 and Theorem 7.3.19), and because the general case is rather complex, as shown in the following proposition:

Proposition 7.3.4 ($\mathcal{F}_\ell(\mathbb{D},\mathbb{O})$) *Let $\mathbb{D}$ be as in Definition 7.3.1. Let $\mathbb{O}$:=(7.58).*

(a) Then $\mathbb{O}$ is admissible with internal loop for $\mathbb{D}$ iff

$$\mathbb{H} := \begin{bmatrix} I_Z & -\mathbb{D}_{11}\mathbb{O}_{11} & -\mathbb{D}_{11}\mathbb{O}_{12} \\ 0 & I_Y - \mathbb{D}_{21}\mathbb{O}_{11} & -\mathbb{D}_{21}\mathbb{O}_{12} \\ 0 & -\mathbb{O}_{21} & I_\Xi - \mathbb{O}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(Z \times Y \times \Xi); \tag{7.66}$$

equivalently, iff

$$\mathbb{R} := \begin{bmatrix} I_U - \mathbb{O}_{11}\mathbb{D}_{21} & -\mathbb{O}_{11}\mathbb{D}_{22} & -\mathbb{O}_{12} \\ 0 & I_W & 0 \\ -\mathbb{O}_{21}\mathbb{D}_{21} & -\mathbb{O}_{21}\mathbb{D}_{22} & I_\Xi - \mathbb{O}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}_\infty(U \times W \times \Xi). \tag{7.67}$$

(b) $\mathbb{O}$ is [exponentially] DPF-stabilizing with internal loop for $\mathbb{D}$ iff (7.27) is [exponentially] stable (equivalently, iff (7.28) is [exponentially] stable).

(c) $\mathbb{O}$ is admissible (resp. [exponentially] DPF-stabilizing) with internal loop for $\mathbb{D}$ iff $\mathbb{O}$ is admissible (resp. [exponentially] DPF-stabilizing) for $\left[\begin{smallmatrix} \mathbb{D}_{11} & 0 & \mathbb{D}_{12} \\ \mathbb{D}_{21} & 0 & \mathbb{D}_{22} \\ 0 & I & 0 \end{smallmatrix}\right]$.

(d) $\mathbb{O}$ is admissible (resp. [exponentially] DPF-stabilizing) with internal loop for $\mathbb{D}$ iff $\mathbb{O}^{\mathrm{d}}$ is admissible (resp. [exponentially] DPF-stabilizing) with internal loop for $\mathbb{D}_{\mathrm{d}} := \left[\begin{smallmatrix} \mathbb{D}_{22}^{\mathrm{d}} & \mathbb{D}_{12}^{\mathrm{d}} \\ \mathbb{D}_{21}^{\mathrm{d}} & \mathbb{D}_{11}^{\mathrm{d}} \end{smallmatrix}\right]$. If this is the case, then $\mathcal{F}_\ell(\mathbb{D},\mathbb{O})^{\mathrm{d}} = \mathcal{F}_\ell(\mathbb{D}_{\mathrm{d}},\mathbb{O}^{\mathrm{d}})$.

(e) If $\mathbb{O}$ is admissible with internal loop for $\mathbb{D}$, then $\mathcal{F}_\ell(\mathbb{D},\mathbb{O}) : w \mapsto z$ is given by

$$\mathcal{F}_\ell(\mathbb{D},\mathbb{O}) = \mathbb{D}_{12} + (\mathbb{H}^{-1})_{12}\mathbb{D}_{22} = \mathbb{D}_{12} + \mathbb{D}_{11}(\mathbb{R}^{-1})_{12} \in \mathrm{TIC}_\infty(W,Z). \tag{7.68}$$

The map $w \mapsto u$ is given by $(\mathbb{R}^{-1})_{12}$.

If $\mathbb{O} = \left[\begin{smallmatrix} \mathbb{Q} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for some $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ (i.e., $\mathbb{O}$ is well-posed), then $\mathcal{F}_\ell(\mathbb{D},\mathbb{O}) = \mathbb{D}_{12} + \mathbb{D}_{11}\mathbb{Q}(I-\mathbb{D}_{21}\mathbb{Q})^{-1}\mathbb{D}_{22}$, by (7.65). For DPF-controllers with d.c. internal loop, a third formula for $\mathcal{F}_\ell$ is given in Corollary 7.3.20(c).

Proof: (a)&(b) Substitute (7.58) for $\mathbb{O}$ in Proposition 7.2.5(a)&(a') to obtain (a)&(b).

(c) By Definition 7.3.1, $\mathbb{O}$ is DPF-admissible with internal loop for $\mathbb{D}$ iff (7.58) is DF-admissible with internal loop for $\mathbb{D}$. By Proposition 7.2.5(c), this is the case iff (7.58) is DF-admissible for $\left[\begin{smallmatrix} \mathbb{D} & 0 \\ 0 & I \end{smallmatrix}\right]$; equivalently, iff

$$\begin{bmatrix} 0 & \mathbb{O} \\ 0 & 0 \end{bmatrix} \text{ is DF-admissible for } \begin{bmatrix} \mathbb{D}_{11} & 0 & \mathbb{D}_{12} \\ \mathbb{D}_{21} & 0 & \mathbb{D}_{22} \\ 0 & I & 0 \end{bmatrix} =: \underline{\mathbb{D}}. \tag{7.69}$$

As noted below Definition 7.3.1, this is the case iff $\mathbb{O}$ is DPF-admissible for $\underline{\mathbb{D}}$.

Replace "admissible" by "[exponentially] stabilizing" above to obtain the proof of remaining claims.

(d) We have $\mathbb{R} \in \mathcal{G}\mathrm{TIC}_\infty$ iff $\mathbb{R}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}_\infty$. Exchange the first and second rows and exchange the first and second columns of $\mathbb{R}^{\mathrm{d}}$ to obtain $\mathbb{H}$ with $\mathbb{D}_{\mathrm{d}}$ in place of $\mathbb{D}$ and $\mathbb{O}^{\mathrm{d}}$ in place of $\mathbb{O}$. This proves the admissibility claim, and from this we also observe that $(\mathbb{R}^{-1}_{\mathbb{D},\mathbb{O}})_{12} = (\mathbb{H}^{-1}_{\mathbb{D}_{\mathrm{d}},\mathbb{O}^{\mathrm{d}}})^{\mathrm{d}}_{12}$, hence (use (c) twice)

$$\mathcal{F}_\ell(\mathbb{D}_{\mathrm{d}},\mathbb{O}^{\mathrm{d}})^{\mathrm{d}} = ((\mathbb{D}_{\mathrm{d}})_{12} + (\mathbb{H}^{-1}_{\mathbb{D}_{\mathrm{d}},\mathbb{O}^{\mathrm{d}}})_{12}(\mathbb{D}_{\mathrm{d}})_{22})^{\mathrm{d}} = \mathbb{D}_{12} + \mathbb{D}_{11}(\mathbb{R}^{-1}_{\mathbb{D},\mathbb{O}})_{12} = \mathcal{F}_\ell(\mathbb{D},\mathbb{O}). \tag{7.70}$$

Similarly, one observes from Proposition 7.2.5(a)&(a') that (7.28) is stable iff (7.27) is stable after the substitutions $\mathbb{D} \mapsto \mathbb{D}_{\mathrm{d}}$, $\mathbb{O} \mapsto \mathbb{O}^{\mathrm{d}}$ (this requires just a bit more reordering).

(Note that except for (7.70), part (d) is also contained in Lemma 6.7.2(f').)

(e) The symbols of Proposition 7.2.5 are now denoted as follows (cf. Definition 7.3.1): we have $u_L \mapsto \begin{bmatrix} u_L \\ w \end{bmatrix}$, $u \mapsto \begin{bmatrix} u \\ w \end{bmatrix}$ and $y \mapsto \begin{bmatrix} z \\ y \end{bmatrix}$. The map "$u_L \mapsto \begin{bmatrix} y \\ \xi \end{bmatrix}$" given by $\mathbb{H}^{-1}\begin{bmatrix} \mathbb{D} \\ 0 \end{bmatrix}$, by (7.27), hence $w \mapsto z$ is given by

$$(\mathbb{H}^{-1}\begin{bmatrix} \mathbb{D} \\ 0 \end{bmatrix})_{12} = (\mathbb{H}^{-1})_{11}\mathbb{D}_{12} + (\mathbb{H}^{-1})_{12}\mathbb{D}_{22} = \mathbb{D}_{12} + (\mathbb{H}^{-1})_{12}\mathbb{D}_{22}, \tag{7.71}$$

since now $(\mathbb{H}^{-1})_{11} = I$, by Lemma A.1.1(b1)&(b2). Analogously, $w \mapsto z$ is given by $\mathbb{D}_{11}(\mathbb{R}^{-1})_{12} + \mathbb{D}_{12}(\mathbb{R}^{-1})_{22} = \mathbb{D}_{11}(\mathbb{R}^{-1})_{12} + \mathbb{D}_{12}$, by (7.28). Obviously, $w \mapsto u$ is given by $(\mathbb{R}^{-1})_{12}$, by (7.28). □

For ease of reference, we collect into a lemma some remarks made above (more or less explicitly):

Lemma 7.3.5 ($\mathbb{O}$ DPF-stabilizes $\mathbb{D}$ $\Rightarrow$ $\mathbb{O}$ DF-stabilizes $\mathbb{D}_{21}$) *Let* $\mathbb{D} = \begin{bmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix} \in \mathrm{TIC}(U \times W, Z \times Y)$ *and* $\mathbb{O} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$. *Let* Σ *and* $\widetilde{\Sigma}$ *be realizations of* $\mathbb{D}$ *and* $\mathbb{O}$*, respectively. Then the following are equivalent:*

(i) $\mathbb{O}$ *is an admissible DPF-controller with internal loop for* $\mathbb{D}$*;*

(ii) $\mathbb{O}$ *is an admissible DF-controller with internal loop for* $\mathbb{D}_{21}$*;*

(iii) $\widetilde{\Sigma}$ *is an admissible DPF-controller with internal loop for* Σ*;*

(iv) $\widetilde{\Sigma}$ *is an admissible DF-controller with internal loop for* $\Sigma_{21} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_2 & \mathbb{D}_{21} \end{array}\right]$*;*

(v) $I - \mathbb{D}^o \in \mathcal{G}\mathrm{TIC}_\infty(U \times W \times Z \times Y \times \Xi)$.

Moreover, if $\mathbb{O}$ *DPF-stabilizes* $\mathbb{D}$*, then* $\mathbb{O}$ *DF-stabilizes* $\mathbb{D}_{21}$*; if* $\widetilde{\Sigma}$ *DPF-stabilizes* Σ*, then* $\widetilde{\Sigma}$ *DF-stabilizes* Σ_{21} *(all prefices apply, because the system "*Σ_I^o*" for* Σ_{21} *and* $\widetilde{\Sigma}$ *(cf. (7.21)) is a part of the system "*Σ_I^o*" for* Σ *and (7.59)).*

The converse to the last claim is not true in general (take, e.g., $\mathbb{O} = 0 = \mathbb{D}_{21}$, $\mathbb{D}$ unstable; cf. also Example 7.3.7), but it is true when, e.g., Σ_{21} is optimizable and estimatable; see Lemma 7.3.6 and Theorem 7.3.19.

Proof: (Naturally, the lemma still remains true if we throughout the lemma remove the phrases "with internal loop", since $\mathbb{O}$ is a well-posed DPF-controller for Σ iff $\mathbb{O}$ is a well-posed DF-controller for Σ_{21} (iff $\mathbb{O} = \begin{bmatrix} \mathbb{O}_{11} & 0 \\ 0 & 0 \end{bmatrix}$). Note also that the "resp." part is not defined for $\widetilde{\Sigma}$ in place of $\mathbb{O}$.)

Let $\widetilde{\mathbb{H}} \in \mathrm{TIC}_\infty(Y \times \Xi)$ be the map $\mathbb{H}$ for $\mathbb{O}$ and $\mathbb{D}_{21}$ from Proposition 7.2.5(a).

1° *Admissibility*: We observe from Proposition 7.3.4(a) (and Lemma A.1.1(b)), that $\mathbb{H} \in \mathcal{G}\mathrm{TIC}_\infty$ iff $\widetilde{\mathbb{H}} \in \mathcal{G}\mathrm{TIC}_\infty$, and that

$$\mathbb{H}^{-1} = \begin{bmatrix} I_Z & \begin{bmatrix} * & * \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \widetilde{\mathbb{H}}^{-1} \end{bmatrix} \in \mathrm{TIC}_\infty(Z \times (Y \times \Xi)) \tag{7.72}$$

(when $\widetilde{\mathbb{H}} \in \mathcal{G}\mathrm{TIC}_\infty(Y \times \Xi)$). Therefore, (i) and (ii) are equivalent. From Lemma 7.2.2 (and Definition 7.3.1) we obtain "(ii)$\Leftrightarrow$(v)", "(ii)$\Leftrightarrow$(iv)" and "(iii)$\Leftrightarrow$(iv)" (since (7.58) is the I/O map of (7.59).

2° *We shall show that $\widetilde{\Sigma}_I^o$ is obtained by removing the fourth and fifth rows and columns (those corresponding to W and Z) from Σ_I^o:* (Here $\widetilde{\Sigma}_I^o$ is the system "Σ_I^o" for Σ_{21} and $\widetilde{\Sigma}$ (cf. (7.21)), and Σ_I^o is the one of Definition 7.3.1, i.e., the closed-loop system of (7.60). Obviously, the above claim holds for $\widetilde{\Sigma}^o$ and Σ^o in place of $\widetilde{\Sigma}_I^o$ and Σ_I^o.)

Assume that $\mathbb{O}$ is admissible for $\mathbb{D}_{21}$. Let $\widetilde{\mathbb{D}}^o \in \mathrm{TIC}_\infty(U \times Y \times \Xi)$ be the map (7.20) (for $\mathbb{D}_{21}$ and $\mathbb{O}$), and define $\mathbb{D}^o$ by (7.63). Set $\widetilde{\mathbb{T}} := (I - \widetilde{\mathbb{D}}^o)^{-1}$. From (7.27) we observe that

$$(I - \mathbb{D}^o)^{-1} = \begin{bmatrix} \widetilde{\mathbb{T}}_{11} & * & 0 & \widetilde{\mathbb{T}}_{12} & \widetilde{\mathbb{T}}_{13} \\ 0 & I & 0 & 0 & 0 \\ * & * & I & * & * \\ \widetilde{\mathbb{T}}_{21} & * & 0 & \widetilde{\mathbb{T}}_{22} & \widetilde{\mathbb{T}}_{23} \\ \widetilde{\mathbb{T}}_{31} & * & 0 & \widetilde{\mathbb{T}}_{32} & \widetilde{\mathbb{T}}_{33} \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W \times Z \times Y \times \Xi). \tag{7.73}$$

(This proves 2° for $\mathbb{D}_I^o := (I - \mathbb{D}^o)^{-1} - I$.) Apply then (6.125) to observe that

$$\mathbb{C}_I^o := (I - \mathbb{D}^o)^{-1}\mathbb{C}^o = \begin{bmatrix} \widetilde{\mathbb{T}}_{12}\mathbb{C}_2 & \widetilde{\mathbb{T}}_{11}\widetilde{\mathbb{C}}_1 + \widetilde{\mathbb{T}}_{13}\widetilde{\mathbb{C}}_2 \\ 0 & 0 \\ * & * \\ \widetilde{\mathbb{T}}_{22}\mathbb{C}_2 & \widetilde{\mathbb{T}}_{21}\widetilde{\mathbb{C}}_1 + \widetilde{\mathbb{T}}_{23}\widetilde{\mathbb{C}}_2 \\ \widetilde{\mathbb{T}}_{32}\mathbb{C}_2 & \widetilde{\mathbb{T}}_{31}\widetilde{\mathbb{C}}_1 + \widetilde{\mathbb{T}}_{33}\widetilde{\mathbb{C}}_2 \end{bmatrix}. \tag{7.74}$$

(Remove the second and third rows to obtain "$\widetilde{\mathbb{C}}_I^o$".) The proof for $\mathbb{B}_I^o$ is analogous. Finally, from (6.125) and (7.4) (for Σ and $\widetilde{\Sigma}'$, so that $\mathbb{B}^o = \left[\begin{smallmatrix} \mathbb{B}_1 & \mathbb{B}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{\mathbb{B}}_1 & \widetilde{\mathbb{B}}_2 \end{smallmatrix}\right]$) we observe that

$$\mathbb{A}_I^o = \mathbb{A}^o + \mathbb{B}^o\tau\mathbb{C}_I^o = \begin{bmatrix} \mathbb{A} & 0 \\ 0 & \widetilde{\mathbb{A}} \end{bmatrix} + \begin{bmatrix} \mathbb{B}_1 & 0 & 0 \\ 0 & \widetilde{\mathbb{B}}_1 & \widetilde{\mathbb{B}}_2 \end{bmatrix} \tau \begin{bmatrix} (\mathbb{C}_I^o)_{11} & (\mathbb{C}_I^o)_{12} \\ (\mathbb{C}_I^o)_{41} & (\mathbb{C}_I^o)_{42} \\ (\mathbb{C}_I^o)_{51} & (\mathbb{C}_I^o)_{52} \end{bmatrix}, \tag{7.75}$$

which equals $\widetilde{\mathbb{A}}_I^o := \mathbb{A}^o + \begin{bmatrix} \mathbb{B}_1 & 0 & 0 \\ 0 & \widetilde{\mathbb{B}}_1 & \widetilde{\mathbb{B}}_2 \end{bmatrix} \tau\widetilde{\mathbb{C}}_I^o$, the semigroup of $\widetilde{\Sigma}_I^o$.

3° *Stabilization:* We observe from 2° that $\widetilde{\Sigma}_I^o \in \mathrm{WPLS}(U \times Y \times \Xi, H \times \widetilde{H}, U \times Y \times \Xi)$ is a part of $\Sigma_I^o \in \mathrm{WPLS}(U \times W \times Z \times Y \times \Xi, H \times \widetilde{H}, U \times W \times Z \times Y \times \Xi)$.

Indeed, $\widetilde{\mathbb{D}}_I^o = (I - \widetilde{\mathbb{D}}^o)^{-1} - I = \widetilde{\mathbb{T}} - I$ is a part of $\mathbb{D}_I^o$, the semigroup $\mathbb{A}_I^o$ is the same for both systems, and $\widetilde{\mathbb{C}}_I^o$ and $\widetilde{\mathbb{B}}_I^o$ are parts of $\mathbb{C}_I^o$ and $\mathbb{B}_I^o$, respectively, as noted above.

Therefore, if $\widetilde{\Sigma}$ is DPF-stabilizing with internal loop for Σ, i.e., Σ_I^o is stable, then also $\widetilde{\Sigma}_I^o$ is stable, since it is a part of Σ_I^o, i.e., then $\widetilde{\Sigma}$ is DPF-stabilizing for Σ_{21}. Analogously, if $\mathbb{O}$ is DPF-stabilizing with internal loop for $\mathbb{D}$, i.e., $\mathbb{D}_I^o$ is stable, then so is $\widetilde{\mathbb{D}}_I^o$. For same reasons, any prefices (e.g., "exponentially", "ω-"; for Σ also "strongly", "internally", "SOS-" etc.) apply. □

It is not exactly the same thing to DPF-stabilize Σ and DF-stabilize Σ_{21}, but pretty close:

Lemma 7.3.6 ($\Sigma \leftrightarrow \Sigma_{21}$)

(a) $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{O} \end{array}\right] \in \mathrm{WPLS}(Y \times \Xi, \widetilde{H}, U \times \Xi)$ *is an exponentially stabilizing DPF-controller with internal loop for* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Z \times Y)$ *iff* $\widetilde{\Sigma}$ *is an exponentially stabilizing DF-controller with internal loop for* Σ_{21}.

(b1) If Σ_{21} *is exponentially jointly stabilizable and detectable, then the following are equivalent:*

(i) $\mathbb{O}$ *DPF-stabilizes* $\mathbb{D}$ *exponentially with internal loop;*
(ii) $\mathbb{O}$ *DF-stabilizes* $\mathbb{D}_{21}$ *exponentially with internal loop;*
(iii) $\mathbb{O}$ *has a realization that DPF-stabilizes* Σ *exponentially with internal loop.*

(b2) If Σ_{21} *and* Σ *are [strongly] jointly stabilizable and detectable, then the following are equivalent:*

(i) $\mathbb{O}$ *DPF-stabilizes* $\mathbb{D}$ *with internal loop;*
(ii) $\mathbb{O}$ *DF-stabilizes* $\mathbb{D}_{21}$ *with internal loop;*
(iii) $\mathbb{O}$ *has a realization that DPF-stabilizes* Σ *[strongly] with internal loop.*

(c) If Σ_{21} *is optimizable and estimatable, then (b1)(i)&(ii) are equivalent, and so are (b2)(i)&(ii).*

Proof: (a) The two closed-loop systems have same semigroup $\mathbb{A}_I^o$, as noted in the proof of Lemma 7.3.5, hence either is exponentially stable iff $\mathbb{A}_I^o$ is exponentially stable, by Lemma 6.1.10(a1).

(b1)&(b2) Implication "(iii)⇒(i)" is trivial (since $\mathbb{D}_I^o$ is a part of Σ_I^o), and "(i)⇒(ii)" follows from Lemma 7.3.5.

To complete the equivalence, we assume that (ii) holds and that Σ_{21} and Σ are [[exponentially] strongly] jointly stabilizable and detectable [[the assumption on Σ is unnecessary]].

By Theorem 6.6.28 [[(shifted; note that we tacitly apply shifting several times below too)]], $\mathbb{D}_{21}$ has a [[exponential]] d.c.f.; therefore, so does $\overline{\mathbb{D}} := \begin{bmatrix} \mathbb{D}_{21} & 0 \\ 0 & I \end{bmatrix}$.

By Proposition 7.2.5(c), $\mathbb{Q}$ DF-stabilizes $\overline{\mathbb{D}}$ [[exponentially]]. By Proposition 7.1.6(d), $\mathbb{Q}$ has a [[exponential]] d.c.f. By Theorem 6.6.28, $\mathbb{Q}$ has an [[exponentially]] strongly jointly stabilizable and detectable realization.

By Theorem 7.3.11(b)(1.)[[(c1)]], this realization stabilizes Σ [[exponentially] strongly] with internal loop. (Here we needed the assumption on Σ, or at least the assumption that Σ is, e.g., q.r.c.-stabilizable [since $\mathbb{A}_I^o$ is the same for Σ and Σ_{21}, "strongly" is not needed here] [[since Σ_{21} is optimizable and estimatable, so is Σ]].)

A remark for (b1): It is not sufficient for (b1)(i)–(iii) that $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ with internal loop: by Corollary 7.3.20, this holds iff $\mathbb{Q}$ is equivalent to $\mathbb{Q}$ of Corollary 7.3.20(ii) for some $\mathbb{U} \in \mathrm{TIC}(U)$, whereas exponential stabilization requires that $\mathbb{U} \in \mathrm{TIC}_{\exp}(U)$.

(c) 1° *The corresponding discrete-time claim holds:* By Lemma 13.3.17(b), Σ_{21} is exponentially jointly stabilizable and detectable, hence so is Σ. Therefore, (b1)(i)–(iii) are equivalent, and so are (b2)(i)–(iii).

2° *The original claim holds:* Use discretization (see Theorem 13.4.4(e1)). □

In Lemma 7.3.6(b2) (compare to (b1)), the condition on Σ is not superfluous:

Example 7.3.7 (Σ_{21} and $\mathbb{D}$ strongly stable but Σ not DPF-stabilizable) Let $\mathbb{A}$ be as in Example 6.1.14(a), so that $\mathbb{A}$ and Σ_{21} are strongly stable but $\mathbb{B}_2$, $\mathbb{C}_1$ and $\mathbb{D}_{12}$ are unstable, where

$$\Sigma := \left(\begin{array}{c|cc} A & 0 & I \\ \hline I & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \in \mathrm{WPLS}. \tag{7.76}$$

Moreover, no DPF-controller [with internal loop] has any effect on Σ; in particular, Σ is not DPF-stabilizable, although $\widetilde{\Sigma} := \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right]$ DF-stabilizes Σ_{21} exponentially.

Note that Σ is exponentially jointly stabilizable and detectable and has bounded "B and C", but Σ_{21} is only strongly jointly stabilizable and detectable. ◁

(All this is straightforward (use Example 6.1.14(a), Proposition 7.3.4(a) and Lemma 6.6.25.))

Lemma 7.3.8 (Equivalent DPF-controllers) *Let* $\mathbb{D} = \begin{bmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix} \in \mathrm{TIC}(U \times W, Z \times Y)$, $\mathbb{Q} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$, *and* $\mathbb{Q}' \in \mathrm{TIC}_\infty(Y \times \Xi', U \times \Xi')$. *Let* Σ, $\widetilde{\Sigma}$ *and* $\widetilde{\Sigma}'$ *be arbitrary realizations of* $\mathbb{D}$, $\mathbb{Q}$ *and* $\mathbb{Q}'$ *respectively. Then the following are equivalent:*

(i) $\mathbb{Q}$ *and* $\mathbb{Q}'$ *are equivalent DPF-controllers with internal loop for* $\mathbb{D}$*;*

(ii) $\mathbb{Q}$ *and* $\mathbb{Q}'$ *are equivalent DF-controllers with internal loop for* $\mathbb{D}_{21}$*;*

(iii) $\mathbb{Q}_{\mathrm{DF}}$ *and* $\mathbb{Q}'_{\mathrm{DF}}$ *are equivalent DF-controllers with internal loop for* $\mathbb{D}$*;*

(iv) $\widetilde{\Sigma}$ *and* $\widetilde{\Sigma}'$ *are equivalent DPF-controllers with internal loop for* Σ*;*

(v) $\widetilde{\Sigma}$ *and* $\widetilde{\Sigma}'$ *are equivalent DF-controllers with internal loop for* Σ_{21}*;*

(vi) $\widetilde{\Sigma}_{\mathrm{DF}}$ *and* $\widetilde{\Sigma}'_{\mathrm{DF}}$ *are equivalent DF-controllers with internal loop for* Σ*;*

(vii) $\begin{bmatrix} I&0&0&0&0\\0&0&0&I&0\end{bmatrix}\mathbb{D}^o_I\begin{bmatrix} I&0&0&0&0\\0&0&0&I&0\end{bmatrix}^{\mathrm{T}}:\begin{bmatrix}u_L\\y_L\end{bmatrix}\mapsto\begin{bmatrix}u\\y\end{bmatrix}$ *is unaffected when* $\mathbb{O}$ *is replaced by* $\mathbb{O}'$ *(equivalently,* $\widetilde{\Sigma}$ *is replaced by* $\widetilde{\Sigma}'$*);*

(viii) $\begin{bmatrix} I&0&0&0&0\\0&I&0&0&0\\0&0&I&0&0\\0&0&0&I&0\end{bmatrix}\mathbb{D}^o_I\begin{bmatrix} I&0&0&0&0\\0&I&0&0&0\\0&0&I&0&0\\0&0&0&I&0\end{bmatrix}^{\mathrm{T}}:\begin{bmatrix}u_L\\w\\z\\y_L\end{bmatrix}\mapsto\begin{bmatrix}u\\w\\z\\y\end{bmatrix}$ *is unaffected when* $\mathbb{O}$ *is replaced by* $\mathbb{O}'$*;*

(ix) the closed-loop maps $\Sigma^o_I,\Sigma^{o\prime}_I : x_0,u_L,w,(z,)y_L\mapsto x,u,(w,),y,z$ *are unaffected when* $\widetilde{\Sigma}$ *is replaced by* $\widetilde{\Sigma}'$*;*

In particular, two admissible DPF-controllers with coprime internal loop are equal iff they are equivalent for $\mathbb{D}$, equivalently, for $\mathbb{D}_{21}$, by Lemma 7.2.12(c).

Recall that any well-posed map is a map with internal loop, and that any well-posed controller having a (right, left or doubly) coprime factorization is a controller with a (right, left or doubly, respectively) coprime internal loop.

The equivalence between (iv) and (v) was expected: if Σ_{21} does not see any difference between $\widetilde{\Sigma}$ and $\widetilde{\Sigma}'$, why would the rest of Σ see any; the rest of the equivalence follows from this.

Condition (ix) says that the two closed-loop maps are equal except possibly for second and seventh rows and columns (those corresponding to $\widetilde{H}$ and Ξ (or $\widetilde{H}'$ and Ξ')), i.e., only the maps concerning $\widetilde{x}$, $\widetilde{x}_0$, ξ and ξ_L may differ for equivalent controllers for Σ; thus, there is no difference from the part of $\widetilde{\Sigma}$ visible for Σ.

Consequently, for $\widetilde{x}_0=0$ and $\xi_L=0$ (or $\widetilde{x}'_0=0$ and $\xi'_L=0$), the signals x,u,y,z in Figure 7.11 are unaffected when $\widetilde{\Sigma}$ is replaced by an equivalent controller (as long as x_0,u_L,w,y_L are fixed).

Proof of Lemma 7.3.8: (See (7.63) for $\mathbb{D}^o$ and note that $\mathbb{D}^o_I=\mathbb{D}^o(I-\mathbb{D}^o)^{-1}=(I-\mathbb{D}^o)^{-1}-I\in\mathrm{TIC}_\infty(U\times W\times Z\times Y\times\Xi)$ for any admissible controller with internal loop for $\mathbb{D}$.)

1° *"(i)⇔(iv)⇔(vii)", "(ii)⇔(v)" and "(iii)⇔(vi)⇔(viii)":* These follow from Definitions 7.3.1, 7.3.1 and 7.3.1, respectively.

2° *"(vii)⇒(iii)":* Assume (vii), i.e., that the maps $\widetilde{\mathbb{T}}_{11},\widetilde{\mathbb{T}}_{12},\widetilde{\mathbb{T}}_{21},\widetilde{\mathbb{T}}_{22}$ in (7.73) are equal for $\mathbb{O}$ and $\mathbb{O}'$. Then also the maps

$$((I-\mathbb{D}^o)^{-1})_{34}=((I-\mathbb{D}^o)^{-1}-I)_{34}=(\mathbb{D}^o(I-\mathbb{D}^o)^{-1})_{34}=\mathbb{D}_{11}\widetilde{\mathbb{T}}_{12}\qquad(7.77)$$

are equal for $\mathbb{O}$ and $\mathbb{O}'$. We conclude from (7.73) that the maps $((I-\mathbb{D}^o)^{-1})_{ij}$ are equal for $\mathbb{O}$ and $\mathbb{O}'$ for $i=1,2,3,4,\ j=3,4$. By Proposition 7.2.5(e) (cf. (7.27)), we obtain (iii).

3° *"(iii)⇒(ix)":* This follows from Lemma 7.2.2.

Remarks on (ix): Here, as elsewhere, Σ^o_I and $\Sigma^{o\prime}_I$ are the combined closed-loop systems corresponding to $\widetilde{\Sigma}$ and $\widetilde{\Sigma}'$, respectively; cf. the proof of Lemma 7.3.5. By (ix), they become equal once we remove their second and last rows and columns.

In (xi), we must include "τ" after $\mathbb{B}^o_I$ and $\mathbb{B}^{o\prime}_I$ (in Σ^o_I and $\Sigma^{o\prime}_I$), cf. the remark below Definition 6.1.5.

We have z and w in parenthesis in (ix), because z does not affect any other signal and w is not affected by any signal.)

4° *"(ix)⇒(viii)⇒(vii)":* This is trivial. Thus, only (ii) and (v) are missing from the equivalence; they are adopted in 5°–6° below.

5° *"(ii)⇒(vii)":* Assume (ii). Then $\widetilde{\mathbb{T}} = (I - \widetilde{\mathbb{D}}^{o})^{-1}$ is unaffected by the replacement $\mathbb{O} \mapsto \mathbb{O}'$. By (7.73), this means that (vii) holds.

6° *"(i)⇒(ii)":* Assume (i). With the notation of the proof of Lemma 7.3.5, we have

$$\begin{bmatrix}\mathbb{O}_{\mathrm{DF}11} & \mathbb{O}_{\mathrm{DF}12}\end{bmatrix}\mathbb{H}^{-1} = \begin{bmatrix}0 & \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\widetilde{\mathbb{H}}^{-1} \\ 0 & \begin{bmatrix}0 & 0\end{bmatrix}\end{bmatrix} \in \mathrm{TIC}_\infty(Z \times Y \times \Xi, U \times W). \tag{7.78}$$

By (i) and Proposition 7.2.5(e), the map $(\begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12}\end{bmatrix}\widetilde{\mathbb{H}}^{-1})_1$ is unaffected by the replacement $\mathbb{O} \mapsto \mathbb{O}'$, hence $(\begin{bmatrix}\mathbb{O}_{\mathrm{DF}11} & \mathbb{O}_{\mathrm{DF}12}\end{bmatrix}\mathbb{H}^{-1})_1$ is unaffected by $\mathbb{O} \mapsto \mathbb{O}'$ (equivalently, by $\mathbb{O}_{\mathrm{DF}} \mapsto \mathbb{O}'_{\mathrm{DF}}$).

From (7.72) we observe that $(\widetilde{\mathbb{H}}^{-1})_{11} \in \mathrm{TIC}_\infty(Y)$ is contained in $(\mathbb{H}^{-1})_{11} \in \mathrm{TIC}_\infty(Z \times Y)$. We conclude from Proposition 7.2.5(e) that (ii) holds. □

Lemma 7.3.9 (Well-posed $\mathbb{Q} = \mathbb{O}_{11} + \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21}$) *Let* $\mathbb{O} = \begin{bmatrix}\mathbb{O}_{11} & \mathbb{O}_{12} \\ \mathbb{O}_{21} & \mathbb{O}_{22}\end{bmatrix} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ *be an admissible DPF-controller with internal loop for* $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$.

Then $\mathbb{O}$ *is equivalent to a well-posed DPF-controller for* $\mathbb{D}$ *iff* $I - \mathbb{O}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$*; if this is the case, then that well-posed DPF-controller is given by* $\mathbb{O}_{11} + \mathbb{O}_{12}(I - \mathbb{O}_{22})^{-1}\mathbb{O}_{21}$ *(in particular, it is unique).* □

(This follows from Lemma 7.2.7 and Lemma 7.3.8(i)&(ii), because a map $\mathbb{O}' \in \mathrm{TIC}_\infty(Y \times \Xi', U \times \Xi')$ is a well-posed DPF-controller for $\mathbb{D}$ iff $\mathbb{O}'$ is well-posed DF-controller for $\mathbb{D}_{21}$, i.e., iff $\mathbb{O}' = \begin{bmatrix}* & 0 \\ 0 & 0\end{bmatrix}$.)

Lemma 7.3.10 *Let* $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ *or* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *be a map with coprime internal loop. Then so is*

$$\begin{bmatrix}0 & \mathbb{Q} \\ 0 & 0\end{bmatrix} = \begin{bmatrix}0 & \mathbb{Y} \\ 0 & 0\end{bmatrix}\begin{bmatrix}I & 0 \\ 0 & \mathbb{X}\end{bmatrix}^{-1} \quad or \quad \begin{bmatrix}0 & \mathbb{Q} \\ 0 & 0\end{bmatrix} = \begin{bmatrix}\widetilde{\mathbb{X}} & 0 \\ 0 & I\end{bmatrix}^{-1}\begin{bmatrix}0 & \widetilde{\mathbb{Y}} \\ 0 & 0\end{bmatrix}, \tag{7.79}$$

respectively. The following are equivalent:

(i) $\mathbb{Q}$ *is an admissible [stabilizing] DPF-controller with coprime internal loop for* $\mathbb{D}$*;*

(ii) $\begin{bmatrix}0 & \mathbb{Q} \\ 0 & 0\end{bmatrix}$ *is an admissible [stabilizing] DF-controller with coprime internal loop for* $\mathbb{D}$*;*

(iii) $\begin{bmatrix}0 & \mathbb{Y} \\ I & I-\mathbb{X}\end{bmatrix}$ *is an admissible [stabilizing] DPF-controller with internal loop for* $\mathbb{D}$*;*

(iv) (7.81) is an admissible [stabilizing] DF-controller with internal loop for $\mathbb{D}$*;*

(v) (7.82) is an admissible [stabilizing] DF-controller with internal loop for $\mathbb{D}$*.*

(Recall Definition 7.2.11 of maps with coprime internal loop.) From "(i)⇔(iii)" we conclude that one need not first extend $\mathbb{Q}$ to $\begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix}$ and then take a representative (such as (7.81)); one can also take first a representative $\mathbb{O} = \begin{bmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X} \end{bmatrix}$ (or $\mathbb{O} = \begin{bmatrix} 0 & I \\ \widetilde{\mathbb{Y}} & I-\widetilde{\mathbb{X}} \end{bmatrix}$, respectively) of $\mathbb{Q}$, and then extend it as in (7.58): $\mathbb{Q}$ is an admissible [stabilizing] DPF-controller with coprime internal loop for $\mathbb{D}$ iff some (hence any) of its representatives is an admissible [stabilizing] DPF-controller with internal loop for $\mathbb{D}$.

Proof: We treat the r.c. case; the l.c. and d.c. cases are analogous.

Suppose that $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ is a map with r.c. internal loop (i.e., $\mathbb{Y},\mathbb{X} \in \mathrm{TIC}$ are r.c.).

1° *(7.79) is a map with coprime internal loop:* This means that $\begin{bmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ 0 & \mathbb{X} \end{bmatrix}$ are r.c. Indeed, $\widetilde{\mathbb{M}}\mathbb{X} - \widetilde{\mathbb{N}}\mathbb{Y} = I$ implies that

$$\begin{bmatrix} I & 0 \\ 0 & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbb{X} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \widetilde{\mathbb{N}} & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{bmatrix} = I. \tag{7.80}$$

2° *The equivalence of (i)–(v):* By Definition 7.2.11, the (canonical) representative of map of form $\begin{bmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \mathbb{X} \end{bmatrix}^{-1}$ with r.c. internal loop is given by

$$\begin{bmatrix} 0 & 0 & 0 & \mathbb{Y} \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & I-\mathbb{X} \end{bmatrix} \in \mathrm{TIC}(Z \times Y \times (Z \times Y), U \times W \times (Z \times Y)) \tag{7.81}$$

(note that here "Ξ"$= Z \times Y$, whereas below "Ξ"$= Y$). We conclude that (ii) is equivalent to (iv), by Definition 7.2.11.

The DF-controller (7.58) corresponding to the canonical representative $\mathbb{O} = \begin{bmatrix} 0 & \mathbb{Y} \\ I & I-\mathbb{X} \end{bmatrix}$ of the map $\mathbb{Q}$ with internal loop is

$$\begin{bmatrix} 0 & 0 & \mathbb{Y} \\ 0 & 0 & 0 \\ 0 & I & I-\mathbb{X} \end{bmatrix} \in \mathrm{TIC}(Z \times Y \times Y, U \times W \times Y). \tag{7.82}$$

By Definition 7.3.1, (iii) stands for (v), and (i) stands for (ii). Thus, we can complete the equivalence by showing (iv) equivalent to (v).

Let $\widetilde{\mathbb{D}}^o$ be the map "$\mathbb{D}^o$" of (7.63) that results from applying the DPF-controller $\mathbb{O}$ to $\mathbb{D}$ (equivalently, the DF-controller (7.82) to $\mathbb{D}$), and let $\mathbb{D}^o$ be the map "$\mathbb{D}^o$" of (7.20) that results from applying the DF-controller (7.81) to $\mathbb{D}$.

Then $\mathbb{D}^o = \begin{bmatrix} \widetilde{\mathbb{D}}^o & 0 \\ 0 & I \end{bmatrix}$ modulo certain permutation of rows and and the same permutation of columns. Therefore, $I - \mathbb{D}^o \in \mathcal{G}\mathrm{TIC}_\infty$ iff $I - \widetilde{\mathbb{D}}^o \in \mathcal{G}\mathrm{TIC}_\infty$, and $(I - \mathbb{D}^o)^{-1} \in \mathrm{TIC}$ iff $(I - \widetilde{\mathbb{D}}^o)^{-1} \in \mathrm{TIC}$. Thus, the admissibility and stabilizability of (7.81) for $\mathbb{D}$ is equivalent to that of (7.82).

(An intuitive proof would go as follows: (7.82) is obtained by deleting the Z part (not Y part) of ξ (7.81), and this Z part is obviously well-posed and stable, and does not affect any other signals.) □

Trivially, $\widetilde{\Sigma}$ I/O-DPF-stabilizes Σ iff $\mathbb{O}$ DPF-stabilizes $\mathbb{D}$ (i.e., iff $\mathbb{D}_I^o$ becomes stable). Under standard assumptions, this is also equivalent to the stronger condition that $\widetilde{\Sigma}$ DPF-stabilizes Σ:

Theorem 7.3.11 ($\widetilde{\Sigma}$ DPF-stabilizes $\Sigma \Leftrightarrow \mathbb{O}$ DPF-stabilizes $\mathbb{D}$) *Let* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Z \times Y)$ *and* $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{O} \end{array}\right] \in \mathrm{WPLS}(Y \times \Xi, \widetilde{H}, U \times \Xi)$.

(a) Suppose that Σ and $\widetilde{\Sigma}$ are SOS-stabilizable. Then $\widetilde{\Sigma}$ SOS-DPF-stabilizes Σ with internal loop iff $\widetilde{\Sigma}$ I/O-DPF-stabilizes Σ with internal loop.

(b) **([Strong] stability)** *Suppose that any of the following conditions holds:*

(1.) both Σ and $\widetilde{\Sigma}$ are [[exponentially] strongly] q.r.c.-stabilizable;
(2.) both Σ and $\widetilde{\Sigma}$ are [[exponentially] strongly] q.l.c.-detectable;
(3.) both Σ and $\widetilde{\Sigma}$ are SOS-stabilizable and [[exponentially] strongly] detectable;
(4.) both Σ and $\widetilde{\Sigma}$ are detectable and [exponentially] stabilizable.

Then $\widetilde{\Sigma}$ [[exponentially] strongly] DPF-stabilizes Σ with internal loop iff $\widetilde{\Sigma}$ I/O-DPF-stabilizes Σ with internal loop.

(c1) **(Exponential stability)** *The system $\widetilde{\Sigma}$ DPF-stabilizes Σ exponentially with internal loop iff $\widetilde{\Sigma}$ I/O-DPF-stabilizes Σ with internal loop and Σ and $\widetilde{\Sigma}$ are optimizable and estimatable.*

(c2) Suppose that any of the following conditions holds:

(1.) both Σ and $\widetilde{\Sigma}$ are optimizable and estimatable;
(2.) both Σ and $\widetilde{\Sigma}$ are optimizable and input-detectable;
(3.) both Σ and $\widetilde{\Sigma}$ are estimatable and output-stabilizable;
(4.) both Σ and $\widetilde{\Sigma}$ are optimizable and q.r.c.-stabilizable;
(5.) both Σ and $\widetilde{\Sigma}$ are estimatable and q.l.c.-detectable.

Then $\widetilde{\Sigma}$ DPF-stabilizes Σ exponentially with internal loop iff $\widetilde{\Sigma}$ I/O-DPF-stabilizes Σ with internal loop.

(d) **(Well-posed controllers)** *Suppose that, instead,* $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{Q} \end{array}\right] \in \mathrm{WPLS}(Y, \widetilde{H}, U)$. *Then (a)–(c2) hold if we delete the words "with internal loop" everywhere.*

Thus, under corresponding assumptions above, all maps between the signals in Figure 7.11 are (SOS-/strongly/exponentially) stable iff the maps from u_L, w, y_L, z_L, ξ_L to u, w, y, z, ξ are stable.

Proof: This follows from Theorem 7.2.3 (and Definition 7.3.1), because (in the well-posed case (d); the case with internal loop is analogous and left to the reader) if $\widetilde{\Sigma} = \left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \mathbb{Q} \end{array}\right]$ is a realization of $\mathbb{Q}$, then

$$\widetilde{\Sigma}_{\mathrm{DPF}} := \left[\begin{array}{c|cc} \widetilde{\mathbb{A}} & 0 & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & 0 & \mathbb{Q} \\ 0 & 0 & 0 \end{array}\right] \tag{7.83}$$

is a realization of $\left[\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right]$ (having the stabilizability and detectability properties of $\widetilde{\Sigma}$, because it is a parallel connection (see Lemma 6.7.18) of $\widetilde{\Sigma}$ and 0). □

We can now almost state that exponential DPF-stabilizability is equivalent to the optimizability and estimatability of Σ_{21}:

Theorem 7.3.12 (Exp. DPF-stabilizable ⇔ opt. & est.) *Let* $\Sigma := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U \times W, H, Z \times Y)$.

(a) If Σ is exponentially DPF-stabilizable with internal loop, then Σ and $\Sigma_{21} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_2 & \mathbb{D}_{21} \end{array}\right]$ are optimizable and estimatable.

(b1) Conversely, if Σ_{21} is exponentially jointly stabilizable and detectable, then [Σ and] Σ_{21} are exponentially D[P]F-stabilizable with internal loop.

(b2) If Σ_{21} and Σ are [strongly] jointly stabilizable and detectable, then Σ is [strongly] DPF-stabilizable with internal loop and Σ_{21} is [strongly] DF-stabilizable with internal loop.

(c) Assume that $\mathbb{A}Bu_0, \mathbb{A}^*C^*y_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+;H)$ for all $u_0 \in U$ and $y_0 \in Y$, and that $\mathbb{D}_{21}$ is ULR. Then the following are equivalent:

(i) Σ is exponentially DPF-stabilizable;
(ii) Σ is exponentially DPF-stabilizable with internal loop;
(iii) Σ_{21} is exponentially DF-stabilizable;
(iv) Σ_{21} is exponentially DF-stabilizable with internal loop;
(v) (A, B_1) is optimizable and (A, C_2) is estimatable (equivalently, Σ_{21} is optimizable and estimatable);
(vi) Σ_{21} is exponentially jointly stabilizable and detectable;
(vii) Σ and Σ_{21} are exponentially jointly stabilizable and detectable by some bounded K and H.

Moreover, if (vii) holds, then (d1) applies with those K and H (hence (6.169) and (7.84) become ULR).

(d1) If K and H are exponentially jointly stabilizing for Σ_{21} and s.t. “(6.169)” (i.e., $\left[\begin{array}{c|c} A & H \; B_1 \\ \hline C_2 & \\ K & \end{array}\right]$) is SR and “$I - \mathbb{G}_L$”$\in \mathcal{G}\mathrm{TIC}_\infty(Y)$ (this holds if “(6.169)” for Σ_{21} is ULR), then

$$\left(\begin{array}{c|c} A + BK_{\mathrm{s}} + HC_{\mathrm{s}} + HD_{21}K_{\mathrm{s}} & -H \\ \hline K & 0 \end{array}\right) \in \mathrm{WPLS}(Y, H, U) \tag{7.84}$$

is an exponentially DPF-stabilizing controller for Σ. Moreover, (7.22) is SR and exponentially jointly stabilizable and detectable.

(d2) If K and H are [strongly] jointly stabilizing for Σ_{21} and s.t. “(6.169)” (i.e., $\left[\begin{array}{c|c} A & H \; B_1 \\ \hline C_2 & \\ K & \end{array}\right]$) is SR and “$I - \mathbb{G}_L$”$\in \mathcal{G}\mathrm{TIC}_\infty(Y)$ (this holds if “(6.169)” for Σ_{21} is ULR), and Σ is jointly stabilizable and detectable, then (7.84) is a [strongly] DPF-stabilizing controller for Σ. Moreover, (7.22) is SR and [strongly] jointly stabilizable and detectable.

(e) Assume that Σ is exponentially DPF-stabilizable with internal loop. Then any map DPF-stabilizes $\mathbb{D}$ [exponentially] with internal loop iff it DF-stabilizes $\mathbb{D}_{21}$ [exponentially] with internal loop.

Obviously, the assumptions of (c) hold if B and C are bounded (or Hypothesis 9.5.1 holds), hence always in discrete time.

Part (d1) is a generalization of a classical result (see Lemma 12.1 of [ZDG] or Lemma A.4.2 of [GL]).

Also claim (e) is a generalization of a classical result (see [Francis], p. 35; in fact, Francis only assumes that $\mathbb{D}$ is exponentially DPF-stabilizable (since a rational H^∞ function is $\mathrm{H}^\infty_{-\exp}$ for some $\varepsilon > 0$), but, by Lemma 7.1.4 and Theorem 6.6.28, this implies that $\mathbb{D}$ has an exponentially jointly stabilizable and detectable realization (assuming that $\mathbb{D}$ is rational, we could also choose any minimal realization), so that (e) applies).

Proof: (a) By Lemma 7.3.5, Σ_{21} is exponentially DF-stabilizable [with internal loop], hence Σ_{21} is optimizable and estimatable, by Theorem 7.2.3(c1). Therefore, also Σ is optimizable and estimatable, by Lemma6.7.4.

(b1)&(b2) This follows from Theorem 7.2.4(b) and Lemma 7.3.6(b1)&(b2) (moreover, from the proofs we observe that (7.23) will do for Σ too).

(Note from Definition 7.3.1 that if Σ is DPF-stabilizable with internal loop, then it is DF-stabilizable with internal loop, by Definition 7.3.1.)

(c) This follows from Theorem 7.2.4(c) and Lemma 7.3.6(a).

(d1)&(d2) (The assumptions on (6.169) and $\mathbb{G}_L$ refer to those corresponding to Σ_{21} in place of Σ in Definition 6.6.21. Note that it suffices that K and H are ULR and exponentially jointly stabilizing for Σ_{21} (and then (7.84) becomes ULR).)

Make the assumptions of (d1) [[(d2)]]. By Theorem 7.2.4(d), (7.84) is SR, [[exponentially] strongly] jointly stabilizable and detectable, and a [[exponentially] strongly] DF-stabilizing controller for Σ_{21}, hence it I/O-DPF-stabilizes Σ, by Lemma 7.3.6(b2)[[(b1)]]. Consequently, (7.84) DPF-stabilizes Σ [[exponentially] strongly], by Theorem 7.3.11(b)(1.).

(e) This follows from (a) and Lemma 7.3.6(c). □

For the rest of the section, we concentrate on I/O-stabilization by DPF-controllers with d.c. internal loop (equivalently, on the stabilization of plants with $\mathbb{D}_{21}$ having a d.c.f., as the lemma below shows), because this seems to cover all the interesting cases (cf. also the preceeding sections and Lemma 6.5.10).

Lemma 7.3.13 *Let $\mathbb{Q}$ DPF-stabilize $\mathbb{D} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ with internal loop.*

Then $\mathbb{Q}$ has a d.c. (resp. r.c., l.c.) internal loop iff $\mathbb{D}_{21}$ has a d.c.f. (resp. l.c.f., r.c.f.) Moreover, if $\mathbb{D}_{21}$ has a d.c.f. (resp. l.c.f., r.c.f.), then so does $\mathbb{D}$.

In particular, if any system $\overline{\Sigma} = \left[\begin{array}{c|c} \overline{\mathbb{A}} & \overline{\mathbb{B}} \\ \hline \overline{\mathbb{C}} & \overline{\mathbb{D}} \end{array}\right] \in \mathrm{WPLS}$ (resp. map $\overline{\mathbb{D}} \in \mathrm{TIC}_\infty$) is DPF-stabilizable by a [exponentially] jointly stabilizable and detectable controller $\left[\begin{array}{c|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} \\ \hline \widetilde{\mathbb{C}} & \widetilde{\mathbb{Q}} \end{array}\right]$ (resp. by a map $\widetilde{\mathbb{Q}}$ with [exponentially] d.c. internal loop), then $\overline{\mathbb{D}}$, $\widetilde{\mathbb{Q}}$ and $\overline{\mathbb{D}}_{21}$ have [exponential] d.c.f.'s.

Remember that a map has a [exponential] d.c.f. iff it has a [exponentially] jointly stabilizable and detectable realization (by Theorem 6.6.28). [See Theorem 7.3.12(c) for several equivalent conditions for smooth systems (in particular, for finite-dimensional ones).]

Proof: By Lemma 7.3.5, $\mathbb{Q}$ stabilizes $\mathbb{D}_{21}$ with internal loop, so the first conclusion follows from Corollary 7.2.13(b).

If $\mathbb{Q}$ has a d.c. (resp. r.c., l.c.) internal loop, then so does $\left[\begin{smallmatrix}0 & \mathbb{Q}\\ 0 & 0\end{smallmatrix}\right]$, by Lemma 7.3.10, and $\left[\begin{smallmatrix}0 & \mathbb{Q}\\ 0 & 0\end{smallmatrix}\right]$ DF-stabilizes $\mathbb{D}$ (by the definition of DPF-stabilization), hence $\mathbb{D}$ has a d.c.f. (resp. l.c.f., r.c.f.). The claim on $\overline{\Sigma}$ follows from this and Theorem 6.6.28. □

Proposition 7.3.14 *Let* $\mathbb{D} = \left[\begin{smallmatrix}\mathbb{D}_{11} & \mathbb{D}_{12}\\ \mathbb{D}_{21} & \mathbb{D}_{22}\end{smallmatrix}\right] \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$.
We have (i)⇔(ii)⇔(iii), where

(i) $\mathbb{D}$ *has a stabilizing DPF-controller with internal loop, and* $\mathbb{D}_{21}$ *has a d.c.f.;*

(ii) $\mathbb{D}$ *has a stabilizing DPF-controller with d.c. internal loop;*

(iii) $\mathbb{D}$ *has a d.c.f. of the form*

$$\mathbb{D} = \begin{bmatrix}\mathbb{N}_{11} & \mathbb{N}_{12}\\ \mathbb{N}_{21} & \mathbb{N}_{22}\end{bmatrix}\begin{bmatrix}\mathbb{M}_{11} & \mathbb{M}_{12}\\ 0 & I\end{bmatrix}^{-1} = \begin{bmatrix}I & \widetilde{\mathbb{M}}_{12}\\ 0 & \widetilde{\mathbb{M}}_{22}\end{bmatrix}^{-1}\begin{bmatrix}\widetilde{\mathbb{N}}_{11} & \widetilde{\mathbb{N}}_{12}\\ \widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{N}}_{22}\end{bmatrix}, \tag{7.85}$$

s.t. $\mathbb{N}_{21}$ *and* $\mathbb{M}_{11}$ *are r.c., and* $\widetilde{\mathbb{N}}_{21}$ *and* $\widetilde{\mathbb{M}}_{22}$ *are l.c.*

If $\dim U, \dim Y < \infty$ *and* $\mathbb{D}$ *has a stabilizing (well-posed) DPF-controller, then (i)–(iii) hold.*

Let $\mathcal{A} \subset \mathrm{TIC}$. *Then we have (i')⇔(ii')⇔(ii*)⇔(iii'), where*

(i') $\mathbb{D}$ *has a stabilizing DPF-controller with internal loop, and* $\mathbb{D}_{21}$ *has a d.c.f. over* $\mathcal{A}$*;*

(ii') $\mathbb{D}$ *has a stabilizing DPF-controller* $\mathbb{Q}$ *with d.c. internal loop over* $\mathcal{A}$*, and* $\mathbb{D}_{21}$ *and* $\mathbb{Q}$ *have a joint d.c.f. over* $\mathcal{A}$*;*

(ii)* $\mathbb{D}$ *has a stabilizing DPF-controller with d.c. internal loop, and* $\mathbb{D}_{21}$ *has a d.c.f. over* $\mathcal{A}$*;*

(iii') $\mathbb{D}$ *has a d.c.f. of the form (7.85), s.t.* $\mathbb{N}_{21}$ *and* $\mathbb{M}_{11}$ *are r.c. over* $\mathcal{A}$*, and* $\widetilde{\mathbb{N}}_{21}$ *and* $\widetilde{\mathbb{M}}_{22}$ *are l.c. over* $\mathcal{A}$*.*

If $\mathbb{D}$ *has a d.c.f. over* $\mathcal{A}$*, then we have (i')⇔(ii')⇔(ii*)⇔(iii')⇔(iii"), where*

(iii") $\mathbb{D}$ *has a d.c.f. over* $\mathcal{A}$ *of the form (7.85), s.t.* $\mathbb{N}_{21}$ *and* $\mathbb{M}_{11}$ *are r.c. over* $\mathcal{A}$*, and* $\widetilde{\mathbb{N}}_{21}$ *and* $\widetilde{\mathbb{M}}_{22}$ *are l.c. over* $\mathcal{A}$*.*

If $\mathcal{B} \subset \mathcal{A} \subset \mathrm{ULR}_0$*, then the stabilizing DPF-controllers in (i'), (ii*) and (ii') can be chosen to be well-posed.*

Proof: Note that the equivalence (i)⇔(ii)⇔(iii) follows from (i')⇔(ii')⇔(iii'), by taking $\mathcal{A} = \mathrm{TIC}$, so we do not need to prove the former. If $\mathbb{D}$ has a stabilizing DPF-controller $\mathbb{Q} \in \mathrm{TIC}_\infty$, then it is a stabilizing DF-controller for $\mathbb{D}_{21}$, hence then $\mathbb{D}_{21}$ and $\mathbb{Q}$ have d.c.f.'s, by Lemma 7.1.4, provided that $\dim U, \dim Y < \infty$.

1° The equivalence "(i')⇔(ii*)" follows from Lemma 7.3.13. Clearly (ii')⇒(ii*).

2° "(iii')⇒(ii')": Because $\mathbb{D}_{21} = \mathbb{N}_{21}\mathbb{M}_{11}^{-1}$ is a r.c.f. over $\mathcal{A}$ and $\mathbb{D}_{21} = \widetilde{\mathbb{M}}_{22}^{-1}\widetilde{\mathbb{N}}_{21}$ is a l.c.f. over $\mathcal{A}$, they can be extended to a d.c.f. over $\mathcal{A}$, by Lemma 6.5.8; in particular, we can find $\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$ s.t.

$$\begin{bmatrix} \mathbb{M}_{11} & \mathbb{Y} \\ \mathbb{N}_{21} & \mathbb{X} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix}^{-1} \in \mathcal{GA}. \tag{7.86}$$

But, by Corollary 7.2.15(i), $\left[\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} I & 0 \\ 0 & \mathbb{X} \end{smallmatrix}\right]^{-1}$ DF-stabilizes $\mathbb{D}$ iff

$$\begin{bmatrix} I & \widetilde{\mathbb{M}}_{12}\mathbb{X} - \widetilde{\mathbb{N}}_{11}\mathbb{Y} \\ 0 & \widetilde{\mathbb{M}}_{22}\mathbb{X} - \widetilde{\mathbb{N}}_{21}\mathbb{Y} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}, \tag{7.87}$$

i.e., iff $\widetilde{\mathbb{M}}_{22}\mathbb{X} - \widetilde{\mathbb{N}}_{21}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}$ (by Lemma A.1.1(b)), and latter is true by (7.86), hence $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ DPF-stabilizes $\mathbb{D}$ with d.c. internal loop over $\mathcal{A}$.

3° *If (iii') holds (e.g., (iii) holds), then a map $\mathbb{Q}$ with an internal loop DPF-stabilizes $\mathbb{D}$ iff $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$:* Indeed, if $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$, then it DF-stabilizes $\mathbb{D}_{21}$, in particular, $\mathbb{Q}$ has a d.c. internal loop in either case, by Corollary 7.2.13. For the converse, in 2° it was noted that $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ iff $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$, where $\widetilde{\mathbb{M}}_{22}\mathbb{X} - \widetilde{\mathbb{N}}_{21}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}$, which in turn is true iff $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$, by Corollary 7.2.15(i).

4° "(ii*)⇒(iii')": Assume (ii*), i.e., that some $\mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ DPF-stabilizes $\mathbb{D}$ with d.c. internal loop. It follows from Lemma 7.2.16(b), that for some d.c.f. $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1} = \widetilde{\mathbb{N}}'(\widetilde{\mathbb{M}}')^{-1}$ we have (see (7.79))

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = \mathbb{M}' \begin{bmatrix} \widetilde{\mathbb{X}} & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{bmatrix} \widetilde{\mathbb{N}}', \text{ i.e.,} \tag{7.88}$$

$$\begin{bmatrix} \mathbb{M}'_{11}\widetilde{\mathbb{X}} & \mathbb{M}'_{12} \\ \mathbb{M}'_{21}\widetilde{\mathbb{X}} & \mathbb{M}'_{22} \end{bmatrix} = \begin{bmatrix} I + \mathbb{Y}\widetilde{\mathbb{N}}'_{21} & \mathbb{Y}\widetilde{\mathbb{N}}'_{22} \\ 0 & I \end{bmatrix}. \tag{7.89}$$

By Lemma 6.5.6(d), $\mathbb{M}'_{22} = I$ implies that $\mathbb{D}$ has a r.c.f. of the form of the first equality in (7.85); from the $(1,1)$-block of Corollary 7.2.15(i') applied to this r.c.f. we see that $\widetilde{\mathbb{X}}\mathbb{M}_{11} - \widetilde{\mathbb{Y}}\mathbb{N}_{21} \in \mathcal{G}\mathrm{TIC}$, hence $\mathbb{M}_{11}$ and $\mathbb{N}_{21}$ are r.c.

Let $\mathbb{D}_{21} = \mathbb{T}\mathbb{S}^{-1}$ be a r.c.f. over $\mathcal{A}$. Then $\begin{bmatrix} \mathbb{N}_{21} & \mathbb{M}_{11} \end{bmatrix} = \begin{bmatrix} \mathbb{T}\mathbb{U} & \mathbb{S}\mathbb{U} \end{bmatrix}$ for some $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.4.5(c). Thus we may multiply r.c.f. in (7.85) by $\left[\begin{smallmatrix} \mathbb{U}^{-1} & 0 \\ 0 & I \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$ to the right, to make $\mathbb{N}_{21}$ and $\mathbb{M}_{11}$ r.c. over $\mathcal{A}$.

The dual part is obtained analogously from $\left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & \mathbb{X} \end{smallmatrix}\right] \widetilde{\mathbb{M}} - \mathbb{N} \left[\begin{smallmatrix} 0 & \widetilde{\mathbb{Y}} \\ 0 & 0 \end{smallmatrix}\right]$, which implies that $\widetilde{\mathbb{M}}_{11} = I$ (a r.c.f. and a l.c.f. form a d.c.f., by Lemma 6.5.8).

5° From ''(i')⇔(ii')⇔(iii')" we obtain directly "(i")⇔(ii")⇐(iii")". Assuming (ii”), we obtain (iii”) from 4° as follows:

The d.c.f. $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1} = \widetilde{\mathbb{N}}'(\widetilde{\mathbb{M}}')^{-1}$ in 4° can be chosen to be over $\mathcal{A}$, hence so can (7.85), by Lemma 6.5.6(d). All the other claims are contained in (iii’), which is equivalent to (ii’), hence a consequence of (ii”).

6° If $\mathcal{B} \subset \mathcal{A} \subset \mathrm{ULR}_0$ and (ii’) holds (recall that (i’)⇔(ii’)⇔(ii*)), then there is a well-posed $\mathbb{Q}$ that has a d.c.f. over $\mathcal{A}$ joint with $\mathbb{D}_{21}$, by Proposition 7.1.10. Therefore, this $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$; by 3°, it DPF-stabilizes $\mathbb{D}$. □

By combining Lemma 7.3.13 and Proposition 7.3.14, we deduce the following: if $\mathbb{D}$ is DPF-stabilizable by a controller $\mathbb{Q}$ with internal loop, and either $\mathbb{D}_{21}$ or $\mathbb{Q}$ has a d.c.f., then so do all of $\mathbb{D}$, $\mathbb{D}_{21}$ and $\mathbb{Q}$, and the following hypothesis holds:

Hypothesis 7.3.15 *We shall assume that* $\mathbb{D} = \begin{bmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$, *and that* $\mathbb{D}$ *has a d.c.f. of the form*

$$\mathbb{D} = \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{N}_{21} & \mathbb{N}_{22} \end{bmatrix} \begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & \widetilde{\mathbb{M}}_{12} \\ 0 & \widetilde{\mathbb{M}}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{\mathbb{N}}_{11} & \widetilde{\mathbb{N}}_{12} \\ \widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{N}}_{22} \end{bmatrix}, \tag{7.90}$$

s.t. $\mathbb{N}_{21}$ *and* $\mathbb{M}_{11}$ *are r.c. and* $\widetilde{\mathbb{N}}_{21}$ *and* $\widetilde{\mathbb{M}}_{22}$ *are l.c.*

Note that $\mathbb{D}_{21} = \mathbb{N}_{21}\mathbb{M}_{11}^{-1}$ is a r.c.f. and $\mathbb{D}_{21} = \widetilde{\mathbb{M}}_{22}^{-1}\widetilde{\mathbb{N}}_{21}$ is a l.c.f.

Under the standard assumptions of the H^∞ Four-Block Problem, Hypothesis 7.3.15 is satisfied (cf. Lemmas 12.5.4 and 12.5.5). Under sufficient regularity, the I/O map of an exponentially DF-stabilizable system satisfies Hypothesis 7.3.15 (exponentially), by Theorem 7.3.12(c)(1)&(6) and Proposition 7.3.14.

As noted just before the hypothesis, this hypothesis is at most slightly stronger than the assumption that $\mathbb{D}$ is DPF-stabilizable [with internal loop]; it excludes only the case where $\mathbb{Q}$ and $\mathbb{D}_{21}$ have no jointly stabilizable and detectable realizations (cf. also Lemma 6.5.10).

At least for finite-dimensional U and Y, any DPF-stabilizable $\mathbb{D} \in \mathrm{TIC}_\infty$ satisfies the hypothesis, by Lemma 7.1.4.

Lemma 7.3.16 ($\mathbb{M} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, $\widetilde{\mathbb{M}} = \left[\begin{smallmatrix} I & * \\ 0 & * \end{smallmatrix}\right]$) *Let* $\mathbb{D} = \mathbb{N}_{\mathrm{u}}\mathbb{M}_{\mathrm{u}}^{-1}$ *be an r.c.f. with* $\mathbb{M}_{\mathrm{u}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}_\infty(U \times W)$. *Then all such r.c.f.’s are given by*

$$\mathbb{N} = \mathbb{N}_{\mathrm{u}}\mathbb{X}, \quad \mathbb{M} = \mathbb{M}_{\mathrm{u}}\mathbb{X}, \quad \mathbb{X} = \begin{bmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} \\ 0 & I \end{bmatrix} \in \mathcal{G}\mathrm{TIC}(U \times W). \tag{7.91}$$

For the dual claim we have $\widetilde{\mathbb{N}} = \widetilde{\mathbb{X}}\widetilde{\mathbb{N}_{\mathrm{y}}}$, $\widetilde{\mathbb{M}} = \widetilde{\mathbb{X}}\widetilde{\mathbb{M}_{\mathrm{y}}}$, $\widetilde{\mathbb{X}} = \left[\begin{smallmatrix} I & \widetilde{\mathbb{X}}_{12} \\ 0 & \widetilde{\mathbb{X}}_{22} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$.

Note that this implies that $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ (by Lemma A.1.1(b2)(2)) and $\mathbb{X}_{12} \in \mathrm{TIC}(W,U)$ are arbitrary.

Proof: Clearly all r.c.f.’s defined by (7.91) satisfy $\mathbb{M} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$. Conversely, by Lemma 6.4.5(c), $\mathbb{N} = \mathbb{N}_{\mathrm{u}}\mathbb{X}$ and $\mathbb{M} = \mathbb{M}_{\mathrm{u}}\mathbb{X} = \left[\begin{smallmatrix} * & * \\ \mathbb{X}_{21} & \mathbb{X}_{22} \end{smallmatrix}\right]$, where $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$. Therefore, $\mathbb{M} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$ implies that $\begin{bmatrix} \mathbb{X}_{21} & \mathbb{X}_{22} \end{bmatrix} = \begin{bmatrix} \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

The dual claim is obtained analogously (or by taking adjoints). □

Corollary 7.3.17 *If Hypothesis 7.3.15 holds, then all d.c.f.'s of $\mathbb{D}$ of form (7.90) satisfy Hypothesis 7.3.15.*

Proof: Let $\mathbb{D} = \mathbb{N}_\mathrm{u}\mathbb{M}_\mathrm{u}{}^{-1} = \widetilde{\mathbb{M}_\mathrm{y}}{}^{-1}\widetilde{\mathbb{N}_\mathrm{y}}$ be as in Hypothesis 7.3.15, and let $\widetilde{\mathbb{P}}\mathbb{M}_{\mathrm{u}\,11} - \widetilde{\mathbb{Q}}\widetilde{\mathbb{N}_\mathrm{y}}{}_{21} = I$. Let also $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ be a d.c.f. with $\mathbb{M} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, $\widetilde{\mathbb{M}} = \left[\begin{smallmatrix} I & * \\ 0 & * \end{smallmatrix}\right]$. Let $\mathbb{X}$ be as in (7.91). Then

$$\mathbb{X}_{11}^{-1}\widetilde{\mathbb{P}}\mathbb{M}_{11} - \mathbb{X}_{11}^{-1}\widetilde{\mathbb{Q}}\mathbb{N}_{21} = \mathbb{X}_{11}^{-1}\mathbb{X}_{11} = I. \tag{7.92}$$

The dual claim is obtained analogously. □

Remark 7.3.18 *The r.c.f. in Hypothesis 7.3.15 says (roughly) that $\mathbb{D}$ can be stabilized by measuring the full output (z and y) and controlling u (i.e., not affecting w). Similarly, the l.c.f. says that $\mathbb{D}$ can be stabilized by measuring y and controlling the full input (u and y).*

Thus is an intuitive explanation of the necessity (at least under certain regularity) of Hypothesis 7.3.15 for one being able to stabilize $\mathbb{D}$ by measuring y and controlling u; by Proposition 7.3.14 these are also sufficient.

It will be shown in Theorem 7.3.19 that $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ iff $\mathbb{Q}$ stabilizes $\mathbb{D}_{21}$. Indeed, "all the poles of $\mathbb{D}$ are shared by $\mathbb{D}_{21}$" (cf. [Francis, p. 34]), because

$$\mathbb{D} = \begin{bmatrix} \mathbb{N}_{11}\mathbb{M}_{11}^{-1} & -\mathbb{N}_{11}\mathbb{M}_{11}^{-1}\mathbb{M}_{12} + \mathbb{N}_{12} \\ \mathbb{N}_{21}\mathbb{M}_{11}^{-1} & -\mathbb{N}_{21}\mathbb{M}_{11}^{-1}\mathbb{M}_{12} + \mathbb{N}_{22} \end{bmatrix}. \tag{7.93}$$

Therefore, the poles of $\widehat{\mathbb{D}}$ are poles of $\widehat{\mathbb{M}}_{11}$, which in turn are exactly the poles of $\widehat{\mathbb{D}}_{21}$, by Lemma 6.5.4, hence all these three maps have same poles (up to multiplicities).

Thus, stabilization of either $\mathbb{D}$ or $\mathbb{D}_{21}$ is equivalent to removing these singularities.

A simple example of non-DPF-stabilizable $\mathbb{D}$ is thus any $\mathbb{D} = \left[\begin{smallmatrix} 0 & \mathbb{D}_{12} \\ 0 & 0 \end{smallmatrix}\right] \in \mathrm{TIC}_\infty \setminus \mathrm{TIC}$.

From the above hypothesis (roughly, the DPF-stabilizability of $\mathbb{D}$), it follows that all stabilizing DPF-controllers with internal loop for $\mathbb{D}$ are exactly the stabilizing
(DF-)controllers with d.c. internal loop for $\mathbb{D}_{21}$, i.e., the ones given by the Youla parametrization of Theorem 7.2.14:

Theorem 7.3.19 (DPF-stabilization with IL) *Assume Hypothesis 7.3.15. Then the following are equivalent for a controller $\mathbb{Q}$ with internal loop:*

(i) $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ with internal loop.

(i') $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ with d.c. internal loop.

(ii) $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$ with internal loop.

(ii') $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$ with d.c. internal loop.

(iii) $\widetilde{\mathbb{M}}_{22}\mathbb{X} - \widetilde{\mathbb{N}}_{21}\mathbb{Y} \in \mathcal{G}\mathrm{TIC}(Y)$ and $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ for some $\mathbb{X}, \mathbb{Y} \in \mathrm{TIC}$.

(iii') $\widetilde{\mathbb{X}}\mathbb{M}_{11} - \widetilde{\mathbb{Y}}\mathbb{N}_{21} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *for some* $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$.

(iv) $\begin{bmatrix} \mathbb{M}_{11} & \mathbb{Y} \\ \mathbb{N}_{21} & \mathbb{X} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$ *and* $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ *for some* $\mathbb{X}, \mathbb{Y} \in \mathrm{TIC}$.

(iv') $\begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$ *and* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *for some* $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathrm{TIC}$.

(v) For any r.c.f. $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ *and l.c.f.* $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$*, there are* $\mathbb{N}_0, \mathbb{M}_0, \widetilde{\mathbb{N}}_0, \widetilde{\mathbb{M}}_0 \in \mathrm{TIC}$ *s.t.* $\begin{bmatrix} \mathbb{M}_0 & \mathbb{Y} \\ \mathbb{N}_0 & \mathbb{X} \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_0 & \widetilde{\mathbb{M}}_0 \end{bmatrix}$ *is a d.c.f. of* $\mathbb{D}_{21}$.

The map $\mathbb{Q}$ *is well-posed iff the "denominator" ($\mathbb{X}$ or $\widetilde{\mathbb{X}}$) is invertible in* $\mathcal{G}\mathrm{TIC}_\infty$ *in any of the above equivalent conditions.*

By (v), $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ iff $\mathbb{Q}$ has a joint d.c.f. with $\mathbb{D}_{21}$. (See Definition 7.2.11 for maps with coprime internal loop.)

Proof: 1° The fact that a DPF-stabilizing controller of $\mathbb{D}$ with internal loop has necessarily a d.c. internal loop, is given by Lemma 7.3.13, hence "(i)⇔(i')" holds.

2° Similarly, "(ii)⇔(ii')" follows from Theorem 7.2.14.

3° "(i')⇔(ii')": This follows from part 3° of the proof of Proposition 7.3.14.

4° By Corollary 7.2.15, all the other conditions are equivalent to (ii'), and the two final claims hold. □

By combining the above theorem and Theorem 7.2.14, we see that all stabilizing DPF-controllers for $\mathbb{D}$ with internal loop are given by the Youla parametrization:

Corollary 7.3.20 (All stabilizing DPF-controllers with IL) *Assume Hypothesis 7.3.15, and choose* $\mathbb{T}, \mathbb{S}, \widetilde{\mathbb{T}}, \widetilde{\mathbb{S}} \in \mathrm{TIC}$ *s.t.*

$$\begin{bmatrix} \mathbb{M}_{11} & \mathbb{T} \\ \mathbb{N}_{21} & \mathbb{S} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{S}} & -\widetilde{\mathbb{T}} \\ -\widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix}^{-1} \in \mathcal{G}\mathrm{TIC}(U \times Y). \tag{7.94}$$

(This is a d.c.f. of $\mathbb{D}_{21}$*.)*

The following parametrizations are alternative (equivalent) parametrizations of all (modulo being equivalent) DPF-controllers $\mathbb{Q}$ *with internal loop that stabilize* $\mathbb{D}$*, and each* parameter *(($\mathbb{X}, \mathbb{Y}$) in (i) and (iii), ($\widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}}$) in (i'), and* $\mathbb{U}$ *in (ii) and (ii'); these all are required to be stable) determines a different (nonequal; see Definition 7.2.11) map* $\mathbb{Q}$ *with d.c. internal loop.*

(i) $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ *such that* $\widetilde{\mathbb{M}}_{22}\mathbb{X} - \widetilde{\mathbb{N}}_{21}\mathbb{Y} = I$.

(i') $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ *such that* $\widetilde{\mathbb{X}}\mathbb{M}_{11} - \widetilde{\mathbb{Y}}\mathbb{N}_{21} = I$.

(ii) **(Youla)** $\mathbb{Q} = (\mathbb{T} + \mathbb{M}_{11}\mathbb{U})(\mathbb{S} + \mathbb{N}_{21}\mathbb{U})^{-1}$ *(i.e.,* $\begin{bmatrix} \mathbb{Y} \\ \mathbb{X} \end{bmatrix} = \begin{bmatrix} \mathbb{M}_{11} & \mathbb{T} \\ \mathbb{N}_{21} & \mathbb{S} \end{bmatrix} \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}$*), where* $\mathbb{U} \in \mathrm{TIC}(U)$.

(ii') $\mathbb{Q} = (\widetilde{\mathbb{S}} + \widetilde{\mathbb{N}}_{21}\mathbb{U})^{-1}(\widetilde{\mathbb{T}} + \widetilde{\mathbb{M}}_{22}\mathbb{U})$ *(i.e.,* $\begin{bmatrix} \widetilde{\mathbb{X}} & \widetilde{\mathbb{Y}} \end{bmatrix} = \begin{bmatrix} I & \mathbb{U} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{S}} & \widetilde{\mathbb{T}} \\ \widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix}$*), where* $\mathbb{U} \in \mathrm{TIC}(U)$.

(iii) $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ *(=* $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$*), where* $\begin{bmatrix} \mathbb{M}_{11} & \mathbb{Y} \\ \mathbb{N}_{21} & \mathbb{X} \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$.

Moreover, the following holds:

(a) The well-posed ones (if any) are exactly those $\mathbb{Q}$ for which the "denominator" is in $\mathcal{G}\mathrm{TIC}_\infty$ (cf. Theorem 7.1.7); they satisfy

$$\widetilde{\mathbb{X}} = (\mathbb{M}_{11} - \mathbb{Q}\mathbb{N}_{21})^{-1}, \ \ \widetilde{\mathbb{Y}} = \widetilde{\mathbb{X}}\mathbb{Q}; \ \ \mathbb{X} = (\widetilde{\mathbb{M}}_{22} - \widetilde{\mathbb{N}}_{21}\mathbb{Q})^{-1}, \ \ \mathbb{Y} = \mathbb{Q}\mathbb{X}. \tag{7.95}$$

(b) For any $\mathbb{U} \in \mathrm{TIC}$ we have (identity as equal maps with coprime internal loop)

$$(\mathbb{T} + \mathbb{M}_{11}\mathbb{U})(\mathbb{S} + \mathbb{N}_{21}\mathbb{U})^{-1} = (\widetilde{\mathbb{S}} + \widetilde{\mathbb{N}}_{21}\widetilde{\mathbb{U}})^{-1}(\widetilde{\mathbb{T}} + \widetilde{\mathbb{M}}_{22}\widetilde{\mathbb{U}}). \tag{7.96}$$

(c) If $\mathbb{Y}, \mathbb{X}, \widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}}$ are as in (i) and (i'), then the closed-loop I/O maps are given by

$$\begin{bmatrix} \widetilde{\mathbb{N}}_{11} + \widetilde{\mathbb{P}}\widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{N}}_{12} + \widetilde{\mathbb{P}}\widetilde{\mathbb{N}}_{22} \\ \mathbb{X}\widetilde{\mathbb{N}}_{21} & \mathbb{X}\widetilde{\mathbb{N}}_{22} \end{bmatrix} = \begin{bmatrix} \mathbb{N}_{11}\widetilde{\mathbb{X}} & \mathbb{N}_{11}\mathbb{P} + \mathbb{N}_{12} \\ \mathbb{N}_{21}\widetilde{\mathbb{X}} & \mathbb{N}_{21}\mathbb{P} + \mathbb{N}_{22} \end{bmatrix} : \begin{bmatrix} u_L \\ w \end{bmatrix} \mapsto \begin{bmatrix} z \\ y \end{bmatrix}, \tag{7.97}$$

where $\mathbb{P} = \widetilde{\mathbb{Y}}\mathbb{N}_{22} - \widetilde{\mathbb{X}}\mathbb{M}_{12}$ and $\widetilde{\mathbb{P}} = \widetilde{\mathbb{N}}_{11}\mathbb{Y} - \widetilde{\mathbb{M}}_{12}\mathbb{X}$; see (7.65) (without the third (y_L) column and top (u) row) for alternative formulae in the well-posed case.

In particular, (cf. (7.64))

$$\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q}) = \mathbb{N}_{11}\mathbb{P} + \mathbb{N}_{12} = \widetilde{\mathbb{N}}_{12} + \widetilde{\mathbb{P}}\widetilde{\mathbb{N}}_{22}. \tag{7.98}$$

Recall from Lemma 7.2.12(c2), that the maps (7.97) depend (of course) on $\mathbb{D}$ and $\mathbb{Q}$ only, not on the particular coprime factors $(\mathbb{X}, \mathbb{Y}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}}, \mathbb{N}, \mathbb{M}, \widetilde{\mathbb{N}}, \widetilde{\mathbb{M}})$ of $\mathbb{Q}$ and $\mathbb{D}$ (though we do require (i), (i') and Hypothesis 7.3.15).

The H^∞ 4BP (see Chapter 12; Section 12.3 in particular) consists of finding, for a given $\mathbb{D}$, a stabilizing DPF-controller $\mathbb{Q}$ s.t. the norm $\|\mathcal{F}_\ell\|$ is less than a given constant γ (or for a given Σ a [exponentially] stabilizing controller $\widetilde{\Sigma}$ s.t. $\|\mathcal{F}_\ell\| < \gamma$).

Proof: By Lemma 6.5.8, it follows from Hypothesis 7.3.15 that $\mathbb{D}_{21}$ has a d.c.f. of form (7.94). By Theorem 7.3.19, the stabilizing DPF-controllers for $\mathbb{D}$ with internal loop are exactly the DF-stabilizing controllers for $\mathbb{D}_{21}$ with d.c. internal loop, and these parametrized by Theorem 7.2.14, which also provides the well-posedness claim and (7.96).

Formula (7.95) follows from (7.8) and (7.10).

(c) From (7.52) and Lemma 7.3.10 we see that the map $\left[\begin{smallmatrix} u_L \\ w \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$ is given by $\mathbb{N}'\widetilde{\mathbb{X}}'$ when $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1}$ is a r.c.f.,

$$\widetilde{\mathbb{X}}' = \begin{bmatrix} \widetilde{\mathbb{X}} & 0 \\ 0 & I \end{bmatrix}, \quad \widetilde{\mathbb{Y}}' = \begin{bmatrix} 0 & \widetilde{\mathbb{Y}} \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad \widetilde{\mathbb{X}}'\mathbb{M}' - \widetilde{\mathbb{Y}}'\mathbb{N}' = I. \tag{7.99}$$

This condition can be satisfied by setting $\mathbb{N}' := \mathbb{N}\mathbb{G}$, $\mathbb{M}' := \mathbb{M}\mathbb{G}$, where $\mathbb{G} := (\widetilde{\mathbb{X}}'\mathbb{M} - \widetilde{\mathbb{Y}}'\mathbb{N})^{-1} \in \mathcal{G}\mathrm{TIC}(U \times W)$. Therefore,

$$\mathbb{G} = \left[\begin{smallmatrix} I & -\mathbb{P} \\ 0 & I \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} I & \mathbb{P} \\ 0 & I \end{smallmatrix}\right], \tag{7.100}$$

(by (i')), where $-\mathbb{P} = \widetilde{\mathbb{X}}\mathbb{M}_{12} - \widetilde{\mathbb{Y}}\mathbb{N}_{22}$. Thus, $\mathbb{N}'\widetilde{\mathbb{X}}' = \mathbb{N}\mathbb{G}\widetilde{\mathbb{X}}'$ is given by (7.97).

Assuming (i), we obtain the dual formula in (7.97) analogously from $\mathbb{X}'\widetilde{\mathbb{N}}' = \begin{bmatrix} I & 0 \\ 0 & \mathbb{X} \end{bmatrix} \widetilde{\mathbb{G}}\widetilde{\mathbb{N}}$ and

$$\widetilde{\mathbb{G}} = (\widetilde{\mathbb{M}}\mathbb{X}' - \widetilde{\mathbb{N}}\mathbb{Y}')^{-1} = \begin{bmatrix} I & \widetilde{\mathbb{M}}_{12}\mathbb{X}' - \widetilde{\mathbb{N}}_{11}\mathbb{Y} \\ 0 & I \end{bmatrix}^{-1} =: \begin{bmatrix} I & \widetilde{\mathbb{P}} \\ 0 & I \end{bmatrix}. \tag{7.101}$$

□

Corollary 7.3.21 ($\mathcal{A}$ case) *Let $\mathcal{A} \underset{a}{\subset}$ TIC be inverse-closed and let Proposition 7.3.14(iii') hold.*

If the elements of (7.94) are chosen from $\mathcal{A}$, then all stabilizing DPF-controllers of $\mathbb{D}$ with a (d.c.) internal loop are the ones parametrized in Corollary 7.3.20, and the ones with d.c. internal loop over $\mathcal{A}$ are exactly those with $\mathbb{U} \in \mathcal{A}$. If, in addition, $\mathcal{B} \underset{a}{\subset} \mathcal{A} \underset{a}{\subset}$ ULR, then the one with $\mathbb{U} = -M_{11}^{-1}T$ is well-posed.

Proof: By Lemma 6.5.8, we can take $\mathbb{T},\mathbb{S},\widetilde{\mathbb{T}},\widetilde{\mathbb{S}} \in \mathcal{A}$. in Corollary 7.3.20; the rest follows by combining Corollary 7.3.20 and Proposition 7.2.17. □

As the final I/O result of this section we note that the following well-known criteria (see Theorem 4.2.1, p. 27 of [Francis] or Theorem 2.1 of [Green]) are valid for general WPLSs too:

Lemma 7.3.22 *Let $\mathbb{D} = \begin{bmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$ and $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ have coprime factorizations $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1} = \widetilde{\mathbb{M}}^{-1}\widetilde{\mathbb{N}}$ and $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$. Then the following are equivalent:*

(i) $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$;

(ii) $\begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} & \mathbb{Y} \\ \mathbb{M}_{21} & \mathbb{M}_{22} & 0 \\ \mathbb{N}_{21} & \mathbb{N}_{22} & \mathbb{X} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}(U \times W \times Y)$.

(iii) $\begin{bmatrix} \widetilde{\mathbb{X}} & 0 & \widetilde{\mathbb{Y}} \\ \widetilde{\mathbb{N}}_{11} & \widetilde{\mathbb{M}}_{11} & \widetilde{\mathbb{M}}_{12} \\ \widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{M}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}(U \times Z \times Y)$.

Even if $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ were merely a map with d.c. internal loop, then (i)–(iii) are still equivalent.

Proof: Let $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1}$ be a map with r.c. internal loop. We prove that (i)⇔(ii); the case (i)⇔(iii) is analogous and the well-posed case (the one where $\mathbb{X},\mathbb{X}^{-1} \in \mathcal{G}\mathrm{TIC}_\infty$) follows from this general case (with coprime internal loop).

By Lemma 7.3.10, $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ iff (7.79) DF-stabilizes $\mathbb{D}$. By Corollary 7.2.15, this holds iff

$$\begin{bmatrix} \begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix} & \begin{bmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{N}_{21} & \mathbb{N}_{22} \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & \mathbb{X} \end{bmatrix} \end{bmatrix} \tag{7.102}$$

is in $\mathcal{G}$TIC. Because (7.102) becomes a triangular matrix by permuting the first and third rows and columns, we may delete its third row and third column to obtain that (7.102) is in $\mathcal{G}$TIC iff (ii) holds, by Lemma A.1.1(b). □

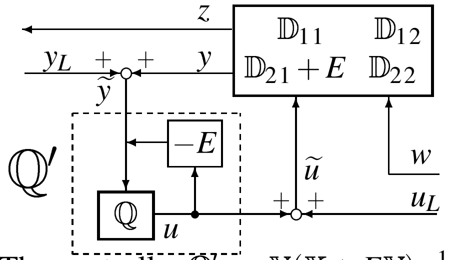


Figure 7.12: The controller $\mathbb{Q}' := \mathbb{Y}(\mathbb{X}+E\mathbb{Y})^{-1} : \widetilde{y} \mapsto u$ for $\mathbb{D}+\begin{bmatrix} 0 & 0 \\ E & 0 \end{bmatrix}$

We sometimes want to remove the feedthrough term from $\mathbb{D}_{21}$, hence we need the following lemma:

Lemma 7.3.23 ($D_{21} = 0$ w.l.o.g.) *Let $\mathbb{D} = \begin{bmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix} \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$. Let $F = \begin{bmatrix} 0 & 0 \\ E & 0 \end{bmatrix} \in \mathcal{B}(U \times W, Z \times Y)$.*

Then $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ is a stabilizing DPF-controller with d.c. internal loop for $\mathbb{D}$ iff $\mathbb{Q}' = \mathbb{Y}(\mathbb{X}+E\mathbb{Y})^{-1} = (\widetilde{\mathbb{X}}+\widetilde{\mathbb{Y}}E)^{-1}\widetilde{\mathbb{Y}}$ is a stabilizing DPF-controller with d.c. internal loop for $\mathbb{D}+F$. The corresponding closed-loop maps $w \mapsto z$ and $w \mapsto u$ (see (7.97)) are identical.

The controller $\mathbb{Q}'$ can be realized by adding to $\mathbb{Q}$ an output feedback through $-E$ in the same way as in Figure 7.12.

Finally, Hypothesis 7.3.15 holds for $\mathbb{D}$ iff it holds for $\mathbb{D}+F$.

Thus, when finding such a controller for a regular $\mathbb{D}$, (possibly under an additional restriction such as "$\|w \mapsto z\| < \gamma$") we may take $D_{21} = 0$ w.l.o.g. (see Lemma 7.2.7 for well-posedness of controllers). See also the remarks below Lemma 7.2.18.

Proof: Let a stabilizing $\mathbb{Q} = \mathbb{Y}\mathbb{X}^{-1} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$ be given. Let $\mathbb{N}, \mathbb{M}, \widetilde{\mathbb{N}}, \widetilde{\mathbb{M}}$ be as in Proposition 7.3.14. Then

$$\begin{bmatrix} \begin{bmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix} & \begin{bmatrix} 0 & \mathbb{Y} \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{N}_{21} & \mathbb{N}_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ E\mathbb{M}_{11} & E\mathbb{M}_{12} \end{bmatrix} & \begin{bmatrix} I & 0 \\ 0 & \mathbb{X} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & E\mathbb{Y} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \widetilde{\mathbb{X}} & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbb{Y}}E & 0 \\ 0 & 0 \end{bmatrix} & -\begin{bmatrix} 0 & \widetilde{\mathbb{Y}} \\ 0 & 0 \end{bmatrix} \\ -\begin{bmatrix} \widetilde{\mathbb{N}}_{11} & \widetilde{\mathbb{N}}_{12} \\ \widetilde{\mathbb{N}}_{21} & \widetilde{\mathbb{N}}_{22} \end{bmatrix} - \begin{bmatrix} \widetilde{\mathbb{M}}_{12}E & 0 \\ \widetilde{\mathbb{M}}_{22}E & 0 \end{bmatrix} & \begin{bmatrix} \widetilde{\mathbb{M}}_{11} & \widetilde{\mathbb{M}}_{12} \\ \widetilde{\mathbb{M}}_{21} & \widetilde{\mathbb{M}}_{22} \end{bmatrix} \end{bmatrix}^{-1} \tag{7.103}$$

is a d.c.f. of $\mathbb{D}+F$, by Lemma 6.5.7(a), hence then $\mathbb{Q}'$ DPF-stabilizes $\mathbb{D}+F$, by Theorem 7.3.19(iii). From this we obtain the equivalence (alternatively, we can obtain it directly by applying Lemma 7.2.18 to (7.79)).

Moreover, $\mathbb{P}' := \widetilde{\mathbb{Y}}(\mathbb{N}_{22}+E\mathbb{M}_{12}) - (\widetilde{\mathbb{X}}+\widetilde{\mathbb{Y}}E)\mathbb{M}_{12} = \mathbb{P}$, hence $\mathbb{N}_{11}\mathbb{P}+\mathbb{N}_{12} = \mathbb{N}_{11}\mathbb{P}'+\mathbb{N}_{12} : w \mapsto z$, by Corollary 7.3.20(c).

As in the proof of Corollary 7.3.20(c), we can verify that $\begin{bmatrix} u_L \\ w \end{bmatrix} \mapsto \begin{bmatrix} \widetilde{u} \\ w \end{bmatrix}$ is given by $\mathbb{M}'\widetilde{\mathbb{X}}' = \mathbb{M}\mathbb{G}\widetilde{\mathbb{X}}' = \begin{bmatrix} \mathbb{M}_{11}\widetilde{\mathbb{X}} & \mathbb{M}_{11}\mathbb{P}+\mathbb{M}_{12} \\ 0 & I \end{bmatrix}$, so that $\mathbb{M}_{11}\mathbb{P}+\mathbb{M}_{12} : w \mapsto u$ is unchanged. (Note that it would be more logical to have w_L in place of w and to have $w = 0$ and hence $\widetilde{w} = w_L$. Due to historical reasons, we denote $\widetilde{w} = w_L$ by w.)

(Alternatively, one can observe that $(w \mapsto u) = (y_L \mapsto u)\mathbb{D}_{22}$ and $(w \mapsto z) = \mathbb{D}_{12}+\mathbb{D}_{11}(y_L \mapsto u)\mathbb{D}_{22}$ are unaffected, by Lemma 7.2.18.)

The final claim follows from Proposition 7.3.14(ii)&(iii) (alternatively, from (7.103)). □

The following remark is obtained in the same way as Remark 7.2.19 was:

Remark 7.3.24 (Exponential DPF-stabilization) *By Remark 6.7.19, for any claims in this section (and others), we can deduce the corresponding results about* ω*-stabilization for some* $\omega \in \mathbf{R}$*, hence also for exponential stabilization.*

For example, if $\widetilde{\mathbb{Q}}$ *DPF-stabilizes* $\mathbb{D}$ *exponentially with internal loop and* $\widetilde{\mathbb{Q}}$ *or* $\mathbb{D}_{21}$ *has an exponential d.c.f., then so do* $\widetilde{\mathbb{Q}}$*,* $\mathbb{D}_{21}$ *and* $\mathbb{D}$*, by Lemma 7.3.13. Assume that this is the case.*

Then Hypothesis 7.3.15 holds and the two r.c.f.'s and l.c.f.'s assumed there are exponential ones, and a map $\mathbb{Q}$ *DPF-stabilizes* $\mathbb{D}$ *exponentially [with internal loop] iff* $\mathbb{Q}$ *DF-stabilizes* $\mathbb{D}_{21}$ *exponentially [with internal loop] (which in turn is equivalent for* $\widetilde{\mathbb{M}}_{22}\mathbb{X} - \widetilde{\mathbb{N}}_{21}\mathbb{Y}$ *being in* $\mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$*).*

Furthermore, all exponentially stabilizing DPF-controllers with internal loop are given by Corollary 7.3.20, if we choose (7.94) to be in $\mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$ *require also the parameters to be exponentially stable.*

For any optimizable and estimatable realizations of $\mathbb{D}$ *and* $\mathbb{Q}$ *(such do exist, by Theorem 6.6.28), the combined closed-loop system in Figure 7.10 becomes exponentially stable, by Theorem 7.3.11(c1). Similar remarks apply to parts (b) and (c) of the theorem and the results not mentioned here.*

Notes

Almost all standard classical results on DPF-stabilization (see, e.g., pp. 26–36 and 42–47 of [Francis]) are special cases or simple corollaries of those presented here. Any book on the H^∞ 4BP contains at least some theory on DPF-stabilization (often under the name "dynamic stabilization" or "chain scattering transformation"); see, e.g., [ZDG], [IOW] or [GL] for further theory on finite-dimensional systems and Section 2.7 of [Keu] on some results on Pritchard–Salamon systems. While this is being written, most of this section and some extended results have been included in [Sbook] (which is restricted to well-posed controllers). Further historical notes can be found in [CZ].

Some of the I/O results of this section have been presented in [Green] for well-posed rational transfer functions and later in [CZ] and [CG97] for the Callier–Desoer class (see Lemma 6.5.10(c)). However, many of their proofs cannot be extended to our generality, because the Corona Theorem (see Theorem 4.1.6(c)) only holds for matrix-valued transfer functions, by Lemma 4.1.10. (Theorem 4.1.6(c) for TIC is from [Tolokonnikov] (see [Nikolsky], p. 293). It is newer than [Vid] and it does not seem to be well known. Therefore, it might be that some of the results of [CZ] are not well-known to hold for general matrix-valued transfer functions.) Nevertheless, the book [CZ] contains also some further theory on dynamic partial feedback and robust control, some of which can be extended to our setting.

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Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations

Volume 2/3 — Optimal Control and Riccati Equations

Kalle Mikkola

$$
\begin{aligned}
K^*SK &= A^*\mathcal{P} + \mathcal{P}A + C^*JC \\
S &= D^*JD + \operatorname*{w-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B \\
K &= -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)
\end{aligned}
$$

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Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations

Volume 2/3 — Optimal Control and Riccati Equations

Kalle Mikkola

Dissertation for the degree of Doctor of Science in Technology to be presented with due permission of the Department of Engineering Physics and Mathematics, for public examination and debate in Council Room at Helsinki University of Technology (Espoo, Finland) on the 18th of October, 2002, at 12 o'clock noon.

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

Kalle Mikkola: *Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations*; Helsinki University of Technology Institute of Mathematics Research Reports A452 (2002).

Abstract: *In this monograph, we solve rather general linear, infinite-dimensional, time-invariant control problems, including the* H^∞ *and LQR problems, in terms of algebraic Riccati equations and of spectral or coprime factorizations. We work in the class of (weakly regular) well-posed linear systems (WPLSs) in the sense of G. Weiss and D. Salamon.*
Moreover, we develop the required theories, also of independent interest, on WPLSs, time-invariant operators, transfer and boundary functions, factorizations and Riccati equations. Finally, we present the corresponding theories and results also for discrete-time systems.

AMS subject classifications: 42A45, 46E40, 46G12, 47A68, 49J27, 49N10, 49N35, 93-02, 93A10, 93B36, 93B52, 93C05, 93C55, 93D15

Keywords: suboptimal H-infinity control, standard H-infinity problem, measurement feedback problem; H-infinity full information control problem, state feedback problem; Nehari problem; LQC, LQR control, H2 problem, minimization; bounded real lemma, positive real lemma; dynamic stabilization, controller with internal loop, strong stabilization, exponential stabilization, optimizability; canonical factorization, (J,S)-spectral factorization, (J,S)-inner coprime factorization, generalized factorization, J-lossless factorization; compatible operators, weakly regular well-posed linear systems, distributed parameter systems; continuous-time, discrete-time; infinite-horizon; time-invariant operators, Toeplitz operators, Wiener class, equally-spaced delays, Popov function; transfer functions, H-infinity boundary functions, Fourier multipliers; Bochner integral, strongly measurable functions, strong Lp spaces; Laplace transform, Fourier transform

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Chapter 8

Optimal Control ($\frac{\mathrm{d}}{\mathrm{du}}\mathcal{J} = 0$)

And if dearly that error hath cost me,
And more that I once could foresee,
I have found that, whatever it lost me,
It could not deprive me of thee.
— Lord Byron (1788–1824), "Stanzas to Augusta"

In this chapter we present an abstract theory on optimization and optimal control in state feedback form and the application of this theory to WPLSs with guidelines to more general or time-varying systems.

We shall study the critical points of a given cost function and the case where such control corresponds to a stabilizing state feedback pair. Such an "optimal" state feedback pair corresponds to a "stabilizing" solution of the Riccati equation, as shown in Chapter 9. The corresponding special control problems are solved in Chapters 10–12.

In Sections 8.1 and 8.2 we work in an abstract setting. The cost function $\mathcal{J}(x,u)$ is a quadratic function of vectors x and u, and we wish to find "controls" u that are critical points of $\mathcal{J}(x,\cdot)$ for a fixed "initial state" x; equivalently, for which the Fréchet derivative of $\mathcal{J}(x,\cdot)$ is zero. Such controls correspond to solutions of optimization problems (e.g., LQR, H^∞ or any other quadratic maximization, minimization or minimax problems). In the sequel, we prefer the word "critical" (or "J-critical") to "optimal", since in general critical points need not be optimums although the converse is always true.

We show that there is a unique J-critical control for each x iff there is a unique J-critical control for $x = 0$ and the abstract system is "stabilizable". Moreover, if this is the case, then the J-critical control can be written in "state feedback form".

In Section 8.2, we define and study "J-coercivity", which is a generalization of the standard nonsingularity assumptions of several control problems (including the "J-coercivity" assumptions defined in [S97b]–[S98d], the "Popov Toeplitz invertibility" assumption in the stable case and the "no transmission zeros" and "no invariant zeros" assumptions in the positive case). We show that any "stabilizable" J-coercive abstract system has a unique critical control for each initial state, so that the results of Section 8.1 can be applied. We also present some related results.

There are three reasons for the use of this abstract setting. First, this can be considered as a short-hand notation for WPLSs as to make the optimization theory simple, clear and neat. Second, this theory can be applied more generally, as indicated in Section 8.5 on time-varying systems and in Section 8.6 on systems whose input operator ("B") is allowed to be more unbounded than those of WPLSs. However, we only give guidelines for these settings since they go beyond the scope of this book.

The third and most important reason is that when working with the H^∞ problem in Section 11.7, we have to optimize the control for a fixed state *and* a fixed disturbance; the WPLS framework does not cover such optimization. Therefore, we solve the H^∞ full-information control problem in the abstract framework of Section 8.1, although the results will be applied to WPLSs only.

Naturally, in Chapters 9–12 we have to work hard on the "raw WPLS solutions" obtained as corollaries of the abstract theory, before we can turn them into direct generalizations of classical control theory results.

In Sections 8.3 and 8.4 we apply our abstract optimization theory to obtain a very general theory on control problems for WPLSs. Given a WPLS $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]$ and a cost operator $J=J^*\in\mathcal{B}(Y)$, we study the *cost function*

$$\mathcal{J}(x_0,u):=\int_0^\infty \langle y(t),Jy(t)\rangle_Y\,dt,\quad \text{where}\quad y:=\mathbb{C}x_0+\mathbb{D}u\quad (x_0\in H,\ u:\mathbf{R}_+\to U) \tag{8.1}$$

(for suitable $\mathbb{C}$ and $\mathbb{D}$ this covers all classical control problems mentioned above) and u is required to be exponentially stabilizing, strongly stabilizing, stabilizing or something similar, depending on how stable one wishes the closed-loop system to be.

Under J-coercivivity and stabilizability assumptions, there is a unique J-critical ("optimal") control for any initial state, and this optimal control can be given in a WPLS form (this generalizes the corresponding result in [FLT]). However, the corresponding feedback need not be well-posed without additional assumptions on the system, as illustrated in Examples 8.4.13 and 11.3.7. This leads to some additional difficulties in the Riccati equation theory (the situation is the same even in the case studied in [FLT]).

In Theorem 8.4.5, we extend the standard result that an optimization problem over exponentially stabilizing controllers can be solved by first finding a preliminary exponentially stabilizing controller and then optimizing over stable controls for the preliminary controlled system. We also give corresponding results over other forms of stabilization, but for them one needs an additional quasi-coprimeness ("q.r.c.-") assumption. Then we give further results on J-coercivity and recall its connection to spectral and coprime factorizations for maps in MTIC classes.

Sections 8.1 and 8.2 are written for a the abstract setting of Hypothesis 8.1.1 (see also Hypothesis 8.2.2). Hypothesis 8.3.1 is assumed throughout this Sections 8.3–8.5, and Hypothesis 8.6.1 is assumed through Sections 8.6.

8.1 Abstract J-critical control ($Jy_{ycrit} \perp \Delta y$)

The more control, the more that requires control.

In this section we define the set $\mathcal{U}$ of admissible controls and the cost function $\mathcal{J}$ and study their basic properties.

Standing Hypothesis 8.1.1 *Throughout this section and Section 8.2 we shall assume that U, X, Y^{s}, Z^{s} and are Banach spaces, that Y and Z are TVSs, and that the embeddings $Y^{\mathrm{s}} \subset Y$ and $Z^{\mathrm{s}} \subset Z$, are continuous. We also assume that $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(X \times U, Z \times Y)$ and $J = J^* \in \mathcal{B}(Y^{\mathrm{s}}, Y^{\mathrm{s}*})$.*

All results given in these two sections are valid whether we use linear or conjugate-linear duals, i.e., $X^* := X^{\mathrm{B}}$ or $X^* := X^{\mathrm{d}}$; see Remark A.3.22 for details. In particular, we may use Hilbert space adjoints instead of Banach space adjoints.

We use sesquilinear duals and adjoints (see Remark A.3.22) and the notation of Lemma A.3.5 (e.g., "$J = J^*$" means that $J^*|_{Y^{\mathrm{s}}} = J$), hence the results look as if the spaces in Standing Hypothesis 8.1.1 were Hilbert spaces.

Outside Sections 8.1 and 8.2, we shall apply these results only in the case where Y^{s} is a Hilbert space. Therefore, we recommend the reader to consider only this Hilbert space setting so that $Y^{\mathrm{s}*} = Y^{\mathrm{s}}$ and hence there is nothing special with inner products or self-adjointness.

Remark 8.1.2 *In this section one may allow $U, X, Y^{\mathrm{s}}, Z^{\mathrm{s}}$ to be arbitrary* F-spaces *(i.e., complete metrizable TVSs, see Theorem 1.24 of [Rud73]), because the Closed Graph Theorem (Theorem 2.15 of [Rud73]) is the only nongeneral TVS property that we use here.*

Given an *input* $u \in U$ and *initial state* $x \in X$, we call $z := Ax + Bu \in Z$ the *state*, $y := Cx + Du \in Y$ the *output* and $\langle y, Jy \rangle$ the *cost* of the "system" for u and x s.t. the state and output are *stable*, i.e., $z \in Z^{\mathrm{s}}$ and $y \in Y^{\mathrm{s}}$ As before, we set $\|y\|_{Y^{\mathrm{s}}} = +\infty$ for $y \notin Y^{\mathrm{s}}$, etc.

We could have dropped A, B, Z^{s} and Z from the theory without reducing generality (replace C by $\left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]$ and D by $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ etc.), but we have chosen this more explicit presentation to make later applications more obvious.

The simplest application of this theory to WPLSs is obtained by the substitutions $U \mapsto \mathrm{L}^2(\mathbf{R}_+; U)$, $Y^{\mathrm{s}} \mapsto \mathrm{L}^2(\mathbf{R}_+; Y)$, $Y \mapsto \mathrm{L}^2_\omega(\mathbf{R}_+; Y)$, $X \mapsto H$, $Z^{\mathrm{s}}, Z \mapsto \mathrm{L}^2_\omega(\mathbf{R}_+; H)$, $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B}\tau \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ and $x \mapsto x_0 \in H$, where U, H, Y are Hilbert spaces, $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U, H, Y)$ and $\omega \in \mathbf{R}$.

It follows that $\mathcal{U}(x) := \{u \in U \,|\, Cx + Du \in Y^{\mathrm{s}},\ Ax + Bu \in Z^{\mathrm{s}}\}$ becomes the space of stable (unless we take $U = \mathrm{L}^2_\omega$) controls that make the output $Cx + Du$ stable. Sometimes we also require the state $\mathbb{A}x_0 + \mathbb{B}\tau u$ to be stable, i.e., we set $Z^{\mathrm{s}} := \mathrm{L}^2(\mathbf{R}_+; H)$. See Remark 8.3.4 for further applications to WPLSs.

Now for the general definitions:

Definition 8.1.3 (*J*-critical control) *For each $x \in X$, we set*

$$\mathcal{U}(x) := \{u \in U \,\big|\, Cx + Du \in Y^{\mathrm{s}} \;\&\; Ax + Bu \in Z^{\mathrm{s}}\} \tag{8.2}$$

$$\mathcal{Y}(x) := \{Cx + Du \,\big|\, u \in \mathcal{U}(x)\}, \tag{8.3}$$

$$\mathcal{J}(x,u) := \langle Cx + Du, J(Cx + Du)\rangle \quad (u \in \mathcal{U}(x)). \tag{8.4}$$

We call $\mathcal{J}$ the cost function. *A control $u \in \mathcal{U}(x)$ (resp. output $Cx + Du \in \mathcal{Y}(x)$) is called J-*critical *for x (w.r.t. $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$) if $\langle Cx + Du, JD\eta\rangle = 0$ for all $\eta \in \mathcal{U}(0)$.*

Note that $\mathcal{U}(x) \subset U$ and $\mathcal{Y}(x) \subset Y^{\mathrm{s}} \subset Y$ for all $x \in X$.

Given an initial state $x \in X$, we often wish to find a *stabilizing control* (i.e., $u \in \mathcal{U}$) s.t. the output $y := Cx + Du \in \mathcal{Y}$ is optimizes the cost $\mathcal{J}(x,u) = \langle y, Jy\rangle$ in some sense. Because optimality (in most reasonable settings) requires that the Fréchet derivative of the cost function is zero, or equivalently, that the control is J-critical (by Lemma 8.1.6), we shall concentrate on J-critical controls in this section. See Sections 8.2 and 11.7 for applications.

In the following few lemmas we list simple algebraic properties of the above concepts for future use:

Lemma 8.1.4 ($\mathcal{U}(x)$ and $\mathcal{Y}(x)$) *The sets $\mathcal{U}(0) \subset U$ and $\mathcal{Y}(0) \subset Y^{\mathrm{s}}$ are linear subspaces. Let $x \in X$, $u \in \mathcal{U}(x)$, $y \in \mathcal{Y}(x)$. Then $\mathcal{U}(x) = u + \mathcal{U}(0)$ and $\mathcal{Y}(x) = y + \mathcal{Y}(0)$.*

Moreover, $\mathcal{U}(\alpha x_0 + \beta x_1) = \alpha\mathcal{U}(x_0) + \beta\mathcal{U}(x_1)$ and $\mathcal{Y}(\alpha x_0 + \beta x_1) = \alpha\mathcal{Y}(x_0) + \beta\mathcal{Y}(x_1)$ ($\alpha, \beta \in \mathbf{C} \setminus \{0\}$, $x_0, x_1 \in X$ s.t. $\mathcal{U}(x_0) \neq \emptyset$). □

(The very easy proof is left to the reader. The last two formulae need not hold for x_0, x_1 s.t. $\mathcal{U}(x_0) = \emptyset = \mathcal{U}(x_1)$.)

The case where $Z^{\mathrm{s}} = Z$ and $Y^{\mathrm{s}} = Y$ is called the stable case, because if(f) $\mathcal{U}(x) = U$ for all $x \in X$ (i.e., if(f) all controls are stabilizing), then we can replace Z^{s} by Z and Y^{s} by Y, w.l.o.g., by the following lemma:

Lemma 8.1.5 (Stable case) *If $Y^{\mathrm{s}} = Y$ and $Z^{\mathrm{s}} = Z$, then $\mathcal{U}(x) = U$ for all $x \in X$. We have $\mathcal{U}(x) = U$ for all $x \in X$ iff $C \in \mathcal{B}(X, Y^{\mathrm{s}})$, $D \in \mathcal{B}(U, Y^{\mathrm{s}})$, $A \in \mathcal{B}(X, Z^{\mathrm{s}})$, $B \in \mathcal{B}(U, Z^{\mathrm{s}})$.* □

(This follows from Standing Hypothesis 8.1.1 and Lemma A.3.6.)

With the standard substitutions mentioned above, the stable case is the case where each x and u produce a stable output, i.e., where $\mathbb{C}$ and $\mathbb{D}$ are stable (that is, $\Sigma \in$ SOS; the alternative substitutions with $Z^{\mathrm{s}} \mapsto \mathrm{L}^2(\mathbf{R}_+; H)$ correspond to exponentially stable Σ).

The J-critical controls are exactly the zeros of the derivative of the cost:

Lemma 8.1.6 (*J*-critical $\Leftrightarrow \frac{\mathrm{d}\mathcal{J}}{\mathrm{d}u} = 0$) *Let $x \in X$.*

A control $u_{\mathrm{crit}} \in \mathcal{U}(x)$ is J-critical for x iff $\frac{\mathrm{d}\mathcal{J}(x,u)}{\mathrm{d}u}(u_{\mathrm{crit}}) = 0$. In particular, if $u_{\mathrm{crit}}(x)$ is a local extremal point of $\mathcal{J}(x,\cdot)$ (on $\mathcal{U}(x)$), then $u_{\mathrm{crit}}(x)$ is J-critical for x.

Here $\frac{\mathrm{d}\mathcal{J}(x,u)}{\mathrm{d}u}$ is the *(real) Fréchet derivative* of $\mathcal{J}(x,\cdot)$ on its domain $\mathcal{U}(x)$. Thus, $\frac{\mathrm{d}\mathcal{J}(x,u)}{\mathrm{d}u}(\widetilde{u}) : \mathcal{U}(0) \to \mathbf{R}$, where $\widetilde{u} \in \mathcal{U}(x)$, means the operator

$$\mathcal{U}(0) \ni \eta \mapsto \frac{\mathrm{d}\mathcal{J}(x,\widetilde{u}+t\eta)}{\mathrm{d}t}(0) = \lim_{t\to 0} \frac{\mathcal{J}(x,\widetilde{u}+t\eta) - \mathcal{J}(x,\widetilde{u})}{t} \in \mathbf{R}. \tag{8.5}$$

Proof: 1° *"Iff":* By Lemma 8.1.4, $\mathcal{U}(x) = u_{\text{crit}} + \mathcal{U}(0)$. By linearity and continuity, $\frac{\mathrm{d}\mathcal{J}(x,u_{\text{crit}}+t\eta)}{\mathrm{d}t}(0) = 2\operatorname{Re}\langle Cx+Du, JD\eta\rangle$. This is zero for all $\eta \in \mathcal{U}(0)$ iff $\langle Cx+Du, JD\eta\rangle = 0$ for all $\eta \in \mathcal{U}(0)$, (apply the right-hand-side to η and $i\eta$), i.e., iff u_{crit} is J-critical for x.

It is obvious that $\mathcal{J}(x,\widetilde{u}+t\eta)$ has an extremum at $t=0$ if $\widetilde{u}$ is an extremum (or a saddle point) of $\mathcal{J}(x,\cdot)$. □

Given a critical control u_{crit}, the cost for $u_{\text{crit}}+\eta$ equals the critical cost plus the cost for η:

Lemma 8.1.7 (Critical cost $\mathcal{J}(x,u_{\text{crit}})$) *Let $x \in X$ and $u_{\text{crit}} \in \mathcal{U}(x)$. Set $y_{\text{crit}} := Cx + Du_{\text{crit}}$. Then the following are equivalent:*

(i) u_{crit} is J-critical for x;

(ii) $\mathcal{J}(x,u_{\text{crit}}+\eta) = \langle y_{\text{crit}}, Jy_{\text{crit}}\rangle + \langle D\eta, JD\eta\rangle \quad (\eta \in \mathcal{U}(0))$;

(iii) $\langle Cx+D(u_{\text{crit}}+\eta_1), J(Cx+D(u_{\text{crit}}+\eta_2))\rangle = \langle y_{\text{crit}}, Jy_{\text{crit}}\rangle + \langle D\eta_1, JD\eta_2\rangle$ $(\eta_1,\eta_2 \in \mathcal{U}(0))$.

Note that (ii) means that $\mathcal{J}(x,u_{\text{crit}}+\eta) = \mathcal{J}(x,u_{\text{crit}}) + \mathcal{J}(0,\eta)$.

Proof: By a direct computation, (i)⇒(iii)⇒(ii) ((i) implies that the cross terms of (iii) are zero).

Assume (ii). Then $2\operatorname{Re}\langle y_{\text{crit}}, JD\eta\rangle = 0$ for all $\eta \in \mathcal{U}(0)$. An application to η and $i\eta$ shows that $\langle y_{\text{crit}}, JD\eta\rangle = 0$ for all $\eta \in \mathcal{U}(0)$, i.e., that (i) holds. □

For positive cost functions ($\mathcal{J}(0,\cdot) \geq 0$), "$J$-critical" is equivalent to "minimizing":

Corollary 8.1.8 (Minimizing ⇔ J-critical & ≥ 0) *A control $u \in \mathcal{U}(x)$ minimizes [strictly] $\mathcal{J}(x,\cdot)$ (on $\mathcal{U}(x)$) iff u is J-critical and $\langle D\eta, JD\eta\rangle \geq 0$ [> 0] for all nonzero $\eta \in \mathcal{U}(0)$.*

Note that $\mathcal{J}(0,\eta) = \langle D\eta, JD\eta\rangle$.

Proof: "If": This follows from Lemma 8.1.7(ii). "Only if": This follows from Lemma 8.1.6 and Lemma 8.1.7(ii). □

All critical controls produce the same cost and the same sensitivity of the cost to a disturbance:

Lemma 8.1.9 ($\mathcal{U}^{\text{crit}}$ and uniqueness) *Denote by $\mathcal{U}^{\text{crit}}(x)$ the set of J-critical controls for each $x \in X$. Then $\mathcal{U}^{\text{crit}}(0)$ is a linear subspace of $\mathcal{U}(0)$. If $u \in \mathcal{U}^{\text{crit}}(x)$, $x, x' \in X$, then $\mathcal{U}^{\text{crit}}(x+x') = u + \mathcal{U}^{\text{crit}}(x')$. Moreover, $\mathcal{J}(x,u) = \mathcal{J}(x,v)$, and $\mathcal{J}(x,u+\eta) = \mathcal{J}(x,v+\eta)$ for $u,v \in \mathcal{U}^{\text{crit}}(x)$, $\eta \in \mathcal{U}(0)$.*

In particular, there is at most one J-critical control for each $x \in X$ iff $\mathcal{U}^{\text{crit}}(0) = \{0\}$.

Proof: 1° $\mathcal{U}^{\mathrm{crit}}(x+x')=u+\mathcal{U}^{\mathrm{crit}}(x')$: Let $u\in\mathcal{U}^{\mathrm{crit}}(x)$. For an arbitrary $u'\in U$, we have $u+u'\in\mathcal{U}^{\mathrm{crit}}(x+x')$ iff

$$\langle Cx+Du+Cx'+Du',JD\eta\rangle=0 \quad (\eta\in\mathcal{U}(0)), \tag{8.6}$$

equivalently, iff $u'\in\mathcal{U}^{\mathrm{crit}}(x')$. Therefore, $\mathcal{U}^{\mathrm{crit}}(x+x')=u+\mathcal{U}^{\mathrm{crit}}(x')$.

2° Obviously, $\mathcal{U}^{\mathrm{crit}}(0)$ is a subspace of $\mathcal{U}(0)$.

3° Let $u,u+\widetilde{u}\in\mathcal{U}^{\mathrm{crit}}(x)$, (so that $\widetilde{u}\in\mathcal{U}^{\mathrm{crit}}(0)$, by 1°). By Lemma 8.1.7(ii), we have $\mathcal{J}(x,u+\widetilde{u})=\mathcal{J}(x,u)+\langle D\widetilde{u},JD\widetilde{u}\rangle=\mathcal{J}(x,u)$. If $\eta\in\mathcal{U}(0)$, then

$$\mathcal{J}(x,u+\widetilde{u}+\eta)=\mathcal{J}(x,u+\widetilde{u})+\langle D\eta,JD\eta\rangle=\mathcal{J}(x,u)+\langle D\eta,JD\eta\rangle=\mathcal{J}(x,u+\eta). \tag{8.7}$$

4° Since always $0\in\mathcal{U}^{\mathrm{crit}}(0)$, the last claim follows from the identity $\mathcal{U}^{\mathrm{crit}}(x+x')=u+\mathcal{U}^{\mathrm{crit}}(x')$ (with $x'=0$). □

We shall later give several sufficient conditions for the existence of a unique J-critical control. Such a control and corresponding state and output are always produced by a kind of an abstract system (in "state feedback form"):

Theorem 8.1.10 ($\boldsymbol{\Sigma_{\mathrm{crit}}}$) *Let there be a unique J-critical control $u_{\mathrm{crit}}(x)$ for each $x\in X$. Define*

$$\Sigma_{\mathrm{crit}}:=\begin{bmatrix}A_{\mathrm{crit}}\\C_{\mathrm{crit}}\\K_{\mathrm{crit}}\end{bmatrix}:x\mapsto\begin{bmatrix}Ax+Bu_{\mathrm{crit}}(x)\\Cx+Du_{\mathrm{crit}}(x)\\u_{\mathrm{crit}}\end{bmatrix}=:\begin{bmatrix}z_{\mathrm{crit}}(x)\\y_{\mathrm{crit}}(x)\\u_{\mathrm{crit}}(x)\end{bmatrix}. \tag{8.8}$$

Then $\Sigma_{\mathrm{crit}}\in\mathcal{B}(X,Z^{\mathrm{s}}\times Y^{\mathrm{s}}\times U)$. Moreover, by setting $\mathcal{P}:=C_{\mathrm{crit}}^*JC_{\mathrm{crit}}\in\mathcal{B}(X,X^*)$ we obtain

$$\mathcal{J}(x,u_{\mathrm{crit}}(x)+\eta)=\langle x,\mathcal{P}x\rangle_{\langle X,X^*\rangle}+\mathcal{J}(0,\eta)=\langle y_{\mathrm{crit}}(x),Jy_{\mathrm{crit}}(x)\rangle_{Y^{\mathrm{s}}}+\langle D\eta,JD\eta\rangle_{Y^{\mathrm{s}}}. \tag{8.9}$$

for $x\in X$ and $\eta\in\mathcal{U}(0)$.

(Stable case) *If $A\in\mathcal{B}(X,Z^{\mathrm{s}})$ and $C\in\mathcal{B}(X,Y^{\mathrm{s}})$, or $B\in\mathcal{B}(U,Z^{\mathrm{s}})$ and $D\in\mathcal{B}(U,Y^{\mathrm{s}})$, then $C\in\mathcal{B}(X,Y^{\mathrm{s}})$ and $\mathcal{P}=C^*JC_{\mathrm{crit}}=C_{\mathrm{crit}}^*JC$.*

Thus, we can consider $K_{\mathrm{crit}}\in\mathcal{B}(X,U)$ as a "state feedback operator", and Σ_{crit} as the left column of the corresponding "closed-loop system".

In Theorem 8.3.9 we shall show that if $\left[\begin{array}{c|c}A&B\\\hline C&D\end{array}\right]$ is a WPLS, then Σ_{crit} is a WPLS. For sufficiently smooth WPLSs (and for all wpls's), Σ_{crit} becomes the left column of the closed-loop system corresponding to a state feedback operator determined by the stabilizing solution of the corresponding Riccati equation (see, e.g., Section 9.2).

Proof: 1° "$C_{\mathrm{crit}},K_{\mathrm{crit}}\in\mathcal{B}$": The map $K_{\mathrm{crit}}:x\to u_{\mathrm{crit}}(x)$ are obviously linear, hence so are $C_{\mathrm{crit}}:x\to y_{\mathrm{crit}}(x)$ and $A_{\mathrm{crit}}:x\to z_{\mathrm{crit}}(x)$, i.e., $\Sigma_{\mathrm{crit}}\in\mathrm{Hom}(X,Z^{\mathrm{s}}\times Y^{\mathrm{s}}\times U)$. To show that Σ_{crit} is bounded, we use the Closed Graph Theorem.

Assume that $x_n\to0$ in X and $\Sigma_{\mathrm{crit}}x_n\to\left[\begin{smallmatrix}z\\y\\u\end{smallmatrix}\right]$ in $Z^{\mathrm{s}}\times Y^{\mathrm{s}}\times U$ (by Lemma A.3.4(E1), we only need to show that $z,y,u=0$).

Then $C_{\mathrm{crit}}x_n \to y \in Y^{\mathrm{s}}$, but $C_{\mathrm{crit}}x_n = Cx_n + DK_{\mathrm{crit}}x_n \to 0 + Du$ in Y, hence $y = Du$. Analogously, $z = Bu$, hence $u \in \mathcal{U}(0)$. But

$$\langle D\eta, JDu\rangle = \lim_{n\to+\infty} \langle D\eta, JC_{\mathrm{crit}}x_n\rangle = \lim_{n\to+\infty} 0 = 0 \tag{8.10}$$

for all $\eta \in \mathcal{U}(0)$, hence $u = u_{\mathrm{crit}}(0) = 0$; consequently, $y = Du = 0$ and $z = Bu = 0$. Thus, $\Sigma_{\mathrm{crit}} \in \mathcal{B}$.

2° *(8.9):* This follows from the definitions of C_{crit} and $\mathcal{J}$, and from Lemma 8.1.7(ii).

3° *Case $A \in \mathcal{B}(X, Z^{\mathrm{s}})$, $C \in \mathcal{B}(X, Y^{\mathrm{s}})$:* Let $x \in X$. Obviously, $\mathcal{U}(x) = \mathcal{U}(0)$. Moreover, $DK_{\mathrm{crit}}x = C_{\mathrm{crit}}x - Cx \in Y^{\mathrm{s}}$, and $\langle DK_{\mathrm{crit}}x, JC_{\mathrm{crit}}x\rangle = 0$, because $K_{\mathrm{crit}}x \in \mathcal{U}(x) = \mathcal{U}(0)$. Therefore,

$$\langle Cx, JC_{\mathrm{crit}}x\rangle = \langle C_{\mathrm{crit}}x, JC_{\mathrm{crit}}x\rangle - 0 = \langle x, \mathcal{P}x\rangle \tag{8.11}$$

Because $x \in X$ was arbitrary, we have $\mathcal{P} = C^* JC_{\mathrm{crit}}$, hence $C_{\mathrm{crit}}^* JC = \mathcal{P}^* = \mathcal{P}$.

4° *Case $B \in \mathcal{B}(U, Z^{\mathrm{s}})$, $D \in \mathcal{B}(U, Y^{\mathrm{s}})$:* Now $Ax = A_{\mathrm{crit}}x - BK_{\mathrm{crit}}x \in Z^{\mathrm{s}}$, $Cx = C_{\mathrm{crit}}x - DK_{\mathrm{crit}}x \in Y^{\mathrm{s}}$, for any $x \in X$. Therefore, $A \in \mathcal{B}(X, Z^{\mathrm{s}})$, $C \in \mathcal{B}(X, Y^{\mathrm{s}})$, by Lemma A.3.6. Thus, the rest follows from 3°. □

(See the notes on p. 362.)

8.2 Abstract J-coercivity ($\mathcal{J} \mapsto \begin{bmatrix} u & Du \end{bmatrix}$)

Coercion. The unpardonable crime.

— Dorothy Miller Richardson (1873–1957)

In this section we aim to give sufficient conditions to the existence of a unique "optimal" (J-critical) control, so that the abstract optimization theory of Section 8.1 can be applied. The most important of such conditions is that the system is "stabilizable" (i.e., $\mathcal{U}(x) \neq \emptyset$ for all $x \in X$) and J-coercive.

In the WPLS framework with $J = I$ (a minimization problem), J-coercivity means the condition $\|\mathbb{D}u\|_2 \geq \varepsilon\|u\|_2$ or something similar, depending on the desired set of admissible controls u. The concept J-coercivity is a generalization of the standard nonsingularity assumptions of several control problems.

Indeed, later in this section and in Section 11.7 (and in their WPLS applications in Chapters 10–12), we shall show that all optimization (sub)problems faced in rather general LQR and H^∞ settings are J-coercive. Moreover, in Propositions 10.3.1 and 10.3.2 we show that (the extensions to WPLSs of) all different classical nonsingularity assumptions on LQR settings are equivalent.

At the end of this section, we shall give applications to stable systems, including the direct formula for the optimal control given in Lemma 8.2.9. Recall the assumptions of Standing Hypothesis 8.1.1 and note also those of Hypothesis 8.2.2.

We start by defining J-coercivity. The map D is J-coercive iff $\mathcal{J}(0,u) \mapsto \begin{bmatrix} u & Du \end{bmatrix}$ is continuous:

Definition 8.2.1 (J-coercive) *We call D J-*coercive (over $\mathcal{U}$) *if there is $\varepsilon > 0$ s.t. for all nonzero $u \in \mathcal{U}(0)$ there is a nonzero $v \in \mathcal{U}(0)$ s.t.*

$$\langle Dv, JDu\rangle \geq \varepsilon\|u\|_D\|v\|_D. \tag{8.12}$$

where $\|u\|_D := \max\{\|u\|_U, \|Bu\|_{Z^s}, \|Du\|_{Y^s}\}$. If, in addition, $\langle Du, JDu\rangle \geq 0$ for each $u \in \mathcal{U}(0)$, then D is called positively J-coercive.

By Lemma 8.2.3(c2), $\mathbb{D}$ is positively J-coercive iff "we can take $v = u$", i.e., iff $\langle Du, JDu\rangle \geq \varepsilon\|u\|_D^2$ for all $u \in \mathcal{U}(0)$ and some $\varepsilon > 0$. Obviously, we can replace $\|\cdot\|_D$ by an equivalent norm in (8.12).

When applying the results of this section, we often take $Z^s = Z$ (or $A = 0 = B$), so that A and B become continuous and hence insignifant and can be dropped from all formulae. In particular, then $\|u\|_D$ becomes equivalent to $\|u\|_D' := \max\{\|u\|_U, \|Du\|_{Y^s}\}$.

We shall use certain techniques that require reflexivity:

Standing Hypothesis 8.2.2 *Throughout the rest of this section, except in Lemmas 8.2.4, 8.2.8 and 8.2.9, we assume that U, Z^s and Y^s are reflexive.*

This hypothesis allows us to fully apply Lemma A.3.5(c2):

Lemma 8.2.3 *Equip $\mathcal{U}(0)$ with norm $\|\cdot\|_D$. Then we have the following:*

(a) **(Stable case)** *Let $Y^s = Y$ and $Z^s = Z$. Then $\mathcal{U}(0)$ and U are (TVS) isomorphic, hence D is J-coercive iff $D^*JD \in \mathcal{GB}(U,U^*)$. Moreover, D is positively J-coercive iff $D^*JD \gg 0$.*

(b) $\mathcal{U}(0)$ is a reflexive Banach space (a Hilbert space if U, Z^s and Y^s are Hilbert spaces), and $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]_{|\mathcal{U}(0)} \in \mathcal{B}(\mathcal{U}(0), Z^s \times Y^s)$.

(c1) D is J-coercive iff $D^*JD \in \mathcal{GB}(\mathcal{U}(0))$.

(c2) D is positively J-coercive iff $D^*JD \gg 0$ on $\mathcal{U}(0)$.

(d1) $\mathbb{D}$ is positively J-coercive iff there is $\varepsilon > 0$ s.t. for all $u \in \mathcal{U}(0)$ we have

$$\langle Du, JDu\rangle \geq \varepsilon(\|u\|_U^2 + \|Bu\|_{Z^s}^2 + \|Du\|_{Y^s}^2). \tag{8.13}$$

(d2) Let $J \gg 0$. Then D is positively J-coercive iff $\|Du\|_{Y^s} \geq \varepsilon'(\|u\|_U + \|Bu\|_{Z^s})$ for some $\varepsilon' > 0$ and all $u \in \mathcal{U}$.

(d3) If D is J-coercive, then D is injective on U.

(d4) Let D be J-coercive. Then $\|JDu\|_{Y^{s*}} \geq \varepsilon'\|u\|_D$ for some $\varepsilon' > 0$ and all $u \in \mathcal{U}(0)$.

(e) Even if Standing Hypothesis 8.2.2 does not hold, claims (d1)–(d4) above hold and J-coercivity implies that there is at most one J-critical control for each $x_0 \in X$.

If $J \geq 0$, $Y^s = Y$ and $Z^s = Z$, then D is J-coercive iff $\langle Du, JDu\rangle \geq \varepsilon\|u\|_U^2$ ($u \in U$) for some $\varepsilon > 0$, by (a). The operator D^*JD can be considered as the Popov Toeplitz operator.

Proof: (a) Obviously, $\|\cdot\|_D$ is now equivalent to $\|\cdot\|_U$ on $U = \mathcal{U}(0)$, hence the claims follow from (c1)&(c2).

(b) By Lemma A.3.15 (with $T := \begin{bmatrix} I^* & B^* & D^* \end{bmatrix}^*$), $\mathcal{U}(0)$ is a reflexive Banach space (a Hilbert space if U, Z^s and Y^s are Hilbert spaces) and $T \in \mathcal{B}(\mathcal{U}(0), U \times Z^s \times Y^s)$.

(c1)&(c2) These follow from Lemma A.3.4(N4)(xi) and Lemma A.3.5(c2)&(d),

(d1) This a reformulation of (c2) (with an equivalent norm).

(d2) Now $D^*JD \gg 0 \Leftrightarrow D^*D \gg 0 \Leftrightarrow \|Du\|_{Y^s} \geq \varepsilon'\|u\|_D$ ($u \in \mathcal{U}(0)$) $\Leftrightarrow$ $\|Du\|_{Y^s} \geq \varepsilon'(\|u\|_U + \|Bu\|_{Z^s})$ ($u \in \mathcal{U}(0)$) (note that this does not hold for all $u \in U$ in general).

(d3) This follows from (c1).

(d4) Let $u \in \mathcal{U}(0)$. Let $\varepsilon > 0$ and v be as in Definition 8.2.1. Then

$$\varepsilon\|u\|_D\|Dv\| \leq \varepsilon\|u\|_D\|v\|_D \leq \langle Dv, JDu\rangle \leq \|JDu\|\|Dv\|. \tag{8.14}$$

Because $Dv \neq 0$, by (d3), we have $\|JDu\|_2 \geq \varepsilon\|u\|_D$. Since u was arbitrary, (d4) holds.

(e) For (d1)–(d4) this is obvious; the uniqueness claim follows from 2° of the proof of Theorem 8.2.5. □

A coordinate change in the input space affects J-coercivity and J-critical control in the expected way:

Lemma 8.2.4 (D vs. DE) *Let $E \in \mathcal{GB}(U)$. Let $\mathcal{U}_{DE}(x) := \{u \in U \mid [\begin{smallmatrix} A & BE \\ C & DE \end{smallmatrix}][\begin{smallmatrix} x \\ u \end{smallmatrix}] \in Y^{\mathrm{s}} \times Z^{\mathrm{s}}\}$. Then we have the following:*

(a) $\mathcal{U}_{DE}(x) = E^{-1}\,\mathcal{U}(x)$ for all $x \in X$.

(b) A control u is J-critical for x and $[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$ iff $E^{-1}u$ is J-critical for x and $[\begin{smallmatrix} A & BE \\ C & DE \end{smallmatrix}]$.

(c) D is J-coercive iff DE is J-coercive.

Naturally, here the J-coercivity of DE refers to the set $\{u \in U \mid DEu \in Y^{\mathrm{s}}\}$ in place of $\mathcal{U}(0)$ and to the norm $\|u\|_{DE} := \max\{\|u\|, \|BEu\|, \|DEu\|\}$ in place of $\|u\|_D$.

Proof: Claims (a) and (b) are trivial. Choose $\delta > 0$ is s.t. $\delta\|u\| \le \|Eu\| \le \delta^{-1}\|u\|$ $(u \in U)$. Then $\delta\|u\|_D \le \|E^{-1}u\|_{DE} \le \delta^{-1}\|u\|_D$ $(u \in U)$, hence these two norms are equivalent. From this and (a) we obtain (c) easily. □

Trivially, $\mathcal{U}(x) \neq \emptyset$ is a necessary control for the existence of a J-critical control for x. For J-coercive systems, this is also sufficient:

Theorem 8.2.5 (J-coercive $\Rightarrow \exists!J$-critical control) *Assume that D is J-coercive. If $x \in X$ is s.t. $\mathcal{U}(x) \neq \emptyset$, then there is a unique J-critical control for x.*

This follows from the fact that J-coercivity allows us to project an element of $J(Cx + D\mathcal{U}(x))$ continuously. However, J-coercivity is not the weakest possible assumption for the above theorem: whenever $J = I$, $C = 0$ and $D > 0$, then there is a unique $\mathcal{J}$-minimizing control for all $x \in X$ even if $D \not\gg 0$.

Nevertheless, even for $J = I$, conditions that are necessary and sufficient do not seem to be useful (e.g., "for some (hence all) $y \in \mathcal{Y}(x)$, the orthogonal projection of y to $\overline{\mathrm{Ran}(D)}$ is in $\mathrm{Ran}(D)$").

Proof: 1° *Existence:* We write $E := D_{|\mathcal{U}(0)}$ to clarify the proof. By Lemma 8.2.3(b)&(c1), we have $E \in \mathcal{B}(\mathcal{U}(0), Y^{\mathrm{s}})$ and $E^*JE \in \mathcal{GB}(\mathcal{U}(0))$.

Let $x \in X$. Choose $u' \in \mathcal{U}(x)$, and set $y' := Cx + Du' \in Y^{\mathrm{s}}$. Set $u'' := -(E^*JE)^{-1}E^*Jy' \in \mathcal{U}(0)$, $u := u' + u'' \in \mathcal{U}(x)$. Then

$$\langle Cx + Du, JD\eta\rangle_{Y^{\mathrm{s}}} = \langle y' + Eu'', JE\eta\rangle_{Y^{\mathrm{s}}} = \langle E^*Jy' + E^*JEu'', \eta\rangle_D = 0 \qquad (8.15)$$

for all $\eta \in \mathcal{U}(0)$, hence u is J-critical for x.

2° *Uniqueness:* If u is J-critical for $x = 0$, then $\langle Dv, JDu\rangle = 0$ for all $v \in \mathcal{U}(0)$, hence then $\|u\|_D = 0$, hence $u = 0$. Thus, the J-critical control is unique for any $x \in X$, by Lemma 8.1.9. □

Thus, we have solved a rather general minimization problem:

Corollary 8.2.6 (Minimization) *Assume that $\mathcal{U}(x) \neq \emptyset$ for all $x \in X$. If there is $\varepsilon > 0$ s.t.*

$$\mathcal{J}(0,u) \geq \varepsilon(\|u\|_U^2 + \|Bu\|_{Z^s}^2 + \|Du\|_{Y^s}^2) \qquad (u \in \mathcal{U}(0)), \tag{8.16}$$

then there is a unique minimizing control for each $x \in X$. The corresponding state, output and cost are given by (8.8) and (8.9).

Proof: By Lemma 8.2.3(c2), D is positively J-coercive, hence Theorems 8.2.5 and 8.1.10 apply. By Corollary 8.1.8, the unique J-critical control is strictly minimizing (on $\mathcal{U}(x)$). □

A special case of this is the standard LQR problem:

Corollary 8.2.7 (Standard LQR problem) *Let $J \gg 0$ and $\mathcal{U}(x) \neq \emptyset$ for all $x \in X$. Let there be $\varepsilon > 0$ s.t. $\|Du\|_{Y^s} \geq \varepsilon(\|u\|_U + \|Bu\|_{Z^s})$ for all $u \in \mathcal{U}(0)$.*

Then there is a unique minimizing control for each $x \in X$. The corresponding state, output and cost are given by (8.8) and (8.9). □

(This follows directly from Corollary 8.2.6.)

Analogously, if $Y^s = Y$ and $Z^s = Z$ (the stable case), $J \geq 0$, and $\|J^{1/2}Du\|_{Y^s} \geq \varepsilon\|u\|$ for all $u \in U$, then there is a unique minimizing control for each $x \in X$, and Theorems 8.2.5 and 8.1.10 and Lemma 8.2.8 apply.

The condition "$\mathcal{U}(x) \neq \emptyset$ for all $x \in X$" is called the *Finite Cost Condition*, because for $J = I$ this condition holds iff for each x there is a control $u \in \mathcal{U}(x)$ s.t. $\mathcal{J}(x,u) = \|Cx + Du\|_{Y^s}^2 < \infty$.

The proof and implications of Theorem 8.2.5 become easy in the stable case:

Lemma 8.2.8 (Stable J-critical control) *Assume that $\mathcal{U}(x) = U$ for all $x \in X$. Assume, in addition, that the "Popov Toeplitz operator" $T := D^*JD$ be invertible ($T \in \mathcal{GB}(U,U^*)$).*

Then there is a the unique J-critical control for each $x \in X$, and the system Σ_{crit} and J-critical cost operator $\mathcal{P}$ of Theorem 8.3.9 are given by

$$\begin{aligned} A_{\text{crit}} &= A - BT^{-1}D^*JC, & (8.17)\\ C_{\text{crit}} &= \left(I - DT^{-1}D^*J\right)C, & (8.18)\\ K_{\text{crit}} &= -T^{-1}D^*JC, & (8.19)\\ \mathcal{P} &= C^*\left(J - JDT^{-1}D^*J\right)C = C^*JC_{\text{crit}}. & (8.20) \end{aligned}$$

The condition on T holds iff D is J-coercive, by Lemma 8.2.3(a).

Proof: Now $\langle Cx + Du, JD\eta\rangle = 0$ for all $\eta \in U = \mathcal{U}(0)$ iff $D^*J(Cx + Du) = 0$ ($\in U^*$), i.e., iff $u = -T^{-1}D^*JCx$. The formulae can be computed from (8.8). □

Standard stable LQR and H^∞ problems are of the following form and hence give us the following direct formulae for the solutions:

Lemma 8.2.9 ("Stable Σ with bounded C") *Assume that $\mathcal{U}(x) = U$ for all $x \in X$, $\widetilde{C} \in \mathcal{B}(Z^s, Y^s)$, "$y = \widetilde{C}z + \widetilde{D}u$", i.e., $C = \widetilde{C}A$, $D = \widetilde{C}B + \widetilde{D}$, and that $S := \widetilde{D}^* J\widetilde{D} \in \mathcal{GB}(U, U^*)$, $\widetilde{D}^* J\widetilde{C} = 0$, $T := D^*JD = S + B^*QB \in \mathcal{GB}(U, U^*)$, where $Q := \widetilde{C}^* J\widetilde{C} \in \mathcal{B}(Z^s)$.*

Then Lemma 8.2.8 applies, $K_{\mathrm{crit}} = -S^{-1}B^*QA_{\mathrm{crit}}$ and $\mathcal{P} = A^*QA_{\mathrm{crit}}$.

Thus, $u_{\mathrm{crit}}(x) = -S^{-1}B^*Qz_{\mathrm{crit}}(x)$. See Proposition 8.3.10 (and equation (8.93)) for a WPLS application of the above two lemmas and Chapter 10 for further LQR results. Below the proposition we describe two methods for obtaining a direct formula for u_{crit} and $\mathcal{P}$ in the unstable case.

Proof: Note that $\widetilde{D} = D - \widetilde{C}B \in \mathcal{B}(U, Y)$ and $D^*JC = B^*QA$. Multiply (8.19) by T to the left to obtain

$$SK_{\mathrm{crit}} + B^*QBK_{\mathrm{crit}} = -B^*QA, \tag{8.21}$$

i.e., $-SK_{\mathrm{crit}} = B^*Q(A + BK_{\mathrm{crit}}) = B^*QA_{\mathrm{crit}}$, as claimed.

Moreover, $C_{\mathrm{crit}} = \widetilde{C}A_{\mathrm{crit}} + \widetilde{D}K_{\mathrm{crit}}$, hence $\mathcal{P} = C^*JC_{\mathrm{crit}} = A^*QA_{\mathrm{crit}}$. A common alternative formula is

$$\mathcal{P} = C_{\mathrm{crit}}^* JC_{\mathrm{crit}} = \mathbb{A}_{\mathrm{crit}}^* Q\mathbb{A}_{\mathrm{crit}} + K_{\mathrm{crit}}^* SK_{\mathrm{crit}}. \tag{8.22}$$

$\square$

Notes for Sections 8.1 and 8.2

For stable WPLSs, the idea to use Fréchet differentiation for optimal control (cf. Lemma 8.1.6) and the stable Popov Toeplitz operator method of Lemma 8.2.8 were first used in [S97b], which also contains a variant of the stable case of Theorem 8.1.10 for WPLSs. These methods seem to have been used in control theory for several decades,

and the same holds for the alternative "completing the square" method (not presented here) for minimization problems; see [Zwart] for a WPLS application. We have not seen earlier unstable versions of Theorem 8.2.5 in any framework.

See the notes for Section 8.4 for "J-coercivity". As noted below Theorem 8.2.5, one could prove the existence of a unique J-critical control under weaker assumptions than J-coercivity. Such "singular control problems" are usually ruled out by the assumptions, because such settings are rarely encountered in practice and they cannot be solved as satisfactorily.

The abstract setting of these two sections would allow for further extension of control theory (e.g., feedback and coprimeness), and one could easily obtain results analogous to those in Chapter 6 (or to those in the other chapters). However, we do not have the need to address these concepts at this abstract level.

Further notes are given in Sections 8.3 and 8.4, where this abstract theory is applied to WPLSs

8.3 J-critical control for WPLSs

To drift with every passion till my soul
Is a stringed lute on which all winds can play,
Is it for this that I have given away
Mine ancient wisdom, and austere control?
— Oscar Wilde (1856–1900)

In this section we apply the abstract optimization theory of Sections 8.1 and 8.2 to WPLSs.

Standing Hypothesis 8.3.1 *Throughout this Sections 8.3–8.5, U, W, H and Y denote Hilbert spaces of arbitrary dimensions, $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, and $J = J^* \in \mathcal{B}(Y)$.*

As explained on pp. 351-352, we consider the cost function (8.1).

In the minimization (LQR) problem, one often takes $J = I$ so that $\mathcal{J}(x_0,u) = \int_0^\infty \|y(t)\|_H^2\,dt$, and one wishes to find, for each initial state $x_0 \in H$, a "stabilizing" control $u : \mathbf{R}_+ \to U$ s.t. the cost $\mathcal{J}(x_0,u)$ is minimized over all $u \in \mathcal{U}_*^*(x_0)$, where $\mathcal{U}_*^*(x_0)$ denotes the set of "stabilizing" controls for the initial state x_0.

We may choose $\mathcal{U}_*^*(x_0)$ to be the set of those $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ for which the output $y := \mathbb{C}x_0 + \mathbb{D}u$ is in L^2; we denote this set by $\mathcal{U}_{\text{out}}(x_0)$. The subset of u's for which also the state $x := \mathbb{A}x_0 + \mathbb{B}\tau u$ belongs to L^2 is denoted by $\mathcal{U}_{\text{exp}}(x_0)$. We also allow for other choices of $\mathcal{U}_*^*$, so that we are able to solve control problems with very different stability restrictions depending on the choice.

In other control problems, one may wish to maximize $\mathcal{J}(x_0,\cdot)$ or to find a minimax point of $\mathcal{J}(x_0,\cdot)$ (as for the H^∞ problem). Therefore, instead of minimums, we look for the critical points (the zeros of the Fréchet derivative) of $\mathcal{J}(x_0,\cdot)$ over $\mathcal{U}_*^*(x_0)$; we call these the "J-critical controls" (cf. Lemma 8.3.6). Naturally, all extremums and other saddle points of $\mathcal{J}(x_0,\cdot)$ are J-critical.

In this section we shall study such controls and show that if there is a unique J-critical control for each initial state $x_0 \in H$, then this control and the corresponding state and output can be represented as the state and output of a WPLS (Theorem 8.3.9). This WPLS is obtained by applying certain kind of state feedback to the original system, but this feedback need not be well-posed (unless the WPLS is sufficiently regular); we study necessary and sufficient conditions for its well-posedness (the maps from the state to the output and J-critical control (feedback) are always well-posed but the sensitivity of the feedback loop to external input/disturbance need not be). Such conditions are treated from another point of view in Section 9.14, and further sufficient regularity conditions are given in other sections of Chapter 9.

The "J-critical" system (the closed-loop system in the well-posed case) becomes output stable if we optimize over $\mathcal{U}_{\text{out}}$ and exponentially stable if we optimize over $\mathcal{U}_{\text{exp}}$; conversely, the control corresponding to an output-stabilizing (resp. exponentially stabilizing) state feedback is necessarily in $\mathcal{U}_{\text{out}}$ (resp. $\mathcal{U}_{\text{exp}}$). Thus, optimization over $\mathcal{U}_{\text{out}}$ (resp. $\mathcal{U}_{\text{exp}}$) corresponds to optimization over all output- (resp. exponentially) stabilizing state feedback pairs (under the regularity conditions mentioned above).

Such a J-critical state feedback pair corresponds to a unique "stabilizing" solution of the Riccati equation, see Chapter 9 for details. The corresponding special control problems are solved in Chapters 10–12.

We also present similar results for the case where the critical controls are not unique.

Now it is the time to define $\mathcal{U}_*^*$. Due to generality, the definition contains awfully many symbols, hence we recommend the reader to just note that $\mathcal{U}_{\mathrm{exp}}$ and $\mathcal{U}_{\mathrm{out}}$ are as explained above, observe the alternative (equivalent) definition of J-critical points (controls) of the cost function and ignore the rest of the definition until there is some need for the other $\mathcal{U}_*^*$'s. The general case will become more clear later with the applications:

Definition 8.3.2 ($\mathcal{U}_*^*$ and J-critical control) *Assume that Z^{s} is a Banach space, Z^{u} is a TVS s.t. $Z^{\mathrm{s}} \underset{c}{\subset} Z^{\mathrm{u}}$, $\mathbb{Q} \in \mathcal{B}(H, Z^{\mathrm{u}})$, $\mathbb{R} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U), Z^{\mathrm{u}})$ and $\vartheta \in \mathbf{R}$. Define*

$$\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}(x_0) := \{u \in \mathrm{L}^2_{\vartheta}(\mathbf{R}_+;U) \,\big|\, \left[\begin{smallmatrix}\mathbb{C} & \mathbb{D}\\ \mathbb{Q} & \mathbb{R}\end{smallmatrix}\right]\left[\begin{smallmatrix}x_0\\ u\end{smallmatrix}\right] \in \mathrm{L}^2 \times Z^{\mathrm{s}}\} \qquad (x_0 \in H); \tag{8.23}$$

$$\|u\|_{\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}} := \max\{\|u\|_{\mathrm{L}^2_{\vartheta}}, \|\mathbb{R}u\|_{Z^{\mathrm{s}}}, \|\mathbb{D}u\|_2\} \le \infty \qquad (u \in \mathrm{L}^2_{\vartheta}(\mathbf{R}_+;U)); \tag{8.24}$$

$$x_{x_0,u} := \mathbb{A}x_0 + \mathbb{B}\tau u \qquad (u \in \mathrm{L}^2(\mathbf{R}_+;U)); \tag{8.25}$$

$$y_{x_0,u} := \mathbb{C}x_0 + \mathbb{D}u \qquad (u \in \mathrm{L}^2(\mathbf{R}_+;U)); \tag{8.26}$$

$$\mathcal{Y}_*^*(x_0) := \{y_{x_0,u} \,\big|\, u \in \mathcal{U}_*^*(x_0)\}; \tag{8.27}$$

$$\mathcal{J}(x_0,u) := \langle y_{x_0,u}, Jy_{x_0,u}\rangle \qquad (u \in \mathcal{U}_*^*(x_0)). \tag{8.28}$$

We use $\mathcal{U}_*^* := \mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}$ when we do not wish to specify $\mathbb{Q}$, $\mathbb{R}$, Z^{u} and Z^{s} and ϑ. We are mainly interested in the following choices of $\mathcal{U}_*^*$:

$$\mathcal{U}_{\mathrm{out}}(x_0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,\big|\, y_{x_0,u} \in \mathrm{L}^2\}, \tag{8.29}$$

$$\mathcal{U}_{\mathrm{sta}}(x_0) := \{u \in \mathcal{U}_{\mathrm{out}}(x_0) \,\big|\, \sup \|x_{x_0,u}\| < \infty\}, \tag{8.30}$$

$$\mathcal{U}_{\mathrm{str}}(x_0) := \{u \in \mathcal{U}_{\mathrm{out}}(x_0) \,\big|\, \|x_{x_0,u}(t)\|_H \to 0 \text{ as } t \to +\infty\}, \tag{8.31}$$

$$\mathcal{U}_{\mathrm{exp}}(x_0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,\big|\, x_{x_0,u} \in \mathrm{L}^2\}; \tag{8.32}$$

i.e., then we assume that $\vartheta = 0$, $\begin{bmatrix}\mathbb{Q} & \mathbb{R}\end{bmatrix} = \begin{bmatrix}\mathbb{A} & \big| & \mathbb{B}\tau\end{bmatrix}$, $Z^{\mathrm{u}} = \mathrm{L}^2_{\omega}(\mathbf{R}_+;H)$, and $Z^{\mathrm{s}} = \mathrm{L}^2_{\omega}(\mathbf{R}_+;H)$, $Z^{\mathrm{s}} = \mathrm{L}^{\infty}(\mathbf{R}_+;H)$, $Z^{\mathrm{s}} = \mathcal{C}_0(\mathbf{R}_+;H)$ or $Z^{\mathrm{s}} = \mathrm{L}^2(\mathbf{R}_+;H)$, respectively, where $\omega := \omega_A + 1$.

We call $\mathcal{J}$ the cost function. *A control $u \in \mathcal{U}_*^*(x_0)$ (resp. output $y_{x_0,u} \in \mathcal{Y}_*^*(x_0)$) is called* J-critical (over $\mathcal{U}_*^*$) *for x_0 (and Σ) if $\langle \mathbb{D}\pi_+\eta, Jy_{x_0,u}\rangle_{\mathrm{L}^2} = 0$ for all $\eta \in \mathcal{U}_*^*(0)$.*

We call an admissible state feedback pair $\begin{bmatrix}\mathbb{K} & \big| & \mathbb{F}\end{bmatrix}$ (or a corresponding state feedback operator) J-critical *(over $\mathcal{U}_*^*$ for Σ) if $\mathbb{K}_\flat x_0 := (I-\mathbb{F})^{-1}\mathbb{K}x_0$ is J-critical for all $x_0 \in H$.*

Thus, we usually write $\mathcal{U}_*^*$ instead of $\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}$ (and we often omit "over $\mathcal{U}_*^*$"). Of course, we still assume that some $\begin{bmatrix}\mathbb{Q} & \mathbb{R}\end{bmatrix}$, Z^{u}, Z^{s} and ϑ are given. The optimal controls for most reasonable control problems are J-critical over $\mathcal{U}_*^*$ where $\mathcal{U}_*^*$ is the set over which one wishes to optimize, as explained above.

Note that $\|\cdot\|_{\mathcal{U}_*^*}$ is a norm on $\mathcal{U}_*^*(0)$; it will be used to define J-coercivity in the next section. Some simplifications of these norms are given in Lemma 8.4.2.

By Lemma 6.7.8, we have

$$\mathcal{U}_{\exp}(x_0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,|\, x_{x_0,u}, y_{x_0,u} \in \mathrm{L}^2\}. \tag{8.33}$$

Thus, $\mathcal{U}_{\exp}(x_0)$ is, indeed, the set of "exponentially stabilizing controls" ($x, y \in \mathrm{L}^2$) described before the Definition 8.3.2. Analogously, $\mathcal{U}_{\mathrm{out}}(x_0)$ is the set of "output stabilizing controls" ($y \in \mathrm{L}^2$); $\mathcal{U}_{\mathrm{sta}}(x_0)$ is the set of "stabilizing controls" ($y \in \mathrm{L}^2$, x bounded); and $\mathcal{U}_{\mathrm{str}}(x_0)$ is the set of "strongly stabilizing controls" ($y \in \mathrm{L}^2$, x strongly stable).

In finite-dimensional problems, one usually optimizes over $\mathcal{U}_{\exp}$. In applications, often physical quantities may determine a natural norm for the state, and this norm might be such that one does not want to require the "optimal" control to be exponentially stabilizing. Therefore, also set $\mathcal{U}_{\mathrm{out}}$ has often been used, particularly for infinite-dimensional problems (see, e.g., [Zwart], [WW], [LT00a]), and there are at least some kind of implicit applications of $\mathcal{U}_{\mathrm{str}}$ [Oostveen] and $\mathcal{U}_{\mathrm{sta}}$ [S97b]–[S98d].

Coercivity assumptions that guarantee the existence of a unique J-critical control over $\mathcal{U}_{\mathrm{out}}$ are very natural whereas their analogies for $\mathcal{U}_{\exp}$ are substantially stronger though still rather commonly used (compare Propositions 10.3.1 to 10.3.2). Nevertheless, in the literature one often uses just the former assumption and obtains an optimal control which is optimal over $\mathcal{U}_{\exp}$ too — how is this trick possible? The secret is to assume exponential detectability, since it implies that the four sets coincide:

Lemma 8.3.3 ($\mathcal{U}_{\exp} = \mathcal{U}_{\mathrm{out}}$) *We have $\mathcal{U}_{\exp}(x_0) \subset \mathcal{U}_{\mathrm{out}}(x_0)$ and $\mathcal{U}_{\mathrm{str}}(x_0) \subset \mathcal{U}_{\mathrm{sta}}(x_0) \subset \mathcal{U}_{\mathrm{out}}(x_0)$ for all $x_0 \in H$.*

If Σ is estimatable or exponentially q.r.c.-stabilizable (e.g., exponentially stable), then $\mathcal{U}_{\exp} = \mathcal{U}_{\mathrm{str}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{out}}$, and then Σ is [positively] J-coercive over $\mathcal{U}_{\mathrm{out}}$ iff Σ is [positively] J-coercive over $\mathcal{U}_{\exp}$.

If Σ is [strongly] q.r.c.-stabilizable (e.g., [strongly] stable), then $[\mathcal{U}_{\mathrm{str}} =]\, \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{out}}$. If u is J-critical for x_0 over $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{str}}$, then u is J-critical for x_0 over $\mathcal{U}_{\mathrm{sta}}$.

(Since $\ell^2 \subset c_0$, we have $\mathcal{U}_{\exp} \subset \mathcal{U}_{\mathrm{str}}$ in discrete time (cf. Theorem 13.3.13).)

Thus, when Σ is estimatable, we may equivalently optimize over any of these sets for main domains for u, and we only have to look for J-coercivity over $\mathcal{U}_{\mathrm{out}}$ (and optimizability) to guarantee the existence of a unique J-critical control (see Definition 8.4.1 and Theorem 8.4.3).

Proof: (By $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}}$ we mean equality as functions of $x_0 \in H$ (to L^2); thus, we could as well write $\mathcal{U}_{\mathrm{out}} \equiv \mathcal{U}_{\mathrm{sta}}$.)

The claims on estimatability follow from Theorem 6.7.7. The claims on q.r.c.-stabilizability are given in Theorem 8.4.5(g2).

The rest is rather obvious. (In fact, whenever two of these four spaces are equal for $x_0=0$, then they have equal norms, by Lemma A.3.6 (with, e.g., $\|u\|_{X_3} := \|u\|_2+\|y\|_2+\|x\|_{\mathrm{L}^2_1}$), and hence [positive] J-coercivity over them become equivalent, by (c1) and (c2) of Lemma 8.2.3.) □

Trivially, the condition that $\mathcal{U}_*^*(x_0)\neq\emptyset$ for all $x_0\in H$ is necessary for the existence of a J-critical control for any initial state, hence for the solvability of any reasonable optimization problem (over $\mathcal{U}_*^*$). This condition is called the *Finite Cost Condition*, since, for $J\gg 0$, it corresponds to the existence of u s.t. "$\mathcal{J}(x_0,u)<\infty$". For $\mathcal{U}_*^*=\mathcal{U}_{\exp}$ this concept obviously coincides with optimizability.

After a "translation of notation", the results of previous sections can be read as results for WPLSs:

Remark 8.3.4 *We can apply the theory of Sections 8.1 and 8.2 by substitutions* $X\mapsto H$, $U\mapsto \mathrm{L}^2_\vartheta(\mathbf{R}_+;U)$, $Y^{\mathrm{s}}\mapsto \mathrm{L}^2(\mathbf{R}_+;Y)$, $Y\mapsto \mathrm{L}^2_\beta(\mathbf{R}_+;Y)$, $Z^{\mathrm{s}}\mapsto \mathrm{L}^2_\beta(\mathbf{R}_+;H)\times Z^{\mathrm{s}}$, $Z\mapsto \mathrm{L}^2_\beta(\mathbf{R}_+;H)\times Z^{\mathrm{u}}$ *and* $\left[\begin{array}{c|c} A & B\end{array}\right]\mapsto\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}\tau \\ \mathbb{Q} & \mathbb{R}\end{array}\right]$, $\left[\begin{array}{c|c} C & D\end{array}\right]\mapsto\left[\begin{array}{c|c} \mathbb{C} & \mathbb{D}\end{array}\right]$, *where* $\beta>\max\{\vartheta,\omega_A\}$.

(Note that Σ *and* $\mathbb{B}\tau$ *are* β*-stable. Above, one can equivalently write explicitly* $B\mapsto\pi_+\mathbb{B}\tau\pi_+$, $D\mapsto\pi_+\mathbb{D}\pi_+$.*)*

Thus, the concepts $\mathcal{J}$, *"J-critical" and "J-coercive" of those two sections coincide with those of this section, and* $\mathcal{U}$ *becomes* $\mathcal{U}_*^*$.

Consequently, we obtain Lemmas 8.3.5, 8.3.6, 8.3.7 and 8.3.8, and Proposition 8.3.10 and Theorem 8.4.3 from corresponding results in Sections 8.1 and 8.2 (see those sections for further results). □

(By Remark 8.1.2, we might also allow completely unstable controls by substitution $U\mapsto \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$.)

Next we list the "translated" auxiliary lemmas mentioned in the remark:

Lemma 8.3.5 ($\mathcal{U}_*^*(x_0)$ and $\mathcal{Y}_*^*(x_0)$) *The sets* $\mathcal{U}_*^*(0)\subset \mathrm{L}^2_\vartheta(\mathbf{R}_+;U)$ *and* $\mathcal{Y}_*^*(0)\subset \mathrm{L}^2(\mathbf{R}_+;Y)$ *are linear subspaces. Let* $x_0\in H$, $u\in\mathcal{U}_*^*(x_0)$, $y\in\mathcal{Y}_*^*(x_0)$. *Then* $\mathcal{U}_*^*(x_0)=u+\mathcal{U}_*^*(0)$ *and* $\mathcal{Y}_*^*(x_0)=y+\mathcal{Y}_*^*(0)$.

Moreover, $\mathcal{U}_*^*(\alpha x_0+\beta x_1)=\alpha\mathcal{U}_*^*(x_0)+\beta\mathcal{U}_*^*(x_1)$ *and* $\mathcal{Y}_*^*(\alpha x_0+\beta x_1)=\alpha\mathcal{Y}_*^*(x_0)+\beta\mathcal{Y}_*^*(x_1)$ *(*$\alpha,\beta\in\mathbf{C}\setminus\{0\}$, $x_0,x_1\in H$ *s.t.* $\mathcal{U}_*^*(x_0)\neq\emptyset$*).* □

As claimed above, a control u_{crit} is J-critical iff the gradient of the cost $u\mapsto\langle y,Jy\rangle$ is zero at u_{crit}:

Lemma 8.3.6 (J-critical $\Leftrightarrow \frac{\mathrm{d}\mathcal{J}}{\mathrm{d}u}=0$) *A control* $u_{\mathrm{crit}}(x_0)\in\mathcal{U}_*^*(x_0)$ *is J-critical for* x_0 *iff* $\frac{\mathrm{d}\mathcal{J}(x_0,u)}{\mathrm{d}u}(u_{\mathrm{crit}}(x_0))=0$.

In particular, if $u_{\mathrm{crit}}(x_0)$ *is a local extremal point or saddle point of* $\mathcal{J}(x_0,u)$ *(over* $\mathcal{U}_*^*(x_0)$*), then* $u_{\mathrm{crit}}(x_0)$ *is J-critical for* x_0. □

The expression $\frac{\mathrm{d}\mathcal{J}(x_0,u)}{\mathrm{d}u}$ denotes the (real) *Fréchet derivative* of $\mathcal{J}(x_0,\cdot)$ on its domain $\mathcal{U}_*^*(x_0)$; see Lemma 8.1.6 for details.

Saddle points correspond to solutions of the H^∞ minimax problem.

Lemma 8.3.7 (Critical cost $\mathcal{J}(x_0, u_{\rm crit})$) *Let $x_0 \in H$ and $u_{\rm crit}(x_0) \in \mathcal{U}_*^*(x_0)$. Set $y_{\rm crit}(x_0) := y_{x_0,u_{\rm crit}(x_0)}$. Then the following are equivalent:*

(i) $u_{\rm crit}(x_0)$ is J-critical for x_0;

(ii) $\mathcal{J}(x_0, u_{\rm crit}(x_0) + \eta) = \langle y_{\rm crit}(x_0), Jy_{\rm crit}(x_0)\rangle + \langle \mathbb{D}\eta, J\mathbb{D}\eta\rangle \quad (\eta \in \mathcal{U}_*^*(0))$;

(iii) $\langle y_{x_0,u_{\rm crit}(x_0)+\eta_1}, Jy_{x_0,u_{\rm crit}(x_0)+\eta_2}\rangle = \langle y_{\rm crit}(x_0), Jy_{\rm crit}(x_0)\rangle + \langle \mathbb{D}\eta_1, J\mathbb{D}\eta_2\rangle$ $(\eta_1, \eta_2 \in \mathcal{U}_*^*(0))$.

□

Note that (ii) means that $\mathcal{J}(x_0, u_{\rm crit}(x_0)+\eta) = \mathcal{J}(x_0, u_{\rm crit}(x_0)) + \mathcal{J}(0,\eta)$. Thus, given a critical control $u_{\rm crit}$, the cost for $u_{\rm crit} + \eta$ equals the critical cost plus the cost for η.

All critical controls produce the same cost and the same sensitivity of the cost to a disturbance:

Lemma 8.3.8 ($\mathcal{U}_*^{*,\rm crit}$ and uniqueness) *Let $\mathcal{U}_*^{*,\rm crit}(x_0)$ be the set of J-critical controls for $x_0 \in H$. Then $\mathcal{U}_*^{*,\rm crit}(0)$ is a linear subspace of $\mathcal{U}_*^*(0)$. If $u \in \mathcal{U}_*^{*,\rm crit}(x_0)$, then $\mathcal{U}_*^{*,\rm crit}(x_0) = u + \mathcal{U}_*^{*,\rm crit}(0)$. Moreover, $\mathcal{J}(x_0,u) = \mathcal{J}(x_0,v)$, and $\mathcal{J}(x_0, u+\eta) = \mathcal{J}(x_0, v+\eta)$ for $u, v \in \mathcal{U}_*^{*,\rm crit}(x_0)$, $\eta \in \mathcal{U}_*^*(0)$.*

In particular, there is at most one J-critical control for each $x \in X$ iff $\mathcal{U}^{\rm crit}(0) = \{0\}$.

□

We shall later meet several sufficient conditions for the existence of a unique J-critical control. Such a control and corresponding state and output are always produced by a WPLS:

Theorem 8.3.9 ($\Sigma_{\rm crit}$) *Assume that there is a unique J-critical control $u_{\rm crit}(x_0)$ over $\mathcal{U}_*^*$ for each $x_0 \in H$, and define*

$$\Sigma_{\rm crit} := \left[\begin{array}{c|c} \mathbb{A}_{\rm crit} & \\ \hline \mathbb{C}_{\rm crit} & \\ \mathbb{K}_{\rm crit} & \end{array}\right] : x_0 \mapsto \left[\begin{array}{c|c} x_{\rm crit}(x_0) & \\ \hline y_{\rm crit}(x_0) & \\ u_{\rm crit}(x_0) & \end{array}\right] := \left[\begin{array}{c|c} \mathbb{A}x_0 + \mathbb{B}\tau u_{\rm crit}(x_0) & \\ \hline \mathbb{C}x_0 + \mathbb{D}u_{\rm crit}(x_0) & \\ u_{\rm crit}(x_0) & \end{array}\right]. \tag{8.34}$$

(Alternatively, we may assume that $\Sigma_{\rm crit}$ is any J-critical control in WPLS form (see Definition 8.3.15).)

Then the following hold except that in (a1)–(a5) and (b2) we assume, in addition, that $\vartheta = 0$ (e.g., that $\mathcal{U}_*^* \in \{\mathcal{U}_{\rm out}, \mathcal{U}_{\rm sta}, \mathcal{U}_{\rm str}, \mathcal{U}_{\rm exp}\}$)).

(a1) The maps $\mathbb{C}_{\rm crit}$ and $\mathbb{K}_{\rm crit}$ are stable, and $\Sigma_{\rm crit} \in$ WPLS$(\{0\}, H, Y \times U)$.

(a2) If $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$, then $\Sigma_{\rm crit}$ is exponentially stable; if $\mathcal{U}_*^* = \mathcal{U}_{\rm str}$, then $\Sigma_{\rm crit}$ is strongly stable; if $\mathcal{U}_*^* = \mathcal{U}_{\rm sta}$, then $\Sigma_{\rm crit}$ is stable.

(a3) If $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is [strongly] stable, then $\Sigma_{\rm crit}$ is [strongly] stable.

(a4) If $\omega \ge 0$ is s.t. $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in$ WPLS$_\omega$, then $\Sigma_{\rm crit} \in$ WPLS$_\omega$.

(a5) If Σ is estimatable, then $\Sigma_{\rm crit}$ is exponentially stable.

(a1') $\mathbb{C}_{\rm crit}$ is stable, $\mathbb{K}_{\rm crit}$ is ϑ-stable, $\mathbb{Q} + \mathbb{R}\mathbb{K}_{\rm crit} \in \mathcal{B}(H, Z^{\rm s})$, and $\Sigma_{\rm crit} \in$ WPLS$(\{0\}, H, Y \times U)$.

(a2') If $\mathcal{U}_*^{*,\mathrm{crit}}(x_0) \subset \mathcal{U}_{\mathrm{exp}}(x_0)$ for all $x_0 \in H$, then Σ_{crit} is exponentially stable.

(a4') If $\omega \geq \vartheta$ is s.t. $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right] \in \mathrm{WPLS}_\omega$, then $\Sigma_{\mathrm{crit}} \in \mathrm{WPLS}_\omega$.

(b1) We call $\mathcal{P} := \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}_{\mathrm{crit}} \in \mathcal{B}(H)$ the J-critical cost operator. *It satisfies*

$$\mathcal{J}(x_0, u_{\mathrm{crit}}(x_0) + \eta) = \langle x_0, \mathcal{P}x_0 \rangle_H + \mathcal{J}(0, \eta) \quad (x_0 \in H,\ \eta \in \mathcal{U}_*^*(0)). \tag{8.35}$$

(b2) **(Stable case)** *If 1. $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$, and $\mathbb{C}$ or $\mathbb{D}$ is stable; 2. $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{sta}}$, and $\mathbb{A}$ and $\mathbb{C}$ (or $\mathbb{B}$ and $\mathbb{D}$) are stable; 3. $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{str}}$, and $\mathbb{A}$ and $\mathbb{C}$ (or $\mathbb{B}$ and $\mathbb{D}$) are strongly stable; or 4. $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, and $\mathbb{A}$ is exponentially stable or $\mathbb{B}\tau$ stable; then $\mathbb{C}$ is stable and*

$$\mathcal{P} := \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}_{\mathrm{crit}} = \mathbb{C}^* J \mathbb{C}_{\mathrm{crit}} = \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}. \tag{8.36}$$

(b2') If $\mathbb{C}$ is stable and $\mathbb{Q}[H] \subset Z^{\mathrm{s}}$, or $\mathbb{D}$ is stable and $\mathbb{R} \in \mathcal{B}(\mathrm{L}^2_\vartheta(\mathbf{R}_+;U), Z^{\mathrm{s}})$, then $\mathbb{C}$ is stable and (8.36) holds.

The (optimal) control $\mathbb{K}_{\mathrm{crit}}x_0$ equals $(K_{\mathrm{crit}})_{\mathrm{w}}x$ a.e., where $x := \mathbb{A}_{\mathrm{crit}}x_0$ is the state of Σ_{crit} with $A_{\mathrm{crit}} = A + BK_{\mathrm{crit}}$, by Lemma 8.3.17(a). Thus, such a control corresponds to some kind of state feedback, but the feedback loop need not be well-posed (indeed, the "maps $\mathbb{K}$, $\mathbb{F}$, $\mathbb{B}_\flat$, $\mathbb{D}_\flat$, $\mathbb{F}_\flat$ of Definition 6.6.10" need not be well posed); see Remark 9.7.7 and Examples 8.4.13 and 11.3.7 (which also cover the stable setting of Proposition 8.3.10 below). This means that any external input (e.g., disturbance or modelling error) might "explode" the system.

The corresponding generalized Riccati equations are treated in Section 9.7; the one in [FLT] is a special case of these. The rest of Sections 9.1–9.12 treat the case where the "optimal state feedback" is well posed, by which we mean that Σ_{crit} is the left column of $\Sigma_\flat$ for some admissible state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ for Σ.

In discrete-time, a unique minimizing control is always of this form, and it corresponds to the unique $\mathcal{U}_*^*$-stabilizing solution of the DARE, by Theorem 14.1.6. If Σ is sufficiently regular, then the same holds in continuous time too (see, e.g., Lemma 8.3.18 or Remark 9.9.14).

Proof of Theorem 8.3.9: (a1') If Σ_{crit} is a general J-critical control in WPLS form, then it is rather obvious that (a1') holds; therefore, we assume below that there is a unique J-critical control over $\mathcal{U}_*^*$.

Since the rest follows from Corollary 8.1.10, we only have to show that $\Sigma_{\mathrm{crit}} \in \mathrm{WPLS}$.

Let $x_0 \in H$, $t \geq 0$. We first show that $\pi_+\tau^t\mathbb{K}_{\mathrm{crit}}x_0$ is J-critical for $\mathbb{A}_{\mathrm{crit}}^t x_0$, i.e., equal to $\mathbb{K}_{\mathrm{crit}}\mathbb{A}_{\mathrm{crit}}^t x_0$: For $\eta \in \mathcal{U}_*^*(0)$ we have $\tau^{-t}\eta \in \mathcal{U}_*^*(0)$, hence

$$\langle J\pi_+\tau^t\mathbb{C}_{\mathrm{crit}}x_0, \mathbb{D}\eta \rangle_{\mathrm{L}^2} = \langle J\mathbb{C}_{\mathrm{crit}}x_0, \mathbb{D}\tau^{-t}\eta \rangle_{\mathrm{L}^2} = 0 \quad (\eta \in \mathcal{U}_*^*(0)). \tag{8.37}$$

But

$$\begin{aligned}\pi_+\tau^t\mathbb{C}_{\mathrm{crit}}x_0 &= \pi_+\tau^t(\mathbb{C}x_0 + \mathbb{D}\mathbb{K}_{\mathrm{crit}}x_0) = \mathbb{C}\mathbb{A}^t x_0 + \pi_+\mathbb{D}(\pi_+ + \pi_-)\tau^t\mathbb{K}_{\mathrm{crit}}x_0 \qquad (8.38)\\ &= \mathbb{C}\mathbb{A}^t x_0 + \mathbb{D}\pi_+\tau^t\mathbb{K}_{\mathrm{crit}}x_0 + \mathbb{C}\mathbb{B}\tau^t\mathbb{K}_{\mathrm{crit}}x_0 = \mathbb{C}\mathbb{A}_{\mathrm{crit}}^t x_0 + \mathbb{D}\pi_+\tau^t\mathbb{K}_{\mathrm{crit}}x_0. \qquad (8.39)\end{aligned}$$

This and (8.37) imply that $\pi_+\tau^t\mathbb{K}_{\rm crit}x_0$ is J-critical for $\mathbb{A}^t_{\rm crit}x_0$; thus

$$\pi_+\tau^t\mathbb{K}_{\rm crit}x_0 = u_{\rm crit}(\mathbb{A}^t_{\rm crit}x_0) = \mathbb{K}_{\rm crit}\mathbb{A}^t_{\rm crit}x_0,\ \ \pi_+\tau^t\mathbb{C}_{\rm crit}x_0 = y_{\rm crit}(\mathbb{A}^t_{\rm crit}x_0) = \mathbb{C}_{\rm crit}\mathbb{A}^t_{\rm crit}x_0. \tag{8.40}$$

By the dynamic programming principle, $\mathbb{A}$ is a semigroup; a detailed proof of this fact goes as follows, using (8.40):

$$\mathbb{A}^s_{\rm crit}\mathbb{A}^t_{\rm crit} = \mathbb{A}^s(\mathbb{A}^t + \mathbb{B}\tau^t\mathbb{K}_{\rm crit}) + \mathbb{B}\tau^s\mathbb{K}_{\rm crit}\mathbb{A}^t_{\rm crit} \tag{8.41}$$
$$= \mathbb{A}^s\mathbb{A}^t + \mathbb{B}\tau^s\pi_-\tau^t\mathbb{K}_{\rm crit} + \mathbb{B}\tau^s\pi_+\tau^t\mathbb{K}_{\rm crit} = \mathbb{A}^s\mathbb{A}^t + \mathbb{B}\tau^{s+t}\mathbb{K}_{\rm crit} = \mathbb{A}^{t+s}_{\rm crit}. \tag{8.42}$$

Obviously, $\mathbb{A}^0_{\rm crit} = \mathbb{A}^0 = I$. By Theorem 6.2.13(a1), $\mathbb{A}_{\rm crit}x_0 = x_{\rm crit}(x_0)$ is continuous for each $x_0 \in H$. Therefore, $\mathbb{A}_{\rm crit}$ is a C_0-semigroup. This and (8.40) imply that $\Sigma_{\rm crit}$ is a WPLS.

(a1)&(a4) These follow from (a1') and (a4'), respectively.

(a2) The exponentially stable case follows from (a2'); the rest is obvious.

(a2') Now $\mathbb{A}_{\rm crit}x_0 \in \mathrm{L}^2$ for all $x_0 \in H$, hence $\mathbb{A}_{\rm crit}$ is exponentially stable, by Lemma A.4.5.

(a3) See the proof of Lemma 6.6.8(a).

(a4') For any $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$, we have $\|\tau^t u\|_{\mathrm{L}^2_\omega} = e^{\omega t}\|u\|_{\mathrm{L}^2_\omega}$ hence $\|\mathbb{B}\tau^t u\|_H \le \|\mathbb{B}\|e^{\omega t}\|u\|_{\mathrm{L}^2_\omega}$. Thus, if $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{K}_{\rm crit}$ are ω-stable, then so is $\mathbb{A}_{\rm crit}$.

(a5) *Exponential stability:* Let $x_0 \in H$. Because $u_{\rm crit}(x_0), y_{\rm crit}(x_0) \in \mathrm{L}^2$, we have $\mathbb{A}_{\rm crit}x_0 = \mathbb{A}x_0 + \mathbb{B}u_{\rm crit}(x_0) \in \mathrm{L}^2$, by Theorem 6.7.7. By Lemma A.4.5, $\mathbb{A}_{\rm crit}$ is exponentially stable, hence so is $\Sigma_{\rm crit}$.

(b1) This follows from Corollary 8.1.10 (or directly form the definitions of $\mathbb{C}_{\rm crit}$, $\mathcal{J}$ and $\mathcal{P}$).

(b2') This follows from Theorem 8.1.10.

(b2) This follows from (b2') except that for $\mathcal{U}_{\rm exp}$ we also used the following: if $\mathbb{B}\tau$ is stable and $\mathcal{U}^*_* = \mathcal{U}_{\rm exp}$, then $\mathbb{A} = \mathbb{A}_{\rm crit} - \mathbb{B}\tau\mathbb{K}_{\rm crit}$ is exponentially stable, by Lemma A.4.5; if $\mathbb{A}$ is exponentially stable, then so is Σ.

(N.B. if $\mathbb{A}$ or $\mathbb{B}$ is exponentially stable, then so is $\mathbb{B}\tau$, by Lemma 6.1.10.)

□

The proof and implications of Theorem 8.4.3 become simple in the stable case:

Proposition 8.3.10 (Stable J-critical control) *Assume that $\mathcal{U}^*_*(x_0) = \mathrm{L}^2(\mathbf{R}_+;U)$ for all $x_0 \in H$. Assume, in addition, that the Popov Toeplitz operator $T := \pi_+\mathbb{D}^*J\mathbb{D}\pi_+$ is invertible (i.e., $T \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_+;U))$).*

Then there is a the unique J-critical control for each $x_0 \in H$, and the system $\Sigma_{\rm crit}$ and J-critical cost operator $\mathcal{P}$ of Theorem 8.3.9 are given by

$$\mathbb{A}_{\rm crit} = \mathbb{A} - \mathbb{B}\tau\pi_+T^{-1}\pi_+\mathbb{D}^*J\mathbb{C}, \tag{8.43}$$
$$\mathbb{C}_{\rm crit} = \left(I - \mathbb{D}\pi_+T^{-1}\pi_+\mathbb{D}^*J\right)\mathbb{C}, \tag{8.44}$$
$$\mathbb{K}_{\rm crit} = -T^{-1}\pi_+\mathbb{D}^*J\mathbb{C}, \tag{8.45}$$
$$\mathcal{P} = \mathbb{C}^*\left(J - J\mathbb{D}\pi_+T^{-1}\pi_+\mathbb{D}^*J\right)\mathbb{C} = \mathbb{C}^*J\mathbb{C}_{\rm crit}. \tag{8.46}$$

If, in addition, $C \in \mathcal{B}(H,Y)$, $D^*JC = 0$ and $R := D^*JD \in \mathcal{GB}(U)$, then

$\mathbb{K}_{\text{crit}} = -R^{-1}(\pi_+\mathbb{B}\tau\pi_+)^*Q\mathbb{A}_{\text{crit}}$, *i.e.*, $u_{\text{crit}}(x_0) = -R^{-1}(\pi_+\mathbb{B}\tau\pi_+)^*Qx_{\text{crit}}(x_0)$ *for all* $x_0 \in H$, *where* $Q := C^*JC$ *(so that* $\mathcal{J} = \langle x, Qx\rangle + \langle u, Ru\rangle$*).*

Note that $\mathcal{U}_{\text{out}} \equiv \mathrm{L}^2(\mathbf{R}_+;U)$ iff $\Sigma \in \mathrm{SOS}$, $\mathcal{U}_{\text{sta}} \equiv \mathrm{L}^2(\mathbf{R}_+;U)$ iff Σ is stable, $\mathcal{U}_{\text{str}} \equiv \mathrm{L}^2(\mathbf{R}_+;U)$ iff Σ is strongly stable, and $\mathcal{U}_{\text{exp}} \equiv \mathrm{L}^2(\mathbf{R}_+;U)$ iff Σ is exponentially stable.

The condition on T means that $\mathbb{D}$ is J-coercive, by Lemma 8.2.3(a). E.g., for $J \gg 0$, we have $T \in \mathcal{GB}$ iff $\|\mathbb{D}u\|_2 \ge \varepsilon\|u\|^2$ ($u \in \mathrm{L}^2$) for some $\varepsilon > 0$.

There are two well-known methods for obtaining a direct formula (as at the end of Proposition 8.3.10) also in the unstable case. One is to first solve the finite-time problem on $[0,T]$ and then take a limit of $x_{\text{crit}}(x_0)$, $y_{\text{crit}}(x_0)$, $u_{\text{crit}}(x_0)$ and $\mathcal{P}x_0$ as $T \to +\infty$; we take a quick glance at this in Section 8.5.

The other method is to derive the corresponding Riccati equation and use it to obtain more information on the solution. This works well when B is bounded (in particular, in the discrete-time case) or when Σ is otherwise regular, but the classical results cannot be completely generalized to the general case (only to the extent of Section 9.7), hence we shall present partial results for different generalities in Chapter 9.

Proof of Lemma 8.3.10: This follows from Lemmas 8.2.8 and 8.2.9. To apply the latter, we must set $\widetilde{C} := \left[\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right]$, $\widetilde{D} := \left[\begin{smallmatrix} D \\ 0 \end{smallmatrix}\right]$, and use substitutions $U \mapsto \mathrm{L}^2(\mathbf{R}_+;U)$ and $Z^{\mathrm{s}} \mapsto Z^{\mathrm{s}}_2$ in Remark 8.3.4, where Z^{s}_2 is the closure of

$$Z^{\mathrm{s}}_0 := \{\mathbb{A}x_0 + \mathbb{B}\tau u \,|\, x_0 \in H,\ u \in \mathrm{L}^2(\mathbf{R}_+;U)\} \quad \text{w.r.t.} \quad \|x\|_{Z^{\mathrm{s}}_2} := \max\{\|x\|_{\mathrm{L}^2_\beta}, \|Cx\|_2\} \tag{8.47}$$

(indeed, $Cx = y - Du \in \mathrm{L}^2$, where $x := \mathbb{A}x_0 + \mathbb{B}\tau u$ and $y := \mathbb{C}x_0 + \mathbb{D}u$, for all $x_0 \in H$ and $u \in \mathrm{L}^2(\mathbf{R}_+;U)$, hence $\|\cdot\|_{Z^{\mathrm{s}}_2}$ is a norm on the vector space Z^{s}_0). It follows that $\widetilde{C}$ becomes continuous, $\mathcal{U}^*_*$ is unchanged, and the assumptions of Lemma 8.2.9 are satisfied. □

If Σ is SOS-stable and $\mathcal{U}^*_* = \mathcal{U}_{\text{out}}$ (or $\mathcal{U}^*_* \in \{\mathcal{U}_{\text{exp}}, \mathcal{U}_{\text{str}}, \mathcal{U}_{\text{sta}}\}$ and Σ has the corresponding stability), then the existence of a spectral factorization leads to the existence of a stable, optimal state feedback pair (the converse holds under J-coercivity, by Corollary 9.9.11):

Corollary 8.3.11 (SpF $\Rightarrow$ J-critical) *Assume that* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{SOS}(U,H,Y)$ *and that* $\mathcal{U}^*_*(x_0) = \mathrm{L}^2(\mathbf{R}_+;U)$ *for all* $x_0 \in H$. *Assume, in addition, that the Popov operator* $\mathbb{D}^*J\mathbb{D}$ *has a spectral factorization* $\mathbb{X}^*S\mathbb{X}$. *Then its Toeplitz operator has the inverse* $T^{-1} = \mathbb{X}^{-1}\pi_+S^{-1}\mathbb{X}^{-*}$, *hence then Proposition 8.3.10 applies.*

In fact, then also (Crit1)–(Crit4) of Theorem 9.9.10 hold; in particular, (9.140) defines a stable, J-critical state feedback pair and (8.43)–(8.46) can be written as and (9.141)–(9.145). □

(This is obvious.)

We now compute the I-critical (minimizing) cost operator and control for the delay line system of Example 6.2.14:

Example 8.3.12 ($\mathcal{J}$ and $\mathcal{P}$) Take again $\Sigma := \left[\begin{array}{c|c} \pi_+\tau & \pi_{[0,1)}\tau(-1) \\ \hline \pi_+ & \tau(-1) \end{array}\right]$ (Σ is strongly stable) with $U = \mathbf{C} = Y$, $H := \mathrm{L}^2(\mathbf{R}_+;Y)$, and $J = I$. By Proposition 8.3.10, we have (note that $T = \pi_+$)

$$\mathcal{P} = \pi_+(I - \tau(1)\pi_+\tau(1)^*)\pi_+ = \pi_{[0,1)}, \qquad \mathbb{K}_{\mathrm{crit}} = -\pi_+\tau(1), \quad (8.48)$$

$$\mathbb{A}_{\mathrm{crit}}(t) = \pi_+\tau - \pi_{[0,1)}\tau\pi_{[1,\infty)} = (\pi_+ - \pi_{[1-t,1)})\tau, \qquad \mathbb{C}_{\mathrm{crit}} = \pi_{[0,1)}. \quad (8.49)$$

Thus, for $x_0 \in H$, the control $u = -\pi_+\tau(1)x_0$ is the unique J-critical (and minimizing) control (over $\mathcal{U}_{\mathrm{out}}$). Now $\mathcal{J}(x_0,u) = \|x_0 + \tau(-1)u\|_2^2$, and the J-critical cost is

$$\mathcal{J}(x_0, u_{\min}(x_0)) = \|\pi_{[0,1)}x_0\|_2^2 = \langle x_0, \mathcal{P}x_0\rangle. \quad (8.50)$$

Naturally, this is the minimal cost, because $\mathcal{J}(x_0,u) = \|x_0 + \tau(-1)u\|_2^2$ and hence u can cancel x_0 on $[1,+\infty)$ only. $\triangleleft$

See Example 9.8.15 for the corresponding Riccati equation.

Note that the cost function $\mathcal{J}$, the J-critical control and state u_{crit} and y_{crit}, and the J-critical cost operator $\mathcal{P}$ depend on $\mathbb{C}$, $\mathbb{D}$ and J only, whereas x_{crit} depends on $\mathbb{A}$ and $\mathbb{B}$ too.

We sometimes need the following useful formula (with terms corresponding to $\pi_{[0,t)}J$ and $\pi_{[t,\infty)}J$):

$$\langle \mathbb{D}v, J\mathbb{D}u\rangle_{\mathrm{L}^2} = \langle \mathbb{D}^t v, J\mathbb{D}^t u\rangle_{\mathrm{L}^2} + \langle \mathbb{C}\mathbb{B}^t v + \mathbb{D}\pi_+\tau^t v, J(\mathbb{C}\mathbb{B}^t u + \mathbb{D}\pi_+\tau^t u)\rangle_{\mathrm{L}^2} \quad (8.51)$$

for all $t > 0$, $u, v \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$ s.t. $\mathbb{D}u, \mathbb{D}v \in \mathrm{L}^2$; (use the fact that $\langle \mathbb{D}v, \pi_{[t,\infty)}J\mathbb{D}u\rangle = \langle \mathbb{D}(\pi_+ + \pi_-)\tau^t v, \pi_+ J\mathbb{D}(\pi_+ + \pi_-)\tau^t u\rangle$). In particular,

$$\mathcal{J}(0,u) = \langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle + \mathcal{J}(\mathbb{B}^t u, \pi_+\tau^t u) \quad (t > 0,\ u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U),\ \mathbb{D}u \in \mathrm{L}^2). \quad (8.52)$$

We now give two necessary and sufficient conditions for a unique J-critical control to be of state feedback form, i.e., for Σ_{crit} to be the left column of some closed-loop system of Σ:

Theorem 8.3.13 ($\Sigma_{\mathrm{crit}} = \Sigma_{\circlearrowleft}$) *Let there be a unique J-critical control over $\mathcal{U}^*_*(x_0)$ for each $x_0 \in H$. Assume that $\vartheta = 0$. Set $\mathbb{T} := \mathbb{K}_{\mathrm{crit}}\mathbb{B}$. Let $0 \le \gamma > \omega_A$. Then the following hold:*

(a) $\mathbb{T} \in \mathcal{B}(\mathrm{L}^2_\gamma(\mathbf{R};U))$, and, for each $v \in \mathrm{L}^2_\gamma(\mathbf{R}_-;U)$, $\mathbb{T}v$ is uniquely defined by the conditions $\mathbb{T}v \in \mathcal{U}^*_*(\mathbb{B}v)$ and $\langle \mathbb{D}(v + \mathbb{T}v), J\mathbb{D}\eta\rangle = 0$ for all $\eta \in \mathcal{U}^*_*(0)$.

If $\mathcal{U}^*_* = \mathcal{U}_{\mathrm{out}}$, then $\mathbb{T}v$ is uniquely defined by $\mathbb{T}v, \pi_+\mathbb{D}(v + \mathbb{T}v) \in \mathrm{L}^2$ and $\langle \mathbb{D}(v + \mathbb{T}v), J\mathbb{D}\eta\rangle = 0$ for all $\eta \in \mathcal{U}_{\mathrm{out}}(0)$; in particular, then $\mathbb{T}$ depends only on $\mathbb{D}$ and J.

If $\mathcal{U}^*_* = \mathcal{U}_{\exp}$ (resp. $\mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}$), then $\mathbb{T}v$ is uniquely defined by $\mathbb{T}v, \pi_+\mathbb{D}(v + \mathbb{T}v) \in \mathrm{L}^2$, $\pi_+\mathbb{B}\tau(v + \mathbb{T}v) \in \mathrm{L}^2$ (resp. $\in \mathrm{L}^\infty$, $\in \mathcal{C}_0$) and $\langle \mathbb{D}(v + \mathbb{T}v), J\mathbb{D}\eta\rangle = 0$ for all $\eta \in \mathcal{U}^*_*(0)$; in particular, then $\mathbb{T}$ depends only on $\mathbb{B}$, $\mathbb{D}$ and J.

(b1) Conditions (i)–(iii) are equivalent:

(i) There is an admissible state feedback pair $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ for Σ s.t. the corresponding closed-loop system $\Sigma_{\circlearrowleft}$ satisfies $\mathbb{K}_{\circlearrowleft} = \mathbb{K}_{\mathrm{crit}}$.

(ii) There is $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ s.t. $-\mathbb{X}\mathbb{T} = \pi_+\mathbb{X}\pi_- \in \mathcal{B}(\mathrm{L}^2_\alpha(\mathbf{R}_-;U), \mathrm{L}^2_\beta(\mathbf{R}_+;U))$ for some $\alpha, \beta \in \mathbf{R}$.

(iii) There is $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ *s.t.* $\mathbb{TM} = \pi_+\mathbb{M}\pi_- \in \mathcal{B}(\mathrm{L}^2_\alpha(\mathbf{R}_-;U), \mathrm{L}^2_\beta(\mathbf{R}_+;U))$ *for some* $\alpha, \beta \in \mathbf{R}$.

(Naturally, it suffices to have the equality on $\mathrm{L}^2_\mathrm{c}(\mathbf{R}_-;U)$ *on (ii) or (iii) if* α *and* β *are big enough to make both sides continuous.)*

(b2) If $\mathbb{X}$ *solves (ii) and* $\mathcal{U}_*^* \in \{\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{str}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{exp}}\}$*, then all solutions of (ii) are given by* $E\mathbb{X}$ *(*$E \in \mathcal{GB}(U)$*).*

(c1) Assume (i). Then $\mathbb{A}_\circlearrowleft = \mathbb{A}_{\mathrm{crit}}$ *and* $\mathbb{C}_\circlearrowleft = \mathbb{C}_{\mathrm{crit}}$*. Moreover,* $\mathbb{X} := I - \mathbb{F}$*,* $\mathbb{M} := \mathbb{X}^{-1} = \mathbb{F}_\circlearrowleft + I$ *and* $\mathbb{N} := \mathbb{D}_\circlearrowleft := \mathbb{DM}$ *satisfy (ii) and (iii) (for any* $\alpha, \beta > \max(0, \omega_A)$*),* $\left[\begin{smallmatrix}\mathbb{N}\\ \mathbb{M}\end{smallmatrix}\right] \in \mathrm{TIC}_\omega$ *for all* $\omega > 0$ *and* $\left[\begin{smallmatrix}\mathbb{N}\\ \mathbb{M}\end{smallmatrix}\right]\mathrm{L}^2_\mathrm{c} \subset \mathrm{L}^2$*. Naturally,* $\mathbb{K}$*,* $\mathbb{F}$ *and* $\mathbb{X}$ *are* ω*-stable for any* $\omega > \omega_A$*. If* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$*, then* $\Sigma_\circlearrowleft$ *is exponentially stable.*

(c2) Conversely, if (ii) holds, then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] := \left[\begin{array}{c|c} \mathbb{XK}_{\mathrm{crit}} & I - \mathbb{X} \end{array}\right]$ *satisfies (i). Moreover,* $\mathbb{X}$ *satisfies (ii) iff* $\mathbb{M} := \mathbb{X}^{-1}$ *satisfies (iii).*

Assume, in addition, that $\mathcal{U}_*^* \in \{\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{str}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{exp}}\}$*. Then also (d)–(f) hold:*

(d) Assume (i). We have $\mathbb{K}_\circlearrowleft x_0 + \mathbb{M}u_\circlearrowleft \in \mathcal{U}_*^*(x_0)$ *for all* $u_\circlearrowleft \in \mathrm{L}^2_\mathrm{c}(\mathbf{R}_+;U)$*. If* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$*, then*

$$\mathcal{U}_{\mathrm{exp}}(x_0) = \{\mathbb{K}_\circlearrowleft x_0 + \mathbb{M}u_\flat \,\big|\, u_\circlearrowleft \in \mathrm{L}^2(\mathbf{R}_+;U)\} \quad (x_0 \in H). \tag{8.53}$$

(e) Assume (i). Then there is a unique $S = S^* \in \mathcal{B}(U)$ *s.t.* $\langle \mathbb{N}u_\circlearrowleft, J\mathbb{N}u_\circlearrowleft\rangle = \langle u_\circlearrowleft, Su_\circlearrowleft\rangle$ *for all* $u_\circlearrowleft \in \mathrm{L}^2_\mathrm{c}(\mathbf{R}_+;U)$*. Moreover,* S *is one-to-one,* $\langle \mathbb{D}v, J\mathbb{DM}u_\circlearrowleft\rangle = \langle \mathbb{M}^{-1}v, Su_\circlearrowleft\rangle$ *for all* $v \in \mathcal{U}_*^*(0)$ *and* $u_\circlearrowleft \in \mathrm{L}^2_\mathrm{c}(\mathbf{R};U)$*, and*

$$\mathcal{J}(x_0, \mathbb{K}_\circlearrowleft x_0 + \mathbb{M}u_\circlearrowleft) = \langle x_0, \mathcal{P}x_0\rangle + \langle u_\circlearrowleft, Su_\circlearrowleft\rangle \quad (x_0 \in H,\ u_\circlearrowleft \in \mathrm{L}^2_\mathrm{c}(\mathbf{R}_+;U)). \tag{8.54}$$

(f) If $(\mathcal{P}, S, \mathbb{K}, \mathbb{X})$ *satisfies (i) and (e), then all such quadruples are given by* $(\mathcal{P}, E^{-*}SE^{-1}, E\mathbb{K}, E\mathbb{X})$*,* $E \in \mathcal{GB}(U)$*.*

We shall show in Theorem 9.9.1(a1) that (i)–(iii) hold iff the Riccati equation (eIARE) for Σ and J has a "$\mathcal{U}_*^*$-stabilizing" solution (and that solution is given by $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$; cf. (f)). As noted above, this is always the case when Σ is sufficiently regular. Conditions (i)–(iii) are treated from another point of view in Section 9.14.

Proof: (a) Both $\mathbb{K}_{\mathrm{crit}}$ and $\mathbb{B}$ are γ-stable, hence so is $\mathbb{T}$. Set $x_0 := \mathbb{B}v$, $u := \mathbb{T}v = \mathbb{K}_{\mathrm{crit}}x_0$, $y := \mathbb{C}x_0 + \mathbb{D}u$. Then $u_{\mathrm{crit}}(x_0)$ is the unique $u \in \mathcal{U}_*^*(x_0)$ satisfying $\langle y, J\mathbb{D}\eta\rangle = 0$ for all $\eta \in \mathcal{U}_*^*(0)$, and $y = \mathbb{C}x_0 + \mathbb{D}u = \mathbb{CB}v + \mathbb{DT}v = \pi_+\mathbb{D}(\pi_- v + \mathbb{T}v)$.

By definition, $u \in \mathcal{U}_{\mathrm{out}}(\mathbb{B}v)$ iff $u \in \mathrm{L}^2$ and $y \in \mathrm{L}^2$; for $\mathcal{U}_{\mathrm{exp}}$ we have the extra condition that $\pi_+(\mathbb{AB}v + \mathbb{B}\tau\mathbb{T}v) = \pi_+\mathbb{B}\tau(v + \mathbb{T}v) \in \mathrm{L}^2(\mathbf{R}_+;U)$.

(b1)&(c2) If (i) holds, then $\mathbb{TM} = \mathbb{K}_\circlearrowleft\mathbb{B}_\circlearrowleft = \pi_+\mathbb{M}\pi_-$ in $\mathcal{B}(\mathrm{L}^2_\alpha(\mathbf{R};U))$ for any $\alpha > \max(\omega_A, 0)$ (note that $\mathbb{B}$, $\mathbb{M}$ and $\mathbb{K}_\circlearrowleft$ are α-stable) hence then (iii) holds.

If $\mathbb{X} = \mathbb{M}^{-1} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, then $\pi_+\mathbb{X}\pi_-\mathbb{M}\pi_- = \pi_+ I\pi_- - \pi_+\mathbb{X}\pi_+\mathbb{M}\pi_- = -\pi_+\mathbb{X}\pi_+\mathbb{M}\pi_-$. Therefore, (iii) implies (ii).

Assume (ii). Let $\omega > \max\{\omega_A, \alpha, \beta\}$. Then $\mathrm{L}^2_\omega(\mathbf{R}_-;U) \subset \mathrm{L}^2_\alpha(\mathbf{R}_-;U)$ and $\mathbb{X}, \mathbb{T}, \pi_+\mathbb{X}\pi_- \in \mathcal{B}(\mathrm{L}^2_\omega)$, hence then $-\mathbb{X}\mathbb{T} = \pi_+\mathbb{X}\pi_- \in \mathcal{B}(\mathrm{L}^2_\omega)$. Set $\mathbb{F} := I - \mathbb{X}$ and $\mathbb{K} := \mathbb{X}\mathbb{K}_{\mathrm{crit}}$. Then $\mathbb{K}\mathbb{B} = \pi_+\mathbb{F}\pi_-$, and

$$\mathbb{K}\mathbb{A}^t = \mathbb{X}\mathbb{K}_{\mathrm{crit}}\mathbb{A}^t = \mathbb{X}\mathbb{K}_{\mathrm{crit}}(\mathbb{A}^t_{\mathrm{crit}} - \mathbb{B}^t\mathbb{K}_{\mathrm{crit}}) = \mathbb{X}\pi_+\tau^t\mathbb{K}_{\mathrm{crit}} - \mathbb{X}\mathbb{K}_{\mathrm{crit}}\mathbb{B}\tau^t\mathbb{K}_{\mathrm{crit}} \tag{8.55}$$

$$= \pi_+\mathbb{X}\pi_+\tau^t\mathbb{K}_{\mathrm{crit}} - \pi_+\mathbb{X}\pi_-\tau^t\mathbb{K}_{\mathrm{crit}} = \pi_+\tau^t\mathbb{X}\mathbb{K}_{\mathrm{crit}} = \pi_+\tau^t\mathbb{K}. \tag{8.56}$$

Therefore, $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{K} & \mathbb{F}\end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,U)$, hence $\Sigma_{\mathrm{ext}} \in \mathrm{WPLS}_{\omega'}(U,H,Y\times U)$ for any $\omega' > \omega_A$. Obviously, Σ_{crit} is the left column of the corresponding closed-loop system $\Sigma_\circlearrowleft$, since $\mathbb{K}_\circlearrowleft = \mathbb{M}\mathbb{K} = \mathbb{K}_{\mathrm{crit}}$.

(b2) This follows from (f).

(c1) Now $\mathbb{C}_\circlearrowleft = \mathbb{C} + \mathbb{D}\mathbb{K}_\circlearrowleft = \mathbb{C}_{\mathrm{crit}}$, $\mathbb{A}_\circlearrowleft = \mathbb{A} + \mathbb{B}^t\mathbb{K}_\circlearrowleft = \mathbb{A}_{\mathrm{crit}}$. The formulae for (ii) and (iii) were shown above. Because $\mathbb{C}_\circlearrowleft$ and $\mathbb{K}_\circlearrowleft$ are stable, the maps $\mathbb{N}$ and $\mathbb{M}$ are as above, by Lemma 6.1.11. The ω_A claim follows from Lemma 6.1.10, and the final claim from Theorem 8.3.9(a2).

(d) Now $u := \mathbb{K}_\circlearrowleft x_0 + \mathbb{M}u_\circlearrowleft \in \mathrm{L}^2$ and $y := \mathbb{C}x_0 + \mathbb{D}u = \mathbb{C}_\circlearrowleft x_0 + \mathbb{N}u_\circlearrowleft \in \mathrm{L}^2$, by (c1). If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then also $x := \mathbb{A}x_0 + \mathbb{B}^t u = \mathbb{A}_\circlearrowleft x_0 + \mathbb{B}^t_\circlearrowleft u_\circlearrowleft \in \mathrm{L}^2$, by Lemma 6.1.10. Analogously, if $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{sta}}$ (resp. $= \mathcal{U}_{\mathrm{str}}$) and $u_\circlearrowleft \in \mathrm{L}^2([0,T);U)$, then $\mathbb{B}_\circlearrowleft\tau^{T+t}u_\circlearrowleft = \mathbb{A}^t_\circlearrowleft\mathbb{B}_\circlearrowleft\tau^T u_\circlearrowleft$ is bounded (resp. goes to zero), as $t \to +\infty$, since $\mathbb{A}_\circlearrowleft$ is stable (resp. strongly stable). Thus, $u \in \mathcal{U}_*^*(x_0)$. Formula (8.53) will be proved in Theorem 8.4.5(e).

(e) Let $u_\circlearrowleft \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+;U)$, $v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_-;U)$. Then $u := \mathbb{M}u_\circlearrowleft \in \mathcal{U}_*^*(0)$, by (d), hence then $\langle \mathbb{N}\pi_- v, J\mathbb{N}u_\circlearrowleft\rangle = \langle \mathbb{C}_{\mathrm{crit}}\mathbb{B}v, J\mathbb{D}u\rangle = 0$. Consequently, we obtain S from Lemma 2.3.1. By (8.35), we have

$$\mathcal{J}(x_0, \mathbb{K}_\circlearrowleft x_0 + \mathbb{M}u_\circlearrowleft) = \langle x_0, \mathcal{P}x_0\rangle + \mathcal{J}(0, \mathbb{M}u_\circlearrowleft) = \langle x_0, \mathcal{P}x_0\rangle + \langle u_\circlearrowleft, Su_\circlearrowleft\rangle \tag{8.57}$$

for all $x_0 \in H$, $u_\circlearrowleft \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+;U)$.

Let $v \in \mathcal{U}_*^*(0)$ and $u_\circlearrowleft \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R};U)$. Because $\langle \mathbb{D}v, J\mathbb{N}\pi_- u_\circlearrowleft\rangle = \langle \mathbb{D}v, J\mathbb{C}_{\mathrm{crit}}\mathbb{B}_\circlearrowleft u_\circlearrowleft\rangle = 0$, we may assume that $u_\circlearrowleft = \pi_+ u_\circlearrowleft$. Choose $T > 0$ s.t. $u_\circlearrowleft = \pi_{[0,T)}u_\circlearrowleft$. Set $v_\circlearrowleft := \mathbb{M}^{-1}v$, $v_1 := \mathbb{M}\pi_{[0,T)}v_\circlearrowleft \in \mathcal{U}_*^*(0)$, $v_2 := v - v_1 \in \mathcal{U}_*^*(0)$. Then $\pi_{[0,T)}v_2 = \pi_{[0,T)}\mathbb{M}\pi_{[T,\infty)}v_\circlearrowleft = 0$, hence

$$\langle \mathbb{D}v, J\mathbb{N}u_\circlearrowleft\rangle = \langle \pi_{[0,T)}v_\circlearrowleft, Su_\circlearrowleft\rangle + \langle \mathbb{D}v_2, J\mathbb{C}_{\mathrm{crit}}\mathbb{B}_\circlearrowleft u_\circlearrowleft\rangle = \langle \pi_{[0,T)}v_\circlearrowleft, Su_\circlearrowleft\rangle = \langle v_\circlearrowleft, Su_\circlearrowleft\rangle. \tag{8.58}$$

By taking $v_\circlearrowleft := \chi_{[0,1)}v_0$, $u_\circlearrowleft := \chi_{[0,1)}u_0$ for arbitrary $u_0, v_0 \in U$, we see that S is unique.

Let $u_0 \in U \setminus \{0\}$ be arbitrary. Set $u_\circlearrowleft := \chi_{[0,1)}u_0$, $u := \mathbb{M}u_\circlearrowleft \in \mathcal{U}_*^*(0) \setminus \mathcal{U}_*^{*,\mathrm{crit}}(0)$ (recall that $\mathcal{U}_*^{*,\mathrm{crit}}(0) = \{0\}$, by uniqueness), hence then $\langle \mathbb{D}v, J\mathbb{D}u\rangle \neq 0$ for some $v \in \mathcal{U}_*^*(0)$. Thus, $\langle \mathbb{M}^{-1}v, Su_\circlearrowleft\rangle \neq 0$. In particular, $Su_0 \neq 0$. Therefore, S is one-to-one.

(f) Let $(\mathcal{P}, \widetilde{S}, \widetilde{\mathbb{K}}, \widetilde{\mathbb{X}})$ also satisfy (i) and (e). Claim (e) and the proof of Lemma 9.10.1(c2) shows that (9.160) is satisfied by both $\mathbb{X}$ and $\widetilde{\mathbb{X}}$, hence

$\widetilde{\mathbb{X}} = E\mathbb{X}$ and $\widetilde{S} = E^{-*}SE^{-1}$ for some $E \in \mathcal{GB}(U)$, by Lemma 2.3.5 (recall that S is one-to-one). By (c2), $\widetilde{\mathbb{K}} = \widetilde{\mathbb{X}}\mathbb{K}_{\mathrm{crit}} = E\mathbb{X}\mathbb{K}_{\mathrm{crit}} = E\mathbb{K}$. □

The above assumption on uniqueness in the theorem is mostly superfluous for a control corresponding to a state feedback pair:

Lemma 8.3.14 *Let $\begin{bmatrix} \mathbb{K} & \big| & \mathbb{F} \end{bmatrix}$ be a J-critical state feedback pair over $\mathcal{U}_*^*$ for Σ and J. Set $\mathbb{T} := \mathbb{K}_{\circlearrowleft}\mathbb{B}$, $\Sigma_{\mathrm{crit}} := \Sigma_{\circlearrowleft}$.*

Then (i)–(iii) and (c1)–(e) of Theorem 8.3.13 hold except that S need not one-to-one in (e). □

(The same proof applies.) In particular, then $\mathbb{A}_{\circlearrowleft} = \mathbb{A}_{\mathrm{crit}}$ and $\mathbb{C}_{\circlearrowleft} = \mathbb{C}_{\mathrm{crit}}$, so that $\Sigma_{\circlearrowleft}$ produces J-critical state, control and output (for zero input).

In fact, most of the rest of Theorem 8.3.13 holds in this more general case too, but we shall return to this in Chapter 9. See Theorem 9.9.1(a1)&(e2)&(f2)–(h) for details.

If $\begin{bmatrix} \mathbb{K} & \big| & \mathbb{F} \end{bmatrix}$ is a J-critical state feedback pair over $\mathcal{U}_*^*$, then the left column of the corresponding closed-loop system is just like the one in (8.34) (except that it need not be unique). We shall call such a column a J-critical control in WPLS form in order to be able to treat both cases simultaneously:

Definition 8.3.15 (J-critical control in WPLS form) *We call the control $x_0 \mapsto \mathbb{K}_{\mathrm{crit}}x_0$ (and Σ_{crit}) a* control for Σ in WPLS form *if $\mathbb{K}_{\mathrm{crit}} : H \to \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$ is s.t.*

$$\Sigma_{\mathrm{crit}} := \left[\begin{array}{c|c} \mathbb{A}_{\mathrm{crit}} & \\ \hline \mathbb{C}_{\mathrm{crit}} & \\ \mathbb{K}_{\mathrm{crit}} & \end{array}\right] := \left[\begin{array}{c|c} \mathbb{A}+\mathbb{B}\mathbb{K}_{\mathrm{crit}} & \\ \hline \mathbb{C}+\mathbb{D}\mathbb{K}_{\mathrm{crit}} & \\ \mathbb{K}_{\mathrm{crit}} & \end{array}\right] \in \mathrm{WPLS}(\{0\},H,Y\times U). \tag{8.59}$$

If $\mathbb{K}_{\mathrm{crit}}x_0$ is J-critical for each x_0, then we say that $\mathbb{K}_{\mathrm{crit}}$ (or Σ_{crit}) is a J-critical control in WPLS form *and that u_{crit} can be given in WPLS form, where* $\begin{bmatrix} x_{\mathrm{crit}}(x_0) \\ y_{\mathrm{crit}}(x_0) \\ u_{\mathrm{crit}}(x_0) \end{bmatrix} := \Sigma_{\mathrm{crit}}x_0$.

If $\begin{bmatrix} \mathbb{K} & \big| & \mathbb{F} \end{bmatrix}$ is an admissible state feedback pair for Σ with closed-loop system $\Sigma_\flat$, then we call $\mathbb{K}_{\circlearrowleft} := (I-\mathbb{F})^{-1}\mathbb{K}$ (an the left column of $\Sigma_\flat$) a control in state feedback form.

As explained below Theorem 8.3.9, a control in WPLS form need not be of state feedback form unless, e.g., B is bounded, as shown in Lemma 8.3.18.

We start with some rather obvious facts:

Lemma 8.3.16

(a1) A unique J-critical control can always be given in WPLS form.

(a2) If there is a J-critical state feedback pair, then the corresponding J-critical control can be given in WPLS form.

(b) A control ("Σ_{crit}") in WPLS form has the properties described in Remark 9.7.7, Theorem 8.3.9, and Theorem 8.3.13(b1)&(c1)&(c2)&(d).

(c) A control in state feedback form is in WPLS form.

Thus, we need not assume a J-critical control to be unique in Theorem 8.3.9 as long as $\Sigma_{\rm crit}$ is a J-critical control for Σ in WPLS form. Therefore, we can and will use the latter (weaker) assumption in several results below to treat both cases simultaneously.

Proof: (a1) This is contained in Theorem 8.3.9.

(a2) In fact, given any admissible state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ for Σ, then the control $\mathbb{K}_{\circlearrowleft} := (I-\mathbb{F})^{-1}\mathbb{K}$ can be given in state feedback form (let $\Sigma_{\rm crit}$ be the left column of the corresponding closed-loop system).

(b) This is contained in Remark 9.7.7, Theorem 8.3.9, and the proof of Theorem 8.3.13(b1)&(c1)&(c2)&(d) (note that in (b1) we only use the fact that $\mathbb{K}_{\rm crit}\mathbb{B}\mathbb{M} = \pi_+\mathbb{M}\pi_-$ (or $-\mathbb{X}\mathbb{K}_{\rm crit}\mathbb{B} = \pi_+\mathbb{X}\pi_-$)).

(c) This is obvious. $\square$

The generators of a control in WPLS form are analogous to those of a state feedback system:

Lemma 8.3.17 *Let $\Sigma_{\rm crit}$ be a control in WPLS form.*

(a) We have $A_{\rm crit} = A + BK_{\rm crit}$ and $C_{\rm crit} = C_{\rm c} + D_{\rm c}K_{\rm crit}$ on $\mathrm{Dom}(A_{\rm crit})$, hence $\mathrm{Dom}(A_{\rm crit}) \subset H_B$, where $\left[\begin{array}{c|cc} A_{\rm crit}^{\rm T} & C_{\rm crit}^{\rm T} & K_{\rm crit}^{\rm T} \end{array}\right]^{\rm T}$ are the generators of $\Sigma_{\rm crit}$, and $(C_{\rm c}, D_{\rm c})$ is any compatible pair for Σ.

(b) Let $K_{\rm c}$ be a compatible admissible state feedback operator for Σ, and let $\Sigma_\flat$ be the corresponding closed-loop system.

Then $K_{\rm c} = K_{\rm crit}$ on $\mathrm{Dom}(A_{\rm crit})$ iff $\mathbb{K}_\flat = \mathbb{K}_{\rm crit}$. If $\mathbb{K}_\flat = \mathbb{K}_{\rm crit}$, then $\mathbb{A}_\flat = \mathbb{A}_{\rm crit}$, $\mathbb{C}_\flat = \mathbb{C}_{\rm crit}$, and $K_{\rm c}$ is the unique compatible operator having $\mathbb{K}_\flat = \mathbb{K}_{\rm crit}$.

Proof: (a) 1° As in the proof of Proposition 6.6 of [W94b], we take the Laplace transform of the equation $\mathbb{A}_{\rm crit}^t x_0 - \mathbb{A}^t x_0 = \mathbb{B}\tau^t\mathbb{K}_{\rm crit}x_0$ ($x_0 \in H$, $t \geq 0$) to obtain

$$(s-A_{\rm crit})^{-1} - (s-A)^{-1} = (s-A)^{-1}BK_{\rm crit}(s-A_{\rm crit})^{-1} \in \mathcal{B}(H). \tag{8.60}$$

Multiply this by $(s-A_{\rm crit}) \in \mathcal{B}(A_{\rm crit}, H)$ to the right and by $s-A \in \mathcal{B}(H, H_{-1})$ to the left to obtain

$$A_{\rm crit} - A = (s-A) - (s-A_{\rm crit}) = BK_{\rm crit} \in \mathcal{B}(\mathrm{Dom}(A_{\rm crit}), H_{-1}). \tag{8.61}$$

(N.B. by duality, $A^* = A_{\rm crit}^* - K_{\rm crit}^* B^*$ on $\mathrm{Dom}(A^*)$.) If $x_0 \in \mathrm{Dom}(A_{\rm crit})$, then $Ax_0 + BK_{\rm crit}x_0 \in H$, hence then $x_0 \in H_B$ (see Definition 6.1.17).

2° By Laplace transforming the equation $\mathbb{C}_{\rm crit} = \mathbb{C} + \mathbb{D}\mathbb{K}_{\rm crit}$ and using Theorem 6.2.11(c1), Lemma 6.3.10(a) and (8.60), we obtain that

$$\widehat{\mathbb{C}_{\rm crit}}(s) = C_{\rm crit}(s-A_{\rm crit})^{-1} = \widehat{\mathbb{C}}(s) + \widehat{\mathbb{D}}(s)\widehat{\mathbb{K}_{\rm crit}}(s) \tag{8.62}$$

$$= C(s-A)^{-1} + D_{\rm c}K_{\rm crit}(s-A_{\rm crit})^{-1}C_{\rm c}(s-A)^{-1}BK_{\rm crit}(s-A_{\rm crit})^{-1} \tag{8.63}$$

$$= (C_{\rm c} + D_{\rm c}K_{\rm crit})(s-A_{\rm crit})^{-1} \in \mathcal{B}(H, Y) \tag{8.64}$$

for $s>\max\{\omega_A,\omega_{A_{\rm crit}}\}$. Because $s-A_{\rm crit}$ maps H onto $\mathrm{Dom}(A_{\rm crit})$, we must have $C_{\rm crit}=C_{\rm c}+D_{\rm c}K_{\rm crit}$.

(b) 1° If $\mathbb{K}_\flat=\mathbb{K}_{\rm crit}$, then $K_{\rm c}=K_{\rm crit}$ on $\mathrm{Dom}(A_{\rm crit})$, by Proposition 6.6.18(d2), because $\mathrm{Dom}(A_{\rm crit})\subset H_B$, by (c).

2° Assume that $K_{\rm c}=K_{\rm crit}$ on $\mathrm{Dom}(A_{\rm crit})$. By Proposition 6.6.18(d2), $A_\flat=A+BK_{\rm c}$ and $K_\flat=K_{\rm c}$ on H_B and $\mathrm{Dom}(A_\flat)\subset H_B$. But $\mathrm{Dom}(A_{\rm crit})\subset H_B$ and $A_{\rm crit}=A+BK_{\rm c}$, by (c), hence $A_\flat=A_{\rm crit}$ on $\mathrm{Dom}(A_{\rm crit})$. Choose $\omega>\max\{\omega_{A_\flat},\omega_{A_{\rm crit}}\}$. Then

$$\mathrm{Dom}(A_{\rm crit})=\{x_0\in H_B\,\big|\,(\omega-A_{\rm crit})x_0\in H\}\subset\{x_0\in H\,\big|\,(\omega-A_\flat)x_0\in H\}=\mathrm{Dom}(A_\flat), \tag{8.65}$$

hence $\mathbb{A}_{\rm crit}=\mathbb{A}_\flat$, by Lemma A.4.2(i). Consequently, $\mathbb{K}_\flat=K_{\rm c}\mathbb{A}_\flat=K_{\rm crit}\mathbb{A}_{\rm crit}=\mathbb{K}_{\rm crit}$ on $\mathrm{Dom}(A_{\rm crit})$, hence on H, by density. In particular, $K_{\rm c}$ is J-critical. It follows that $\mathbb{C}_\flat=\mathbb{C}+\mathbb{D}\mathbb{K}_\flat=\mathbb{C}_{\rm crit}$.

3° *Uniqueness:* If also $K_{\rm c}'$ is compatible and admissible and $\mathbb{K}_\flat'=\mathbb{K}_{\rm crit}$ (i.e., $K_{\rm c}'=K_{\rm crit}$ on $\mathrm{Dom}(A_{\rm crit})$), then $K_{\rm c}'=K_{\rm c}$ on H_B (i.e., $\begin{bmatrix}\mathbb{K}' & \mathbb{F}'\end{bmatrix}=\begin{bmatrix}\mathbb{K} & \mathbb{F}\end{bmatrix}$), by Proposition 6.6.18(g), i.e., $K_{\rm c}'=K_{\rm c}$. (Recall that we consider $K_{\rm c}$ and $K_{\rm c}'$ equal when $K_{\rm c}'=K_{\rm c}$ on H_B, since the values of $K_{\rm c}$ outside H_B do not affect $\Sigma_{\rm ext}$, nor $\Sigma_\flat$.) □

As stated above, when B is bounded, a control in WPLS form corresponds to an ULR state feedback operator:

Lemma 8.3.18 *Let $\Sigma_{\rm crit}$ be a control in WPLS form and let B be bounded.*

Then $\Sigma_{\rm crit}$ is of state feedback form.

Moreover, $\mathrm{Dom}(A_{\rm crit})=H_B=\mathrm{Dom}(A)$, $K_{\rm crit}$ *is an ULR state feedback operator for Σ, and $\Sigma_{\rm crit}$ is the left column of the corresponding closed-loop system $\Sigma_\circlearrowleft$.*

Since, in discrete time, B is always bounded, any control in "wpls form" is necessarily induced by state feedback to the original system (the proof below applies mutatis mutandis; in fact, the discrete-time form of the proof is contained at the beginning of the proof of Theorem 14.1.6).

Proof: (Note that $\Sigma_{\rm crit}$ is not assumed to be J-critical.)

Let $A_{\rm crit},C_{\rm crit},K_{\rm crit}$ be the generators of $\Sigma_{\rm crit}$. By Lemma 6.3.16(b), $\left[\begin{array}{c|c}A_{\rm crit} & B\\ \hline K_{\rm crit} & 0\end{array}\right]$ generate an ULR WPLS $\left[\begin{array}{c|c}\mathbb{A}_{\rm crit} & \mathbb{B}_{\rm crit}\\ \hline \mathbb{K}_{\rm crit} & \mathbb{F}_{\rm crit}\end{array}\right]$. Because $F_{\rm crit}=0$ and $\mathbb{F}_{\rm crit}\in\mathrm{ULR}$, we have $I+\mathbb{F}_{\rm crit}\in\mathcal{G}\mathrm{TIC}_\infty$, by Proposition 6.3.1(c). Therefore, $\begin{bmatrix}-\mathbb{K}_{\rm crit} & -\mathbb{F}_{\rm crit}\end{bmatrix}$ is an admissible state feedback pair for $\begin{bmatrix}\mathbb{A}_{\rm crit} & \mathbb{B}_{\rm crit}\end{bmatrix}$,

By (6.145), the corresponding closed-loop system $\Sigma_\flat$ is generated by $\left[\begin{array}{c|c}A & B\\ \hline -(K_{\rm crit})_{\rm w} & 0\end{array}\right]$, hence of form $\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline -\mathbb{K} & -\mathbb{F}\end{array}\right]$ for some $\begin{bmatrix}\mathbb{K} & \mathbb{F}\end{bmatrix}$, where $\mathbb{F}=I-(I+\mathbb{F}_{\rm crit})^{-1}$. Indeed, $H_{B_{\rm crit}}:=\mathrm{Dom}(A_{\rm crit})+(s-A_{\rm crit})^{-1}BU=\mathrm{Dom}(A_{\rm crit})$ with equivalent norms, by Lemma A.3.6. But

$$H_B=\mathrm{Dom}(A)=\{x_0\in H\,\big|\,Ax_0\in H\}=\mathrm{Dom}(A_{\rm crit})=H_{B_{\rm crit}}, \tag{8.66}$$

(by the formula $A_{\rm crit}=A+BK_{\rm crit}$ from Lemma 8.3.17(a)) with equivalent norms, by Lemma A.3.6, hence $A_\flat:=A_{\rm crit}-BK_{\rm crit}=A$ (with same domains).

Consequently, $K = (K_{\rm crit})_{\rm w}|_{{\rm Dom}(A)} = K_{\rm crit}$ is an admissible ULR state feedback operator for Σ, and $\Sigma_{\rm crit}$ is the left column of the corresponding closed-loop system (due to same generators). □

As noted in [FLT], optimizability is almost equivalent to exponential stabilizability:

Proposition 8.3.19 *Let Σ be optimizable. Then there is an exponentially stable WPLS of form (8.59).*

The difference is that the "exponentially stabilizing state feedback" of Theorem 8.3.9 need not be well-posed in general; see Theorem 9.2.12 for sufficient conditions.

Proof: Define $\Sigma_{\rm ext} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C}_{\rm ext} & \mathbb{D}_{\rm ext} \end{array}\right]$ by

$$\mathbb{C}_{\rm ext} := \begin{bmatrix} \mathbb{C} \\ \mathbb{A} \\ 0 \end{bmatrix}, \quad \mathbb{D}_{\rm ext} := \begin{bmatrix} \mathbb{D} \\ \mathbb{B} \\ I \end{bmatrix}, \quad J = I \tag{8.67}$$

to have $\mathcal{J}_{\rm ext}(x_0,u) = \|u\|_2^2 + \|x\|_2^2 + \|y\|_2^2$, where $x := \mathbb{A}x_0 + \mathbb{B}\tau u$, $y := \mathbb{C}x_0 + \mathbb{D}u$.

Because $\Sigma_{\rm ext}$ is J-coercive over $\mathcal{U}_{\rm exp}$, we obtain an exponentially stable system $(\Sigma_{\rm ext})_{\rm crit} = \left[\begin{array}{c} \mathbb{A}_{\rm crit} \\ \hline \mathbb{C}_{\rm ext} + \mathbb{D}_{\rm ext}\mathbb{K}_{\rm crit} \\ \mathbb{K}_{\rm crit} \end{array}\right] \in {\rm WPLS}(U,H,Y \times H \times U)$ from Theorems 8.4.3 and 8.3.9; and $\mathbb{C}_{\rm ext} + \mathbb{D}_{\rm ext}K_{\rm crit} = \begin{bmatrix} \mathbb{C} + \mathbb{D}\mathbb{K}_{\rm crit} \\ \mathbb{A}_{\rm crit} \\ \mathbb{K}_{\rm crit} \end{bmatrix}$, so that we only have to drop the second and third row from $(\Sigma_{\rm ext})_{\rm crit}$ to obtain an exponentially stable WPLS of form (8.59). □

Now we can give a postponed proof:

Lemma 8.3.20 *Theorem 6.7.7 holds.*

Proof: Let $\Sigma_\sharp := \left[\begin{array}{c|cc} \mathbb{A}_\sharp & \mathbb{B}_\sharp & \mathbb{H}_\sharp \end{array}\right]$ be the exponentially stable system of Proposition 8.3.19 for $\Sigma^{\rm d}$ (which is optimizable), and set $M := \|\mathbb{A}_\sharp\|_{\mathcal{B}(H,{\rm L}^2)} + \|\mathbb{B}_\sharp\tau\|_{\rm TIC} + \|\mathbb{H}_\sharp\tau\|_{\rm TIC} < \infty$. One easily verifies that $x := \mathbb{A}x_0 + \mathbb{B}\tau u = \mathbb{A}_\sharp x_0 + \mathbb{B}_\sharp \tau u - \mathbb{H}_\sharp \tau y$. Since $\Sigma_\sharp$ is strongly stable, we have that $x \in C_0(\mathbf{R}_+;H)$. □

Notes

Much of Lemmas 8.3.5–8.3.8 and Theorem 8.3.9 has been used for decades in some special cases; see [Zwart] for the standard unstable LQR (minimization) problem and [S98c] for the stable indefinite setting of Proposition 8.3.10 and Corollary 8.3.11; both articles only treat J-coercive WPLSs and only the latter treats the closed-loop system.

Section 3 of [S98c] essentially contains Proposition 8.3.10 except for its last paragraph, whose results resemble what is a key formula in several articles by Irena Lasiecka, Roberto Triggiani and others. They first use such a stable case result (a special case of Lemma 8.2.9, as in Section 8.5) to solve the finite-time LQR problem and then let the length of the time interval approach infinity to obtain the solution for the infinite-time problem as the limit of the finite-time solution (they use a very coercive cost function to guarantee the convergence). Their results in [FLT] include Proposition 8.3.19. See also [LT00a]–[LT00b] (also a third part of the trilogy is supposed to appear). This finite-time method has been used by several authors at least since seventies.

All the articles mentioned above optimize over $\mathcal{U}_{\mathrm{out}}$ ($= \mathcal{U}_{\mathrm{sta}}$ in [S97b] and [S98c], and $= \mathcal{U}_{\mathrm{exp}}$ in much of [FLT] and [LT00a]–[LT00b], by Lemma 8.3.3), and so does, e.g., [CZ]. Nevertheless, $\mathcal{U}_{\mathrm{exp}}$ is the most commonly used class for finite-dimensional systems (see, e.g., [IOW], [LR], [GL]) and possibly also for infinite-dimensional systems (see, e.g., [Keu], [Pandolfi] and [WR00]).

For finite-dimensional systems, strong stability is equivalent to exponential stability, but in the infinite-dimensional case the requirement of strong stability ($\mathcal{U}_{\mathrm{str}}$) has often been used since the seventies; see, e.g., [Slemrod] and [Balakrishnan]. This case has been studied for WPLSs with bounded input and output operators (B and C) by Ruth Curtain and Job Oostveen in several articles (see [OC98]), and the monograph [Oostveen] contains a rather mature theory, further historical remarks on this case and examples where $\mathcal{U}_{\mathrm{str}}$ is the most natural choice for $\mathcal{U}_*^*$. See p. 501 for a comparison of $\mathcal{U}_{\mathrm{exp}}$, $\mathcal{U}_{\mathrm{str}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{out}}$.

We give several sufficient conditions of a unique J-critical control to be of (regular) state feedback form in Remark 9.9.14 and in the notes to Section 9.9.

8.4 J-coercivity and factorizations

A person who is wise does nothing against their will, nothing with sighing or under coercion.
— Marcus Tullius Cicero (106 B.C. – 43 B.C.)

In this section, we apply J-coercivity to WPLSs and explore its connection to optimal (J-critical) control and spectral and inner coprime factorizations. This concept generalizes several general nonsingularity assumptions of control problems.

We shall list several equivalent conditions for positive J-coercivity in Section 10.3, such as the popular "no transmission zeros" ($\mathcal{U}_{\rm out}$) and "no invariant zeros" ($\mathcal{U}_{\rm exp}$) conditions. Most of these equivalent conditions have been used in classical minimization problems; we also show there that several other classical minimization assumptions are stronger than positive J-coercivity.

In the stable case, J-coercivity is equivalent to the condition that the *Popov Toeplitz operator* $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+$ is invertible, by Lemma 8.4.11(a1). The general definition below requires that "$\pi_+\mathbb{D}^*J\mathbb{D}\pi_+$" is invertible on $\mathcal{U}_*^*(0)$, by Lemma 8.2.3(c1), hence also the general condition can be considered as a Popov Toeplitz invertibility condition.

General J-coercivity with the minimal stabilizability assumption $\mathcal{U}_*^*(x_0) \neq 0$ $(x_0 \in H)$ will be shown to be a sufficient condition for the existence of an optimal (i.e., J-critical) control and for the existence of a unique "stabilizing" solution of the Riccati equation (under sufficient regularity); in fact, these three are equivalent in some cases (see, e.g., Theorem 9.2.16).

At the end of this section, we shall show that J-coercivity over $\mathcal{U}_{\rm out}$ is implied by the existence of an inner coprime factorization (spectral factorization in the stable case) of the I/O map, with equivalence under sufficient regularity and stabilizability assumptions.

We also show that optimization over $\mathcal{U}_{\rm exp}$ can be reduced to the stable case, whereas for $\mathcal{U}_{\rm out}$, $\mathcal{U}_{\rm sta}$ and $\mathcal{U}_{\rm str}$, we need quasi-coprimeness for an analogous reduction.

In accordance to Definition 8.2.1, we generalize J-coercivity as follows:

Definition 8.4.1 (J-coercive) *We call* $\mathbb{D}$ J-coercive (over $\mathcal{U}_*^*$) *if there is* $\varepsilon > 0$ *s.t. for each nonzero* $u \in \mathcal{U}_*^*(0)$ *there is a nonzero* $v \in \mathcal{U}_*^*(0)$ *s.t.*

$$\langle \mathbb{D}v, J\mathbb{D}u\rangle \geq \varepsilon \|u\|_{\mathcal{U}_*^*}\|v\|_{\mathcal{U}_*^*}. \tag{8.68}$$

If, in addition, $\langle \mathbb{D}u, J\mathbb{D}u\rangle \geq 0$ *for each* $u \in \mathcal{U}_*^*(0)$*, then* $\mathbb{D}$ *is called* positively J-coercive (over $\mathcal{U}_*^*$).

(Note that for $\mathcal{U}_{\rm out}$, [positive] J-coercivity depends on $\mathbb{D}$ and J only, not on the rest of Σ.)

If $\widehat{\mathbb{D}}$ is a rational matrix-valued function or stable, then $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\rm out}$ iff $\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}} \geq \varepsilon I$ a.e. on $i\mathbf{R}$ for some $\varepsilon > 0$ ("$\widehat{\mathbb{D}}$ has a full column rank on $i\mathbf{R} \cup \{\infty\}$"), by Proposition 10.3.1(b)&(c), unless U is unseparable (see (c)).

By Lemma 8.2.3, we have the following when Z^{s} is a Hilbert space (here $\mathbb{D}$ stands for $\mathbb{D}|_{\mathcal{U}_*^*(0)}$):

(b) $\mathbb{D} \in \mathcal{B}(\mathcal{U}_*^*(0), \mathrm{L}^2(\mathbf{R}_+;Y))$ and $\mathcal{U}_*^*(0)$ is a Hilbert space (under an equivalent norm);

(c1) $\mathbb{D}$ is J-coercive iff $\mathbb{D}^* J\mathbb{D} \in \mathcal{GB}(\mathcal{U}_*^*(0))$

(c2) $\mathbb{D}$ is positively J-coercive iff $\mathbb{D}^* J\mathbb{D} \gg 0$ on $\mathcal{U}_*^*(0)$, i.e., iff $\langle \mathbb{D}u, J\mathbb{D}u\rangle \geq \varepsilon \|u\|_{\mathcal{U}_*^*}^2$ for all $u \in \mathcal{U}_*^*(0)$ and some $\varepsilon > 0$.

We can simplify the $\|\cdot\|_{\mathcal{U}_*^*}$ norms as follows:

Lemma 8.4.2 *The norm $\|\cdot\|_{\mathcal{U}_*^*}$ is a norm on $\mathcal{U}_*^*(0)$. The following norms are equivalent to $\|\cdot\|_{\mathcal{U}_{\mathrm{out}}}$, $\|\cdot\|_{\mathcal{U}_{\mathrm{sta}}}$, $\|\cdot\|_{\mathcal{U}_{\mathrm{str}}}$ and $\|\cdot\|_{\mathcal{U}_{\mathrm{exp}}}$, respectively:*

$$\|u\|'_{\mathcal{U}_{\mathrm{out}}} := \max\{\|u\|_2, \|\mathbb{D}u\|_2\}, \tag{8.69}$$

$$\|u\|'_{\mathcal{U}_{\mathrm{str}}} := \max\{\|u\|_2, \|\mathbb{D}u\|_2, \|\mathbb{B}\tau u\|_\infty\} =: \|u\|'_{\mathcal{U}_{\mathrm{sta}}}, \tag{8.70}$$

$$\|u\|'_{\mathcal{U}_{\mathrm{exp}}} := \max\{\|u\|_2, \|\mathbb{B}\tau u\|_2\}. \tag{8.71}$$

□

(For $\|u\|'_{\mathcal{U}_{\mathrm{exp}}}$, this follows from Lemma 6.7.8; the other claims are obvious.)

If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then Z^{s} is a Hilbert space, so that then (b)–(c2) above hold and J-coercivity implies the existence of a unique J-critical control:

Theorem 8.4.3 (J-coercive $\Rightarrow \exists! J$-critical control) *Assume that Z^{s} is a reflexive Banach space and $\mathbb{D}$ is J-coercive. If $x_0 \in H$ is s.t. $\mathcal{U}_*^*(x_0) \neq \emptyset$, then there is a unique J-critical control over $\mathcal{U}_*^*$ for x_0.* □

(This follows from Theorem 8.2.5. J-coercivity is not the weakest possible assumption, e.g., let $C = 0$, $D > 0$, $J = I$ (but not $D \gg 0$). However, with reasonable additional assumptions, we obtain the converse for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, as in, e.g., Theorem 9.2.16.)

Since $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{exp}}$ are the most common sets of admissible controls, and J-coercivity implies all standard classical coercivity assumptions for control problems (see Section 10.3), the above theorem suffices for most applications. However, we often obtain results for $\mathcal{U}_{\mathrm{str}}$ and $\mathcal{U}_{\mathrm{sta}}$ from those for $\mathcal{U}_{\mathrm{out}}$ by suitable strong stabilizability assumptions that make these three equal.

The uniqueness part of the above theorem does not require reflexivity:

Lemma 8.4.4 *If $\mathbb{D}$ is J-coercive, then there is at most one J-critical control for each $x_0 \in H$.* □

(This follows from Lemma 8.2.3.)

If there is a J-critical state feedback pair over $\mathcal{U}_{\mathrm{exp}}$, then Σ is exponentially stabilizable. On the other hand, if Σ is exponentially stabilizable, then optimization over $\mathcal{U}_{\mathrm{exp}}$ can be reduced to optimization of the corresponding exponentially stable closed-loop system:

Theorem 8.4.5 (Reduce $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}$ to $\mathcal{U}_{\mathrm{exp}}$) *Let $\begin{bmatrix} \widetilde{\mathbb{K}} \mid \widetilde{\mathbb{F}} \end{bmatrix}$ be admissible for Σ, let $\Sigma_\flat$ be the corresponding closed loop system, and set $\widetilde{\mathbb{X}} := I - \widetilde{\mathbb{F}}$, $\widetilde{\mathbb{M}} := \widetilde{\mathbb{X}}^{-1}$. Set*

$$\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(x_0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,|\, \mathbb{A}_\flat x_0 + \mathbb{B}_\flat \tau u \in \mathrm{L}^2\}, \tag{8.72}$$

$$\mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}(x_0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,|\, \mathbb{C}_\flat x_0 + \mathbb{D}_\flat \tau u \in \mathrm{L}^2\} \quad (x_0 \in H). \tag{8.73}$$

Then the following hold:

(a) The system $\Sigma_\flat$ has a J-critical pair over $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}$ iff Σ has a J-critical pair over $\mathcal{U}_{\mathrm{exp}}$.

Moreover, if $\begin{bmatrix} \mathbb{K}_\natural \mid \mathbb{F}_\natural \end{bmatrix}$ is J-critical over $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}$ for $\Sigma_\flat$, then $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix} := \begin{bmatrix} \mathbb{K}_\natural + \mathbb{X}_\natural \widetilde{\mathbb{K}} \mid I - \mathbb{X}_\natural \widetilde{\mathbb{X}} \end{bmatrix}$ (here $\mathbb{X}_\natural := I - \mathbb{F}_\natural$) is J-critical over $\mathcal{U}_{\mathrm{exp}}$ for Σ.

Conversely, if $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is J-critical over $\mathcal{U}_{\mathrm{exp}}$ for Σ, then $\begin{bmatrix} \mathbb{K}_\natural \mid \mathbb{F}_\natural \end{bmatrix} := \begin{bmatrix} \mathbb{K} - \mathbb{X}\mathbb{K}_\flat \,|\, I - \mathbb{X}\widetilde{\mathbb{M}} \end{bmatrix}$ (here $\mathbb{X} := I - \mathbb{F}$) is J-critical over $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}$ for $\Sigma_\flat$.

The corresponding closed-loop systems correspond to each other as in Lemma 6.7.12.

(b) $\mathbb{K}_{\mathrm{crit}}$ is a J-critical control over $\mathcal{U}_{\mathrm{exp}}$ in WPLS form for Σ iff $\mathbb{K}_{\mathrm{crit}}^\flat := \widetilde{\mathbb{X}}\mathbb{K}_{\mathrm{crit}} - \widetilde{\mathbb{K}}$ is a J-critical control over $\mathcal{U}_{\mathrm{exp}}$ in WPLS form for $\Sigma_\flat$.

(c1) Let $x_0 \in H$. If $u_\flat \in \mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(x_0)$, then $u := \mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}} u_\flat \in \mathcal{U}_{\mathrm{exp}}(x_0)$ and $u_\flat = -\widetilde{\mathbb{K}} x_0 + \widetilde{\mathbb{X}} u$.

Conversely, if $u \in \mathcal{U}_{\mathrm{exp}}(x_0)$, then $u_\flat := -\widetilde{\mathbb{K}} x_0 + \widetilde{\mathbb{X}} u \in \mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(x_0)$ and $u = \mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}} u_\flat$.

Thus, $\mathcal{U}_{\mathrm{exp}}(x_0) = \mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}}[\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(x_0)]$ and $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(x_0) = -\widetilde{\mathbb{K}} x_0 + \widetilde{\mathbb{X}}[\mathcal{U}_{\mathrm{exp}}(x_0)]$; in particular, $\mathcal{U}_{\mathrm{exp}}(0) = \widetilde{\mathbb{M}}\,\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(0)$. Moreover, $y := \mathbb{C} x_0 + \mathbb{D} u = \mathbb{C}_\flat x_0 + \mathbb{D}_\flat u_\flat$ and $x := \mathbb{A} x_0 + \mathbb{B}\tau u = \mathbb{A}_\flat x_0 + \mathbb{B}_\flat \tau u_\flat$ in either case.

(c2) If $u = \mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}} u_\flat$, then u is J-critical over $\mathcal{U}_{\mathrm{exp}}(x_0)$ iff $u_\flat$ is J-critical over $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(x_0)$.

(c3) If(f) there is a unique J-critical control over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for Σ and for each $x_0 \in H$, then the same holds for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ and $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}$, and $\mathbb{A}_{\mathrm{crit}} = \mathbb{A}_{\mathrm{crit}}^\flat$, $\mathbb{C}_{\mathrm{crit}} = \mathbb{C}_{\mathrm{crit}}^\flat$ and $\mathcal{P}$ are common for Σ and $\Sigma_\flat$, but $\mathbb{K}_{\mathrm{crit}} = \mathbb{K}_\flat + \widetilde{\mathbb{M}}\mathbb{K}_{\mathrm{crit}}^\flat$, $\mathbb{K}_{\mathrm{crit}}^\flat = \widetilde{\mathbb{X}}\mathbb{K}_{\mathrm{crit}} - \widetilde{\mathbb{K}}$.

(c4) If(f) there is a J-critical control over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for Σ and for all $x_0 \in H$, then the same holds for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ and $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}$, and $\mathcal{P}$ is common for Σ and $\Sigma_\flat$.

(c5) There is $\varepsilon > 0$ s.t. $\varepsilon\|u\|_{\mathcal{U}_{\mathrm{exp}}} \le \|u_\flat\|_{\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}} \le \varepsilon^{-1}\|u\|_{\mathcal{U}_{\mathrm{exp}}}$ whenever $u \in \mathcal{U}_{\mathrm{exp}}(0)$ and $u_\flat \in \mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(0)$ are as in (c1).

(d) The map $\mathbb{D}$ is [positively] J-coercive over $\mathcal{U}_{\mathrm{exp}}$ iff $\mathbb{D}_\flat$ is [positively] J-coercive over $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat} = \mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}$.

(e) If $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is exponentially stabilizing, then*

$$\mathcal{U}_{\rm exp}(x_0) = \{\mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}} u_\flat \,|\, u_\flat \in \mathrm{L}^2(\mathbf{R}_+;U)\} \tag{8.74}$$

and $\mathcal{U}_{\rm exp}^{\Sigma_\flat}(x_0) = \mathcal{U}_{\rm out}^{\Sigma_\flat}(x_0) = \mathrm{L}^2(\mathbf{R}_+;U)$*, for all* $x_0 \in H$.

(f) Claims (a)–(e) also hold with replacements $\mathcal{U}_{\rm exp} \mapsto \mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}$, $\mathcal{U}_{\rm exp}^{\Sigma_\flat}, \mathcal{U}_{\rm out}^{\Sigma_\flat} \mapsto \mathcal{U}^{\gamma,\Sigma_\flat}_{[\mathbb{Q}_\flat\ \mathbb{R}_\flat]}$ *and* $\mathrm{L}^2(\mathbf{R}_+;U) \mapsto \mathcal{U}^{\gamma,\Sigma_\flat}_{[\mathbb{Q}_\flat\ \mathbb{R}_\flat]}(x_0)$*, where* $\begin{bmatrix}\mathbb{Q}_\flat & \mathbb{R}_\flat\end{bmatrix} := \begin{bmatrix}\mathbb{Q} & \mathbb{R} \\ 0 & I\end{bmatrix}\begin{bmatrix} I & 0 \\ \mathbb{K}_\flat & \widetilde{\mathbb{M}}\end{bmatrix}$, $Z_\flat^{\rm u} := Z^{\rm u} \times \mathrm{L}^2_\gamma$, $Z_\flat^{\rm s} := Z^{\rm s} \times \mathrm{L}^2_\vartheta$ *and* $\gamma > \max\{\omega_A, \omega_{A_\flat}, \vartheta\}$.

(g1) If $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is q.r.c.-SOS-stabilizing, then (a)–(e) also hold with replacements* $\mathcal{U}_{\rm exp} \mapsto \mathcal{U}_{\rm out}$ *and* $\mathcal{U}_{\rm exp}^{\Sigma_\flat} \mapsto \mathcal{U}_{\rm out}^{\Sigma_\flat} = \mathrm{L}^2(\mathbf{R}_+;U)$.

Moreover, then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *(in (a)) is q.r.c.-SOS-stabilizing iff* $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ *is q.r.c.-SOS-stabilizing (equivalently, stable and [r.c.-]SOS-stabilizing).*

(g2) If $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is [[exponentially] strongly] q.r.c.-stabilizing, then* $\mathcal{U}_{\rm out} = \mathcal{U}_{\rm sta}[= \mathcal{U}_{\rm str}[= \mathcal{U}_{\rm exp}]]$.

Thus, if we are optimizing over $\mathcal{U}_{\rm exp}$, we only need to stabilize Σ exponentially and then find an J-critical control for $\Sigma_\flat$ w.r.t. $\mathcal{U}_{\rm out} = \mathcal{U}_{\rm exp}$ corresponding to $\Sigma_\flat$. If the original system is J-coercive, then we end up with the situation of Proposition 8.3.10, by (d).

The key to the Theorem is (c1), the fact that $u, x \in \mathrm{L}^2 \Leftrightarrow u_\flat, x \in \mathrm{L}^2$ (this follows from Lemma 6.1.10). As shown by Example 9.13.2 (for $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$), an analogous reduction cannot be made for general $\mathcal{U}_*^*$. Indeed, we do not have a similar equivalence "$u, y \in \mathrm{L}^2 \Leftrightarrow u_\flat, y \in \mathrm{L}^2$" for $\mathcal{U}_{\rm out}$ unless $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ is q.r.c.-SOS-stabilizing (cf. Theorem 9.9.10). Fortunately, part (f) is helpful in certain technical proofs.

Proof of Theorem 8.4.5: (c1) 1° Let $u_\flat \in \mathcal{U}_{\rm exp}^{\Sigma_\flat}(x_0)$, $x_0 \in H$. Then $u, x \in \mathrm{L}^2$, by Lemma 6.1.10, where $u := \mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}} u_\flat = \mathbb{K}_\flat x_0 + \mathbb{F}_\flat u_\flat + u_\flat$ and $x := \mathbb{A} x_0 + \mathbb{B}\tau u = \mathbb{A}_\flat x_0 + \mathbb{B}_\flat \tau u_\flat$. Thus, $u \in \mathcal{U}_{\rm exp}(x_0)$.

2° By exchanging the roles of Σ and $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ (note that the pair $-\left[\begin{array}{c|c} \mathbb{K}_\flat & \mathbb{F}_\flat \end{array}\right]$ is admissible for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$, and Σ with the added row $-\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is the corresponding closed-loop system, by Lemma 6.6.14), we note that if $u \in \mathcal{U}_{\rm exp}(x_0)$, then $u_\flat \in \mathcal{U}_{\rm exp}^{\Sigma_\flat}(x_0)$, where $u_\flat := -\mathbb{K} x_0 - \mathbb{F} u + I = -\mathbb{K} x_0 + \mathbb{X} u$.

3° We noted above that "$x = x$"; the same holds for y: $\mathbb{C} x_0 + \mathbb{D} u = \mathbb{C} x_0 + \mathbb{D}\mathbb{K}_\flat x_0 + \mathbb{D}\widetilde{\mathbb{M}} u_\flat = \mathbb{C}_\flat x_0 + \mathbb{D}_\flat u_\flat$.

(c2) Now $u_\flat$ is J-critical over $\mathcal{U}_{\rm exp}^{\Sigma_\flat}(x_0)$ iff $\langle y, J\mathbb{D}_\flat \eta_\circlearrowleft\rangle = \langle y, J\mathbb{D}\eta\rangle = 0$ for all $\eta_\circlearrowleft \in \mathcal{U}_{\rm exp}(0)$, i.e., for all $\eta := \widetilde{\mathbb{M}}\eta_\circlearrowleft \in \mathcal{U}_{\rm exp}(x_0)$, i.e., iff u is J-critical over $\mathcal{U}_{\rm exp}(x_0)$.

(c3) By (c1)–(c2), there is a unique J-critical control over $\mathcal{U}_{\rm exp}^{\Sigma_\flat}(x_0)$ for each x_0, and (c3) holds. (Exchange the roles of Σ and $\Sigma_\flat$ for the converse.)

(c4) The first claim follows from (c2). By (c1), $\langle y, Jy\rangle$ $(=: \langle x_0, \mathcal{P}x_0\rangle)$ is the same for $\mathcal{U}_{\exp}$ and $\mathcal{U}_{\exp}^{\Sigma_\flat}$, hence so is $\mathcal{P}$.

(c5) Set $x := \mathbb{B}\tau u = \mathbb{B}_\flat \tau u_\flat$ (by (c1), we have $u = \widetilde{\mathbb{M}} u_\flat$). Then $\|u_\flat\|_{\mathcal{U}_{\exp}^{\Sigma_\flat}} := \max(\|u_\flat\|_2, \|x\|_2) \le 2M \max(\|u\|_2, \|x\|_2) =: \|u\|_{\mathcal{U}_{\exp}}$ (here we have used (8.71); for an equivalent norm we need to divide ε by an equivalence constant) for some $M := M_{\Sigma'} < \infty$, by Lemma 6.7.8, where $\Sigma' := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\widetilde{\mathbb{K}} & \widetilde{\mathbb{X}} \end{array}\right]$; analogously, $\max(\|u\|_2, \|x\|_2) \le 2M' \max(\|u_\flat\|_2, \|x\|_2)$ for some fixed $M' < \infty$; take $\varepsilon := \min\{(2M)^{-1}, (2M')^{-1}\}$.

(a) This follows from Lemma 6.7.12 (either directly (since $\mathbb{C}_\circlearrowleft$ and $\mathbb{D}_\circlearrowleft$ are common for both closed-loop systems) or from the fact that $\mathbb{K}_\circlearrowleft = \widetilde{\mathbb{M}}\mathbb{K}_\circlearrowleft^\flat + \mathbb{K}_\flat$, $\mathbb{K}_\circlearrowleft^\flat = -\widetilde{\mathbb{K}} + \widetilde{\mathbb{X}}\mathbb{K}_\circlearrowleft$, as in (c3)).

(b) Set $\mathbb{K}_{\text{crit}}^\flat := \widetilde{\mathbb{M}}^{-1}(\mathbb{K}_{\text{crit}} - \mathbb{K}_\flat)$, $\mathbb{A}_{\text{crit}} := \mathbb{A} + \mathbb{B}\tau\mathbb{K}_{\text{crit}}$, and $\mathbb{C}_{\text{crit}} := \mathbb{C} + \mathbb{D}\mathbb{K}_{\text{crit}}$. Then $\mathbb{A}_\flat + \mathbb{B}_\flat\tau\mathbb{K}_{\text{crit}}^\flat = \mathbb{A}_{\text{crit}}$ and $\mathbb{C}_\flat + \mathbb{D}_\flat\mathbb{K}_{\text{crit}}^\flat = \mathbb{C}_{\text{crit}}$. By a straightforward computation using the above formulae (and the identity $\widetilde{\mathbb{X}}\pi_+\widetilde{\mathbb{M}}\pi_- = -\pi_+\widetilde{\mathbb{X}}\pi_-\widetilde{\mathbb{M}}$), one verifies that $\mathbb{K}_{\text{crit}}^\flat \mathbb{A}_{\text{crit}}^t = \pi_+\tau^t\mathbb{K}_{\text{crit}}^\flat$ for any $t \ge 0$, so that $\left[\begin{array}{c|cc} \mathbb{A}_{\text{crit}}^{\mathrm{T}} & \mathbb{C}_{\text{crit}}^{\mathrm{T}} & {\mathbb{K}_{\text{crit}}^\flat}^{\mathrm{T}} \end{array}\right]^{\mathrm{T}} \in$ WPLS; thus, $\mathbb{K}_{\text{crit}}^\flat$ is a control in WPLS form.

Now $\langle J\mathbb{C}_{\text{crit}}x_0, \mathbb{D}\eta\rangle = 0$ for all $\eta \in \mathcal{U}_{\exp}(0)$ iff $\langle J\mathbb{C}_{\text{crit}}x_0, \mathbb{D}_\flat\eta_{\flat'}\rangle = 0$ for all $\eta_{\flat'} \in \mathcal{U}_{\exp}^{\Sigma_\flat}(0) = \widetilde{\mathbb{M}}^{-1}\mathcal{U}_{\exp}(0)$, hence (b) holds (since we can interchange Σ and $\Sigma_\flat$ for the converse).

(d) (Note that the J-coercivity of $\mathbb{D}_\flat$ over $\mathcal{U}_{\exp}^{\Sigma_\flat}$ is equivalent to the J-coercivity of $\mathbb{D}_\flat$ over $\mathcal{U}_{\text{out}}^{\Sigma_\flat}$, by Lemma 8.3.3.)

This follows from (c5) and (c1): Let $\mathbb{D}$ be J-coercive over $\mathcal{U}_{\exp}$, and let $\varepsilon > 0$ as in Definition 8.4.1. Given a nonzero $u_\flat \in \mathcal{U}_{\exp}^{\Sigma_\flat}(0)$, set $u := \widetilde{\mathbb{M}}u_\flat \in \mathcal{U}_{\exp}(0) \setminus \{0\}$ (by (c1)), $x := \mathbb{B}\tau u = \mathbb{B}_\flat\tau u_\flat$. Choose a nonzero $v \in \mathcal{U}_{\exp}(0)$ as in Definition 8.4.1, and set $v_\flat := \widetilde{\mathbb{X}}v \in \mathcal{U}_{\exp}^{\Sigma_\flat}$, $\widetilde{x} := \mathbb{B}\tau v$. Then $\langle \mathbb{D}_\flat v_\flat, J\mathbb{D}_\flat u_\flat\rangle = \langle \mathbb{D}v, J\mathbb{D}u\rangle \ge \varepsilon\|u\|\|v\| \ge \varepsilon'\|u_\flat\|\|v_\flat\|$. Since $u_\flat$ was arbitrary, $\mathbb{D}_\flat$ is J-coercive. Exchange Σ and $\Sigma_\flat$ for the converse. [By (c1), $\langle \mathbb{D}\cdot, J\mathbb{D}\cdot\rangle \ge 0 \Leftrightarrow \langle \mathbb{D}_\flat\cdot, J\mathbb{D}_\flat\cdot\rangle \ge 0$.]

(e) By Lemma 6.1.10, we have $\mathcal{U}_{\exp}^{\Sigma_\flat}(x_0) = \mathcal{U}_{\text{out}}^{\Sigma_\flat}(x_0) = \mathrm{L}^2(\mathbf{R}_+; U)$ for all $x_0 \in H$. By (c1), we obtain (8.74).

(f) 1° *The definition of* $\begin{bmatrix} \mathbb{Q}_\flat & \mathbb{R}_\flat \end{bmatrix}$ *implies that (c1) holds:* Indeed, $\widetilde{\mathbb{K}}$, $\widetilde{\mathbb{X}}$, $\mathbb{K}_\flat$ and $\widetilde{\mathbb{M}}$ are γ-stable, the upper row of $\begin{bmatrix} \mathbb{Q}_\flat & \mathbb{R}_\flat \end{bmatrix}$ corresponds to condition $\begin{bmatrix} \mathbb{Q} & \mathbb{R} \end{bmatrix}\begin{bmatrix} x_0 \\ u \end{bmatrix} \in Z^{\mathrm{s}}$ and the lower row to condition $u \in \mathrm{L}^2_\vartheta$, so that $\mathcal{U}^{\gamma,\Sigma_\flat}_{[\mathbb{Q}_\flat \ \mathbb{R}_\flat]}$ is independent on γ (since $u := \mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}}u_\flat \in \mathrm{L}^2_\vartheta \Rightarrow u_\flat = \widetilde{\mathbb{K}}x_0 + \widetilde{\mathbb{X}}u \in \mathrm{L}^2_\gamma$).

2° *The rest:* The proofs of (a)–(e) above apply with slight changes ((c5) becomes easier).

(g1) By Lemma 6.5.6(f)&(a1), we have $\mathcal{U}_{\text{out}}(0) = \mathbb{M}\mathrm{L}^2(\mathbf{R}_+; U)$. Since $\mathbb{K}_\flat x_0 \in \mathcal{U}_{\text{out}}(x_0)$ for all $x_0 \in H$, we obtain "$\mathcal{U}_{\text{out}} = \{\mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}}u_\flat \mid u_\flat \in \mathrm{L}^2(\mathbf{R}_+; U)\}$" (cf. (8.74) from Lemma 8.3.5. The proofs of (a)–(d) above apply with slight changes (use, e.g., Lemma 8.4.11(b1) for J-coercivity). The last claim follows from Lemma 6.7.11(a2) and Lemma 6.6.17(b).

(g2) Now the proof and conlusion of (g1) applies also to $\mathcal{U}_{\text{sta}}$ [and $\mathcal{U}_{\text{str}}$ [and

$\mathcal{U}_{\mathrm{exp}}$]] in place of $\mathcal{U}_{\mathrm{out}}$. [[Note that it suffices that $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ is exponentially stabilizing and q.r.c.-stabilizing.]] $\square$

We will often need the assumption that J-coercivity implies the existence of a spectral factorization. It is well-known that this is true for any rational transfer function (hence for any stable I/O map of a system with $\dim H < \infty$); in fact, this is true for any element of $\mathrm{MTIC}_{\mathbf{TZ}}$, as we shall show in Theorem 8.4.9. Since there is a wide variety of classes satisfying this assumption, we shall write below three hypotheses with differing strengths, and then use these as the assumptions of our results in optimal control theory, to avoid dependence on the current state of spectral factorization theory (or on the part included in this book).

We start by the weakest formulation:

Definition 8.4.6 (J-coercive ⇒ SpF) *Let $\mathbb{D} \in \mathrm{TIC}(U,Y)$ and $J = J^* \in \mathcal{B}(Y)$. We write $(\mathbb{D}, J) \in \mathrm{SpF}$ iff either $\mathbb{D}$ is not J-coercive or $\mathbb{D}^* J\mathbb{D}$ has a spectral factorization.*

Thus, $(\mathbb{D}, J) \in \mathrm{SpF}$ means that if $\pi_+\mathbb{D}^* J\mathbb{D}\pi_+$ is invertible, then $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ and $S \in \mathcal{GB}(U)$. This (and the stronger requirement below) is satisfied by any of the classes in Theorem 8.4.9 (alternatively, by any $\mathbb{D} \in \mathrm{TIC}$ if $J \gg 0$). Recall from Lemma 6.4.7(b) that the converse holds for any $\mathbb{D} \in \mathrm{TIC}(U,Y)$ and $J = J^* \in \mathcal{B}(Y)$.

However, often we also need to know that $\mathbb{D}$ belongs not only to the class TIC but also to some subclass "$\mathcal{A}$" whose every element is ULR and has the above property (for each J) and which is closed w.r.t. spectral factorization. We formulate this as follows (see Definition 6.2.4 for "$\underset{a}{\subset}$"):

Hypothesis 8.4.7 (ULR classes $\mathcal{A}(U)$ that admit spectral factorization)

(1.) We have $\mathcal{B} \subset \mathcal{A} \underset{a}{\subset} \mathrm{TIC} \cap \mathrm{ULR}$;

(2.) if Y is an arbitrary Hilbert space, $\mathbb{D} \in \mathcal{A}(U,Y)$, $J = J^* \in \mathcal{B}(Y)$, and the Popov Toeplitz operator $\pi_+\mathbb{D}^* J\mathbb{D}\pi_+$ is invertible on $\pi_+\mathrm{L}^2$, then $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$ for some $S = S^* \in \mathcal{GB}(U)$ and $\mathbb{X} \in \mathcal{GA}(U)$.

If Hypothesis 8.4.7 holds for $\mathcal{A}(U)$, then, trivially, $(\mathbb{D}, J) \in \mathrm{SpF}$ for any Y, $\mathbb{D} \in \mathcal{A}(U,Y)$ and $J = J^* \in \mathcal{B}(Y)$ See Hypothesis 10.6.6 and Lemma 10.6.7 for the positive case.

A sufficient condition for (2.) is that if $\mathbb{E} = \mathbb{E}^* \in \mathcal{A}(U)$ and $\pi_+\mathbb{E}\pi_+$ is invertible on $\pi_+\mathrm{L}^2(U)$, then $\mathbb{E}$ has a spectral factorization over $\mathcal{A}(U)$. However, the weaker formulation above has the advantage to cover also exponentially stable classes (cf. Theorem 8.4.9) and still be strong enough for applications.

Much of our theory is valid even without the assumption (1.), but because all classes listed in Theorem 8.4.9 satisfy (1.), we have assumed it to simplify the presentation.

Sometimes we also wish to have $D^*JD = X^*SX$ (cf. Example 6.3.7):

Hypothesis 8.4.8 (Classes $\mathcal{A}(U)$ that admit spectral factorization with $D^*JD = X^*SX$) *We require that $\mathcal{A}(U)$ satisfies Hypothesis 8.4.7 with*

(3.) $D^*JD = X^*SX$.

By Lemma 6.4.5(a), condition (3.) holds for some $\mathbb{X}$ and S satisfying (2.) iff (3.) holds for all such $\mathbb{X}$ and S (for fixed $\mathbb{D}$ and J). A sufficient condition is that $\mathcal{A} \subset \mathrm{SHPR}$, by Lemma 6.3.6(b).

Now we cite the main results of Chapter 5:

Theorem 8.4.9 (Classes satisfying Hypothesis 8.4.7) *Let U be a Hilbert space, let $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$, and let ($\alpha$), ($\beta$) or ($\gamma$) hold, where*

- *(α) $\mathcal{A}$ is one of the classes* $\mathrm{MTIC}^{\mathrm{L}^1}$, $\mathrm{MTIC}^{\mathrm{L}^1,\mathcal{BC}}$, $\mathrm{MTIC}_{T\mathbf{Z}}$, $\mathrm{MTIC}^{\mathcal{BC}}_{T\mathbf{Z}}$, $\mathrm{MTIC}_{\mathrm{d},T\mathbf{Z}}$, *and* $\mathrm{MTIC}^{\mathcal{BC}}_{\mathrm{d},T\mathbf{Z}}$*;*
- *(β)* $\dim U < \infty$ *and $\mathcal{A}$ is one of the classes* MTIC, $\mathrm{MTIC}_{\mathrm{d}}$, $\mathrm{MTIC}_{\mathbf{S}}$, *and* $\mathrm{MTIC}_{\mathrm{d},\mathbf{S}}$.
- *(γ)* $\mathcal{A}(U,Y) = \mathcal{B}(U,Y) + \{\mathbb{D} \,|\, \widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$ *for some* $\omega < 0\}$ *(this is the set of maps having an exponentially stable realization with a bounded input operator).*

Then Hypothesis 8.4.7 holds for $\mathcal{A}(U)$ and for the class $\mathcal{A}_{\mathrm{exp}}(U)$ of exponentially stable $\mathcal{A}(U)$ maps, *where*

$$\mathcal{A}_{\mathrm{exp}}(U,Y) := \cap_{r<0}\{\mathrm{e}^{r\cdot}\mathbb{D}\mathrm{e}^{-r\cdot} \,|\, \mathbb{D} \in \mathcal{A}(U,Y)\} \tag{8.75}$$

Moreover,

- *(a) If* $\mathcal{A} = \mathrm{MTIC}^{\mathrm{L}^1}$, $\mathcal{A} = \mathrm{MTIC}^{\mathrm{L}^1,\mathcal{BC}}$ *or (γ) holds, then also Hypothesis 8.4.8 holds for $\mathcal{A}(U)$ as well as for $\mathcal{A}_{\mathrm{exp}}(U)$ (for any Hilbert space U).*
- *(b) We have $X \in \mathcal{GB}(U)$, and we can choose $\mathbb{X}$ and S s.t. $X = I$.*
- *(c) If (α) or (β) holds, then $\mathcal{A} = \mathcal{A}^{\mathrm{d}}$. We have $\mathcal{A} = \mathcal{A}^{\mathrm{d}}$ also for the class*
  - *(γ')* $\mathcal{A}(U,Y) = \mathcal{B}(U,Y) + \{\mathbb{D} \,|\, \widehat{\mathbb{D}}, \widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(*,*))$ *for some* $\omega < 0\}$ *(this is the set of I/O maps having an exponentially stable PS-realization).*

Thus, Hypothesis 8.4.7 holds for $\mathcal{A}(U) = \mathrm{MTIC}_{T\mathbf{Z}}(U)$ and $\mathcal{A}(U) = \mathrm{MTIC}(\mathbf{C}^n)$ (and for their subclasses of exponentially stable maps) for any $n \in \mathbf{N}$ and any Hilbert space U (we hope that the future study will show the reference to U superfluous for MTI (and its subclasses), i.e., that Hypothesis 8.4.7 holds for $\mathrm{MTI}(U)$ for any Hilbert space U).

See Lemma 14.3.5 for four more classes (the Cayley images of discrete ℓ^1 classes). Also Theorem 9.2.14 contains analogous results, with requirements posed on the whole system (e.g., we may allow for any $\mathbb{D}, \mathbb{B}\tau \in \mathrm{SMTIC}^{\mathrm{L}^1}_{-\varepsilon}$ if C is bounded and $D^*JC = 0$, to obtain "(1.)–(3.)").

Proof of Theorem 8.4.9: *Case (γ):* Now Hypothesis 8.4.8 holds, by Theorem 9.2.14(c2) (by its proof we have $\mathbb{X} \in \mathcal{A}(U)$).

(It is not a problem that we refer here to later results; part (γ) of this lemma is not used in this monograph before Chapter 12 except in phrases like "if $\mathcal{A}$ is any of the classes of Theorem 8.4.9"; in particular, part (γ) is not used in derivation of any earlier result of this monograph, we just want to record it next to (α) and (β).)

The assumption means that $\widehat{\mathbb{D}} : \mathbf{C}^+_\omega \to \mathcal{B}(U,Y)$ is s.t. $\widehat{\mathbb{D}}u_0 \in \mathrm{H}^2_\omega(\mathbf{C}^+;Y)$ for all $u_0 \in U$ and $y_0 \in Y$ (see Lemma F.3.2(a)); the number $\omega < 0$ may depend on $\mathbb{D}$. (Thus, $\mathcal{A}_{\mathrm{exp}} = \mathcal{A}$.)

The correspondence to bounded B was shown in Theorem 6.9.1.

Case (γ′): (The remarks of case (γ) apply. The correspondence to PS-systems was shown in Theorem 6.9.6.)

We obtain the result from case (γ) (since obviously $\mathcal{A} = \mathcal{A}^{\mathrm{d}}$) except for the fact that $\widehat{\mathbb{X}}(\bar{\cdot})^* - X^* \subset \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\varepsilon;\mathcal{B}(U))$ for some $\varepsilon < 0$, which was recorded in the proof of Theorem 9.2.14(c2).

Cases (α) and (β):

For the rest of the proof, we assume that (α) or (β) holds.

Hypotheses 8.4.7(1.)&(2.) (and Hypothesis 8.4.8(4.) if $\mathcal{A} = \mathrm{MTIC}^{\mathrm{L}^1}$ or $\mathcal{A} = \mathrm{MTIC}^{\mathrm{L}^1,\mathcal{BC}}$) is satisfied by Theorem 2.6.4.

Moreover, if $\mathbb{D}$ and J are as in Hypothesis 8.4.7(3.) and we set $\mathbb{E} := \mathbb{D}^* J \mathbb{D} \in \mathcal{A}'(U)$, where $\mathcal{A}' := \mathcal{A} + \mathcal{A}^*$ is the corresponding noncausal (MTI) class, then $\mathbb{E}$ has a spectral factorization in $\mathcal{A}(U)$, by Theorem 5.2.7 (and Lemma 5.2.1(d)), hence (3.) holds for $\mathcal{A}(U)$.

If, in addition, $\mathbb{D} \in \mathcal{A}_{\mathrm{exp}}(U)$, i.e., $\mathbb{D} \in \mathcal{A}_\omega(U)$ for some $\omega < 0$, then $\mathbb{D} \in \mathcal{A}_\omega(U) \cap \mathcal{A}_{-\omega}(U)$ and $\mathbb{D}^* \in \mathcal{A}'_{-\omega} \cap \mathcal{A}'_\omega(U)$, by Theorem 2.6.4(g1)&(g2), hence then $\mathbb{E} := \mathbb{D}^* J \mathbb{D} \in \mathcal{A}'_{-\omega} \cap \mathcal{A}'_\omega(U)$; thus, then the spectral factorization of $\mathbb{E}$ is in fact a spectral factorization in $\mathcal{A}_{\mathrm{exp}}(U)$, by Theorem 5.2.2. Therefore, (3.) holds for $\mathcal{A}_{\mathrm{exp}}(U)$ too.

(a) This was noted above.

(b) This follows from Proposition 6.3.1(c). □

The reason for mentioning also subclasses of classes mentioned above is that in many theorems using Hypothesis 8.4.7 some kind of controllers are constructed within the same class, hence stricter conditions guarantee smoother controllers.

By [Treil94], the class $\mathrm{CTIC}(\mathbf{C})$ does not satisfy Hypothesis 8.4.7: There is $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 \\ \mathbb{D}_2 \end{bmatrix} \in \mathrm{CTIC}(\mathbf{C},\mathbf{C}^2)$ s.t. the spectral factor $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(\mathbf{C})$ satisfying $\mathbb{X}^*\mathbb{X} = \mathbb{D}^*\mathbb{D}$ (i.e., $|\widehat{\mathbb{X}}|^2 = |\widehat{\mathbb{D}}_1|^2 + |\widehat{\mathbb{D}}_2|^2$ on $i\mathbf{R}$) does not belong to CTIC. We can even take $\widehat{\mathbb{D}}_1$ and $\widehat{\mathbb{D}}_2$ to have no zeros on $\overline{\mathbf{C}^+} \cup \{\infty\}$.

Lemma 8.4.10 *Let $\mathcal{A}(U)$ satisfy Hypothesis 8.4.7. Then $\mathcal{A}(U)$ is inverse closed in* $\mathrm{TIC}(U)$*, i.e.,* $\mathbb{X} \in \mathcal{A}(U) \cap \mathcal{G}\mathrm{TIC}(U) \Rightarrow \mathbb{X} \in \mathcal{G}\mathcal{A}(U)$.

(Analogously, $\mathcal{A}(U,Y)$ is inverse closed.)

Proof: Because the Toeplitz operator $\pi_+\mathbb{X}^*\mathbb{X}\pi_+$ has the inverse $\mathbb{X}^{-1}\pi_+\mathbb{X}^{-*}$ on $\pi_+\mathrm{L}^2$, we have $\mathbb{X}^*\mathbb{X} = \mathbb{Z}^* S \mathbb{Z}$ for some $S = S^* \in \mathcal{G}\mathcal{B}(U)$, $\mathbb{Z} \in \mathcal{G}\mathcal{A}(U)$, by (2.). By Lemma 6.4.5(a), $\mathbb{X} = E\mathbb{Z} \in \mathcal{G}\mathcal{A}(U)$ for some $E \in \mathcal{G}\mathcal{B}(U)$. □

In the stable or r.c.-stabilizable case (see Lemma 8.3.3), J-coercivity can be easily verified:

Lemma 8.4.11 (J-coercive) *Let $J = J^* \in \mathcal{B}(Y)$ and $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$. The following holds for $\mathcal{U}_*^* := \mathcal{U}_{\mathrm{out}}$:*

(a1) Let $\mathbb{D} \in \mathrm{TIC}$. Then $\mathbb{D}$ is J-coercive iff the Popov Toeplitz operator $\pi_+\mathbb{D}^* J\mathbb{D}\pi_+$ is invertible on $\mathrm{L}^2(\mathbf{R}_+;U)$.

(a2) Let $\mathbb{D} \in \mathrm{TIC}$. Then $\mathbb{D}$ is positively J-coercive iff $\mathbb{D}^* J\mathbb{D} \gg 0$.

(b1) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a q.r.c.f. Then $\mathcal{U}_{\mathrm{out}}(0) = \mathbb{M}\mathrm{L}^2(\mathbf{R}_+;U)$. Moreover, $\mathbb{D}$ is [positively] J-coercive iff $\mathbb{N}$ is [positively] J-coercive.

(b2) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ be a q.r.c.f. and $S = \mathbb{N}^* J\mathbb{N}$. Then $\mathbb{D}$ is [positively] J-coercive over $\mathcal{U}_{\mathrm{out}}$ iff $S \in \mathcal{GB}$ [$S \gg 0$].

(c) The space $\mathcal{U}_{\mathrm{out}}(0)$ is a Hilbert space, and $\mathbb{D} \in \mathcal{B}(\mathcal{U}_{\mathrm{out}}(0), \mathrm{L}^2(\mathbf{R}_+;Y))$.

Moreover, $\mathbb{D}$ is [positively] J-coercive iff $\mathbb{D}^* J\mathbb{D}$ is invertible [$\gg 0$] on $\mathcal{U}_{\mathrm{out}}(0)$.

(d1) $\mathbb{D}$ is positively J-coercive iff there is $\varepsilon > 0$ s.t. for all $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ we have

$$\langle \mathbb{D}u, J\mathbb{D}u\rangle \geq \varepsilon(\|u\|_2^2 + \|\mathbb{D}u\|_2^2) \tag{8.76}$$

(d2) Let $J \gg 0$. Then $\mathbb{D}$ is positively J-coercive iff $\|\mathbb{D}u\|_2 \geq \varepsilon\|u\|_2$ for some $\varepsilon > 0$ and all $u \in \mathrm{L}^2(\mathbf{R}_+;U)$.

(d3) If $\mathbb{D}$ is J-coercive, then $\mathbb{D}$ is injective on $\mathrm{L}^2(\mathbf{R}_+;U)$.

(d4) Let $\mathbb{D}$ be J-coercive. Then $\|J\mathbb{D}u\|_2 \geq \varepsilon\|u\|_{\mathbb{D}}$ for some $\varepsilon > 0$ and all $u \in \mathcal{U}_{\mathrm{out}}(0)$.

Recall that $\mathcal{U}_{\mathrm{out}}(0) = \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,|\, \mathbb{D}u \in \mathrm{L}^2\}$, and that the [positive] J-coercivity of $\mathbb{D}$ (over $\mathcal{U}_{\mathrm{out}}$) depends only on $\mathbb{D}$ and J. Thus, we can define the [positive] J-coercivity of $\mathbb{N}$ analogously (we used this implicitly in (b1)). Consequently, $\mathbb{N}$ is [positively] J-coercive iff $\pi_+\mathbb{N}^* J\mathbb{N}\pi_+ \in \mathcal{GB}$ [$\gg 0$], by (a1) [(a2)].

Obviously, when $\mathbb{D} \in \mathrm{TIC}(U,Y)$, the space $\mathcal{U}_{\mathrm{out}}(0)$ equals $\mathrm{L}^2(\mathbf{R}_+;U)$ with an equivalent norm ($\|\cdot\|_{\mathcal{U}_{\mathrm{out}}}$). Contrary to (b1), we have no control on zeros of $\widehat{\mathbb{N}}$ if $\mathbb{N}$ and $\mathbb{M}$ are not required to be q.r.c. (e.g., take $\widehat{\mathbb{N}}(s) = s/(s+1) = \widehat{\mathbb{M}}(s)$, $\mathbb{D} = I$).

The condition $\mathbb{D}^* J\mathbb{D} \gg 0$ in (a2) holds iff $\pi_+\mathbb{D}^* J\mathbb{D}\pi_+ \gg 0$ on $\mathrm{L}^2(\mathbf{R}_+;U)$, or equivalently, iff $\widehat{\mathbb{D}}^* J\widehat{\mathbb{D}} \geq \varepsilon I$ a.e. on $i\mathbf{R}$ (in $\mathrm{L}^\infty_{\mathrm{strong}}$). See Propositions 10.3.1 and 10.3.2 for further equivalent conditions for positive J-coercivity and Lemma 2.2.2 for the invertibility of $\pi_+\mathbb{D}^* J\mathbb{D}\pi_+$.

Proof of Lemma 8.4.11: Part (b1) follows from Lemma 6.5.6(f)&(a1)&(a2). Part (b2) follows from (a1)–(b1), because $\pi_+ S\pi_+ \in \mathcal{G}\mathrm{L}^2(\mathbf{R}_+;U)$ iff $S \in \mathcal{GB}$. The rest follows from Lemma 8.2.3. □

For classes satisfying Hypothesis 8.4.7, the existence of a spectral factorization and the invertibility of the Popov Toeplitz operator (that is, the J-coercivity of $\mathbb{D}$) are equivalent:

Theorem 8.4.12 (MTI spectral factorization) *Let $\mathcal{A} \subset \mathrm{TIC}$ and $J = J^* \in \mathcal{B}(Y)$. For $\mathbb{D} \in \mathrm{TIC}(U,Y)$ we have (iv)⇒(iii)⇔(ii)⇒(i), where* *(i)* $\mathbb{D}$ *is* J*-coercive over* $\mathcal{U}_{\mathrm{out}}$ *(i.e.,* $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+$ *is invertible on* $\mathrm{L}^2(\mathbf{R}_+;U)$*);*

(ii) $\mathbb{D}^*J\mathbb{D}=\mathbb{X}^*S\mathbb{X}$ *for some* $\mathbb{X}\in\mathcal{G}\mathrm{TIC}(U)$, $S\in\mathcal{G}\mathcal{B}(U)$*;*

(iii) $\mathbb{D}^*J\mathbb{D}=\mathbb{Y}^*\mathbb{X}$ *for some* $\mathbb{X},\mathbb{Y}\in\mathcal{G}\mathrm{TIC}(U)$*;*

(iv) $\mathbb{D}^*J\mathbb{D}=\mathbb{X}^*S\mathbb{X}$ *for some* $\mathbb{X}\in\mathcal{G}\mathcal{A}(U)$, $S\in\mathcal{G}\mathcal{B}(U)$*.*

If $\mathbb{D}\in\mathcal{A}(U,Y)$ *and* $\mathcal{A}(U)$ *satisfies Hypothesis 8.4.7, then (i)–(iv) are equivalent and any spectral factorization of* $\mathbb{D}^*J\mathbb{D}$ *is over* $\mathcal{A}(U)$*.*

If $\mathbb{D}^*J\mathbb{D}\geq 0$*, then (i)*$\Leftrightarrow$*(ii)*$\Leftrightarrow$*(iii).*

In fact, in discrete-time, (i)–(iv) are equivalent for $\mathbb{D}\in\mathcal{A}:=\mathrm{tic}_{\mathrm{exp}}$ (the set of exponentially stable discrete-time maps), by Theorem 14.3.2. Unfortunately, the Cayley images of $\widehat{\mathrm{tic}_{\mathrm{exp}}}$ cover only those continuous-time maps which are H^∞ outside some disc in the left half-plane, and that requirement is rather strong. By Example 8.4.13, (i) does not imply (ii) for general $\mathbb{D}\in\mathrm{TIC}$.

Proof: "(iii)$\Leftrightarrow$(ii)" holds by Lemma 6.4.7(a3). "(iv)$\Rightarrow$(ii)" is trivial and "(ii)$\Rightarrow$(i)" follows from $\pi_+\mathbb{X}^{-1}S^{-1}\mathbb{X}^{-*}\pi_+=(\pi_+\mathbb{X}^*S\pi_+\mathbb{X}\pi_+)^{-1}$.

The missing implication (i)$\Longrightarrow$(iv) is contained in Hypothesis 8.4.7, and the last sentences follow from Lemma 6.4.5(a) Lemma 6.4.7(a). □

The solvability of several control problems implies the invertibility of the corresponding Popov Toeplitz operator (condition (i) above) and is implied by the existence of a spectral factorization of the Popov operator (condition (ii) above). Thus, the above equivalence makes all these equivalent. Even better, the regularity of $\mathcal{A}$ (see Hypothesis 8.4.7(1.)) makes a complete Riccati equation theory possible.

This is why we obtain complete solutions for the classes of Theorem 8.4.9, but only sufficient conditions (in terms of spectral factorizations and Riccati equations) in the general case (e.g., compare Theorem 11.3.3 to Proposition 11.3.4(f)). See Remark 9.9.14 for other classes of systems for which similar (even better) optimality, factorization and Riccati equation results can be established.

Ilya Spitkovsky has constructed an example showing that the invertibility of the Toeplitz operator (i.e., J-coercivity over $\mathcal{U}_{\mathrm{out}}$) does not imply the existence of a (bounded) spectral factorization in the indefinite case, as mentioned in [S98c, Remark 4.8]. We give here an extended version of that example:

Example 8.4.13 ((minimax) J-coercive $\not\Rightarrow$ SpF) (We give this example for discrete time, use Cayley transform (see Lemma 13.2.1–Theorem 13.2.3) for the continuous-time counterpart.)

(a) Let $J:=J_\gamma^{2,1}:=\mathrm{diag}(1,1,-\gamma^2)$, where $\gamma:=\sqrt{2}$. By Lemma 6.4.7(a), there is $h\in\mathcal{G}\mathrm{H}^\infty(\mathbf{D})$ s.t. $|h|$ equals $1/2$ on the left hemicircle and $\sqrt{3}/2$ on the right hemicircle. Set

$$\widehat{\mathbb{D}}:=\begin{bmatrix} ih & h\\ h(-\cdot) & h(-\cdot)\\ 0 & 1\end{bmatrix}\in\mathrm{H}^\infty(\mathbf{D};\mathbf{C}^{3\times 2}).\quad\text{Then}\tag{8.77}$$

$$\widehat{\mathbb{E}}:=\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}=\begin{bmatrix}1 & f\\ \bar{f} & -1\end{bmatrix}\in\mathrm{L}^\infty(\partial\mathbf{D};\mathbf{C}^{2\times 2}),\tag{8.78}$$

where $f \in \mathrm{L}^\infty(\partial\mathbf{D};\mathbf{C})$ assumes exactly two values, $f_r := \frac{1}{4} - i\frac{3}{4}$ on the right hemicircle and $f_l := \frac{3}{4} - i\frac{1}{4}$ on the left hemicircle.

Therefore, $\widehat{\mathbb{E}}_{11} = 1 \gg 0$ and $\widehat{\mathbb{E}}_{22} - \widehat{\mathbb{E}}_{21}\widehat{\mathbb{E}}_{11}^{-1}\widehat{\mathbb{E}}_{12} = -1 - |f|^2 \ll 0$, so that $\mathbb{D}$ is minimax J-coercive, hence J-coercive over $\mathcal{U}_{\text{out}}$, by Lemma 11.4.2. By Theorem 9.15.3 and Lemma 9.15.2, there is a unique $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}^2(\partial\mathbf{D};\mathbf{C}^{2\times 2})$ modulo a constant s.t. $\widehat{\mathbb{X}}^* \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \widehat{\mathbb{X}} = \widehat{\mathbb{E}}$ a.e. on $\partial\mathbf{D}$.

Let E_r and E_l be the two values of $\widehat{\mathbb{E}}$. Since the eigenvalues of $E_r^{-1}E_l$ are not positive (not even real), it follows that $\widehat{\mathbb{X}} \notin \mathcal{G}\mathrm{H}^\infty(\mathbf{D};\mathbf{C}^{2\times 2})$; in fact, one can show that both $\widehat{\mathbb{X}}$ and $\widehat{\mathbb{X}}^{-1}$ are unbounded. By uniqueness, there can be no ($\mathcal{G}\mathrm{H}^\infty$) spectral factorization of $\mathbb{D}^* J \mathbb{D}$.

(b) Furthermore, there is a minimax $\widetilde{J}$-coercive $\widehat{\mathbb{D}_0} \in \mathrm{H}^\infty(\mathbf{D};\mathbf{C}^{6\times 4})$, where $\widetilde{J} := J_\gamma^{4,2} := \left[\begin{smallmatrix} I_{4\times 4} & 0 \\ 0 & -2I_{2\times 2} \end{smallmatrix}\right]$, s.t. $\widehat{\mathbb{D}_0}$ is of form $\left[\begin{smallmatrix} * & * \\ 0 & I_{2\times 2} \end{smallmatrix}\right]$, and $\widehat{\mathbb{D}_0}^* \widetilde{J}\widehat{\mathbb{D}_0} = \widehat{\mathbb{X}_0}^* S \widehat{\mathbb{X}_0}$, where $S := J_1^{2,2} = \left[\begin{smallmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{smallmatrix}\right]$, $\widehat{\mathbb{X}_0} \in \mathcal{G}\mathrm{H}^2(\mathbf{D};\mathbf{C}^4)$ and both $\widehat{\mathbb{X}_0}$ and $\widehat{\mathbb{X}_0}^{-1}$ are unbounded near -1.

Consequently, the corresponding continuous-time $\widetilde{J}$-critical "state feedback controller" over $\mathcal{U}_{\text{out}}$ (the H^∞ full-information minimax controller over $\mathcal{U}_{\text{out}}$) is non-well-posed (alternatively, unstable, by (c)) in both its open-loop and closed-loop forms, as shown in Example 11.3.7.

(c) Set $K := \overline{\mathbf{D}} \setminus \{z \,\big|\, |z \pm i| < \varepsilon\}$ for some $\varepsilon > 0$. Then there is a neighborhood $\Omega \supset \mathbf{D}$ of K s.t. $\widehat{\mathbb{D}}, \widehat{\mathbb{X}}, \widehat{\mathbb{X}}^{-1} \in \mathrm{H}^\infty(\Omega;\mathcal{B}(\mathbf{C}^2,*))$ in (a). In particular, the corresponding continuous-time maps (Cayley inverses) are uniformly half-plane regular.

Consequently, if we drop the rotation from the proof of (b), the maps $\mathbb{D}_0$, $\mathbb{X}_0, \mathbb{X}_0^{-1}$ become (well-posed and) uniformly half-plane-regular (but $\mathbb{X}_0^{\pm 1}$ become unstable, althought $\mathbb{X}_0^{\pm 1}\mathrm{L}^2_{\mathrm{c}} \subset \mathrm{L}^2$). ◁

The above example shows that the J-critical control is not always of state feedback form (in continuous time); cf. Remark 9.7.7(a3). (Due to minimax J-coercivity, the example shows that the problem cannot be avoided even in connection with H^∞ problems.)

In discrete-time that cannot happen, but the implication (i)⇒(iii) in Theorem 8.4.12 is nevertheless false in general in discrete-time too, as shown by the above example (see also Section 9.15 and Example 11.3.7) unless $\mathbb{D}$ is exponentially stable (in discrete time).

Proof of Example 8.4.13: (a) (Ilya Spitkovsky has skecthed the proof; this is a modified version of that sketch.)

1° *Constructing* $\mathbb{D}$ *s.t. (8.78) holds:* By Theorem 3.1.3(a1)&(e1), any $\widehat{\mathbb{F}} \in \mathrm{L}^\infty(i\mathbf{R})$ satisfying $\widehat{\mathbb{F}} \geq \varepsilon$ a.e. on $i\mathbf{R}$ corresponds to some $\mathbb{F} \in \mathrm{TI}(\mathbf{C})$ with $\mathbb{F} \geq \varepsilon I$, so that $\mathbb{F} = |\mathbb{Z}|^2$ for some $\mathbb{Z} \in \mathcal{G}\mathrm{TIC}(\mathbf{C})$ (i.e., some $\widehat{\mathbb{Z}} \in \mathrm{H}^\infty(\mathbf{C}^+)$), by Lemma 6.4.7(a). Apply Cayley transform to this result to show the existence of $h \in \mathrm{H}^\infty(\mathbf{D})$ (see the example above). (This also shows that h is invertible in H^∞, but we do not need the invertibility of h.) One easily verifies that the eigenvalues of

$$E_r^{-1}E_l = \frac{-1}{1+|f_r|^2}\begin{bmatrix} -1 - f_r\bar{f}_l & f_r - f_l \\ \bar{f}_l - \bar{f}_r & -1 - \bar{f}_r f_l \end{bmatrix} \tag{8.79}$$

are given by $\lambda_\pm := \mathrm{Re}\,t \pm \sqrt{(\mathrm{Re}\,t)^2-4(|t|^2+|s|^2)}$, where $t := -1-f_r\bar{f}_l$, $s := f_r - f_l$, hence these values are not real.

2° *The other claims:* The other claims are explained in the example except the fact that that $\widehat{\mathbb{X}}, \widehat{\mathbb{X}}^{-1} \notin \mathrm{H}^\infty(\mathbf{D};\mathbf{C}^{2\times 2})$.

If we had $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}^\infty$, then the factorization would exist in all L^p spaces; however, it does not exist for $p = 2\pi/\arg\lambda_\pm$ (since $\lambda_\pm$ are not real; see [LS] for details).

Choose $z_0 \in \partial\mathbf{D}$ is s.t. $\widehat{\mathbb{X}}$ or $\widehat{\mathbb{X}}^{-1}$ is unbounded in each neighborhood of z_0. (N.B. $z_0 = \pm i$, because $\widehat{\mathbb{X}}$ and $\widehat{\mathbb{X}}^{-1}$ have holomorphic extensions around each point of $\overline{\mathbf{D}}\setminus\{\pm i\}$, by Lemma 9.15.5. For the same reason it seems that the Cayley inverse of the function $\widehat{\mathbb{D}_0}$ constructed below will not be (weakly) regular.)

Set $\widehat{\mathbb{X}}^{\mathrm{d}}(z) := \widehat{\mathbb{X}}(\bar{z})^*$ $(z \in \mathbf{D})$ (cf. Lemma 13.1.8). Obviously, $\widehat{\mathbb{X}}^{\mathrm{d}} \in \mathcal{G}\mathrm{H}^2(\mathbf{D};\mathbf{C}^{2\times 2})$ and

$$(\widehat{\mathbb{X}}^{-\mathrm{d}}(z))^* J_1^{-1}\widehat{\mathbb{X}}^{-\mathrm{d}}(z) = (\widehat{\mathbb{X}}(\bar{z})^* J_1 \widehat{\mathbb{X}}(\bar{z}))^{-1} = \widehat{\mathbb{E}}(\bar{z})^{-1} = \widehat{\mathbb{E}}(z)^{-1} \tag{8.80}$$

(since $\widehat{\mathbb{E}}(\bar{z}) = \widehat{\mathbb{E}}(z)$, because $z \mapsto \bar{z}$ maps the right and left hemicircles onto themselves).

Set $a := \sqrt{8/13}$, so that $a^*a = (1+|f|^2)^{-1} = 8/13$. One easily verifies that $\widehat{\mathbb{E}}^{-1} = a^*a\widehat{\mathbb{E}}$, hence $(a\widehat{\mathbb{X}}^*)J_1(a\widehat{\mathbb{X}}) = \widehat{\mathbb{E}}^{-1}$. We conclude from Lemma 6.4.5(a) that $a\widehat{\mathbb{X}} = E\widehat{\mathbb{X}}^{-\mathrm{d}} = E\widehat{\mathbb{X}}(\bar{\cdot})^{-*}$ for some $E \in \mathcal{G}\mathbf{C}^{2\times 2}$. Therefore, both $\widehat{\mathbb{X}}$ and $\widehat{\mathbb{X}}^{-1}$ are unbounded on $\mathbf{D}$ (one near z_0 and the other near $\overline{z_0}$ (at least)).

(b) 1° *Constructing $\widehat{\mathbb{D}_a}$ and $\widehat{\mathbb{X}_a}$:* Set $\widehat{\mathbb{F}}(z) := \widehat{\mathbb{D}}(-z)$, $\widehat{\mathbb{Z}}(z) := \widehat{\mathbb{X}}(-z)$ $(z \in \mathbf{D})$, so that $\widehat{\mathbb{F}}^* J\widehat{\mathbb{F}} = \widehat{\mathbb{Z}}^* J_1\widehat{\mathbb{Z}}$. By (a)2°, $\widehat{\mathbb{X}_a} = \begin{bmatrix}\widehat{\mathbb{X}} & 0\\ 0 & \widehat{\mathbb{Z}}\end{bmatrix} \in \mathcal{G}\mathrm{H}^2(\mathbf{D};\mathbf{C}^{4\times 4})$ and its inverse are both unbounded at $\pm z_0 = \pm i$. Obviously, $\widehat{\mathbb{X}_a}^* \begin{bmatrix}J_1 & 0\\ 0 & J_1\end{bmatrix}\widehat{\mathbb{X}_a} = \widehat{\mathbb{D}_a}^* J'\widehat{\mathbb{D}_a}$, where $\widehat{\mathbb{D}_a} := \begin{bmatrix}\mathbb{D} & 0\\ 0 & \mathbb{F}\end{bmatrix}$, $J' := \begin{bmatrix}J & 0\\ 0 & J\end{bmatrix}$, $J_1 := \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.

2° *Rotation to -1:* Replace $\widehat{\mathbb{D}_a}$ by $\widehat{\mathbb{D}_a}(i\cdot)$ (and hence $\widehat{\mathbb{X}_a}$ by $\widehat{\mathbb{X}_a}(i\cdot)$). Then both $\widehat{\mathbb{X}_a}$ and $\widehat{\mathbb{X}_a}^{-1}$ become unbounded near 1 and near -1. (Recall that (our) Cayley transform maps -1 to ∞; this is why this is important.)

3° *Constructing $\widehat{\mathbb{D}_0}, \widehat{\mathbb{X}_0}$:* Set

$$\widehat{\mathbb{D}_0} := T'\widehat{\mathbb{D}_a}T, \quad \widehat{\mathbb{X}_0} := T\widehat{\mathbb{X}_a}T, \quad \text{where} \tag{8.81}$$

$$T := \begin{bmatrix} I&0&0&0\\ 0&0&I&0\\ 0&I&0&0\\ 0&0&0&I\end{bmatrix} = T^* = T^{-1} \in \mathbf{C}^{4\times 4}, \quad T' := \begin{bmatrix} I&0&0&0&0&0\\ 0&I&0&0&0&0\\ 0&0&0&0&I&0\\ 0&0&0&I&0&0\\ 0&0&I&0&0&0\\ 0&0&0&0&0&I\end{bmatrix} \in \mathbf{C}^{6\times 6} \tag{8.82}$$

(note that $T\cdot$ (resp. $\cdot T$) permutes second and third rows (resp. columns) and $T'\cdot$ (resp. $\cdot T'$) permutes third and fifth rows (resp. columns)), so that

$$S := J_1^{2,2} := \mathrm{diag}(1,1,-1,-1) = T\begin{bmatrix}J_1 & 0\\ 0 & J_1\end{bmatrix}T, \quad \widetilde{J} = T'J'T', \quad \text{and} \tag{8.83}$$

$$\widehat{\mathbb{D}_0}^*\widetilde{J}\widehat{\mathbb{D}_0} = T\widehat{\mathbb{D}_a}^* J'\widehat{\mathbb{D}_a}T = T\widehat{\mathbb{X}_a}^*\begin{bmatrix}J_1 & 0\\ 0 & J_1\end{bmatrix}\widehat{\mathbb{X}_a}T = \widehat{\mathbb{X}_0}^* S\widehat{\mathbb{X}_0}. \tag{8.84}$$

Thus, $\mathbb{X}_0^* S\mathbb{X}_0$ is a spectral factorization of $\mathbb{D}_0^*\widetilde{J}\mathbb{D}_0$. Moreover, $\widehat{\mathbb{D}_0}$ is of form

$\begin{bmatrix} * & * \\ 0 & I_{2\times 2}\end{bmatrix}$ and

$$\widehat{\mathbb{D}_0}^* \widetilde{J}\widehat{\mathbb{D}_0} = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} \\ \begin{bmatrix} \bar{f} & 0 \\ 0 & \bar{f} \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix}, \tag{8.85}$$

hence $\mathbb{D}_0$ is obviously minimax $\widetilde{J}$-coercive w.r.t. $\mathbf{C}^2 \times \mathbf{C}^2$ (see Definition 11.4.1). See Example 11.3.7 for the final claims. For clarity, we write out $\widehat{\mathbb{D}_0}$ (here $g := h(i\cdot)$, $G := h(-i\cdot)$):

$$\widehat{\mathbb{D}_0} = \begin{bmatrix} ig & 0 & g & 0 \\ G & 0 & G & 0 \\ 0 & ig & 0 & g \\ 0 & G & 0 & G \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8.86}$$

(c) 1° *Constructing* Ω*:* Since $|h|^2$ has a holomorphic extension (namely $1/2$ and $\sqrt{3}/2$ on two disjoint open sets) to a neighborhood of $K \cap \partial\mathbf{D}$, so has its spectral factor, that is, h (and its inverse h^{-1}), by Lemma 9.15.5.

Consequently, $\widehat{\mathbb{D}}$ and hence also $\widehat{\mathbb{X}}$ and $\widehat{\mathbb{X}}^{-1}$ have holomorphic extensions to a neighborhood of $K \cap \partial\mathbf{D}$. Let Ω be the union of that neighborhood and $\mathbf{D}$.

2° *The rest:* Take Cayley transforms to obtain that for any $\varepsilon > 0$, there is a compact set $K_2 \subset \mathbf{C}^- \cup \{z \in \mathbf{C} \,\big|\, |z \pm i| < \varepsilon\}$ s.t. $\widehat{\mathbb{D}}, \widehat{\mathbb{X}}, \widehat{\mathbb{X}}^{-1} \in \mathrm{H}^\infty(K_2^c; \mathcal{B}(\mathbf{C}^2, *))$. In particular, these functions are continuous at infinity, hence ULR, even uniformly half-plane-regular (to be exact, $\mathbb{X}$ and/or $\mathbb{X}^{-1}$ is not stable (at $\pm i$) but it is otherwise uniformly half-plane-regular). Consequently, $\widehat{\mathbb{D}_0}, \widehat{\mathbb{X}_0}^{\pm 1} \in \mathrm{H}^\infty(K_2^c; \mathcal{B}(\mathbf{C}^4, *))$. □

Next we state the unstable version of Theorem 8.4.12:

Corollary 8.4.14 (MTI (J,S)-inner r.c.f.) *Let $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$, $\mathcal{A} \subset \mathrm{TIC}$, $J = J^* \in \mathcal{B}(Y)$ and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$.*

(a) If $\mathbb{D}$ has a $(J,*)$-inner q.r.c.f. $\mathbb{N}\mathbb{M}^{-1}$, then $\mathbb{D}$ and $\mathbb{N}$ are J-coercive.

Let, in addition, $\mathbb{D}$ have a [q.]r.c.f. $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1}$. Then we have the following:

(b1) We have (iii)⇔(ii)⇒(i), where

(i) $\mathbb{N}'$ is J-coercive (i.e., $\mathbb{D}$ is J-coercive);
(ii) $\mathbb{N}'^* J \mathbb{N}'$ has a spectral factorization;
(ii') $\mathbb{N}'^* J \mathbb{N}'$ has a spectral factorization over $\mathcal{A}$;
(iii) $\mathbb{D}$ has a $(J,*)$-inner [q.]r.c.f.
(iii') $\mathbb{D}$ has a $(J,*)$-inner [q.]r.c.f. $\mathbb{N}\mathbb{M}^{-1}$ with $\mathbb{N}, \mathbb{M} \in \mathcal{A}$.

(b2) If $\mathbb{N}', \mathbb{M}' \in \mathcal{A}$, then (ii')⇒(iii')⇒(iii)⇔(ii)⇒(i).

(b3) Assume that $\mathbb{N}', \mathbb{M}' \in \mathcal{A}$*, and that* $\mathcal{A}(U)$ *satisfies Hypothesis 8.4.7.*

Then (i)–(iii') are equivalent. Moreover, if some factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *satisfies (ii) or (iii), then* $\mathbb{N}, \mathbb{M} \in \mathcal{A}$*,* $\mathbb{D}, \mathbb{M}^{-1} \in \mathrm{ULR}$*, and we can choose them so that* $\widehat{\mathbb{M}}(+\infty) = I$*.*

(b4) Assume that Hypothesis 8.4.7 holds for $\mathcal{A}(U)$*, and that (iii') holds. Set* $\mathbb{X} := \mathbb{M}^{-1}$*. Then* $N^*JN = S$ *and* $D^*JD = X^*SX$*.*

(b5) If $J \geq 0$*, then (i)⇔(ii)⇔(iii).*

Theorem 9.9.10 gives further equivalent conditions in terms of Riccati equations. Example 8.4.13 shows that implication (i)⇒(iii) does not hold in general.

The above corollary will be applied in Chapters 9 and 11 to reduce inner coprime factorizations of unstable maps to spectral factorizations of their stabilized counterparts.

Note that, by (b3), any $(J,*)$-inner r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ with $\mathbb{N}, \mathbb{M} \in \mathcal{A}$ is actually a r.c.f. *over* $\mathcal{A}$ (i.e., also $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}} \in \mathcal{A}$ for some stable $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}}$ satisfying $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$).

If $\mathbb{N}$ and $\mathbb{M}$ are exponentially stable discrete-time maps (cf. the remark after Theorem 8.4.12), then (i)–(iii) are equivalent; see Corollary 14.3.3.

Proof of Corollary 8.4.14: (a) This follows from "(iii)⇒(i)" of (b1) (take $\mathbb{N}' = \mathbb{N}$, $\mathbb{M}' = \mathbb{M}$) and Lemma 8.4.11(b1).

(b1) We have (ii')⇒(ii)⇒(i) (even (ii')⇔(ii)⇔(i) provided Hypothesis 8.4.7 is satisfied), by Theorem 8.4.12.

"(ii)⇔(iii)": If (ii) holds, i.e., $\mathbb{U}^*S\mathbb{U} = \mathbb{N}'^*J\mathbb{N}'$ for some $\mathbb{U} \in \mathcal{G}\mathrm{TIC}(U)$, $S \in \mathcal{GB}(U)$, then $\mathbb{N}'\mathbb{U}^{-1}$ is (J,S)-inner; thus, $\mathbb{D} = (\mathbb{N}'\mathbb{U}^{-1})(\mathbb{M}'\mathbb{U}^{-1})^{-1}$ is as in (iii).

Conversely, if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a $(J,*)$-inner [q.]r.c.f., then, by Lemma 6.4.5(c), $\mathbb{N}' = \mathbb{N}\mathbb{U}$ for some $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$, hence then $\mathbb{N}'^*J\mathbb{N}' = \mathbb{U}^*\mathbb{N}^*J\mathbb{N}\mathbb{U} = \mathbb{U}^*S\mathbb{U}$ is a spectral factorization of $\mathbb{N}'$.

(b2) Implication "(ii')⇒(iii')" follows from the proof of "(ii)⇔(iii)". Obviously, (iii')⇒(iii); the rest follows from (b1).

(b3) By Theorem 8.4.12, we have (i)⇔(ii'), hence all five claims are equivalent, by (b2). Theorem 8.4.12 also implies that necessarily $\mathbb{U} \in \mathcal{GA}$ in (ii), which provides the $\mathbb{N}, \mathbb{M} \in \mathcal{A}$ claim, by the proof of "(ii)⇔(iii)" above.

Because $\mathcal{A} \subset \mathrm{ULR}$, we have $\mathbb{M}^{-1}, \mathbb{N}\mathbb{M}^{-1} \in \mathrm{ULR}$, and $M := \widehat{\mathbb{M}}(+\infty) \in \mathcal{GB}$, by Lemma 6.2.5 and Proposition 6.3.1(c). Therefore, we can take $M = I$, by Lemma 6.4.5(a).

(b4) This follows from Lemma 6.3.6(b).

(b5) This follows from (b1) and the last claim in Theorem 8.4.12. □

Notes for Section 8.4

The Popov Toeplitz invertibility condition of Proposition 8.3.10 and Lemma 8.4.11(a1) is very common in the literature of stable control problems [WW] [S98c]. Our definition of J-coercivity generalizes this concept to general control problems for WPLSs. In a sense, it is the weakest assumption that guarantees the existence of a unique solution (with equivalence for smooth systems with $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ and $\dim U < \infty$ or $D^*JD \in \mathcal{GB}(U)$); see Theorems 9.2.16 and 10.2.11 and Corollaries 10.2.3 and 9.2.19. However there are singular counterexamples at least when U is infinite-dimensional (e.g., take $C = 0$, $D > 0$, $D \not\gg 0$, $J = I$). Singular control problems are rare in practice and in the literature, and they cannot be solved as satisfactorily; see, e.g., [Stoorvogel] for the finite-dimensional case.

Propositions 10.3.1 and 10.3.2 provide several conditions that are equivalent to J-coercivity. Many popular assumptions in the literature are special cases of these assumptions [ZDG] [Keu] and most others are stronger [FLT]. We have seen nothing similar in the indefinite unstable case.

The name "J-coercivity" is from [S97b]–[S98c], where it means the invertibility of the Popov Toeplitz operator for a stable system, and for a jointly stabilizable and detectable system it means the J-coercivity of the corresponding stabilized subsystem. From Lemma 8.4.11(a1)–(b1) and Theorem 6.6.28 we observe that these definitions are special cases of that of ours.

In the stable case, the method of Theorem 8.4.3 (as the first part of Proposition 8.3.10) is very old. The same holds for the reduction of unstable $\mathcal{U}_{\mathrm{exp}}$ problems to the stable case, as in Theorem 8.4.5; its coprimeness method for $\mathcal{U}_{\mathrm{sta}}$ has been used in [S98b] and in [S98c, Section 7] for jointly stabilizable and detectable systems. See Chapter 5 for notes on the spectral factorization results on which Theorems 8.4.9 and 8.4.12 are based.

Lemma 8.4.11(a1)–(b1) are essentially from [S98c]. Part of Example 8.4.13 was mentioned in [S98c, Remark 4.8], and its original form is due to Ilya Spitkovsky in a communication with Joseph Ball and Olof Staffans.

8.5 Problems on a finite time interval

Lord of the far horizons,
Give us the eyes to see
Over the verge of the sundown
The beauty that is to be.
— Bliss Carman (1861–1929)

In this section, we swiftly review how the abstract optimization of Sections 8.1–8.2 can be applied to finite-horizon problems. The derivation of further details and differential Riccati equation theory is analogous to that in the infinite-horizon case, but it requires a lengthy treatment, hence we omit it.

Throughout this section, the letters U, H, and Y denote (complex) Hilbert spaces of arbitrary dimensions, $T>0$, $J=J^*\in\mathcal{B}(Y)$ and $\left[\begin{array}{c|c}\mathbb{A}&\mathbb{B}\\\hline\mathbb{C}&\mathbb{D}\end{array}\right]\in$ WPLS(U,H,Y). In finite time interval (finite-horizon) problems, the cost to be optimized is given by

$$\mathcal{J}(x_0,u)=\int_0^T\langle y(t),Jy(t)\rangle_Y\,dt, \tag{8.87}$$

(as opposed to $T=\infty$ in other sections, i.e., in infinite-horizon problems), where $x_0\in H$ is the initial state, $u\in\mathrm{L}^2([0,T);U)$ is the control and $y:=\mathbb{C}x_0+\mathbb{D}u$ is the corresponding output. Often one wishes to add to $\mathcal{J}$ an end cost $\langle x(T),Gx(T)\rangle_H$, where $x(T):=\mathbb{A}(T)x_0+\mathbb{B}\tau(T)u$ is the (terminal) state at time T.

The abstract optimization theory can be applied also to finite-horizon problems by using the following substitutions:

Remark 8.5.1 (Sections 8.1–8.2 apply also to finite-horizon problems) *Standing Hypothesis 8.1.1 is satisfied with substitutions*

$$U\mapsto\mathrm{L}^2([0,T);U),\qquad X\mapsto H \tag{8.88}$$

$$Y,Y^{\mathrm{s}}\mapsto\mathrm{L}^2([0,T);Y),\qquad Z,Z^{\mathrm{s}}\mapsto\mathrm{L}^2([0,T);H), \tag{8.89}$$

$$\left[\begin{array}{c|c}A&B\\\hline C&D\end{array}\right]\mapsto\left[\begin{array}{c|c}\pi_{[0,T)}\mathbb{A}&\pi_{[0,T)}\mathbb{B}\tau\pi_{[0,T)}\\\hline\pi_{[0,T)}\mathbb{C}&\pi_{[0,T)}\mathbb{D}\pi_{[0,T)}\end{array}\right]\qquad J\mapsto J. \tag{8.90}$$

It follows that $\mathcal{J}(x_0,u)=\int_0^T\|y(t)\|_Y^2\,dt$, where $y:=\mathbb{C}x_0+\mathbb{D}\tau u$, for all $u\in\mathcal{U}(x_0)=$ $\mathrm{L}^2([0,T);U)$ and $x_0\in H$.

To add an end cost $\langle x(T),Gx(T)\rangle_H$, $G\in\mathcal{B}(H)$, one could substitute $C\mapsto\begin{bmatrix}\pi_{[0,T)}\mathbb{C}\\\mathbb{A}^T\end{bmatrix}$, $D\mapsto\begin{bmatrix}\pi_{[0,T)}\mathbb{D}\pi_{[0,T)}\\\mathbb{B}^T\end{bmatrix}$, $J\mapsto\begin{bmatrix}J&0\\0&G\end{bmatrix}$, $Y,Y^{\mathrm{s}}\mapsto\mathrm{L}^2([0,T);Y)\times H$.

In particular, "the stable case" applies (in both cases, although we refer below to the case $G=0$). (The substitution $\mathcal{U}(x_0)=\mathrm{L}^2([0,T);U)$ above corresponds to "$\mathcal{U}_{\mathrm{exp}}=\mathcal{U}_{\mathrm{str}}=\mathcal{U}_{\mathrm{sta}}=\mathcal{U}_{\mathrm{out}}$".)

Assume that "D^*JD"$=\pi_{[0,T)}\mathbb{D}^*J\pi_{[0,T)}\mathbb{D}\pi_{[0,T)}$ is invertible (this is the case if, e.g., $J\geq0$ and there is a quadratic cost on u). Then the system is J-coercive, so that Lemma 8.2.8 provides us direct formulae for $\mathbb{A}_{\mathrm{crit},T}$, $\mathbb{C}_{\mathrm{crit},T}$, $\mathbb{K}_{\mathrm{crit},T}$, and "$\mathcal{P}=C^*JC_{\mathrm{crit}}$".

In particular, $\mathcal{P}$ becomes (we shall set $\mathcal{P}_T := \mathcal{P}$ to distinguish between cost operators for problems on different intervals, i.e., for different values of T)

$$\langle x_1, \mathcal{P}_T x_0\rangle = \int_0^T \langle J(\mathbb{C}x_1)(t), ((\mathbb{C}+\mathbb{D}\mathbb{K}_{\mathrm{crit},T})x_0)(t)\rangle_H \, dt. \tag{8.91}$$

Assume, in addition, that $C \in \mathcal{B}(H,U)$, $D^*JC = 0$. Then, as in the proof of Lemma 8.2.9, we have

$$\langle x_1, \mathcal{P}_T x_0\rangle = \int_0^T \langle C\mathbb{A}^t x_1, JC\mathbb{A}^t_{\mathrm{crit},T} x_0\rangle_H \, dt, \tag{8.92}$$

and $\mathbb{A}_{\mathrm{crit}} = \mathbb{A} + \mathbb{B}\tau\mathbb{K}_{\mathrm{crit}} \in \mathcal{B}(H, \mathcal{C}([0,T];H))$.

Again by Lemma 8.2.9, with the additional assumption that $D^*JD \in \mathcal{GB}(U)$, we also have

$$\mathbb{K}_{\mathrm{crit},T} x_0 = -(D^*JD)^{-1}(\pi_{[0,T)}\mathbb{B}\tau\pi_{[0,T)})^* C^* JC\mathbb{A}_{\mathrm{crit},T} x_0. \tag{8.93}$$

It seems that one can rewrite Sections 9.7 and 9.2 for finite time; in particular, that a unique J-critical control is of the state feedback form whenever $\mathbb{D} \in \mathrm{MTIC}_\infty^{\mathrm{L}^1}$ and C is bounded, or $\mathbb{D} \in \mathcal{B} + \mathrm{L}^2_\infty *$ (we do not have to pose any requirements on $\mathbb{A}B$ (or $\mathbb{B}\tau$), since on a finite time horizon we always have "the stable case", as in Hypothesis 9.2.2(6.)–(7.)), and that the J-critical state feedback operator corresponds then to a unique solution of the differential Riccati equation.

At least when $\mathcal{J}(x_0,u) = \|y\|^2$, $y = \left[\begin{smallmatrix} y_1 \\ u \end{smallmatrix}\right]$, the solution converges strongly to the infinite-horizon solution, as $T \to +\infty$ (possibly we have to restrict T to a sequence converging to infinity), and $\mathcal{P}_T \to \mathcal{P}$ strongly. This follows by standard arguments (see Lemma 4.2 of [FLT] in the case of a bounded C; the same arguments combined with Lemma A.3.1(i1)&(i4) also apply to a general WPLS).

The situation is analogous also in the case of an extended linear system (see Section 8.6).

However, we leave the details to the reader, because it seems that this approach does not provide us with any additional information $\mathcal{P}$ (e.g., we do not know whether $\mathcal{P}$ converges in the way that $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ even for the special $\mathbb{D}$'s mentioned above). This is an interesting open problem.

Notes

It is well-known that, under certain regularity (see, e.g., [GL] for the finite-dimensional and [Jacob98], [Jacob01], [Jacob99], [FLT] or [LT00b] for the infinite-dimensional case), 1. one can show that $\mathcal{P}_T \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$ and $(\mathbb{K}_{\mathrm{crit},T} x_0)(s) = -B_{\mathrm{w}}^* \mathcal{P}_{T-s} \mathbb{A}^s_{\mathrm{crit},T} x_0$, i.e., that $u_{\mathrm{crit}} = -B_{\mathrm{w}}^* \mathcal{P}_{T-s} x_{\mathrm{crit}}$; moreover, $\mathcal{P}_T$ satisfies the corresponding differential Riccati equation (as a function of $T > 0$); 2. by letting $T \to +\infty$, we obtain that $\mathcal{P}_T$, u_{crit}, x_{crit} and y_{crit} converge (for a suitable cost function; for our techniques one sufficient condition seems to be that $J \geq 0$).

Analogously, one could apply the abstract optimization theory of Sections 8.1–8.2 to time-variant systems. Birgit Jacob [Jacob98] has formulated "tv-systems", which are a natural (time-variant) generalization of WPLSs, and solved the standard optimal control problem on a finite time interval (the same is done for a generalization of PS-systems in [Jacob01] and for a generalization of WPLSs with bounded input and output operators in [Jacob99]; thus, [Jacob98] is the most general one). Her solution coincides with the implications of Lemma 8.2.8 combined with an equivalent of Remark 8.5.1 for tv-systems, but assuming the equivalent of weak regularity (with $D^*JC=0$), she obtains the formula $(\mathbb{K}_{\mathrm{crit},T}x_0)(t) = -B_{\mathrm{w}}^*\mathcal{P}_{T-t}\mathbb{A}^t_{\mathrm{crit},T}x_0$ (where B_{w}^* is allowed to be time-varying). In [Jacob99], Jacob has solved the finite-horizon minimization problem for a nonstandard (general) cost function for time-variant systems but under the assumption that the input and output operators are bounded. For time-invariant WPLSs with bounded output operators, the nonstandard minimization problem is solved in [BP] in terms of an integral Riccati inequality, by Francesca Bucci and Luciano Pandolfi.

Historical remarks on the LQR problem are given in the notes to Chapter 6 of [CZ]; also finite-horizon problems are treated therein.

Over a finite time horizon we have "$\mathcal{U}_{\mathrm{exp}}=\mathcal{U}_{\mathrm{out}}$", because $\mathrm{L}^2_{\mathrm{loc}}([0,T];H)=\mathrm{L}^2([0,T];H)$, hence the minimization results in something resembling more $\mathcal{U}_{\mathrm{out}}$ than $\mathcal{U}_{\mathrm{exp}}$ but having the nicest properties of both settings.

In particular, if we let $T\to+\infty$, we end up with a minimizing (hence J-critical) control over $\mathcal{U}_{\mathrm{out}}$ (not over $\mathcal{U}_{\mathrm{exp}}$, and the closed-loop system need not be exponentially stable), and the state feedback formula $u=-B_{\mathrm{w}}^*\mathcal{P}_T$ converges to $u=-B_{\mathrm{w}}^*\mathcal{P}$ (or to some other operator, denoted by "$-B^*\mathcal{P}$" in [FLT], if we reduce regularity assumptions), where $\mathcal{P}$ refers to the J-critical cost operator over $\mathcal{U}_{\mathrm{out}}$. Naturally, this can be fixed by assuming the system to be estimatable (see Lemma 8.3.3).

It seems that the methods of this monograph could be generalized to tv-systems. This might be an interesting subject for future studies.

8.6 Extended linear systems (ELS)

Human beliefs, like all other natural growths, elude the barrier of systems.
— George Eliot (1819–1880)

In this section, we give guidelines on how to extend our optimization and Riccati equation results for more general systems than WPLSs. Most readers probably wish to skip this section (the results are not used elsewhere in this monograph).

The optimization results given above, such as Theorem 8.3.9 (often combined with Theorem 8.4.3) require that $\mathcal{U}_*^*(x_0) \neq \emptyset$ for all $x_0 \in H$. What if this holds only for all $x_0 \in H_{1/2}$, for some Hilbert space $H_{1/2}$ s.t. $H_1 \underset{c}{\subset} H_{1/2} \underset{c}{\subset} H$? If "$B \in \mathcal{B}(U, H_{1/2-1})$", then we could replace H by $H_{1/2}$ and go on — except that $\mathbb{B}$ need not be well-posed anymore.

Despite the generality of WPLSs, there are some interesting PDE-based systems for which there are no known finite cost condition results for any choice of state space that would make $\mathbb{B}$ well-posed, as noted in [LT93]. Therefore, in this section, we shall give guidelines on how to relax the well-posedness requirement of $\mathbb{B}$.

For systems mentioned above, to guarantee the finite cost condition the state space must be chosen to be very small (this causes the high unboundedness of B), and hence the output operator becomes bounded; therefore we assume this, i.e., that $C \in \mathcal{B}(H, Y)$, although basic results of this section hold for general well-posed C too (see Remark 8.6.8).

Thus, we are again solving the problem "$x' = Ax + Bu$, $y = Cx + Du$, $x(0) = x_0$", but this time we do not require $\mathbb{B}$ to be well-posed, i.e., to have values in H (equivalently, to satisfy $\mathbb{B}\pi_{[-t,0)} \in \mathcal{B}(\mathrm{L}^2([-t,0);U), H)$). The solution $\Sigma := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] : \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ is called an *Extended Linear System (ELS)* under the assumptions stated below:

Standing Hypothesis 8.6.1 *Throughout this section, we shall assume that U, H and Y are Hilbert spaces, that $\mathbb{A}$ is a C_0-semigroup on H, $B \in \mathcal{B}(U, H_{-1})$, $C \in \mathcal{B}(H, Y)$, $D \in \mathcal{B}(U, Y)$, $J = J^* \in \mathcal{B}(Y)$ and* $0 \leq \beta > \omega_A$.

By Lemma 6.3.16, $\left(\begin{smallmatrix} A \\ C \end{smallmatrix}\right)$ generate $\left[\begin{smallmatrix} \mathbb{A} \\ \mathbb{C} \end{smallmatrix}\right] \in \mathrm{WPLS}_\beta(\{0\}, H, Y)$ (this is trivial, with $\mathbb{C} = C\mathbb{A} : H \to \mathcal{C}(\mathbf{R}_+; U)$) and $\left(\begin{smallmatrix} A & B \\ I & 0 \end{smallmatrix}\right)$ generate $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{A} & \mathbb{B}\tau \end{smallmatrix}\right] \in \mathrm{WPLS}_\beta(U, H_{-1}, H_{-1})$. Thus, we can and will have this section based heavily on WPLS results and on the abstract optimization theory Sections 8.1 and 8.2.

Moreover, $\mathbb{B}\tau$ maps $\mathrm{L}^2_\beta + \pi_+\mathrm{L}^2_{\mathrm{loc}}$ to $\mathcal{C}(\mathbf{R}; H_{-1})$ (because $\pi_-\tau u \in \mathcal{C}(\mathbf{R}; \mathrm{L}^2_\beta)$ and $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2_\beta; H_{-1})$) and $\mathrm{W}^{1,2}_\beta(\mathbf{R}; U) \to \mathcal{C}(\mathbf{R}; H)$ (by Theorem 6.2.13(b1), because $(H_{-1})_B = H$).

We now define some spaces of allowable inputs u with graph norms:

Definition 8.6.2 *For all $T \in [0,\infty)$ we define*

$$\widetilde{U}_{\mathrm{loc}} := \{u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U) \,|\, \mathbb{B}\tau u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};H)\}, \tag{8.94}$$

$$\mathcal{U}_{\mathrm{exp}}(x_0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,|\, \mathbb{A}x_0 + \mathbb{B}\tau u \in \mathrm{L}^2(\mathbf{R}_+;H)\} \quad (x_0 \in H), \tag{8.95}$$

$$\mathcal{U}^{\mathcal{C}}_{\mathrm{exp}}(x_0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,|\, \mathbb{A}x_0 + \mathbb{B}\tau u \in \mathrm{L}^2(\mathbf{R}_+;H) \cap \mathcal{C}(\mathbf{R}_+;H)\} \quad (x_0 \in H), \tag{8.96}$$

$$\widetilde{U}_{\beta} := \{u \in \mathrm{L}^2_{\beta}(\mathbf{R};U) \,|\, \mathbb{B}\tau u \in \mathrm{L}^2_{\beta}(\mathbf{R};H)\}, \tag{8.97}$$

$$\widetilde{U}_{[0,T)} := \{u \in \mathrm{L}^2([0,T);U) \,|\, \mathbb{B}\tau u \in \mathrm{L}^2([0,T);H),\ \mathbb{B}\tau^T u \in H\}, \tag{8.98}$$

$$\|u\|^2_{\widetilde{U}_{\beta}} := \|u\|_2^2 + \|\mathbb{B}\tau u\|^2_{\mathrm{L}^2_{\beta}}, \tag{8.99}$$

$$\|u\|^2_{\widetilde{U}_{[0,T)}} := \|u\|_2^2 + \|\mathbb{B}\tau u\|^2_{\mathrm{L}^2([0,T);H)} + \|\mathbb{B}\tau^T u\|^2_H, \tag{8.100}$$

$$\mathbb{D}u := (C\mathbb{B}\tau + D)u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};Y) \quad (u \in \widetilde{U}_{\mathrm{loc}}). \tag{8.101}$$

As for WPLSs, we extend $\mathbb{B}$ to $\mathcal{B}(\mathrm{L}^2_{\omega}(\mathbf{R};U),H_{-1})$ for those ω for which $\mathbb{B}$ is continuous; note that $\mathrm{L}^2_{\omega'}(\mathbf{R}_-;U) \underset{c}{\subset} \mathrm{L}^2_{\omega}$ for all $\omega' > \omega$. (Thus, $u \in \widetilde{U}_{\mathrm{loc}}$ implies that $\pi_- u \in \mathrm{L}^2_{\omega}$ for some such ω.)

Obviously, $\mathbb{D}$ is time-invariant and causal $\mathrm{Dom}(\mathbb{D}) \to \mathrm{L}^2_{\mathrm{loc}}$, and $\mathrm{W}^{1,2}_{\beta}(\mathbf{R};U) \to \mathcal{C}(\mathbf{R};H)$, although $\mathrm{L}^2_{\beta} \setminus \mathrm{Dom}(\mathbb{D})$ may be nonempty (and $\mathbb{D} \notin \mathrm{TIC}_{\infty}$ is possible; take, e.g., $U = H$, $C = I$, $B = A$ s.t. $\widehat{\mathbb{D}}(s) = \widehat{\mathbb{B}\tau}(s) = (s-A)^{-1}A = s(s-A)^{-1} - I$ is unbounded on each right half-plane (i.e., A must be non-analytic; see Section 9.4).

Now we list the basic properties of ELSs:

Lemma 8.6.3 *Let $x_0 \in H$ and $T \in \mathbf{R}_+$. Then*

(a1) $\mathbb{B}\tau \in \mathcal{B}(\widetilde{U}_{\beta}, \mathrm{L}^2_{\beta}(\mathbf{R};H))$, $\mathbb{D} \in \mathcal{B}(\widetilde{U}_{\beta}, \mathrm{L}^2_{\beta}(\mathbf{R};Y))$, *and* $\mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2(\mathbf{R}_+;Y)$ *for all* $u \in \mathcal{U}_{\mathrm{exp}}(x_0)$.

(b1) $\left[\frac{\mathbb{A}}{\mathbb{C}}\right] \in \mathrm{WPLS}_{\beta}(\{0\},H,Y)$ *and* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{A} & \mathbb{B}\tau \end{array}\right] \in \mathrm{WPLS}_{\beta}(U,H_{-1},H_{-1})$.

(b2) $\pi_+\mathbb{D}\pi_- = \mathbb{C}\mathbb{B}$ *on* $\widetilde{U}_{\beta}$.

(c1) $\mathbb{B}\tau \in \mathcal{B}(\mathrm{W}^{n+1,2}_{\omega}(\mathbf{R};U), \mathrm{W}^{n,2}_{\omega}(\mathbf{R};H))$, *hence* $\mathbb{B}\tau : \mathrm{W}^{n+1,2}_{\omega}(\mathbf{R};U) \to \mathcal{C}^n(\mathbf{R};H)$, *and* $\|(\mathbb{B}\tau u)^{(n)}(t)\|_H \le M e^{\omega t}\|u\|_{\mathrm{W}^{n+1,2}}$.

(c2) $\mathbb{C}x_0 \in \mathcal{C}(\mathbf{R}_+;Y)$ *($\in \mathcal{C}^k(\mathbf{R}_+;Y)$ if $x_0 \in H_k := \mathrm{Dom}(A^k)$).*

(d) $\widehat{\mathbb{B}\tau u}(s) = (s-A)^{-1}B\widehat{u}(s)$, $\widehat{\mathbb{C}x_0}(s) = C(s-A)^{-1}x_0$, *and* $\widehat{\mathbb{D}u}(s) = (D + C(s-A)^{-1}B)\widehat{u}(s)$ *(if $\mathbb{B}\tau u \in \mathrm{L}^2_{\mathrm{loc}}$) for all $s \in \mathbf{C}^+_{\omega_A}$, and $u \in \mathrm{L}^2_{\omega}(\mathbf{R}_+;U)$, $\omega < \mathrm{Re}\, s$.*

(e) When $\omega > \mathrm{Re}\, s > \omega_A$ and $u_0 \in U$, we have

$$\mathbb{B}\tau\pi_+ e^{s\cdot}u_0 = (e^{s\cdot} - \mathbb{A})(s-A)^{-1}Bu_0 \in \mathcal{C}(\mathbf{R}_+;H) \cap \mathrm{L}^2_{\omega}, \tag{8.102}$$

$$\mathbb{D}\pi_+ e^{s\cdot}u_0 = e^{s\cdot}\widehat{\mathbb{D}}(s)u_0 - \mathbb{C}(s-A)^{-1}Bu_0 \in \mathcal{C}(\mathbf{R}_+;Y) \cap \mathrm{L}^2_{\omega}. \tag{8.103}$$

In particular, $\pi_+ e^{s\cdot}u_0 \in \widetilde{U}_{[0,T)}$.

(f) $\pi_{[0,T)}\mathcal{U}^{\mathcal{C}}_{\mathrm{exp}}(x_0) \subset \widetilde{U}_{[0,T)} \underset{c}{\subset} \widetilde{U}_{\beta} \subset \widetilde{U}_{\mathrm{loc}}$ *and* $\widetilde{U}_{[0,T)}$ *and* $\widetilde{U}_{\beta}$ *are Hilbert spaces.*

Proof: (a1) Obviously, $\mathbb{B}\tau \in \mathcal{B}$; hence $\mathbb{D} := C\mathbb{B}\tau + D \in \mathcal{B}$. The last claim is also obvious.

(b1) This was noted above.

(b2) Now $(\pi_+\mathbb{D}\pi_- u)(t) = \pi_+ C\mathbb{B}\tau^t\pi_- u = \pi_+ C\mathbb{A}^t\mathbb{B}u = (\mathbb{C}\mathbb{B}u)(t)$, by 2. and 3. of Definition 6.1.1.

(c1) This follows from Theorem 6.2.13(c2) and Lemma 6.3.19 (because now $(H_{-1})_B = H$, since $B \in \mathcal{B}(U, H_{-1})$, i.e., B is bounded to H_{-1}).

(c2) This follows from Theorem 6.2.13(c1).

(d) This follows from (b1) and the fact that $\mathbb{D} := C\mathbb{B}\tau + D$.

(e) This follows from (b1), Lemma 6.2.10, and the fact that $\mathbb{D} := C\mathbb{B}\tau + D$.

(f) Let $x_0 \in H$ and $T \geq 0$. Because $\mathbb{A}x_0 \in \mathcal{C} \cap \mathrm{L}^2_\beta$, we have $\pi_{[0,T)}\,\mathcal{U}^{\mathcal{C}}_{\exp}(x_0) \subset \widetilde{U}_{[0,T)}$. Obviously, $\widetilde{U}_\beta \subset \widetilde{U}_{\mathrm{loc}}$.

Assume that $u \in \widetilde{U}_{[0,T)}$. Set $x(t) := \mathbb{B}^t u = \mathbb{B}^t u$ $(t \geq 0)$. Then

$$x(T+t) = \mathbb{B}\tau^{T+t}u = \mathbb{B}\tau^t\pi_-\tau^T u = \mathbb{A}^t\mathbb{B}\tau^T u = \mathbb{A}x(T) \quad (t \geq 0). \tag{8.104}$$

hence $\|x(T+\cdot)\|_{\mathrm{L}^2_\beta} \leq M\|x(T)\|_H$, where $M := \|\mathbb{A}\|_{\mathcal{B}(H,\mathrm{L}^2_\beta(\mathbf{R};H))}$. Therefore, $\widetilde{U}_{[0,T)} \underset{c}{\subset} \widetilde{U}_\beta$. $\square$

The fact that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ and $\left[\begin{array}{c} \mathbb{A} \\ \hline \mathbb{C} \end{array}\right]$ are WPLSs offers us a wide range of additional useful facts. Note that if $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ generate a WPLS (i.e., $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2([-\varepsilon,0);U),H)$, $\mathbb{D} \in \mathcal{B}(\mathrm{L}^2([0,\varepsilon);U),\mathrm{L}^2([0,\varepsilon);Y))$ for some $\varepsilon > 0$), then all above definitions coincide with corresponding WPLS definitions.

Remark 8.6.4 *Substituting $X \mapsto H$, $U \mapsto \widetilde{U}_\beta$, $Y^{\mathrm{s}}, Z^{\mathrm{s}} \mapsto \mathrm{L}^2$, $Y, Z \mapsto \mathrm{L}^2_\beta$ and $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \mapsto \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}\tau \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$, (cf. Remark 8.3.4), we obtain that $\mathcal{U} = \mathcal{U}_{\exp}$ and that the results of Sections 8.1 and 8.2 become applicable.* $\square$

For example, the finite cost condition together with J-coercivity imply the existence of a unique J-critical control, and $\mathcal{J}(0,\cdot) \geq 0$ implies that a J-critical control is minimizing. Also most of Section 8.3 and much of further WPLS theory remain valid.

To give some examples, we mention explicitly below two more results from Sections 8.1 and 8.2:

Theorem 8.6.5 (LQR) *Assume that $\mathcal{U}_{\exp}(x_0) \neq 0$ for all $x_0 \in H$. Let $C = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] \in \mathcal{B}(H, H \times U)$, $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, $J = \left[\begin{smallmatrix} Q & 0 \\ 0 & S \end{smallmatrix}\right] \in \mathcal{B}(H \times U)$, $0 \ll Q \in \mathcal{B}(H)$, $0 \ll S \in \mathcal{B}(U)$.*

Then, for each $x_0 \in H$, there is a unique J-critical control, and this control is strictly minimizing over $\mathcal{U}_{\exp}(x_0)$. $\square$

(This follows from Remark 8.6.4 and Corollary 8.2.7.) See Theorem 8.6.6 and Remark 8.6.7 for corresponding closed-loop systems and Riccati equations.

The assumptions correspond to $D^*JC = 0$, $C^*JC = Q \gg 0$, $D^*JD = S \gg 0$, and the corresponding cost is given by $\mathcal{J}(x_0,u) = \langle x, Qx\rangle_{\mathrm{L}^2} + \langle u, Su\rangle_{\mathrm{L}^2}$. Thus, standard LQR problems are covered; for the indefinite case and H^∞ problems we need slight changes.

The ELS variant of Theorem 8.3.9 takes the following form:

Theorem 8.6.6 ($\boldsymbol{\Sigma_{\mathrm{crit}}}$) *Assume that there is a unique J-critical control $u_{\mathrm{crit}}(x_0)$ over $\mathcal{U}_*^*$ for each $x_0 \in H$. Define*

$$\Sigma_{\mathrm{crit}} := \left[\begin{array}{c|c} \mathbb{A}_{\mathrm{crit}} & \\ \hline \mathbb{C}_{\mathrm{crit}} & \\ \mathbb{K}_{\mathrm{crit}} & \end{array}\right] : x_0 \mapsto \left[\begin{array}{c|c} x_{\mathrm{crit}}(x_0) & \\ \hline y_{\mathrm{crit}}(x_0) & \\ u_{\mathrm{crit}}(x_0) & \end{array}\right] := \left[\begin{array}{c|c} \mathbb{A}x_0 + \mathbb{B}\tau u_{\mathrm{crit}}(x_0) & \\ \hline \mathbb{C}x_0 + \mathbb{D}u_{\mathrm{crit}}(x_0) & \\ u_{\mathrm{crit}}(x_0) & \end{array}\right]. \tag{8.105}$$

Then the following hold:

(a) We have $\Sigma_{\mathrm{crit}} \in \mathrm{WPLS}_0^{-1}(\{0\}, H_{-1}, Y \times U)$ and $\Sigma_{\mathrm{crit}} \in \mathcal{B}(H, \mathrm{L}^2(\mathbf{R}_+;H) \times \mathrm{L}^2(\mathbf{R}_+;Y) \times \mathrm{L}^2(\mathbf{R}_+;U))$.

(b) By setting $\mathcal{P} := \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}_{\mathrm{crit}} \in \mathcal{B}(H)$ we obtain

$$\mathcal{J}(x_0, u_{\mathrm{crit}}(x_0) + \eta) = \langle x_0, \mathcal{P}x_0 \rangle_H + \mathcal{J}(0, \eta) \quad (x_0 \in H,\ \eta \in \mathcal{U}_{\mathrm{exp}}(0)). \tag{8.106}$$

(c) $\mathbb{C}_{\mathrm{crit}} = C\mathbb{A}_{\mathrm{crit}} + D\mathbb{K}_{\mathrm{crit}}$.

(d) If $x_{\mathrm{crit}}(x_0) \in \mathcal{C}(\mathbf{R}_+;H)$ (equivalently, $u_{\mathrm{crit}}(x_0) \in \mathcal{U}_{\mathrm{exp}}^{\mathcal{C}}(x_0)$) for all $x_0 \in H$, then $\Sigma_{\mathrm{crit}} \in \mathrm{WPLS}_{-\varepsilon}(\{0\}, H, Y \times U)$ for some $\varepsilon > 0$, $A_{\mathrm{crit}} = A + BK_{\mathrm{crit}}$ and $C_{\mathrm{crit}} = C + DK_{\mathrm{crit}}$ on $\mathrm{Dom}(A_{\mathrm{crit}})$, hence then $\mathrm{Dom}(A_{\mathrm{crit}}) \subset H_B$, where $\left[\begin{smallmatrix} A_{\mathrm{crit}} \\ \hline C_{\mathrm{crit}} \\ K_{\mathrm{crit}} \end{smallmatrix}\right]$ are the generators of Σ_{crit}.

Here $\mathrm{WPLS}_\omega^{-1}$ is defined as WPLS_ω, except that the semigroup $\mathbb{A}$ needs to be strongly continuous (i.e., "C_0") in H_{-1} norm only, i.e., $\|\mathbb{A}x_0 - x_0\|_{H_{-1}} \to 0$ for all $x_0 \in H$.

However, from the fact that $\mathbb{A}_{\mathrm{crit}}x_0 \in \mathrm{L}^2(\mathbf{R}_+;H) \cap \mathcal{C}(\mathbf{R}_+;H_{-1})$ one can usually derive the fact that actually $\mathbb{A}_{\mathrm{crit}}x_0 \in \mathcal{C}(R_+;H)$, i.e., that (d) applies; see Examples 1.4.1–1.4.4 of [LT93] for details.

Proof of Theorem 8.6.6: By Corollary 8.1.10 (and Remark 8.6.4), $\Sigma_{\mathrm{crit}} \in \mathcal{B}(H, \mathrm{L}^2)$.

Also the rest of the proof goes as the proof of Theorem 8.3.9 (use Lemma 8.6.3) except (c), which obvious, and (d), which is given below.

(d) Assume that $x_{\mathrm{crit}}(x_0) \in \mathcal{C}(\mathbf{R}_+;H)$ for all $x_0 \in H$, equivalently, that $\mathbb{A}_{\mathrm{crit}}$ is strongly continuous. Then $\mathbb{A}_{\mathrm{crit}}$ is a C_0-semigroup. Now the exponential stability of $\mathbb{A}_{\mathrm{crit}}$ and the rest of (c) follows as in the proof of Theorem 8.3.9. □

Having given the above basic theory, we make two remarks on natural extensions:

Remark 8.6.7 *It seems that most of the Riccati equation theory "on $\mathrm{Dom}(A_{\mathrm{crit}})$", as given in Section 9.7, can be extended to this more general setting. (The operators $\mathbb{B}^t$ and $\mathbb{D}^t$ are defined on $\widetilde{U}_{[0,T)}$ and the equations only need to hold on $\widetilde{U}_{[0,T)}$.) Moreover, one can go on to derive "extended B_{w}^*-CARE theory", the ELS extension of the theory of Section 9.2 under similar assumptions.*

Finally, by using Remark 8.6.4, one can write ELS counterparts for several of the optimization results of Chapters 10 and 11 (at least). In particular, instead of the special (standard LQR) cost function of Theorem 8.6.5, we can use any J-coercive cost function, over, e.g., $\mathcal{U}_{\mathrm{out}}$ too. □

(The original proofs will do, mutatis mutandis.)

Remark 8.6.8 (More general systems than ELSs) *Instead of Hypothesis 8.6.1, the theory of this section can be adapted for general well-posed $\mathbb{C}$ (i.e., s.t. $\left[\frac{\mathbb{A}}{\mathbb{C}}\right] \in \mathrm{WPLS}$) and for general unbounded B (e.g., $B \in \mathcal{B}(U, H_{-827})$).* □

(We omit the details; the methods are roughly the same.)

Notes

Some special cases of the theory are given in [LT93], by Irena Lasiecka and Roberto Triggiani, who treats the standard LQR problem and present examples of physical ELSs that are not WPLSs. Luciano Pandolfi [Pandolfi] has treated minimization of a more general cost function, although under a special coercivity condition and the assumptions that $\mathbb{A}$ is analytic with the unboundedness of B being less than 1 ("$\beta > -1$") and (A, B) having a bounded exponentially stabilizing state feedback operator.

Chapter 9

Riccati Equations and J-Critical Control

Jacopo Francesco, Count Riccati, born at Venice on May 28, 1676, and died at Trèves on April 15, 1754, did a great deal to disseminate a knowledge of the Newtonian philosophy in Italy. Besides the equation known by his name, certain cases of which he succeeded in integrating, he discussed the question of the possibility of lowering the order of a given differential equation.

— 'A Short Account of the History of Mathematics' (4th edition, 1908) by W. W. Rouse Ball.

In this chapter, we shall establish the connection between optimal control and stabilizing solutions of Riccati equations (read "optimal" as "J-critical"). A summary of the main results is given on pp. 28–31, so here we only list the contents of each section.

Throughout this chapter, we shall assume that Standing Hypothesis 9.0.1 holds. Moreover, Standing Hypothesis 9.1.2 is assumed in Section 9.1, Standing Hypothesis 9.4.1 in Section 9.4, and Standing Hypothesis 9.5.1 in Sections 9.5 and 9.6.

In Section 9.1, we establish the equivalence mentioned on p. 9 of (I) J-critical control, (II) spectral or coprime factorization, and (III) stabilizing solutions of Riccati equations. Under further (e.g., MTIC type) regularity, (IV) the standard coercivity assumption is shown to be a fourth equivalent condition. Some further results in this direction are given in Sections 9.2 and 9.9.

In general, the connection between optimization and Riccati equation is trickier than in the case of a bounded input operator B: a J-critical control (when it exists) need not be given by a regular state feedback operator (nor by any well-posed state feedback). In Section 9.1, this difficulty is overcome by using the special classes of Theorem 8.4.9 (with partial results for the general case). In Section 9.2, we list several additional cases where this difficulty disappears, namely smoothing semigroups, bounded input maps or smooth I/O maps (see Hypothesis 9.2.2). We also the Riccati equation and corresponding results. A summary of most of our sufficient conditions is given in Remark

9.9.14. Applications to parabolic-type problems (i.e., to systems with an analytic semigroup) are given in Section 9.5.

A casual reader might be satisfied with the three sections mentioned above and skip most of the rest of the chapter, which contains a more general and hence necessarily less satisfactory theory and results on which the above is based.

In Section 9.7, we treat the most general case, where a unique J-critical control for any WR (Weakly Regular) system is shown to correspond to a solution of a *generalized Riccati equation* given on $\mathrm{Dom}(A_{\mathrm{crit}})$, the domain of the generator of the "(state-feedback controlled) closed-loop semigroup". We do not require the optimal control to be well-posed nor regular. This generalized Riccati equation is a rigorous extension of the Riccati equation of F. Flandoli et al. [FLT]. An integral version of the equation is given for arbitrary (even irregular) systems.

In Section 9.9, we show that for a WR system, there is a (well-posed) WR J-critical state feedback operator iff there is a "stabilizing" solution to the *extended Continuous-time Algebraic Riccati Equation (eCARE)*. In the general (possibly irregular) case the eCARE has to be replaced by the *extended Integral Algebraic Riccati Equation (eIARE)*, which also allows us to reduced several results to the substantially simpler discrete-time theory. The word "extended" corresponds to possibly noninvertible signature operators and is redundant under standard coercivity (and regularity) assumptions. Further results on Riccati equations are given in Sections 9.8–9.12.

In Section 9.13, we present examples that illustrate various pathological cases, including those mentioned above. In Section 9.14, we show that the J-critical control is given by (well-posed) state feedback iff certain factorization condition is satisfied, and we use this to extend a part of the "H^2 (generalized) canonical factorization theory" of [CG81] and [LS] to maps with infinite-dimensional input and output spaces. Positive Riccati equations will be treated in Sections 10.6 and 10.7.

A reader interested mainly in results can read the sections linearly. A more technically oriented reader may wish to read Sections 9.7 and 9.8–9.11 before 9.2–9.6, 9.1, 9.12, 9.13 and 9.14, although a brief glance at Section 9.1 before the start might nevertheless be a good idea. The reader wishing to verify all proofs rigorously may follow the order described in the proof of Theorem 14.1.3.

The results of this chapter hold when we optimize under any decent restrictions on the stability of the input, state, output and/or additional output:

Standing Hypothesis 9.0.1 *Throughout this chapter and Chapter 10, we assume that* $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $J = J^* \in \mathcal{B}(Y)$*. The letters* U*,* H*,* Y *and* Z *denote Hilbert spaces of arbitrary dimensions.*

We also assume that $\begin{bmatrix}\mathbb{Q} & \mathbb{R}\end{bmatrix}$*,* Z^{u} *and* Z^{s} *are as in Definition 8.3.2,* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{Q} & \mathbb{R}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,\widetilde{Y})$ *for some Hilbert space* $\widetilde{Y}$*, and that* $\pi_+\tau^t z \in Z^{\mathrm{s}} \Leftrightarrow z \in Z^{\mathrm{s}}$ $(z \in Z^{\mathrm{u}},\ t > 0)$.

The reader may ignore the latter paragraph of the hypothesis and read $\mathcal{U}_*^*$ as any of $\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}, \mathcal{U}_{\mathrm{exp}}$ (see Definition 8.3.2), since these obviously satisfy the latter assumption (with $\widetilde{Y} = H$, because $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{A} & \mathbb{B}\tau\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,H)$ and $x :=$

$\mathbb{A}x_0 + \mathbb{B}\tau u \in \mathcal{C}(\mathbf{R}_+;H) \subset \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;H)$, by Theorem 6.2.13(a1)). Sometimes we also require that Z^{s} is reflexive; this is satisfied by $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{exp}}$.

In fact, if $Z^{\mathrm{u}} \subset \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;\widetilde{Y})$, then $Z^{\mathrm{s}} = \mathrm{L}^p_\omega$ will do for any $p \in [1,\infty]$, $\omega \in \mathbf{R}$; and if $Z^{\mathrm{u}} \subset \mathcal{C}(\mathbf{R}_+;\widetilde{Y})$, then $Z^{\mathrm{s}} = \mathrm{e}^{-\omega\cdot}\mathcal{C}_{\mathrm{b}}$ or $Z^{\mathrm{s}} = \mathrm{e}^{-\omega\cdot}\mathcal{C}_0$ will do for any $\omega \in \mathbf{R}$ (assuming that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{Q} & \mathbb{R} \end{array}\right] \in \mathrm{WPLS}(U,H,\widetilde{Y})$).

By capital letters we again denote the generators or feedthrough operators of integral maps (see Lemma 6.1.16 and Definition 6.2.3).

9.1 The Riccati Equation: A summary for $\mathcal{U}_{\text{out}}$ (r.c.f.↔CARE)

It is not your fault, it is the fault of the mathematics.
— George Weiss, about the CARE at MTNS'98

In this section, we show that, for a regular strongly q.r.c.-stabilizable systems, the conditions (I)–(III) below are equivalent:

(I) the existence of regular optimal state feedback,

(II) the existence of a regular $(J,*)$-inner coprime factorization,

(III) the existence of a stabilizing solution of the Riccati equation,

(IV) the J-coercivity of the I/O map

(see, e.g., Theorem 9.1.7 for details). Moreover, we show that from the solution of the Riccati equation one can compute the optimal cost, feedback and factorization ("(III)↦(I),(II)"). Similarly, we also give formulae (I)↦(II),(III) as well as (II)↦(I),(III). Furthermore, the solution of (III) and the optimal state feedback operator are shown to be unique.

The standard coercivity assumption (IV) is necessary for (I)–(III). As noted in Section 8.4, (IV) is not sufficient in general, but for sufficiently regular systems we have the equivalence (I)–(IV), as stated in Corollary 9.1.11(iv) and Corollary 9.1.12(iv').

Condition (I) refers to a J-critical state feedback for the generalized optimal control problem of Section 8.3 (or Definition 9.1.3); see Chapters 10–12 for applications, such as LQR, H^2 and H^∞ control problems.

Thus, this is a summary of one aspect of the continuous-time algebraic Riccati equation (CARE) theory of this chapter, featuring the equivalence "(I)–(III)" on page 9 under various assumptions. The reader may wish to delay the verification of the proofs till the end of this chapter.

For strongly stable systems, the above requirements become simpler, as shown in Corollary 9.1.9, and the factorization condition (II) holds iff the Popov function has a regular spectral factorization.

Conditions (I)–(III) are simplified also in Proposition 9.1.15, assuming that the semigroup is smoothing, the input operator is bounded or the I/O map is smooth. The Riccati equation is compared to those existing in the literature in Remark 9.1.14. In Corollary 9.1.13 we solve the classical problem of finding a $(J,*)$-inner coprime factorization for a given I/O map by solving the Riccati equation corresponding to a realization of the map.

All of the above refers to the rather complicated case $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$, which shall be the subject of this section, but we first take a look at the simpler case of $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, where a solution of the CARE is J-critical iff it is exponentially stabilizing:

Theorem 9.1.1 *Let Σ be WR. Then the following are equivalent:*

(i) There is a J-critical WR state feedback operator K for Σ over $\mathcal{U}_{\text{exp}}$.

(ii) The eCARE (9.110) has an exponentially stabilizing solution $\mathcal{P} = \mathcal{P}^*$.

Moreover, if (ii) holds, then $\mathcal{P}$ is the J-critical cost operator, hence unique, K is a WR state feedback operator K for Σ, the J-critical control is given by the state feedback $u = K_{\rm L,s}x$, and Theorem 9.9.1 applies. □

(This follows from Corollary 9.9.2; in the coercive case the eCARE is reduced to the CARE. However, both (i) and (ii) may be false even for a coercive system.)

Recall also from Lemma 8.3.3 that $\mathcal{U}_{\rm exp} = \mathcal{U}_{\rm out}$ when the system is exponentially stable or exponentially detectable. In Sections 9.9 and 9.2, we give analogous results further results for $\mathcal{U}_{\rm exp}$ and other $\mathcal{U}_*^*$'s, but for a general treatment of the equivalence of "(I)–(IV)", we use $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$. Therefore, we make the following assumption:

Standing Hypothesis 9.1.2 *Throughout the rest of this section, we assume that $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$.*

This makes the identification of the J-critical solution of the CARE more complicated than in the case $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$ illustrated above. Instead of exponential stabilization, we should now check that the controlled (closed-loop) semigroup is output-stable and satisfies the condition (PB) in order to know that the control corresponding to a solution of the CARE truely optimizes over all $u \in \mathcal{U}_{\rm out}$, i.e., over all stable controls ($u \in \mathrm{L}^2$) that make the output stable ($y := \mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$). (In case $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$ it suffices to verify that the solution is exponentially stabilizing, as noted above.)

To avoid the verification of (PB), we (mostly) assume that the system is strongly stable or strongly right-coprime stabilizable. But before we go into this, we recall some definitions by simplifying special cases of Definitions 8.3.2, 6.6.10 and 9.8.1.

A state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is called J-critical (i.e., optimal) if the resulting (closed-loop) contorl $u = \mathbb{K}_\circlearrowleft x_0$ is J-critical for each initial state $x_0 \in H$:

Definition 9.1.3 (Critical control) *Set $\mathcal{U}_{\rm out}(0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,|\, \mathbb{D}u \in \mathrm{L}^2\}$.*

We call a control $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ J-critical *for $x_0 \in H$ (and Σ) if $y := \mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$, and $\langle \mathbb{D}\eta, Jy\rangle_{\mathrm{L}^2} = 0$ for all $\eta \in \mathcal{U}_{\rm out}(0)$.*

Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be an admissible state feedback pair for Σ, and set[1]

$$\mathbb{X} := I - \mathbb{F}, \qquad \mathbb{M} := \mathbb{X}^{-1} = \mathbb{F}_\circlearrowleft + I, \qquad \mathbb{N} := \mathbb{D}\mathbb{M} = \mathbb{D}_\circlearrowleft; \tag{9.1}$$

$$\Sigma_{\rm ext} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F} \end{array}\right], \quad \Sigma_\circlearrowleft := \left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right] := \left[\begin{array}{c|c} \mathbb{A} + \mathbb{B}\tau\mathbb{M}\mathbb{K} & \mathbb{B}\mathbb{M} \\ \hline \mathbb{C} + \mathbb{D}\mathbb{M}\mathbb{K} & \mathbb{D}\mathbb{M} \\ \mathbb{M}\mathbb{K} & \mathbb{M} - I \end{array}\right] \tag{9.2}$$

(so that $\Sigma_\circlearrowleft = (\Sigma_{\rm ext})_{[0\ I]}$ is the corresponding closed-loop system; cf. Figure 9.1).

We call $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ J-critical *for Σ if $u = \mathbb{K}_\circlearrowleft x_0$ is J-critical for each $x_0 \in H$ (and Σ). In this case, we call the equation $u = \mathbb{K}_\circlearrowleft x_0$ the* J-critical control in the feedback form.

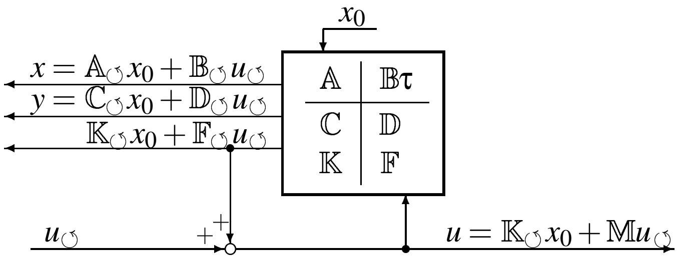

Figure 9.1: State feedback connection

(See Definition 8.3.2 for the general case.)

If $\mathbb{F}$ is weakly regular (WR), then feedback $\mathbb{K}x_0 + \mathbb{F}u$ can be written as $K_{\mathrm{w}}x(t) + Fu(t)$ for a.e. $t \geq 0$; we remind the reader that if the feedthrough is zero ($F = 0$), then K (or its weak Weiss extension K_{w}) is called a state feedback operator:

Definition 9.1.4 (J-critical K) *We call $K \in \mathcal{B}(H_1, U)$ a* (WR) admissible state feedback operator *if $\left[\, K \;\middle|\; 0 \,\right]$ generates a WR admissible state feedback pair $\left[\, \mathbb{K} \;\middle|\; \mathbb{F} \,\right]$ for Σ. We call K* stabilizing *(resp.* J-critical, r.c.-stabilizing, stable, UR, ...*) if $\left[\, \mathbb{K} \;\middle|\; \mathbb{F} \,\right]$ is stabilizing (resp. J-critical, r.c.-stabilizing, stable, UR, ...) (see Definition 6.6.10).*

(See Definition 8.3.2 for the general $\mathcal{U}_*^*$'s in place of $\mathcal{U}_{\mathrm{out}}$.) Thus, $K \in \mathcal{B}(H_1, U)$ is admissible iff $\left[\begin{smallmatrix} A & B \\ K & 0 \end{smallmatrix}\right]$ generate a WR WPLS $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{K} & \mathbb{F} \end{smallmatrix}\right]$ with $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty$ (cf. Lemma 6.3.13), or equivalently, iff $H_B \subset \mathrm{Dom}(K_{\mathrm{w}})$ and the feedback $u = K_{\mathrm{w}}x + u_\circlearrowleft$ is admissible (see Proposition 6.2.8(a2), Theorem 6.2.13(a1) and Definition 6.6.10; for ULR K, admissibility is redundant, by Proposition 6.3.1(c)).

If this is the case, then $\left[\begin{smallmatrix} A & B \\ C & D \\ K & 0 \end{smallmatrix}\right]$ [$\left[\begin{smallmatrix} A+BK_{\mathrm{w}} & B \\ C+DK_{\mathrm{w}} & D \\ K_{\mathrm{w}} & 0 \end{smallmatrix}\right]$] are [compatible] generating operators of Σ_{ext} [$\Sigma_\circlearrowleft$] (see Proposition 6.6.18(d2) and recall that we denote the generating operators of maps by the corresponding capital letters, e.g., $M := \widehat{\mathbb{M}}(+\infty)$).

Now we can define the CAREs. As noted in Remark 9.1.6, their admissible solutions lead to WR state feedback operators, and such operators are J-critical iff they are $\mathcal{U}_{\mathrm{out}}$-stabilizing, or equivalently, q.r.c.-stabilizing, provided that Σ is strongly q.r.c.-stabilizable. These things will soon be clarified.

Definition 9.1.5 (CARE) *We call $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ (or $(\mathcal{P}, S, K)$) a* solution *(of the Continuous-time Algebraic Riccati Equation (CARE) induced by Σ and J) iff Σ is WR and $\mathcal{P}$ satisfies*

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC & \text{on } \mathrm{Dom}(A)\,, \\ S = D^*JD + \operatorname*{w-lim}_{s \to +\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B & \text{on } U\,, \\ K = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) & \text{on } \mathrm{Dom}(A)\,, \end{cases} \tag{9.3}$$

[1]In this chapter, we usually denote the closed-loop system corresponding to $\left[\, \mathbb{K} \;\middle|\; \mathbb{F} \,\right]$ by $\Sigma_\circlearrowleft$ (instead of $\Sigma_\flat$), and leave $\Sigma_\flat$ for preliminarily closed systems (as in Theorem 9.1.10). In most applications, the second output of $\Sigma_\circlearrowleft$ equals the J-critical control (for $u_\circlearrowleft = 0$).

and $S = S^* \in \mathcal{GB}(U)$ (for some S and K).

A solution $\mathcal{P}$ of the CARE is called stabilizing *(resp.* J-critical, admissible, r.c.-stabilizing, stable, UR, ...*) if K is a stabilizing (resp. J-critical, admissible, r.c.-stabilizing, stable, UR, ...) state feedback operator for Σ.*

We call S the signature operator *corresponding to $\mathcal{P}$.*

(More general AREs will be formulated in Definition 9.8.1. See Proposition 9.1.15 for smooth cases where the w-lim term disappears, and recall that one often assumes that $D^*JD = I$ and $D^*JC = 0$ to obtain $S = I$, $K = -B_{\rm w}^*\mathcal{P}$.)

Note that, given $\mathcal{P}$, the operators S and K are uniquely determined and can be eliminated from the first equation. By Remark 9.1.6, we have $H_B \subset \mathrm{Dom}(K_{\rm w})$, hence any extended WPLS of Σ with generators $\left[\begin{array}{c|c} A & B \\ \hline C & \\ K & \end{array}\right]$ is WR, by Lemma 9.11.5(b).

Thus, a solution $\mathcal{P}$ is admissible iff the operators $\mathbb{K} : x_0 \mapsto K\mathbb{A}(\cdot)x_0$ and $\mathbb{F} := K_{\rm w}\mathbb{B}\tau$ extend Σ to another WPLS (see Lemma 6.3.13 and Remark 9.1.6) satisfying $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty$ (if $\mathbb{F} \in$ ULR, then the latter condition is redundant, by Proposition 6.3.1(c)). In Remark 9.8.3 we give the necessary and sufficient conditions in details. However, in many applications K is bounded ($K \in \mathcal{B}(H,U)$), hence necessarily ULR and admissible, by Lemma 6.6.11 (see also Lemma 6.3.17).

In (9.3), as elsewhere, the weak limit of an operator function $s \mapsto \mathbb{T}(s) \in \mathcal{B}(U)$ means the map $u_0 \mapsto \text{w-lim}\,\mathbb{T}(s)u_0$ (see Lemma A.3.1(h)), whereas "lim" refers to limit in $\mathcal{B}(U)$, i.e., to a uniform limit. Note that we require this weak limit to exist (and be self-adjoint, equivalently, that $S = S^*$); such a weak limit is necessary bounded, by Lemma A.3.1(h3).

The equations of the CARE (9.3) are given in $\mathcal{B}(H_1, H_{-1}^*)$, $\mathcal{B}(U)$ and $\mathcal{B}(H_1, U)$, respectively; e.g., the first one is equivalent to

$$\langle Ax_0, \mathcal{P}x_1\rangle_H + \langle x_0, \mathcal{P}Ax_1\rangle_H = -\langle Cx_0, JCx_1\rangle_Y + \langle Kx_0, SKx_1\rangle_U \quad \text{for all } x_0, x_1 \in \mathrm{Dom}(A) \tag{9.4}$$

(by Lemma A.3.1(g3), it is enough to verify this for $x_1 = x_0$).

Note that CAREs cover the Riccati equations presented earlier for WPLSs in [WW, Theorem 12.8], [S98b, Theorem 6.1(v)], [Mik97b] and [Mik98]. To get an example where $S \neq D^*JD$, apply Corollary 9.1.12 to $I = \tau(-1)^*I\tau(-1)$ (note that $\widehat{\tau(-1)}(+\infty) = 0$); see Example 9.8.15 for details.

The weak regularity of K (see Proposition 6.2.8(a2)) is inherent in the CARE:

Remark 9.1.6 (Implicit assumptions of the CARE) *The third equation of the CARE requires that $\mathcal{P}\mathrm{Dom}(A) \subset \mathrm{Dom}(B_{\rm w}^*)$ and the second equation that $\mathcal{P}(\alpha - A)^{-1}Bu_0 \in \mathrm{Dom}(B_{\rm w}^*)$.*

These hold iff $\mathcal{P}H_B \subset \mathrm{Dom}(B_{\rm w}^*)$, equivalently, iff $\mathcal{P} \in \mathcal{B}(H_B, \mathrm{Dom}(B_{\rm w}^*))$.

It follows that $B_{\rm w}^*\mathcal{P} \in \mathcal{B}(H_B, U)$, $B_{\rm w}^*\mathcal{P}(\cdot - A)^{-1} \in \mathrm{H}(\mathbf{C}^+_{\omega_A}; \mathcal{B}(H,U))$, and $B_{\rm w}^*\mathcal{P}(\cdot - A)^{-1}B \in \mathrm{H}(\mathbf{C}^+_{\omega_A}; \mathcal{B}(U))$. This implies that $K_{\rm w} \in \mathcal{B}(H_B, U)$ and hence $K \in \mathcal{B}(H_1, U)$.

Note also that we have required S to be invertible (cf. Definition 9.8.1) and $\mathcal{P}$ and S to be self-adjoint.

If B is bounded, then any $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ is SR (see Definition 9.1.5) and $S = D^*JD = S^*$; in particular, the above implicit requirements mentioned in Theorem 9.1.6 are satisfied, see Proposition 9.1.15.

Proof: In order to have the term $B_{\mathrm{w}}^*\mathcal{P}$ well-defined, we must require $\mathcal{P}\mathrm{Dom}(A) \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$, and for each $u_0 \in U$ there must exist $\alpha > \omega_A$ s.t. $\mathcal{P}(\alpha - A)^{-1}Bu_0 \in \mathrm{Dom}(B_{\mathrm{w}}^*)$. The latter implies that $\mathcal{P}(\alpha - A)^{-1}BU \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ for all $\alpha \in \sigma(A)^c$, by the resolvent equation and the fact that $\mathcal{P}\mathrm{Dom}(A) \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$. Combining these two inclusions, we get $\mathcal{P}H_B \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$, hence $\mathcal{P} \in \mathcal{B}(H_B, \mathrm{Dom}(B_{\mathrm{w}}^*))$, by Lemma A.3.6 and Proposition 6.2.8(b1). The converse implications are trivial.

Now $\widehat{\mathbb{H}} := B_{\mathrm{w}}^*\mathcal{P}(\cdot - A)^{-1}B \in \mathrm{H}(\mathbf{C}^+_{\omega_A}; \mathcal{B}(U))$, because $\widehat{\mathbb{H}}(s) - \widehat{\mathbb{H}}(s_0) = (s_0 - s)B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}(s_0 - A)^{-1}B \in \mathrm{H}(\mathbf{C}^+_{\omega_A}; \mathcal{B}(U))$, by the resolvent equation, Lemma A.4.4(a).

Because $H_B \subset \mathrm{Dom}(C_{\mathrm{w}})$ and $H_B \subset \mathrm{Dom}(B_{\mathrm{w}}^*\mathcal{P})$, we obtain $H_B \subset \mathrm{Dom}(K_{\mathrm{w}})$. This inclusion is continuous, by Lemma A.3.6, and $K_{\mathrm{w}} \in \mathcal{B}(\mathrm{Dom}(K_{\mathrm{w}}), U)$, by Lemma 6.2.8(b2). □

Thus, the CARE is well-defined only for $\mathcal{P} \in \mathcal{B}(H) \cap \mathcal{B}(H_B, \mathrm{Dom}(B_{\mathrm{w}}^*))$. This is not a restriction, because if a J-critical control can be given by a WR stabilizing state feedback operator K, then the *J-critical cost* $\mathcal{P} := \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ is in $\mathcal{B}(H) \cap \mathcal{B}(H_B, \mathrm{Dom}(B_{\mathrm{w}}^*))$, and $\mathcal{P}$ satisfies the CARE, by Theorem 9.9.1(a1) and Proposition 9.8.10.

Conversely, any strongly stabilizing solution of the CARE is unique and corresponds to control $u = K_{\mathrm{w}}x$ that is J-critical w.r.t. the closed-loop system; if we require the K to be strongly q.r.c.-stabilizing, then this control is J-critical w.r.t. the original system too. This is illustrated in the rest of this section; the remaining sections of this chapter study the (extended) CARE further.

Theorem 9.1.7 (r.c.f.⇔CARE) *Let Σ be WR and strongly q.r.c.-stabilizable. Then the following are equivalent:*

(i) There is a J-critical WR q.r.c.-stabilizing state feedback operator K for Σ.

(ii) The I/O-map $\mathbb{D}$ has a $(J,*)$-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ with $\mathbb{X} := \mathbb{M}^{-1}$ WR and $X = I$.

(iii) The CARE (9.3) has a q.r.c.-stabilizing solution $\mathcal{P} = \mathcal{P}^*$.

Moreover, the following holds for the (possible) solutions of (i)–(iii):

(a1) The solutions of (i)–(iii) are unique and correspond to each other as follows: K is the state feedback operator “K” of the CARE, $\mathbb{X} = I - \mathbb{F}$ (and $\mathbb{N} = \mathbb{D}\mathbb{M}$), where $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is the pair generated by $\begin{bmatrix} K \mid 0 \end{bmatrix}$; $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ and $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ are obtained from $\mathbb{N}$ and $\mathbb{M}$ as in Theorem 9.9.10(g).

(a2) The corresponding operators $\mathcal{P}$, $S = \mathbb{N}^* J \mathbb{N}$ and the pair $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ generated by $\begin{bmatrix} K \mid 0 \end{bmatrix}$ satisfy (Crit1)–(Crit4) Theorem 9.9.10; in particular, Theorem 9.9.10, Proposition 9.11.4 and Lemma 9.11.5 apply.

(a3) The J-critical SOS-stabilizing pairs for Σ are the ones generated by $\left[\begin{array}{c|c} EK & I-E \end{array}\right]$ ($E \in \mathcal{GB}(U)$); in particular, they are WR and strongly q.r.c.-stabilizing.

(b) The operators K and $\mathcal{P}$ are strongly q.r.c.-stabilizing.

(c) The critical control for $x_0 \in H$ is given by $u_{\text{crit}}(t) = \mathbb{K}_{\circlearrowleft} x_0 = K_{\text{w}} x(t)$ (a.e.), and the closed-loop cost function $\mathcal{J}_{\circlearrowleft}(x_0, u_{\circlearrowleft})$ for $y = \mathbb{C}_{\circlearrowleft} x_0 + \mathbb{D}_{\circlearrowleft} u_{\circlearrowleft}$, $u_{\circlearrowleft} \in \mathrm{L}^2(\mathbf{R}_+; U)$ is given by

$$\mathcal{J}_{\circlearrowleft}(x_0, u_{\circlearrowleft}) := \langle y, Jy \rangle_{\mathrm{L}^2(\mathbf{R}_+;Y)} = \langle x_0, \mathcal{P} x_0 \rangle_H + \langle u_{\circlearrowleft}, S u_{\circlearrowleft} \rangle_{\mathrm{L}^2(\mathbf{R}_+;U)}. \tag{9.5}$$

(d1) If $\mathbb{X} \in \mathrm{SR}$, then $\mathbb{M} \in \mathrm{SR}$, $\mathbb{N} \in \mathrm{WR}$ and $u_{\text{crit}}(t) = K_{\text{s}} x(t)$ (a.e.).

(d2) Assume that $\mathbb{D} \in \mathrm{SR}$. Then we may replace "WR" by "SR" (resp. "UR") in (i) and (ii) if the weak limit in the CARE is replaced by a strong (resp. uniform) limit.

In the uniform case, the requirement $X = I$ can be removed if we allow multiple solutions in (ii) ($\mathbb{D} = (\mathbb{N}E)(\mathbb{M}E)^{-1}$ where $E \in \mathcal{GB}(U)$); cf. (a1). Note also the implications

$$\mathbb{D}, \mathbb{X} \in \mathrm{SR} \Rightarrow \mathbb{N}, \mathbb{M} \in \mathrm{SR}, \quad \mathbb{X} \in \mathrm{SR} \Leftrightarrow \mathbb{M} \in \mathrm{SR}; \tag{9.6}$$

$$\mathbb{D}, \mathbb{X} \in \mathrm{UR} \Rightarrow \mathbb{N}, \mathbb{M} \in \mathrm{UR}, \quad \mathbb{X} \in \mathrm{UR} \Leftrightarrow \mathbb{M} \in \mathrm{UR}. \tag{9.7}$$

(d3) We have $\widehat{\mathbb{X}}(s) = I - K_{\text{w}}(s-A)^{-1}B$, and $\widehat{\mathbb{M}}(s) = I + K_{\text{w}}(s - A - BK_{\text{w}})^{-1}B$, and $\mathbb{K}_{\circlearrowleft} x_0 = K_{\text{w}} \mathbb{A}_{\circlearrowleft}(\cdot) x_0$ a.e.

(e) If (i) has a solution [with $S \gg 0$], then $\mathbb{D}$ is [positively] J-coercive.

The condition $X = I$ (equivalently, $X \in \mathcal{GB}(U)$; cf. Lemma 6.4.5(e)) is not restrictive (at least) when $\dim U < \infty$ or $\mathbb{X} \in \mathrm{UR}$; cf. (d2), Lemma 9.9.7(d) and Corollary 9.1.11.

All theorems and corollaries of this section are special cases of Theorem 9.1.7 (with further details or simplifications). See Section 10.2 for the positive case.

Proof: The equivalence (i)⇔(ii)⇔(iii) is the one in Theorem 9.9.10(d1); in particular, (a2) holds.

(a1)-(a3),(c) These follow from (a1)–(b) and (e1)&(f1) of Theorem 9.9.10, respectively.

(b) By Theorem 6.7.15(a2), any q.r.c.-SOS-stabilizing K is strongly q.r.c.-stabilizing.

(d1) We have $\mathbb{M} \in \mathrm{SR}$, by Proposition 6.3.1(a3), and $\mathbb{N} \in \mathrm{WR}$, by Lemma 6.2.5.

Set $x = \mathbb{A}_{\circlearrowleft} x_0$, so that $u_{\text{crit}} = \mathbb{K}_{\circlearrowleft} x_0 = (K_{\circlearrowleft})_{\text{s}} x$ a.e., by (6.46), and $(K_{\circlearrowleft})_{\text{s}} = K_{\text{s}}$, by Lemma 6.6.18(d3). Therefore, $u_{\text{crit}} = K_{\text{s}} x$ a.e.

(The J-critical control corresponds to $x = \mathbb{A}_{\circlearrowleft} x_0 + \mathbb{B}_{\circlearrowleft} \tau u_{\circlearrowleft}$ with $u_{\circlearrowleft} = 0$ (see Definition 9.1.4), although the formula $u = K_{\text{s}} x$ holds for arbitrary $u_{\circlearrowleft}$.)

(d2) The first claim follows from Lemma 9.11.5(e); the second from Proposition 6.3.1(b1) and Lemma 6.4.5(e) (the latter implies that the q.r.c.f. parametrization applies in WR and SR cases too if we replace "$X = I$" by "$X \in$

$\mathcal{GB}(U)$" in (ii)). The implications follow from (d1), the equation $\mathbb{N} = \mathbb{DM}$, and Lemma 6.2.5.

(d3) This follows from Proposition 6.6.18(d1).

(e) This follows from (Crit1) and Theorem 9.9.10(e2). □

From the above proof we note the following facts:

Remark 9.1.8 (Reduction of assumptions)

(a) We can remove the assumption in Theorem 9.1.7 that Σ is strongly q.r.c.-stabilizable if we add this condition as an additional requirement to condition (ii) and replace "q.r.c.-stabilizing" by "strongly q.r.c.-stabilizing" in (i) and (iii).

(b) Alternatively, we can use P-stabili[zabili]ty or P-SOS-stabili[zabili]ty (see Definition 9.8.1) instead of strong stabilizability (either in assumptions, as in the theorem, or in requirements, as in (a)) if we alter part (b) of the theorem accordingly.

Claims (a) and (b) applies also Theorem 9.1.10 and Corollary 9.1.11; claim (b) applies also Corollaries 9.1.9 and 9.1.12. □

On the other hand, Theorem 6.7.15 allows us often to reduce further the stabilization assumption; e.g., if Σ is exponentially q.r.c.-stabilizable, then any I/O-stabilizing (or input stabilizing) solution is exponentially q.r.c.-stabilizing. In the standard LQR problem for an exponentially q.r.c.-stabilizable or estimatable system, any nonnegative solution is a unique and minimizing, by Proposition 10.7.3(c3). See Sections 10.1 and 9.2 for further simplifications.

Theorem 9.9.1(a1) and Corollary 9.9.2 show the necessary and sufficient conditions for the existence of a J-critical state feedback pair in terms of solutions of Riccati equations, without additional stability or stabilizability assumptions.

If Σ is strongly stable, then an admissible solution $(\mathcal{P}, S, K)$ of the CARE is (strongly) r.c.-stabilizing if it is stable and stabilizing; in fact, it suffices that $\mathbb{K}$ is stable and $\mathbb{X} \in \mathcal{G}$TIC, by Proposition 9.8.11. Thus we get additional equivalent conditions in this case:

Corollary 9.1.9 (Stable Σ: SpF⇔CARE) *Let Σ be WR and strongly stable.*

Then the assumptions of Theorems 9.1.7 are satisfied, and each of the following conditions is equivalent to (i)–(iii) of Theorem 9.1.7

(i) There is a J-critical WR stable, stabilizing state feedback operator K for Σ.

(ii) The Popov operator $\mathbb{D}^* J \mathbb{D}$ has a WR spectral factorization $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* S \mathbb{X}$ with $X = I$.

(iii) The CARE (9.3) has a stable, stabilizing solution $\mathcal{P} = \mathcal{P}^*$.

Moreover, the solutions of (i)–(iii) solve the corresponding conditions of Theorem 9.1.7, and vice versa; in particular, (a)–(d) of that theorem apply, and "stable, stabilizing" is equivalent to "q.r.c.-stabilizing" (and to "stable and strongly r.c.-stabilizing") in (i) and (iii). Furthermore, *(a) If Σ is exponentially stable, then "stable, stabilizing" is equivalent to "exponentially stabilizing", and to "I/O-stabilizing", and to "$\mathbb{M}$-stabilizing".*

(b) If (iii) holds, then $\mathcal{P} = \mathbb{C}^*(J - J\mathbb{D}\pi_+(\pi_+\mathbb{D}^*J\mathbb{D}\pi_+)^{-1}\pi_+\mathbb{D}^*J)\mathbb{C}$.

Recall from Definition 6.4.4 that (ii) requires that $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ and $S \in \mathcal{G}\mathcal{B}(U)$. See Theorem 10.6.3 for the positive case ($S \gg 0$, or equivalently, $\mathbb{D}^*J\mathbb{D} \gg 0$).

Proof: Because Σ is stable, now a stabilizing state feedback pair is stable iff it is q.r.c.-stabilizing, by Lemma 6.6.17(a). Therefore, (i) and (iii) are equivalent to those of Theorem 9.1.7, in particular, any solutions are stable and strongly stabilizing, by (b) of the theorem.

Finally, (ii) is equivalent to (ii) of Theorem 9.1.7, by Lemma 6.4.8(b) (use the r.c.f. $\mathbb{D} = \mathbb{D}I^{-1}$).

(a) This follows from Proposition 9.8.11.

(b) This follows from Theorem 9.9.10(g1). □

In Theorem 9.1.7 we assumed the existence of a preliminary strongly stabilizing feedback pair. Assuming that this pair is regular, we have three more equivalent conditions, namely conditions (i)–(iii) for the preliminarily stabilized system:

Theorem 9.1.10 (SR stabilized Σ: r.c.f.⇔CARE)
Let Σ be SR and have a SR strongly q.r.c.-stabilizing state feedback operator K' (i.e., pair $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ with $F' = 0$). Let $\Sigma_\flat^1 := \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ be the two top rows of the corresponding SR strongly stable closed-loop system. Let $J = J^* \in \mathcal{B}(Y)$.

Then the following are equivalent:

(i) There is a J-critical SR q.r.c.-stabilizing state feedback operator K for Σ.

(ii) The I/O-map $\mathbb{D}$ has a $(J,*)$-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ with $\mathbb{M}$ SR and $M = I$.

(iii) The CARE (9.3) has a q.r.c.-stabilizing solution $\mathcal{P} = \mathcal{P}^*$ admitting a strong limit in the CARE.

($\mathrm{i}_\flat$) The J-critical control for $\Sigma_\flat^1$ can be given by a SR stable, stabilizing state feedback operator $K_\natural$.

($\mathrm{ii}_\flat$) The Popov operator $\mathbb{D}_\flat^*J\mathbb{D}_\flat$ has a SR spectral factorization $\mathbb{D}_\flat^*J\mathbb{D}_\flat = \mathbb{X}_\natural^*S\mathbb{X}_\natural$ with $X_\natural = I$.

($\mathrm{iii}_\flat$) The CARE

$$\begin{cases} K_\natural^*SK_\natural = A_\flat^*\mathcal{P} + \mathcal{P}A_\flat + C_\flat^*JC_\flat, \\ \qquad S = D^*JD + \operatorname*{s-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B \in \mathcal{G}\mathcal{B}(U), \\ \qquad K_\natural = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC_\flat). \end{cases} \tag{9.8}$$

(on $\mathrm{Dom}(A_\flat)$ for $\Sigma_\flat^1$ and J) has a stable, stabilizing solution $\mathcal{P} = \mathcal{P}^*$.

(Note that the first and third equation of (9.8) are given on $\mathrm{Dom}(A_\flat)$*, and that* $A_\flat = A + BK'_{\mathrm{s}}$ *and* $C_\flat = C + DK'_{\mathrm{s}}$ *on* $H_B \supset \mathrm{Dom}(A_\flat)$*, by Proposition 6.6.18(d3).)*

If some (hence all) of (i)–(iii$_\flat$) have a solution, then the following holds:

(a) The above assumptions and conditions (i)–(iii) are stronger than those of Theorem 9.1.7; in particular, (a1)–(d2) of that theorem apply for these solutions.

In particular, the solutions are unique and correspond to each other as in that theorem. (The corresponding claim holds also for (i$_\flat$)–(iii$_\flat$), by Corollary 9.1.9).

(b) The solutions of (i) and (i$_\flat$) are connected by $K_{\mathrm{s}} = K'_{\mathrm{s}} + (K_\natural)_{\mathrm{s}}$*, which holds on* $H_B = H_{B_\flat} = H_{B_\circlearrowleft}$

(The operators K_{s} *and* K'_{s} *are the strong Yosida extensions of* K *and* K' *w.r.t.* A*, and* $(K_\natural)_{\mathrm{s}}$ *is the strong Yosida extension of* $K_\natural$ *w.r.t.* $A_\flat$*. However, if one extends the restriction (to* $\mathrm{Dom}(A)$*,* $\mathrm{Dom}(A_\flat)$ *or* $A_\circlearrowleft$*) of any of these three (extended) operators w.r.t. any of* A*,* $A_\flat$ *and* $A_\circlearrowleft$*, then the extension coincides with the original one on* H_B*.)*

The solutions of (ii) and (ii$_\flat$) are connected by $\mathbb{M} = \mathbb{M}'\mathbb{X}_\natural^{-1}$*,* $\mathbb{N} = \mathbb{D}\mathbb{M}$*, where* $\mathbb{M}' := (I - \mathbb{F}')^{-1}$*,* $\mathbb{N}' := \mathbb{D}\mathbb{M}' = \mathbb{D}_\flat$*. The solutions of (iii) and (iii$_\flat$) are equal.*

Recall from Corollary 9.1.9, that "stable, stabilizing" in (iii$_\flat$) is equivalent to "q.r.c.-stabilizing".

Proof: We shall again use the implication $\mathbb{X} \in \mathrm{SR}$ & $X \in \mathcal{GB} \Rightarrow \mathbb{X}^{-1} \in \mathrm{SR}$ (from Proposition 6.3.1(a3)) and the fact that SR is closed under compositions. E.g., we note that $\Sigma_\flat$ is SR.

Conditions (i)–(iii) are equivalent, by Theorem 9.1.7(d2); likewise, conditions (i$_\flat$)–(iii$_\flat$) are equivalent, by Corollary 9.1.9 and Theorem 9.1.7(d2). In particular, (a) holds. Equivalence (ii)$\Leftrightarrow$(ii$_\flat$) follows from Lemma 6.4.8(b) (by its proof, $\mathbb{M} = \mathbb{M}'\mathbb{X}_\natural^{-1}$).

(a) This was noted above.

(b) The formula $\mathbb{M} = \mathbb{M}'\mathbb{X}_\natural^{-1}$ was noted above. By Theorem 9.9.10(g2), the unique solution of (i$_\flat$) corresponds to the same $\mathcal{P}$ as that of (i), hence also the solutions of (iii) and (iii$_\flat$) are identical. The rest of the third paragraph follows from these. The first and second paragraphs follow from Proposition 6.6.18(a1)&(e)&(f). $\square$

If, e.g., Σ is stable and $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$, then we get further equivalent conditions:

Corollary 9.1.11 (MTIC Σ: r.c.f.$\Leftrightarrow$CARE) *Let Hypothesis 8.4.7 hold for* $\widetilde{\mathcal{A}}(U)$*. Let* Σ *be strongly q.r.c.-stabilizable in* $\widetilde{\mathcal{A}}$*, i.e., let it have a strongly stabilizing state feedback pair* $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ *with* $\mathbb{N}', \mathbb{M}' \in \widetilde{\mathcal{A}}$ *and q.r.c., where* $\mathbb{M}' := (I - \mathbb{F}')^{-1} = I + \mathbb{F}_\flat$*,* $\mathbb{N}' := \mathbb{D}\mathbb{M}' = \mathbb{D}_\flat$*.*

Then each of the following conditions if equivalent to (i)–(iii) of Theorem 9.1.7 as well as to (i)–(iii$_\flat$) of Theorem 9.1.10:

(ii') $\mathbb{D}$ has a $(J,*)$-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$.

(ii$'_\flat$) The Popov operator $\mathbb{D}_\flat^* J\mathbb{D}_\flat$ has a spectral factorization $\mathbb{X}_\natural^* S\mathbb{X}_\natural$.

(iv) $\mathbb{D}$ is J-coercive.

(iv$_\flat$) $\mathbb{D}_\flat$ is J-coercive.

(iv$'_\flat$) The Popov Toeplitz operator $\pi_+\mathbb{D}_\flat^* J\mathbb{D}_\flat\pi_+$ is invertible in $\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))$.

Moreover, we have the following:

(a) The above equivalence holds even if the "SR" (resp. "s-lim") in any of (i)–(iii$_\flat$) of Theorem 9.1.10 are replaced by "WR" (resp. w-lim) or by "UR" (resp. lim).

(b) The assumptions of Theorems 9.1.7 and Theorem 9.1.10 are satisfied.

Assume that some (hence all) of (i)–(iv$_\flat$) holds. Then we have the following:

(c1) The solutions of (ii') (resp. (ii$'_\flat$)) are equal to the unique solution of (ii) (resp. (ii$_\flat$)) modulo a $\mathcal{GB}(U)$ operator.

Consequently, we necessarily have $\mathbb{D},\mathbb{F},\mathbb{X}\in$ ULR and $\mathbb{M},\mathbb{N},\mathbb{X}_\natural\in\widetilde{\mathcal{A}}\subset$ ULR.

(c2) Theorem 9.1.10 shows how the (unique) solutions of the other conditions relate to each other.

(d) If Hypothesis 8.4.8 holds for $\mathcal{A}(U)$, and $\mathbb{D}=\mathbb{N}\mathbb{M}^{-1}$ is a (J,S)-inner [q.]r.c.f., then $N^*JN=S$, i.e., $D^*JD=X^*SX$, where $\mathbb{X}:=\mathbb{M}^{-1}$.

Moreover, any $(J,*)$-inner-right factorization of $\mathbb{D}$ is a $(J,*)$-inner [q.]r.c.f.

Proof: Obviously, (b) holds. By Corollary 8.4.14(b3), conditions (ii'), (ii$'_\flat$), (iv) and (iv$_\flat$) are equivalent to each other and to (ii)'s of both theorems, and the solutions are necessarily in $\widetilde{\mathcal{A}}$. By Lemma 8.4.11(a1), (iv$'_\flat$)⇔(iv$_\flat$). The whole equivalence follows from this.

(a) This follows, because the solutions are in $\widetilde{\mathcal{A}}\subset$ ULR $\subset$ UR.

(c1) This follows from Lemma 6.4.5(e) and Lemma 6.4.8(a).

(c2) This follows from the above.

(d) This follows from Corollary 8.4.14(b4) and Lemma 6.4.5(e) □

In the stable case, we get still more equivalent conditions:

Corollary 9.1.12 (Stable MTIC Σ: SpF⇔CARE) *Let Hypothesis 8.4.7 hold for $\widetilde{\mathcal{A}}(U)$. Let Σ be strongly stable, and let $\mathbb{D}\in\widetilde{\mathcal{A}}$.*

Then the assumptions of Corollaries 9.1.11 and 9.1.9 are satisfied, and we have two more equivalent conditions:

(ii") $\mathbb{D}^* J\mathbb{D}$ has a spectral factorization $\mathbb{X}^* S\mathbb{X}$.

(iv') The Popov Toeplitz operator $\pi_+\mathbb{D}^* J\mathbb{D}\pi_+$ is invertible in $\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))$.

(a) Assume (ii'). Then $\mathbb{X}\in\mathcal{G}\widetilde{\mathcal{A}}$. If, in addition, Hypothesis 8.4.8 holds for $\mathcal{A}(U)$, then $D^*JD=X^*SX$.

(b) The solutions of (ii") correspond to those of (ii') through $\mathbb{M} = \mathbb{X}^{-1}$, $\mathbb{N} = \mathbb{D}\mathbb{M}$, $S = S$.

See Example 9.8.15 for an example with $X^*SX \neq D^*JD$.

Proof: Take $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \end{array}\right]$ in order to have $\Sigma_\flat^1 = \Sigma$ in Theorem 9.1.10. □

In the classical theory, one is often given just an I/O map and uses some (e.g., minimal) realization to get a Riccati equation for solving the problem.

This can be done in the infinite-dimensional case too: if $\mathbb{D} \in \mathrm{TIC}$ (resp. $\mathbb{D} \in \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ is as above), we can choose any strongly stable realization of $\mathbb{D}$ (e.g., the strongly stable exactly observable realization (6.11)) and apply Corollary 9.1.9 (resp. Corollary 9.1.12); if, instead, $\mathbb{D} \in \mathrm{TIC}_\infty \setminus \mathrm{TIC}$, we can proceed as follows:

Corollary 9.1.13 (I/O-result) *Let* $\mathbb{D} \in \mathrm{TIC}_\infty(U,Y)$ *have a q.r.c.f.* $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1}$, *and let* $J = J^* \in \mathcal{B}(Y)$. *Take a strongly stable realization* $\Sigma_\flat$ *of* $\begin{bmatrix} \mathbb{N}' \\ \mathbb{M}'-I \end{bmatrix}$ *(e.g., the one of Lemma 6.6.29), and close it with the output feedback* $L = \begin{bmatrix} 0 & -I \end{bmatrix}$ *to obtain another system*

$$\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K}' & I - \mathbb{M}'^{-1} \end{array}\right] := (\Sigma_\flat)_L \in \mathrm{WPLS}; \tag{9.9}$$

Then $\left[\begin{array}{c|c} \mathbb{K}' & I - \mathbb{M}'^{-1} \end{array}\right]$ *is strongly q.r.c.-stabilizing for* $\Sigma := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$.

Therefore, we can apply Theorems 9.1.7 and 9.1.10 (and Corollary 9.1.11 if $\mathbb{N}, \mathbb{M} \in \widetilde{\mathcal{A}}$*) for the realization* Σ *of* $\mathbb{D}$ *in order to find a* $(J,*)$*-inner q.r.c.f. for* $\mathbb{D}$. □

(This is obvious. Here $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ and Σ refer to components of 9.9.)

We now take a look at cases where the CARE becomes simpler.

Remark 9.1.14 *The CARE takes the form*

$$A^*\mathcal{P} + \mathcal{P}A + C_1^*QC_1 = (B_w^*\mathcal{P} + NC_1)^*(X^*X)^{-1}(B_w^*\mathcal{P} + NC_1), \tag{9.10}$$

of M. Weiss and G. Weiss [WW, Theorem 12.8], if we make (some of) the assumptions of Section 2 of [WW], namely that $Y := Y_1 \times U$, $J := \begin{bmatrix} Q & N^* \\ N & R \end{bmatrix}$ $\mathbb{D} := \begin{bmatrix} \mathbb{D}_1 \\ I \end{bmatrix}$, $\mathbb{C} := \begin{bmatrix} \mathbb{C}_1 \\ 0 \end{bmatrix}$, *where* Y_1 *is a Hilbert space,* $\mathbb{D}_1 \in \mathrm{TIC}(U,Y_1)$ *is the unique* TIC*-extension of "*$\mathbb{F}$*",* $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \gg 0$, $D = \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathcal{B}(Y_1 \times U)$, *and* $\mathbb{X}$ *and* S *are replaced by* $S^{1/2}\mathbb{X}$ *and* I*, respectively. In this case, the cost function takes the form*

$$\mathcal{J}(x_0,u) = \int_0^\infty \left\langle \begin{bmatrix} Q & N^* \\ N & R \end{bmatrix} \begin{bmatrix} y_1(t) \\ u(t) \end{bmatrix}, \begin{bmatrix} y_1(t) \\ u(t) \end{bmatrix} \right\rangle dt. \tag{9.11}$$

as in [WW, equation (2.8)]; here $\begin{bmatrix} y_1 \\ u \end{bmatrix} = \mathbb{C}x_0 + \mathbb{D}u$. □

When C is bounded (e.g., for finite-dimensional H), one often writes the above cost function in form $\mathcal{J} = \langle \begin{bmatrix} x \\ u \end{bmatrix}, J' \begin{bmatrix} x \\ u \end{bmatrix} \rangle$, where $J' = \begin{bmatrix} C & D \end{bmatrix}^* J \begin{bmatrix} C & D \end{bmatrix}$, $\mathbb{C} = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $\mathbb{D} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ (note that here $y = \begin{bmatrix} x \\ u \end{bmatrix}$).

In several special cases, the CARE can be simplified:

Proposition 9.1.15 (Special classes of systems) *If B is bounded (this includes the Pritchard–Salamon class), then each solution of the CARE is ULR, and we can formulate (iii) (the CARE) as follows:*

(iii') $A^*\mathcal{P}+\mathcal{P}A+C^*JC=(B^*\mathcal{P}+D^*JC)^*(D^*JD)^{-1}(B^*\mathcal{P}+D^*JC)$, $\mathcal{P}=\mathcal{P}^*\in\mathcal{B}(H)$, $D^*JD\in\mathcal{GB}$, and $K:=(D^*JD)^{-1}(B^*\mathcal{P}+D^*JC)$ is q.r.c.-stabilizing.

If $\mathbb{D}\in\mathrm{TIC}(U,Y)$ and Hypothesis 9.2.2 is satisfied, then the can formulate (iii) (the CARE) as follows:

(iii") $A^*\mathcal{P}+\mathcal{P}A+C^*JC=(B_{\rm w}^*\mathcal{P}+D^*JC)^*(D^*JD)^{-1}(B_{\rm w}^*\mathcal{P}+D^*JC)$, $\mathcal{P}=\mathcal{P}^*\in\mathcal{B}(H)$, $\mathcal{P}[H]\subset\mathrm{Dom}(B_{\rm w}^*)$, $D^*JD\in\mathcal{GB}$, and $K:=(D^*JD)^{-1}(B_{\rm w}^*\mathcal{P}+D^*JC)$ is q.r.c.-stabilizing.

If Σ is strongly stable and $\mathbb{D}\in\widetilde{\mathcal{A}}(U,Y)$ (resp. Σ is strongly q.r.c.-stabilizable in $\widetilde{\mathcal{A}}$), where $\widetilde{\mathcal{A}}$ satisfies Hypothesis 8.4.8 (e.g., $\widetilde{\mathcal{A}}=\mathrm{MTIC}^{\mathrm{L}^1}$), then the can formulate (iii) (the CARE) in Corollary 9.1.12 (resp. in Corollary 9.1.11) as follows:

(iii"') $A^*\mathcal{P}+\mathcal{P}A+C^*JC=(B_{\rm w}^*\mathcal{P}+D^*JC)^*(D^*JD)^{-1}(B_{\rm w}^*\mathcal{P}+D^*JC)$, $\mathcal{P}=\mathcal{P}^*\in\mathcal{B}(H)$, $D^*JD\in\mathcal{GB}$, $\text{w-lim}_{s\to+\infty}B_{\rm w}^*\mathcal{P}(s-A)^{-1}B=0$, and $K:=(D^*JD)^{-1}(B_{\rm w}^*\mathcal{P}+D^*JC)$ is q.r.c.-stabilizing.

Moreover, in each of these cases, $\Sigma_{\rm ext}$, $\Sigma_{\rm ext}^{\rm d}$, $\Sigma_{\circlearrowleft}$, $\Sigma_{\circlearrowleft}^{\rm d}$ become ULR and we may use $B_{\rm s}^*$ and s-lim instead of $B_{\rm w}^*$ and w-lim.

(In (iii') and (iii"), the operators K is necessarily admissible and ULR even if it were not q.r.c.-stabilizing.)

In the standard LQR (minimization) problem, we have $J=I$, hence then for these special classes the CARE (9.11) takes the familiar form

$$A^*\mathcal{P}+\mathcal{P}A+C_1^*C_1=(B_{\rm w}^*\mathcal{P})^*B_{\rm w}^*\mathcal{P} \tag{9.12}$$

and the corresponding (J-critical) control is given by $u(t)=K_{\rm w}x(t)=-B_{\rm w}^*\mathcal{P}x(t)$ a.e.; when B is bounded, then so is K, the CARE becomes $A^*\mathcal{P}+\mathcal{P}A+C_1^*C_1=\mathcal{P}BB^*\mathcal{P}$, and $u=-B^*\mathcal{P}x$, as in the finite-dimensional case. See also Theorem 9.2.14 and Corollary 9.2.15 for variants of (iii"), and also Corollary 10.2.3 and Theorem 9.9.6 for bounded B.

Proof: (As noted in Remark 6.9.3, a PS-system has a bounded B w.r.t. the larger of the two state spaces.)

1° *$\widetilde{\mathcal{A}}$: (iii"')⇔(iii):* Obviously, (iii"') holds iff (iii) holds and $S=D^*JD$. But if (iii) holds, then $\mathbb{D}^*J\mathbb{D}=\mathbb{X}^*S\mathbb{X}$ (resp. $\mathbb{D}_\flat^*J\mathbb{D}_\flat=\mathbb{X}_\natural^*S\mathbb{X}_\natural$) is a spectral factorization in $\widetilde{\mathcal{A}}$, by the corollary, hence then $D^*JD=I^*SI=S$, by the hypothesis.

2° *Assuming Hypothesis 9.2.2:* Use Theorem 9.2.9.

3° *Bounded B:* This follows from 2° and the fact that $B_{\rm w}^*=B$ with $\mathrm{Dom}(B_{\rm w}^*)=H$. □

Notes

Most of this section was contained in [Mik97b], partially also in [Mik98]. See the notes on p. 502 for earlier partial results for WPLSs (mainly special cases of implications "(ii)⇒(i)⇒(iii)"). Many analogous results are well-known for finite-dimensional systems (see [IOW]) and for Pritchard–Salamon systems (see [Weiss97]), particularly for exponentially stable systems (so that $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$). Note that "Wiener–Hopf factorizations" and (J,S)-inner-outer factorizations are equivalent to spectral factorizations in the stable case, by Lemma 6.4.8.

We have defined the CAREs only for WR systems with WR optimal state feedback pairs. See Section 9.7 for "CAREs" (on $\text{Dom}(A_{\text{crit}})$ instead of $\text{Dom}(A)$) for arbitrary WR systems, and Remark 9.12.1 and Definition 9.8.4 (the IARE) for arbitrary WPLSs. The corresponding equivalences (for IAREs) for arbitrary WPLSs are given in Theorem 9.9.1 and Corollary 9.9.11.

In the finite-dimensional case, the theory of algebraic Riccati equations and inequalities is very mature, and an excellent reference on the theory is [LR], by Peter Lancaster and Leiba Rodman. Several aspects of the finite-dimensional theory still await generalizations.

9.2 Riccati equations when $\mathbb{A}Bu_0 \in \mathrm{L}^1$

Everything should be made as simple as possible, but not simpler.
— Albert Einstein (1879–1955)

In this section, we shall establish a simplified CARE theory for systems of the form studied in Section 6.8 (see Hypothesis 9.2.2). For them, a unique J-critical control is always of uniformly line-regular (ULR) state feedback form, i.e., it corresponds to a ULR $\mathcal{U}_*^*$-stabilizing solution of the CARE (we assume that $D^*JD \in \mathcal{GB}(U)$). Moreover, we can remove the limit term from the CARE (and hence $S = D^*JD$). We may, instead, require that $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(U,H)$ (this is not the case for general CAREs); see Definition 9.2.6 for details. As a result, for this class we can and will formulate most results in this book to look like their finite-dimensional counterparts.

Main results of this section include Theorems 9.2.9–9.2.18 and 9.2.3. Several minimization results for these systems are given in Chapter 10 and H^∞ results in Chapters 11–12.

In practical applications of the theory of this section, one uses conditions such as those in Hypothesis 9.2.2, but to make room for future extensions of the theory, we often use the following, weaker and more abstract hypothesis in our results:

Hypothesis 9.2.1 (Σ is smooth) *The system $\Sigma \in \mathrm{WPLS}(U,H,Y)$ is ULR, $J = J^* \in \mathcal{B}(Y)$, and if there is a J-critical control for Σ over $\mathcal{U}_*^*$ in WPLS form, then $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$.*

(Here $\mathcal{P}$ is the J-critical cost operator, it is defined in Theorem 8.3.9(b1). If we say that "$\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ satisfies Hypothesis 9.2.1 for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$", we naturally mean that Hypothesis 9.2.1 holds with $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ replaced by $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ (also in the definition of $\mathcal{U}_{\exp}$) and B_{w}^* replaced by $(B_\flat^*)_{\mathrm{w}}$. Naturally, the last assumption above can be read as "either $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ or there is no J-critical control for Σ over $\mathcal{U}_*^*$ in WPLS form".)

Not even all stable ULR systems satisfy Hypothesis 9.2.1, by the counterexample given in Example 9.8.15 (with $\mathbb{D} = \tau^{-1} \in \mathrm{MTIC}$). However, Hypothesis 9.2.1 is satisfied in the following cases (and others):

Hypothesis 9.2.2 *At least one of (1.)–(7.) holds, where*

(1.) B is bounded (i.e., $B \in \mathcal{B}(U,H)$);

(2.) **(Analytic $\mathbb{A}$)** *Hypotheses 9.5.1 and 9.5.7 hold;*

(3.) $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1];H)$ for all $u_0 \in U$, $C \in \mathcal{B}(H,Y)$ and $D^*JC = 0$;

(4.) $\mathbb{A}B \in \mathrm{L}^1([0,1];\mathcal{B}(U,H))$, $C \in \mathcal{B}(H,Y)$, $D^*JD \in \mathcal{GB}(U)$;

(5.) $\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1];H)$ and $C_{\mathrm{w}}\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1];Y)$ for all $u_0 \in U$;

(6.) **(Stable case)** *$C \in \mathcal{B}(H,Y)$, $D^*JC = 0$, $\mathbb{D} \in \mathcal{B}(U,Y) + \mathcal{B}(U,\mathrm{L}^1(\mathbf{R}_+;Y))*$ and the assumptions in (b2) or (b2') of Theorem 8.3.9 hold;*

(7.) **(Stable case)** $\mathbb{D} \in \mathcal{B}(U,Y) + \mathcal{B}(U, \mathrm{L}^2(\mathbf{R}_+;Y))*$ *and the assumptions in (b2) or (b2') of Theorem 8.3.9 hold.*

As in Section 6.8, by "$\mathbb{A}B \in \mathrm{L}^1([0,1],*)$" we mean that $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1],*)$, etc. See Lemma 6.8.1–6.8.3 for equivalent conditions for (1.)–(7.) (e.g., (5.) holds iff $(\widehat{\mathbb{D}} - D)u_0 \in \mathrm{H}^2(\mathbf{C}^+_\omega;Y)$ and $(\cdot - A)^{-1}Bu_0 \in \mathrm{H}^2(\mathbf{C}^+_\omega;H)$ for all $u_0 \in U$ and some (hence all) $\omega > \omega_A$, by Lemma 6.8.1(a)&(d1)). Recall from Theorem F.2.1(g), that $\mathrm{L}^p_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(U,Y)) \subset \mathcal{B}(U,\mathrm{L}^p(\mathbf{R}_+;Y))$.

Concerning (6.) and (7.), we note that for $\mathcal{U}^*_* = \mathcal{U}_{\mathrm{out}}$ (resp. $\mathcal{U}_{\mathrm{sta}}$, $\mathcal{U}_{\mathrm{str}}$, $\mathcal{U}_{\mathrm{exp}}$), the assumptions in Theorem 8.3.9(b2) hold if $\left[\frac{\mathbb{A}}{\mathbb{C}}\right]$ is output stable (resp. stable, strongly stable, exponentially stable). The convolutions in (6.) and (7.) are explained in Proposition 6.3.4(a3) and Lemma F.2.2(d1)–(d3).

Theorem 9.2.3 *If Hypothesis 9.2.2 holds, then Hypothesis 9.2.1 holds (even with* $\mathrm{Dom}(B^*_{\mathrm{L,w}})$ *in place of* $\mathrm{Dom}(B^*_{\mathrm{w}})$*).*

Note that most cases of Hypothesis 9.2.2 are independent of $\mathcal{U}^*_*$, whereas Hypothesis 9.2.1 depends on $\mathcal{U}^*_*$.

Moreover, we have $\mathcal{P} \in \mathcal{B}(U, \mathrm{Dom}(B^*_{\mathrm{L,s}}))$ in cases (1.), (2.) and (4.); this also holds in (3.)–(7.) whenever the conditions of Lemma 9.2.8(c1)&(c2) are satisfied, by Theorem 9.2.9 (and Lemma 9.6.2).

Proof of Theorem 9.2.3: This follows from Lemmas 9.3.2 and 9.3.4. □

All assumptions (1.)–(7.) guarantee certain regularity of $\mathbb{D}$. Roughly, (1.)–(5.) also require that B is bounded or $\mathbb{A}$ is smoothing (any of them implies that $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1);H)$ for all $u_0 \in U$), and (6.) and (7.) require that Σ is somewhat stable. Indeed, under any of (1.)–(5.), we shall use (9.56) to obtain that $\mathcal{P}[H] \subset \mathrm{Dom}(B^*_{\mathrm{w}})$; under any of (6.)–(7.), we shall use (8.36) to obtain that $\mathcal{P}[H] \subset \mathrm{Dom}(B^*_{\mathrm{w}})$; from these formulae one observes why it is hard to weaken the above assumptions without giving up $\mathcal{P}[H] \subset \mathrm{Dom}(B^*_{\mathrm{w}})$ (and hence B^*_{w}-CARE's). See Lemma 9.3.4 for details.

Any of (2.)–(7.) allows B to be highly unbounded when $\mathbb{A}$ is highly smoothing (e.g., analytic), but (3.)–(5.) require B to be bounded when $\mathbb{A}$ is nonsmoothing (e.g., invertible).

An alternative approach is to give up "$\mathcal{P}[H] \subset \mathrm{Dom}(B^*_{\mathrm{w}})$ and assume that $\mathbb{D} \in \mathrm{MTIC}$ (or that Σ is exponentially or q.r.c.-stabilizable in MTIC), and require J-coercivity to obtain a factorization of the Popov operator. See Section 9.1 and Remark 9.9.14 for details.

In the stable case with $\dim Y < \infty$, a bounded C is enough for Hypothesis 9.2.2:

Proposition 9.2.4 *If C is bounded,* $\dim Y < \infty$ *and $\mathbb{B}$ is stable, then $\widehat{\mathbb{D}} - D \in \mathrm{H}^2(\mathbf{C}^+;\mathcal{B}(U,Y))$; in particular, then $\mathbb{D} \in \mathcal{B}(U,Y) + \mathcal{B}(U,\mathrm{L}^2(\mathbf{R}_+;Y))*$ (cf. (7.) above).*

If C is bounded, $\dim Y < \infty$, $\mathcal{U}^*_* = \mathcal{U}_{\mathrm{exp}}$ (resp. $\mathcal{U}^*_* = \mathcal{U}_{\mathrm{out}}$) and there is an exponentially (resp. q.r.c.-)stabilizing bounded state feedback operator $K \in$

$\mathcal{B}(H,U)$ for Σ, then we can reduce the problem to the stable case, which can be solved by Theorem 9.2.9, the above proposition, and (7.) of the hypothesis; indeed, the solution corresponds to the original one through the formulae of Proposition 6.6.18(f), by Theorem 8.4.5.

Naturally, if $C = \left[\begin{smallmatrix} C_1 \\ 0 \end{smallmatrix}\right] \in \mathcal{B}(U, Y_1 \times Y_2)$, then it suffices that $\dim Y_1 < \infty$, since then $\mathbb{D}_2 = D_2 \in \mathcal{B}(U,Y_2)$.

Proof: By Theorem 6.2.11(c2), we have $B^*(\cdot - A^*)^{-1}C^* y_0 \in \mathrm{H}^2$ for all $y_0 \in Y$, hence $\widehat{\mathbb{D}}^*(\bar{\cdot}) - D \in \mathrm{H}^2$, hence $\widehat{\mathbb{D}} - D \in \mathrm{H}^2$, hence $\mathbb{D} - D \in \mathcal{B}(U, \mathrm{L}^2(\mathbf{R}_+;Y))$, by Lemma F.3.4(d). □

The standing assumption that $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ is often implied by the other assumptions:

Remark 9.2.5 (Sufficient conditions for (3.) or (5.)) *Drop, for a moment, the standing assumption that* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}$. *Assume, instead, that $\mathbb{A}$ is a C_0-semigroup on H, $B \in \mathcal{B}(U,H_{-1})$, $C \in \mathcal{B}(H_1,Y)$ and $D \in \mathcal{B}(U,Y)$ (recall that $H_1 := \mathrm{Dom}(A)$ with graph norm, and that $H_{-1} := \mathrm{Dom}(A^*)^*$ w.r.t. the pivot space H).*

(a) If $B \in \mathcal{B}(U,H)$, and $C_{\mathrm{w}}\mathbb{A}x_0 \in \mathrm{L}^2([0,1);Y)$ for each $x_0 \in H$, then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ for any $\omega > \omega_A$, and (1.) and (5.) are satisfied.

(b) If $\mathbb{A}B, C_{\mathrm{w}}\mathbb{A}, C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^2([0,1);\mathcal{B})$, then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ for any $\omega > \omega_A$, and (5.) is satisfied.

(c) If $B^*\mathbb{A}^* \in \mathrm{L}^2([0,1);\mathcal{B}(H,U))$, $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1);H)$ for all $u_0 \in U$, $C \in \mathcal{B}(H,Y)$, and $D^*JC = 0$, then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$ for any $\omega > \omega_A$, and (3.) is satisfied.

(d) We have $C(\cdot - A)^{-1}, (\cdot - A)^{-1}B, \widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong},\infty}$ iff (5.) holds.

Analogously, $\mathbb{D} \in \mathcal{B}(U,Y) + \mathcal{B}(U,\mathrm{L}^2(\mathbf{R}_+;Y))*$ iff $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U,Y))$, by Lemma F.3.4(d).

Proof: Claims (a)–(c) follow easily from, e.g., Definition 6.1.1, Lemma 6.8.1, Lemma 6.3.16(b)&(c) and duality.

By Lemma 6.8.1(a)&(d1)&(e1)&(e2), claim (d) holds (and we may use $C_{\mathrm{L,s}}$ in place of C_{w} everywhere in the proposition). □

Under Hypothesis 9.2.1 (with $D^*JD \in \mathcal{GB}(U)$), one can replace the CARE by the following simplified form, as shown in Theorem 9.2.9:

Definition 9.2.6 (B_{w}^*-CARE) *Let $\mathbb{D}$ be WR. An operator $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$ is called a* solution of the B_{w}^*-CARE *if $D^*JD \in \mathcal{GB}(U)$ and $\mathcal{P}$ satisfies*

$$(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^*(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) = A^*\mathcal{P} + \mathcal{P}A + C^*JC \tag{9.13}$$

We call a solution $\mathcal{P}$ stabilizing (resp. ULR, ...) *if $K := -(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)$ is stabilizing (resp. ULR, ...), and we call $\mathcal{P}$* $\mathcal{U}_{\exp}$-stabilizing *iff K is exponentially stabilizing (see Definition 9.8.1 for other $\mathcal{U}_*^*$, $\Sigma_{\circlearrowleft}$, $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, $\mathbb{X}$, $\mathbb{N}$ and $\mathbb{M}$).*

We often call $(\mathcal{P},S,K)$ a *solution of the B_{w}^*-CARE*, where $K := -(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P}+D^*JC)$, $S := D^*JD$. In most standard forms of LQR and H$^\infty$ problems, we have $D^*JC = 0$ and hence then (9.13) reduces even further and $K = -(D^*JD)^{-1}B_{\mathrm{w}}^*\mathcal{P}$.

The equation (9.13) is given in $\in \mathcal{B}(\mathrm{Dom}(A),\mathrm{Dom}(A)^*)$, i.e., it holds iff

$$\langle Kx_0,(D^*JD)Kx_1\rangle_U = \langle Ax_0,\mathcal{P}x_1\rangle_H + \langle \mathcal{P}x_0,A\mathcal{P}x_1\rangle_H + \langle Cx_0,JCx_1\rangle_Y \quad \text{for all } x_0,x_1 \in \mathrm{Dom}(A) \tag{9.14}$$

(equivalently, whenever $x_0 \in \mathrm{Dom}(A)$, $x_1 = x_0$, by Lemma A.3.1(g3)).

Note that the condition "$\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H,\mathrm{Dom}(B_{\mathrm{w}}^*))$" may be replaced by "$\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ & $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$" (by Lemma A.3.6). Therefore, $\mathcal{P} \in \mathcal{B}(H,\mathrm{Dom}(B_{\mathrm{w}}^*))$ iff $\mathcal{P} \in \mathcal{B}(H)$ and $\langle r(r-A)^{-1}Bu_0,\mathcal{P}x_0\rangle_H$ converges as $r \to +\infty$, for all $u_0 \in U$ and $x_0 \in H$. In most cases we may replace B_{w}^* by $B_{\mathrm{L,w}}^*$, often even by $B_{\mathrm{L,s}}^*$, as noted in Lemma 9.2.8(c).

We first note that the solutions of the B_{w}^*-CARE are admissible and ULR solutions of the CARE:

Proposition 9.2.7 (B_{w}^*-CARE⇒CARE&IARE) *Assume that $\mathbb{D}$ is* ULR *and that $D^*JD \in \mathcal{GB}(U)$.*

(a) An operator $\mathcal{P} = \mathcal{P}^*$ is a solution of the B_{w}^*-CARE iff it is a solution of the CARE and $\mathcal{P} \in \mathcal{B}(H,\mathrm{Dom}(B_{\mathrm{w}}^*))$.

(b) Any solution of the B_{w}^*-CARE is admissible and ULR, hence a solution of the IARE (with $S := D^*JD$).

(c) If the B_{w}^*-CARE has a $\mathcal{U}_*^*$-stabilizing solution, then Hypothesis 9.2.1 holds.

Note that the CARE only requires that $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H_B,\mathrm{Dom}(B_{\mathrm{w}}^*))$. Obviously, a solution of the CARE cannot be a solution of the B_{w}^*-CARE when $S \neq D^*JD$, and this can happen even if C is bounded, $D^*JC = 0$ and $\mathbb{D},\mathbb{F} \in$ ULR, even for the $\mathcal{U}_{\mathrm{out}}$-stabilizing solution as shown in Example 9.13.8.

Fortunately, under the assumptions of Hypothesis 9.2.2, the "optimizing (i.e., $\mathcal{U}_*^*$-stabilizing, or equivalently, J-critical) solution of the CARE is always a solution of the B_{w}^*-CARE too.

Proof of Proposition 9.2.7: (a) Either assumption implies that $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H,U)$, by Lemma A.3.6. It follows that $S = D^*JD$ in the CARE, by Proposition 9.11.4(b3). The rest of the equivalence follows directly from the definitions. (Claim (a) holds whenever $\mathbb{D}$ is WR and $D^*JD \in \mathcal{GB}(U)$.)

(b) By Lemma 6.3.17, K is an ULR admissible state feedback operator. By Proposition 9.8.10, any WR solution of the CARE is a solution of the IARE.

(c) By (b) and Theorem 9.9.1, a $\mathcal{U}_*^*$-stabilizing solution $\mathcal{P}$ of the B_{w}^*-CARE is the J-critical cost operator; by assumption, $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$. □

We usually have additional regularity for $\mathcal{P}$ and $\Sigma_{\circlearrowleft}$. Some of this is given in the lemma below, more can be observed from the proofs and Lemmas 6.8.1–6.8.4.

Lemma 9.2.8 *Let $\mathbb{D}$ be* ULR *and $D^*JD \in \mathcal{GB}(U)$, and let $\mathcal{P}$ be a solution of the B_{w}^*-CARE. Then*

(a) If $C \in \mathcal{B}(H,Y)$ or $D^*JC = 0$, then $K \in \mathcal{B}(H,U)$.

(b1) If (5.) of Hypothesis 9.2.2 holds, then $\widehat{\mathbb{F}} := K_{\mathrm{w}}(s-A)^{-1}B \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U))$ for any $\omega > \omega_{\mathbb{A}}$, and $(s-A_\circlearrowleft)^{-1}B, \widehat{\mathbb{D}_\circlearrowleft} - D, \widehat{\mathbb{M}} - I \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;*)$ for any $\omega > \omega_{\mathbb{A}_\circlearrowleft}$.

(b2) If $\widehat{\mathbb{D}} - D \in \mathrm{H}^2(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ and $\dim U < \infty$, then $\widehat{\mathbb{F}} := K_{\mathrm{w}}(s-A)^{-1}B \in \mathrm{H}^2(\mathbf{C}^+_\alpha;\mathcal{B}(U))$ and $\widehat{\mathbb{D}_\circlearrowleft} - D, \widehat{\mathbb{M}} - I \in \mathrm{H}^2(\mathbf{C}^+_\alpha;*)$, for $\alpha \geq \omega$ s.t. $\alpha > \max\{\omega_A, \omega_{A_\circlearrowleft}\}$.

(c1) If $\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$ and $C \in \mathcal{B}(H,Y)$, or $C_{\mathrm{L,w}}\mathbb{A}B \in \mathrm{L}^2([0,1);\mathcal{B}(U,H))$ and $\mathbb{A}B \in \mathcal{B}(U,H)$ a.e., then $\mathcal{P} \in \mathcal{B}(H;\mathrm{Dom}(B^*_{\mathrm{L,s}}))$.

(c2) If $\mathcal{P}$ is $\mathcal{U}^*_*$-stabilizing and either (6.) holds and $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}(U,Y)$, or (7.) holds and $\mathbb{D} - D \in \mathrm{L}^2(\mathbf{R}_+;\mathcal{B}(U,Y))*$, then $\mathcal{P} \in \mathcal{B}(H,\mathrm{Dom}(B^*_{\mathrm{L,s}}))$.

Proof: (a) This follows from the fact that $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H,U)$.

(b1) By Lemma 6.3.17, we have

$$\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{array}{c|c} K_{\mathrm{w}}\mathbb{A} & -S^{-1}(B_{\mathrm{w}}^*\mathcal{P}\mathbb{B}\tau + D^*J\mathbb{D}) \end{array}\right]. \tag{9.15}$$

Because $C(s-A)^{-1}B, (s-A)^{-1}B \in \mathrm{H}^2_{\mathrm{strong},\infty}$ and $B_{\mathrm{w}}^*\mathcal{P}$ is bounded, we have $K_{\mathrm{w}}(s-A)^{-1}B \in \mathrm{H}^2_{\mathrm{strong},\infty}$, hence in $\mathrm{H}^2_{\mathrm{strong},\omega}$ for any $\omega > \omega_A$, by Lemma 6.8.1(d1).

By Proposition 6.3.3(b1), we have $\widehat{\mathbb{M}} - I \in \mathrm{H}^2_{\mathrm{strong},\infty}$, where $\mathbb{M} := (I-\mathbb{F})^{-1}$. Consequently, $\widehat{\mathbb{B}_\circlearrowleft\tau} = \widehat{\mathbb{B}\tau}\widehat{\mathbb{M}} \in \mathrm{H}^2_{\mathrm{strong},\infty}$ and $\widehat{\mathbb{D}_\circlearrowleft} = \widehat{\mathbb{D}}\widehat{\mathbb{M}} \in \mathcal{B} + \mathrm{H}^2_{\mathrm{strong},\infty}$, by Proposition 6.3.3(c). The rest follows from this and Lemma 6.8.1(a)&(d1).

(b2) (Note that $\mathrm{H}^2_{\mathrm{strong}} = \mathrm{H}^2$, because $\dim U < \infty$. Note also that (b2) corresponds to assumption (7.).) Apply Lemma 6.3.16(b) to $\left[\begin{array}{c|c} A^* & T^* \\ \hline B^* & 0 \end{array}\right]$, where $T := -S^{-1}B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H,U)$, to obtain (since $\dim U < \infty$ implies that weak=uniform) that

$$T\mathbb{B}\tau = -S^{-1}B_{\mathrm{w}}^*\mathcal{P}\mathbb{B}\tau \in \mathrm{H}^2(\mathbf{C}^+_\alpha;\mathcal{B}(U)), \tag{9.16}$$

because $\mathbb{B}$ is α-stable. Now the claim on $\widehat{\mathbb{F}}$ follows from (9.15).

Because $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\alpha(U)$, we have $\widehat{\mathbb{M}} - I \in \mathrm{H}^2(\mathbf{C}^+_\alpha;\mathcal{B}(U))$, by Proposition 6.3.3(b1). Because $\widehat{\mathbb{D}} - D \in (\mathrm{H}^2 \cap \mathrm{H}^\infty)(\mathbf{C}^+_\alpha;\mathcal{B}(U,Y))$, we have $\widehat{\mathbb{D}}\widehat{\mathbb{M}} - D \in \mathrm{H}^2(\mathbf{C}^+_\alpha;\mathcal{B}(U,Y))$.

(c1) This follows as in the proof of Lemma 9.3.4 (see its 2° and 4°) with $\Sigma_\circlearrowleft$ in place of Σ_{crit} except that in 6° we need to use the fact that $\mathcal{P}$ is also a solution of the B_{w}^*-CARE for $\Sigma_\flat$, by Lemma 9.12.3 and Proposition 9.2.7(a). We need to use equation (9.156), and it follows from Lemma 9.10.1(b4) and Proposition 9.2.7(b).

(c2) This was shown in 5° of the proof of Lemma 9.3.4 (combined with, e.g., Theorem 9.2.9(i)&(iii)). $\square$

Now we are ready for the main result, the equivalence between the existence of a stabilizing solutions of the B_{w}^*-CARE and the existence of a unique optimal control:

Theorem 9.2.9 (B_{w}^*-CARE $\Leftrightarrow$ J-critical) *Assume that $S := D^*JD \in \mathcal{GB}(U)$ and that Hypothesis 9.2.1 holds. Then the following are equivalent:*

(i) there is a unique J-critical control over $\mathcal{U}_*^*(x_0)$ for each $x_0 \in H$;

(ii) there is a J-critical state feedback pair over $\mathcal{U}_*^*$;

(iii) the B_{w}^*-CARE has a $\mathcal{U}_*^*$-stabilizing solution;

(iv) the CARE has a $\mathcal{U}_*^*$-stabilizing solution;

(v) the eIARE has a $\mathcal{U}_*^*$-stabilizing solution.

Assume, in addition, that any (hence all) of (i)–(v) has a solution. Then

(a1) The solutions $\mathcal{P}$ of (iii)–(v) are unique and equal, with the same S, K and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ (modulo (9.114) for (v)) $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H,U)$, and the corresponding triple is given by $(\mathcal{P},S,K)$, where $K := -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)$.

All J-critical state feedback pairs over $\mathcal{U}_*^*$ are generated by $\left[\begin{array}{c|c} EK & I-E \end{array}\right]$ ($E \in \mathcal{GB}(U)$).

(a2) Σ and its closed-loop system $\Sigma_\circlearrowleft$ corresponding to the state feedback operator K are ULR*, and $\Sigma_\circlearrowleft$ has generators*

$$\left[\begin{array}{c|c} A+BK_{\mathrm{s}} & B \\ \hline C+DK_{\mathrm{s}} & D \\ K_{\mathrm{s}} & 0 \end{array}\right] \tag{9.17}$$

(b1) Theorem 8.3.9 applies to the left column of $\Sigma_\circlearrowleft$.

(b2) If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then $\Sigma_\circlearrowleft$ is exponentially stable and, for any initial state x_0 and closed-loop input $u_\circlearrowleft$, the corresponding (J-critical if $u_\circlearrowleft = 0$) closed-loop cost is given by (cf. Figure 9.1)

$$\mathcal{J}(x_0, \mathbb{K}_\circlearrowleft x_0 + (\mathbb{F}_\circlearrowleft + I)u_\circlearrowleft) = \langle x_0, \mathcal{P}x_0\rangle_H + \langle u_\circlearrowleft, Su_\circlearrowleft\rangle \quad (x_0 \in H,\ u_\circlearrowleft \in \mathrm{L}^2(\mathbf{R}_+;U)). \tag{9.18}$$

(c1) Assume that (2.) or (4.) of Hypothesis 9.2.2 holds (or that $\mathbb{A}B \in \mathrm{L}^1([0,1];\mathcal{B}(U,H))$, $C \in \mathcal{B}(H,Y)$ and $D^*JC = 0$).

Then $\mathbb{B}_\circlearrowleft\tau, \mathbb{D}_\circlearrowleft, \mathbb{F}_\circlearrowleft, \mathbb{M} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1} \subset \mathrm{ULR} \cap \mathrm{UVR}$, in particular, $\mathbb{A}_\circlearrowleft B \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$, for any $\omega > \omega_{A_\circlearrowleft}$.

(c2) Assume that (1.) or (5.) of Hypothesis 9.2.2 holds. Then $\mathbb{B}_\circlearrowleft\tau, \mathbb{D}_\circlearrowleft - D, \mathbb{F}_\circlearrowleft, \mathbb{M} - I \in \mathcal{L}^{-1}\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}) \subset \mathrm{ULR} \cap \mathrm{SVR}$; in particular, $\mathbb{A}_\circlearrowleft Bu_0 \in \mathrm{L}^2_\omega(\mathbf{R}_+;H)$, for all $\omega > \omega_{A_\circlearrowleft}$ and $u_0 \in U$.

(c3) If Σ satisfies (1.), (2.), (4.) or (5.) of Hypothesis 9.2.2, then so does $\Sigma_\circlearrowleft$.

(d1) If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ (or $\left[\frac{\mathbb{A}}{\mathbb{C}}\right]$ is estimatable and $\mathcal{U}_*^* \in \{\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}\}$), then $\omega_{A_\circlearrowleft} < 0$.

(d2) If $\omega_{A_\circlearrowleft} < 0$ and the assumptions of (c1) or (c2) hold, then $\mathbb{B}_\flat\tau, \mathbb{D}_\flat, \mathbb{F}_\flat \in$ SHPR $\cap$ ULR.

Recall that $\mathbb{D} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ means that $\mathbb{D}u = D + f * u$ $(u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U))$ for some $f \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,Y))$ and $D \in \mathcal{B}(U,Y)$.

See Theorem 9.9.1 for further details (e.g., for "(b2)" for $\mathcal{U}_*^* \neq \mathcal{U}_{\exp}$). Recall that our cost function is

$$\mathcal{J}(x_0,u) := \int_0^\infty \langle y(t), Jy(t)\rangle_Y \, dt, \tag{9.19}$$

where $y := C_{\mathrm{w}}x + Du$ a.e., $x := \mathbb{A}x_0 + \mathbb{B}\tau u$ (the latter is the strong solution of $x' = Ax_0 + Bu_0$).

Proof: "(iii)$\Rightarrow$(iv)" follows from Proposition 9.2.7(a), "(iv)$\Rightarrow$(v)" from Proposition 9.8.10, "(v)$\Rightarrow$(ii)" from Theorem 9.9.1(a1), "(ii)$\Rightarrow$(iii)$\Leftarrow$(i)" from Theorem 9.2.3 and Proposition 9.3.1, "(iv)$\Rightarrow$(i)" from Theorem 9.9.1(f2).

(a1) By Proposition 9.2.7(a), a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ of the B_{w}^*-CARE solves also the CARE and hence the eIARE; by Theorem 9.8.12(e)&(s2)&(s3), there are no other $\mathcal{U}_*^*$-stabilizing solutions of the eIARE nor of the CARE (hence nor of the B_{w}^*-CARE). By Theorem 9.9.1(f2), all J-critical state feedback pairs over $\mathcal{U}_*^*$ are generated by $\left[\begin{array}{c|c} EK & I-E \end{array}\right]$ $(E \in \mathcal{GB}(U))$.

(a2) See Proposition 6.6.18(d4) for the generators of $\Sigma_\circlearrowleft$. By Lemma 9.3.2, $\mathbb{D} \in$ ULR. By Proposition 9.2.7(b), $\mathbb{F}$ is ULR, hence so are $\mathbb{M}$ and $\mathbb{D}_\circlearrowleft = \mathbb{D}\mathbb{M}$, by Proposition 6.3.1(b2).

(b1) By (a1), $\mathbb{K}_\circlearrowleft x_0$ is J-critical for each $x_0 \in H$, hence (b1) holds.

(b2) This follows from Theorem 8.3.9(a2) and (9.139).

(c1) Note first that $\mathrm{MTIC}_\infty^{\mathrm{L}^1} \subset$ ULR $\cap$ UVR, by Proposition 6.3.4(a1).

1° *Case (2.):* This follows from Lemma 9.6.1.

2° *Cases (4.):* By Lemma 6.8.4(a1) (note that (a2) would provide the sharp result $\omega = \omega_{A_\circlearrowleft}$ in some cases), we have $\mathbb{A}_\circlearrowleft B \in \mathrm{L}^1_\omega(\mathbf{R}_+;\mathcal{B}(U,H))$, i.e., $\mathbb{B}_\circlearrowleft\tau \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$. Because K and $C+DK$ are bounded, we have $\mathbb{D}_\circlearrowleft, \mathbb{F}_\circlearrowleft, \mathbb{M} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ (recall that $\mathbb{F}_\circlearrowleft = \mathbb{M} - I$).

(c2) Note first that $\mathcal{L}^{-1}\mathrm{H}^2_{\mathrm{strong}} \subset$ ULR $\cap$ SVR, by Proposition 6.3.3(a).

1° *Case (1.):* Combine (9.17) and Theorem 6.9.1(a).

2° *Case (5.):* This follows from Lemma 9.2.8(b1). (Also (7.) will do for $\omega > \max\{\omega_A, \omega_{A_\circlearrowleft}\}$ if $\omega \geq 0$ and $\mathcal{U}_*^* = \mathcal{U}_{\exp}$, by Lemma 9.2.8(b2).)

(c3) For (1.) this is obvious. For (2.) this follows from Lemma 9.6.1, for (4.) from Lemma 6.8.4(a1), and for (5.) from (c2).

(d1) This follows from Theorem 8.3.9(a2)&(a5) (cf. Lemma 8.3.3).

(d2) The ULR property was established in (a2). We have $\mathrm{MTIC}^{\mathrm{L}^1} \subset$ UHPR, by Theorem 2.6.4(f), and $\mathcal{L}^{-1}\mathrm{H}^2_{\mathrm{strong}} \in$ SHPR, by Proposition 6.3.3(a). □

In coercive minimization problems, we usually do not have the check whether a solution of the CARE is stabilizing:

Theorem 9.2.10 *Assume Hypothesis 9.2.1. Assume also that $J \gg 0$ and that there is $\varepsilon > 0$ s.t. $\begin{bmatrix} C & D \end{bmatrix}^* J \begin{bmatrix} C & D \end{bmatrix} \geq \varepsilon \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ on $H_1 \times U$. Then* *(a) A minimizing (see Definition 10.2.1) solution of the $B_{\rm w}^*$-CARE is nonnegative.*

(b) If $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$, then the following are equivalent:

(i) There is a minimizing control over $\mathcal{U}_*^*(x_0)$ for each $x_0 \in H$;
(ii) There is a minimizing ULR state feedback operator over $\mathcal{U}_*^*$;
(iii) $\mathcal{U}_*^*(x_0) \neq \emptyset$ for all $x_0 \in H$
(iv) $B_{\rm w}^*$-CARE has a nonnegative solution.

Moreover, if (i)–(iv) hold, then the smallest nonnegative solution $\mathcal{P}$ of the $B_{\rm w}^*$-CARE is the unique $\mathcal{U}_{\rm out}$-stabilizing (and SOS-stabilizing) solution of the $B_{\rm w}^*$-CARE (and of the CARE), and strictly minimizing over $\mathcal{U}_{\rm out}$.

(c) Assume that $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$. Then (i) and (ii) are equivalent, and they hold iff the $B_{\rm w}^*$-CARE has an exponentially stabilizing solution.

Moreover, if such a solution exists, then it is the greatest solution of the $B_{\rm w}^*$-CARE and strictly minimizing over $\mathcal{U}_{\rm exp}$.

(d) **($\mathcal{P}$ is unique)** *Assume that Σ is strongly stable (resp. estimatable; e.g., C is bounded and $C^*C \gg 0$). Then $\mathcal{U}_{\rm out} = \mathcal{U}_{\rm sta} = \mathcal{U}_{\rm str}$ (resp. $= \mathcal{U}_{\rm exp}$), hence then (i)–(iv) are equivalent also for any of these.*

Moreover, then the $B_{\rm w}^*$-CARE has at most one nonnegative solution, and such a solution is strongly (resp. exponentially) q.r.c.-stabilizing and strictly minimizing over $\mathcal{U}_{\rm out}$, $\mathcal{U}_{\rm sta}$ and $\mathcal{U}_{\rm str}$ (resp. and $\mathcal{U}_{\rm exp}$).

Note that $J \gg 0$, $D^*JC = 0$ and $D^*D \gg 0$ (or $J = \left[\begin{smallmatrix} Q & 0 \\ 0 & R \end{smallmatrix}\right] \gg 0$, $\mathbb{C} = \left[\begin{smallmatrix} \mathbb{C}_1 \\ 0 \end{smallmatrix}\right]$ and $\mathbb{D} = \left[\begin{smallmatrix} \mathbb{D}_1 \\ I \end{smallmatrix}\right]$) imply that the coercivity conditions of the theorem are satisfied.

Assume that $\mathcal{U}_{\rm out}(x_0) \neq \emptyset$. Then (b) and (c) show that for the standard LQR cost function, there is a smallest nonnegative solution of the $B_{\rm w}^*$-CARE, and it corresponds to the minimizing control over $\mathcal{U}_{\rm out}$, and if there is a minimizing control over $\mathcal{U}_{\rm exp}$, then it corresponds to the greatest solution of the $B_{\rm w}^*$-CARE (and such a solution exists). A more detailed treatment of this phenomenom is given in Theorem 3.0.5 of [Dumortier] (assuming that B and C are bounded), which couples the nonnegative solutions of the CARE with the non-observable poles of Σ.

Proof of Theorem 9.2.10: Note first that $\mathbb{D}$ is ULR, by Theorem 9.2.3, and that (10.87) holds (see the proof of Proposition 10.7.3(c2)).

(a) This follows from equation $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ (see Theorem 9.9.1(a2)&(g2)).

(b) 1° *(ii)⇒(i)⇒(iii)⇒(i)⇒(ii)&(iv) and (iv)⇒(i):* Directly from the definitions we obtain that (ii)⇒(i)⇒(iii). By (the proofs of) Proposition 10.7.3(c2)&(d), we have (iii)⇒(i) and any minimizing control over $\mathcal{U}_{\rm out}$ is unique.

By Theorem 9.2.9(i)&(ii)&(a1)&(a2), a unique minimizing control corresponds to a $\mathcal{U}_{\rm out}$-stabilizing ULR solution $\mathcal{P}$ of the $B_{\rm w}^*$-CARE. Thus, (i)⇒(ii)&(iv) (because $\mathcal{P} = \mathbb{C}_{\rm crit}^* J \mathbb{C}_{\rm crit} \geq 0$).

Finally, assume (iv), so that there is an admissible nonnegative solution of the IARE, by Proposition 9.2.7(b). Then there is a unique minimizing control over $\mathcal{U}_{\rm out}(x_0)$ for each $x_0 \in H$, by Proposition 10.7.3(c2)&(d), so that (i) holds.

2° Let $\mathcal{P}$ be as in 1°. By Proposition 10.7.3(d1), $\mathcal{P}$ is the smallest nonnegative solution of the eIARE, hence it is the smallest nonnegative solution of the B_{w}^*-CARE(and of the CARE).

(c) The equivalence follows from Theorem 9.2.9(i)&(ii)&(iii)&(a1)&(a2), and the fact that $S = D^*JD \geq \varepsilon I \gg 0$ (which also implies that a minimizing control is necessarily unique, by Theorem 9.9.1(f2)).

Any exponentially stabilizing solution is the greatest solution of the B_{w}^*-CARE, by Corollary 9.2.11.

(d) (Bounded C with $C^*C \gg 0$ implies exponential detectability, by Lemma 6.6.25.) This follows from Proposition 10.7.3(d2)&(d3). □

For the B_{w}^*-CARE with $S \gg 0$, a strongly stabilizing solution the greatest of all (self-adjoint) solutions, not just of nonnegative ones (cf. Theorem 9.8.13 and Corollary 15.5.3):

Corollary 9.2.11 (Greatest solution $\mathcal{P}_+$ of the B_{w}^*-CARE) *If $D^*JD \gg 0$ and the B_{w}^*-CARE has a strongly ($\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$-)stabilizing solution, then this solution is the greatest solution of the B_{w}^*-CARE.* □

(This follows from Proposition 9.2.7(b) and Theorem 9.8.13. Recall that "greatest" is always "maximal".)

The example $A = i$, $B = 0 = C$, $D = 1 = J$, $\mathcal{P} \in \mathbf{R}$ shows that "strongly" is not redundant in the above corollary; by Example 9.13.12(b), "strongly" cannot be replaced by "weakly" (take, e.g., $B = 0 = C$, $\mathbb{A} = \tau$, $H := \mathrm{L}^2$, $D = 1 = J$, $\widetilde{\Sigma}$ as in the example; however, "strongly" and "weakly" coincide in the finite-dimensional case).

G. Weiss and R. Rebarber [WR97] [WR00] have posed the question whether optimizability is equivalent to exponential stabilizability. We give here a positive answer for a special case:

Theorem 9.2.12 (Optimizable $\Leftrightarrow$ exp. stabilizable) *Assume that $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1];H)$ for all $u_0 \in U$. Then the following are equivalent:*

(i) $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable;

(ii) Σ is exponentially stabilizable;

(iii) Σ has an exponentially stabilizing bounded state feedback operator $K \in \mathcal{B}(H,U)$;

(iv) There is $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$ s.t. $\mathcal{P} \geq 0$ and

$$(B_{\mathrm{w}}^*\mathcal{P})^* B_{\mathrm{w}}^*\mathcal{P} = A^*\mathcal{P} + \mathcal{P}A + I. \tag{9.20}$$

By Lemma 6.8.4, it follows that if $\mathbb{D}, \mathbb{B}\tau \in \mathrm{MTIC}_\infty^{\mathrm{L}^1}$ and Σ is optimizable, then $\mathbb{D}$ is exponentially stabilizable in $\mathrm{MTIC}_\infty^{\mathrm{L}^1}$.

Analogously, if $\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1];H)$ for all $u_0 \in U$ and $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable, $\mathbb{A}_\flat Bu_0 \in \mathrm{L}^2_{-\varepsilon}(\mathbf{R}_+;H)$ for all $u_0 \in U$ and some $\varepsilon > 0$, by Lemma 6.8.4(b).

In the theorem, we showed that $\mathcal{U}_{\exp}(x_0) \neq \emptyset$ for each $x_0 \in H$ iff $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is exponentially stabilizable. We note that one can analogously show that Σ

is output-stabilizable iff $\mathcal{U}_*^*(x_0) \neq \emptyset$ for each $x_0 \in H$, by using substitutions $\mathbb{C} \mapsto \begin{bmatrix} \mathbb{C} \\ 0 \end{bmatrix}$, $\mathbb{D} \mapsto \begin{bmatrix} \mathbb{D} \\ I \end{bmatrix}$, $J := I$.

Proof of Theorem 9.2.12: *(iii)⇒(ii)⇒(i)⇒(iv)⇒(iii):* Obviously, (iii)⇒(ii)⇒(i). Assume (i). Set $C := \begin{bmatrix} I \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $J = I$ (i.e., "$\mathbb{D} = \begin{bmatrix} x \\ u \end{bmatrix}$, $\mathcal{J} = \|x\|^2 + \|u\|^2$"). Because $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{exp}}$, there is a unique J-critical control over $\mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$ iff (i) holds, by Theorem 8.4.3.

But this is the case iff (iv) holds, by Theorem 9.2.9(i)&(iii) and Theorem 9.2.10(a). Thus, (iv)⇔(i). Moreover, (iv) implies (iii), because the operator K in Theorem 9.2.9(a1) is $\mathcal{U}_{\text{exp}}$-stabilizing, i.e., exponentially stabilizing. Thus, (iv)⇒(iii), and equivalence is established. □

By combining the above with its dual, we observe that optimizability and estimatability are equivalent to exponential joint stabilizability and detectability:

Corollary 9.2.13 *Assume that $\mathbb{A}^*C^*y_0 \in \mathrm{L}^1([0,1];H)$ for all $y_0 \in Y$. Then*

(a) Σ is estimatable iff Σ has an exponentially detecting (bounded) output injection operator $\mathtt{H} \in \mathcal{B}(Y,H)$.

(b) If, in addition, $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1];H)$ for all $u_0 \in U$, Σ is optimizable and estimatable, and $\mathtt{H}$ and K are as in (a) and Theorem 9.2.12, then K and $\mathtt{H}$ are exponentially (r.c.- and l.c.-) jointly stabilizing.

(c) If $\mathbb{A}B, C_{\mathrm{w}}\mathbb{A}, C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,1);*)$, and Σ is optimizable and estimatable, then $\mathbb{D}$ has a d.c.f. over $\mathrm{MTIC}^{\mathrm{L}^1}_{\text{exp}}$.

Thus, if the two strong L^1 assumptions are satisfied, then Σ is exponentially [jointly stabilizable and] detectable iff Σ is [optimizable and] estimatable, by (b).

Note that Hypothesis 9.5.1 implies the assumptions of (c), by Lemma 9.5.2.

Proof: (a) This is the dual of Theorem 9.2.12.

(b) This follows from Lemma 6.6.26.

(c) Now both closed-loop systems of (6.169) have their I/O maps in $\mathrm{MTIC}^{\mathrm{L}^1}_{-\varepsilon}(Y \times U)$ for some $\varepsilon > 0$, by Lemma 6.8.4(c1) (and its dual). □

Next we show that the invertibility of the Popov Toeplitz operator (i.e., J-coercivity over $\mathcal{U}_{\text{out}}$) is now equivalent to the existence of a spectral factorization. We assume Σ to be exponentially stable to guarantee the stability of the spectral factor (this is not needed for most MTIC classes):

Theorem 9.2.14 (Popov⇔SpF)

(a) Let Σ be exponentially stable. Assume that (1.), (2.), (4.) or (5.) of Hypothesis 9.2.2 holds, or that $C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \in \mathcal{B}(H, Y_1 \times Y_2)$ and $\dim Y_1 < \infty$. Then the following are equivalent:

(i) $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_+;U))$ (i.e., $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\text{out}}(= \mathcal{U}_{\text{exp}})$);

(i') $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_+;U))$ and $D^*JD \in \mathcal{GB}(U)$;

(ii) $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $S \in \mathcal{GB}(U)$*;*

(ii') $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $S = D^*JD \in \mathcal{GB}(U)$*, and* $\mathbb{D}, \mathbb{X} \in \mathrm{ULR}$ *and* $X = I$*;*

(iii) the B_{w}^**-CARE has an exponentially stabilizing solution.*

(iii') the B_{w}^**-CARE has an* $\mathbb{M}$*-stabilizing solution.*

(iv) the IARE (or CARE) has an $\mathbb{M}$*-stabilizing solution.*

(b) Assume that Σ *is exponentially stable and Hypothesis 9.2.1 holds for* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ $(= \mathcal{U}_{\mathrm{exp}})$*.*

Then (i)–(iv) are still equivalent provided that $\mathbb{D}$ *and* $\mathbb{D}^{\mathrm{d}}$ *are strongly half-plane-regular or that we assume that* $D^*JD \in \mathcal{GB}(U)$*.*

(c1) Conditions (i) and (ii) are equivalent (and imply that $\mathbb{D}, \mathbb{X} \in \mathrm{ULR}$*) if* $\mathbb{D} \in \widetilde{\mathcal{A}}$*.*

(c2) If $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_{-\varepsilon}; \mathcal{B}(U,Y))$ *for some* $\varepsilon > 0$ *or* $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$*, then (i)–(ii') are equivalent.*

(c3) If $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$*, and* Σ *is exponentially stable, then (i)–(iv') are equivalent once we replace the* B_{w}^**-CARE by the CARE.*

(d) Assume that Σ *is exponentially stable and ULR. Then (ii)* $\Leftarrow$ *(iv)* $\Leftarrow$ *(iii')* $\Leftrightarrow$ *(iii)* $\Rightarrow$ *(ii')* $\Rightarrow$ *(ii)* $\Rightarrow$ *(i).*

As one observes from the results of this section, $\mathbb{X}$ shares most properties of $\mathbb{D}$. If $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \gg 0$, then we can allow for $\Sigma \in \mathrm{SOS}$ and weaken the assumptions as shown in Theorem 10.6.3 and Lemma 10.6.2(b)–(d) (in the indefinite case, Σ may be even strongly stable with $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+$ invertible and still $\mathbb{X}, \mathbb{X}^{-1}$ unstable, by Example 8.4.13).

Proof of Theorem 9.2.14: (Naturally, "$\mathbb{M}$-stabilizing" means "s.t. $\mathbb{M}$ is stable" (cf. Definition 9.8.1).)

Note that by (d), we only have to prove "(i)$\Rightarrow$(iii)" in (a) and (b).

(a) 1° *Cases (2.)&(4.):* These follow from Lemma 9.3.2 and Hypotheses 8.4.7 and 8.4.8.

2° *Case (1.):* There is a unique J-critical control over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$, for each $x_0 \in H$, by Proposition 8.3.10. By Theorem 9.9.6(d), it follows that the eCARE has an exponentially stabilizing solution with $S = D^*JD$ and $\widehat{\mathbb{D}} - D, \widehat{\mathbb{X}} - I \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_{\omega}; \mathcal{B}(*,*))$.

Moreover, $S \in \mathcal{GB}(U)$, by Lemma 9.10.3 and Lemma 8.4.11(a1). By Theorem 9.9.1(g2), we have $\mathbb{N}^*J\mathbb{N} = S$, hence $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$.

3° *Case (5.):* This follows from (c2), since now $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_{-\varepsilon}; \mathcal{B}(U,Y))$ for some $\varepsilon > 0$, by Lemma 6.8.1(d1).

4° *Bounded* $\left[\begin{smallmatrix} C_1 \\ 0 \end{smallmatrix}\right]$ *with* $\dim Y_1 < \infty$*:* As in the proof of Proposition 9.2.4, one can show that $\widehat{\mathbb{D}} - D \in \mathrm{H}^2(\mathbf{C}^+_{-\varepsilon}; \mathcal{B}(U,Y))$ for some $\varepsilon > 0$. Thus, also this follows from (c2).

(b) 1° $D^*JD \in \mathcal{GB}(U)$*:* If $\mathbb{D}, \mathbb{D}^{\mathrm{d}} \in \mathrm{SHPR}$ and (i) (or (ii)) holds, then $D^*JD \in \mathcal{GB}(U)$, by Lemma 6.3.6(c1), hence we may assume that $D^*JD \in \mathcal{GB}(U)$ (since it is contained in the other conditions).

2° *(i)$\Rightarrow$(iii):* Assume (i). Then there is a unique J-critical control over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$, for each $x_0 \in H$, by Proposition 8.3.10. By Theorem

9.2.9(i)&(iii)&(a2), it follows that the B_{w}^*-CARE has an exponentially stabilizing solution with $X = I$, $S = D^*JD$ and $\mathbb{D}, \mathbb{X} \in \mathrm{ULR}$.

(c1) This follows from Hypothesis 8.4.7.

(c2) For $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$, this follows from Theorem 8.4.9(a)&(b)

Assume that $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_{-\varepsilon}; \mathcal{B}(U,Y))$ for some $\varepsilon > 0$. Then $\widehat{\mathbb{D}}$ has a realization $\Sigma_{\mathbb{D}}$ of type (1.), by Theorem 6.9.1(a)&(d1), hence then this follows from (a).

Remark: If also $\widehat{\mathbb{D}}(\bar{\cdot})^* - D^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_{-\varepsilon}; \mathcal{B}(Y,U))$, then (apply Lemma 6.3.17 with $R := -S^{-1}D^*J$ and $T := -S^{-1}B_{\mathrm{w}}^*\mathcal{P}$; here we refer to $\Sigma_{\mathbb{D}}$, not to Σ)

$$\widehat{\mathbb{X}}(\bar{\cdot})^* - X^* = \widehat{\mathbb{F}}(\bar{\cdot})^* = (R(\widehat{\mathbb{D}} - D) + T\mathbb{B}\tau)(\bar{\cdot})^* \tag{9.21}$$

$$= (\widehat{\mathbb{D}}(\bar{\cdot})^* - D^*)R^* + B^*(\cdot - A^*)T^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_{-\varepsilon'}; \mathcal{B}(Y,U)), \tag{9.22}$$

by Lemma A.4.5(v)&(i)&(vi) (choose $\varepsilon' \in (0,\varepsilon]$ s.t. $-\varepsilon' > \omega_{A_\circlearrowleft}$).

(c3) By (c2), (i)–(ii') are equivalent. The rest follow from Corollary 9.1.12 and Proposition 9.8.11(d1).

(d) Trivially, (iii)⇒(iii'), and (ii')⇒(ii). By Proposition 9.2.7(b), we have (iii')⇒(iv). By Proposition 9.8.11(c)&(d), we have (iv)⇒(ii). By Proposition 9.2.7(b) and Proposition 9.8.11(c)&(d), we have (iii)⇒(ii'). By Theorem 8.4.12, we have (ii)⇒(i). □

In the unstable case, a corresponding result can be formulated in the following way:

Corollary 9.2.15 (*J*-coercive⇔RCF) *Assume that Σ is optimizable and estimatable. Assume that (1.), (2.), (4.) or (5.) of Hypothesis 9.2.2 holds (or that Hypothesis 9.2.1 holds and $D^*JD \in \mathcal{GB}$). Then the following are equivalent:*

(i) $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{out}} (= \mathcal{U}_{\exp})$;

(ii) $\mathbb{D}$ has a $(J,*)$-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$;

(iii) the B_{w}^*-CARE has an exponentially stabilizing solution.

(iii') the B_{w}^*-CARE has an I/O-stabilizing solution.

Let K correspond to (iii). Then K is ULR, J-critical over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\exp}$ and exponentially stabilizing, and $\mathbb{N} = \mathbb{D}_\circlearrowleft$ and $\mathbb{M} = \mathbb{F}_\circlearrowleft + I$.

In particular, then $\mathbb{N}$ and $\mathbb{M}$ are exponentially stable, $\mathbb{D}, \mathbb{X}, \mathbb{N}, \mathbb{M} \in \mathrm{ULR}$ (here $\mathbb{X} := \mathbb{M}^{-1}$), $M = I = X$ and $S = D^*JD$.

(Note that $\mathbb{M}, \mathbb{N} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_{-\varepsilon}; \mathcal{B})$ for some $\varepsilon > 0$ in cases (1.) and (5.).)

Cf. Corollary 8.4.14 and Theorem 9.9.10.

Proof: Recall from Lemma 8.3.3 that $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\exp}$. By Theorem 6.7.15(c1), we may replace "I/O-stabilizing" by "exponentially stabilizing", "input-stabilizing" or "output-stabilizing" in (iii').

1° *(ii)⇒(i):* This follows from Corollary 8.4.14(b1).

2° *(iii)⇒(ii):* This follows from Theorem 9.9.10(d2).

3° *(iii)⇔(iii'):* By Theorem 6.7.15(c1), "I/O-stabilizing" is equivalent to "exponentially q.r.c.-stabilizing".

4° *(i)⇒ $D^*JD \in \mathcal{GB}(U)$:* Assume (i) and any of (1.), (2.), (4.) and (5.). By Lemma 9.3.2, $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1];H)$ for all $u_0 \in U$. By Theorem 9.2.12, Σ has a bounded exponentially stabilizing state feedback operator $K' \in \mathcal{B}(U,H)$.

By Lemma 9.3.3, also $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ (corresponding to K') satisfies (1.), (2.), (4.) or (5.). By Theorem 8.4.5(g1), $\mathbb{D}_\flat$ is J-coercive over $\mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}$, so that Theorem 9.2.14(a)(ii') holds; in particular $D^*JD = D_\flat^*JD_\flat \in \mathcal{GB}(U)$.

5° *(i)⇒(iii):* Assume (i), so that $D^*JD \in \mathcal{GB}(U)$, by 4° (or by assumption). By Theorem 8.4.3, there is a unique J-critical control over $\mathcal{U}_{\mathrm{exp}}$. Therefore, (iii) holds, by Theorem 9.2.9(iii),

5° The rest follows easily from Theorem 9.2.9.

6° *Remark:* In addition to (1.), (2.), (4.) or (5.) and optimizability, estimatability, assume that (ii) holds and that Σ is also input-detectable (by Corollary 9.2.13, the latter holds in case (2.)).

Then Σ is exponentially detectable, by the dual of Theorem 6.7.15(c1). Therefore, $\mathbb{D}_\flat$ and $\mathbb{F}_\flat + I$ are part of an exponential d.c.f., by Lemma 6.6.26, hence so are $\mathbb{N} = \mathbb{D}_\flat\mathbb{W}$ and $\mathbb{M} = (\mathbb{F}_\flat + I)\mathbb{W}$, by Lemma 6.5.9(d), where $\widetilde{\mathbb{X}}^*S\widetilde{\mathbb{X}} = \mathbb{D}_\flat^*J\mathbb{D}_\flat$ as in Theorem 9.2.14(a)(ii) and $\mathbb{W} := \widetilde{\mathbb{X}}^{-1} \in \mathcal{G}\mathrm{TIC}$. In particular, then $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a (exponential) r.c.f. (We do not know whether the q.r.c.f. has to be a r.c.f. in general.) □

We can now show that for $\mathcal{U}_{\mathrm{exp}}$ (assuming that $D^*JD \in \mathcal{GB}(U)$), J-coercivity is equivalent to the existence of a unique optimal control:

Theorem 9.2.16 ($\mathcal{U}_{\mathrm{exp}}$: Unique optimum ⇔ B_{w}^*-CARE⇔ J-coercive) *Assume Hypothesis 9.2.1 for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, and that $D^*JD \in \mathcal{GB}(U)$. Then conditions (i)–(iii) are equivalent.*

(i) There is a unique J-critical control over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for each $x_0 \in H$.

(ii) The B_{w}^*-CARE has an exponentially stabilizing solution.

(iii) $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{exp}}$, and Σ is exponentially stabilizable. □

(The proof of Theorem 14.2.7 applies mutatis mutandis.)

The assumption $D^*JD \in \mathcal{GB}(U)$ is not superfluous neither redundant in general (in (i); probably neither in (iii)), by, e.g., Example 9.13.3. However, in (iii) it is often redundant:

Lemma 9.2.17 ($\mathcal{U}_{\mathrm{exp}}$: J-coercive ⇒ $\exists(D^*JD)^{-1}$) *Assume that 1. $\mathbb{D}$ is ULR [or SLR] and that $\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$, (this is the case if (1.), (2.) or (4.) of Hypothesis 9.2.2 holds), or 2. $(\cdot - A)^{-1}B, C(\cdot - A)^{-1}B, (\cdot - A^*)^{-1}C^*, B_{\mathrm{w}}^*(\cdot - A^*)^{-1}C^* \in \mathrm{H}^2_{\mathrm{strong},\infty}$ [or 3. that $\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1);H)$ for all $u_0 \in U$].*

If Σ is optimizable and $\mathbb{D}$ is [positively] J-coercive over $\mathcal{U}_{\mathrm{exp}}$, then $D^*JD \in \mathcal{GB}(U)$ [$\gg 0$].

See Corollary 9.2.19 for the "converse" (where $D^*JD \in \mathcal{GB}(U)$ is equivalent to J-coercivity when there is a J-critical control).

Proof: (Set $\mathcal{U}_*^* =: \mathcal{U}_{\mathrm{exp}}$.)

1° *Case 1.* *($\mathbb{D} \in$ ULR etc.):* By Theorem 9.2.12(iii), Σ has a bounded exponentially stabilizing $K \in \mathcal{B}(H,U)$. As noted in the proof of Lemma 6.8.4(d), we have $\mathbb{B}_\flat\tau, \mathbb{M} \in \mathrm{UHPR} \cap \mathrm{TIC}$, hence $\mathbb{D}_\flat = \mathbb{D}\mathbb{M} \in \mathrm{ULR} \cap \mathrm{TIC}$.

But $\mathbb{D}_\flat$ is J-coercive, by Theorem 8.4.5(d), hence $D_\flat^* J D_\flat \in \mathcal{GB}(U)$, by Lemma 6.3.6(d1). Since $D = D_\flat$, we have $D^*JD \in \mathcal{GB}(U)$.

2° *Case 2.:* (Note that "2." holds iff Σ and Σ^{d} satisfy (5.) of Hypothesis 9.2.2, by Lemma 6.8.1(a)&(d1).)

By Theorem 9.2.12(iii), Σ has a bounded exponentially stabilizing $K \in \mathcal{B}(H,U)$; choose some $\omega \in (\omega_{A_\flat}, 0)$. By Lemma 6.8.4(b)&(c3), we have $\mathbb{B}_\flat\tau, \mathbb{F}_\flat, \mathbb{D}_\flat - D \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(U,*))$ (since $\mathbb{F}_\flat = K\mathbb{B}_\flat\tau$). Since $\mathbb{C}^{\mathrm{d}}_\flat = \mathbb{C}^{\mathrm{d}} + \mathbb{K}^{\mathrm{d}}_{\circlearrowleft}\tau\mathbb{D}^{\mathrm{d}}$, we have

$$\widehat{\mathbb{C}^{\mathrm{d}}_\flat\tau} = \widehat{\mathbb{C}^{\mathrm{d}}\tau} + \widehat{\mathbb{K}^{\mathrm{d}}_{\circlearrowleft}\tau}\widehat{\mathbb{D}^{\mathrm{d}}} \in \mathrm{H}^2_{\mathrm{strong},\infty} + \mathrm{H}^\infty_{\mathrm{strong},\infty}\mathrm{H}^2_{\mathrm{strong},\infty} \subset \mathrm{H}^2_{\mathrm{strong},\infty}. \tag{9.23}$$

By Lemma 6.8.1(a)&(d1) (applied to Σ^{d}), it follows that $\widehat{\mathbb{C}^{\mathrm{d}}_\flat\tau}, \widehat{\mathbb{D}^{\mathrm{d}}_\flat} \subset \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(Y,*))$. But $\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(Y,*)) \subset \mathrm{ULR} \cap \mathrm{SHPR}$, by Proposition 6.3.3(a). We conclude that the assumptions of Lemma 6.3.6(d1) are satisfied by $\Sigma_\flat$, hence $D^*JD \in \mathcal{GB}(U)$ (as in 1°).

3° *Positive case:* (Note that here we have allowed also assumption 3.) Replace "ULR" by "SLR", "UHPR" by "SHPR" and "(d1)" by "(d2)" in 1°. □

We finish this section by presenting two "generalizations" of Theorem 9.2.16, based on B^*_{w}-CARE theory, which allow one to use weaker assumptions than in the above results, at the cost of having to use the CARE instead of the B^*_{w}-CARE:

Theorem 9.2.18 ($\mathcal{U}_{\mathbf{exp}}$: J-coercive $\Rightarrow$ CARE) *Assume that $\mathbb{A}B \in \mathrm{L}^1([0,1); \mathcal{B}(U,H))$, $C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,1); \mathcal{B}(H,Y))$ and $C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,1); \mathcal{B}(U,Y))$, and that $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{exp}}$. Then the following are equivalent:*

(i) there is a J-critical control over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for each $x_0 \in H$;

(ii) there is a [unique] exponentially stabilizing solution $(\mathcal{P}, S, K)$ of the CARE;

(iii) $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable.

If $(\mathcal{P}, S, K)$ is as in (ii), then K is ULR and J-critical over $\mathcal{U}_{\mathrm{exp}}$, $S = D^*JD \in \mathcal{GB}(U)$, $\mathbb{B}\tau, \mathbb{D}, \mathbb{F} \in \mathrm{MTIC}^{\mathrm{L}^1}_\infty$ and $\mathbb{B}_{\circlearrowleft}\tau, \mathbb{N}, \mathbb{M} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega \subset \mathrm{UHPR}$ for some $\omega < 0$.

Recall from Lemma 8.3.3 that if Σ is estimatable, then J-coercivity over $\mathcal{U}_{\mathrm{exp}}$ is equivalent to J-coercivity over $\mathcal{U}_{\mathrm{out}}$, and that then (i)–(iii) become equivalent to the existence of a (unique) J-critical control over $\mathcal{U}_{\mathrm{str}}$ (or $\mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_{\mathrm{sta}}$ or $\mathcal{U}_{\mathrm{exp}}$).

If $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\mathrm{exp}}$, then "$J$-critical" becomes equivalent to "minimizing", by Lemma 10.2.2.

Proof: We note first that $C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1_\omega(\mathbf{R}_+; \mathcal{B}(U,Y))$ for any $\omega > \omega_A$, by Lemma 6.8.3(c), hence $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\infty(U,Y) \subset \mathrm{ULR}$, by Lemma 6.8.1(e1).

1° *(ii)⇒(i)⇒(iii):* Trivially, we have (i)⇒(iii). If $(\mathcal{P}, S, K)$ solves (ii), then K is J-critical, by Theorem 9.8.5, hence (ii) implies (i).

2° *(iii)⇒(ii):* Assume (iii). By Theorem 9.2.12(iii), Σ has a bounded exponentially stabilizing state feedback operator $\widetilde{K} \in \mathcal{B}(H,U)$. Let $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ and $\Sigma_\flat$ be corresponding state feedback pair and closed-loop system, so that $\omega_{A_\flat} < 0$ and $D_\flat = D$.

By Lemma 6.8.4(c1)&(a1), we have $\mathbb{B}_\flat\tau, \mathbb{D}_\flat, \widetilde{\mathbb{M}} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ for all $\omega > \omega_{A_\flat}$, where $\widetilde{\mathbb{M}} := \widetilde{\mathbb{X}}^{-1} \in \mathcal{G}\mathrm{MTIC}_\infty^{\mathrm{L}^1}(U)$, $\widetilde{\mathbb{X}} := I - \widetilde{\mathbb{F}}$. Thus, (ii) follows from Proposition 9.9.5.

3° *The rest:* By Proposition 9.9.5 (see 2°), the solution of (ii) is unique, K is J-critical over $\mathcal{U}_{\mathrm{exp}}$, and $S = D^*JD \in \mathcal{GB}(U)$.

By Theorem 8.4.9, we have $\mathbb{X}_\natural \in \mathcal{G}\mathrm{MTIC}_\omega^{\mathrm{L}^1}$ for some $\omega \in (\omega_{A_\flat}, 0)$ (since $\mathbb{D}_\flat \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ for any such ω), where $\mathbb{X}_\natural^* S \mathbb{X}_\natural = \mathbb{D}_\flat^* J \mathbb{D}_\flat$ (cf. the proof of Proposition 9.9.5); fix such an ω.

Then, $\mathbb{B}_\circlearrowleft\tau, \mathbb{D}_\circlearrowleft, \mathbb{M} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ (since they are equal to $\mathbb{B}_\flat\tau\mathbb{X}_\natural^{-1}, \mathbb{D}_\flat\mathbb{X}_\natural^{-1}, \widetilde{\mathbb{M}}\mathbb{X}_\natural^{-1}$), hence $\mathbb{F}, \mathbb{X} \in \mathrm{MTIC}_\infty^{\mathrm{L}^1} \subset \mathrm{ULR}$; in particular, K is ULR. □

By strengthening the assumption on $\mathbb{B}\tau$, we can show that J-coercivity is also necessary when D^*JD is invertible:

Corollary 9.2.19 ($\mathcal{U}_{\mathrm{exp}}$: Unique optimum ⇔ CARE ⇔ J-coercive) *Assume that $\mathbb{A}B \in \mathrm{L}^2([0,1);\mathcal{B}(U,H))$, $C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,1);\mathcal{B}(H,Y))$, $C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,Y))$. Then the following are equivalent:*

(i) there is a [unique] J-critical control over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for each $x_0 \in H$, and $D^*JD \in \mathcal{GB}(U)$;

(ii) there is a [unique] exponentially stabilizing solution $(\mathcal{P}, S, K)$ of the CARE;

(iii) $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable and $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{exp}}$.

Any solution of (ii) is as in Theorem 9.2.18.

Note that any solution of (i) or (ii) are unique. See Corollary 10.2.10 for case $D^*JD \gg 0$.

Proof: Set $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. By Theorem 9.2.18, we have (iii)⇒(ii).

1° *(ii)⇒(i):* Assume (ii). Then K is J-critical, by Theorem 9.8.5, and $\mathbb{D}$ is J-coercive, by Proposition 9.9.12(b). By Lemma 9.2.17, we have $D^*JD \in \mathcal{GB}(U)$. Thus, (i) holds.

2° *(i)⇒(iii):* Assume (i). Then $\mathcal{U}_{\mathrm{exp}}(x_0) \neq \{0\}$ for each $x_0 \in H$, i.e., $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable. By Lemma 9.3.7(4), there is at most one (hence exactly one) J-critical control for Σ over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for each $x_0 \in H$. By Proposition 9.9.12(c)(3.), $\mathbb{D}$ is J-coercive, hence (iii) holds.

3° If some triple $(\mathcal{P}, S, K)$ solves (ii), then the assumptions of Theorem 9.2.18 are satisfied (since (ii) implies (iii), by the above), hence its conclusions hold. □

Notes for Sections 9.2 and 9.3

In the case of bounded B and C, the Riccati equation theory for WPLSs is rather well-known (see, e.g., [CZ]). For Pritchard–Salamon systems (which are a special case of Hypothesis 9.2.2(1.)), many of the results of this section are known (see, e.g., [Keu] and [Weiss97]). See also the notes on pp. 465 and 520. In the generality of these two sections, our results seem to be new.

See the notes on p. 418 for Theorem 9.2.14 and Corollary 9.2.15, and the notes on p. 853 for Corollary 9.2.11. Most of Proposition 9.2.4 is contained in [Sal89].

9.3 Proofs for Section 9.2

A witty saying proves nothing.
— Voltaire (1694–1778)

Now we shall show that, under Hypothesis 9.2.1, a J-critical control over $\mathcal{U}_*^*$ in WPLS form (if any) is necessarily of state feedback form:

Proposition 9.3.1 (Case $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$) *Let $\Sigma_{\mathrm{crit}} \in \mathrm{WPLS}(\{0\}, H, Y)$ be a J-critical control for Σ in WPLS form. Assume that $\mathbb{D}$ is ULR and $D^*JD \in \mathcal{GB}(U)$, and that $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$.*

Then $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$, $\mathcal{P}$ satisfies the B_{w}^*-CARE, and

$$K := -(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) \tag{9.24}$$

is the unique ULR J-critical state feedback operator for Σ.

Thus, then Σ_{crit} is of state feedback form.

Proof: Set $S := D^*JD$.

1° *$B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H, U)$:* By Lemma A.3.6, $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{L,w}}^*))$, hence $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H, U)$, because $B_{\mathrm{w}}^* \in \mathcal{B}(\mathrm{Dom}(B_{\mathrm{w}}^*), U)$, by Proposition 6.2.8(b1).

2° *K is ULR and admissible:* By 1°, we have $K := -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) \in \mathcal{B}(H_1, U)$. By Lemma 6.3.17, K is an ULR admissible state feedback operator for Σ,

3° *K is J-critical and unique:* Obviously, $K_{\mathrm{w}} = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC_{\mathrm{w}}) \in \mathcal{B}(\mathrm{Dom}(C_{\mathrm{w}}), U)$. By regularity, $H_B \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{w}})$. Therefore, $K_{\mathrm{w}} = K_{\mathrm{crit}}$ on $\mathrm{Dom}(A_{\mathrm{crit}})$, by (9.66). By Lemma 8.3.17(b), it follows that $\left[\begin{array}{c|c} \Sigma_{\mathrm{crit}} & * \end{array}\right]$ is the corresponding closed-loop system; in particular, K is J-critical. From Lemma 8.3.17(b) we also obtain that K_{w} is the unique J-critical compatible state feedback operator.

4° *$\mathcal{P}$ is a solution of the B_{w}^*-CARE:* By 3° and Corollary 9.9.2, $\mathcal{P}$ is a $\mathcal{U}_*^*$-stabilizing solution of the CARE, hence of the B_{w}^*-CARE, by Proposition 9.2.7(a),

5° *Remark: Case $S := D^*JD \notin \mathcal{GB}(U)$?* If $S^*S \gg 0$ on $\mathrm{Ker}(S)^\perp$ (this is the case whenever $\dim U < \infty$), then the above procedure produces an ULR (hence admissible) K s.t. $SK_{\mathrm{w}} = SK_{\mathrm{crit}}$ on $\mathrm{Dom}(A_{\mathrm{crit}})$ (set, e.g., $K_2 := K|_{\mathrm{Ker}(S)} := 0$).

However, we would in general have $\mathrm{Dom}(A_{\circlearrowleft}) \neq \mathrm{Dom}(A_{\mathrm{crit}})$, (it is not clear whether we could even find a new definition for K_2 s.t. K were admissible (it is for any bounded K_2) and $K_2 = (K_{\mathrm{crit}})_2$ on $\mathrm{Dom}(A_{\mathrm{crit}})$). Consequently, we should somehow prove that the WPLS $\Sigma_{\circlearrowleft}$ is still $\mathcal{U}_*^*$-stabilizing and that (9.65) holds (alternatively, we could more directly verify the equations of Theorem 9.7.1). □

We list here some basic consequences of Hypothesis 9.2.2:

Lemma 9.3.2 *Assume Hypothesis 9.2.2. Then $\mathbb{D}$ is ULR and SVR; in cases (1.)–(5.) we also have $\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1]; H)$ for all $u_0 \in U$, hence then also $\mathbb{B}\tau$ is ULR and SVR.*

If Σ is exponentially stable, then $\mathbb{D}$ is strongly half-plane-regular except possibly in case (7.), $\mathbb{D} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ for some $\omega < 0$ in cases (2.) and (4.). and $\mathbb{D} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ for some $\omega < 0$ in cases (1.)–(5.).

Proof: (The claim on $\mathbb{B}\tau$ follows from Lemma 6.3.16(c) and Lemma 6.8.1(e1).)

For (5.) and (7.), this follows from Proposition 6.3.3(a) and Lemma 6.8.1(d1); for (1.), from Lemma 6.3.16(b); for (2.), from Lemma 9.5.2.

In case (3.), (4.) or (6.), $\mathbb{D}$ is ULR, by Lemma 6.3.16(c), and SVR (and SHPR if Σ is exponentially stable, by Lemma 6.8.1(d1)), by Proposition 6.3.4(a3) or Lemma 6.8.1(a). □

Most classes are closed w.r.t. bounded state feedback:

Lemma 9.3.3 *If(f) Σ satisfies (1.), (2.), (4.) or (5.) of Hypothesis 9.2.2, then the closed-loop system $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ corresponding to any bounded state feedback operator ($K \in \mathcal{B}(H,U)$) satisfies the same condition.* □

(This follows from Lemma 6.8.4(a1)&(c3) and Lemma 9.5.4.)

Now we establish the sufficiency of Hypothesis 9.2.2:

Lemma 9.3.4 *Let Σ_{crit} be a J-critical control for Σ in WPLS form. Assume that Hypothesis 9.2.2 holds. Then $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{L,w}}^*)) \subset \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$.*

Thus, Proposition 9.3.1 applies if, in addition, $D^*JD \in \mathcal{GB}(U)$.

The key to the theory behind Section 9.2 is the method used in 1°–2° below:

Proof: In case (1.) we have $\mathrm{Dom}(B^*) = H$, hence then trivially $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B^*))$ (and $B^* = B_{\mathrm{w}}^* = B_{\mathrm{L,s}}^*$). For case (2.), this will be shown in Lemma 9.6.2. For the others, we go on as follows:

Let $x_0 \in H$ be arbitrary. Choose some $t > 0$. We shall use (9.56) to show that $\mathcal{P}x_0 \in \mathrm{Dom}(B_{\mathrm{L,w}})$.

1° *We have $\mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}_{\mathrm{crit}}^t x_0 \in \mathrm{Dom}(B_{\mathrm{L,s}}) \subset \mathrm{Dom}(B_{\mathrm{L,w}})$ in cases (1.)–(5.):* By Lemma 6.8.1(b2)&(b1), we have $\mathbb{A}^{t\,*} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{L,s}}^*))$. Therefore, (set $z_0 := \mathcal{P}\mathbb{A}_{\mathrm{crit}}^t x_0$)

$$\frac{1}{r} B^* \int_0^r (\mathbb{A}^q)^* \mathbb{A}^{t\,*} z_0 \, dq = \frac{1}{r} B_{\mathrm{L,s}}^* \mathbb{A}^{t\,*} \int_0^r (\mathbb{A}^q)^* z_0 \, dq \to B_{\mathrm{L,s}}^* \mathbb{A}^{t\,*} z_0, \tag{9.25}$$

by continuity. Thus, $\mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}_{\mathrm{crit}}^t x_0 \in \mathrm{Dom}(B_{\mathrm{L,s}}) \subset \mathrm{Dom}(B_{\mathrm{L,w}})$, by Proposition 6.2.8(c1)&(c4)&(d).

2° *Assume (3.):* Set $f := JC\mathbb{A}_{\mathrm{crit}} x_0 \in \mathcal{C}(\mathbf{R}_+; Y)$, $F := C\mathbb{A}Bu_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+; Y)$, so that $\mathbb{D}\chi_{[-r,0)}u_0 = F * \chi_{[-r,0)} = \int_0^r \tau F \, dm$, by Lemma 6.8.1(f). Then

$$\frac{1}{r}\langle \mathbb{B}\chi_{[-r,0)}u_0, \mathbb{C}^{t\,*} J \mathbb{C}_{\mathrm{crit}}^t x_0 \rangle_H = \frac{1}{r}\langle \mathbb{B}\chi_{[-r,0)}u_0, \mathbb{C}^* \pi_{[0,t)} f \rangle_H \tag{9.26}$$

$$= \frac{1}{r}\langle \mathbb{C}\mathbb{B}\chi_{[-r,0)}u_0, \pi_{[0,t)} f \rangle_Y = \frac{1}{r}\int_0^t \Big\langle \int_0^r (\tau^s F)(q)\, dq, f(s) \Big\rangle_Y ds \tag{9.27}$$

$$= \frac{1}{r}\int_0^r \int_0^t \langle (\tau^q F)(s), f(s) \rangle_Y \, ds\, dq = \frac{1}{r}\int_0^r \langle \tau^q F, \pi_{[0,t)} f(s) \rangle_{\mathrm{L}^2} dq \tag{9.28}$$

$$\to \langle F, \pi_{[0,t)} f(s) \rangle_{\mathrm{L}^2}, \tag{9.29}$$

as $r \to 0+$, by continuity, because $\pi_{[0,t)}\tau^q F \to \pi_{[0,t)}F$ in L^1, as $q \to 0+$, and $\pi_{[0,t)}f \in \mathcal{C} \subset \mathrm{L}^\infty([0,t);Y)$ (hence the last integrand in (9.28) is continuous in q). (Obviously, the use of the Fubini Theorem in the beginning of (9.28) was justified.)

Because $(\mathbb{B}^{\mathrm{d}})^*\chi_{[0,r)}u_0 = \mathbb{B}\chi_{[-r,0)}u_0$, and $u_0 \in U$ was arbitrary, it follows from Proposition 6.2.8(c2) that $\mathbb{C}^{t\,*}J\mathbb{C}^t_{\mathrm{crit}}x_0 \in \mathrm{Dom}(B^*_{\mathrm{L,w}})$ ($\in \mathrm{Dom}(B^*_{\mathrm{L,s}})$ if $\mathbb{A}B \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+;\mathcal{B}(U,H))$ (e.g., if (4.) holds), because then $\pi_{[0,t)}\tau^q F \to \pi_{[0,t)}F$ in L^1 independently of u_0 (as long as $\|u_0\|_U \le 1$)).

Combine this with 1° to observe that $\mathcal{P}x_0 \in \mathrm{Dom}(B^*_{\mathrm{L,w}})$ (even $\mathcal{P}x_0 \in \mathrm{Dom}(B^*_{\mathrm{L,s}})$ if $\mathbb{A}B \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+;\mathcal{B}(U,H))$), by (9.56).

3° *Assume (4.):* With the additional assumption that $D^*JC = 0$, this is contained in case (3.). In 6°, we shall remove this assumption.

4° *Assume (5.):* We can work as in case (3.), except that we have to set $f := J\mathbb{C}_{\mathrm{crit}}x_0 \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;Y)$, but we have $F := C_{\mathrm{L,s}}\mathbb{A}Bu_0 \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;Y)$. Now $\pi_{[0,t)}\tau^q F \to \pi_{[0,t)}F$ in L^2, so we again get a convergence as $r \to 0+$, by the Hölder Inequality.

Thus, we again have $\mathcal{P}x_0 \in \mathrm{Dom}(B^*_{\mathrm{L,w}})$ (even $\mathcal{P}x_0 \in \mathrm{Dom}(B^*_{\mathrm{L,s}})$ if $C_{\mathrm{L,s}}\mathbb{A}B \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;\mathcal{B}(U,Y))$, as in 2°).

5° *Assume (6.) or (7.)* Now $\mathbb{C}$ is stable and $\mathcal{P} = \mathbb{C}^*J\mathbb{C}_{\mathrm{crit}}$, by Theorem 8.3.9(b2), so that we may take $t = +\infty$ and skip 1°. In (6.), we have $f \in \mathcal{C}_{\mathrm{b}} \subset \mathrm{L}^\infty$ (by Theorem 8.3.9(a2)&(a3)) and we may work as in 2° (and replace $B^*_{\mathrm{L,w}}$ by $B^*_{\mathrm{L,s}}$ if $\mathbb{D} - D \in \mathrm{L}^1(\mathbf{R}_+;\mathcal{B}(U,Y))*$, i.e., if $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}(U,Y)$). In (7.), we have $f := J\mathbb{C}_{\mathrm{crit}}x_0 \in \mathrm{L}^2(\mathbf{R}_+;Y)$, $F \in \mathrm{L}^2(\mathbf{R}_+;Y)$, and we may work as in 4° (and replace $B^*_{\mathrm{L,w}}$ by $B^*_{\mathrm{L,s}}$ if $\mathbb{D} - D \in \mathrm{L}^2(\mathbf{R}_+;\mathcal{B}(U,Y))*$).

(N.B. We could replace the assumptions of Theorem 8.3.9(b2)&(b2') in (6.) and (7.) by the slightly weaker assumptions that $\mathbb{C}$ is stable and $\mathcal{P} = \mathbb{C}^*J\mathbb{C}_{\mathrm{crit}}$ for any J-critical control in WPLS form, since that assumption is never used for anything else in this monograph.)

6° *Case (4.) when $D^*JC \ne 0$:* Set $K' := -S^{-1}D^*JC \in \mathcal{B}(H,U)$, and let $\Sigma_\flat$ be the corresponding closed-loop system as in Lemma 6.8.4. Then both (3.) and (4.) of Hypothesis 9.2.2 are satisfied with $\Sigma_\flat$ in place of Σ, because $D_\flat = D$ and $C_\flat = C + DK' \in \mathcal{B}(H,Y)$, hence $D^*JC_\flat = D^*J(C + DK') = 0$.

By Theorem 8.4.5(f)&(b), there is a J-critical control over $\mathcal{U}^{\gamma,\Sigma_\flat}_{[\mathbb{Q}_\flat \ \mathbb{R}_\flat]}$ in WPLS form for $\Sigma_\flat$ (and Standing Hypothesis 9.0.1 is obviously satisfied for $\Sigma_\flat$, J and $\begin{bmatrix}\mathbb{Q}_\flat & \mathbb{R}_\flat\end{bmatrix}$ too).

Thus, we obtain a J-critical state feedback operator $K_\natural$ for $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\end{array}\right]$ over $\mathcal{U}^{\gamma,\Sigma_\flat}_{[\mathbb{Q}_\flat \ \mathbb{R}_\flat]}$ from case (3.) of this lemma and Proposition 9.3.1 (which is already known to hold in case (3.)), and then $K := K' + K_\natural \in \mathcal{B}(H,U)$ is J-critical for Σ, by Theorem 8.4.5(f)&(b) and Proposition 6.6.18(f).

Because $\mathcal{P} = \mathbb{C}^*_{\circlearrowleft}J\mathbb{C}_{\circlearrowleft}$ is the same for both systems, $\mathcal{P}[H] \subset \mathrm{Dom}((B_\flat)^*_{\mathrm{L,s}})$, and $B^*_{\mathrm{L,s}} = (B_\flat)^*_{\mathrm{L,s}}$ (with same domains), by Proposition 6.6.18(c6) (since $\widetilde{\mathbb{X}} := I - \mathbf{K}'\mathbb{B}\tau = I - K'\mathbb{A}B* \in \mathrm{MTIC}_\infty$), we have $\mathcal{P}[H] \subset \mathrm{Dom}(B^*_{\mathrm{L,s}})$. $\square$

One might be tempted to try to remove the $(\cdot - A)^{-1}B \in \mathrm{H}^2_{\mathrm{strong},\infty}$ assumption from (5.) in case $\mathcal{U}^*_* = \mathcal{U}_{\mathrm{out}}$ by replacing Σ by a suitable realization of $\mathbb{D}$. However, if we choose, e.g., the shift realization (6.11), and ω is big enough to allow that $\mathbb{B}u := \pi_+\mathbb{D}\pi_- u \in H := \mathrm{L}^2_\omega(\mathbf{R}_+;U)$ for each $u \in \mathrm{L}^2_\alpha$ (for some $\alpha \in \mathbf{R}$), then we do no longer know whether $\mathcal{U}_{\mathrm{out}}(x_0) \neq \emptyset$ for each $x_0 \in H$ (unless $\omega < 0$, in which case this reduces to (7.)). In general, if we weaken $\|\cdot\|_H$ enough to get $\|Bu_0\|_H < \infty$ (or $\mathbb{A}^t Bu_0 \in H$ for some $t = t_{u_0}$) for all $u_0 \in U$, the closure of $B[U]$ (or $\mathrm{Ran}(\mathbb{B})$) in H grows, and we do no longer know whether $\mathcal{U}_{\mathrm{out}}(x_0) \neq \emptyset$ for each $x_0 \in H$.

In case $\mathcal{U}^*_* = \mathcal{U}_{\mathrm{exp}}$, such a removal seems even harder. Analogous problems are faced when one tries to weaken (3.) to the assumption that $\mathbb{D} - D \in \mathcal{B}(U, \mathrm{L}^1_\omega(\mathbf{R}_+;Y))*$, $C \in \mathcal{B}(H,Y)$ and $D^*JC = 0$.

Next we show how a B^*_{w}-CARE can be reduced to a stabilized one:

Proposition 9.3.5 (Σ-B^*_{w}-CARE $\cong$ $\Sigma_\flat$-B^*_{w}-CARE) *Let K' be an ULR state feedback operator for Σ. Then the solutions $(\mathcal{P}_\natural, S_\natural, K_\natural)$ of the B^*_{w}-CARE for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ correspond to the solutions $(\mathcal{P}, S, K)$ of the B^*_{w}-CARE for Σ through*

$$K = K' + K_\natural, \;\; S = S_\natural, \;\; \mathcal{P} = \mathcal{P}_\natural. \tag{9.30}$$

Let K' and $(\mathcal{P}, S, K)$ be as above and $K_\natural = K - K'$. Then

(a) The two top rows ($\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \end{array}\right]$) of the corresponding closed-loop systems are equal, and Lemma 6.7.11(a') and Lemma 9.12.3(a)–(d2) apply.

(b) If K' is [q.]r.c.-SOS-stabilizing, then K is [q.]r.c.-SOS-stabilizing for Σ iff $K_\natural$ is q.r.c.-SOS-stabilizing (equivalently, stable and r.c.-SOS-stabilizing) for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$.

Proof: This follows from Proposition 9.12.4:

0° Let $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ be the pair generated by K', $\mathbb{M}' := (I - \mathbb{F}')^{-1} \in \mathcal{G}\mathrm{TIC}_\infty$, so that $\mathbb{M}' \in \mathcal{G}\mathrm{ULR}$. Then $\mathbb{D}$ is ULR iff $\mathbb{D}_\flat := \mathbb{D}\mathbb{M}'$ is ULR, and $D^*JD = D_\flat^* JD_\flat$; in particular the B^*_{w}-CAREs are well-defined (if either is).

1° Let $(\mathcal{P}, S, K)$ be a solution of the B^*_{w}-CARE for Σ. Then $(\mathcal{P}, S, K_\natural)$ is a solution of the CARE for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$, Proposition 9.12.4. But $\mathrm{Dom}((B^*_\flat)_{\mathrm{w}}) = \mathrm{Dom}(B^*_{\mathrm{w}})$, by Proposition 6.6.18(c5), hence $(\mathcal{P}, S, K_\natural)$ is a solution of the B^*_{w}-CARE too, by Proposition 9.2.7(a).

2° Conversely, by Proposition 9.12.4, all solutions of the B^*_{w}-CARE for $\Sigma_\flat$ are of this form.

3° Exchange the roles of Σ and $\Sigma_\flat$ for the converse.

(a)&(b) These follow from Lemma 9.12.3. □

Also Hypothesis 9.2.1 can be reduced to the stable case:

Lemma 9.3.6

(a) Assume that Σ has an exponentially stabilizing ULR state feedback operator K', $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\end{array}\right]$ satisfies Hypothesis 9.2.1 for $\mathcal{U}_{\text{exp}}^{\Sigma_\flat}$, and $D^*JD \in \mathcal{GB}(U)$. Then Σ satisfies Hypothesis 9.2.1 for $\mathcal{U}_{\text{exp}}$.

(b) Part (a) also holds with the replacements of Theorem 8.4.5(f).

In particular, then the B_{w}^*-CARE for Σ has an exponentially stabilizing solution iff the B_{w}^*-CARE for $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\end{array}\right]$ does (with same $\mathcal{P}$, whereas $K = K' + K_\natural$, where $K_\natural$ corresponds to $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\end{array}\right]$ and K' is the preliminary state feedback operator).

Proof: (a) Since K' and the corresponding closed-loop system $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat\end{array}\right]$ are ULR, by the assumption, Σ is ULR.

By Theorem 8.4.5(c4), $\mathcal{P}$ (if any) is common for $\mathcal{U}_{\text{exp}}$ and $\mathcal{U}_{\text{exp}}^{\Sigma_\flat}$. But $\text{Dom}((B_\flat^*)_{\text{w}}) = \text{Dom}(B_{\text{w}}^*)$, by Proposition 6.6.18(c5), hence also Σ satisfies Hypothesis 9.2.1 for $\mathcal{U}_{\text{exp}}$.

(b) This follows as in 1°. □

The operator D^*JD does not necessarily contain any information on the signature properties of a problem (see Example 9.13.7), but under sufficient regularity it does:

Lemma 9.3.7 ($\exists(D^*JD)^{-1} \Rightarrow$ *J*-critical control is unique) *If any of (1+)–(4) holds and $D^*JD \in \mathcal{GB}(U)$, then there is at most one J-critical control for each $x_0 \in H$.*

(1+) $J \geq 0$ *and* $\mathbb{D} \in \text{UR}$.

(2+) $\mathcal{J} \geq 0$, $\mathbb{D} \in \text{MTIC}_\infty$ *and* $D^*JD \gg 0$.

(3) $\Sigma \in \text{SOS}$, $\mathbb{D} \in \text{MTIC}_\infty$, $\mathbb{A}B \in \text{L}^2([0,1);\mathcal{B}(U,H))$ *and* $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$.

(4) $\mathbb{D} \in \text{MTIC}_\infty$, $\mathbb{A}B \in \text{L}^2([0,1);\mathcal{B}(U,H))$, Σ *is optimizable and* $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$.

Proof: Assume that $u \in \mathcal{U}_*^*(0) \setminus \{0\}$. We shall below construct $v \in \mathcal{U}_*^*(0)$ (the construction depends on the additional assumption; in (1+) and (2+) we shall take $v = u$) s.t. $\langle \mathbb{D}v, J\mathbb{D}u \rangle \neq 0$. Since this shows that u is not J-critical (by definition), it follows 0 is the unique J-critical control for $x_0 = 0$. By Lemma 8.3.8, it follows that there is at most one J-critical control for any $x_0 \in H$. W.l.o.g., we assume that

$$T := \inf\{t \geq 0 \,|\, \|\pi_{[0,t)}u\|_2 = 0\} = 0, \quad \text{i.e., } \|\pi_{[0,t)}u\|_2 > 0 \text{ for all } t > 0. \tag{9.31}$$

(by Lemma 9.7.9, $\tau^T u = \pi_+ \tau^T u \in \mathcal{U}_*^*(0)$, and $\tau^T u$ is J-critical (since $\langle \mathbb{D}\tau^T u, J\mathbb{D}\eta \rangle = \langle \mathbb{D}u, J\mathbb{D}\tau^{-T}\eta \rangle = 0$ for all $\eta \in \mathcal{U}_*^*(0)$), hence u can be replaced by $\tau^T u$ (and later v by $\tau^{-T}v$) to satisfy (9.31)).

(1+) Choose $\varepsilon > 0$ s.t. $D^*JD \gg \varepsilon^2 I$. Then $\|J^{1/2}Du_0\| \geq \varepsilon\|u_0\|$ for all $u_0 \in U$. Choose $\omega > 0$ s.t. $\|J^{1/2}\widehat{\mathbb{D}}(s) - J^{1/2}D\|_{\mathcal{B}} < \varepsilon/2$ for all $s \in \mathbf{C}_\omega^+$. Choose $s \in \mathbf{C}_\omega^+$ s.t. $\widehat{u}(s) \neq 0$. Then $\|J^{1/2}\widehat{\mathbb{D}}(s)\widehat{u}(s)\|_Y > \varepsilon\|\widehat{u}(s)\| - \varepsilon\|\widehat{u}(s)\|/2 > 0$, hence $J^{1/2}\mathbb{D}u \neq 0$, hence $0 < \|J^{1/2}\mathbb{D}u\|_2^2 = \langle \mathbb{D}u, J\mathbb{D}u \rangle$.

(2+) Set $\varepsilon := \|(D^*JD)^{-1/2}\|^{-1} > 0$, so that $\langle u_0, D^*JDu_0\rangle = \|(D^*JD)^{1/2}u_0\|_U^2 \geq \varepsilon^2\|u_0\|_U^2$ for all $u_0 \in U$. By Theorem 2.6.4(i1), there is $t > 0$ s.t.

$$\|\pi_{[0,t)}(\mathbb{D}^*\pi_{[0,t)}J\mathbb{D} - D^*JD)\pi_{[0,t)}\| < \frac{\varepsilon^2}{2}. \tag{9.32}$$

Consequently,

$$\langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle \geq \langle Du, \pi_{[0,t)}JDu\rangle - \frac{\varepsilon^2}{2}\|\pi_{[0,t)}u\|_2^2 \geq \varepsilon^2\|\pi_{[0,t)}u\|_2^2 - \frac{\varepsilon^2}{2}\|\pi_{[0,t)}u\|_2^2 > 0. \tag{9.33}$$

Thus, $\langle \mathbb{D}u, J\mathbb{D}u\rangle = \langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle + \mathcal{J}(\mathbb{B}^t u, \pi_+\tau^t u) \geq \langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle > 0$, by (8.52), hence u is not J-critical.

(3) Set $\varepsilon := \|(D^*JD)^{-1}\|^{-1} > 0$, $M := \|D^*JD\|$, $M' := \|\mathbb{D}^*J\|\|\mathbb{C}\|$, so that $\|D^*JDu_0\| \geq \varepsilon\|u_0\|$ for all $u_0 \in U$. By Theorem 2.6.4(i1), there is $t > 0$ s.t.

$$\|\pi_{[0,t)}D^*JD(\mathbb{D}^*\pi_{[0,t)}J\mathbb{D} - D^*JD)\pi_{[0,t)}\| < \varepsilon^2/2 \quad\text{and}\quad \|\mathbb{B}^t\| < \varepsilon^2/3MM' \tag{9.34}$$

(take $\|\pi_{[0,t)}(J\mathbb{D} - JD)\pi_{[0,t)}\|$ and $\|\pi_{[0,t)}(\mathbb{D} - D)D^*JD\pi_{[0,t)}\|$ small enough). Set $v := \pi_{[0,t)}D^*JDu$. Then

$$\begin{aligned}\operatorname{Re}\langle \mathbb{D}v, J\mathbb{D}u\rangle_{L^2} &= \operatorname{Re}\langle \pi_{[0,t)}\mathbb{D}D^*JDu, J\mathbb{D}u\rangle + \operatorname{Re}\langle \tau^{-t}\mathbb{C}\mathbb{B}^t D^*JDu, J\mathbb{D}u\rangle && (9.35)\\ &\geq \langle DD^*JDu, \pi_{[0,t)}JDu\rangle - \frac{\varepsilon^2}{2}\|\pi_{[0,t)}u\|_2^2 - \frac{\varepsilon}{3}\|\pi_{[0,t)}u\|_2^2 && (9.36)\\ &\geq \|\pi_{[0,t)}D^*JDu\|_2^2 - \frac{5\varepsilon^2}{6}\|\pi_{[0,t)}u\|_2^2 \geq \varepsilon^2\|\pi_{[0,t)}u\|_2^2 - \frac{5\varepsilon^2}{6}\|\pi_{[0,t)}u\|_2^2, && (9.37)\end{aligned}$$

by (9.31).

(4) By Theorem 9.2.12(iii), Σ has a bounded exponentially stabilizing $K \in \mathcal{B}(H,U)$. By Lemma 6.8.4(b), we have $\mathbb{A}_\flat B \in L^2(\mathbf{R}_+; \mathcal{B}(U,H))$ and $\mathbb{D}_\flat \in \text{MTIC}_\infty$.

Since $D_\flat^*JD_\flat = D^*JD \in \mathcal{GB}(U)$, there is at most one J-critical control for $\Sigma_\flat$ over $\mathcal{U}_{\text{exp}}^{\Sigma_\flat}(x_0) = \mathcal{U}_{\text{out}}^{\Sigma_\flat}(x_0)$ for each $x_0 \in H$, by (3).

By Theorem 8.4.5(c2)&(c1) (or the "iff" in (c3)), it follows that there is at most one J-critical control for Σ over $\mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$. □

(See the notes on p. 434.)

9.4 Analytic semigroups

A modern revolutionary group heads for the television station.
— Abbie Hoffman (1936–)

In this section we shall list the basic properties of analytic semigroups. These will be applied in Section 9.5 to WPLSs with an analytic semigroup.

Standing Hypothesis 9.4.1 *Throughout this section, $\mathbb{A}$ is assumed to be an analytic semigroup. We also assume that $\zeta > \omega_A$.*

An *analytic semigroup* is a C_0-semigroup having numbers $\omega > \omega_A$ and $M_{A,\omega} < \infty$ s.t. $\|(s-A)^{-1}\| \le M_{A,\omega}/|s-\omega|$ for all $s \in \mathbf{C}^+_\omega$ (see, e.g., Section 2.5 of [Pazy] for equivalent definitions). (Note from Lemma A.4.4(c1) that any semigroup satisfies $\|(s-A)^{-1}\| \le M_{A,\omega}/\operatorname{Re}(s-\omega)$.)

For all $\beta > 0$, we define the fractional powers of $(\cdot - A)$ by setting

$$(\zeta - A)^{-\beta} := \Gamma(\beta)^{-1}\int_0^\infty t^{\beta-1}\mathrm{e}^{-\zeta t}\mathbb{A}(t)\,dt \in \mathcal{B}(H) \tag{9.38}$$

$$(\zeta - A)^{\beta} := ((\zeta - A)^{-\beta})^{-1}, \quad H_\beta := \operatorname{Dom}((\zeta-A)^\beta) := \operatorname{Ran}((\zeta-A)^{-\beta}). \tag{9.39}$$

We let H_β be the closure of H w.r.t. $\|\cdot\|_{H_\beta} := \|(\zeta-A)^\beta\cdot\|_H$ for $\beta < 0$, and set $\|x\|_{H_\beta} := \|(\zeta-A)^\beta x\|_H$ for all $\beta \in \mathbf{R}$ (these definitions coinside with spaces H_n ($n \in \mathbf{Z}$) defined in Lemma 6.1.16; in particular, $H_0 = H$). These spaces are independent of ζ up to an equivalent norm, by Lemma 9.4.2(f2).

Lemma 9.4.2 (Properties of analytic semigroups)

(a) For any $\omega > \omega_A$, there are $\theta \in (\pi/2, \pi]$ and $M = M_{A,\omega,\theta} < \infty$ s.t. $\|(s-A)^{-1}\| \le M/|s-\omega|$ for all s in

$$\Sigma_{\theta,\omega} := \{s \in \mathbf{C} \mid s \ne \omega,\ |\arg(s-\omega)| < \theta\}. \tag{9.40}$$

(Note that $\mathbf{C}^+_{\omega'} \subset \Sigma_{\theta,\omega}$ for any $\omega' > \omega$.)

(b) $H_\gamma \underset{c}{\subset} H_\beta$ densely ($\gamma \ge \beta$), the map $(\zeta-A)^\beta \in \mathcal{B}(H_{\gamma+\beta}, H_\gamma)$ is an isometric isomorphism, and $(\zeta-A)^\beta(\zeta-A)^\gamma = (\zeta-A)^{\beta+\gamma}$ ($\beta,\gamma \in \mathbf{R}$).

(c1) Also $\mathbb{A}^*$ is analytic, and $(H_\beta)^* = (H^*)_{-\beta} =: H^*_{-\beta}$ ($\beta \in \mathbf{R}$).

(c2) Also $\mathrm{e}^{s\cdot}\mathbb{A}$ (with generator $s + A$) is analytic for any $s \in \mathbf{C}$.

(d) The semigroup $\mathbb{A}_\beta := (\zeta-A)^{-\beta}\mathbb{A}(\zeta-A)^\beta$ on H_β is isometrically isomorphic to $\mathbb{A}$ ($\beta \in \mathbf{R}$); we denote all these semigroups on $\mathbb{A}$ and their generators by A.

(e) $(\omega-A)^\beta\mathbb{A}^t = \mathbb{A}^t(\omega-A)^\beta$ and $(\omega-A)^\beta(s-A)^{-1} = (s-A)^{-1}(\omega-A)^\beta$ ($t \ge 0,\ \beta \in \mathbf{R},\ \omega > \omega_A,\ s \in \sigma(A)^c$).

(f1) $(s-A)^{-1} \in \mathcal{GB}(H_\beta, H_{\beta+1})$ for any $s \in \sigma(A)^c$, $\beta \in \mathbf{R}$, by the resolvent equation.

(f2) $(\omega - A)^{-\beta} \in \mathcal{GB}(H_\alpha, H_{\alpha+\beta})$ *(*$\alpha, \beta \in \mathbf{R}$, $\omega > \omega_A$*).*

(f3) $(\cdot - A)^{-1} \in \mathrm{H}^\infty(\mathbf{C}^+_\zeta; \mathcal{B}(H_\beta, H_{\beta+1}))$, $(s \mapsto s(s-A)^{-1}) \in \mathrm{H}^\infty(\mathbf{C}^+_\zeta; \mathcal{B}(H_\beta))$ *(*$\beta \in \mathbf{R}$*).*

(f4) $A \in \mathcal{GB}(H_\beta, H_{\beta-1})$, $(\beta \in \mathbf{R})$.

(g) $(s-A)^{-1} \to 0$ *strongly in* $\mathcal{B}(H_\beta, H_{\beta+1})$, *as* $s \in \Sigma_{\theta,\omega}$, $|s| \to +\infty$, *for any* ω *and* θ *as in (a).*

(h1) For each $\omega > \omega_A$, *there is* $M' < \infty$ *s.t.*

$$\|(\zeta - A)^\beta \mathbb{A}^t\|_{\mathcal{B}(H)} \le M'(1 + t^{-\beta})\mathrm{e}^{\omega t} \quad (t > 0,\ \beta \in [0,1]). \tag{9.41}$$

(h2) We have $\mathbb{A}^t \in \mathcal{B}(H_\alpha, H_\beta)$ *and* $\|\mathbb{A}^t\|_{\mathcal{B}(H_\alpha, H_{\alpha+\beta})} \le M_{\beta,\zeta} t^{-\beta} \mathrm{e}^{\zeta t}$ $(\alpha, \beta \in \mathbf{R},\ t > 0)$.

(i) $\mathbb{A} \in \mathcal{C}^\infty((0,+\infty); \mathcal{B}(H_\beta))$ *(*$\beta \in \mathbf{R}$*).*

(j) $(\zeta - A)^\beta \mathbb{A} \in \mathrm{L}^p_\omega(\mathbf{R}_+; \mathcal{B}(H_\alpha))$ *(*$\beta p < 1$, $\omega > \omega_A$, $p \in [1,\infty]$, $\beta \le 1$, $\alpha \in \mathbf{R}$*).*

(k) $\|(\omega - A)^\alpha (s-A)^{-1}\|_{\mathcal{B}(H)} \le M(1 + |s - \omega_0|^\alpha)/|s - \omega_0|$ *(*$s \in \Sigma_{\theta,\omega_0}$, $\omega > \omega_0 > \omega_A$, $0 \le \alpha \le 1$*) for* $\theta \in (\pi/2, \pi]$ *as in (a), where* $M < \infty$ *depends only on A,* ω_0 *and* θ.

(l) $(\omega - A)^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty s^{-\alpha}(s + \omega - A)^{-1}\, ds$ *(*$\omega > \omega_A$, $\alpha \in (0,1)$*).*

Proof: (a) One obtains this from Theorem 2.5.2 of [Pazy] (move the sector to the left and decrease the angle simultaneously, so that the sector is contained in the union of the old sector plus a compact subset of $\mathbf{C}^+_{\omega_A}$).

(c1) By, e.g., Theorem 5.2 of [Pazy] (shifted by ζ_A), $\mathbb{A}^*$ is analytic with uniformly bounded growth bound ζ_A.

Define the spaces $(H^*)_\beta =: H^*_\beta$ with A^* in place of A. For all $x \in H_\beta$, $z \in H$, we have

$$|\langle x, z\rangle_H| = |\langle (\zeta - A)^\beta x, (\zeta - A^*)^{-\beta} z\rangle_H| \le \|x\|_{H_\beta} \|z\|_{H^*_{-\beta}}, \tag{9.42}$$

hence $H^*_{-\beta}$ is the closure of H w.r.t. $\|\cdot\|_{(H_\beta)^*}$, hence it can be identified to $(H_\beta)^*$.

(f2) Use, e.g., the sine formula (2.6.4) of [Pazy] to show that the ranges of $(\zeta - A)^{-\beta}$ and $(r - A)^{-\beta}$ are equal for $\beta \in (0,1)$. Integral powers agree by the resolvent equation (i.e., by (f1)), and negative powers follow from the positive ones. The topologies coincide by Lemma A.3.6.

(f3) Use the resolvent equation (boundedness follows from the definition of an analytic semigroup).

(g) This was shown in Lemma A.4.4(e2) (which provides us several convergence results) in case $\beta = 0$; the other cases follow from (d).

(i) This follows from Corollary 2.4.4 and Theorem 2.5.2(d) of [Pazy].

(j) By (i), $(\zeta - A)^\beta \mathbb{A} \in \mathrm{L}(\mathbf{R}_+; \mathcal{B}(H_\beta))$. By (h) (with some $\alpha \in (\omega_A, \omega)$ in place of ω), we have $\|(\zeta - A)^\beta \mathbb{A}\| \in \mathrm{L}^p_\omega$.

(b)&(c2)–(f4)&(h1)&(h2)&(k)&(l) All this is well-known or almost obvious; see, e.g., [Pazy], [Sbook], [Lunardi] and/or [HP]. □

If the "discontinuity" of a perturbation (e.g., "feedback operator") T is less than one, the resulting ("closed-loop") semigroup is also analytic:

Lemma 9.4.3 (Properties of interpolation spaces H_α $(\alpha \in \mathbf{R})$)

(a1) Let $T \in \mathcal{B}(H_\gamma, H_\alpha)$, $\gamma - \alpha < 1$. Then the operator $A+T$ with domain $H_{\beta+1}$ generates an analytic semigroup on H_β, for any $\beta \in [\gamma-1, \alpha+1]$.

In particular, if we define the spaces $\widetilde{H}_\beta$ ($r \in \mathbf{R}$) as H_β with $A+T$ in place of A, then $H_\beta = \widetilde{H}_\beta$ (with equivalent norms), i.e., $(\omega - A)^{-\beta}[H] = (\omega - A - T)^{-\beta}[H]$, for all $\beta \in [\gamma-1, \alpha+1]$.

(a2) Even if we shifted the indices by replacing H by H_{β_0} (i.e., by considering A as a semigroup on H_{β_0}, not on H), for any $\beta_0 \in [\gamma-1, \alpha+1]$), before defining the spaces H_β and $\widetilde{H}_\beta$ (in (a1)), the results and the spaces H_β and $\widetilde{H}_\beta$ would be unaltered (except for the shift in the index).

(b1) Let A and $\widetilde{A}$ generate analytic semigroups on H, and $\mathrm{Dom}(A) = \mathrm{Dom}(\widetilde{A})$. Define the spaces $\widetilde{H}_\alpha$ ($\alpha \in \mathbf{R}$) as H_α with $\widetilde{A}$ in place of A. Then

$$\widetilde{H}_1 = H_1 \underset{c}{\subset} H_\alpha \underset{c}{\subset} \widetilde{H}_\beta \underset{c}{\subset} H_\gamma \underset{c}{\subset} H = \widetilde{H}_0 \quad (1 \geq \alpha \geq \beta \geq \gamma \geq 0). \tag{9.43}$$

(b2) We have $(\omega - A)^{-\alpha} \in \mathcal{B}(\widetilde{H}_\beta, \widetilde{H}_\gamma)$ and $(r - \widetilde{A})^{-\alpha} \in \mathcal{B}(H_\beta, H_\gamma)$ in (b1) when $\alpha > \gamma - \beta$, $\beta, \gamma \in [0,1]$, $\omega > \omega_A$, $r > \omega_{\widetilde{A}}$.

Note that if $0 \in [\gamma-1, \alpha+1]$, then $A+T$ generates an analytic semigroup on H, and the definition of $\widetilde{H}_\alpha$ can be based on H (instead of some H_β).

It seems that if we would replace the spaces H_r ($r \in (0,1)$) by the interpolation spaces $\mathcal{D}(r,p)$ or $\mathcal{D}(p)$ of [Lunardi], we would obtain analogous results even more easily. However, we have chosen the spaces H_r to make the comparison to earlier results easier. Because of the interpolation properties, the difference in smoothness is less than ε for any $\varepsilon > 0$.

Proof of Lemma 9.4.3: (a1) *Part I: Assume that $T \in \mathcal{B}(H_1, H_\alpha)$, $\alpha \in (0,1]$.*

1° *We have $(\omega - A - T)^{-\beta} \in \mathcal{B}(H, H_\beta)$ for $\beta \in [0,1)$:* Assume, w.l.o.g., that $\beta > 0$. By Propositions 2.4.1(ii) and 2.2.13 of [Lunardi], $A+T$ with $\mathrm{Dom}(A+T) := \mathrm{Dom}(A)$ generates an analytic semigroup on H. Obviously,

$$(s-A)^{-1}(I + T(s-A-T)^{-1}) = (s-A-T)^{-1} \quad (s \in \sigma(A)^c \cap \sigma(A+T)^c). \tag{9.44}$$

Let $\omega > \omega_1 := \max\{\omega_A, \omega_{A+T}\}$ and $\beta \in (0,1)$. Then, by (9.44), we have (here $c := \pi^{-1} \sin \pi\alpha$)

$$(\omega - A - T)^{-\beta} - (\omega - A)^{-\beta} = c \int_0^\infty s^{-\beta} (\omega + s - A)^{-1} T (\omega + s - A - T)^{-1} \, ds. \tag{9.45}$$

By Lemma A.4.4(c2), $\|T(\omega + s - A - T)^{-1}\|_{\mathcal{B}(H, H_\alpha)} \leq M_\omega$ and $\|(\omega + s - A)^{-1} T (\omega + s - A - T)^{-1}\|_{\mathcal{B}(H, H_{\alpha+1})} \leq M'_\omega$ for $s \geq 0$. Thus, part $\int_0^1$ of the integral (9.45) converges in $H_{\alpha+1}$, so we only need to show that also $(\omega - A)^\beta \int_1^\infty$

belongs to $\mathcal{B}(H)$ in order to establish that $(\omega - A - T)^{-\beta} \in \mathcal{B}(H, H_\beta)$. By Lemma 9.4.2(k), this follows from the fact that (choose any $\omega_0 \in (\omega_A, \omega)$)

$$\int_1^\infty s^{-\beta} M(1 + |s + \omega - \omega_0|^{\beta-\alpha})/|s + \omega - \omega_0|\, ds < \infty. \tag{9.46}$$

2° *We have $H_\beta = \widetilde{H}_\beta$ for $\beta \in [0,1)$:* By 1°, we have $\widetilde{H}_\beta := \operatorname{Ran}((\omega - A - T)^{-\beta}) \subset H_\beta$. Exchange the roles of A and $A+T$ to obtain that $H_\beta \subset \widetilde{H}_\beta$. By Lemma A.3.6, also the topologies coincide (i.e., the norms are equivalent).

Part II: Assume that $T \in \mathcal{B}(H_\alpha, H)$, $\alpha < 1$. W.l.o.g., we assume that $\alpha \geq 0$, because $H \underset{c}{\subset} H_\alpha$, hence $\mathcal{B}(H_\alpha, H) \subset \mathcal{B}(H_0, H)$ for $\alpha < 0$.

3° *We have $H_\beta = \widetilde{H}_\beta$ for $\beta \in [\alpha - 1, \alpha]$:* Apply 2° to the state space $H_{\alpha-1}$.

4° *We have $H_\beta = \widetilde{H}_\beta$ for $\beta \in [\alpha - 1, 1]$:* Let $\beta \in [\alpha - 1, 0]$. Then $(I + T(s - A - T)^{-1}) \in \mathcal{GB}(H_\beta)$ for s big enough, hence the two resolvents have same range, i.e., $H_{\beta+1} = \widetilde{H}_{\beta+1}$, by (9.44). Thus, we have covered the values in $[\alpha, 1]$; the others were covered in 3°.

Part III: Assume that $T \in \mathcal{B}(H_\gamma, H_\alpha)$, $\gamma - \alpha < 1$: Just apply Part II with H_α in place of H.

5° *Remarks:* Note that we again identify the analytic semigroup $\widetilde{\mathbb{A}}$ on $H_{\alpha-1}$ generated by $A+T$ with domain $\operatorname{Dom}(A+T) := H_\alpha$ and its restriction onto H_β (which is generated by $A+T$ with domain $\operatorname{Dom}(A+T) := H_{\beta+1}$) for any $\beta \in [\alpha - 1, 1]$.

If $\beta > 1$, then we may have $\widetilde{H}_\beta \neq H_\beta$: Let $\beta \in (1,2]$. Unless A is bounded, we can choose $x_0 \in H_2$, $\alpha \in [0,1)$, $z \in H \setminus H_{\beta-1}$ and $\Lambda \in H_\alpha^*$ s.t. $\Lambda x_0 = 1$, and set $Tx := (Lax)z$ $(x \in H_\alpha^*)$, so that $T \in \mathcal{B}(H_\alpha, H)$ and $Tx_0 = z \notin H_{\beta-1}$. It follows that $(A+T)x_0 \notin H_{\beta-1}$, hence $x_0 \in H_\beta \setminus \widetilde{H}_\beta$ (since $(A+T)[\widetilde{H}_\beta] \subset \widetilde{H}_{\beta-1} = H_{\beta-1}$).

(a2) (You can consider A (or its restriction or extension) as an analytic semigroup on any H_β, and it is not always obvious, which one you should use. Here we stated that you may use any of them and still get the same results and spaces.)

Set $G := H_{\beta_0}$, and define the spaces G_β and $\widetilde{G}_\beta$ as in (a1) (by (a1), this is possible and $G_\beta = \widetilde{G}_\beta$ for all $\beta \in [\gamma - 1 - \beta_0, \alpha + 1 - \beta_0]$).

By Lemma 9.4.2(f2), we have $G_\beta = H_{\beta_0+\beta}$ and $\widetilde{G}_\beta = \widetilde{H}_{\beta_0+\beta}$ $(\beta \in \mathbf{R})$. Thus, we obtained again the same fractional power spaces, even though we had the starting point G in place of H.

(b1) By Lemma A.3.6, the spaces $H_1 := \operatorname{Dom}(A)$ and $\widetilde{H}_1 := \operatorname{Dom}(\widetilde{A})$ have equivalent norms. If $\alpha = 1$, then $H_\alpha = \widetilde{H}_1 \underset{c}{\subset} \widetilde{H}_\beta$, so we assume that $\alpha < 1$ (and $\beta > 0$). By Propositions 2.2.13 and 1.2.3 of [Lunardi], we have

$$H_\alpha \underset{c}{\subset} (H, H_1)_{\alpha,\infty} \underset{c}{\subset} (H, H_1)_{\beta,1} \underset{c}{\subset} \widetilde{H}_\beta. \tag{9.47}$$

Analogously, $\widetilde{H}_\beta \underset{c}{\subset} H_\gamma$.

(b2) This follows from (9.43) and Lemma 9.4.2(b)&(f2). □

Notes

As obvious from the proofs, most of the above is well known. Classical references on semigroups include [Pazy] and [HP]; see [Lunardi] and [Sbook] for further results on analytic semigroups.

9.5 Parabolic problems and CAREs

All animals are equal, but some animals are more equal than others
— George Orwell (1903–1950), "Animal Farm", 1945

In this section, we apply our results for systems having analytic (see Section 9.4) semigroups. The proofs only use the fact that the smoothness of $\mathbb{A}$ compensates the unboundedness of B and C, so that the results of this section (with same proofs, mutatis mutandis) can be applied whenever $\mathbb{A}$ is smoothing.

In Theorem 9.5.9, we show that a smooth Riccati equation has a stabilizing solution iff there is a unique optimal control. The resulting closed-loop system is also analytic. A corollary of this for minimization problems is given in In Corollary 9.5.10 we interpret this for minimization problems, and in Corollaries 9.5.11–9.5.12 for H^∞ problems. E.g., there is a suboptimal H^∞ state feedback pair for the system iff the Riccati equation (11.15) has a nonnegative solution s.t. $A+BK$ is exponentially stable (assuming the standard signature and nonsingularity conditions therein); moreover, the Riccati equation can be replaced by the smoother on in Theorem 9.5.9 and the closed-loop system is analytic.

We assume that $\mathbb{A}$ is analytic and that the unboundedness the input and output operators is less than $1/2$ each:

Standing Hypothesis 9.5.1 *Throughout this section and Section 9.6, letters U, H and Y denote Hilbert spaces of arbitrary dimensions, $\mathbb{A}$ is an analytic semigroup, $\zeta > \omega_A$, $\beta > -1/2$, $\gamma < 1/2$, $B \in \mathcal{B}(U, H_\beta)$, $C \in \mathcal{B}(H_\gamma, Y)$, $D \in \mathcal{B}(U,Y)$, $J = J^* \in \mathcal{B}(Y)$.*

(By Lemma 9.5.2, it follows that $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ generate a WPLS (also without Standing Hypothesis 9.0.1).)

By spaces H_r we denote the spaces $(\zeta - A)^{-r}H$ ($r \in \mathbf{R}$), as in Section 9.4. Note that we may assume that $\gamma \ge 0 \ge \beta$, w.l.o.g. (i.e., if the hypothesis holds and we replace γ by $\max\{0,\gamma\}$ and β by $\min\{0,\beta\}$, then the hypothesis still holds, by Lemma 9.4.2(b)).

Drop the assumptions $\beta > -1/2$, $\gamma < 1/2$ for a moment. Whenever $\gamma, \beta \in \mathbf{R}$ and $\gamma - \beta < 1$, we can replace H by $H_{(\gamma+\beta)/2}$ above to obtain a WPLS, by Lemma 9.5.2. The only difference between different state spaces is that in applications we must require that the finite cost condition is satisfied in H (not in some H_r), hence one might wish to take H as small as possible. Therefore, in some applications it might be more suitable to realize $\mathbb{D}$ on, e.g., H_γ, in which case Σ need no longer be a WPLS (unless $\gamma - \beta < 1/2$); cf. Section 8.6.

The system $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a smooth Pritchard–Salamon system w.r.t. $\mathcal{W} := H_\gamma$ and $\mathcal{V} := H_\beta$ whenever $\gamma - \beta < 1/2$, as one can deduce from the following:

Lemma 9.5.2 *Let $\omega > \omega_A$. Then the operators $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ generate $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}_\omega(U,H,Y)$. Moreover, $\mathbb{A}B \in \mathrm{L}^2_\omega(\mathbf{R}_+; \mathcal{B}(U,H))$, $C\mathbb{A} \in \mathrm{L}^2_\omega(\mathbf{R}_+; \mathcal{B}(U,H))$, and $C\mathbb{A}B \in \mathrm{L}^1_\omega(\mathbf{R}_+; \mathcal{B}(U,Y))$, hence $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}_\omega \subset \mathrm{ULR} \cap \mathrm{UVR}$. Finally, $C \subset (C_{|H_1})_{\mathrm{L,s}}$.*

Note that the "standard" C (that of Chapter 6) is given by $C|_{H_1}$. Analogously, the "standard" B is given by $B \in \mathcal{B}(U, H_{-1})$ with adjoint $B^* \in \mathcal{B}(H_1^*, U)$, but we shall write B^* for the adjoint $B^* \in \mathcal{B}(H_{-\beta}^*, U)$ of $B \in \mathcal{B}(U, H_\beta)$; we still have $B^* \subset B^*_{\mathrm{L,s}}$, where $B^*_{\mathrm{L,s}}$ can be computed from either B^*.

Proof: (This holds also without Standing Hypothesis 9.0.1.) The "moreover" claims follow from Lemma 9.4.2(j), and they imply that $\Sigma \in$ WPLS, by Lemmas D.1.7 and 6.3.13. We have $C \subset (C|_{H_1})_{\mathrm{L,s}}$, because $\frac{1}{t}\int_0^t \mathbb{A}^r x_0\,dr \to x_0$ in H_γ, for any $x_0 \in H_\gamma$, by Lemma 9.4.2(d), □

Remark 9.5.3 (Optimizable iff exponentially stabilizable) *The* L^1 *assumptions of Theorem 9.2.12 and Corollary 9.2.13 are satisfied (under Standing Hypothesis 9.5.1). In particular,* Σ *is optimizable [and estimatable] iff* Σ *is exponentially [jointly] stabilizable [and detectable].* □

The class of analytic WPLSs is invariant under smooth feedback, and the spaces H_α remain unaffected (for α's sufficiently close to zero):

Lemma 9.5.4 *If* $K \in \mathcal{B}(H_r, U)$, $r < 1/2$, *then* K *is an ULR admissible state feedback operator for* Σ. *Moreover, if* $\Sigma_\flat$ *is the closed-loop system corresponding to* $\begin{pmatrix} K & \big| & I - X \end{pmatrix}$ *for some* $X \in \mathcal{GB}(U)$, *then* $\mathbb{A}_\flat$ *is analytic on* $H_\alpha = (H_\flat)_\alpha$ *for* $\alpha \in [r-1, \beta+1]$, *and also* $\Sigma_\flat$ *satisfies Hypothesis 9.5.1 (with* $\gamma_{\flat'} := \max\{\gamma, r\} < 1/2$ *in place of* γ*).*

From Proposition 6.6.18 it follows that the closed-loop generators are given by $\left[\begin{array}{c|c} A+BMK & BM \\ \hline C+DMK & DM \\ K & M-I \end{array}\right]$, where $M := X^{-1}$.

Proof: 1° *Case* $X = I$*:* By Lemma 9.5.2, $\left[\begin{array}{c|c} A & B \\ \hline K & 0 \end{array}\right]$ generate a WPLS $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{K} & \mathbb{F} \end{array}\right]$ s.t. $\mathbb{F} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1} \subset$ ULR and $F = 0$, hence $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ (even $\mathcal{G}\mathrm{MTIC}_\infty^{\mathrm{L}^1}(U)$), by Proposition 6.3.1(c). The last claim follows from Lemma 9.4.3(a) (recall that $C_\flat = C + DK$).

2° *General case:* First apply 1° to $\begin{pmatrix} MK & \big| & 0 \end{pmatrix}$ (note that $MK \in \mathcal{B}(H_r, U)$), and then apply Lemma 6.6.12 to replace $\begin{pmatrix} MK & \big| & 0 \end{pmatrix}$ by $\begin{pmatrix} XMK & \big| & I - X \end{pmatrix}$. □

Lemma 9.5.5 *If* $\mathcal{P} \in \mathcal{B}(H)$, *then* $\|sB^*(s-A)^{-*}\mathcal{P}(s-A)^{-1}B\| \to 0$, *as* $s \in \Sigma_{\theta,\omega_0}$, $|s| \to \infty$. □

(This follows from Lemma 9.4.2(k) with $\alpha \mapsto -\beta < 1/2$.)

Corollary 9.5.6 *If* $(\mathcal{P}, S, \begin{bmatrix} K & \big| & F \end{bmatrix})$ *is a SR solution of the eCARE, then* $X^*SX = D^*JD$. □

(This follows from Lemma 9.5.5 and Propositions 9.11.3(b) and 9.8.10.)

We shall often make the following, stronger hypothesis:

Hypothesis 9.5.7 *Let at least one of (1.)–(3.) hold, where*

(1.) $\gamma < 1/4$, $\beta > -1/2$ *and* $D^*JC = 0$*;*

(2.) $\gamma < 1/4$, $\beta > -1/2$ *and* $D^*JD \in \mathcal{GB}(U)$*;*

(3.) $\gamma - \beta < 1/2$*;*

By Theorem 9.2.3, Hypothesis 9.5.7 implies Hypothesis 9.2.1, hence it implies that Theorems 9.2.9, 9.2.10 and 9.2.14, Corollary 9.2.15 etc. apply (under the additional assumptions of those results).

Naturally, whenever $\gamma, \beta \in \mathbf{R}$, $\gamma - \beta < 1/2$, we can replace H by H_r for any $r \in (\gamma - 1/2, \beta + 1/2)$ to satisfy (3.). Analogously, whenever $\gamma, \beta \in \mathbf{R}$, $\gamma - \beta < 3/4$ and $D^*JC = 0$ or $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ and $D^*JD \in \mathcal{GB}(U)$, we can replace H by H_r for any $r \in (\gamma - 1/4, \beta + 1/2)$ to satisfy (1.) or (2.). However, we need additional assumptions for guaranteeing that this does not change the problem (the most important case is the stable case for $\mathcal{U}_{\text{out}}$ or $\mathcal{U}_{\exp}$):

Lemma 9.5.8 (Shifting $H \mapsto H_r$) *Let Σ' be the system Σ with state space H_r, where $r \in (\gamma - 1/2, \beta + 1/2)$, and that $J = J^* \in \mathcal{B}(Y)$. Assume that $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\text{out}}$ (so that $\mathbb{D}'$ is J-coercive over $\mathcal{U}'_{\text{out}}$). Assume that $\mathcal{U}_{\text{out}}(x_0) \neq \emptyset$ for all $x_0 \in H$ and $\mathcal{U}'_{\text{out}}(x_0) \neq \emptyset$ for all $x_0 \in H_r$ (e.g., that $\mathbb{A}$ is exponentially stable).*

If $r > 0$, then $\mathcal{P}' = \mathcal{P}|_{H_r}$ and $\Sigma'_{\text{crit}} = \Sigma_{\text{crit}}|_{H_r}$ (if K' is as in Lemma 9.6.1, then K' is the unique J-critical state feedback operator for Σ too). Moreover, $\mathcal{U}_{\text{out}}(x_0) = \mathcal{U}'_{\text{out}}(x_0)$ for all $x_0 \in H \cap H_r$.

(Exchange the roles of H and H_r for the case $r < 0$.)

Proof: (Here $\mathcal{U}'_{\text{out}}$ means $\mathcal{U}_{\text{out}}$ for Σ' etc., and $\mathcal{P}$ (resp. $\mathcal{P}'$) is the J-critical cost operator over $\mathcal{U}_{\text{out}}$ (resp. $\mathcal{U}'_{\text{out}}$). Note that $r \in (\gamma - 1/2, \beta + 1/2)$ is equivalent to the condition that Standing Hypothesis 9.5.1 holds for Σ'.)

Note that $\mathcal{U}_{\text{out}}(x_0)$ depends on x_0, $\mathbb{D}$ and J only (not on H, nor on the rest of Σ).

Recall from Lemma 8.4.2 that J-coercivity over $\mathcal{U}_{\text{out}}$ depends only on J and $\mathbb{D}$, hence $\mathbb{D}'$ is J-coercive over $\mathcal{U}'_{\text{out}}$.

Under the assumptions, there is a unique J-critical control over $\mathcal{U}_{\text{out}}$ for each $x_0 \in H$ and over $\mathcal{U}'_{\text{out}}$ for each $x_0 \in H_r$, by Theorem 8.4.3, thus $\mathcal{P}$, Σ_{crit} and Σ'_{crit} are well-defined, by Theorem 8.3.9.

Assume that $r > 0$. Since ($\mathbb{A}' = \mathbb{A}|_{H_r}$ and) $\mathbb{C}' = \mathbb{C}|_{H_r}$, we have $\mathcal{U}_{\text{out}}(x_0) = \mathcal{U}'_{\text{out}}(x_0)$ for all $x_0 \in H_r = H_r \cap H$ (because $\mathbb{C}x_0 + \mathbb{D}u = \mathbb{C}'x_0 + \mathbb{D}u$ for all $u \in \mathrm{L}^2(\mathbf{R}_+; U)$).

Given $x_0 \in H_r$, a vector $u \in \mathcal{U}_{\text{out}}(x_0) = \mathcal{U}'_{\text{out}}(x_0)$ is J-critical over $\mathcal{U}_{\text{out}}$ iff it is J-critical over $\mathcal{U}'_{\text{out}}$ (iff $0 = \langle \mathbb{C}'x_0 + \mathbb{D}u, J\mathbb{D}\eta \rangle = \langle \mathbb{C}x_0 + \mathbb{D}u, J\mathbb{D}\eta \rangle$ for all $\eta \in \mathcal{U}_{\text{out}}(0) = \mathcal{U}'_{\text{out}}(0)$). It follows that $\Sigma'_{\text{crit}} = \Sigma_{\text{crit}}|_{H_r}$, hence $\mathcal{P}' := (\mathbb{C}'_{\text{crit}})^* J \mathbb{C}'_{\text{crit}} = \mathcal{P}|_{H_r}$.

If K' is as in Lemma 9.6.1, then $K' \in \mathcal{B}(H_\gamma, U)$, hence then K' is admissible for Σ too; let $\Sigma_\circlearrowleft$ be the corresponding closed-loop system. The restriction of

K_{crit} to H_{r+1} is equal to $K'_{\text{crit}} = K'$, hence $K_{\text{crit}} = K'$ on H_1, by continuity. We conclude that $\mathbb{K}_{\circlearrowleft} = \mathbb{K}_{\text{crit}}$, i.e., K' is J-critical for Σ over $\mathcal{U}_{\text{out}}$. Uniqueness follows from Lemma 8.3.17(b). □

Under Hypothesis 9.5.7 we have more smoothness than in the B_{w}^*-CARE theory:

Theorem 9.5.9 *Assume that* $S := D^*JD \in \mathcal{GB}(U)$, $\gamma < 1/4$ *and* $\beta > -1/2$. *Then the assumptions of Theorem 9.2.9 are satisfied and we have one more equivalent condition:*

(vi) There is $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H, (H_\beta)^*)$ *s.t.*

$$(B^*\mathcal{P} + D^*JC)^*(D^*JD)^{-1}(B^*\mathcal{P} + D^*JC) = A^*\mathcal{P} + \mathcal{P}A + C^*JC \qquad (9.48)$$

on $H_{\gamma+\varepsilon}$ *for some (equivalently, all)* $\varepsilon > 0$, *and* $K := -(D^*JD)^{-1}(B^*\mathcal{P} + D^*JC)$ *is* $\mathcal{U}_*^*$*-stabilizing.*

Moreover, also Lemmas 9.6.1 and 9.5.4 are applicable for $\mathcal{P}$ *and* K.

Proof: 1° *The proof:* The assumptions of Theorem 9.2.9 are now satisfied, by Theorem 9.2.3 (use Hypothesis 9.2.2(2.)). One easily verifies that (vi) implies (iii). Conversely, (i) implies (vi), by Lemma 9.6.1.

2° *Remarks:* Recall that $(H_\beta)^* = H_{-\beta}^*$. Condition $\mathcal{P} \in \mathcal{B}(H, (H_\beta)^*)$ implies that $B^*\mathcal{P} \in \mathcal{B}(H, U)$, hence necessarily $K \in \mathcal{B}(H_\gamma, U)$, so that Lemma 9.5.4 applies for the closed-loop system (in particular, K is necessarily admissible and $\mathbb{A}_{\circlearrowleft}$ is analytic).

"On $H_{\gamma+\varepsilon}$" means that $\langle Kx_0, D^*JDKx_1\rangle = \langle Ax_0, \mathcal{P}x_1\rangle + \langle \mathcal{P}x_0, Ax_1\rangle + \langle Cx_0, JCx_1\rangle$ for all $x_0, x_1 \in H_{\gamma+\varepsilon}$, equivalently, that $K^*D^*JDK = A^*\mathcal{P} + \mathcal{P}A + C^*JC$ in $\mathcal{B}(H_{\gamma+\varepsilon}, (H_{\gamma+\varepsilon})^*)$.

For $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, the last condition means that the (analytic) semigroup generated by $A + BK$ is exponentially stable. □

The above remarks lead to applications of the theory in other sections, such as the ones below:

Corollary 9.5.10 (LQR — minimization)

(a) The assumptions of Corollary 10.2.10 are satisfied.

(b) Assume that Hypothesis 9.5.7 holds and that $D^*JD \gg 0$. *Then the assumptions of Corollary 10.2.9 are satisfied, and Lemma 9.6.1 applies; in particular, the* B_{w}^**-CARE in (ii) is satisfied on* $H_{\gamma+\varepsilon}$ *too (iff it is satisfied on* H_1*), for any* $\varepsilon > 0$, *and* $\mathbb{A}_{\circlearrowleft}$ *is analytic.*

A similar comment applies to Theorem 10.1.4(b3)&(b4)&(b6).

(c) Assume that Σ *is estimatable. Then there is a nonnegative solution of the LQR-CARE iff* Σ *is optimizable. If this is the case, then this solution is unique and strictly minimizing over* $\mathcal{U}_{\text{exp}}$ *(and* $\mathcal{U}_{\text{out}}$*).*

If $\gamma < 1/4$ *and* $\beta > -1/2$, *then Theorem 9.5.9 applies.*

Note that the proof and the remarks below Hypothesis 9.5.7 lead to further simplifications of the results mentioned in the corollary and several others.

See the comments below Corollary 10.4.4 for the parabolic H^2 problem.

Proof: (a) This follows from Lemma 9.5.2.

(b) The assumptions follow from Theorem 9.2.3; the rest follows from Lemma 9.6.1.

(c) See Theorem 10.1.4(c1) and the above, and note that now $D^*JD \gg 0$. □

Under Standing Hypothesis 9.5.1, we can strengthen our H^∞ Full-Information Control Problem results:

Corollary 9.5.11 (H^∞ FICP)

(a) Assumption (2.) of Theorem 11.1.4 is satisfied.

(b) Assume Hypothesis 9.5.7.

Then assumption (2.) of Theorems 11.1.3 and 11.1.6 is satisfied, and Lemma 9.6.1 applies; in particular, the B_{w}^*-CARE in (iii) is satisfied on $H_{\gamma+\varepsilon}$ too (iff it is satisfied on H_1), for any $\varepsilon > 0$, and $\mathbb{A}_\circlearrowleft$ is necessarily analytic.

(c) The assumptions of Theorem 11.2.7 are satisfied for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ iff (A, B_1) is optimizable and (1.) of Theorem 11.1.4 holds.

(Do not mix the γ ("γ_C") of this section (and (b) above) with that ("challenge number") of Chapter 11. This applies Corollary 9.5.12 too.)

Proof: (a) This follows from Lemma 9.5.2.

(b) Assumption (2.) follows from Theorem 9.2.3. The rest follows from Lemma 9.6.1.

(c) Now there is an exponentially stabilizing state feedback operator $\widetilde{K} \in \mathcal{B}(H, U)$ iff (A, B_1) is optimizable, by Remark 9.5.3. By Lemma 9.5.4, corresponding (exponentially stable) closed-loop system $\Sigma_\flat$ (and $\Sigma_\circlearrowleft$ if any) also satisfies Hypothesis 9.5.1, in particular, $\mathbb{D}_\flat \in \mathrm{MTIC}(U, Y)$, by Lemma 9.5.2. See 0.1° of the proof of Theorem 11.1.4 for (1.). □

Our results for the standard H^∞ problem (the H^∞ "measurement-feedback" or "four-block" problem) can also be strengthened:

Corollary 9.5.12 (H^∞ 4BP) *Assume Hypothesis 12.1.1.*

(a) Assumption (A1) of Theorem 12.1.5 is satisfied.

(b) Assume (A1)(I) and (A2) of Theorem 12.1.4. Then Lemma 9.6.1 applies for the three CAREs. In particular, the $\mathcal{P}_{\mathrm{X}}$-CARE (1.) is satisfied on $H_{\gamma+\varepsilon}$ too (iff it is satisfied on H_1), for any $\varepsilon > 0$, and $\mathbb{A}_\circlearrowleft$ is necessarily analytic. Analogously, the $\mathcal{P}_{\mathrm{Y}}$-CARE (2.) (and the $\mathcal{P}_{\mathrm{Z}}$-CARE (4.) if (1.) is satisfied) is satisfied on $H^*_{-\beta+\varepsilon}$ (iff it is satisfied on H_1^*), for any $\varepsilon > 0$.

Proof: (Note from Lemma 12.5.4 that Hypothesis 12.5.1 is satisfied iff (A, B_1) is optimizable, (A, C_2) is estimatable and (A2) of Theorem 12.1.5 is satisfied, by (a).)

(a) This follows from Lemma 9.5.2.

(b) It was shown in 4° of the proof of Theorem 12.1.4 that (2.) or (3.) of Hypothesis 9.5.7 is satisfied by Σ_X and Σ_Y; the same was observed for Σ_Z (assuming that Σ_X is satisfied) close to the end of the proof. Therefore, Lemma 9.6.1 applies to these CAREs. □

With our (weaker) standing hypothesis that $\gamma < 1/2$ and $\beta > -1/2$, we obtain the following:

Theorem 9.5.13 ($\mathcal{U}_{\text{exp}}$: Unique optimum ⇔ CARE ⇔ J-coercive) *Assume that $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$. Then the following are equivalent:*

(i) there is a [unique] J-critical control over $\mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$, and $D^*JD \in \mathcal{GB}(U)$;

(ii) there is a [unique] exponentially stabilizing solution $(\mathcal{P}, S, K)$ of the CARE;

(iii) $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable and $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\text{exp}}$.

If $(\mathcal{P}, S, K)$ is as above, then K is J-critical over $\mathcal{U}_{\text{exp}}$, $S = D^*JD \in \mathcal{GB}(U)$, $\mathbb{B}\tau, \mathbb{D}, \mathbb{F} \in \mathrm{MTIC}_\infty^{\mathrm{L}^1}$ and $\mathbb{B}_{\circlearrowleft}\tau, \mathbb{N}, \mathbb{M} \in \mathrm{MTIC}_{-\varepsilon}^{\mathrm{L}^1}$ for some $\varepsilon > 0$.

Note that any solutions of (i) or (ii) are unique. If $D^*JD \gg 0$, then "J-critical" becomes equivalent to "minimizing" and "J-coercive" equivalent to "positively J-coercive", by Proposition 9.9.12. Thus, then Corollary 10.2.10 (and Theorem 10.1.4(b6)&(b4)) applies.

Proof: This follows from Lemma 9.5.2 and Corollary 9.2.19. □

Notes for Sections 9.5 and 9.6

A state-of-art treatment on parabolic systems is given in [LT00a] by Irena Lasiecka and Roberto Triggiani, by using a p.d.e. approach. They only treat the LQR and H^∞ FICP problems, in the case of $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ with a standard cost function (whose form is a special case of that of Proposition 9.7.6), but they allow for any $\beta > -1$ (for $\gamma = 0$, i.e., they take $H = H_\gamma$, so that C becomes bounded), hence their results also apply to systems with very unbounded input operators, thus covering a wide range of important applications. Since their proofs are based on the convergence of the finite-horizon solutions, they might be inappropriate for more general cost functions, as explained in the notes on p. 465.

The history and status quo of research in the p.d.e. approach to analytic systems is well documented in the notes of [LT00a], with references to works of the authors, G. Da Prato, F. Flandoli, X. Li, C. McMillan and others, including articles on singular control.

The first application of WPLS theory to parabolic systems seems to be [St97a] (and [S98e]), which uses the theory of [S97b] to establish a solution to the positively J-coercive stable quadratic minimization problem and show that its J-critical cost operator satisfies the Riccati equation. Its proof leans heavily on the results of Lasiecka and Triggiani, so that its contribution is mainly the WPLS formulation and the characterization of the closed-loop sensibility to external input (i.e., the second column of $\Sigma_{\circlearrowleft}$). That article and [Sbook] essentially contain Lemmas 9.5.2, 9.5.8 and 9.6.3.

9.6 Parabolic problems: proofs

The voice of the majority is no proof of justice.

— Friedrich von Schiller (1759–1805)

In this section, we prove three lemmas that were used for the results of the above section. We still assume Standing Hypothesis 9.5.1.

We first show that a J-critical control in WPLS form (e.g., a unique J-critical control, by Lemma 8.3.16(a1)), is always of state feedback form and corresponds to a solution of the B_{w}^*-CARE:

Lemma 9.6.1 *Let Σ_{crit} be a J-critical control for Σ in WPLS form. Assume that Hypothesis 9.5.7 holds and that $S := D^*JD \in \mathcal{GB}(U)$.*

Then $\mathcal{P} \in \mathcal{B}(H,(H_r)^*)$ for any $r < 1-2\gamma$; in particular, $\mathcal{P}[H] \subset (H_\beta)^* = \mathrm{Dom}(B^*) \subset \mathrm{Dom}(B_{\mathrm{L,s}}^*) \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$. Thus, $B^*\mathcal{P} \in \mathcal{B}(H,U)$, and

$$K := -S^{-1}(B^*\mathcal{P} + D^*JC) \in \mathcal{B}(H_\gamma, U) \tag{9.49}$$

is a unique ULR J-critical state feedback operator for Σ. Moreover $\mathcal{P}$ satisfies the B_{w}^*-CARE

$$A^*\mathcal{P} + \mathcal{P}A + C^*JC = K^*SK \in \mathcal{B}(H_1,(H_1)^*) \tag{9.50}$$

(even $\in \mathcal{B}(H_{\gamma+\varepsilon},(H_{\gamma+\varepsilon})^*)$ for all $\varepsilon > 0$).

Finally, the corresponding closed-loop system $\Sigma_\circlearrowleft$ is generated by $A+BK$, and $\Sigma_\circlearrowleft$ is analytic on $H_\alpha = (H_\circlearrowleft)_\alpha$ for $\alpha \in [\gamma-1, \beta+1]$. Consequently, $\mathbb{B}_\circlearrowleft\tau, \mathbb{D}_\circlearrowleft, \mathbb{F}_\circlearrowleft \in \mathrm{MTIC}_\omega^{\mathrm{L}^1}$ for all $\omega > \omega_{A_\circlearrowleft}$.

See the remark for B^* below Lemma 9.5.4. Note that K is bounded (i.e., $K \in \mathcal{B}(H,U)$) if $D^*JC = 0$.

Proof: 1° *K is ULR and J-critical:* By Lemma 9.6.2, we have $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{L,s}}^*))$. By Proposition 9.3.1, $K := -S^{-1}(B_{\mathrm{L,s}}^*\mathcal{P} + D^*JC) \in \mathcal{B}(H_\gamma, U)$ is a unique ULR J-critical state feedback operator for Σ (in particular, $\mathbb{K}_\circlearrowleft = \mathbb{K}_{\mathrm{crit}}$, where $\Sigma_\circlearrowleft$ is the corresponding closed-loop system). We obtain "$\mathcal{P}(H,H_r)$" from 2° and the rest of the claim from Lemmas 9.5.4 and 9.5.2 and 3°.

2° *We have $\mathcal{P} \in \mathcal{B}(H_{\gamma-r},(H_{\gamma-s})^*)$ when $r,s \geq 0$, $r+s < 1$:* (For $\gamma - r = 0$ this becomes $\gamma - s > 2\gamma - 1$, hence $\gamma - s = \beta$ is allowed, so that $\mathcal{P} \in \mathcal{B}(H,(H_\beta)^*)$.) W.l.o.g., we assume that $\gamma \geq 0$.

Choose $p,q \in [1,\infty]$ s.t. $p^{-1} > r$, $q^{-1} > s + 2\gamma$ and $p^{-1} + q^{-1} = 1$. Then $C_{\mathrm{crit}}\mathbb{A}_{\mathrm{crit}} \in \mathrm{L}_\omega^p(\mathbf{R}_+;\mathcal{B}(H_{\gamma-r},Y))$, by Lemma 9.4.2(j) (since $C_{\mathrm{crit}} = C + DK_{\mathrm{crit}} \in \mathcal{B}(H_\gamma, Y)$), and $C\mathbb{A} \in \mathrm{L}_\omega^q(\mathbf{R}_+;\mathcal{B}(H_{\gamma-s},Y))$, hence $\mathbb{C}^*\pi_{[0,t)}J\mathbb{C}_{\mathrm{crit}} \in \mathcal{B}(H_{\gamma-r},(H_{\gamma-s})^*)$.

But $\mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}_{\mathrm{crit}}^t \in \mathcal{B}(H_{\gamma-1},(H_v)^*)$, for any $v \in \mathbf{R}$, by Lemma 9.5.4 and Lemma 9.4.2(h2). Therefore, $\mathcal{P} \in \mathcal{B}(H_{\gamma-r},(H_{\gamma-s})^*)$.

3° *The B_{w}^*-CARE on $H_{\gamma+\varepsilon}$:* Let $\varepsilon > 0$. We have $A \in \mathcal{B}(H_{\gamma+\varepsilon}, H_{\gamma+\varepsilon-1})$, $\mathcal{P} \in \mathcal{B}(H_{\gamma+\varepsilon-1},(H_\gamma)^*)$ (take $s = 0$, $r := 1-\varepsilon$), and $(H_\gamma)^* \underset{c}{\subset} (H_{\gamma+\varepsilon})^*$, hence $\mathcal{P}A \in \mathcal{B}(H_{\gamma+\varepsilon},(H_{\gamma+\varepsilon})^*)$.

By taking adjoints, we obtain that $A^*\mathcal{P} \in \mathcal{B}(H_{\gamma+\varepsilon},(H_{\gamma+\varepsilon})^*)$. Since obviously $C^*JC, K^*SK \in \mathcal{B}(H_{\gamma+\varepsilon},(H_{\gamma+\varepsilon})^*)$, and H_1 is dense in $H_{\gamma+\varepsilon}$, the B_{w}^*-CARE holds also on $H_{\gamma+\varepsilon}$, i.e.,

$$\langle Ax_0,\mathcal{P}x_1\rangle_{\langle H_{\gamma+\varepsilon-1},(H_{\gamma+\varepsilon-1})^*\rangle}+\langle \mathcal{P}x_0,Ax_1\rangle_{\langle (H_{\gamma+\varepsilon-1})^*,H_{\gamma+\varepsilon-1}\rangle}+\langle Cx_0,JCx_1\rangle_Y=\langle Kx_0,SKx_1\rangle. \tag{9.51}$$

□

The above proof was based on the following result:

Lemma 9.6.2 *Let Σ_{crit} be a J-critical control for Σ in WPLS form. Assume that Hypothesis 9.5.7 holds. Then $\mathcal{P} \in \mathcal{B}(H,\mathrm{Dom}(B_{\mathrm{L,s}}^*))$.*

Proof: 1° *Assumption (1.):* Now we have to alter 2° of the proof of Lemma 9.3.4 as follows:

Let $\omega > \omega_A$. By Lemma 9.4.2(j), we have $C\mathbb{A}B \in \mathrm{L}^p_\omega(\mathbf{R}_+;\mathcal{B}(U,Y))$ for any $p \in [1,(\beta+\gamma)^{-1})$; set $p := 4/3$. Analogously, $\mathbb{C} = C\mathbb{A} \in \mathrm{L}^4_\omega(\mathbf{R}_+;\mathcal{B}(H,Y))$.

Fix $x_0 \in H$ and $u_0 \in U$. Then $\mathbb{K}_{\mathrm{crit}}x_0 \in \mathrm{L}^2(\mathbf{R}_+;Y)$, hence $C\mathbb{B}\tau\mathbb{K}_{\mathrm{crit}}x_0 = C\mathbb{A}B * \mathbb{K}_{\mathrm{crit}}x_0 \in \mathrm{L}^4_{\mathrm{loc}}(\mathbf{R}_+;Y)$, because $p^{-1}+2^{-1} = 1+4^{-1}$ (see Lemma D.1.7). Consequently, $f := C\mathbb{A}_{\mathrm{crit}}x_0 = C(\mathbb{A}+\mathbb{B}\tau\mathbb{K}_{\mathrm{crit}})x_0 \in \mathrm{L}^4_{\mathrm{loc}}(\mathbf{R}_+;Y)$. Thus,

$$\mathbb{C}_{\mathrm{crit}}x_0 - D\mathbb{K}_{\mathrm{crit}}x_0 = C\mathbb{A}_{\mathrm{crit}}x_0 \in \mathrm{L}^4_{\mathrm{loc}}(\mathbf{R}_+;Y). \tag{9.52}$$

Set $F := C\mathbb{A}Bu_0 \in \mathrm{L}^p_\omega$. Because $p^{-1}+4^{-1} = 1$ (and $C^*JD = 0$), we can again work as in 2° of the proof of Lemma 9.3.4 to obtain that $\mathcal{P}x_0 \in \mathrm{Dom}(B_{\mathrm{L,s}}^*)$.

(The moral of the proof: the "p" in "L^p" does not always mean 1, 2 or ∞.)

2° *Assumption (2.):* Part 6° of the proof of Lemma 9.3.4 applies here too, mutatis mutandis; we sketch this below:

By Lemma 9.5.4, $K' := -(D^*JD)^{-1}D^*JC \in \mathcal{B}(H_\gamma,U)$ is admissible and ULR for Σ, and assumption (1.) is satisfied by the corresponding closed-loop system (which has the input operator B and the output operator $C+DK' \in \mathcal{B}(H_\gamma,U)$; the spaces H_β, H_γ and $\mathrm{Dom}(B_{\mathrm{L,s}}^*)$ remain unchanged).

3° *Assumption (3.):* This follows from Lemma 9.3.4, since Hypothesis 9.2.2(5.) is satisfied, by Lemma 9.4.2(j). □

Lemma 9.6.3 *If $\mathbb{D} \in$ TIC, then $\mathbb{D}$ is uniformly half-plane-regular.*

(Naturally, this is not true without Standing Hypothesis 9.5.1.)

Proof: We shall show that we have $\widehat{\mathbb{D}}(s) \to D$, as $s \in \mathbf{C}^+$, $|s| \to \infty$ regardless of the stability of $\mathbb{D}$. By Definition 6.2.3, it follows that $\mathbb{D}$ is UHPR if $\mathbb{D} \in$ TIC (or $\mathbb{D}[\mathrm{L}^2_{\mathrm{c}}] \subset \mathrm{L}^2$).

Let $\omega > \omega_0 > \omega_A$. Choose θ and M for ω_0 as in Lemma 9.4.2(a). Then

$$\widehat{\mathbb{D}}(s) - D = C(\omega-A)^{-\gamma}\cdot(\omega-A)^{\gamma+\beta}(s-A)^{-1}\cdot(\omega-A)^{-\gamma}B \to 0, \tag{9.53}$$

as $s \in \Sigma_{\theta,\omega_0}$, $|s| \to \infty$, by Lemma 9.4.2(k). □

(See the notes on p. 451.)

9.7 Riccati equations on $\mathrm{Dom}(A_{crit})$

Just because they are called 'forbidden' transitions does not mean that they are forbidden. They are less allowed than allowed transitions, if you see what I mean.

— From a Part 2 Quantum Mechanics lecture.

In Theorem 8.3.9, we observed that a unique J-critical control is always of WPLS form. In most earlier theory (e.g., for bounded B), such a control is necessarily of state feedback form, and corresponding cost, signature and state feedback operators $(\mathcal{P}, S, K)$ satisfy the CARE. Conversely, the K-operator of any "stabilizing" solution of the CARE is a J-critical state feedback operator (also in our setting). We shall extend this equivalence to sufficiently regular systems with unbounded B in Theorem 9.2.9, based on the results of this section (see also Remark 9.9.14).

In this section, we shall study the situation for an arbitrary WPLS with a unique J-critical control (or more generally, with a control in WPLS form) without assuming the J-critical control to be of (well-posed) state feedback form (cf. Remark 9.7.7(a1)–(a3)), and we shall derive certain Riccati-like equations.

Thus, given, e.g., a J-coercive ($\mathcal{U}_*^*$-) stabilizable regular system, we can obtain the optimal state feedback by solving the generalized Riccati equation (9.67). However, since these equations are given on $\mathrm{Dom}(A_{\mathrm{crit}})$ (for WR systems), which is unknown a priori, it seems very hard to solve such equations and thus find the optimal control. The integral version given below (for general WPLSs) seems even less applicable.

Our results rigourously extend the equation in [FLT] (contained in Proposition 9.7.6), where similar equations are given in a coercive, positive setting with a bounded output operator. A more general setting, still with bounded C, is treated in Lemma 9.7.5, and for general WR systems the equations are given in Theorem 9.7.3.

However, for the above reasons and others explained in this chapter (e.g., D^*JD cannot serve as the signature operator, as explained in the notes to Section 9.8), we consider the "regular Riccati equations" due to M. Weiss, G. Weiss and O. Staffans more applicable than these "closed-loop domain Riccati equations", and prefer developing their theory to cover standard control problems. Neverthless, the latter can be build on the former, as we partially do, and in some cases the two theories coincide (see the parabolic theory of Chapter 9.5).

If there is a J-critical control in WPLS form (e.g., a unique J-critical control, see Lemma 8.3.16(a1)), then Σ_{crit} is "$\mathcal{U}_*^*$-stable" and (9.55)–(9.57) hold:

Theorem 9.7.1 (Generalized IARE) *Assume that* $\Sigma_0 := \begin{bmatrix} \mathbb{A}_0 \\ \mathbb{C}_0 \\ \mathbb{K}_0 \end{bmatrix}$ *is a control in WPLS form, and that* $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$.

Then $\mathbb{K}_0 x_0$ *is J-critical for* x_0 *for each* $x_0 \in H$ *and* $\mathcal{P} = \mathbb{C}_0^* J \mathbb{C}_0$ *iff* $\mathbb{K}_0 x_0 \in$

$\mathcal{U}_*^*(x_0)$ *for all* $x_0 \in H$ *and the following hold:*

$$\langle \mathbb{B}^t u + \mathbb{A}_0^t x_0, \mathcal{P}\mathbb{A}_0^t x_0 \rangle \to 0, \text{ as } t \to +\infty \ \ (x_0 \in H,\ u \in \mathcal{U}_*^*(0)), \tag{9.54}$$

$$0 = (\mathbb{D}^t)^* J \mathbb{C}_0^t + (\mathbb{B}^t)^* \mathcal{P}\mathbb{A}_0^t \in \mathcal{B}(H, \mathrm{L}^2([0,t];U)), \tag{9.55}$$

$$\mathcal{P} = \mathbb{A}^{t\,*} \mathcal{P}\mathbb{A}_0^t + \mathbb{C}^{t\,*} J \mathbb{C}_0^t \in \mathcal{B}(H). \tag{9.56}$$

We can make the following enhancements above:

(a) We may replace (9.56) above by

$$\mathcal{P} = \mathbb{A}_0^{t\,*} \mathcal{P}\mathbb{A}_0^t + \mathbb{C}_0^{t\,*} J \mathbb{C}_0^t \in \mathcal{B}(H). \tag{9.57}$$

(b) Condition (9.54) is redundant if $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ *or* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{str}}$.

(c) If $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ *(resp.* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{str}}$*), then* Σ_0 *is J-critical iff* Σ_0 *is exponentially (resp. strongly) stable and (9.55)–(9.56) hold.*

Thus, we can consider (9.55)–(9.56) as a generalized IARE, whose solution $\mathcal{P}$ is "stabilizing" iff $\mathbb{K}_{\circlearrowleft} x_0 \in \mathcal{U}_*^*(x_0)$ and (9.54) holds. When applied to the left column of a (state feedback) closed-loop system, these conditions determine whether the corresponding state feedback pair is J-critical.

Since the above conditions are hard to verify, we go on to develop further conditions, but first we make an observation from part (c) above:

Corollary 9.7.2 ($\mathcal{U}_{\exp} \Rightarrow \mathcal{U}_{\mathrm{str}}$) *If there is a unique J-critical control over* $\mathcal{U}_{\exp}$*, then this control is J-critical over* $\mathcal{U}_{\mathrm{str}}$. □

(The converse does not hold even for exponentially stabilizable systems, by Example 9.13.14, which also illustrates that a J-critical state feedback operator over $\mathcal{U}_{\mathrm{str}}$ need not be strongly stabilizing, although it must stabilize the left column strongly).

Proof of Theorem 9.7.1: Trivially, condition $\mathbb{K}_{\circlearrowleft} x_0 \in \mathcal{U}_*^*(x_0)$ $(x_0 \in H)$ is necessary. For the rest of the proof, we assume that this condition holds. Consequently, $\mathbb{C}_{\circlearrowleft}$ is stable and Theorem 8.3.9(a2) holds.

1° *"Only if":* Given $\widetilde{\eta} \in \mathrm{L}^2([0,t);U)$, we have for $\eta := \widetilde{\eta} + \tau^{-t}\mathbb{K}_0\mathbb{B}^t\widetilde{\eta}$ ($\in \mathcal{U}_*^*(0)$, by Lemma 9.7.10) and any $x_0 \in H$ that

$$0 = \langle J\mathbb{C}x_0 + \mathbb{D}\mathbb{K}_0 x_0, \mathbb{D}\eta \rangle = \langle (\pi_{[0,t)} + \tau^{-t}\tau^t\pi_{[t,\infty)}) J\mathbb{C}_0 x_0, \mathbb{D}\eta \rangle \tag{9.58}$$

$$= \langle \pi_{[0,t)} J\mathbb{C}_0 x_0, \mathbb{D}\eta \rangle + \langle J\pi_+ \tau^t \mathbb{C}_0 x_0, \mathbb{D}\tau^t \eta \rangle \tag{9.59}$$

$$= \langle \pi_{[0,t)} J\mathbb{C}_0 x_0, \mathbb{D}\eta \rangle + \langle J\mathbb{C}_0 \mathbb{A}_0^t x_0, \mathbb{C}_0\mathbb{B}^t\widetilde{\eta} \rangle \tag{9.60}$$

$$= \langle J\mathbb{C}_0^t x_0, \mathbb{D}^t\widetilde{\eta} \rangle + \langle \mathbb{A}_0^t x_0, \mathcal{P}\mathbb{B}^t\widetilde{\eta} \rangle. \tag{9.61}$$

Thus, (9.55) holds. Equation (9.57) and the convergence $(\mathbb{A}_0^t)^*\mathcal{P}\mathbb{A}_0^t x_0 \to 0$ are obtained as in the proof of Lemma 9.10.1(d1).

Let now $x_0 \in H$ and $\eta \in \mathcal{U}_*^*(0)$ be arbitrary. Because $\langle \pi_{[0,t)}\mathbb{D}\eta, J\mathbb{C}_0 x_0 \rangle \to \langle \mathbb{D}\eta, J\mathbb{C}_0 x_0 \rangle$, as $t \to \infty$, by Corollary B.3.8, equation (9.55) implies that

$$\langle \mathbb{B}\tau^t\eta, \mathcal{P}\mathbb{A}_0^t x_0 \rangle \to -\langle \mathbb{D}\eta, J\mathbb{C}_0 x_0 \rangle, \quad \text{as } t \to +\infty. \tag{9.62}$$

Because $\eta \in \mathcal{U}_*^*(0)$ was arbitrary, J-criticality implies that (9.54) holds.

2° *"If":* Assume that Σ_0 is $\mathcal{U}_*^*$-stable and that (9.54)–(9.57) hold.

Identity $\mathcal{P} = \mathbb{C}_0^* J \mathbb{C}_0$ is obtained as in the proof of Lemma 9.10.1(d1). From (9.62) and (9.54) we obtain that $\langle \mathbb{D}\eta, J\mathbb{C}_0 x_0\rangle = 0$.

3° *Remark:* If the equivalent conditions hold, then $\mathbb{A}_0^t \mathcal{P} \mathbb{A}_0^t x_0, \mathcal{P}\mathbb{A}_0^t x_0 \to 0$, as $t \to \infty$, as one observes from the proof of Lemma 9.10.1(d1).

(a) Equation (9.57) is equivalent to (9.56), because $(9.56)^* - (9.57) = \mathbb{K}_0^*((\mathbb{B}^t)^* \mathcal{P} \mathbb{A}_0^t + (\mathbb{D}^t)^* J \mathbb{C}_0^t) = 0$ when (9.55) holds.

(b) Let $x_0 \in H$ and $\eta \in \mathcal{U}_*^*(0)$. If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{str}}$, then $\mathbb{A}_0 x_0, \mathbb{B}\tau\eta \in C_0(\mathbf{R}_+; H)$, by Theorem 8.3.9(a2), hence then (9.54) obviously holds.

If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then $\mathbb{A}_0 x_0, \mathbb{B}\tau\eta \in \mathrm{L}^2(\mathbf{R}_+; H)$, hence then the limit in (9.62) cannot be nonzero in any case, so it must be zero (since it exists, by (9.62)).

(c) "Only if" follows from Theorem 8.3.9(a2), and "if" from (b). □

If $\mathbb{D}$ is regular, then it follows that certain Riccati equation is satisfied on $\mathrm{Dom}(A_{\mathrm{crit}})$:

Theorem 9.7.3 ($\mathbf{Dom}(A_{\mathbf{crit}})$-CARE) *Let Σ_{crit} be a J-critical control for Σ in WPLS form. Then*

$$-A_{\mathrm{crit}}^* \mathcal{P} = \mathcal{P} A_{\mathrm{crit}} + C_{\mathrm{crit}}^* J C_{\mathrm{crit}} \qquad \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}), \mathrm{Dom}(A_{\mathrm{crit}})^*), \tag{9.63}$$
$$-A_{\mathrm{crit}}^* \mathcal{P} = \mathcal{P} A + C_{\mathrm{crit}}^* J C \qquad \in \mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A_{\mathrm{crit}})^*), \tag{9.64}$$
$$-A^* \mathcal{P} = \mathcal{P} A_{\mathrm{crit}} + C^* J C_{\mathrm{crit}} \qquad \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}), \mathrm{Dom}(A)^*). \tag{9.65}$$

Recall that $C_{\mathrm{crit}} = C_{\mathrm{c}} + D_{\mathrm{c}} K_{\mathrm{crit}}$ and $A_{\mathrm{crit}} = A + BK_{\mathrm{crit}}$ on $\mathrm{Dom}(A_{\mathrm{crit}}) \subset H_B$.

Assume, in addition, that $\mathbb{D}$ is WR. Then

(a) **("$K_{\mathrm{crit}} = -B^*\mathcal{P}$")** *$\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}), \mathrm{Dom}(B_{\mathrm{L,w}}^*))$, $B_{\mathrm{L,w}}^* \mathcal{P} = -D^* J C_{\mathrm{crit}}$ on $\mathrm{Dom}(A_{\mathrm{crit}})$, and*

$$(D^* J D) K_{\mathrm{crit}} = -B_{\mathrm{L,w}}^* \mathcal{P} - D^* J C_{\mathrm{L,w}} \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}), U). \tag{9.66}$$

(b) **("$u_{\mathrm{crit}} = -B^*\mathcal{P}x_{\mathrm{crit}}$")** *$(K_{\mathrm{crit}})_{\mathrm{L,s}} x(t) = -(D^*JD)^{-1}(B_{\mathrm{L,w}}^* \mathcal{P} + D^* J C_{\mathrm{L,w}}) x(t) = (\mathbb{K}_{\mathrm{crit}} x_0)(t)$ for a.e. $t \geq 0$ and all $x_0 \in H$, where $x := x_{\mathrm{crit}}(x_0) := \mathbb{A}_{\mathrm{crit}} x_0$, if $D^*JD \in \mathcal{GB}(U)$. In particular, $\mathcal{P}x(t) \in \mathrm{Dom}(B_{\mathrm{L,w}}^*)$ a.e.*

(c) **("CARE" on $\mathbf{Dom(A_{crit})}$)** *If $\mathbb{D}$ and $\mathbb{D}^{\mathrm{d}}$ are SR and $D^*JD \in \mathcal{GB}(U)$, then we can replace $B_{\mathrm{L,w}}^*$ by $B_{\mathrm{L,s}}^*$ and $C_{\mathrm{L,w}}$ by $C_{\mathrm{L,s}}$ in (a) and (b), and we have*

$$A^* \mathcal{P} + \mathcal{P} A + C^* J C_{\mathrm{L,s}} = (\mathcal{P} B + C^* J D)(D^* J D)^{-1}(D^* J C_{\mathrm{L,s}} + B_{\mathrm{L,s}}^* \mathcal{P}) \tag{9.67}$$

in $\mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}), \mathrm{Dom}(A_{\mathrm{crit}})^*)$.

(d) **("CARE" ⇔ J-critical)** *Assume, instead, that Σ_{crit} is a control in WPLS form, $\mathbb{D} \in \mathrm{WR}$, and $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$.*

Then Σ_{crit} is J-critical and $\mathcal{P} = \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}_{\mathrm{crit}}$ iff (9.65) and (9.66) hold and Σ_{crit} is "$\mathcal{U}_*^*$-stabilizing" (i.e., $\mathbb{K}_{\circlearrowleft} x_0 \in \mathcal{U}_*^*(x_0)$ for all $x_0 \in H$ and (9.54) holds).

It would be more useful to obtain a Riccati equation on $\mathrm{Dom}(A)$, and we would like to know that K_{crit} extends to a (regular) state feedback operator for Σ (e.g., that K_{crit} would have an extension to H_B, and that this extension would equal $K_{\mathrm{w}}|_{H_B}$ for some WR state feedback operator K for Σ). Unfortunately, both of these require additional assumptions, by Example 11.3.7.

Proof: (In fact, $\Sigma_{\rm crit}$ need not be J-critical, it suffices that it is a control in WPLS form for Σ (see Definition 8.3.15) s.t. (9.55) and (9.56) hold. Thus, $\Sigma_{\rm crit}$ need not be "$\mathcal{U}_*^*$-stable" provided that it is otherwise as in Theorem 9.7.1.)

Apply Lemma 9.7.8 to (9.57), (9.56) and (9.56)* to obtain (9.63), (9.64) and (9.65). Formulae for $A_{\rm crit}$ and $C_{\rm crit}$ are from Lemma 8.3.17(a).

(b) Let $x_0 \in H$ and $T > 0$. We "connect $\Sigma_{\rm crit}$ to $\Sigma^{\rm d}$ through $\mathcal{P}$, J and time-inversion", as in [S98b, Section 5], [WW, Section 8] or Lemma 9.11.1: Set

$$x_0^* := \mathcal{P}\mathbb{A}_{\rm crit}^T x_0 \in H, \ \ y^* := \pi_{[0,T)}\text{Я}\tau^T J\mathbb{C}_{\rm crit}x_0 \in \mathrm{L}^2([0,T);Y). \tag{9.68}$$

Then, for any $s \in [0,T]$, we have

$$x^*(s) := \mathbb{A}^*(s)x_0^* + \mathbb{C}^{\rm d}\tau^s y^* = \mathbb{A}^{s*}\mathcal{P}\mathbb{A}_{\rm crit}^s\mathbb{A}_{\rm crit}^{T-s}x_0 + \mathbb{C}^*J\pi_{[0,s)}\tau^{T-s}\mathbb{C}_{\rm crit}x_0 \tag{9.69}$$

$$= \mathbb{A}^{s*}\mathcal{P}\mathbb{A}_{\rm crit}^s\mathbb{A}_{\rm crit}^{T-s}x_0 + \mathbb{C}^*J\pi_{[0,s)}\mathbb{C}_{\rm crit}\mathbb{A}_{\rm crit}^{T-s}x_0 = \mathcal{P}\mathbb{A}_{\rm crit}^{T-s}x_0, \tag{9.70}$$

by (9.56). Trivially, $\pi_{[0,T)}\mathbb{B}^{\rm d}x_0^* = \text{Я}\tau^T(\mathbb{B}^T)^*x_0^*$, and $\pi_{[0,T)}\mathbb{D}^{\rm d}y^* = \text{Я}\tau^T(\mathbb{D}^T)^*J\mathbb{C}_{\rm crit}^T x_0$. Therefore, (9.55) implies that

$$0 = \pi_{[0,T)}\mathbb{B}^{\rm d}x_0^* + \pi_{[0,T)}\mathbb{D}^{\rm d}y^*. \tag{9.71}$$

Since $\mathbb{D}$ is WR, we have for a.e. $t \in [0,T]$ that $x^*(T-t) = \mathcal{P}\mathbb{A}_{\rm crit}^t x_0 \in \mathrm{Dom}(B_{\rm L,w}^*)$, by (9.70) and Theorem 6.2.13(a2), and

$$0 = B_{\rm L,w}^* x^*(T-t) + D^*y^*(T-t) = B_{\rm L,w}^*\mathcal{P}\mathbb{A}_{\rm crit}^t x_0 + D^*J(\mathbb{C}_{\rm crit}x_0)(t) \tag{9.72}$$

$$= B_{\rm L,w}^*\mathcal{P}x(t) + D^*J(C_{\rm L,w}x(t) + Du(t)), \tag{9.73}$$

by (9.71) and (6.46) where $x(t) := \mathbb{A}_{\rm crit}^t x_0$, $u := \mathbb{K}_{\rm crit}x_0$. (recall that $\mathbb{C}_{\rm crit}x_0 = \mathbb{C}x_0 + \mathbb{D}\mathbb{K}_{\rm crit}x_0 = \mathbb{C}x_0 + \mathbb{D}u$). Because $u = (K_{\rm crit})_{\rm L,s}x(t)$ a.e., by Lemma 6.2.12(a), and $T > 0$, $t \in [0,T]$ and $x_0 \in H$ were arbitrary, (b) follows.

(a) Let $x_0 \in \mathrm{Dom}(A_{\rm crit})$. Then $x := \mathbb{A}_{\rm crit}x_0 \in \mathcal{C}(\mathbf{R}_+;\mathrm{Dom}(A_{\rm crit})) \cap \mathrm{W}_{\rm loc}^{1,2}(\mathbf{R}_+;H)$, hence $\mathbb{C}_{\rm crit}x_0 = C_{\rm crit}\mathbb{A}_{\rm crit}x_0$ and

$$A^*x_0^* + C^*y^*(0) = A^*\mathcal{P}x(T) + C^*JC_{\rm crit}x(T) = -\mathcal{P}A_{\rm crit}x(T) \in H, \tag{9.74}$$

by (9.65) and (9.68). Therefore, y^* and $\mathbb{B}^{\rm d}x_0 + \mathbb{D}^{\rm d}y^*$ are $\mathrm{W}_{\rm loc}^{1,2}$, hence continuous, by Theorem 6.2.13(b1), so that (9.72)–(9.73) hold everywhere on $[0,T]$.

Since $(C_{\rm L,w}, D)$ is a compatible pair for Σ, by Lemma 6.3.10(e), we have $C_{\rm crit} = C_{\rm L,w} + DK_{\rm crit}$ on $\mathrm{Dom}(A_{\rm crit})$, by Lemma 8.3.17(a). Thus, (a) follows from (9.72) at $t = 0$.

(c) 1° *(a) and (b):* From the proofs of (a) and (b) we see that we can replace $B_{\rm L,w}^*$ by $B_{\rm L,s}^*$ if $\mathbb{D}^{\rm d} \in$ SR and $C_{\rm L,w}$ by $C_{\rm L,s}$ if $\mathbb{D}$ is SR.

2° *The "CARE":* By (a), we have $\mathcal{P} \in \mathcal{B}(Z^*, \mathrm{Dom}(A_{\rm crit})^*)$, where $Z := \mathrm{Dom}(B_{\rm L,s}^*)$.

Let $x_0, x_1 \in \mathrm{Dom}(A_{\rm crit})$. Then $\mathcal{P}x_k \in Z$ and $Ax_k, BK_{\rm crit}x_k \in Z^*$, by Proposition 6.2.8(e)&(f), because $\mathrm{Dom}(A_{\rm crit}) \subset H_B$, by Lemma 8.3.17(a). (Here we needed the strong regularity of $\mathbb{D}^{\rm d}$; that of $\mathbb{D}$ could be dropped without essential changes.) Therefore,

$$\langle A_{\mathrm{crit}}x_0, \mathcal{P}x_1\rangle_H = \langle Ax_0 + BK_{\mathrm{crit}}x_0, \mathcal{P}x_1\rangle_{\langle Z^*,Z\rangle} = \langle Ax_0, \mathcal{P}x_1\rangle_{\langle Z^*,Z\rangle} + \langle BK_{\mathrm{crit}}x_0, \mathcal{P}x_1\rangle_{\langle Z^*,Z\rangle} \tag{9.75}$$

$$= \langle Ax_0, \mathcal{P}x_1\rangle_{\langle Z^*,Z\rangle} + \langle K_{\mathrm{crit}}x_0, B^*_{\mathrm{L,s}}\mathcal{P}x_1\rangle_U, \tag{9.76}$$

by Proposition 6.2.8(e). Now we obtain from (9.64) and the formulae $A_{\mathrm{crit}} = A + BK_{\mathrm{crit}}$ and $C_{\mathrm{crit}} = C_{\mathrm{w}} + DK_{\mathrm{crit}}$ that

$$0 = \langle Ax_0, \mathcal{P}x_1\rangle + \langle \mathcal{P}x_0, Ax_1\rangle + \langle K_{\mathrm{crit}}x_0, B^*_{\mathrm{L,s}}\mathcal{P}x_1\rangle + \langle B^*_{\mathrm{L,s}}\mathcal{P}x_0, K_{\mathrm{crit}}x_1\rangle \tag{9.77}$$

$$+ \langle C_{\mathrm{w}}x_0, JC_{\mathrm{w}}x_1\rangle + \langle C_{\mathrm{w}}x_0, JDK_{\mathrm{crit}}x_1\rangle + \langle DK_{\mathrm{crit}}x_0, JC_{\mathrm{w}}x_1\rangle + \langle DK_{\mathrm{crit}}x_0, JDK_{\mathrm{crit}}x_1\rangle \tag{9.78}$$

$$= \langle Ax_0, \mathcal{P}x_1\rangle + \langle \mathcal{P}x_0, Ax_1\rangle + \langle C_{\mathrm{w}}x_0, JC_{\mathrm{w}}x_1\rangle + \langle K_{\mathrm{crit}}x_0, (B^*_{\mathrm{L,s}}\mathcal{P} + D^*JC_{\mathrm{w}})x_1\rangle \tag{9.79}$$

$$+ \langle (B^*_{\mathrm{L,s}}\mathcal{P} + D^*JC_{\mathrm{w}})x_0, K_{\mathrm{crit}}x_1\rangle + \langle B^*_{\mathrm{L,s}}\mathcal{P}x_0, K_{\mathrm{crit}}x_1\rangle + \langle K_{\mathrm{crit}}x_0, D^*JDK_{\mathrm{crit}}x_1\rangle \tag{9.80}$$

$$= \langle Ax_0, \mathcal{P}x_1\rangle_{\langle Z^*,Z\rangle} + \langle \mathcal{P}x_0, Ax_1\rangle_{\langle Z,Z^*\rangle} + \langle C_{\mathrm{w}}x_0, JC_{\mathrm{w}}x_1\rangle_Y \tag{9.81}$$

$$- \langle (B^*_{\mathrm{L,s}}\mathcal{P} + D^*JC_{\mathrm{w}})x_0, (D^*JD)^{-1}(B^*_{\mathrm{L,s}}\mathcal{P} + D^*JC_{\mathrm{w}})x_1\rangle_U. \tag{9.82}$$

Assume now that also $\mathbb{D}$ is SR. Then $H_B \subset \mathrm{Dom}(C_{\mathrm{L,s}})$, so that we can replace C_{w} by $C_{\mathrm{L,s}}$ above, by Proposition 6.2.8(c1)&(c4)&(d1). Apply Proposition 6.2.8(e) to B and C^* to obtain that

$$K^*_{\mathrm{crit}} = -(\mathcal{P}B + C^*JD)(D^*JD)^{-1} \in \mathcal{B}(U, \mathrm{Dom}(C_{\mathrm{L,s}})^*). \tag{9.83}$$

(actually, the right-hand-side is an element of $\mathcal{B}(U, \mathrm{Dom}(C_{\mathrm{L,s}})^*)$, but $\mathrm{Dom}(A_{\mathrm{crit}}) \underset{c}{\subset} \mathrm{Dom}(C_{\mathrm{L,s}})$ so that K^*_{crit} is the restriction of the right-hand-side onto $\mathrm{Dom}(A_{\mathrm{crit}})$).

(d) By Lemma 9.7.8, we obtain (9.56). By going backwards the proofs of (a) and (b), we obtain that (9.71) holds for any T and x_0, hence (9.55) holds. The rest follows from Theorem 9.7.1. □

If D^*JD is one-to-one, then we obtain the following uniqueness results:

Corollary 9.7.4 *Let* $\mathbb{D}$ *be WR and* $\mathrm{Ker}(D^*JD) = \{0\}$.

(a) There is at most one J-critical control in WPLS form.

(b) There is at most one J-critical SR state feedback operator for Σ.

Although sufficient, the condition $\mathrm{Ker}(D^*JD) = \{0\}$ is not necessary for uniqueness, by, e.g., Example 9.13.10 in which $D = 0$ but the J-critical control over $\mathcal{U}_{\mathrm{out}}$ is nevertheless unique, by J-coercivity. (This might at first seem strange, but (9.66) is not the only condition on K_{crit} in Theorem 9.7.3, also A_{crit} and C_{crit} restrict K_{crit}, hence so does also $\mathcal{U}^*_*$-stabilization.)

Whenever there is a J-critical control over $\mathcal{U}^*_*$ in state feedback form, then the uniqueness of the J-critical control is equivalent to $\mathrm{Ker}(S) = \{0\}$, where S is the signature operator of the problem (which is equal to D^*JD under stronger regularity assumptions), by Theorem 9.9.1(a1)&(f2). See also the notes to Section 9.8.

Proof of Corollary 9.7.4: (a) Now K_{crit} is uniquely defined by the unique (by Lemma 8.3.8) J-critical cost operator $\mathcal{P}$, through (9.66). Now $A_{\text{crit}} = A + BK_{\text{crit}}$ on $\text{Dom}(A_{\text{crit}}) = \{x_0 \in H \,|\, (A + BK_{\text{crit}})x_0 \in H\}$, by Lemma A.4.6, and $C_{\text{crit}} = C + DK_{\text{crit}}$, hence $\mathcal{P}$ determines Σ_{crit} uniquely.

(b) Let K and $\widetilde{K}$ be J-critical SR state feedback operators for Σ. Let $\Sigma_{\circlearrowleft}$ and $\Sigma_\flat$ be corresponding closed-loop systems. By (6.145) and Proposition 6.6.18(d4), K_{crit} is SR and $K_{\text{s}} = (K_{\text{crit}})_{\text{s}} = \widetilde{K}_{\text{s}}$, hence $K = \widetilde{K}$. □

For bounded C, the above "generalized Riccati equations" can be simplified:

Lemma 9.7.5 (Bounded C) *Let Σ_{crit} be a J-critical control for Σ in WPLS form. Assume that $C \in \mathcal{B}(H,Y)$. Then $(\mathcal{P}, D^*JD, K_{\text{crit}})$ satisfy*

$$K_{\text{crit}}^*D^*JDK_{\text{crit}} = A^*\mathcal{P} + \mathcal{P}A + C^*JC \quad \in \mathcal{B}(\text{Dom}(A_{\text{crit}}), \text{Dom}(A_{\text{crit}})^*), \tag{9.84}$$
$$(D^*JD)K_{\text{crit}} = -B^*\mathcal{P} - D^*JC \quad \in \mathcal{B}(\text{Dom}(A_{\text{crit}}), U), \tag{9.85}$$

$$\mathcal{P}^* = \mathcal{P} \in \mathcal{B}(H) \cap \mathcal{B}(\text{Dom}(A_{\text{crit}}), \text{Dom}(A^*)) \cap \mathcal{B}(\text{Dom}(A^*)^*, \text{Dom}(A_{\text{crit}})^*). \tag{9.86}$$

Proof: 1° *$\mathcal{P} \in \mathcal{B}(\text{Dom}(A_{\text{crit}}), \text{Dom}(A^*))$:* From (9.65) we obtain that (here $\alpha \in \sigma(A)^c$ is as in Definition 6.1.17)

$$(\bar\alpha - A^*)\mathcal{P} = \bar\alpha\mathcal{P} + \mathcal{P}A_{\text{crit}} + C^*JC_{\text{crit}} \in \mathcal{B}(\text{Dom}(A_{\text{crit}}), H), \quad \text{hence} \tag{9.87}$$
$$\mathcal{P} = (\bar\alpha - A^*)^{-1}(\bar\alpha\mathcal{P} + \mathcal{P}A_{\text{crit}} + C^*JC_{\text{crit}}) \in \mathcal{B}(\text{Dom}(A_{\text{crit}}), \text{Dom}(A^*)). \tag{9.88}$$

2° *The other claims:* Because $B \in \mathcal{B}(H_1^*, U)$, we have $B^*\mathcal{P} \in \mathcal{B}(\text{Dom}(A_{\text{crit}}), U)$. Consequently, (9.85) follows from (9.66). Now we can get (9.84) from (9.67), but we give a direct proof below:

Part 1° implies that $\mathcal{P} \in \mathcal{B}(\text{Dom}(A^*)^*, \text{Dom}(A_{\text{crit}})^*)$, by Lemma A.3.24, and $H_{-1} = \text{Dom}(A^*)^*$, $A \in \mathcal{B}(H, H_{-1})$ and $B(U, H_{-1})$. Therefore, we can substitute identities $A_{\text{crit}} = A + BK_{\text{crit}}$ and $C_{\text{crit}} = C + DK_{\text{crit}}$ (from Lemma 8.3.17(a)) into (9.65) and separete the terms to obtain

$$0 = A^*\mathcal{P} + \mathcal{P}A + \mathcal{P}BK_{\text{crit}} + C^*JC + C^*JDK_{\text{crit}}. \tag{9.89}$$

Combine this with (9.85) to obtain (9.84). □

The proposition below corresponds to standard control problems, where D and C have been simplified. One example of such is the stabilization or LQR problem, where $\mathcal{J}(x_0, u) := \|x\|_2^2 + \|u\|_2^2$, i.e., $C = \begin{bmatrix} I \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $J = I$, $Y = H \times U$. Also in simplified H^∞ problems, one often has $D^*JC = 0$ and $D^*JD \in \mathcal{GB}(U)$ (e.g., $C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $J = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$, $\gamma > 0$). In these settings, we can simplify the above "$\text{Dom}(A_{\text{crit}})$ CARE" even more to obtain the one in [FLT]:

Proposition 9.7.6 (Bounded C, standard cost) *Let there be a unique J-critical control $u_{\mathrm{crit}}(x_0)$ over $\mathcal{U}_*^*$ for each $x_0 \in H$. Assume that $C \in \mathcal{B}(H,Y)$, $Q := C^*JC \in \mathcal{B}(H)$, $D^*JC = 0$, $R := D^*JD \in \mathcal{GB}(U)$. Then*

$$K_{\mathrm{crit}} = -R^{-1}B^*\mathcal{P} \qquad \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),U), \tag{9.90}$$

$$-A^*\mathcal{P} = \mathcal{P}A - \mathcal{P}BR^{-1}B^*\mathcal{P} + Q \qquad \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),H), \tag{9.91}$$

$$-(A - BR^{-1}B^*\mathcal{P})^*\mathcal{P} = \mathcal{P}A + Q \qquad \in \mathcal{B}(\mathrm{Dom}(A),H), \tag{9.92}$$

$$\mathcal{P}^* = \mathcal{P} \in \mathcal{B}(H) \cap \mathcal{B}(\mathrm{Dom}(A),\mathrm{Dom}(A_{\mathrm{crit}}^*)) \cap \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),\mathrm{Dom}(A^*)) \tag{9.93}$$

$$\cap\, \mathcal{B}(\mathrm{Dom}(A^*)^*,\mathrm{Dom}(A_{\mathrm{crit}})^*) \cap \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}^*)^*,\mathrm{Dom}(A)^*). \tag{9.94}$$

Therefore, $B^*\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),U)$ and $K_{\mathrm{crit}}^* = -\mathcal{P}BR^{-1} \in \mathcal{B}(U,\mathrm{Dom}(A_{\mathrm{crit}})^*)$.

Unfortunately, also this “CARE” is given on $\mathrm{Dom}(A_{\mathrm{crit}})$ instead of $\mathrm{Dom}(A)$. We would like to remove the parenthesis on the left-hand-side of (9.92), but this cannot be done in general, since it would require, e.g., that $\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A))$, which is not the case. By replacing B^* by B_{w}^* we could slightly relax the requirement, but one still need an additional condition to get further (cf. Section 9.2).

Even when C is bounded, $D^*JD = I$, $D^*JC = 0$ and $J = I$, the operator $K_{\mathrm{crit}} = -B^*\mathcal{P}$ need not be bounded (nor $B_{\mathrm{w}}^*\mathcal{P}$), by Example 9.13.8. This shows that while the boundedness of B makes things easy (see Theorem 9.9.6), that of C is not as helpful.

Under the assumptions of the proposition, one easily obtains from (9.55) that $\mathbb{K}_{\mathrm{crit}}^t + \mathbb{B}^{t\,*}(Q+\mathcal{P})\mathbb{A}_{\mathrm{crit}}^t = 0$ (use the fact that $\mathbb{D}^t = \pi_{[0,t)}(D + C\mathbb{B}^t)$, by Theorem 6.2.13), or equivalently, that $u_{\mathrm{crit}} = -\mathbb{B}^{t\,*}(Q+\mathcal{P})x_{\mathrm{crit}}$ a.e., hence this case is essentially simpler than the general case.

Proof of Proposition 9.7.6: 1° By direct substitutions to Theorem 9.7.3, we obtain that

$$-A_{\mathrm{crit}}^*\mathcal{P} = \mathcal{P}A_{\mathrm{crit}} + Q + K_{\mathrm{crit}}^*RK_{\mathrm{crit}} \qquad \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),\mathrm{Dom}(A_{\mathrm{crit}})^*), \tag{9.95}$$

$$-A_{\mathrm{crit}}^*\mathcal{P} = \mathcal{P}A + Q \qquad \in \mathcal{B}(\mathrm{Dom}(A),H), \tag{9.96}$$

$$-A^*\mathcal{P} = \mathcal{P}A_{\mathrm{crit}} + Q \qquad \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),H). \tag{9.97}$$

From (9.96) we obtain that (here $\alpha \in \sigma(A)^c$ and $\beta \in \sigma(A_{\mathrm{crit}})^c$ are as in Definition 6.1.17)

$$(\bar{\beta} - A_{\mathrm{crit}}^*)\mathcal{P} = \bar{\beta}\mathcal{P} + \mathcal{P}A + Q \in \mathcal{B}(\mathrm{Dom}(A),H), \quad \text{hence} \tag{9.98}$$

$$\mathcal{P} = (\bar{\beta} - A_{\mathrm{crit}}^*)^{-1}(\bar{\beta}\mathcal{P} + \mathcal{P}A + Q) \in \mathcal{B}(\mathrm{Dom}(A),\mathrm{Dom}(A_{\mathrm{crit}}^*)). \tag{9.99}$$

Analogously, from (9.97) (or (9.86)) we obtain that $\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),\mathrm{Dom}(A^*))$. Consequently, $B_{\mathrm{w}}^*\mathcal{P} = B^*\mathcal{P}$ on $\mathrm{Dom}(A_{\mathrm{crit}})$, and hence (9.90) follows from (9.66).

2° Because $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ and hence $\mathcal{P} = \mathcal{P}^*$ on any subset of H (e.g., $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}),\mathrm{Dom}(A^*))$), also the adjoints $\mathcal{P},\mathcal{P}^* \in \mathcal{B}(\mathrm{Dom}(A^*)^*,\mathrm{Dom}(A_{\mathrm{crit}})^*)$ become equal (recall that $\mathrm{Dom}(A^*)^* = H_{-1}$), cf. Lemma A.3.24. Thus, one easily verifies (9.93)–(9.94).

3° Now we obtain (9.91) and (9.92) by direct substitutions. Obviously, $K_{\rm crit}^* = -\mathcal{P}BR^{-1} \in \mathcal{B}(U, \mathrm{Dom}(A_{\rm crit})^*)$. □

Remark 9.7.7 *Let $\Sigma_{\rm crit}$ be a J-critical control for Σ in WPLS form.*

(a1) Given $x_0 \in \mathrm{Dom}(A_{\rm crit})$, the J-critical control $u = u_{\rm crit}(x_0)$ and state $x = x_{\rm crit}(x_0)$ are given in the "state feedback form" in the weak sense that

$$u(t) = K_{\rm crit}x(t) \qquad (t \in \mathbf{R}_+); \tag{9.100}$$

$$x'(t) = Ax(t) + Bu(t) = (A + BK_{\rm crit})x \qquad (t \in \mathbf{R}_+) \tag{9.101}$$

(note that u and x are continuous in this case). Furthermore, for an arbitrary $x_0 \in H$, we have $u(t) = (K_{\rm crit})_{\rm s}x(t)$ for almost every $t \in \mathbf{R}_+$

(a2) The operator $A_{\rm crit} := A + BK_{\rm crit}$ generates the C_0-semigroup $\mathbb{A}_{\rm crit}$, and, for any $x_0 \in H$, the critical control $x = x_{\rm crit}(x_0)$ is the strong solution of $x' = A_{\rm crit}x,\ x(0) = x_0$ (here $A_{\rm crit}$ is the extension of the original operator, as in Lemma 6.1.16).

(a3) The "state feedback" of (a1)–(a2) is not well posed if $\Sigma_{\rm crit}$ is not of state feedback form (equivalently, does not correspond to a solution of the eIARE).

In fact, such "non-well-posed state feedback" might "explode" under any external input ("$u_\circlearrowleft$"; e.g., disturbance or modelling error); cf. Figure 9.1 (p. 408) and Example 11.3.7.

(b1) **(CARE)** *The "Riccati" equations of Theorem 9.7.3 are satisfied; in particular, for WR $\mathbb{D}$ with $D^*JD \in \mathcal{GB}(U)$, $\mathcal{P}$ determines $K_{\rm crit}$ uniquely on* $\mathrm{Dom}(A_{\rm crit})$ *and $u_{\rm crit} = -(D^*JD)^{-1}(B_{\rm w}^*\mathcal{P} + D^*JC_{\rm L,w})x_{\rm crit}$ a.e., for any $x_0 \in H$.*

Moreover, $\mathcal{J}(x_0, u_{\rm crit}) = \langle x_0, \mathcal{P}x_0\rangle$.

(b2) **(Uniqueness)** *A $\mathcal{U}_*^*$-stabilizing solution $\mathcal{P}$ (if any) of the "eIARE" of Theorem 9.7.1 is equal to the J-critical cost operator, hence it is unique, by Lemma 8.3.8.*

If $\mathbb{D}$ is WR, then we can use the "$\mathrm{Dom}(A_{\rm crit})$*-CARE" (9.65) and (9.66) instead of the "eIARE", by Theorem 9.7.3(d). In particular, a "$\mathcal{U}_*^*$-stabilizing" solution $\mathcal{P}$ of this "*$\mathrm{Dom}(A_{\rm crit})$*-CARE" is unique (and so is $K_{\rm crit}$ if $D^*JD \in \mathcal{GB}(U)$).*

Thus, the ("$\mathcal{U}_*^*$-stabilizing") solution $\mathcal{P}$ of the "CARE" leads to the "state feedback" formula of (b1) and to the J-critical cost $\langle x_0, \mathcal{P}x_0\rangle$.

If Σ is stable (or suitably (well-posedly) stabilizable) and $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\rm out}$ or $\mathcal{U}_{\rm exp}$, then minimization corresponds to a well-posed (and stable) state feedback, as shown in Section 10.2. However, in, e.g., the stable J-coercive indefinite case, the J-critical state feedback pair might be unstable, or even non-well-posed (i.e., $\widehat{\mathbb{F}}, \widehat{\mathbb{F}_\circlearrowleft} \notin \mathrm{H}^\infty_\infty$), although $\mathbb{K}_\circlearrowleft$ is always well-posed and stable (cf. (a3)); see Example 11.3.7.

Recall that by state feedback we mean the (well-posed) state feedback of Definition 6.6.10, not that of (a3).

Proof of Remark 9.7.7: (a1) This follows from (a2), Lemma A.4.2(c1) and Lemma 6.2.12.

(a2) By Lemma 8.3.17(a), $A_{\mathrm{crit}} = A + BK_{\mathrm{crit}}$ on $\mathrm{Dom}(A_{\mathrm{crit}})$. The rest follows from Lemma 6.1.16(b).

(a3) 1° This is intuitively rather obvious. We shall show in 2° that if the map $u_{\circlearrowleft} \to u$ is well posed, where $u_{\circlearrowleft}$ is an external input and u is the effective input corresponding to $u_{\circlearrowleft}$ and initial state $x_0 = 0$, as in Figure 9.1, then Σ_{crit} is the left column of the corresponding closed-loop system; this proves our first claim. If $u \mapsto y$ is coercive (e.g., $\mathbb{D} = \left[\begin{smallmatrix} \mathbb{D}_1 \\ I \end{smallmatrix}\right]$, as in the standard setting), then this means that also the map $u_{\circlearrowleft} \mapsto y$ becomes ill-posed; the same applies to $u_{\circlearrowleft} \mapsto x$.

2° If $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\omega(U)$, where $\mathbb{M}u_{\circlearrowleft} := u$ (a time-invariant operator $\mathbb{M} \in \mathcal{GB}(\mathrm{L}^2_\omega(\mathbf{R}_+;U))$ can be extended to $\mathcal{G}\mathrm{TIC}_\omega(U)$, by Lemma 2.1.3), then $\mathbb{M}$ defines an admissible state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & I - \mathbb{M}^{-1} \end{array}\right]$ for Σ, and Σ_{crit} is the left column of the corresponding closed-loop system.

Indeed, if $u_{\circlearrowleft} \in \mathrm{L}^2(\mathbf{R}_-;U)$ and $u = \mathbb{M}u_{\circlearrowleft}$, then, obviously, $\pi_+ u = \mathbb{K}_{\mathrm{crit}}\mathbb{B}u = \mathbb{K}_{\mathrm{crit}}\mathbb{B}\mathbb{M}u_{\circlearrowleft}$ (this is the control corresponding to $\pi_+ u_{\circlearrowleft} = 0$ and $x_0 = \mathbb{B}u$). Now we obtain $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ from Theorem 8.3.13(b1) (see Lemma 8.3.16(b)).

(b1) This follows from Theorem 9.7.3.

(b2) This follows from Theorems 9.7.1 and 9.7.3.

Note that the definition of a "$\mathcal{U}^*_*$-stabilizing" solution $\mathcal{P} \in \mathcal{B}(H)$ is rather clumsy; e.g., we require K_{crit} to generate a WPLS ("$\left[\begin{smallmatrix} \mathbb{A}_0 \\ \mathbb{C}_0 \\ \mathbb{K}_0 \end{smallmatrix}\right]$") with $A + BK_{\mathrm{crit}}$ and $C_{\mathrm{w}} + DK_{\mathrm{crit}}$, and pose the conditions $\mathbb{K}_0 x_0 \subset \mathcal{U}^*_*(x_0)$ and (9.54); the latter two conditions become equivalent to $\mathbb{A}_0$ being exponentially stable if $\mathcal{U}^*_* = \mathcal{U}_{\exp}$, by Theorem 9.7.1(c). This can be simplified in the positively coercive case (as done for the CARE in Section 10.7, in particular, Proposition 10.7.3(d3)).

However, a CARE defined on $\mathrm{Dom}(A_{\mathrm{crit}})$ does not seem to be useful, therefore we will not go into these simplifications; we just mention that in Theorem 5.3 of [FLT] it is shown that if $C = \left[\begin{smallmatrix} R \\ 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, $J = I$ and $R \gg 0$ (thus $\mathcal{J}(x_0,u) = \|u\|_2^2 + \langle x, Rx\rangle_{\mathrm{L}^2}$), then (9.67) has at most one self-adjoint solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ s.t. $B^*\mathcal{P} \in \mathcal{B}(\mathrm{Dom}(A_{\mathrm{crit}}), H)$. □

The rest of this section consists of auxiliary lemmas only. Some algebraic (instantaneous) Riccati-like equations can be equivalently written in integral forms and vice versa, as described below:

Lemma 9.7.8 *(a) Let* $\left[\begin{smallmatrix} \mathbb{A}_k \\ \mathbb{C}_k \end{smallmatrix}\right] \in \mathrm{WPLS}(0,H,Y)$ $(k = 1,2)$, $P \in \mathcal{B}(H)$ *and* $\widetilde{J} \in \mathcal{B}(Y)$. *Then*

$$\langle A_1 x_1, P x_2\rangle_{H_1} + \langle x_1, P A_2 x_2\rangle_{H_1} + \langle C_1 x_1, \widetilde{J} C_2 x_2\rangle_{Y_1} \ge 0 \qquad (x_1 \in \mathrm{Dom}(A_1),\ x_2 \in \mathrm{Dom}(A_2)) \tag{9.102}$$

$$\iff \qquad (\mathbb{A}_1^t)^* P \mathbb{A}_2^t + (\mathbb{C}_1^t)^* \widetilde{J} \mathbb{C}_2^t \ge P \qquad (t \in [0,+\infty)). \tag{9.103}$$

Moreover, we can, equivalently, replace "$(t \in [0,+\infty))$" by "$(t \in (0,\varepsilon))$", for *any* $\varepsilon > 0$*, or require (9.102) only for* $x_k \in \cap_{n\in\mathbf{N}} \mathrm{Dom}(A_k^n)$*. All this also holds with "=" or "≤" in place of "≥".*

(b) Part (a) also holds if we replace " $\begin{bmatrix} \mathbb{A}_k \\ \mathbb{C}_k \end{bmatrix} \in \mathrm{WPLS}(0,H,Y)$*" by "*$\mathbb{A}_k$ *is a* C_0*-semigroup on H and* $C_k \in \mathcal{B}(H_1,Y)$*" and set* $(\mathbb{C}_k x_0)(t) = C_k\mathbb{A}_k(t)x_0$ $(t \geq 0)$ *to define* $\mathbb{C}_k : H_1 \to \mathcal{C}(\mathbf{R}_+;Y)$*, except that (9.103) must be applied to elements of* $\mathrm{Dom}(A)$ *only.*

Equation (9.102) is equivalent to

$$A_1^*P + PA_2 + C_1^*\widetilde{J}C_2 \geq 0 \quad (\text{in } \mathcal{B}(\mathrm{Dom}(A_2), \mathrm{Dom}(A_1)^*))\,, \tag{9.104}$$

where $\mathrm{Dom}(A_k)$ is equipped with the graph topology and $\mathrm{Dom}(A_k)^*$ is its dual w.r.t. the pivot space H as in Lemma A.3.24 (or in Lemma 6.1.16 and Definition 6.1.17). However, we encourage the reader to always write such formulae into the longer form (9.102) when he or she has problems to verify them in their short forms.

For equations containing I/O maps ("$\mathbb{D}$"), we need some regularity assumption (for "D" to exist), and for equations containing input maps ("$\mathbb{B}$"), the equivalence becomes rather complicated, and they need special extensions of B^* and $\overline{B^*P}$, as illustrated in Section 9.8 and corresponding proofs in Section 9.11.

Proof: 1° *"⇐":* Let $x_k \in \mathrm{Dom}(A_k)$ $(k = 1,2)$. By Lemma A.4.2(c1), we have

$$(\mathbb{A}_k x_k)' = A_k\mathbb{A}_k x_k = \mathbb{A}_k A_k x_k \in \mathcal{C}(\mathbf{R}_+;H_k) \quad (k = 1,2), \tag{9.105}$$

in particular, $\mathbb{A}_k x_k \in \mathcal{C}(\mathbf{R}_+;\mathrm{Dom}(A_k))$. Consequently, $\mathbb{C}_k x_k = C_k\mathbb{A}_k^{\cdot} x_k \in \mathcal{C}(\mathbf{R}_+;Y)$ $(k = 1,2)$, by Lemma 6.2.12.

Since $f := \langle x_1, gx_2\rangle$, where $g := (\mathbb{A}_1^t)^*P\mathbb{A}_2^t + (\mathbb{C}_1^t)^*\widetilde{J}\mathbb{C}_2^t - P$, satisfies $f(0) = 0$, $f \geq 0$ and $f \in \mathcal{C}^1(\mathbf{R}_+)$, we have $f'(0) \geq 0$, which implies that (9.102) holds.

2° *"⇒":* Assume that (9.102) holds on $\mathrm{Dom}(A_1^\infty) \times \mathrm{Dom}(A_2^\infty)$. Let $a_k \in \mathrm{Dom}(A_k^\infty) := \cap_{n\in\mathbf{N}} \mathrm{Dom}(A_k^n)$ and $t \geq 0$. Set $x_k := \mathbb{A}_k^t a_k \in \mathrm{Dom}(A_k^\infty)$, so that $\mathbb{C}_k a_k = C_k x_k$ and $(\mathbb{A}_k a_k)'(t) = A_k x_k$ $(k = 1,2)$, as in 1°. By substituting these into (9.102), we obtain

$$\begin{aligned}
0 &\leq \langle \mathbb{A}_1'(t)a_1, P\mathbb{A}_2(t)a_2\rangle + \langle \mathbb{A}_1(t)a_1, P\mathbb{A}_2'(t)a_2\rangle + \langle C_1\mathbb{A}_1(t)a_1, \widetilde{J}C_1\mathbb{A}_2(t)a_2\rangle \\
&= \frac{d}{dt}\left[\langle \mathbb{A}_1(t)a_1, P\mathbb{A}_2(t)a_2\rangle_H + \int_0^t \langle C_1\mathbb{A}_1(t)a_1, \widetilde{J}C_2\mathbb{A}_2(t)a_2\rangle_Y\, dt\right] \\
&= \frac{d}{dt}\left[\langle a_1, \mathbb{A}_1(t)^*P\mathbb{A}_2(t)a_2\rangle_H + \langle a_1, (\mathbb{C}_1^t)^*\widetilde{J}\mathbb{C}_2^t a_2\rangle_H\right].
\end{aligned}$$

Thus, the expression in brackets must be increasing, hence for any $t > 0$, we have

$$\begin{aligned}
&\langle a_1, \mathbb{A}_1(t)^*P\mathbb{A}_2(t)a_2\rangle + \langle a_1, \mathbb{C}_1^*\widetilde{J}\pi_{[0,t]}\mathbb{C}_2 a_2\rangle \\
&\qquad \geq \langle a_1, \mathbb{A}_1(0)^*P\mathbb{A}_2(0)a_2\rangle + \langle a_1, \mathbb{C}_1^*\widetilde{J}\pi_{[0,0]}\mathbb{C}_2 a_2\rangle = \langle a_1, Pa_2\rangle - 0.
\end{aligned}$$

The same holds for $a_1, a_2 \in H \times H$ too, because $\mathrm{Dom}(A_k^\infty)$ is dense in H, by Lemma A.4.2(b).

3° The "moreover" claim can be observed from the above proofs; the claim on "$\le$" follows by replacing P by $-P$ and $\widetilde{J}$ by $-\widetilde{J}$; the claim on "$=$" follows "$\le$" and "$\ge$".

(b) The proof of (a) applies here too. Note that here (9.103) means that $\langle x_0, \left((\mathbb{A}_1^t)^* P \mathbb{A}_2^t + (\mathbb{C}_1^t)^* \widetilde{J} \mathbb{C}_2^t - P\right) x_0\rangle_H \ge 0$ for all $x_0 \in \mathrm{Dom}(A) =: H_1$ and all $t \in [0, +\infty)$. □

We shall sometimes need the following lemma:

Lemma 9.7.9 *Let $x_0 \in H$ and $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+; U)$. Then $u \in \mathcal{U}_*^*(x_0)$ iff $\pi_+ \tau^t u \in \mathcal{U}_*^*(\mathbb{A}^t x_0 + \mathbb{B}^t u)$ for some (equivalently, all) $t \ge 0$.*

This says that u is admissible for some initial state $x(0)$ iff at some (hence any) moment t the rest of u is admissible for the current state $x(t)$.

Proof: Given $t \ge 0$, set $u' := \pi_{[0,t)} u$, $u'' := \pi_+ \tau^t u$, so that $u = u' \Diamond_t u''$ (see p. 158) and $x_t := \mathbb{A}^t x_0 + \mathbb{B}^t u = \mathbb{A}^t x_0 + \mathbb{B}^t u'$, Obviously, $u \in \mathrm{L}^2_\vartheta \Leftrightarrow u'' \in \mathrm{L}^2_\vartheta$. We have (recall that $\tau^t u'' = \pi_- \tau^t u$)

$$(\mathbb{C}x_t) + \mathbb{D}u'' = (\pi_+ \tau^t \mathbb{C}x_0 + \pi_+ \mathbb{D}\tau^t u'') + \pi_+ \mathbb{D}\pi_+ \tau^t u = \pi_+ \tau^t (\mathbb{C}x_0 + \mathbb{D}u) \quad (9.106)$$

hence $\mathbb{C}x_t + \mathbb{D}u'' \in \mathrm{L}^2$ iff $\mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$.

Analogously, we can show that $\mathbb{Q}x_t + \mathbb{R}u'' \in Z^{\mathrm{s}}$ iff $\pi_+ \tau^t (\mathbb{Q}x_0 + \mathbb{R}u) \in Z^{\mathrm{s}}$, i.e., iff $\mathbb{Q}x_0 + \mathbb{R}u \in Z^{\mathrm{s}}$ (by Standing Hypothesis 9.0.1). Thus, we have shown that $u \in \mathcal{U}_*^*(x_0) \Leftrightarrow u'' \in \mathcal{U}_*^*(x_t)$. Since $t \ge 0$ was arbitrary, this establishes the claim. □

We finish this section by a result that was used in the proof of Theorem 9.7.1:

Lemma 9.7.10 *Assume that $\begin{bmatrix} \mathbb{A}_0 \\ \hline \mathbb{C}_0 \\ \mathbb{K}_0 \end{bmatrix}$ is a control in WPLS form s.t. $\mathbb{K}_0 x_0 \in \mathcal{U}_*^*(x_0)$ for all $x_0 \in H$.*

Then, for any $t \ge 0$ and $\widetilde{\eta} \in \mathrm{L}^2([0,t); U)$, we have $\eta := \pi_{[0,t)}\widetilde{\eta} + \tau^{-t}\mathbb{K}_0 \mathbb{B}^t \widetilde{\eta} \in \mathcal{U}_*^*(0)$ and $\pi_{[0,t)}\eta = \widetilde{\eta}$. □

(This follows from Lemma 9.7.9 by setting $x_0 = 0$, since $\mathbb{K}_0 \mathbb{B}^t \widetilde{\eta} \in \mathcal{U}_*^*(\mathbb{B}^t \widetilde{\eta})$.)

Notes

As necessary conditions, equations (9.55)–(9.57) and (9.63)–(9.65) are well-known for some special cases; see, e.g., [S97b] for the case of well-posed minimizing state feedback for WPLSs.

Most of Proposition 9.7.6 was established in [FLT] by solving first the finite-time interval problem and then obtaining the infinite-time results as a limit. This method requires a rather coercive cost function (see the notes below the proposition) and is hence not applicable to the more general results of Theorem 9.7.3 and Lemma 9.7.5, whose formulae are well-known at least for systems with bounded B and C [CZ, Exercise 6.21c]. The integrations and differentiations used in Lemma 9.7.8 are well-known [WW] [S97b].

We shall now treat two questions on [FLT] (or on [LT00b]) that have caused some controversy. One observes from (6.52) (with $C = I$) that $(\pi_{[0,t)} \mathbb{B} \tau \pi_{[0,t)})^* v =$

$B^*_{\mathrm{L,s}} \int_t^T \mathbb{A}(s)^* v(s)\,ds$ for all $v \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;H)$. In (1.3b) of [FLT], the same formula is given with "B^*" in place of "$B^*_{\mathrm{L,s}}$", and the results of that article are derived using this formula. According to the authors, "B^*" refers to a "maximal, possibly nonunique extension of the original $B^* \in \mathcal{B}(\mathrm{Dom}(A^*),U)$", whose existence should follow (nonconstructively) directly from the (standard) assumptions that $A^{-1}B \in \mathcal{B}(U,H)$ (use $(s-A)^{-1}$ instead if $0 \in \sigma(A)$) and that $B^*\mathbb{A}^* : \mathrm{Dom}(A^*) \to \mathcal{C}$ extends to an operator $\mathbb{B}^{\mathrm{d}} : H \to \mathrm{L}^2([0,T);U)$ $(T > 0)$. We cannot follow this argument, nor can the experts that we have contacted. However, the constructive, highly nontrivial proof of [W89a] (see (6.52)) can be used. Nevertheless, in several applications, such as in the parabolic setting of Section 9.5, the proof is rather simple.

The second controversial thing is the following: If $D^*JD = I$, then we obtain from Proposition 9.7.6 that

$$\langle B^*\mathcal{P}x_0, B^*\mathcal{P}z_0\rangle_U = \langle \mathcal{P}Ax_0, z_0\rangle_{\langle \mathrm{Dom}(A_{\mathrm{crit}})^*, \mathrm{Dom}(A_{\mathrm{crit}})\rangle} - \langle \mathcal{P}A_{\mathrm{crit}}x_0, z_0\rangle_H \ \ (x_0, z_0 \in \mathrm{Dom}(A_{\mathrm{crit}})). \tag{9.107}$$

By (9.94), the expression

$$\langle \mathcal{P}Ax_0, z_0\rangle_H - \langle \mathcal{P}A_{\mathrm{crit}}x_0, z_0\rangle_{\langle \mathrm{Dom}(A)^*, \mathrm{Dom}(A)\rangle} \ \ (x_0, z_0 \in \mathrm{Dom}(A)), \tag{9.108}$$

is continuous $\mathrm{Dom}(A) \times \mathrm{Dom}(A) \to \mathbf{C}$. In [FLT, Corollary 4.9], Flandoli et al. define $\overline{\langle B^*\mathcal{P}x_0, B^*\mathcal{P}z_0\rangle_U}$ for $x_0, z_0 \in \mathrm{Dom}(A)$ by (9.108) (even if $\mathrm{Dom}(A) \cap \mathrm{Dom}(A_{\mathrm{crit}}) = \{0\}$), so that $A_{\mathrm{crit}}\mathcal{P} = A^*\mathcal{P} - (\overline{B^*\mathcal{P}})^*\overline{B^*\mathcal{P}} \in \mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A)^*)$ and hence (9.96) becomes

$$(\overline{B^*\mathcal{P}})^*\overline{B^*\mathcal{P}} = A^*\mathcal{P} + \mathcal{P}A + Q \in \mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A)^*), \tag{9.109}$$

by the definition of $(\overline{B^*\mathcal{P}})^*\overline{B^*\mathcal{P}}$. Thus, (9.109) is actually an equivalent way of writing (9.96).

G. Weiss and H. Zwart (Section 8 of [WZ]) have constructed a simple example (our Example 9.13.8) satisfying the assumptions of Proposition 9.7.6 and having a ULR unique minimizing (hence J-critical) state feedback operator K that differs from $\overline{B^*\mathcal{P}}$ (as defined by (9.108); see [WZ] for details). In fact, since $\mathrm{Dom}(A) \cap \mathrm{Dom}(A_{\mathrm{crit}}) = \{0\} = \mathrm{Ker}(\mathcal{P})$ in [WZ], one could extend $B^*\mathcal{P}$ to $\mathrm{Dom}(A)$ arbitrarily. The extension $B^*_{\mathrm{L,s}}\mathcal{P} = B^*_{\mathrm{w}}\mathcal{P}$ is the only one leading to the minimizing K.

Therefore, the extension of $B^*\mathcal{P}$ used for (9.109) would seem artificial and not having a connection to the minimizing state feedback operator $-K = B^*_{\mathrm{w}}\mathcal{P} = B^*_{\mathrm{L,s}}\mathcal{P}$ (this formula is valid on $H_B \supset \mathrm{Dom}(A) \cup \mathrm{Dom}(A_{\mathrm{crit}})$, see Section 9.9).

It is not obvious whether this $\overline{B^*\mathcal{P}}$ always corresponds to $\overline{B^*}\mathcal{P}$ for any extension $\overline{B^*}$ of B^*. However, after finishing this chapter, we learned abouve [BLT], in which the authors succeed in settling this last problem for several special cases by proving the existence of an extension $\overline{B^*}$ of B^* on $\mathcal{P}[\mathrm{Dom}(A)]$ s.t. $\overline{B^*}\mathcal{P}$ equal "$\overline{B^*\mathcal{P}}$". In these constructions, their methods resemble the definition of B_{s}.

However, in several applications, such as the parabolic problems of [LT00a], equation (9.109) coincides with the CARE (9.3) and hence becomes useful for the computation of the Riccati operator and the optimizing state feedback operator. In particular, in these cases the signature operator S of the problem equals D^*JD

(See the notes to Section 9.8 for a comparison between D^*JD and the signature operator.) One of the merits of [FLT] is the results that if $Q \gg 0$, then $\mathcal{P}$ is unique (see the proof of Remark 9.7.7(c)).

9.8 Algebraic and integral Riccati equations (CARE↔IARE)

For every complex problem, there is a solution that is simple, neat, and wrong.
— H. L. Mencken

In this section, we shall divide the correspondence between optimal (i.e., J-critical) control and CAREs into two parts, by introducing "Integral Algebraic Riccati Equation (IARE)" in between these two concepts.

The IARE is essentially an integral of the CARE, but it can be formulated regardless of the regularity of the system and J-critical control (as long as it can be given in the state feedback form). Therefore, this new concept allows us to extend the classical one-to-one correspondence between the optimal control and the stabilizing solution of the Riccati equation to general WPLSs in Section 9.9. In this section, we shall establish the equivalence of IAREs and CAREs under weak regularity assumptions (Proposition 9.8.10), and show the uniqueness of their stabilizing solutions (Theorem 9.8.12).

In addition, we shall study the extended forms of these Riccati equations, "eIAREs" and "eCAREs", where the signature operators need not be invertible, since a J-critical control need not correspond to an invertible signature operator unless we assume a coercive cost function (J-coercivity).

We shall also observe that the eIARE is a reformulation of the *extended Discrete-time Algebraic Riccati Equation (eDARE)*, see Proposition 9.8.7 and Theorem 13.4.4. Thus, we may use the eDARE theory from Section 14.1 to solve continuous-time problems in discrete time. In particular, if one wishes to verify the proofs, one should read first Section 14.1. The equivalence of eCAREs and eIAREs then extends the discrete-time results also to (e)CAREs.

Because of the appearance of the feedthrough operators D and $X = I - F$ in the eCAREs, we can define them only for weakly regular $\mathbb{D}$ and $\mathbb{F}$ (their feedthrough operators are often assumed to be zero in classical theory; this cannot be done in general). Nevertheless, also part of the "CARE" theory can be applied to more general systems, as shown in Remark 9.12.1 and Section 9.7.

Further results on Riccati equations will be given in the following sections, and on positive Riccati equations (roughly, with a nonnegative signature operator S) in Section 10.6. The proofs of Proposition 9.8.11(ii) and Theorem 9.8.12(d)&(e) depend on Section 9.9.

Recall our standing assumptions that $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ and that $J = J^* \in \mathcal{B}(Y)$, and the fact that we denote generators by same letters as corresponding operators ($\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$).

We start by extending Definition 9.1.5 to cover the case where $S \notin \mathcal{GB}(U)$ and $F \neq 0$. Solutions of the eCARE will be called "$\mathcal{U}_*^*$-stabilizing" if the corresponding control $u := \mathbb{K}_{\circlearrowleft} x_0$ belongs to the class over which we are optimizing ($u \in \mathcal{U}_*^*(x_0)$) and the residual cost condition (PB) is satisfied (the latter is redundant for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ and for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{str}}$, by Theorem 9.8.5). In Corollary 9.9.2 we shall show that such solutions correspond one-to-one to the optimal state feedback operators.

Definition 9.8.1 (eCARE) *We call $\mathcal{P}$ (or $(\mathcal{P}, S, \left[\begin{array}{c|c} K & F \end{array}\right])$) a* solution of the extended Continuous-time Algebraic Riccati Equation (eCARE)

(induced by Σ and J) iff Σ is WR and $\mathcal{P}$ satisfies

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC & \in \mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A)^*) \\ X^*SX = D^*JD + \operatorname*{w-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B & \in \mathcal{B}(U) \\ X^*SK = -(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) & \in \mathcal{B}(\mathrm{Dom}(A), U) \end{cases} \tag{9.110}$$

(here $X := I - F$) and $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ $F, S \in \mathcal{B}(U)$, $S = S^*$ and $K \in \mathcal{B}(H_1, U)$.

A solution $\mathcal{P}$ of the eCARE is called WR *(resp. SR, UR, admissible, stabilizing, stable, ...) if generators $\left[\begin{array}{c|c} K & F \end{array}\right]$ extend Σ to another WR WPLS Σ_{ext} and the resulting pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is WR (resp. SR, UR, admissible, stabilizing, stable, ...). See Definition 6.6.10 for further prefices and suffices.*

If $\mathcal{P}$ is admissible, then we denote the corresponding closed-loop system by $\Sigma_\circlearrowleft$ and set $\mathbb{X} := I - \mathbb{F} \in \mathrm{TIC}_\infty(U)$, $\mathbb{M} := \mathbb{X}^{-1} \in \mathrm{TIC}_\infty(U)$, $\mathbb{N} := \mathbb{D}_\circlearrowleft := \mathbb{D}\mathbb{M} \in$ $\mathrm{TIC}_\infty(U, Y)$.

We add the prefix "P-" or "PB-" if $\mathcal{P}$ is admissible and satisfies the corresponding residual cost condition *below:*

(P) $\quad\langle \mathbb{A}_\circlearrowleft^t x_0, \mathcal{P}\mathbb{A}_\circlearrowleft^t x_0\rangle \to 0$, *as $t \to +\infty$, for all $x_0 \in H$.*

(PB) $\quad\langle \mathbb{B}^t u, \mathcal{P}\mathbb{A}_\circlearrowleft^t x_0\rangle \to 0$, *as $t \to +\infty$, for all $x_0 \in H$ and $u \in \mathcal{U}_*^*(0)$, and (P) holds.*

We call $\mathcal{P}$ $\mathcal{U}_*^*$-stabilizing *if (PB) is satisfied and $\mathbb{K}_\circlearrowleft x_0 \in \mathcal{U}_*^*(x_0)$ for all $x_0 \in H$.*

The eCARE with additional requirements $X = I$ and $S \in \mathcal{GB}(U)$ is called CARE.

(Note that $\left[\begin{array}{c|c} K & F \end{array}\right]$ extends Σ to a WR WPLS iff $\left[\begin{array}{c|c} A & B \\ \hline K & F \end{array}\right]$ generate a WR WPLS. Note also that when we say that the eCARE has a solution, we tacitly say that $\mathbb{D}$ is WR. Admissible pairs are described in Definition 6.6.10; also here we set $\Sigma_{\mathrm{ext}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F} \end{array}\right]$, so that $\Sigma_\circlearrowleft = (\Sigma_{\mathrm{ext}})_{[0\ I]}$.)

A less abstract formulation of the first of the equations in (9.110) is given in (9.4) (cf. Lemma 9.7.8 and inequality (9.104)).

We shall show in Theorem 9.9.1(a1)&(e1)&(e2) that $\mathcal{U}_*^*$-stabilizing solutions of the eCARE correspond to WR J-critical state feedback pairs over $\mathcal{U}_*^*$. We are mainly interested in WR J-critical state feedback operators, equivalently, in the case $F = 0$ (or $X = I$).

The prefix "P-" (resp. "PB-", "$\mathcal{U}_*^*$-") does not affect other prefices and vice versa, it just adds the requirement (P) (resp. (PB), $\mathcal{U}_*^*$-stability of $\Sigma_\circlearrowleft$) (e.g., by "exponentially PB-stabilizing" we mean "exponentially stabilizing and satisfying (PB)").

(For analogy with the traditional concept "stabilizing solution", we say "$\mathcal{U}_*^*$-stabilizing", not "$\mathcal{U}_*^*$-admissible", although this term does not imply that $\Sigma_\circlearrowleft$ is stable (similarly, $\mathbb{C}$-stabilizing means that $\mathbb{C}_{\circlearrowleft}$ is stable but the whole $\Sigma_{\circlearrowleft}$ need not be, by Definition 6.6.10).)

Remark 9.1.6 applies also to eCAREs, except that K need not be WR unless $X, S \in \mathcal{GB}(U)$; cf. Lemma 9.9.7.

Thus, a solution $\mathcal{P}$ is WR if $\left[\begin{smallmatrix} A & B \\ C & D \\ K & F \end{smallmatrix}\right]$ generate a WR WPLS, i.e., if the operators $\mathbb{K} : x_0 \mapsto K\mathbb{A}(\cdot)x_0$ and $\mathbb{F} := I - X + K_{\mathrm{w}}\mathbb{B}\tau$ extend Σ to another WR WPLS (see Lemma 6.3.13 and Remark 9.1.6). A WR solution is admissible iff $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty$, by Definition 6.6.10; if $\mathbb{F} \in \mathrm{ULR}$, then this holds iff $X \in \mathcal{GB}$, by Proposition 6.3.1(c). By Lemma 6.6.11, any solution leading to a bounded K is ULR (and admissible iff $I - F \in \mathcal{GB}(U)$).

Condition (P) is satisfied by any strongly stabilizing solution. It follows from Proposition 9.8.10, that for a $\left[\begin{array}{c|c} \mathbb{C} & \mathbb{D} \end{array}\right]$-stabilizing solution of the eCARE, condition (P) is equivalent to any of (P1)–(P4) of Lemma 9.10.1(d1).

Remark 9.8.2 (eCARE vs. CARE) *A WR solution $\mathcal{P}$ of eCARE is a WR solution of the CARE (with $S' = X^*SX$ and $K' = X^{-1}K$ in place of S and K) iff $S, X \in \mathcal{GB}(U)$.* □

(This is obvious.) Moreover, the admissibility [stabilizability] of $\left[\begin{array}{c|c} K' & 0 \end{array}\right]$ is the same as that of $\left[\begin{array}{c|c} K & I - X \end{array}\right]$ (see the formula for $\Sigma_{\circlearrowleft}$ in Theorem 9.8.12(s1)) and the same applies for other attributes of $\mathcal{P}$, hence we can consider $\mathcal{P}$ alone a solution of the eCARE when $S, X \in \mathcal{GB}(U)$ (because then $\mathcal{P}$ determines the S and K corresponding to $X = I$ uniquely).

Moderate regularity and coercivity assumptions force X and S to be invertible, by Lemma 9.9.7, hence the eCARE is equivalent to the CARE in most applications.

Next we give necessary and sufficient conditions for a solution of the CARE to be admissible, both in state-space and frequency-domain terms. Fortunately, we only rarely need these conditions.

Remark 9.8.3 (Admissibility of a solution of the CARE) *Let $(\mathcal{P}, S, K)$ be a solution of the CARE.*

(a1) By Lemma 6.3.13 and Remark 9.1.6, $\left[\begin{smallmatrix} A & B \\ K & 0 \end{smallmatrix}\right]$ generate a WR WPLS iff for some $\varepsilon > 0$ and $\omega > \omega_A$, $B_{\mathrm{w}}^*\mathcal{P}\mathbb{A}(\cdot) : \mathrm{Dom}(A) \to \mathcal{C}(\mathbf{R}_+; U)$ extends to a continuous map $H \to \mathrm{L}^2([0,\varepsilon); Y)$, and $B_{\mathrm{w}}^*\mathcal{P}\mathbb{B}\tau : \mathcal{C}_{\mathrm{c}}^\infty([0,\varepsilon); U) \to \mathcal{C}_{\mathrm{b}}([0,\varepsilon); Y)$ extends to a continuous map $\mathrm{L}^2_\omega([0,\varepsilon); U) \to \mathrm{L}^2_\omega([0,\varepsilon); Y)$ for some $\varepsilon > 0$ and $\omega > \omega_A$.

(a2) The corresponding frequency-domain conditions are as follows: There are $\varepsilon > 0$ and $\omega > \omega_A$ s.t. for each $x_0 \in H$ we have $B_{\mathrm{w}}^*\mathcal{P}(s - A)^{-1}x_0 \in \mathrm{H}^2(\mathbf{C}_\omega^+; U)$ and $B_{\mathrm{w}}^*\mathcal{P}(s - A)^{-1}B \in \mathrm{H}^\infty(\mathbf{C}_\omega^+; \mathcal{B}(U, Y))$ (as functions of s).

(b) Assume (a1) (equivalently, (a2)). If $\mathbb{F}$ is ULR, then $\mathcal{P}$ is admissible (i.e., $I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}_\infty(U)$).

(c) The solution $\mathcal{P}$ is admissible iff $\mathbb{X} := I - \mathbb{F} = I - K_{\mathrm{w}}\mathbb{B}\tau$ is invertible on $\mathrm{TIC}_\infty(U)$, or equivalently, iff $I - K_{\mathrm{w}}(\cdot - A)^{-1}B \in \mathcal{G}\mathrm{H}^\infty_\infty$, or equivalently, iff $\pi_{[0,\varepsilon)}\mathbb{X}\pi_{[0,\varepsilon)}$ is invertible on $\mathrm{L}^2([0,\varepsilon); U)$ for some $\varepsilon > 0$.

For the eCARE, corresponding conditions can be given, e.g., as follows: $H_B \subset \mathrm{Dom}(K_w)$, $K_w\mathbb{A}(\cdot) : \mathrm{Dom}(A) \to \mathcal{C}(\mathbf{R}_+;U)$ extends to a continuous map $H \to \mathrm{L}^2([0,\varepsilon);Y)$ for some $\varepsilon > 0$, $K_w\mathbb{B}\tau : \mathcal{C}_c^\infty([0,\varepsilon);U) \to \mathcal{C}_b([0,\varepsilon);Y)$ extends to a continuous map $\mathrm{L}^2_\omega([0,\varepsilon);U) \to \mathrm{L}^2_\omega([0,\varepsilon);Y)$ for some $\varepsilon > 0$ and $\omega > \omega_A$, $A_\flat := A + BK_w$ generates a (C_0-)semigroup on H and $K_w(s - A_\flat)^{-1}B \in \mathcal{G}\mathrm{H}^\infty_\infty$.

Proof: (a1) This follows from the definition of K, Remark 9.1.6 and Lemma 6.3.13 (use the fact that $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ is a WPLS).

(a2) Use Lemmas 6.3.15 instead of Lemma 6.3.13 in (a1).

(b) Use Proposition 6.3.1(c).

(c) Use Theorem 6.2.1 and Lemma 2.2.8. □

We define the solutions of the eIARE analogously to those of the eCARE (recall the notation from (6.5)):

Definition 9.8.4 (A P-stabilizing solution of the eIARE) *We call $\mathcal{P}$ (or $(\mathcal{P}, S, \left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right])$) a* solution of the extended Integral Algebraic Riccati Equation (eIARE) *(induced by Σ and J) iff the eIARE*

$$\begin{cases} \mathbb{K}^{t*}S\mathbb{K}^t = \mathbb{A}^{t*}\mathcal{P}\mathbb{A}^t - \mathcal{P} + \mathbb{C}^{t*}J\mathbb{C}^t \quad (\in \mathcal{B}(H)),\\ \mathbb{X}^{t*}S\mathbb{X}^t = \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t,\\ \mathbb{X}^{t*}S\mathbb{K}^t = -\left(\mathbb{D}^{t*}J\mathbb{C}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{A}^t\right)\end{cases} \tag{9.111}$$

(here $\mathbb{X} := I - \mathbb{F}$) is satisfied for all $t > 0$, and $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $S = S^* \in \mathcal{B}(U)$, $\mathbb{K} \in \mathcal{B}(H, \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U))$, and $\mathbb{F} \in \mathrm{TIC}_\infty(U)$.

A solution $\mathcal{P}$ is called well-posed *(resp. WR, SR, UR, admissible, stabilizing, stable, ...) if $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ extends Σ to another WPLS Σ_{ext} (resp. -"- and the pair $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is WR, SR, UR, admissible, stabilizing, stable, ...).*

We use prefices "$\mathcal{U}_*^*$-", "P-" and "PB-" as in Definition 9.8.1.

The eIARE with additional requirement $S \in \mathcal{GB}(U)$ is called IARE.

If $\mathcal{P}$ is admissible, then we denote the corresponding closed-loop system by $\Sigma_\circlearrowleft$ and set $\mathbb{X} := I - \mathbb{F} \in \mathrm{TIC}_\infty(U)$, $\mathbb{M} := \mathbb{X}^{-1} \in \mathrm{TIC}_\infty(U)$, $\mathbb{N} := \mathbb{D}_\circlearrowleft := \mathbb{D}\mathbb{M} \in \mathrm{TIC}_\infty(U,Y)$.

(Note that $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ extends Σ iff $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{K} & \mathbb{F}\end{smallmatrix}\right] \in$ WPLS.) One explanation of the eIARE is that it specifies the value of the terms in $\langle u_\circlearrowleft, S\pi_{[0,t)}u_\circlearrowleft\rangle$, where $u_\circlearrowleft := -\mathbb{K}x_0 + \mathbb{X}u$ is the closed-loop input (the disturbance to the J-critical state feedback) corresponding to open-loop input u and initial state x_0 (indeed, $\mathbb{K}x_0 + \mathbb{F}u = u - u_\circlearrowleft$ in Figure 9.1 (p. 408)).

The IARE should not be mistaken for "IRE", the integral of the "Differential Riccati Equation", which both correspond to finite-time interval problems (see Section 8.5).

By Lemma 9.10.1(b4), equations (9.153)–(9.161) are satisfied by any admissible solution of the eIARE.

In case of $\mathcal{U}_{\exp}$ or $\mathcal{U}_{\mathrm{str}}$, the attribute "$\mathcal{U}_*^*$-stabilizing" can be reduced substantially:

Theorem 9.8.5 ($\mathcal{U}_*^*$-stabilizing solution) *Let $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ be an admissible solution of the eIARE, with closed-loop system $\Sigma_{\circlearrowleft}$. Then $\mathcal{P}$ is $\mathcal{U}_*^*$-stabilizing iff $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is J-critical over $\mathcal{U}_*^*$.*

Moreover, $\mathcal{P}$ is $\mathcal{U}_{\mathrm{exp}}$-stabilizing iff $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is exponentially stabilizing; and $\mathcal{P}$ is $\mathcal{U}_{\mathrm{str}}$-stabilizing iff $\begin{bmatrix} \mathbb{A}_{\circlearrowleft}^{\mathrm{T}} \mid \mathbb{C}_{\circlearrowleft}^{\mathrm{T}} \quad \mathbb{K}_{\circlearrowleft}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ is strongly stable.

Finally, $\mathcal{P}$ is $\mathcal{U}_{\mathrm{out}}$-stabilizing iff $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is PB-output-stabilizing; and $\mathcal{P}$ is $\mathcal{U}_{\mathrm{sta}}$-stabilizing iff $\begin{bmatrix} \mathbb{A}_{\circlearrowleft}^{\mathrm{T}} \mid \mathbb{C}_{\circlearrowleft}^{\mathrm{T}} \quad \mathbb{K}_{\circlearrowleft}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ is stable and (PB) holds.

(Theorem 9.9.1 develops this further.) In particular, if $\mathcal{P}$ is $\mathcal{U}_{\mathrm{exp}}$-stabilizing, then $\mathcal{P}$ is $\mathcal{U}_{\mathrm{str}}$-stabilizing; the converse is not true, by Example 9.13.14. However, a feedback being $\mathcal{U}_{\mathrm{str}}$-stabilizing does not mean that $\mathbb{B}_{\circlearrowleft}$, $\mathbb{D}_{\circlearrowleft}$ or $\mathbb{F}_{\circlearrowleft}$ is stable, hence it need not be strongly stabilizing in general; in Theorem 9.9.1 we use q.r.c.-stabilizability to provide strongly stabilizing optimal controls.

Proof: The first claim follows from Proposition 9.10.2(i)&(ii)&(c). The other claims follow from (d), (e1) and (f1) of Proposition 9.10.2 (use also Theorem 8.3.9(a1)&(a2) for the "only if" part). □

Condition (PB) is, unfortunately, not redundant for $\mathcal{U}_{\mathrm{out}}$ (nor $\mathcal{U}_{\mathrm{sta}}$) in general, by Example 9.13.2 (or Example 9.13.9). However, in several special cases (PB) can be relaxed also for $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{sta}}$; see, e.g., Theorem 9.9.1(c1)–(c3) and Theorem 9.2.10.

Even in the general case, the conditions can be slightly weakened:

Lemma 9.8.6 (Simplifications) *The following are equivalent:*

(i) $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ is a $\mathcal{U}_*^*$-stabilizing solution of the eIARE;

(ii) $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $S \in \mathcal{B}(U)$, $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is an admissible state feedback pair for Σ, the eIARE (9.111) has a solution for one fixed $T := t > 0$, $\mathbb{K}_{\circlearrowleft} x_0 := (I - \mathbb{F})^{-1}\mathbb{K} \in \mathcal{U}_*^*(x_0)$ $(x_0 \in H)$ and the limits in (P) and (PB) are zero when we restrict t to $T\mathbf{N}$ (or to some of its unbounded subsets).

(Thus, (P) and (PB) may be replaced by the condition that $\langle \mathbb{A}_{\circlearrowleft}^{n_k T} x_0 + \mathbb{B}^{n_k T} u, \mathcal{P}\mathbb{A}_{\circlearrowleft}^{n_k T} x_0 \rangle \to 0$, as $k \to +\infty$, for some sequence $\{n_k\} \subset \mathbf{N}$ s.t. $\lim_{k\to+\infty} n_k = +\infty$).

Proof: This follows from "(ii)⇔(iii)" and (h) Proposition 9.10.2. □

The name IARE instead of "IRE" reflects the fact that we can and (most often) will treat the IARE as an algebraic equation of the integrated terms $\mathbb{A}^t$, $\mathbb{B}^t$, $\mathbb{C}^t$, $\mathbb{D}^t$ and of the operators J and $\mathcal{P}$ (see Remark 9.8.8 for how to eliminate S and $\mathbb{K}^t$). In fact, it equals the Discrete-time Algebraic Riccati Equation (DARE) (for $t = 1$; see Definition 14.1.1 and Definition 13.4.2 for the DARE and for the discretization operator Δ^{S}):

Proposition 9.8.7 (eIARE⇔eDARE) *Let $t = 1$ (or let $t > 0$ be arbitrary and use Remark 13.4.6). Then*

(a) If $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is an [admissible [stabilizing]] solution of the eIARE (for Σ and J), then $(\mathcal{P}, S, \Delta^{\mathrm{S}} \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is an [admissible [stabilizing]] solution of the (discrete) eIARE for $\Delta^{\mathrm{S}}\Sigma$ and J. All prefices and suffices apply.

(b) A triple $(\mathcal{P}, S, \Delta^{\mathrm{S}} \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is an admissible [stabilizing] solution of the (discrete) eIARE for $\Delta^{\mathrm{S}}\Sigma$ and J iff $\mathbb{X}^t := I - \mathbb{F}^t \in \mathcal{GB}(\mathrm{L}^2([0,t);U))$ and $(\mathcal{P}, S_{\mathrm{d}}, K_{\mathrm{d}})$ is an admissible [stabilizing] solution of the eDARE for $\Delta^{\mathrm{S}}\Sigma$ and J, where $S_{\mathrm{d}} := (\mathbb{X}^t)^* S \mathbb{X}^t$, $K_{\mathrm{d}} = (\mathbb{X}^t)^{-1}\mathbb{K}^t$. All prefices and suffices apply.

(c1) Let $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ be an admissible [stabilizing] solution of the eIARE. Then $(\mathcal{P}, S_{\mathrm{d}}, K_{\mathrm{d}})$ is an admissible [stabilizing] solution of the eDARE, by (a) and (b). All prefices and suffices apply (including "$\mathcal{U}_*^*$-").

(c2) Conversely, let $(\mathcal{P}, S_{\mathrm{d}}, K_{\mathrm{d}})$ be a $\mathbb{C}$-P-stabilizing solution of the eDARE for $\Delta^{\mathrm{S}}\Sigma$ and J; let $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{d}} & \mathbb{F}_{\mathrm{d}} \end{array}\right]$ be the corresponding state feedback pair, and set $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] := (\Delta^{\mathrm{S}})^{-1} \left[\begin{array}{c|c} \mathbb{K}_{\mathrm{d}} & \mathbb{F}_{\mathrm{d}} \end{array}\right]$.

Then $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{K} & \mathbb{F} \end{array}\right] \in \mathrm{WPLS}$ iff $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is a $\mathbb{C}$-P-stabilizing solution of the eIARE, where $S := (\mathbb{X}^{t*})^{-1} S_{\mathrm{d}} (\mathbb{X}^t)^{-1}$.

(c3) Let $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ be an admissible solution of the eIARE. Then $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is $\mathcal{U}_{\mathrm{out}}$-stabilizing (resp. $\mathcal{U}_{\mathrm{exp}}$-stabilizing) for Σ iff $(\mathcal{P}, S_{\mathrm{d}}, K_{\mathrm{d}})$ is $\mathcal{U}_*^*$-stabilizing (resp. $\mathcal{U}_{\mathrm{exp}}$-stabilizing) for $\Delta^{\mathrm{S}}\Sigma$.

Also most other attributes are invariant (e.g., "J-critical over $\mathcal{U}_{\mathrm{out}}$ (or over $\mathcal{U}_{\mathrm{exp}}$)", "[strongly] internally stabilizing", "P-output-stabilizing", "P-SOS-q.r.c.-stabilizing", "stable", and "exponentially q.r.c.-stabilizing"); see Theorem 13.4.4(d2)&(e)&(f2) for further attributes and details.

(Here continuous-time and discrete-time $\mathcal{U}_*^*$'s corresponds to each other as in (13.73); in particular, $\mathcal{U}_{\mathrm{exp}}$ and $\mathcal{U}_{\mathrm{out}}$ are "invariant".)

Thus, for admissible $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, the triple $(\mathcal{P}, S, \Delta^{\mathrm{S}} \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is a $\mathcal{U}_*^*$-stabilizing solution of the discrete-time eIARE iff $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is a $\mathcal{U}_*^*$-stabilizing solution of the eIARE.

We have stated (a)–(c1) in one direction only, and the converse (c2)–(c3) contains only a partial result (sufficient for most applications). The reader might wish to consult the discretization theory of Section 13.4 for details and for tools for further results.

Note from (b) that S_{d} is invertible (resp. one-to-one) iff S is invertible (resp. one-to-one); in particular, the equivalence of IARE and DARE is analogous to that of eIARE and eDARE.

Proof of Proposition 9.8.7: (a)&(b) These are obvious (see also Theorem 13.4.4(d)&(e1)); we can obviously include the prefix "$\mathcal{U}_*^*$-".

(c1) This follows from (a) and (b).

(c2) By inverse discretization we obtain from the discrete-time eIARE the continuous-time eIARE for time values in $t\mathbf{N}$. By definition, $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{K} & \mathbb{F} \end{array}\right] \in \mathrm{WPLS}$ is necessary for $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ to be admissible, hence we may assume that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{K} & \mathbb{F} \end{array}\right] \in$ WPLS.

Then $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is P-$\mathbb{C}$-stabilizing iff $(\mathcal{P}, S_{\mathrm{d}}, \left[\begin{array}{c|c} \mathbb{K}_{\mathrm{d}} & \mathbb{F}_{\mathrm{d}} \end{array}\right])$ is P-$\mathbb{C}$-stabilizing, by Proposition 9.10.2(a1)(i)&(ii) (which is valid in discrete time too, by Theorem 14.1.3).

(c3) See Theorem 13.4.4(d2)&(e)&(f2) for the properties. In particular, "J-critical over $\mathcal{U}_*^*$" is preserved, hence also "$\mathcal{U}_*^*$-stabilizing", by Theorem 9.8.5.

E.g., "[strongly] internally", "output-" etc. are preserved by (d2) (or (e1)) of Theorem 13.4.4(d2), and "q.r.c.-" etc. are preserved by (e1).

By (c2), "P-output-stabilizing" is preserved (and "P-" in connection to anything stronger than "$\mathbb{C}$-stabilizing").

We do not know whether "P-admissible" is preserved (from discrete to continuous time), but usually even the discrete form of (P) is enough (e.g., the discrete form of "internally P-stabilizing" is enough to guarantee uniqueness, by Theorem 14.1.4(b)).

(Although we can use (f1) of Theorem 13.4.4 for $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{exp}}$, note that we have to use (f2) for $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$; in particular, we cannot write explicitly "$\mathcal{P}$ is $\mathcal{U}_{\mathrm{sta}}^{\Sigma}$-stabilizing" iff "$\mathcal{P}$ is $\mathcal{U}_{\mathrm{sta}}^{\Delta^{S}\Sigma}$-stabilizing" due to the reasons explained in the proof. Fortunately, this does not hinder us from applying certain discrete-time results for $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$ in the same way as for general $\mathcal{U}_*^*$'s.) □

One can speak of a solution $\mathcal{P}$ of the eIARE without mentioning S and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, because the S and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ can be eliminated:

Remark 9.8.8 *The maps $\mathbb{K}^t$, $\mathbb{X}^t$ and S can be eliminated from the eIARE as follows (this is trivial for $S \in \mathcal{GB}$):*

Set $S' := \mathbb{X}^{t*} S \mathbb{X}^t$, $\mathbb{K}^{t\prime} := \mathbb{M}^t \mathbb{K}^t$, so that the second and third equation of the eIARE determine $S' = (S')^* \in \mathcal{B}(\mathrm{L}^2([0,t);U))$ and $S'\mathbb{K}^{t\prime}$. Let $P \in \mathcal{B}(U)$ be the orthogonal projection onto $\operatorname{Ker}(S')^{\perp}$. Then $S'\mathbb{K}^{t\prime}x_0 = S'P\mathbb{K}^{t\prime}x_0$ determines $P\mathbb{K}^{t\prime}x_0$ uniquely a.e., for any $x_0 \in H$. Consequently,

$$\langle \mathbb{K}^t x_0', S\mathbb{K}^t x_0 \rangle = \langle \mathbb{K}^{t\prime} x_0', S'\mathbb{K}^{t\prime} x_0 \rangle = \langle P\mathbb{K}^{t\prime} x_0', S'P\mathbb{K}^{t\prime} x_0 \rangle \tag{9.112}$$

is uniquely determined by S' and $S'\mathbb{K}^{t\prime}$, for any $x_0, x_0' \in H$.

To establish the equivalence between the eIARE and the eCARE, we first have to show that the latter is well defined. Since $B_{\mathrm{w}}^* \in \mathcal{B}(H_{C,K}^*, U)$ if(f) Σ_{ext} is WR, by Proposition 6.2.8(a1), the following shows that the term $B_{\mathrm{w}}^*\mathcal{P}$ in the eCARE is defined on H_B (cf. Remark 9.1.6):

Lemma 9.8.9 ($\mathcal{P} \in \mathcal{B}(H_B, H_{C,K}^*)$) *Let $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_{\omega}(U,H,Y)$ and $J = J^* \in \mathcal{B}(Y)$. Let $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ be a solution of the eIARE s.t. $\Sigma_{\mathrm{ext}} \in$ WPLS. Then $\mathcal{P} \in \mathcal{B}(H_B, H_{C,K}^*)$.*

Proof: Let $x_0 \in H$, $u_0 \in U$ be s.t. $Ax_0 + Bu_0 \in H$. Choose $\omega \in \mathbf{R}$ s.t. $\Sigma, \Sigma_{\circlearrowleft} \in \mathrm{WPLS}_{\omega}$, and choose $a > \omega$. Set $u := \pi_+ \mathrm{e}^{-a\cdot} u_0 \in \mathrm{W}_{\omega}^{1,2}$.

By Theorem 6.2.13(b), $x' = Ax + Bu \in \mathcal{C}(\mathbf{R}_+; H)$ and $y, z \in \mathrm{W}_{\omega}^{1,2}$, where $z := -\mathbb{K}x_0 + \mathbb{X}u$. Therefore, $y^{\mathrm{d}}, z^{\mathrm{d}} \in \mathrm{W}^{1,2}$ in Lemma 9.11.1, so that also x^{d} is

$\mathcal{C}^1$ on $[0,t]$, by Lemma 9.11.1. By Theorem 6.2.13(b1), we have

$$H_{-1}^* \ni A^*x^{\mathrm{d}} + C^*y^{\mathrm{d}} + K^*z^{\mathrm{d}} = -x^{\mathrm{d}'}(t-\cdot) = -\mathcal{P}x' \in H. \tag{9.113}$$

But $-\mathcal{P}x' \in \mathcal{C}([0,t];H)$ implies at 0 that $A^*\mathcal{P}x_0 + C^*Jy(0) + K^*z(0) \in H$, hence $\mathcal{P}x_0 \in H_{C,K}^*$ (see Definition 6.1.17). Because $x_0 \in H_B$ was arbitrary, we have $\mathcal{P}H_B \subset H_{C,K}^*$; the boundedness follows from Lemma A.3.6. □

Equations eIARE and eCARE are equivalent if(f) $\mathbb{D}, \mathbb{X} \in \mathrm{WR}$:

Proposition 9.8.10 (eIARE⇔eCARE) *Let* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *be WR, and let* $J = J^* \in \mathcal{B}(Y)$. *Then the following problems are equivalent:*

(i) The eIARE has a WR solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$.

(ii) The eCARE has a WR solution $(\mathcal{P}, S, \left[\begin{array}{c|c} K & F \end{array}\right])$.

Moreover, every solution of (i) is a solution of (ii) and vice versa (here $\left[\begin{array}{c|c} K & F \end{array}\right]$ *generate* $\left[\begin{array}{c|c} \mathbb{K} & I-\mathbb{F} \end{array}\right]$*).*

Thus, the WR solutions of the CARE are exactly the WR solutions of the IARE having $F = 0$.

Proof: 1° (i)⇒(ii): The eCARE holds by Lemma 9.11.2, and Proposition 9.11.4(b1)&(d) (where K and $I-X$ are the generators of $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$); in particular, the weak limit converges.

2° (ii)⇒(i): Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be the pair generated by $\left[\begin{array}{c|c} K & I-X \end{array}\right]$. The eIARE holds by Lemmas 9.11.2, 9.11.7 and 9.11.6.

(a) This follows from 1° and 2°. □

The usual q.r.c.-SOS-P-stabilizability requirement (cf. Theorem 9.9.10) becomes simple for $\Sigma \in \mathrm{SOS}$:

Proposition 9.8.11 (Stable CARE/IARE) *If* $\Sigma \in \mathrm{SOS}$, $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$, *and* $\mathcal{P}$ *is an admissible solution of the [e]IARE, then the following are equivalent:*

(i) $\mathcal{P}$ *is q.r.c.-SOS-P-stabilizing;*

(ii) $\mathcal{P}$ *is r.c.-SOS-PB-stabilizing and* $\mathcal{U}_*^*$*-stabilizing;*

(iii) $\mathbb{K}$ *is stable,* $\mathbb{X} := I - \mathbb{F} \in \mathcal{G}\mathrm{TIC}$, *and (P) holds.*

Moreover:

(a) In (i)–(iii), we may replace (P) by

(P') $\langle \mathbb{A}^t x_0, \mathcal{P}\mathbb{A}^t x_0 \rangle \to 0$, *as* $t \to +\infty$, *for all* $x_0 \in H$.

(b) If $\mathcal{P}$ *satisfies (P'),* $S \gg 0$, $\mathbb{M} \in \mathrm{TIC}$ *and* Σ *is stable, then (i)–(iii) hold.*

(c) If (i)–(iii) hold, then $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$.

(d1) If Σ *is exponentially stable, then we have three more equivalent conditions:*

(iv) $\mathcal{P}$ *is exponentially stabilizing;*

(v) $\mathcal{P}$ is exponentially stable and exponentially r.c.-stabilizing;

(vi) $\mathbb{M}$ is stable.

(d2) If Σ is exponentially stable and the IARE has a $\mathcal{U}_{\rm out}$-stabilizing solution, then we have one more equivalent condition:

(viii) $\mathcal{P}$ is I/O-, input-, output- or internally stabilizing.

(d3) If Σ is strongly stable and the IARE has a q.r.c.-SOS-P-stabilizing solution, then each of (iv') and (v') is equivalent to (i)–(iii):

(iv') $\mathcal{P}$ is internally stabilizing (i.e., $\mathbb{A}_{\circlearrowleft}$ is stable);

(v') $\mathcal{P}$ is stable and strongly r.c.-stabilizing;

For the CARE or IARE (i.e., when $S \in \mathcal{GB}(U)$) it follows from (c) that $\mathbb{X}^* S\mathbb{X}$ is a spectral factorization of $\mathbb{D}^* J\mathbb{D}$. By Lemma 9.10.1(b5)&(b6), this holds whenever $\mathcal{P}$ is a P-admissible solution and $\mathbb{D}, \mathbb{X}, \mathbb{X}^{-1} \in$ TIC. See also Corollary 9.9.11.

However, even for a strongly stable system (with $\mathcal{U}_*^* = \mathcal{U}_{\rm out} = \mathcal{U}_{\rm sta} = \mathcal{U}_{\rm str}$), a $\mathcal{U}_*^*$-stabilizing solution need not satisfy any of (i)–(iii), by Example 11.3.7 (since $\mathbb{X}$ and $\mathbb{M}$ may be unstable), and a J-critical control can exist even if there is no $\mathcal{U}_*^*$-stabilizing solution (and hence no J-critical state feedback pair over $\mathcal{U}_*^*$), by the same example. Moreover, even if (i)–(iii) hold, there may also be other (non PB-) r.c.-stabilizing solutions, even if Σ is weakly stable and minimal, by Example 9.13.9.

Proof of Proposition 9.8.11: (See Definition 9.8.4 for the eIARE.)

By Lemma 6.6.17(a), $\mathcal{P}$ is [q.]r.c.-SOS-stabilizing iff it is stable and I/O-P-stabilizing (as in (iii)). By Theorem 9.9.1(b)&(g), (i) implies (PB), hence also "$\mathcal{U}_*^*$-".

(a) This follows from Proposition 9.10.2(a2) (and Lemma 9.10.1(d2)).

(b) This follows from Proposition 10.7.1 and (iii).

(c) This follows from Lemma 9.10.1(f2).

(d1) Now $\mathcal{P}$ is necessarily exponentially stable, by Lemma 6.1.10. If $\mathbb{M}$ is stable, $\Sigma_{\circlearrowleft}$ is exponentially stable, by Corollary 6.6.9, hence $\mathbb{M}$ and $\mathbb{X}$ are then exponentially stable, so that $\mathcal{P}$ is exponentially r.c.-stabilizing. Thus, (vi)$\Rightarrow$(v). Obviously, (vi)$\Leftarrow$(i)$\Leftarrow$(v)$\Rightarrow$(iv)$\Rightarrow$(vi).

(d2) This follows from (c1), (c3)(iv') and Theorem 6.7.15(c1), since a $\mathcal{U}_{\rm out}$-stabilizing solution is (exponentially stable and) exponentially (q.)r.c.-stabilizing, by Theorem 6.7.15(c1).

(d3) Obviously, (v')$\Rightarrow$(i)$\Rightarrow$(iv'). Let $\widetilde{\mathcal{P}}$ be a q.r.c.-SOS-P-stabilizing solution, hence stable and strongly stabilizing, by (ii) and Theorem 6.7.15(a2). Assume (iv'). Then $\mathcal{P} = \widetilde{\mathcal{P}}$, by Theorem 9.8.12(a), hence (v') holds. □

A strongly internally stabilizing (i.e., s.t. $\mathbb{A}_{\circlearrowleft}^t x_0 \to 0$ as $t \to +\infty$, for all $x_0 \in H$) solution of the eIARE, eCARE or CARE is unique:

Theorem 9.8.12 ($\mathcal{P}$ is unique) *We have the following uniqueness results for a solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ (not for S and $\begin{bmatrix} \mathbb{K} & \big| & \mathbb{F} \end{bmatrix}$) of the eIARE:*

(a) If the eIARE has a strongly internally stabilizing solution, then that solution is unique among internally stabilizing solutions.

(b) The eIARE has at most one internally P-stabilizing solution.

(c) If the eIARE has an internally ω*-stabilizing solution for some* $\omega < 0$*, then any other solution is (internally) at most* $-\omega$*-stabilizing.*

(d) The eIARE has at most one P-q.r.c.-SOS-stabilizing solution.

(e) The eIARE has at most one $\mathcal{U}_*^*$*-stabilizing solution.*

In the case of an IARE, the corresponding S and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *unique modulo an invertible operator:*

(s1) Let $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ *be an admissible solution of the eIARE. Then so are the triples*

$$(\mathcal{P}, E^{-*}SE^{-1}, \left[\begin{array}{c|c} E\mathbb{K} & E\mathbb{F}+I-E \end{array}\right]). \quad (E \in \mathcal{GB}(U)); \tag{9.114}$$

The corresponding closed-loop systems are given by $\Sigma_{\circlearrowleft E} = \left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft E^{-1} \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft E^{-1} \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft E^{-1} \end{array}\right]$.

All admissible solutions of form $(\mathcal{P}, *, *)$ *are given by (9.114) iff* $\operatorname{Ker}(S) = \{0\}$.

(s2) Let $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ *be an admissible solution of the eIARE. Then* $(\mathcal{P}, \widetilde{S}, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ *is an admissible solution of the eIARE iff* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is admissible for* Σ *and there is* $E \in \mathcal{GB}(U)$ *s.t.* $\widetilde{S} = E^{-*}SE^{-1}$*,* $\widetilde{S}\widetilde{\mathbb{K}} = \widetilde{S}E\mathbb{K}$ *and* $\widetilde{S}\widetilde{\mathbb{X}} = \widetilde{S}E\mathbb{X}$.[2]

(s3) Let $(\mathcal{P}, S, K)$ *and* $(\mathcal{P}, \widetilde{S}, \widetilde{K})$ *be solutions of the CARE. Then* $\widetilde{S} = S$ *and* $SK = S\widetilde{K}$*; in particular, K is unique if* $\operatorname{Ker}(S) = \{0\}$.

The results (a)–(s2) apply to solutions of eCARE and CARE too, by Proposition 9.8.10.

(s4) Let $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ *be a* $\mathcal{U}_*^*$*-stabilizing solution of the eIARE and* $\operatorname{Ker}(S) = \{0\}$*. Then all* $\mathcal{U}_*^*$*-stabilizing solutions of the eIARE are given by (9.114).*
(In particular, $\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}} \in \mathcal{C}(\mathbf{C}^+_{\omega_A}; \mathcal{B}(U))$ *is independent of the solution.)*

Note that when we call the solution unique, we mean that $\mathcal{P}$ is unique; see (s1)–(s4) for the uniqueness of S and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ (corresponding to a fixed $\mathcal{P}$).

By Theorem 9.9.1(b)&(g), we have "(d)⊂(e)" in the sense that if $\mathcal{P}$ is as in (d), then it is as in (e). Obviously, we also have "(c)⊂(a)⊂(b)". However, in 1° of Example 9.13.2, the solution $(0,1,0)$ is as in (d)–(e) (it is the J-critical cost over $\mathcal{U}_{\text{out}}$, by Theorem 9.9.1(e2)), whereas the $\mathcal{U}_{\text{exp}}$-P-stabilizing solution $(2,1,-2)$ is as in (a)–(c) (the J-critical cost over $\mathcal{U}_{\text{exp}}$; note that "P-" is here redundant) — thus, we can have two "unique" solutions.

Therefore, the condition of (e) is the one to be watched (and that of (d) is sufficient) for $\mathcal{U}_{\text{out}}$, whereas for $\mathcal{U}_{\text{exp}}$ we can use either (e) (for $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$)

[2]This formula is dedicated to Sari.

or exponential stabilization. By Lemma 8.3.3, the $\mathcal{U}_{\mathrm{out}}$-stabilizing and $\mathcal{U}_{\mathrm{exp}}$-stabilizing solutions (if either exists) coincide when Σ is estimatable, which is often the case in classical problems.

An intuitive explanation for the uniqueness is the following: a J-critical control need not be unique (see also Theorem 9.9.1(f2) and Example 9.13.6 on uniqueness of $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ (or K)), but the J-critical cost is always unique, by Lemma 8.3.8, hence the J-critical cost operator $\mathcal{P} := \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}_{\mathrm{crit}}$ is unique.

In the positive case with suitable assumptions, there is at most one solution of the eIARE (and of the CARE), see Theorem 10.1.4(c1)&(c2) and Section 10.7.

Proof of Theorem 9.8.12: (a)–(e) This follows from Theorem 14.1.4. and Proposition 9.8.7(c1). (Alternatively, we could write the same proofs for $\mathcal{P}$, S, $\mathbb{K}^t$ and $\mathbb{F}^t$ in continuous time, even though the "no-feedthrough state feedback pair" need not be "well-posed".) Solutions of the eCARE and the CARE are solutions of the eIARE, hence the uniqueness result applies them too.

(s2) 1° *"If"*: This follows by a direct computation.

2° *"Only if"*: Let $(\mathcal{P}, \widetilde{S}, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ be an admissible solution of the eIARE. From (9.160) it follows that $\mathbb{X}^{t*} S \mathbb{X}^t = \widetilde{\mathbb{X}}_+ \widetilde{S} \widetilde{\mathbb{X}}_+$ for $t > 0$. By Lemma 2.3.5, we have $\widetilde{S} = E^{-*} S E^{-1}$ and $\widetilde{S}\widetilde{\mathbb{X}} = \widetilde{S} E \mathbb{X}$ for some $E \in \mathcal{GB}(U)$. From the eIARE it then follows that $\widetilde{S}\widetilde{\mathbb{K}} = \widetilde{S} E \mathbb{K}$.

(Note that if we split U as $U = U_1 \times U_2$, where $U_1 := \mathrm{Ker}(\widetilde{S})$, $U_2 := U_1^{\perp}$, and P_k is the orthogonal projection of U onto U_k ($k = 1, 2$), then $\mathbb{K} = \begin{bmatrix} \widetilde{\mathbb{K}}_1 \\ \widetilde{\mathbb{K}}_2 \end{bmatrix}$, $\mathbb{X} = \begin{bmatrix} \widetilde{\mathbb{X}}_1 \\ \widetilde{\mathbb{X}}_2 \end{bmatrix}$, where $\widetilde{\mathbb{K}}_2 = P_2 E \mathbb{K}$, $\widetilde{\mathbb{X}}_2 = P_2 E \mathbb{X}$, but $\widetilde{\mathbb{K}}_1$ and $\widetilde{\mathbb{X}}_1$ are arbitrary as long as $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ is admissible for Σ. Equivalently, $K_2 = P_2 E K$ and $\widehat{\widetilde{\mathbb{X}}}_2 = P_2 E \widehat{\mathbb{X}}$, but K_1 is arbitrary as long as $\left[\begin{array}{c|c} A & B \\ \hline -K & * \end{array}\right]$ generate WPLSs, some of which have an I/O map in $\mathcal{G}\mathrm{TIC}_\infty$.)

(s1) Obviously, (9.114) defines a solution for all $E \in \mathcal{GB}$. If $\mathrm{Ker}(S) = \{0\}$, then there are no others, because then the E in (s2) determines $\widetilde{S}$ and $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ uniquely.

For the converse, let $\mathrm{Ker}(S) \neq \{0\}$. Choose $T \in \mathcal{B}(\mathrm{Ker}(S)) \setminus \{0\}$ so small that $\widetilde{\mathbb{X}} := \mathbb{X} + T \in \mathcal{GB}(U)$. Then $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & I - \widetilde{\mathbb{X}} \end{array}\right])$ is an admissible solution of the eIARE, by (s2) (with $E = I$).

(s3) This is obvious from the CARE.

(s4) This follows from (e) and (s1). □

We now note the continuous-time counterpart of Corollary 15.5.3:

Theorem 9.8.13 (Greatest solution $\mathcal{P}_+$ of the CARE/IARE) *If the CARE (resp. IARE) has a strongly ($\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$-)stabilizing solution s.t. $S \gg 0$, then this solution is the greatest admissible solution of the eCARE (resp. eIARE) having $S \geq 0$.* □

(This follows from Corollary 15.5.3 and Propositions 9.8.10 and 9.8.7. Note that it suffices that $\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \end{array}\right]$ is strongly stable; $\left[\begin{array}{c|c} \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right]$ need not be.)

Analogously, one can deduce from Theorem 15.5.2 that if Σ is strongly $\left[\frac{\mathbb{A}}{\mathbb{C}}\middle|\frac{\mathbb{B}}{\mathbb{D}}\right]$-stabilizable and the IARE has a solution s.t. $S \gg 0$, then there is an upper bound $\mathcal{P}_+$ for all solutions $(\mathcal{P}, S, \left[\ \mathbb{K}\ \middle|\ \mathbb{F}\ \right])$ of the eIARE having $S \geq 0$. However, we do not know whether $\mathcal{P}_+$ itself is a solution of the IARE, i.e., whether the corresponding dediscretized $\left[\ \mathbb{K}_+\ \middle|\ \mathbb{F}_+\ \right]$ is well-posed in continuous time (though $(\mathcal{P}_+, S_+, \left[\ \mathbb{K}_+\ \middle|\ \mathbb{F}_+\ \right])$ solves the discretized IARE).

Recall that "the greatest admissible" means that if $\mathcal{P}'$ is an admissible solution of the eIARE or eCARE (or of the corresponding (extended) Riccati inequality "eIARI", see Theorem 15.5.2) s.t. $S' \geq 0$, then $\mathcal{P}' \leq \mathcal{P}$. Recall that we require any solution of any ARE to be self-adjoint.

The CARE of Example 9.13.9 has several stabilizing solutions s.t. $S \gg 0$, but it does not have a maximal (hence not a greatest) solution; therefore, the system cannot be strongly stabilizable (by Theorem 15.5.2 and discretization). On the other hand, Example 9.13.12(b) shows that "strongly" cannot be replaced by "weakly" in the theorem.

In connection with H^∞ control problems, it is common to speak of lossless factorizations instead of Riccati equations. This is due to the fact that if $\mathcal{P} \geq 0$ is a $\mathcal{U}_*^*$-stabilizing solution with $\mathbb{D}_\circlearrowleft \in \mathrm{TIC}$, then $\mathbb{D}_\circlearrowleft$ is (J,S)-lossless:

Lemma 9.8.14 ($\mathcal{P} \geq 0 \Longrightarrow \mathbb{D}_\circlearrowleft$ is (J,S)-lossless) *Let $\mathcal{P} \geq 0$ be an admissible solution of the IARE s.t. $\mathbb{C}_\circlearrowleft$ and $\mathbb{D}_\circlearrowleft$ are stable and (P) holds. Then $\mathbb{D}_\circlearrowleft$ is (J,S)-lossless.*

Thus, we obtain a (J,S)-lossless right factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ when $\mathcal{P} \geq 0$ is P-SOS-stabilizing.

Proof: Indeed, if $\mathcal{P} \geq 0$ is admissible and $\mathbb{N} := \mathbb{D}_\circlearrowleft$ is stable, then $S - \mathbb{N}^{t\,*} J \mathbb{N}^t = \mathbb{B}^{t\,*}_\circlearrowleft \mathcal{P} \mathbb{B}^t_\circlearrowleft \geq 0$ for all $t > 0$, hence $\mathbb{N}^* \pi_- \mathbb{N} \leq \pi_- S$, by Lemma 2.2.4(b1). When also $\mathbb{C}_\circlearrowleft$ is stable and (P) holds, we have $\mathbb{N}^* J \mathbb{N} = S$, by Lemma 9.10.1(f2), hence then $\mathbb{N}$ is (J,S)-lossless.

(In fact, $\mathbb{D}_\circlearrowleft$ is (J,S)-lossless iff $\mathcal{P} \geq 0$ on the reachable subspace $H_\mathbb{B}$ of Σ.) □

Since we have let B be highly unbounded, we meet several phenomena that are not present in classical results. The generality of regular WPLSs allows a wide range of discontinuities, in particular, all discrete systems can be written in the form of a WPLS. Thus is feels somewhat natural that we must add the "$B^*\mathcal{P}B$-term" to the formula for S as in the (classical) discrete case (see, e.g., Section 14.1 or equation (B.2.27) of [GL]). Of course, with certain additional regularity assumptions one can guarantee that $S = D^*JD$ (see Remark 9.9.14(b)).

We give below an example, where $S \neq D^*JD$; see [S96], [WZ], the notes below and Section 9.13 for more examples and a further discussion on this phenomenon.

Example 9.8.15 ($S \neq D^*JD$) Let $U = \mathbf{C} = Y$, $\mathbb{D} = \tau(-1) \in \mathrm{MTIC_d} \subset \mathrm{ULR}$, $J = I$, as in Examples 6.2.14, 6.3.7 and 8.3.12. Then $\mathbb{D}^* J \mathbb{D} = I = \mathbb{X}^* S \mathbb{X}$ with $S = I = \mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$; $D = 0, X = I, D^*JD \neq X^*SX$. In particular, $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{out}}$.

Let Σ be any strongly stable realization of $\mathbb{D}$. Then the CARE has a unique stable, stabilizing solution $(\mathcal{P}, S, K)$ (which is $\mathcal{U}_{\mathrm{out}}$-stabilizing), by Corollary

9.1.9, and $\mathbb{X}$ is the operator corresponding to this solution, i.e., $\widehat{\mathbb{X}}(s) = I - K_{\rm w}(s-A)^{-1}B$. The corresponding control $\mathbb{K}_{\circlearrowleft}x_0$ is the unique J-critical control over $\mathcal{U}_{\rm out}$ for each $x_0 \in H$.

Since $\mathbb{X}$ and $\mathbb{D}$ are SR, we have $K_{\rm w} = K_{\rm L,w} = K_{\rm s} = K_{\rm L,s}$ on H_B and $B_{\rm w}^* = B_{\rm L,w}^* = B_{\rm s}^* = B_{\rm L,s}^*$ on $H_{C,K}^*$, by Proposition 6.2.8. By Proposition 9.11.4(a), we have (take $x_0 := (s-A)^{-1}Bu_0$ so that $Ax_0 + Bu_0 \in H$))

$$B_{\rm w}^*\mathcal{P}(s-A)^{-1}Bu_0 = (X^*SX - D^*JD - X^*SK_{\rm w}(s-A)^{-1}B)u_0 \tag{9.115}$$

$$= (I - 0 - X^*S(\widehat{\mathbb{X}} - X))u_0 = u_0 \to u_0 \tag{9.116}$$

for all $u_0 \in U$, as s goes to $+\infty$ (trivially). By the CARE, we have $K = -B_{\rm w}^*\mathcal{P}$. Thus, again by Proposition 9.11.4(a), we have

$$-(B_{\rm w}^*\mathcal{P})_{\rm w}x_0 = K_{\rm w}x_0 = -B_{\rm w}^*\mathcal{P}x_0 + u_0 \quad \text{for } x_0, u \text{ s.t. } Ax_0 + Bu_0 \in H. \tag{9.117}$$

Let us write out the Riccati equation for the strongly stable realization

$$\Sigma := \left[\begin{array}{c|c} \pi_+\tau & \pi_{[0,1]}\tau(-1) \\ \hline \pi_+ & \tau(-1) \end{array}\right] \in \mathrm{WPLS}_0(U,H,Y) \tag{9.118}$$

of Example 6.2.14; here $H := \mathrm{L}^2(\mathbf{R}_+;Y)$, $U = \mathbf{C} = Y$.

By Example 8.3.12, we have $\mathcal{P} = \pi_{[0,1)} \in \mathcal{B}(H)$ and $-\pi_+\tau^1 = \mathbb{K}_{\circlearrowleft} = \mathbb{K}$ (although $\mathbb{A}_{\circlearrowleft} \neq \mathbb{A}$), hence

$$Kx_0(t+\cdot) = K\mathbb{A}^t x_0 = (\mathbb{K}x_0)(t) = -x_0(t+1) = -\delta_1^* x_0(t+\cdot) \quad (x_0 \in H_1,\ t \geq 0); \tag{9.119}$$

consequently, $K = -\delta_1^*$. Using the results of Example 6.2.14, we get that $H_1 = \mathrm{W}^{1,2}((0,\infty))$ and

$$(s-A^*)^{-1}C^* = \mathrm{e}^{-s\cdot} \in \mathcal{B}(Y,H_C^*), \quad (s-A^*)^{-1}K^* = \mathrm{e}^{-s(\cdot-1)}\pi_{[1,\infty)} \in \mathcal{B}(Y,H_K^*), \tag{9.120}$$

$$\text{hence } H_C^* = \mathrm{W}^{1,2}(\mathbf{R}) \text{ and} \tag{9.121}$$

$$H_{C,K}^* = \mathrm{W}_0^{1,2} + \mathbf{C}\mathrm{e}^{-\cdot} + \mathbf{C}\mathrm{e}^{-(\cdot-1)}\pi_{[1,\infty)} = \{x_0 \in H \,|\, x_0' \in H + \mathbf{C}\delta_1\} \tag{9.122}$$

$$= \mathrm{W}^{1,2}((0,1)) + \mathrm{W}^{1,2}([1,\infty)) = H_B. \tag{9.123}$$

Thus, $\mathcal{P}H_B = \mathrm{W}^{1,2}((0,1)) \subset H_{C,K}^*$ as expected, and (recall that $B_{\rm w}^* = \delta_{1-}^*$)

$$Su_0 = B_{\rm w}^*\mathcal{P}(s-A)^{-1}Bu_0 = \delta_{1-}^*\pi_{[0,1)}\mathrm{e}^{-s(1-\cdot)}u_0 = u_0 \quad (s \in \mathbf{C}^+), \tag{9.124}$$

by (6.61, as proved above for any strongly stable realization of $\mathbb{D}$. By Proposition 6.2.8(c3)&(c1)&(c4)&(d1), we have

$$K_{\rm w}x_0 = K_{\rm L,s}x_0 = \lim_{s\to+\infty} \frac{1}{t}\int_0^t -x_0(1+r)\,dr = -x_0(1+) =: -\delta_{1+}^*x_0 \quad (x_0 \in H_B), \tag{9.125}$$

$$\text{hence } \quad K_{\rm w}(s-A)^{-1}B = 0 \ \text{ for all } \ s \in \mathbf{C}^+, \tag{9.126}$$

as expected. This agrees with the CARE, since $B_{\rm w}^*\mathcal{P} = \delta_{1-}^*$ on H_B, so that

$-B_{\mathrm{w}}^* \mathcal{P} = -\delta_1^* = K$ on H_1. Thus we obtain from (9.125) that

$$(B_{\mathrm{w}}^* \mathcal{P})_{\mathrm{w}} = \delta_{1+}^* = -K_{\mathrm{w}} \text{ and } B_{\mathrm{w}}^* \mathcal{P} = \delta_{1-}^* \quad \text{on } H_B \tag{9.127}$$

(and $-K = B_{\mathrm{w}}^* \mathcal{P} = \delta_1^*$ on H_B). By combining (9.124), (9.127) and (9.126) we obtain for $z_0 \in H_1$, $u_0 \in U$, $x_0 = z_0 + (s-A)^{-1} B u_0$ (i.e., for arbitrary $x_0 \in H_B$) that $K_{\mathrm{w}} x_0 = K_{\mathrm{w}} z_0$ and

$$B_{\mathrm{w}}^* \mathcal{P} x_0 = \delta_{1-}^* z_0 + u_0 = -K_{\mathrm{w}} x_0 + u_0 = (B_{\mathrm{w}}^* \mathcal{P}) x_0 + u_0, \tag{9.128}$$

again as shown above.

Above, we have derived $\mathcal{P}$, S and K from the solution of the J-critical control problem (minimization problem) over $\mathcal{U}_{\text{out}}$. By Corollary 9.1.9, $(\mathcal{P}, S, K)$ is the unique stable, stabilizing solution of the CARE. To verify this, we note that the second and third equations of the CARE hold by (9.124) and (9.127), respectively, and the first one given by

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P} + \mathcal{P}\frac{\mathrm{d}}{\mathrm{d}t} + \delta_0 \delta_0^* = (\delta_{1-}^* \mathcal{P})^* \delta_{1-}^* \mathcal{P}, \quad \text{equivalently,} \tag{9.129}$$

$$\int_{\mathbf{R}_+} x_0' \mathcal{P} \overline{x_1} + \int_{\mathbf{R}_+} x_0 \mathcal{P} \overline{x_1'} = \delta_{1-}^* \mathcal{P} x_0 \overline{\delta_{1-}^* \mathcal{P} x_1} - x_0(0) \overline{x_1(0)} \qquad (x_0, x_1 \in H_1). \tag{9.130}$$

With our $\mathcal{P} = \pi_{[0,1)}$, this becomes $\int_0^1 (x_0' \overline{x_1} + x_0 \overline{x_1'}) = \big|_0^1 x_0 \overline{x_1}$, which confirms that $(\mathcal{P}, S, K)$ indeed solves the CARE.

Obviously, the integral (6.67) does not converges for every $x_0 \in \mathrm{L}^2([0,1); Y) = \mathcal{P}[H]$, hence $\mathcal{P}[H] \not\subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ (although $\mathcal{P}[H_B] \subset H_{C,K}^* \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$, as shown above), so that Hypothesis 9.2.1 is not satisfied. $\triangleleft$

Notes for Section 9.8

The CARE (9.3) is only a slightly extended version of the CARE presented independently by M. Weiss and G. Weiss [WW] and O. Staffans [S97b] (which contained the first and third equations in the setting of Proposition 8.3.10; the formula for S was published in [S98b]).

Our contributions to the theory contain the converse direction — the fact that a stabilizing solution of the Riccati equation leads to the optimal state feedback pair — and the generalization of these results to general cost functions (instead of J-coercive ones), for general regular WPLSs (instead of stable or jointly stabilizable and detectable ones), to general $\mathcal{U}_*^*$'s (instead of $\mathcal{U}_{\text{out}}$), to nonunique optimal control (and the eCARE), and to WR state feedback pairs (instead of SR operators); in fact, the IARE theory also allows for arbitrary (irregular) WPLSs. These will be applied to further control problems in Chapters 10–12.

The equations that constitute the IARE have appeared among the equations in Section 5 of [S98b] and in Sections 7–11 of [WW]; at least some of them can be found in the older literature (e.g., a variant of (9.155) for a standard LQR cost function is contained in Section 5 of [Sal87] for WPLSs and in Lemma 4.3 of [CP78] and Corollary 4.1 of [Gibson] for systems with bounded B and C).

We have not seen such equations treated in the literature as sufficient (and necessary) conditions for optimal control, nor as a discrete-time Riccati equation

(DARE).

Lemma 9.8.9 and Proposition 9.8.10 are based on the methods used in Sections 5–7 of [S98b] (partially also in [WW]). We published an early version of the results of this chapter in [Mik97b] (the stable case); it also contained some of Proposition 9.8.11.

The proof of (a)–(c) Theorem 9.8.12 (in the proof of Theorem 14.1.4) is a generalization of the classical proof for the uniqueness of the exponentially stabilizing solution of DARE (see, e.g., Proposition 13.5.1 of [LR]). See the notes to Section 15.5 for Theorem 9.8.13.

Our proof of Lemma 9.8.14 follows that of Theorem 6.5 of [S98c]. Part of Example 9.8.15 is contained in [WZ] and [S95].

Much attention has been paid to systems with bounded input and output operators (B and C) and to Pritchard–Salamon systems, both of which have the signature operator S equal to $S = D^*JD$ (which is often taken to be the identity in the positive case and to $\left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right]$ in the indefinite case). Indeed, whenever, Hypothesis 9.2.2 holds (e.g., $B \in \mathcal{B}(U,H)$) we have $S = D^*JD$, by Section 9.2; other sufficient conditions are given in Remark 9.9.14(b).

However, in general *our signature operator "S" takes the role of D^*JD exactly as in discrete-time*. Indeed, S is the signature operator of the control problem corresponding to the CARE, by Theorem 9.9.1(h) and (9.139), whereas D^*JD need not contain any information on the signature properties of the problem: in Example 9.13.7 $\mathbb{D}$ and $\mathbb{X}$ are ULR (even MTIC) but the operator D^*JD may have any signature (as long as its norm is less that 4) and still the unique $\mathcal{U}_{\text{out}}$-stabilizing solution of the CARE is maximizing over $\mathcal{U}_{\text{out}}$ (because $S \ll 0$).

In a strictly definite problem, as in Example 9.13.7, the operator S tells us all signature properties (definiteness) of the problem. However, S is not unique, but E^*SE is also a signature operator for Σ and J for any $E \in \mathcal{GB}(U)$, by Theorem 9.9.1(f2). Therefore, S only contains the information on the nature of the problem, not on corresponding directions (for which we need also the corresponding $\mathbb{X} := I - \mathbb{F}$ or K).

Nevertheless, when we consider solutions of the CARE, S becomes unique (since it fixes $X = I$), and then S contains all information on the signature properties of the problem (also on an infinitesimal moment). Thus, then the situation is analogous to the discrete time (see Chapter 14 or some classical textbook), where the signature operator $S := D^*JD + B^*\mathcal{P}B$ takes the role of D^*JD even for finite-dimensional systems. These facts are illustrated in Proposition 11.2.19, where in (b2) we can only report the dimensions of the positive and negative eigenspaces of S (of the IARE), whereas in (d1)&(d2) we can also tell the directions.

A notable special case of the signature properties of S is that $\operatorname{Ker}(S) = \{0\}$ is necessary and sufficient for the J-critical control to be unique (for general WPLSs), whereas $\operatorname{Ker}(D^*JD) = \{0\}$ is sufficient (for WR ones) but not necessary, as noted below Corollary 9.7.4.

Finally, in general (for irregular systems) we do not even have the operator D, whereas any WPLS having a J-critical state feedback pair has a signature operator S, by Theorem 9.9.1. For stable J-coercive systems this equals the signature operator of the spectral factorization, by Corollary 9.9.11.

If we do not require the existence of (well-posed) J-critical state feedback, i.e., if we use the setting of Section 9.7, then we could still define another signature operator, namely $\mathbb{S}^t := \mathbb{D}^{t*} J \mathbb{D}^t + \mathbb{B}^{t*} \mathcal{P} \mathbb{B}^t$ (for some $t > 0$). The map $\mathbb{S}^t$ is the signature operator of the eDARE obtained by discretizing the eIARE as in Proposition 9.8.7, hence it tells us about the signature properties of the problem even if the eIARE would have no solutions; see also Proposition 9.9.12 for the properties of $\mathbb{S}^t$. If there is a J-critical control in state feedback form, then $\mathbb{S}^t = \mathbb{X}^{t*} S \mathbb{X}^t$ and hence then $\mathbb{S}^t$ tells us practically the same information on the problem that S does, by Lemma 2.3.5.

It remains an important open problem to find a decent formula for S in terms of the generating operators when $\mathbb{D}$ and $\mathbb{X}$ are not known to be regular (the one in the (e)IARE is rather complicated and the one in the (e)CARE is not applicable in the irregular case).

Another open problem is the exact connection between S and the signature properties of the problem in the general case. Usually (e.g., when $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, or when $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ and $\mathcal{P}$ is q.r.c.-SOS-stabilizing), the set $\mathcal{U}_*^*(x_0)$ of admissible inputs corresponds exactly to the closed-loop inputs $u_\circlearrowleft \in \mathrm{L}^2(\mathbf{R}_+;U)$, by Theorem 9.9.1(k), so that the dimensions of the positive and negative eigenspaces and kernel of S exactly describe the definiteness of the problem, through equation (9.139).

Even for general $\mathcal{U}_*^*$, the equation holds for (compactly supported) $u \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+;U)$, by Theorem 9.9.1(i3). Thus, e.g., S is [strictly] nonnegative if the problem has a [strict] minimum, but it remains an open problem whether the converse holds for general $\mathcal{U}_*^*$; naturally, a similar situation is met also in the indefinite case (Chapter 11). Fortunately, for most of the time, we only have to treat the case $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ or the quasi-coprime setting.

In applications to certain kinds of systems, one may wish to write the CARE on some larger space than $\mathrm{Dom}(A)$ and/or avoid the Weiss extensions ("B_{w}^*"). One example of this is given in Theorem 9.9.6, for systems with bounded input operator (B), and another in Section 9.5, for parabolic systems. In both examples, the proofs were based in establishing the equivalence of this "smoother CARE" to the original one. In cases where this cannot be done, one may, alternatively, rewrite our proofs of the equivalence CARE⇔IARE in Section 9.11 for this setting; most other results of this monograph are based on IAREs and are hence directly applicable also for such "modified CAREs" after this equivalence has been verified.

9.9 J-Critical control $\leftrightarrow$ Riccati Equation

There is a theory which states that if ever anyone discovers exactly what the Universe is for and why it is here, it will instantly disappear and be replaced by something even more bizarre and inexplicable. There is another theory which states that this has already happened.

— Douglas Adams, "The Hitchhiker's Guide to the Galaxy"

This section provides most of the Riccati equation theory needed for solving the different control problems in Chapters 10–12; in some special cases the results of Sections 9.1–9.2 will suffice, and in some cases we need additional results from the other sections.

We establish the equivalence of the existence of a J-critical state feedback pair and the existence of a $\mathcal{U}_*^*$-stabilizing solution to the Riccati equation (Theorem 9.9.1), as mentioned in preceding sections. We also develop some further results and simplifications under different stabilizability or regularity assumptions.

Most of the latter part of this section (Remark 9.9.9–Corollary 9.9.11) corresponds to quasi-coprime stabilization, which allows us to considerable simplifications when finding optimal [SOS/strongly] stabilizing state feeback; the case with exponentially stabilizing state feedback ($\mathcal{U}_{\mathrm{exp}}$) is originally simpler and described in Corollary 9.9.3–Proposition 9.9.5.

In Proposition 9.9.12 we treat the "signature" operator in the general case of possibly ill-posed optimal "state feedback". In Remark 9.9.14 we summarize several cases in which a unique J-critical control corresponds to a (well-posed) regular state feedback operator.

Recall from Section 8.3 that a J-critical control is one that makes the (Fréchet) derivative of the cost function vanish, any optimal control is usually J-critical and usually also the converse holds.

We start by the equivalence. We first give equivalent conditions under different stabilizability assumptions ((a1)–(d)), and then we note that that the solutions correspond to each other as in classical results ((e1)–(e2)). Parts (f1)–(i) list some facts that will be needed later.

Theorem 9.9.1 (J-Critical control$\Leftrightarrow$eIARE) *The following statements hold:*

(a1) **(J-critical)** *There is a J-critical state feedback pair over $\mathcal{U}_*^*$ for Σ iff the eIARE has a $\mathcal{U}_*^*$-stabilizing solution.*

(a2) **(Min)** *There is a minimizing state feedback pair $\left[\begin{array}{c|c} \mathbb{K}_{\min} & \mathbb{F}_{\min} \end{array}\right]$ over $\mathcal{U}_*^*$ for Σ iff $\mathcal{J}(0,\cdot) \geq 0$ and the eIARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$.*

Assume that $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is as above. Then $S \geq 0$, and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is minimizing (the control $\mathbb{K}_\circlearrowleft := (I-\mathbb{F})^{-1}\mathbb{K}$ is strictly minimizing iff $S > 0$).

If $\mathcal{P} \geq 0$ (e.g., $J \geq 0$) and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ then $\mathcal{P}$ is the smallest nonnegative output-stabilizing solution of the eIARE.

If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, $J \geq 0$ and $S \gg 0$, then $\mathcal{P}$ is the greatest nonnegative admissible solution of the eIARE.

(b) There is a J-critical q.r.c.-SOS-stabilizing state feedback pair over $\mathcal{U}_{\text{out}}$ for Σ iff the eIARE has a P-q.r.c.-SOS-stabilizing solution.

(c1) Let $\mathbb{B}$ be stable. Then there is a J-critical strongly stabilizing state feedback pair for Σ over $\mathcal{U}_{\text{out}}$ [and $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{str}}$] iff the eIARE has a strongly stabilizing solution.

(c2) Let Σ be strongly stable. Then there is a J-critical state feedback pair for Σ over $\mathcal{U}_{\text{out}}$ [and $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{str}}$] iff the eIARE has an output-stabilizing solution.

(c3) Let Σ be strongly q.r.c.-stabilizable. Then there is a J-critical state feedback pair for Σ over $\mathcal{U}_{\text{out}}$ [and $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{str}}$] iff the eIARE has a $\mathcal{U}_{\text{str}}$-stabilizing solution.

(d) Let Σ be estimatable. Then $\mathcal{U}_{\text{exp}} = \mathcal{U}_{\text{str}} = \mathcal{U}_{\text{sta}} = \mathcal{U}_{\text{out}}$. Moreover, there is a J-critical state feedback pair over $\mathcal{U}_{\text{out}}$ for Σ iff the eIARE has an output-stabilizing solution.

Such a solution is exponentially P-q.r.c.-stabilizing and it is the unique internally stabilizing solution.

(e1) If $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ is a solution of the eIARE of the form required in some of (a1)–(d), then the corresponding state feedback pair $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is of the required form (i.e., J-critical or minimizing in the required sense).

(e2) Conversely, if $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is of the required form in some of (a1)–(d), then so is the solution $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ of eIARE, where $\mathcal{P} := \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$, $\mathbb{X} = I - \mathbb{F}$, $\mathbb{M} := \mathbb{X}^{-1}$, $\mathbb{C}_{\circlearrowleft} := \mathbb{C} + \mathbb{D}\mathbb{M}\mathbb{K}$, $\mathbb{B}_{\circlearrowleft} := \mathbb{B}\mathbb{M}$, $\pi_{[0,t)} S = \mathbb{N}^{t\,*} J \mathbb{N}^t + \mathbb{B}_{\circlearrowleft}^{t\,*} \mathcal{P} \mathbb{B}_{\circlearrowleft}^t$, and $\mathbb{N} := \mathbb{D}\mathbb{M}$.

Assume that (at least) one of (a1)–(d) is satisfied by $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ (by $\Sigma_{\circlearrowleft}$ we denote the corresponding closed-loop system). Then the following statements hold:

(f1) **(Uniqueness)** *$\mathcal{P}$ is unique, $\mathcal{U}_*^*$-stabilizing and equal to the J-critical cost operator (naturally, $\mathcal{P}$ may depend on the choice of $\mathcal{U}_*^*$).*

(f2) The J-critical control is unique (for each $x_0 \in H$) iff S is one-to-one. If S is one-to-one, then all J-critical feedback pairs $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ are given by (9.114); the converse is not true:

Assume that S is not one-to-one. Then the pair $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ solving the eIARE with $\mathcal{P}$ and S is not unique modulo (9.114) (but it is unique modulo Theorem 9.8.12(s2)). However, it may still be that only one $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ (modulo (9.114)) is $\mathcal{U}_*^*$-stabilizing (see Example 9.13.6 for details).

(g1) $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ is $\mathcal{U}_*^*$-stabilizing (cf. (a1)) and satisfies (9.153)–(9.163), (P1)–(P4), (P) and (PB) (see Lemma 9.10.1(d1) and Definition 9.8.1).

(g2) $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$. If $\mathbb{N} := \mathbb{D}_{\circlearrowleft} := \mathbb{D}(I - \mathbb{F})^{-1}$ is stable, then $S = \mathbb{N}^* J \mathbb{N}$ and $\pi_+ \mathbb{N}^* J \mathbb{C} = 0$.

(h) Equation (9.139) holds for all $x_0 \in H$ and all $u_{\circlearrowleft} \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+; U)$ (all $u_{\circlearrowleft} \in \mathrm{L}^2(\mathbf{R}_+; U)$ if $\mathbb{D}_{\circlearrowleft}$ is stable).

(i1) If $\mathbb{X}$ is WR and $\exists X_{\text{left}}^{-1}$, then the J-critical control $u_{\text{crit}}(x_0) := (\mathbb{K}_{\circlearrowleft} x_0)$ is given by

$$u_{\text{crit}}(x_0)(t) := (\mathbb{K}_{\circlearrowleft} x_0)(t) = X_{\text{left}}^{-1} K_{\text{w}} x_{\text{crit}}(t) \text{ a.e.} \tag{9.131}$$

(i2) We have $\mathcal{P} \in \mathcal{B}(H) \cap \mathcal{B}(H_B, H_{C,K}^*)$, and $\mathcal{P}\mathbb{A}_{\circlearrowleft}(t)x_0 \to 0$ as $t \to +\infty$, for all $x_0 \in H$.

(i3) $\mathbb{K}_{\circlearrowleft} x_0 + \mathbb{M}\mathrm{L}^2_{\text{c}}(\mathbf{R}_+;U) \subset \mathcal{U}_*^*(x_0)$ for all $x_0 \in H$ (see also (k)).

(i4) $\langle \mathbb{D}u, J\mathbb{D}\mathbb{M}\eta \rangle = \langle \mathbb{M}^{-1}u, S\eta \rangle$ for all $u \in \mathcal{U}_*^*(0)$ and $\eta \in \mathrm{L}^2_{\text{c}}(\mathbf{R};U)$.

(j) Theorem 8.3.9 applies for $\Sigma_{\text{crit}} := \begin{bmatrix} \mathbb{A}_{\circlearrowleft} \\ \hline \mathbb{C}_{\circlearrowleft} \\ \mathbb{K}_{\circlearrowleft} \end{bmatrix}$.

(k) Assume that $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, or that $\mathcal{P}$ is q.r.c.-stabilizing (resp. strongly-q.r.c.-stabilizing, SOS-q.r.c.-stabilizing) and $\mathcal{U}_*^*$ equals $\mathcal{U}_{\text{sta}}$ (resp. $\mathcal{U}_{\text{str}}$, $\mathcal{U}_{\text{out}}$).

Then $\mathbb{K}_{\circlearrowleft} x_0 + \mathbb{M}\mathrm{L}^2(\mathbf{R}_+;U) = \mathcal{U}_*^*(x_0)$ for all $x_0 \in H$, and (9.139) holds for all $u_{\circlearrowleft} \in \mathrm{L}^2(\mathbf{R}_+;U)$. In particular, $\mathbb{K}_{\circlearrowleft}$ is [strictly] minimizing iff $S \ge 0$ [$S > 0$].

Also Lemma 9.8.6 and Theorem 9.8.5 apply to (a1)–(d); note that a solution of any of (a1)–(d) is a solution of (a1). We remind that a solution of the eIARE is required to be self-adjoint, by Definition 9.8.4.

Further simplifications in the positive case are given in Section 10.7.

To give a better understanding of criteria (P) and (PB), we note from Lemma 9.10.1(d) that if the eIARE has an output-stabilizing solution $\mathcal{P}$, then (P) holds iff $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ (which is the J-critical cost iff (PB) holds); (P) is also needed to get "$\pi_+ \mathbb{D}_{\circlearrowleft}^* J\mathbb{C} = 0$", which roughly says that the closed-loop system is "J-critical w.r.t. stable closed-loop inputs", and the second condition in (PB) then extends this to "$\pi_+ \mathbb{D}^* J\mathbb{C} = 0$", i.e., it makes $\mathbb{K}_{\circlearrowleft}$ J-critical (w.r.t. to open-loop inputs in $\mathcal{U}_*^*$).

Proof of Theorem 9.9.1: (a1) This follows from "(i)⇔(ii)" and (c) of Proposition 9.10.2.

(a2) 1° Now $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$, as in (a1) (see also (d)&(e)), and $\mathbb{K}_{\circlearrowleft}$ is cost-minimizing, by Lemma 10.2.2. By (f) and (9.139) we have $S \ge 0$. (A control is minimizing iff it is J-critical, hence $\mathbb{K}_{\circlearrowleft} x_0$ is strictly minimizing iff S is one-to-one, by (e2).)

2° Obviously, the minimal cost $\langle x_0, \mathcal{P} x_0 \rangle$ is ≥ 0 iff $\mathcal{J}(x_0, \cdot) \ge 0$, and $J \ge 0$ suffices for this.

3° If $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ and $J \ge 0$, then $\widetilde{\mathbb{X}}_+^* \widetilde{S} \widetilde{\widetilde{\mathbb{X}}}_+ \ge 0$ (by (9.160)) and hence $\widetilde{S} \ge 0$ for any nonnegative admissible solution $(\widetilde{\mathcal{P}}, \widetilde{S}, \begin{bmatrix} \widetilde{\mathbb{K}} \mid \widetilde{\mathbb{F}} \end{bmatrix})$ of the eIARE. Since $\mathcal{P} \ge 0$ and $S \gg 0$ (the latter is redundant if $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{exp}}$, by Lemma 9.10.3), $\mathcal{P}$ is the greatest nonnegative admissible solution, by Theorem 9.8.13.

(Even without the assumption that $J \ge 0$, we would know that $\mathcal{P}$ were the greatest admissible solution having $S \ge 0$, by Theorem 9.8.13.)

4° Let $\mathcal{P} \ge 0$ and $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$. Assume that also $\mathcal{P}' \ge 0$ is output-stabilizing, so that $\mathcal{P}' \ge \mathbb{C}_{\circlearrowleft}'^* J \mathbb{C}_{\circlearrowleft}'$, by (9.155). If $x_0 \in H$, then $\mathbb{K}_{\circlearrowleft}' x_0 \in \mathcal{U}_{\text{out}}(x_0)$, hence then

$$\langle x_0, \mathcal{P}' x_0 \rangle \ge \langle y, Jy \rangle = \mathcal{J}(x_0, \mathbb{K}_{\circlearrowleft}' x_0) \ge \mathcal{J}(x_0, u_{\min}) = \langle x_0, \mathcal{P} x_0 \rangle, \tag{9.132}$$

where $u_{\min} := \mathbb{K}_{\circlearrowleft}x_0$, $y := \mathbb{C}x_0 + \mathbb{D}\mathbb{K}'_{\circlearrowleft}x_0 = \mathbb{C}'_{\circlearrowleft}x_0 \in \mathrm{L}^2$. Because $x_0 \in H$ was arbitrary, we have $\mathcal{P}' \geq \mathcal{P}$. Because $\mathcal{P}'$ was arbitrary, $\mathcal{P}$ is the smallest one.

(b) This follows from "(i)$\Leftrightarrow$(ii)" and (f2) [(f2)/(e2)] of Proposition 9.10.2.

(c1) If $\mathbb{B}$ is stable and $\mathbb{A}_{\circlearrowleft}$ is strongly stable, then (P) and (PB) obviously hold. Therefore, this follows from (a1) (see also Theorem 9.8.5).

(c2) The equivalence follows from (a1), because an output-stabilizing solution makes $\mathbb{A}_{\circlearrowleft}$ strongly stable, by Theorem 8.3.9(a3), hence (P) and (PB) hold (as in (c1)).

(c3) For $\mathcal{U}_{\mathrm{str}}$, this follows from (a1). For $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{sta}}$, this follows from Lemma 8.3.3. (We do not know whether a $\mathcal{U}_{\mathrm{str}}$-stabilizing solution has to be q.r.c.-stabilizing or even stabilizing.)

(d) By Lemma 8.3.3, we have $\mathcal{U}_{\mathrm{exp}} = \mathcal{U}_{\mathrm{str}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{out}}$. By Theorem 6.7.15, a solution of the eIARE is output-stabilizing iff it is exponentially q.r.c.-stabilizing, hence iff it is $\mathcal{U}_{\mathrm{exp}}$-stabilizing (see Theorem 9.8.5). Thus, the equivalence follows from (a). Condition (P) follows from exponential stability of $\mathbb{A}_{\circlearrowleft}$,

(Note from Theorem 9.8.12(a) that a solution of (c1), (c2) or (d) is a unique internally stabilizing solution.)

(e1)&(e2)&(g1)&(g2) These follow from the above proofs, Proposition 9.10.2(a1)(iii)&(b2) and Lemma 9.10.1(f2).

(f1) The J-critical cost $\langle x_0, \mathcal{P}x_0\rangle$ is independent of the J-critical control $\mathbb{K}_{\circlearrowleft}x_0$, by Lemma 8.3.8, hence $\mathcal{P}$ is unique.

(f2) 1° If $\mathrm{Ker}(S) \neq \{0\}$, then the J-critical control is not unique, by Proposition 9.10.2(g). Assume then that $\mathrm{Ker}(S) = \{0\}$. Let u be J-critical for $x_0 = 0$. By (9.175), we then have $S\mathbb{M}^{-1}u \equiv 0$, hence $u \equiv 0$. By Lemma 8.3.8, the J-critical control is unique for each $x_0 \in H$.

2° If $\mathrm{Ker}(S) = \{0\}$, then all J-critical pairs are given by (9.114), by (a1)&(e2) and Theorem 9.8.12(s1).

3° By Theorem 9.8.12(s1), the pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is not unique when $\mathrm{Ker}(S) \neq \{0\}$. See Example 9.13.6 for the example.

(h) Now $y = \mathbb{C}x_0 + \mathbb{D}\mathbb{K}_{\circlearrowleft}x_0 + \mathbb{D}\mathbb{M}u_{\circlearrowleft} = \mathbb{C}_{\circlearrowleft}x_0 + \mathbb{D}_{\circlearrowleft}u_{\circlearrowleft}$, hence (9.139) follows from (9.162) and (9.163). The proofs of parts (f1) and (f2) are valid in this case too.

(i1) (Recall that $\mathbb{K}_{\circlearrowleft}x_0 = (K_{\mathrm{crit}})_{\mathrm{L,s}}x_{\mathrm{crit}}$ a.e., where $x_{\mathrm{crit}} := \mathbb{A}_{\circlearrowleft}x_0$.) By Proposition 6.6.18(d1), we have

$$u_{\mathrm{crit}}(x_0) = (\mathbb{K}_{\circlearrowleft}x_0) = X_{\mathrm{left}}^{-1}K_{\mathrm{w}}x_{\mathrm{crit}}(x_0) \ \text{ a.e.} \tag{9.133}$$

(i2) This follows from Lemma 9.8.9 and Lemma 9.10.1(d1).

(i3) This follows from Proposition 9.10.2(b1).

(i4) This follows from (9.175).

(j) See Theorem 8.3.9.

(k) The first claim follows from Proposition 9.10.2(d)&(e2)&(f2). The rest follows from the first and (h) (we do not know whether the same holds for general $\mathcal{U}_*^*$, as explained in the notes the Section 9.8). □

By Proposition 9.8.10, eIARE is equivalent to eCARE, hence we obtain the following corollary:

Corollary 9.9.2 (Critical control⇔eCARE) *Let $\mathbb{D}$ be WR. Then there is a J-critical WR state feedback operator for Σ iff the eCARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} K & 0 \end{array}\right])$.*

A similar remark applies to (a2)–(d) of Theorem 9.9.1 too; in particular, there is a cost-minimizing WR state feedback operator for Σ iff $\mathcal{J}(0,\cdot) \geq 0$ and the eCARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} K & 0 \end{array}\right])$.

Assume that such a solution exists. Then K is a J-critical state feedback operator for Σ, $\mathcal{P}$ is unique, and the J-critical control $u_{\text{crit}}(x_0)$ is given by $u_{\text{crit}}(x_0)(t) = K_{\text{L,s}}x(t)$, where $x = \mathbb{A}x_0 + \mathbb{B}\tau u_{\text{crit}}(x_0)$ is the corresponding state. Moreover, (e1)–(k) of Theorem 9.9.1 apply. □

(This follows directly from Theorem 9.9.1, Proposition 9.8.10 and Lemma 6.2.12(a).)

Here the zero in $\left[\begin{array}{c|c} K & 0 \end{array}\right]$ refers to $X = I$ (i.e., $F = 0$), i.e., the eCARE becomes a CARE except that S need not be invertible (it is if, e.g., $\dim U < \infty$ and the J-critical control is unique). Of course, we could allow above an arbitrary WR $\mathcal{U}_*^*$-stabilizing $\left[\begin{array}{c|c} K & F \end{array}\right]$, though it would not significantly increase generality, cf. Lemma 9.9.7.

In the exponentially stable case, we obtain the following formulae for $\mathcal{P}$:

Corollary 9.9.3 (Exponentially stable $\boldsymbol{\Sigma}$) *Let Σ be exponentially stable. Then there is a J-critical state feedback pair for Σ over $\mathcal{U}_{\text{exp}}$ iff the eIARE has an exponentially stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$. If this is the case, then*

$$\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft} = \mathbb{C}^* J \mathbb{C}_{\circlearrowleft} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C} = \mathbb{C}^* J \mathbb{C} - \mathbb{K}^* S \mathbb{K}. \tag{9.134}$$

Moreover, Theorem 9.9.10(e1) and Theorem 9.9.1 apply.

Recall from Lemma 8.3.3 that here $\mathcal{U}_{\text{exp}} = \mathcal{U}_{\text{str}} = \mathcal{U}_{\text{sta}} = \mathcal{U}_{\text{out}}$.

(An analogous result (except for Theorem 9.9.10(e1)) holds for (left-column-)strongly stable systems and $\mathcal{U}_{\text{str}}$ too, whereas Theorem 8.4.5 and hence Corollary 9.9.4 cannot be generalized to $\mathcal{U}_{\text{str}}$ (nor for $\mathcal{U}_{\text{sta}}$ or $\mathcal{U}_{\text{out}}$), by Example 9.13.2.)

Proof: The equivalence follows from Theorem 9.9.1(a1). Since $\mathbb{C}$ and $\mathbb{K}$ are necessarily exponentially stable, we obtain (9.134) from (8.36) and Lemma 9.10.1(d2). □

For exponentially stabilizable systems, optimization over $\mathcal{U}_{\text{exp}}$ can be reduced to the stable case:

Corollary 9.9.4 (Exponentially stabilizable $\boldsymbol{\Sigma}$) *Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be exponentially stabilizing for Σ, with closed-loop system $\Sigma_\flat$. Then $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right])$ is an exponentially stabilizing (equivalently, output-stabilizing) solution of the eIARE for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ iff $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right])$ is a $\mathcal{U}_{\text{exp}}$-stabilizing solution of the eIARE (for Σ), i.e., J-critical for Σ over $\mathcal{U}_{\text{exp}}$, where*

$$\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right] = \left[\begin{array}{c|c} \mathbb{K}_\natural + \mathbb{X}_\natural \mathbb{K} & \mathbb{F} + \mathbb{F}_\natural - \mathbb{F}_\natural \mathbb{F} \end{array}\right] = \left[\begin{array}{c|c} \mathbb{X}' \mathbb{K}_\flat + \mathbb{K}_\natural & \mathbb{F}' \end{array}\right]. \tag{9.135}$$

The corresponding closed-loop systems relate as in (6.194), $S = \mathbb{D}_{\circlearrowleft}^* J \mathbb{D}_{\circlearrowleft}$, and

$$\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft} = \mathbb{C}_\flat^* J \mathbb{C}_{\circlearrowleft} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_\flat = \mathbb{C}_\flat^* J \mathbb{C}_\flat - \mathbb{K}_\natural^* S \mathbb{K}_\natural. \tag{9.136}$$

Moreover, Theorem 9.9.10(e1) and Theorem 9.9.1 apply.

Proof: The equivalence follows from Lemma 6.7.9 and Theorem 8.4.5(a). By Theorem 9.9.1(g), $S = \mathbb{D}_{\circlearrowleft}^* J \mathbb{D}_{\circlearrowleft}$. By Lemma 6.7.12, $\mathbb{C}_{\circlearrowleft}$ and $\mathbb{D}_{\circlearrowleft}$ are the same for both systems, hence so are $\mathcal{P} := \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ and S. We obtain (9.136) from Corollary 9.9.3. □

If Σ is smoothly exponentially stabilizable and J-coercive, then the optimal control is given by a CARE:

Proposition 9.9.5 ($\mathcal{U}_{\text{exp}}$: $\mathbb{D}_\flat \in \widetilde{\mathcal{A}} \Rightarrow$ CARE) *Assume that Σ has a SR exponentially stabilizing state feedback operator K' s.t. $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ satisfies Hypothesis 8.4.7. Assume that $\mathbb{D}$ or $\mathbb{D}_\flat$ is J-coercive over $\mathcal{U}_{\text{exp}}$.*

Then there is a unique exponentially stabilizing solution $(\mathcal{P}, S, K)$ of the CARE, K is J-critical over $\mathcal{U}_{\text{exp}}$, and $\mathbb{D}, \mathbb{X}, \mathbb{N}, \mathbb{M} \in$ SR.

Note from Proposition 9.12.4 that any strong or uniform regularity property of K' and $\widetilde{\mathcal{A}}$ is shared by K. An analogous result for $\mathcal{U}_{\text{out}}$ is given in Theorem 9.9.10(d3).

Proof: By Theorem 8.4.5(d), also $\mathbb{D}_\flat$ is J-coercive over $\mathcal{U}_{\text{exp}}^{\Sigma_\flat}$. Therefore, the CARE for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ has an ULR SOS-stabilizing (hence exponentially stabilizing, by Theorem 6.7.15(b1)) solution $(\mathcal{P}, S, K_\natural)$ with $\mathbb{F}_\natural \in \widetilde{\mathcal{A}}$, by Corollary 9.1.12.

By Proposition 9.12.4, $(\mathcal{P}, S, K' + K_\natural)$ is an exponentially stabilizing solution of the CARE for Σ, hence J-critical over $\mathcal{U}_{\text{exp}}$, by Theorem 9.8.5. By Theorem 9.8.12(e)&(s3), $\mathcal{P}$, S and K are unique. By Proposition 6.6.18(f), we have $\mathbb{D}, \mathbb{X}, \mathbb{N}, \mathbb{M} \in$ SR. □

In discrete-time, a unique minimizing control is always of state feedback form, and it corresponds to the unique $\mathcal{U}_*^*$-stabilizing solution of the DARE, by Theorem 14.1.6. If B is bounded, then the same holds in continuous time too:

Theorem 9.9.6 (Bounded B) *Let B be bounded. Then there is a unique J-critical control for each $x_0 \in H$ iff the eCARE*

$$\begin{cases} K^* S K = A^* \mathcal{P} + \mathcal{P} A + C^* J C, \\ \quad S = D^* J D, \\ \quad SK = -(D^* J C + B^* \mathcal{P}). \end{cases} \tag{9.137}$$

has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ with S one-to-one.

Assume that this is the case. Then the following hold:

(a) The $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ is unique and ULR.

(b1) The J-critical control is determined by $u_{\rm crit}(x_0) = \mathbb{K}_\circlearrowleft x_0$*, i.e., by* $u_{\rm crit}(x_0)(t) = K_{\rm L,s}x(t)$ *for almost all* $t \ge 0$*, where* $x = \mathbb{A}x_0 + \mathbb{B}u_{\rm crit}(x_0)$ *and* $\Sigma_\circlearrowleft$ *is the closed-loop system corresponding to* $\begin{bmatrix} K & | & 0 \end{bmatrix}$*; in particular, the left column of* $\Sigma_\circlearrowleft$ *is equal to* $\Sigma_{\rm crit}$*.*

(b2) Conversely, $Kx_0 = u_{\rm crit}(x_0)(0)$ *for* $x_0 \in \mathrm{Dom}(A) = \mathrm{Dom}(A_{\rm crit}) = H_B$*,* $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$*, and* $S = D^*JD$ *is the corresponding signature operator (see (9.139)).*

(c) Theorem 9.9.1(d1)&(f) and Theorem 8.3.9 apply. If $S \in \mathcal{GB}(U)$*, then also the results of Section 9.2 apply.*

(d) We have $X = I$ *and* $\mathbb{D}, \mathbb{X}, \mathbb{M}, \mathbb{N} \in \mathrm{ULR}$*.*

In fact, $\widehat{\mathbb{D}} - D, \widehat{\mathbb{F}} \in \mathrm{H}^2_{\rm strong}(\mathbf{C}^+_\omega; \mathcal{B})$ *for any* $\omega > \omega_A$*, and* $\widehat{\mathbb{N}} - D, \widehat{\mathbb{M}} - I \in \mathrm{H}^2_{\rm strong}(\mathbf{C}^+; \mathcal{B})$ *(*$\in \mathrm{H}^2_{\rm strong}(\mathbf{C}^+_{-\varepsilon}; \mathcal{B})$ *for some* $\varepsilon > 0$*, hence strongly half-plane-regular, if* $\mathcal{P}$ *is exponentially stabilizing).*

(e1) Any J-critical control in WPLS form ($\Sigma_{\rm crit}$*) is actually of (ULR) state feedback form (even if S is not one-to-one).*

(e2) The J-critical state feedback operators correspond to $\mathcal{U}^*_*$*-stabilizing solutions of the eCARE (9.137) and conversely, as in (b1)–(b2), and (a)–(d) hold for such solutions.*

(Parts (e1) and (e2) holds even if the CARE does not have a $\mathcal{U}^*_*$*-stabilizing solution with S one-to-one.)*

Recall that $\mathcal{P}$ is exponentially stabilizing if $\mathcal{U}^*_* = \mathcal{U}_{\rm exp}$. Condition $B \in \mathcal{B}(U,H)$ can be relaxed to Hypothesis 9.2.2 when $S \in \mathcal{GB}(U)$, by Theorem 9.2.9.

For $x_0 \in H_1$, we have $x(t) \in \mathrm{Dom}(\mathbb{A}_\circlearrowleft) = H_1$ for all $t \ge 0$, hence $u(t) \equiv Kx(t)$ for such initial states.

Proof: 1° *The equivalence of the eCARE and a unique J-critical control:* The equivalence follows from Theorem 9.9.1(a1)&(g2), because any unique (for each x_0) J-critical control $u_{\rm crit}$ corresponds to an ULR J-critical state feedback operator $K = K_{\rm crit}$, by Lemma 8.3.18 and Theorem 8.3.9.

2° *The eCARE becomes (9.137):* By 1°, one choice of $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ is given by $K = K_{\rm crit}$, $F = 0$ (i.e., $X = I$), which corresponds to (9.137), since $X^*SX = D^*JD$, as shown below.

Recall that "$B \in \mathcal{B}(U,H)$" means that $B \in \mathcal{B}(U,H_{-1})$ is such that $Bu_0 = B_0u_0$ for all $u_0 \in U$ for some $B_0 \in \mathcal{B}(U,H)$. Clearly $B^* = B_0|_{\mathrm{Dom}(A^*)}$, hence $B^*_{\rm w} = B^*_0 \in \mathcal{B}(H,U)$. Thus we may write $B = B_{\rm w} = B_0 \in \mathcal{B}(U,H)$ and $B^* = B^*_{\rm w} = B^*_0 \in \mathcal{B}(H,U)$ without misconceptions. The boundedness of B and Lemma A.4.4(d3) imply that $S = D^*JD$.

(a) The uniqueness follows from Theorem 9.9.1(f1); K is ULR by Lemma 6.3.16(b).

(b1) This follows from the formula $u_{\rm crit}(x_0) = \mathbb{K}_{\rm crit}x_0 = \mathbb{K}_\circlearrowleft x_0$ and 2° (see also Lemma 8.3.18).

(b2) If $x_0 \in \mathrm{Dom}(A_{\rm crit})$, then $u_{\rm crit}(x_0) = K_{\rm L,s}\mathbb{A}_{\rm crit}x_0 = K\mathbb{A}_{\rm crit}x_0 \in \mathcal{C}(\mathbf{R}_+;U)$, because $\mathbb{A}_{\rm crit}$ is a C_0-semigroup on $\mathrm{Dom}(A_{\rm crit})$ too. (see Lemma 6.1.16), and

$K = K_{\rm crit} \in \mathcal{B}(\mathrm{Dom}(A_{\rm crit}), U)$. Therefore, $Kx_0 = K\mathbb{A}_{\rm crit}(0)x_0 = u_{\rm crit}(x_0)(0)$. See Theorem 9.9.1(e2) and (h) for the other claims.

(c) This follows from the above and Theorem 9.9.1 (note that (1.) and (5.) of Hypothesis 9.2.2 are satisfied).

(d) This follows from Lemma 6.3.16(b)&(d) and Theorem 6.9.1(a), because $X = I$ in 1° above (note that all possible $\mathbb{X}$'s are given by $E\mathbb{X}$, $E \in \mathcal{GB}(U)$).

(e1)&(e2) The assumption that S is one-to-one was used above only for the uniqueness of $K_{\rm crit}$ and for the existence of a J-critical control in WPLS form, hence (e1)&(e2) hold.

Remark: If we use the actual eCARE (see Definition 9.8.1, we obtain all state feedback pairs; these are exactly the pairs generated by $\left[\begin{array}{c|c} XK & I-X \end{array}\right]$ $(X \in \mathcal{GB}(U))$, where K is a solution of the eCARE (9.137). In particular, the solutions of (9.137) are solutions of the eCARE. □

See Remark 10.2.18 for a different formulation for the cost function when C is bounded. However, the case with a bounded C is not at all as easy as that with a bounded B; cf. Example 9.13.8.

Often $S, X \in \mathcal{GB}(U)$, i.e., the eCARE is equivalent to the CARE (see Remark 9.8.2):

Lemma 9.9.7 ($S, X \in \mathcal{GB}(U)$) *We often require the* signature operator *S to be one-to-one, or even invertible. This is often the case with standard assumptions on $\mathbb{D}$ and J, see, e.g., Section 10.1. We make here some additional remarks on this, assuming that $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is a $\mathcal{U}_*^*$-stabilizing solution of the eIARE (cf. Theorem 9.9.1(a1)).*

(a1) S is one-to-one [invertible] iff $\mathbb{X}^{t\,*} S \mathbb{X}^t$ is one-to-one [invertible] (for any $t > 0$).

(a2) S is one-to-one iff the J-critical control for Σ is unique.

(b1) Assume that $S = S^*$. Then S is invertible iff $S^*S \gg 0$.

(b2) If $\dim U < \infty$, then $S \in \mathcal{GB}(U)$ iff S is one-to-one.

(c1) If $\mathcal{J}(0,\cdot) > 0$, then $S > 0$.

(c2) If $\mathbb{D}$ is [positively] J-coercive over $\mathcal{U}_*^*$, then $S \in \mathcal{GB}(U)$ [$S \gg 0$].

(c3) If $\mathcal{P} \ge 0$, $t > 0$, and $\mathbb{D}^{t\,*} J \mathbb{D}^t > 0$ [$\gg 0$] on $\mathrm{L}^2([0,t);U)$, then $S > 0$ [$\gg 0$].

(c4) If $\langle \mathbb{D}u, J\mathbb{D}u \rangle \ge \varepsilon \|u\|^2_{\mathrm{L}^2_\omega}$ for all $u \in \mathcal{U}_*^*(0)$ and some $\varepsilon > 0$, $\omega \in \mathbf{R}$, then $S \gg 0$.

(c5) Assume that there are $\varepsilon > 0$ and $\omega \in \mathbf{R}$ s.t. for all nonzero $u \in \mathcal{U}_*^*(0)$ there is a nonzero $v \in \mathcal{U}_*^*(0)$ s.t. $\langle \mathbb{D}v, J\mathbb{D}u \rangle \ge \varepsilon \|u\|_{\mathrm{L}^2_\omega} \|v\|_{\mathrm{L}^2_\omega}$. Then $S \in \mathcal{GB}(U)$.

If, in addition, $\mathbb{X} := I - \mathbb{F}$ is WR, then the following hold:

(d) If $\mathbb{X} \in$ UR (e.g., when $\dim U, \dim Y < \infty$), or $\mathbb{X}, \mathbb{X}^{\rm d} \in$ SR, then $X \in \mathcal{GB}(U)$.

However, we do not know, whether X can be noninvertible for WR $\mathbb{X}$.

(e) If $\dim U < \infty$ and X^*SX is one-to-one, then $X, S \in \mathcal{GB}(U)$.

Proof: (Note that most of this holds with weaker assumptions too.)

(a1) Now $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty$ (the admissibility of $\mathcal{P}$) implies that $\mathbb{X}^t \in \mathcal{G}\mathcal{B}(\mathrm{L}^2([0,t);U))$, by Lemma 2.2.8), hence (a1) holds.

(a2) This follows from Theorem 9.9.1(a1)&(e2).

(b1)&(b2) See Lemma A.3.1(c4)&(c3).

(c1) This follows from Theorem 9.9.1(a2).

(c2) See Lemma 9.10.3.

(c3) This follows from (9.160).

(c4)&(c5) These follow from the proof of Lemma 9.10.3 (set $M := \|\pi_{[0,1)}\mathbb{M}^{-1}\pi_{[0,1)}\|_{\mathcal{B}(\mathrm{L}^2_\omega)}$ etc.).

(d) See Lemma 6.3.2(a1)&(a2) and Proposition 6.3.1(a2)&(b1).

(e) Now $X^*SX \in \mathcal{G}\mathcal{B}(U)$ implies that X^*, S, X must be invertible matrices, hence $X, S \in \mathcal{G}\mathcal{B}$. □

Thus, by altering K and F, a smooth solution of the IARE can be converted into a solution of the CARE:

Corollary 9.9.8 *Let $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ be a $\mathcal{U}^*_*$-stabilizing solution of the IARE. If $\mathbb{D} \in \mathrm{WR}$ and $\mathbb{F} \in \mathrm{UR}$, then there are unique $\widetilde{S} \in \mathcal{G}\mathcal{B}(U)$ and $\widetilde{K} \in \mathcal{B}(H_1, U)$ s.t. $(\mathcal{P}, \widetilde{S}, \widetilde{K})$ is a $\mathcal{U}^*_*$-stabilizing solution of the CARE; moreover, then $\widetilde{S} = X^*SX$ and $\widetilde{\mathbb{X}} = X^{-1}\mathbb{X}$.*

Proof: (Here $\widetilde{\mathbb{X}} = I - \widetilde{\mathbb{F}}$, where $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ is the pair generated by $\widetilde{K}$.)

Uniqueness follows from Theorem 9.8.12(b)&(s1); the existence follows from Proposition 9.8.10, Lemma 9.9.7(d) and Remark 9.8.2. □

As noted above (Corollary 9.9.4 and Theorem 8.4.5), optimization over $\mathcal{U}_{\mathrm{exp}}$ can be reduced to optimization over a preliminarily exponentially stabilized system. If Σ is q.r.c.-SOS-stabilizable, then the situation is analogous for optimization over $\mathcal{U}_{\mathrm{out}}$. This case and its special cases will be studied in Theorem 9.9.10 below (see (c1) for $\mathcal{U}_{\mathrm{str}}$ and $\mathcal{U}_{\mathrm{sta}}$), but we first motivate it briefly:

Remark 9.9.9 (q.r.c.-stabilization and $\mathcal{U}_{\mathrm{out}}$) *The assumption that Σ is q.r.c.-SOS-stabilizable (cf. Corollary 6.7.16) and the use of q.r.c.-SOS-stabilizing solutions of the Riccati equation (Theorem 9.9.10) have several advantages:*

(1.) The theory for $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$ become roughly as easy as that for $\mathcal{U}_{\mathrm{exp}}$.

(2.) The control and output of $\Sigma_\circlearrowleft$ depend continuously on closed-loop input (the signal $u_\circlearrowleft$ in Figure 9.1, p. 408); i.e., $\mathbb{D}_\circlearrowleft$ and $\mathbb{F}_\circlearrowleft$ become stable.

(3.) If Σ is assumed to be [strongly] q.r.c.-stabilizable, then the closed-loop system becomes [strongly] stable (see (c1) below).

(4.) We can establish the standard equivalence on optimization, coprime factorizations and Riccati equations.

In general, a J-critical state feedback pair over $\mathcal{U}_{\mathrm{out}}$ (resp. $\mathcal{U}_{\mathrm{sta}}$ or $\mathcal{U}_{\mathrm{str}}$) stabilizes the output (resp. and state) w.r.t. the initial state, but arbitrarily small disturbances in the (closed-loop) input may cause arbitrarily big perturbations in the state and output (as in Example 11.3.7(b)). □

Under certain assumptions, such solutions become the “correct” ones; see, e.g., (d3) below, Theorem 11.1.5 and Corollary 10.2.12(Crit3+).

We give below a variant of the equivalence of (I)–(III) on p. 9 (with (Crit2) and (Crit3) corresponding to (III) and (Crit4) to (II)); the equivalence will be enhanced in parts (d1)–(d4) below, in Section 9.1, and in certain later results.

Theorem 9.9.10 (eIARE$\Leftrightarrow$ $(J,*)$-inner r.c.f.) *Let $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$. Then (Crit1)$\Leftrightarrow$(Crit2)$\Leftrightarrow$(Crit3)$\Leftrightarrow$(Crit4), where*

(Crit1) **(J-critical $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$)** *There is a J-critical q.r.c.-SOS-stabilizing state feedback pair for Σ and $\mathbb{D}$ is J-coercive.*

(Crit2) **(IARE)** *The IARE has a q.r.c.-SOS-P-stabilizing solution.*

(Crit3) **(IARE/DARE)** *There are $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $S \in \mathcal{GB}$ and a q.r.c.-SOS-stabilizing pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ satisfying the “DARE” (9.111) for some $t > 0$, s.t. $\langle \mathbb{A}_{\circlearrowleft}^{nt} x_0, \mathcal{P}\mathbb{A}_{\circlearrowleft}^{nt} x_0\rangle \to 0$ as $n \to +\infty$, $n \in \mathbf{N}$.*

(Crit4) **(R.c.f.)** *The map $\mathbb{D}$ has a $(J,*)$-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, and Σ is q.r.c.-SOS-stabilizable.*

Moreover, we have the following:

(a1) Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ solve (Crit1). Then $S := \mathbb{N}^* J\mathbb{N} = S^* \in \mathcal{GB}(U)$, $\mathcal{P} := \mathbb{C}_{\circlearrowleft}^* J\mathbb{C}_{\circlearrowleft} = \mathcal{P}^* \in \mathcal{B}(H)$, and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ solve (Crit2) and $\mathbb{N}$, $\mathbb{M}$ solve (Crit4), where $\mathbb{M} := (I-\mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$, $\mathbb{C}_{\circlearrowleft} := \mathbb{C} + \mathbb{D}\mathbb{M}\mathbb{K}$.

(a2) Let $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ solve (Crit2) or (Crit3). Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ solves (Crit1) and S and $\mathcal{P}$ are as in (a1).

(a3) Let $\mathbb{N}, \mathbb{M}$ solve (Crit4). Then a solution of (Crit1) and (Crit2) can be constructed as in (g2) (also (g1) applies if $\Sigma \in \mathrm{SOS}$).

(a4) This theorem also holds with “r.c.” or “p.r.c.” in place of “q.r.c.”.

(b) A solution $\mathcal{P}$ of (Crit2) is unique. Given one solution $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ of (Crit1) or (Crit2) or $(\mathbb{N}, \mathbb{M})$ and S of (Crit4), all solutions are given by

$$\left[\begin{array}{c|c} E\mathbb{K} & E\mathbb{F} + I - E \end{array}\right], \quad (\mathbb{N}E^{-1}, \mathbb{M}E^{-1}), \quad E^{-*}SE^{-1}, \qquad E \in \mathcal{GB}(U). \tag{9.138}$$

The corresponding closed-loop systems are given by $\Sigma_{\circlearrowleft E} := \left[\begin{array}{c|c} \mathbb{A}_{\circlearrowleft} & \mathbb{B}_{\circlearrowleft} E^{-1} \\ \hline \mathbb{C}_{\circlearrowleft} & \mathbb{D}_{\circlearrowleft} E^{-1} \\ \mathbb{K}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft} E^{-1} \end{array}\right]$.

(c1) Assume that Σ is [strongly [exponentially]] q.r.c.-stabilizable.

Then any q.r.c.-SOS-stabilizing state feedback pair is [strongly [exponentially]] q.r.c.-stabilizing. Consequently, then this theorem holds with $\mathcal{U}_{\mathrm{sta}}$ [or $\mathcal{U}_{\mathrm{str}}$ [or $\mathcal{U}_{\mathrm{exp}}$]] in place of $\mathcal{U}_{\mathrm{out}}$.

[Moreover, if $J \geq 0$ and (Crit2) has a solution $\mathcal{P}$, then $\mathcal{P}$ is the greatest nonnegative admissible and the unique nonnegative output-stabilizing solution of the eIARE.]

(c2) Assume that Σ is exponentially q.r.c.-stabilizable. Then any I/O-stabilizing or input-stabilizing solution or the IARE (i.e., with stable $\mathbb{N}$ and $\mathbb{M}$ or $\mathbb{B}\mathbb{M}$) is exponentially q.r.c.-stabilizing.

Moreover, then any $(J,*)$-inner q.r.c.f. of $\mathbb{D}$ is exponentially q.r.c.

(c3) Assume that Σ is estimatable. Then any output-stabilizing solution of the IARE is exponentially q.r.c.-stabilizing (hence the greatest nonnegative admissible solution if $J \geq 0$).

(d1) Let $\mathbb{D}$ be WR. If we add to (Crit1)–(Crit4) the requirement that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ must correspond to some WR $\left[\begin{array}{c|c} K & 0 \end{array}\right]$ (equivalently, that $\mathbb{X} = \mathbb{M}^{-1} = I - \mathbb{F}$ is WR *and $X = I$), then a fifth equivalent condition is:*

(Crit5) **(CARE)** *The CARE has a q.r.c.-SOS-P-stabilizing solution.*

Moreover, with these extra requirements any solutions of (Crit1)–(Crit5) are unique and equal (cf. (a1)–(a3)); the same applies to (d2).

(d2) Assume Hypothesis 9.2.1 and that $D^*JD \in \mathcal{GB}(U)$. Then (Crit1)–(Crit6) are equivalent (they are also equivalent to (Crit7) if Σ is optimizable and estimatable), where

(Crit6) **(B_{w}^*-CARE)** *The B_{w}^*-CARE has a q.r.c.-SOS-P-stabilizing solution.*

(d3) **(MTIC)** *Assume that 1. Σ is q.r.c.-SOS-stabilizable in $\widetilde{\mathcal{A}}$, or that 2. Σ has a q.r.c.-SOS-stabilizing SR state feedback operator s.t. $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$, or that 3. $D^*JD \in \mathcal{GB}(U)$ and Σ has an exponentially q.r.c.-stabilizing SR state feedback operator s.t. $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ satisfies Hypothesis 9.2.1, where $\widetilde{\mathcal{A}}$ satisfies Hypothesis 8.4.7.*

Then (Crit1)–(Crit5) and (Crit7) are equivalent and imply that $\mathbb{N}, \mathbb{M} \in \widetilde{\mathcal{A}}$, where

(Crit7) The map $\mathbb{D}$ is J-coercive.

(d4) In the general case we still have (Crit6)⇒(Crit5)⇒(Crit1–4)⇒(Crit7).

Assume that (Crit1) has a solution, and use the notation of Definition 9.1.3. Then the following hold:

(e1) The closed loop cost function $\mathcal{J}_\circlearrowleft(x_0, u_\circlearrowleft)$ for $y = \mathbb{C}_\circlearrowleft x_0 + \mathbb{D}_\circlearrowleft u_\circlearrowleft$, $u_\circlearrowleft \in$ $\mathrm{L}^2(\mathbf{R}_+; U)$ and $x_0 \in H$ is given by

$$\mathcal{J}_\circlearrowleft(x_0, u_\circlearrowleft) := \langle y, Jy \rangle_{\mathrm{L}^2(\mathbf{R}_+;Y)} = \langle x_0, \mathcal{P} x_0 \rangle_H + \langle u_\circlearrowleft, S u_\circlearrowleft \rangle_{\mathrm{L}^2(\mathbf{R}_+;U)}. \tag{9.139}$$

(e2) **(Minimization)** *The pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is minimizing $\Leftrightarrow S \gg 0 \Leftrightarrow \langle \mathbb{D}u, J\mathbb{D}u \rangle \geq$ 0 for all $u \in \mathcal{U}_{\mathrm{out}}(0) \Leftrightarrow \mathbb{D}$ is positively J-coercive.*

(f) Theorem 9.9.1(g)–(j) and Theorem 8.3.9 apply, and (PB) is satisfied.

(g1) **(Stable case)** *Assume that $\mathbb{C}$ and $\mathbb{D}$ are stable (i.e., that $\Sigma \in$* SOS*). Assume (Crit4) (see (Crit1SOS)–(Crit4SOS) of Corollary 9.9.11 for further equivalent conditions). Then the corresponding solution of (Crit2) is*

$$\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] := \left[\begin{array}{c|c} -S^{-1}\pi_+ \mathbb{N}^* J\mathbb{C} & I - \mathbb{M}^{-1} \end{array}\right]. \tag{9.140}$$

Moreover, this $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ *is stable and q.r.c.-SOS-stabilizing. The corresponding* $\mathcal{P}$ *and* $\Sigma_{\circlearrowleft}$ *satisfy*

$$\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J\mathbb{C}_{\circlearrowleft} = \mathbb{C}^* J\mathbb{C}_{\circlearrowleft} = \mathbb{C}_{\circlearrowleft}^* J\mathbb{C} = \mathbb{C}^* \left(J - J\mathbb{N}S^{-1}\pi_+\mathbb{N}^* J\right)\mathbb{C} \tag{9.141}$$

$$= \mathbb{C}^* J\mathbb{C} - \mathbb{K}^* S\mathbb{K} = \mathbb{C}^* (J - J\mathbb{D}\pi_+(\pi_+\mathbb{D}^* J\mathbb{D}\pi_+)^{-1}\pi_+\mathbb{D}^* J)\mathbb{C}, \tag{9.142}$$

$$\mathbb{A}_{\circlearrowleft} = \mathbb{A} + \mathbb{B}\mathbb{K}_{\circlearrowleft}, \tag{9.143}$$

$$\mathbb{C}_{\circlearrowleft} = \left(I - \mathbb{N}S^{-1}\pi_+\mathbb{N}^* J\right)\mathbb{C}, \tag{9.144}$$

$$\mathbb{K}_{\circlearrowleft} = -\mathbb{M}S^{-1}\pi_+\mathbb{N}^* J\mathbb{C}. \tag{9.145}$$

(g2) Given $\mathbb{N}$ *and* $\mathbb{M}$ *as in (Crit4), the pair* $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$*, the Riccati operator* $\mathcal{P}$*, and the closed-loop system* $\Sigma_{\circlearrowleft}$ *can be constructed as follows:*

Choose any q.r.c.-SOS-stabilizing pair $\begin{bmatrix} \mathbb{K}' \mid \mathbb{F}' \end{bmatrix}$ *for* Σ*, and let* $\Sigma_\flat^1 := \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ *be the two top (block) rows of the corresponding closed-loop system* $\Sigma_\flat$*. Set* $\mathbb{M}' := (I - \mathbb{F}')^{-1}$*,* $\mathbb{X}_\natural := \mathbb{M}^{-1}\mathbb{M}'$*, so that* $\mathbb{X}_\natural \in \mathcal{G}\mathrm{TIC}(U)$*.*

Set $\begin{bmatrix} \mathbb{K}_\natural \mid \mathbb{F}_\natural \end{bmatrix} := \begin{bmatrix} -S^{-1}\pi_+\mathbb{N}^* J\mathbb{C}_\flat \mid I - \mathbb{X}_\natural \end{bmatrix}$*. Then a solution* $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ *of (Crit2) and corresponding* $\Sigma_{\circlearrowleft}$ *and* $\mathcal{P}$ *are obtained as follows:*

$$\mathbb{K} = \mathbb{M}^{-1}\mathbb{K}_\flat' + \mathbb{K}_\natural = \mathbb{X}_\natural\mathbb{K}' + \mathbb{K}_\natural, \quad \mathbb{F} = I - \mathbb{M}^{-1}, \tag{9.146}$$

$$\left[\begin{array}{c|c} \mathbb{A}_{\circlearrowleft} & \mathbb{B}_{\circlearrowleft} \\ \hline \mathbb{C}_{\circlearrowleft} & \mathbb{D}_{\circlearrowleft} \\ \mathbb{K}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft} \end{array}\right] = \left[\begin{array}{c|c} \mathbb{A}_\flat + \mathbb{B}\mathbb{M}\tau\mathbb{K}_\natural & \mathbb{B}\mathbb{M} \\ \hline \mathbb{C}_\flat + \mathbb{N}\mathbb{K}_\natural & \mathbb{N} \\ \mathbb{K}_\flat + \mathbb{M}\mathbb{K}_\natural & \mathbb{M} - I \end{array}\right] \tag{9.147}$$

$$\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J\mathbb{C}_{\circlearrowleft} = \mathbb{C}_\flat^* J\mathbb{C}_{\circlearrowleft} = \mathbb{C}_{\circlearrowleft}^* J\mathbb{C}_\flat \tag{9.148}$$

$$= \mathbb{C}_\flat^* J\mathbb{C}_\flat - \mathbb{K}_\natural^* S\mathbb{K}_\natural^* = \mathbb{C}_\flat^* \left(J - J\mathbb{N}S^{-1}\pi_+\mathbb{N}^* J\right)\mathbb{C}_\flat. \tag{9.149}$$

(Recall that $\mathbb{A}_\flat = \mathbb{A} + \mathbb{B}\tau\mathbb{K}_\flat'$*,* $\mathbb{C}_\flat = \mathbb{C} + \mathbb{D}\mathbb{K}_\flat'$*, and* $\mathbb{K}_\flat' = (I - \mathbb{F}')^{-1}\mathbb{K}'$*.)*

Note that $\mathbb{K}_\natural$*,* $\mathbb{C}_\flat$*,* $\mathbb{D}_\flat$*,* $\mathbb{K}_\flat'$*,* $\mathbb{C}_{\circlearrowleft}$ *and* $\mathbb{D}_{\circlearrowleft} = \mathbb{N}$ *are stable, and that* $\mathcal{P}$ *and* $\begin{bmatrix} \mathbb{K}_\natural \mid \mathbb{F}_\natural \end{bmatrix}$ *correspond to this theorem (including (g1)) applied to* $\Sigma_\flat^1$ *and* J*.*

(g3) The constructions (g1), (g2) and (a1) can be used in (c1) and (c2) too.

See Section 9.1 for related results and further equivalent conditions. Further equivalent conditions for optimizable and estimatable systems are given in Corollary 9.2.15.

Quasi-coprimeness is essential in the above theorem; an arbitrary SOS-stabilizing (even exponentially stabilizing solution) need not correspond to the minimizing control over $\mathcal{U}_{\mathrm{out}}$ (though necessarily to that over $\mathcal{U}_{\mathrm{exp}}$), as illustrated in Example 9.13.2.

Indeed, the q.r.c.-property guarantees that $\mathcal{U}_{\mathrm{out}}(x_0)$ corresponds one-to-one and onto to $\mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}(x_0)$; i.e., we obtain the situation of Theorem 8.4.5(e), indeed, if $\begin{bmatrix} \widetilde{\mathbb{K}} \mid \widetilde{\mathbb{F}} \end{bmatrix}$ is q.r.c.-SOS-stabilizing, then

$$\mathcal{U}_{\mathrm{out}}(x_0) = \{\mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}}u_\flat \,|\, u_\flat \in \mathrm{L}^2(\mathbf{R}_+;U) = \mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}(x_0)\} \quad (x_0 \in H). \tag{9.150}$$

Indeed, obviously such u and $y := \mathbb{C}x_0 + \mathbb{D}u = \mathbb{C}_\flat x_0 + \mathbb{D}_\flat u_\flat$ are stable. Conversely, if $u, y \in \mathrm{L}^2$, then $\mathbb{D}_\flat u_\flat = y - \mathbb{C}_\flat x_0 \in \mathrm{L}^2$ and $\widetilde{\mathbb{M}} u_\flat = -\widetilde{\mathbb{K}}_\flat x_0 + u \in \mathrm{L}^2$, where $u_\flat := -\widetilde{\mathbb{K}} x_0 + \widetilde{\mathbb{X}} u$, so that $u_\flat \in \mathrm{L}^2$, since $\mathbb{D}_\flat := \mathbb{D}\widetilde{\mathbb{M}}$ and $\widetilde{\mathbb{M}}$ are q.r.c.

For general SOS-stabilizing state feedback pairs, some elements of $\mathcal{U}_{\mathrm{out}}(x_0)$ may correspond to unstable inputs $u_\flat$ for $\Sigma_\flat$, so that a P-SOS-stabilizing solution optimizes over a too small class of inputs; cf. the (non-q.r.c.-)exponentially (P-)stabilizing state feedback operator $K = -2$ of Example 9.13.2. If $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ were merely output-stabilizing, then also some stable inputs $u_\flat$ for $\Sigma_\flat$ might correspond to unstable inputs u for Σ.

Also the construction formulae of (g1)–(g2) (in particular, (9.140)) base on quasi-coprimeness, hence we cannot give such formulae for non-q.r.c.-SOS-stabilizable systems. Indeed, if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ corresponds to a WR state feedback operator K, then $\widehat{\mathbb{M}}^{-1}$ may only have the singularities of $(s-A)^{-1}$, as noted below Definition 6.6.10; thus we must somehow guarantee that $\widehat{\mathbb{M}}^{-1}$ does not have too many poles (cf. Lemma 6.5.4).

Proof of Theorem 9.9.10: (Note that the stability of $\mathbb{K}_\flat$ and $\mathbb{K}_\circlearrowleft$ might be omitted from the requirements and conclusions, whereas the invertibility of S is essential for, e.g., (Crit4)⇒(Crit2).)

1° "(Crit1)⇔(Crit1$\frac{1}{2}$)": We have one more equivalent condition:

(Crit1$\frac{1}{2}$) $\mathbb{D}$ is J-coercive, and there is a q.r.c.-SOS-stabilizing state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ for Σ s.t. $\pi_+\mathbb{D}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft = 0$, where $\Sigma_\circlearrowleft$ is the corresponding closed-loop system.

Now $\mathcal{U}_{\mathrm{out}}(0) = \mathbb{M}\pi_+\mathrm{L}^2$, by Proposition 9.10.2(e2), hence $\langle \mathbb{D}_\circlearrowleft \pi_+ u, \mathbb{C}_\circlearrowleft x_0\rangle = 0$ for all $x_0 \in H$, $u \in \pi_+\mathrm{L}^2$ iff $\langle \mathbb{D}\pi_+ u, \mathbb{C}_\circlearrowleft x_0\rangle = 0$ for all $x_0 \in H$, $u \in \mathcal{U}_{\mathrm{out}}(0)$ (because $\mathbb{D}_\circlearrowleft \pi_+ u = \mathbb{D}\mathbb{M}\pi_+ u = \mathbb{D}\pi_+\mathbb{M}\pi_+ u$), i.e., iff $\mathbb{K}_\circlearrowleft$ is J-critical.

2° "(Crit1)⇔(Crit2)⇔(Crit3)"&(a1)&(a2) This follows Theorem 9.9.1(b)&(d), and Lemma 8.4.11(b2) (note that (Crit2)–(Crit4) require $S \in \mathcal{GB}(U)$).

3° "(Crit1)⇒(Crit4)"&(a1): These follow from equations $\mathbb{N} = \mathbb{D}\mathbb{M}$ and $\mathbb{N}^* J\mathbb{N} = S$ (see Theorem 9.9.1(b)&(g)) and Lemma 9.10.3.

4° "(Crit4)⇒(Crit1$\frac{1}{2}$)" for stable $\mathbb{C}$ and $\mathbb{D}$: By Lemma 8.4.14(a), $\mathbb{D}$ is J-coercive. Now $\mathbb{M} \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.5.6(b). Obviously, the pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ from (9.140) are stable. By using (9.140) and equation $S^{-1}\mathbb{N}^* J\mathbb{D} = \mathbb{M}^{-1}$, it is straightforward to verify that $\mathbb{K}\mathbb{A} = \pi_+\tau\mathbb{K}$ and $\pi_+\mathbb{F}\pi_- = \mathbb{K}\mathbb{B}$ (see the proof of Theorem 27 of [S97b] for details), hence $\Sigma_{\mathrm{ext}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F} \end{array}\right] \in \mathrm{SOS}(U, H, Y \times U)$. By Corollary 6.6.9, we have $\Sigma_\circlearrowleft \in \mathrm{SOS}$. Now

$$\pi_+\mathbb{N}^* J\mathbb{C}_\circlearrowleft = \pi_+\mathbb{N}^* J\mathbb{N}\mathbb{K} + \pi_+\mathbb{N}^* J\mathbb{C} = \pi_+\mathbb{N}^* J\left(-\mathbb{N}S^{-1}\pi_+\mathbb{N}^* J\mathbb{C} + J\mathbb{C}\right) = 0 \tag{9.151}$$

as required.

(N.B. We do not know whether $S^{-1}\mathbb{N}^* J\mathbb{D} = \mathbb{M}^{-1}$ holds for unstable $\mathbb{M}^{-1}$, therefore we have required the (J,S)-inner right factorization to be q.r.c.)

5° "(Crit4)⇒(Crit1$\frac{1}{2}$)": By Lemma 8.4.14(a), $\mathbb{D}$ is J-coercive. By Lemma 6.4.5(c), $\mathbb{X}_\natural \in \mathcal{G}\mathrm{TIC}(U)$. Therefore, $\mathbb{D}_\flat = \mathbb{N}\mathbb{X}_\natural$ is a (J,S)-inner r.c.f.

By 4°, the state feedback pair $\left[\ \mathbb{K}_\natural \ \middle|\ \mathbb{F}_\natural\ \right]$ is stable and q.r.c.-SOS-stabilizing for $\Sigma_\flat^1$, and the corresponding closed-loop system satisfies $\pi_+\mathbb{D}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft = 0$. Moreover, $\mathbb{D}_\circlearrowleft = \mathbb{D}_\flat \mathbb{X}_\natural^{-1} = \mathbb{N}$.

Apply Lemma 6.7.12 (with $\mathbb{K}'$ and $\mathbb{K}$ interchanged, etc.) to obtain (9.146)–(9.147). Then $\pi_+\mathbb{D}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft = 0$, and $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ q.r.c.-SOS-stabilizes Σ into $\Sigma_\circlearrowleft$ (indeed, by (9.147), $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ is q.r.c.-SOS-stabilizing). Thus, (Crit1$\frac{1}{2}$) holds. The remaining formulae of (g2) follow from those of (g1).

(a1)–(a3) Parts (a1)&(a2) were proved above; see (g1) and (g2) for (a3).

(a4) If any of (Crit1)–(Crit4) holds with "r.c." in place of "q.r.c.", then so do the others, by (a1)–(a3). This property is then inherited in by (b)–(g3). The proof for "p.r.c." is analogous.

(b) This follows from Theorem 9.9.1(f).

(c1) The first claims follows from Theorem 6.7.15(a1)[(a2)[(b1)]], the second claim is a consequence of the first one.

[By Theorem 9.9.1(a2), $\mathcal{P}$ is the smallest nonnegative output-stabilizing solution of the eIARE. But $S \gg 0$, by (e2), and $\mathcal{P}$ is strongly stabilizing, hence $\mathcal{P}$ is the greatest nonnegative admissible solution of the eIARE, by Theorem 9.8.13 (since $S \geq 0$ for admissible nonnegative solutions, by (9.160)). Since $\mathcal{P} \geq \mathcal{P}' \geq \mathcal{P}$ for any nonnegative output-stabilizing solution $\mathcal{P}'$, $\mathcal{P}$ must be unique.]

(c2) This follows from Theorem 6.7.15(b1) and Lemma 6.4.5(e).

(c3) This follows from Theorem 6.7.15(c2) (and from the last claim of (c1)).

(d1) This follows from Proposition 9.8.10 (and (a1)–(a3)).

(d2) We obtain (Crit2)$\Leftrightarrow$(Crit5)$\Leftrightarrow$(Crit6) from Theorem 9.2.9 (the claim on (Crit7) will be proved in Corollary 9.2.15).

(d3) By (d4), we only have to establish (Crit7)$\Rightarrow$(Crit5). Assume (Crit7).

SOS-stabilizability implies that $\mathbb{K}_\flat x_0 \in \mathcal{U}_{\rm out}(x_0) \neq 0$ for all $x_0 \in H$, hence there is a unique J-critical control over $\mathcal{U}_{\rm out}$ for each $x_0 \in H$, by Theorem 8.4.3. Thus, (Crit5) follows from (7.) or (8.) of Remark 9.9.14 (note that "1." is a special case of "2.", by Proposition 6.3.1(c) and Lemma 6.6.12).

(d4) This follows from Proposition 9.2.7(a), Proposition 9.8.10 and Corollary 8.4.14(a).

(e1) This follows from $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$, $\pi_+\mathbb{D}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft = 0$ and $\mathbb{D}_\circlearrowleft^* J\mathbb{D}_\circlearrowleft = S$.

(e2) See the proof of Corollary 10.2.12.

(f) This follows from (b) and Theorem 9.9.1(b).

(g1) This was proved in 4° except for the $\mathcal{P}$ formula, which follows from formulae $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$, $\mathbb{K}\mathbb{N}^* J\mathbb{C}_\circlearrowleft = 0$ (see (Crit1)), $\mathbb{K}\mathbb{N}^* J\mathbb{N}\mathbb{K} = \mathbb{K}S\mathbb{K}$, and Lemma 6.4.7(b), in that order.

(g2) This follows from 5° and (g1) with straightforward computations.

(g3) This follows from the proof of (c) above. □

Note that a q.r.c.-I/O-P-stabilizing solution of the IARE (or CARE) determines a (J,S)-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, by Lemma 9.10.1(b5)&(b6). If Σ is approximately reachable, then this defines $\mathbb{K}$ uniquely modulo (9.138) (because $\pi_+\mathbb{M}^{-1}\pi_- = \mathbb{K}\mathbb{B}$), hence then $\mathcal{P}$ is unique (a q.r.c.-SOS-P-stabilizing solution is always unique, by Theorem 9.9.10(b)).

In the stable case, "q.r.c.-SOS-stabilizing" is equivalent to "stable, SOS-stabilizing" (cf. also Corollary 8.3.11):

Corollary 9.9.11 (SOS-stable IARE) *Let $\Sigma \in$ SOS and $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$.*

Then conditions (Crit1)–(Crit5) in Theorem 9.9.10 can be written in following forms:

(Crit1SOS) There is a J-critical stable, SOS-stabilizing state feedback pair for Σ, and $\mathbb{D}$ is J-coercive.

(Crit2SOS) The IARE has a stable, SOS-P-stabilizing solution.

(Crit3SOS) **(IARE/DARE)** *There are $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $S \in \mathcal{GB}$ and a stable, SOS-stabilizing pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ satisfying the "DARE" (9.111) for some $t > 0$, s.t. $\langle \mathbb{A}_{\circlearrowleft}^{nt} x_0, \mathcal{P}\mathbb{A}_{\circlearrowleft}^{nt} x_0\rangle \to 0$ as $n \to +\infty$, $n \in \mathbf{N}$.*

(Crit4SOS) There is a spectral factorization $\mathbb{X}^* S\mathbb{X}$ of $\mathbb{D}^* J\mathbb{D}$.

(Crit5SOS) The CARE has a stable, SOS-P-stabilizing solution.

We may replace "stable, SOS-P-stabilizing solution" above by "P-admissible solution s.t. $\mathbb{D}, \mathbb{F}, \mathbb{M} \in$ TIC", as well as by "[q.]r.c.-I/O-stabilizing" (and by "exponentially stabilizing" or by "$\mathbb{M}$-stabilizing" if Σ is exponentially stable, and by "stable, P-stabilizing" if Σ is stable).

Moreover, the solutions of (Crit1SOS)–(Crit5SOS) (if any) are the ones of (Crit1)–(Crit5), with $\mathbb{X} = I - \mathbb{F}$. □

(This follows from Lemma 6.6.17(a) and Lemma 6.4.8(a); note that corresponding pairs $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ are the same for each condition. Recall from Definition 9.8.4 that the solution being stable or stabilizing means that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is stable (which is redundant if Σ is exponentially stable) or stabilizing, respectively.)

Proposition 9.8.11 contains related results and the positive case is given in Corollary 10.2.13. We remind that (Crit5SOS) (or (Crit5)) is stronger than (Crit1SOS)–(Crit4SOS), which are equivalent.

J-coercivity is roughly equivalent to the existence of a unique J-critical control:

Proposition 9.9.12 ($\mathcal{U}_{\text{exp}}$: IARE$\Rightarrow$unique optimum$\Leftrightarrow$ J-coercive) *We have (i)$\Leftrightarrow$(ii).*

(i) There is a unique J-critical control over $\mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$, and $\mathbb{S}^t := \mathbb{D}^{t*} J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t \in \mathcal{GB}(\mathrm{L}^2([0,t);U)$ for some (hence all) $t > 0$.

(ii) $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\text{exp}}$, and Σ is optimizable.

Moreover,

(a) Assume (i). Then $\mathbb{S}^t \gg 0 \Leftrightarrow$ the J-critical control is minimizing $\Leftrightarrow \mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{exp}}$.

(b) Assume that the IARE has an exponentially stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$. Then (i) and (ii) hold. Moreover, then $S \gg 0 \Leftrightarrow \mathbb{S}^t \gg 0 \Leftrightarrow \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is minimizing $\Leftrightarrow \mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{exp}}$.

(c) Condition "$\mathbb{S}^t \in \mathcal{GB}$" is redundant in (i) if any of (1.)–(4.) holds, where

(1.) $J \geq 0$ and $\mathbb{D}^{t*}J\mathbb{D}^t \gg 0$ for some $t > 0$;
(2.) $J \geq 0$, $D^*JD \gg 0$ and $\mathbb{D} \in \mathrm{MTIC}_\infty$;
(3.) $\mathbb{D} \in \mathrm{MTIC}_\infty$, $\mathbb{A}B \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;\mathcal{B}(U,H))$ and $D^*JD \in \mathcal{GB}(U)$.
(4.) Hypothesis 9.2.1 holds for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$, and $D^*JD \in \mathcal{GB}(U)$.

Thus, when minimizing over $\mathcal{U}_{\exp}$ with a some coercivity or regularity, the cost must be J-coercive over $\mathcal{U}_{\exp}$. See, e.g., Theorems 9.2.16 and 9.2.18 and Corollary 9.2.19 for enhanced versions of the proposition, and Section 10.2 for a positive variants.

Proof: (Naturally, $\mathcal{P} := \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}_{\mathrm{crit}}$ in (i).)

The equivalence, (a), (b) and (c)(1.) follow from Theorems 14.2.7 and 13.4.4 and Remark 13.4.6.

(c) (2.) Now $\mathcal{P} := \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}_{\mathrm{crit}} \geq 0$, hence $\mathbb{S}^t \geq \mathbb{D}^{t*}J\mathbb{D}^t$. But $\mathbb{D}^{t*}J\mathbb{D}^t \geq D^*JD - \varepsilon \gg 0$ for $t, \varepsilon > 0$ small enough, by Theorem 2.6.4(i1), hence $\mathbb{S}^t \gg 0$ for such t.

(3.) By Lemma A.3.1(c4)&(c1), there is $\varepsilon > 0$ s.t. $\|D^*JDu_0\|_U \geq \varepsilon \|u_0\|_U$ for all $u_0 \in U$. It follows that

$$\|\pi_{[0,t)} D^*JDu\|_2 \geq \varepsilon \|u\|_2 \quad (u \in \mathrm{L}^2([0,t);U)). \tag{9.152}$$

By Theorem 2.6.4(i1)&(i2), $\pi_{[0,t)}(\mathbb{D} - D)\pi_{[0,t)} \to 0$ and $\mathbb{B}^t \to 0$ on L^2, as $t \to 0+$, hence there is $t > 0$ s.t. $\|\pi_{[0,t)}(\mathbb{S}^t - D^*JD)\pi_{[0,t)}\|_{\mathcal{B}(\mathrm{L}^2)} < \varepsilon/2$ (note that $\mathbb{S}^t = \pi_{[0,t)}\mathbb{S}^t\pi_{[0,t)}$). Consequently, $\|\mathbb{S}^t u\|_2 \geq \varepsilon/2\|u\|_2$ for all $u \in \mathrm{L}^2([0,t);U)$. By Lemma A.3.1(c4)&(c1), this means that $\mathbb{S}^t \in \mathcal{GB}(\mathrm{L}^2([0,t);U))$.

(4.) This follows from Theorem 9.2.16 and (b). □

However, uniqueness is sometimes possible under weaker conditions:

Remark 9.9.13 (Unique J-critical control vs. $S \in \mathcal{GB}(U)$ vs. J-coercivity)
Assume that $\mathcal{U}_*^*(x_0) \neq \emptyset$ for all $x_0 \in H$ (this is obviously necessary for the existence of a J-critical control) and that Z^{s} is a Hilbert space (e.g., $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_*^* = \mathcal{U}_{\exp}$).

By Theorem 8.4.3 and Lemma 9.10.3, J-coercivity implies the existence of a unique J-critical control, and also the invertibility of S when the eIARE has a solution (these three are equivalent for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ if Hypothesis 9.2.2 holds (or if Σ is a wpls), by Theorem 9.2.16).

However, the invertibility of S is not necessary, nor is J-coercivity over $\mathcal{U}_*^*$, for the existence of a unique J-critical control (take Σ exponentially stable (so that $\mathcal{U}_{\exp} = \mathcal{U}_{\mathrm{out}}$), $0 < D^*D \not\gg 0$, $B = 0 = C$, $J = I$).

On the other hand, Example 9.13.4 shows that even for very regular ($\mathbb{D} = D \in \mathcal{B}(U,Y)$) exponentially stable systems, $S = D^*D > 0$ is not sufficient for the existence of a J-critical control for all $x_0 \in H$ (over $\mathcal{U}_{\exp} = \mathcal{U}_{\mathrm{out}}$).

It is easy to formulate shortly the necessary and sufficient condition for the existence of a unique J-critical control, but we have found no useful formulations (see the comments below Theorem 8.2.5).

As mentioned above, when B is bounded, any unique J-critical control corresponds to a $\mathcal{U}_*^*$-stabilizing solution of the CARE (and of the $B_{\rm w}^*$-CARE if $D^*JD \in \mathcal{GB}$ or $\mathbb{D}$ is J-coercive). Unfortunately, this is not the case in general, by Example 11.3.7. Therefore, we summarize several sufficient conditions below:

Remark 9.9.14 (Necessity of the CARE) *We write* $(\Sigma,J) \in$ coerciveCARE *(over $\mathcal{U}_*^*$) if 1. Σ is WR, and 2. if Σ is J-coercive and there is a J-critical control for each $x_0 \in H$, then the CARE has a SR $\mathcal{U}_*^*$-stabilizing solution (equivalently, then $K_{\rm crit}$ corresponds to a SR state feedback operator).*

If any of the following conditions holds, then $(\Sigma,J) \in$ coerciveCARE*:*

(1.) $B \in \mathcal{B}(U,H)$;

(2.) Hypothesis 9.2.1 holds and $D^*JD \in \mathcal{GB}(U)$;

(3.) Hypothesis 9.2.1 holds, $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$ and $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$;

(4.) Hypothesis 9.5.1 holds and $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$;

(5.) $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$, $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$, $\pi_{[0,1)}C_{\rm w}\mathbb{A} \in \mathrm{L}^1([0,1);\mathcal{B}(H,Y))$, and $\pi_{[0,1)}C_{\rm w}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,Y))$;

(6.) $\Sigma \in$ SOS, $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$ and $\mathbb{D} \in \widetilde{\mathcal{A}}$, where $\widetilde{\mathcal{A}}$ satisfies Hypothesis 8.4.7.

(7.) $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$, and Σ has a SR q.r.c.-SOS-stabilizing state feedback operator s.t. $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$.

(8.) $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$, and Σ has a SR exponentially q.r.c.-stabilizing state feedback operator s.t. $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ some of (1.)–(6.).

Moreover, the following holds:

(a) In cases (1.)–(6.), we have $\mathbb{D} \in$ ULR (and $\mathbb{F} \in$ ULR for the $\mathcal{U}_*^*$-stabilizing solution).

(b) In cases (1.)–(5.), we necessarily have $S = D^*JD \in \mathcal{GB}(U)$ for the $\mathcal{U}_*^*$-stabilizing solution.

(c) In cases (1.)–(3.), the CARE becomes a $B_{\rm w}^*$-CARE.

(d) In cases (1.)–(4.), "Σ is J-coercive and there is a" can be replaced by "there is a unique" if we assume that $D^*JD \in \mathcal{GB}(U)$.

By Lemma 8.4.4, the control mentioned in "2." is necessarily unique. By Theorem 14.1.6 and Lemma 9.9.7(c2), "$(\Sigma,J) \in$ coerciveCARE" is redundant in discrete time (i.e., it is true for any Σ and J).

Recall from Theorem 8.4.3 that "there is a J-critical control for each $x_0 \in H$" can usually (e.g., for $\mathcal{U}_{\rm out}$ and $\mathcal{U}_{\rm exp}$) be replaced by the finite cost condition "$\mathcal{U}_*^*(x_0) \neq \emptyset$ for all $x_0 \in H$". If Σ is exponentially stable (or estimatable), then $\mathcal{U}_{\rm exp} = \mathcal{U}_{\rm out}$, by Lemma 8.3.3.

See Theorems 9.5.13 and 9.2.18 and Corollary 9.1.11 for more explicit variants of (4.) and (5.) and (7.), respectively.

Proof: (Note that (1.) and (4.)–(7.) are independent of J, and that the $\mathcal{U}_*^*$-stabilizing solution is unique. The requirement "SR" could be "WR" for most but not all applications of the above definition.)

Note first that $\mathbb{D} \in \mathrm{ULR}$ if any of (1.)–(6.) holds and $\mathbb{D} = \mathbb{D}_\flat \mathbb{X}^{-1} \in \mathrm{SR}$ when (7.) or (8.) holds, so that condition "1." is satisfied by any of (1.)–(8.).

Case (1.) is treated in Theorem 9.9.6 (the eCARE becomes a CARE, because J-coercivity implies the invertibility of S $(= D^*JD)$, by Lemma 9.9.7(c2)).

Case (2.) is treated in Theorem 9.2.9. Condition (3.) implies condition (2.), by Lemma 9.2.17 (which contains also alternative assumptions).

Condition (4.) implies condition (5.), by Lemma 9.5.2.

Case (5.) follows from Theorem 9.2.18. Case (6.) follows from "(iv)⇒(iii)" of Corollary 9.1.12.

Case (7.) follows from (6.) and Theorem 8.4.5(g1)&(d)&(a)&(c2) (and Lemma 6.2.5 and Corollary 9.9.8; Proposition 9.12.4 would lead to an alternative proof and (9.226) holds). Analogously, case (8.) can be reduced (1.)–(6.) (use the fact that $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$, by Lemma 8.3.3; naturally, by "$\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ satisfies (n.)" we mean that "(n.)" is satisfied with $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ in place of $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$).

(a)&(b) The proof sketched above applies for (a)&(b) too. (See Theorem 8.4.9(a) for extending (b) to case (6.).)

(c) See Theorem 9.2.9 (use the fact that $\mathrm{Dom}(B_{\mathrm{w}}^*) = H$ in case (1.)).

(d) The above proofs for cases (1.)–(3.) did not use J-coercivity. In cases (4.) and (5.), we can remove "Σ is J-coercive and there is a" completely if we assume that $\mathbb{A}B \in \mathrm{L}^2_{\mathrm{loc}}$ (this is redundant in case (4.), by Lemma 9.5.2) and that $D^*JD \in \mathcal{GB}(U)$, by Corollary 9.2.19.

(N.B. if we assume (1.) and replace "Σ is J-coercive and there is a" by "there is a unique", then the CARE might become an eCARE (instead of a B_{w}^*-CARE) if we would not explicitly assume D^*JD to be invertible.) □

A comparison of $\mathcal{U}_{\mathrm{exp}}$, $\mathcal{U}_{\mathrm{str}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{out}}$

In principle, the above theory on optimization and Riccati equations can be applied over any $\mathcal{U}_*^*$. However, it is not always clear a priori whether a control problem is coercive enough to guarantee the existence of a unique solution (particularly in the case of $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$), and when a solution of the corresponding Riccati equation is found, it is not always easy to verify that it is a correct one ($\mathcal{U}_*^*$-stabilizing).

For optimization over $\mathcal{U}_{\mathrm{out}}$, it is often easy to find sufficient coercivity conditions. For $\mathcal{U}_{\mathrm{exp}}$ one needs stronger assumptions, and for $\mathcal{U}_{\mathrm{str}}$ (resp. $\mathcal{U}_{\mathrm{sta}}$), we have to optimize over $\mathcal{U}_{\mathrm{out}}$ and make suitable stabilizability or other assumptions to guarantee that the closed-loop system actually becomes strongly stable (resp. stable) (see the comments below Theorem 8.4.3).

On the other hand, J-critical state feedback pairs over $\mathcal{U}_{\mathrm{exp}}$ correspond to exponentially stabilizing solutions of the Riccati equation, and the situation with $\mathcal{U}_{\mathrm{str}}$ is analogous, whereas the situation with $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{out}}$ requires the complicated residual condition (PB) (see Theorem 9.8.5); therefore, for these two sets it is not easy to verify for a solution of the Riccati equation that it corresponds to optimal control, unless additional assumptions are made.

By the above, the closed-loop system corresponding to $\mathcal{U}_{\mathrm{exp}}$ is exponentially stable. For $\mathcal{U}_{\mathrm{str}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{out}}$, we only know about the stability of the left column of the closed-loop system (see Theorem 9.8.5). If we wish to pose stability requirements also on the right column, we have to make additional detectability or q.r.c.-stabilizability assumptions. An LQR application with the former assumption is given by Theorem 10.1.4(c2), and an H^∞ application of the latter in Theorem 11.1.5.

This latter "q.r.c." approach is based on the fact that whereas optimization over $\mathcal{U}_{\mathrm{exp}}$ can always be reduced to the stable case, the analogous reduction for $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$ requires quasi-coprimeness; see Theorem 8.4.5(g1) and Remark 9.9.9. Although q.r.c.-stabilizability is trivially possessed by stable systems, it is often difficult to verify for unstable ones, although still popular in articles based on fractional representations of the I/O maps. See Theorem 9.9.1(b)–(c3) and Theorem 9.9.10 for more on this approach.

A third approach for $\mathcal{U}_{\mathrm{out}}$ is used in Theorem 10.1.4(b1), where we only have to study the minimal nonnegative solution.

Job Oostveen has developed a rather extensive optimization theory over $\mathcal{U}_{\mathrm{str}}$ for WPLSs with bounded input and output operators (B and C), and he avoids some of the problems described above by using suitable detectability assumptions (a most elegant example of his results is the one extended in Theorem 10.1.4(c2)). It seems that most of his results can be generalized to more general WPLSs in the same way; we recommend this for a reader interested in $\mathcal{U}_{\mathrm{str}}$.

We conclude that the theory on optimization and Riccati equations becomes most elegant for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, even more beautiful when one assumes estimatability (e.g., a cost on the state, by Lemma 6.6.25), so that $\mathcal{U}_{\mathrm{exp}} = \mathcal{U}_{\mathrm{str}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{out}}$. However, often it is too restricting to require an exponentially stabilizing controller, and in several settings also the theory for $\mathcal{U}_{\mathrm{out}}$ (or $\mathcal{U}_{\mathrm{sta}}$ or $\mathcal{U}_{\mathrm{str}}$) can be substantially simplified.

Notes

The necessity part of Corollary 9.9.2 is contained in [S98b] and [WW] to some extent, in the generality explained in the notes on p. 520. In the same setting, the implication from (Crit4) to (Crit5) (for regular $\mathbb{D}$ and $\mathbb{F}$) and the formulae of (b), (e1), (g1) and (g2) of Theorem 9.9.10 are contained in [S98b, Sections 5–7].

The earlier history of infinite-dimensional Riccati equations is documented in the notes to Section 6 of [CZ]. For future research, a very important task is to determine further sufficient assumptions for a unique optimal control to exist in regular state feedback form, besides those presented in Remark 9.9.14 or elsewhere in this monograph.

One interesting candidate is the condition that $\mathbb{D} \in \mathrm{MTIC}_\infty$ (and it might imply that $\mathbb{F} \in \mathrm{MTIC}_\infty$). By Example 9.8.15, the approach of Section 9.2 does not work for this assumption, not even in the stable case, although that approach might be useful for some other candidate conditions.

9.10 Proofs for Section 9.9: Crit↔eIARE

When eating an elephant take one bite at a time.
— Gen. C. Abrams

In this section, we shall establish the equivalence between the existence of J-critical state feedback pairs and the existence of $\mathcal{U}_*^*$-stabilizing solutions of the eIARE, and also state some related results that are needed for further results. See Definition 9.8.4 (and Definition 6.6.10) for Σ_{ext} and $\Sigma_\circlearrowleft$.

First we explore in detail the connection between J-critical control and the admissible solutions of the eIARE (cf. (b4)):

Lemma 9.10.1 *Let $S \in \mathcal{B}(U)$, and $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$. Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be an admissible state feedback pair for Σ, and let $\Sigma_\circlearrowleft := \left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right] \in \mathrm{WPLS}(U,H,Y \times U)$ be the corresponding closed-loop system. Set $\mathbb{M} := (I-\mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M} = \mathbb{D}_\circlearrowleft$.*

We consider, for $t \geq 0$, the equations

$$0 = \mathbb{D}^{t\,*} J \mathbb{C}^t_\circlearrowleft + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{A}^t_\circlearrowleft, \tag{9.153}$$

$$0 = \mathbb{D}^{t}_\circlearrowleft{}^* J \mathbb{C}^t_\circlearrowleft + \mathbb{B}^{t}_\circlearrowleft{}^* \mathcal{P} \mathbb{A}^t_\circlearrowleft, \tag{9.154}$$

$$\mathcal{P} = \mathbb{A}^{t}_\circlearrowleft{}^* \mathcal{P} \mathbb{A}^t_\circlearrowleft + \mathbb{C}^{t}_\circlearrowleft{}^* J \mathbb{C}^t_\circlearrowleft, \tag{9.155}$$

$$\mathcal{P} = \mathbb{A}^{t}_\circlearrowleft{}^* \mathcal{P} \mathbb{A}^t + \mathbb{C}^{t}_\circlearrowleft{}^* J \mathbb{C}^t, \tag{9.156}$$

$$\pi_{[0,t)} S = \mathbb{N}^{t\,*} J \mathbb{N}^t + \mathbb{B}^{t}_\circlearrowleft{}^* \mathcal{P} \mathbb{B}^t_\circlearrowleft, \tag{9.157}$$

$$S\mathbb{K}^t = -\left(\mathbb{N}^{t\,*} J \mathbb{C}^t + \mathbb{M}^{t\,*} \mathbb{B}^{t\,*} \mathcal{P} \mathbb{A}^t\right), \tag{9.158}$$

$$\mathbb{K}^{t\,*} S \mathbb{K}^t = \mathbb{A}^{t\,*} \mathcal{P} \mathbb{A}^t - \mathcal{P} + \mathbb{C}^{t\,*} J \mathbb{C}^t, \tag{9.159}$$

$$\mathbb{X}^{t\,*} S \mathbb{X}^t = \mathbb{D}^{t\,*} J \mathbb{D}^t + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{B}^t, \tag{9.160}$$

$$\mathbb{X}^{t\,*} S \mathbb{K}^t = -\left(\mathbb{D}^{t\,*} J \mathbb{C}^t + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{A}^t\right). \tag{9.161}$$

Claims (a1)–(b4) hold:

(a1) For any $t \geq 0$ we have (9.154)⇔(9.153), as well as (9.160)⇔(9.157), and (9.161)⇔(9.158).

(b1) Let $t \geq 0$ and let (9.158) hold. Then (9.156)⇔(9.159).

(b2) Let $t \geq 0$ and let (9.160) hold. Then (9.154)⇔(9.158).

(b3) Let $t \geq 0$ and let (9.154) hold. Then (9.155)⇔(9.156).

(b4) For each $t \geq 0$, conditions (i)–(iv) are equivalent, where

> *(i) Equations (9.159)–(9.161) (the eIARE) are satisfied;*
> *(ii) Equations (9.153)–(9.161) are satisfied;*
> *(iii) Equations (9.154), (9.155) and (9.157) hold.*
> *(iv) Equations (9.158), (9.155) and (9.157) hold.*

(b5) If (i), (ii) or (iii) holds for some $t > 0$, then (i)–(iii) hold for nt ($n \in \mathbf{N}$).

(b6) If (P4) of (d1) holds, (9.157) holds for each $t = t_n$ ($n \in \mathbf{N}$), and $\mathbb{N} \in \mathrm{TIC}$, then $\mathbb{N}^* J \mathbb{N} = S$.

If $\mathbb{C}_\circlearrowleft$ is stable, then (c1)–(d2) hold:

(c1) We have $\mathbb{N}\pi_{[0,t)} \in \mathcal{B}(\mathrm{L}^2)$ *for all* $t \geq 0$.

(c2) Assume that $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$. *Then (9.160) is equivalent to*

$$\langle \mathbb{D}_\circlearrowleft u, J\mathbb{D}_\circlearrowleft v\rangle_{\mathrm{L}^2(\mathbf{R}_+;U)} = \langle u, Sv\rangle_{\mathrm{L}^2(\mathbf{R}_+;U)} \quad (u,v \in \mathrm{L}^2([0,t);U)). \tag{9.162}$$

Moreover, if either holds for all $t > 0$, *then (9.162) holds for all* $u,v \in \mathrm{L}^2_\omega(\mathbf{R}_+;U) + \mathrm{L}^2_\mathrm{c}$ *and all* $\omega < 0$.

(c3) If $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$, *then (9.154) is equivalent to*

$$\langle \mathbb{D}_\circlearrowleft \pi_+ u, J\mathbb{C}_\circlearrowleft x_0\rangle_{\mathrm{L}^2(\mathbf{R}_+;U)} = 0 \quad (u \in \mathrm{L}^2([0,t);U),\ x_0 \in H). \tag{9.163}$$

Moreover, if either holds for all $t > 0$, *then (9.163) holds for all* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U) + \mathrm{L}^2_\mathrm{c}$ *and all* $\omega < 0$.

(c4) Assume that $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$, *and* $\langle \mathbb{D}_\circlearrowleft \pi_+ u, J\mathbb{C}_\circlearrowleft x_0\rangle = 0$ *for all* $u \in \mathrm{L}^2_\mathrm{c}$.

Then there is a unique $\widetilde{S} \in \mathcal{B}(U)$ *s.t.* $\langle \mathbb{N}u, J\mathbb{N}u\rangle = \langle u, \widetilde{S}u\rangle$ $(u \in \mathrm{L}^2_\mathrm{c})$.

Moreover, $\widetilde{S} = \widetilde{S}^* \in \mathcal{B}(U)$, *all of (P1)–(P4) hold, and (9.153)–(9.163) are satisfied for all* $t \geq 0$ *with* $\widetilde{S}$ *in place of S.*

(d1) We have

(P1) $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$

iff (9.155) holds for all $t \geq 0$ *and any (hence all) of (P2)–(P4) holds, where*

(P2) $\langle \mathbb{A}^t_\circlearrowleft x_0, \mathcal{P}\mathbb{A}^t_\circlearrowleft x_0\rangle \to 0$, *as* $t \to +\infty$, *for all* $x_0 \in H$;

(P3) ${\mathbb{A}^t_\circlearrowleft}^* \mathcal{P}\mathbb{A}^t_\circlearrowleft x_0 \to 0$, *as* $t \to +\infty$, *for all* $x_0 \in H$;

(P4) There is a sequence $\{t_n\}$ *s.t.* $t_n \to +\infty$ *as* $n \to +\infty$ *and* $\langle \mathbb{A}^{t_n}_\circlearrowleft x_0, \mathcal{P}\mathbb{A}^{t_n}_\circlearrowleft x_0\rangle \to 0$ *for all* $x_0 \in H$.

All this holds even if we restrict t to an unbounded set $R \subset \mathbf{R}_+$. *If (P1) holds, then* $\mathcal{P}\mathbb{A}^t_\circlearrowleft x_0 \to 0$, *as* $t \to +\infty$, *for all* $x_0 \in H$. *(Note that if* $\mathbb{A}_\circlearrowleft$ *is strongly stable, then (P2)–(P4) hold.)*

(d2) Let $\mathbb{C}$ *and* $\mathbb{K}$ *be stable. Then* $\mathcal{P} = \mathbb{C}^* J\mathbb{C} - \mathbb{K}^* S\mathbb{K}$ *iff (9.159) holds for all* $t \geq 0$ *and any (hence all) of (P2)–(P4) holds with* $\mathbb{A}$ *in place of* $\mathbb{A}_\circlearrowleft$. *Moreover,* $\mathcal{P} = \mathbb{C}^* J\mathbb{C} - \mathbb{K}^* S\mathbb{K}$ *implies that* $\mathcal{P}\mathbb{A}^t x_0 \to 0$ *for all* $x_0 \in H$.

If both $\mathbb{C}_\circlearrowleft$ *and* $\mathbb{N} = \mathbb{D}_\circlearrowleft$ *are stable, then (e1)–(f2) hold:*

(e1) Assume that $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$. *Then (9.154) holds for all* $t \geq 0$ *iff* $\pi_+\mathbb{D}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft = 0$. *In fact, it is sufficient that (9.154) holds for* $t = t_n$, $n \in \mathbf{N}$, *where* $t_n \to +\infty$ *as* $n \to +\infty$.

(e2) If $\pi_+\mathbb{D}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft = 0$, *then* $\widetilde{S} := \mathbb{N}^* J\mathbb{N} = \widetilde{S}^* \in \mathcal{B}(U)$.

(f1) If $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft$ *and* $\mathbb{N}^* J\mathbb{N} = S$, *then (9.160) holds for all* $t \geq 0$.

(f2) Conversely, assume that (9.160) holds for all $t \geq 0$. Then the following hold:

If $\mathbb{N}^* J \mathbb{N} = S$, then (P1)–(P4) hold on $H_{\mathbb{B}}$ (instead of H). If (P1), (P2), (P3) or (P4) holds on $H_{\mathbb{B}}$, then $\mathbb{N}^* J \mathbb{N} = S$.

For (f2), we recall from Lemma 6.3.26(b3), that the *reachability subspace* $H_{\mathbb{B}_\circlearrowleft}$ of $\Sigma_\circlearrowleft$ equals that of $H_{\mathbb{B}}$, i.e., the closure (in H) of

$$\{\mathbb{B}^t u \,\big|\, u \in \pi_{[0,t)}\mathrm{L}^2,\ t \geq 0\}. \tag{9.164}$$

Proof:

(a) Multiply by $\mathbb{M}^t$ or $\mathbb{X}^t$ to the left.

(b1) Insert (9.158) into (9.159) to obtain (9.156) (recall that $\mathbb{C}^t_\circlearrowleft = \mathbb{C}^t + \mathbb{N}^t \mathbb{K}^t$ and $\mathbb{A}^t_\circlearrowleft = \mathbb{A}^t + \mathbb{B}^t \mathbb{M}^t \mathbb{K}^t$).

(b2) Use equations $\mathbb{C}^t_\circlearrowleft = \mathbb{C}^t + \mathbb{D}^t_\circlearrowleft \mathbb{K}^t$, $\mathbb{A}^t_\circlearrowleft = \mathbb{A}^t + \mathbb{B}^t_\circlearrowleft \mathbb{K}^t$, and (9.160) to obtain that (cf. Lemma 5.5 of [S98b])

$$\mathbb{D}^t_\circlearrowleft{}^* J \mathbb{C}^t_\circlearrowleft + \mathbb{B}^t_\circlearrowleft{}^* \mathcal{P} \mathbb{A}^t_\circlearrowleft = S\mathbb{K}^t + \mathbb{N}^{t\,*} J \mathbb{C}^t + \mathbb{M}^{t\,*} \mathbb{B}^{t\,*} \mathcal{P} \mathbb{A}^t. \tag{9.165}$$

(b3) By (6.132), the difference (9.155)−(9.156)* is equal to

$$\mathbb{K}^{t\,*}\left(\mathbb{D}^t_\circlearrowleft{}^* J \mathbb{C}^t_\circlearrowleft + \mathbb{B}^t_\circlearrowleft{}^* \mathcal{P} \mathbb{A}^t_\circlearrowleft\right) = \mathbb{K}^* 0 = 0. \tag{9.166}$$

(b4) "(i)⇒(ii)"&"(iv)⇒(iii)⇒(ii)": These follow easily from (a)–(b3); trivially (ii) implies (i), (iii) and (iv).

(b5) Apply Lemma 14.2.1 for $A := \mathbb{A}^t$, $B := \mathbb{B}^t$, ...; equivalently, use discretization (cf. Proposition 9.8.7) to see that if (i) holds for some t, then (i) holds for $t\mathbf{N}$. Combine this with (b4) to obtain (b5).

(b6) From (9.157) we obtain that $\langle \mathbb{N}u, J\mathbb{N}u\rangle = \langle u, Su\rangle$ for $u \in \pi_{[0,T)}\mathrm{L}^2$, because $\mathbb{B}^t_\circlearrowleft u = \mathbb{A}^{t-T}_\circlearrowleft \mathbb{B}^T_\circlearrowleft u$. Because T was arbitrary, $\mathbb{N}^* J \mathbb{N} = S$, by density.

(c1) This follows from Lemma 6.1.11 and the stability of $\mathbb{C}_\circlearrowleft$ (alternatively, from (9.167)).

(c2) (In discrete time, we have (9.162) for all $u, v \in \ell^1(\mathbf{Z}; U)$, by Lemma 13.3.8(b3); cf. Theorem 14.1.3.) Now

$$\pi_{[t,\infty)} \mathbb{N} \pi_{[0,t)} = \tau^{-t} \pi_+ \mathbb{N} \pi_- \tau^t \pi_{[0,t)} = \tau^{-t} \mathbb{C}_\circlearrowleft \mathbb{B}_\circlearrowleft \tau^t \pi_{[0,t)} \in \mathcal{B}(\mathrm{L}^2), \tag{9.167}$$

hence

$$\langle \pi_{[t,\infty)} \mathbb{N} \pi_{[0,t)} u, J \pi_{[t,\infty)} \mathbb{N} \pi_{[0,t)} v\rangle = \langle \tau^{-t} \mathbb{C}_\circlearrowleft \mathbb{B}_\circlearrowleft \tau^t \pi_{[0,t)} u, J \tau^{-t} \mathbb{C}_\circlearrowleft \mathbb{B}_\circlearrowleft \tau^t \pi_{[0,t)} v\rangle \tag{9.168}$$

$$= \langle \mathbb{B}^t_\circlearrowleft u, \mathcal{P} \mathbb{B}^t_\circlearrowleft v\rangle \quad (u, v \in \mathrm{L}^2). \tag{9.169}$$

Consequently, $\langle u, \pi_{[0,t)} S v\rangle = \langle \mathbb{N}\pi_{[0,t)} u, (\pi_{[0,t)} + \pi_{[t,\infty)}) J \mathbb{N} \pi_{[0,t)} v\rangle$ for $u \in \mathrm{L}^2$ iff (9.157) holds (equivalently, (9.160) holds, by (a1)).

The last claim follows from Lemma 2.1.13 (both sides are valid on $\mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+; U)$ and continuous functions on $\mathrm{L}^2_\omega(\mathbf{R}_+; U) \times \mathrm{L}^2_\omega(\mathbf{R}_+; U)$).

(c3) This follows from

$$\pi_{[0,t)}\tau(-t)\mathbb{B}^*_\circlearrowleft(\mathbb{C}^*_\circlearrowleft J\mathbb{C}_\circlearrowleft)\mathbb{A}_\circlearrowleft(t) = \pi_{[0,t)}\tau(-t)\pi_-\mathbb{D}^*_\circlearrowleft\pi_+J\pi_+\tau(t)\mathbb{C}_\circlearrowleft \tag{9.170}$$
$$= \pi_{[0,t)}\mathbb{D}^*_\circlearrowleft J\tau(-t)\pi_+\tau(t)\mathbb{C}_\circlearrowleft = \pi_{[0,t)}\mathbb{D}^*_\circlearrowleft J\pi_{[t,\infty)}\mathbb{C}_\circlearrowleft. \tag{9.171}$$

The last claim follows from Lemma 2.1.13 as in (c2).

(c4) Now $\langle\mathbb{N}\pi_+v, J\mathbb{N}\pi_-u\rangle = \langle\mathbb{N}\pi_+v, J\mathbb{C}_\circlearrowleft\mathbb{B}_\circlearrowleft u\rangle = 0$ for all $u, v \in \mathrm{L}^2_{\mathrm{c}}$, hence there is a unique $\widetilde{S} = \widetilde{S}^* \in \mathcal{B}(U)$ s.t. $\langle\mathbb{N}u, J\mathbb{N}u\rangle = \langle u, Su\rangle$ $(u \in \mathrm{L}^2_{\mathrm{c}})$, by (c1) and Lemma 2.3.1.

By (9.167), we have $\mathbb{B}^t_\circlearrowleft{}^*\mathcal{P}\mathbb{B}^t_\circlearrowleft = (\pi_{[t,\infty)}\mathbb{N}\pi_{[0,t)})^*J\pi_{[t,\infty)}\mathbb{N}\pi_{[0,t)}$. It follows that (9.157) holds with $\widetilde{S}$ in place of S; this for all $t \ge 0$.

From (d1) it follows that (P1)–(P4) and (9.155) hold for all $t \ge 0$; by (c3), (9.154) holds for all $t \ge 0$.

By (a1), (b3), (b2) and (b1), (9.153)–(9.158) hold for all $t \ge 0$ with $\widetilde{S}$ in place of S.

(d1) 1° *Equivalence:* Assume (9.155). For any $x_0 \in H$, we have $\pi_{[0,t)}\mathbb{C}_\circlearrowleft x_0 \to \mathbb{C}_\circlearrowleft x_0$ in L^2, by Corollary B.3.8. Therefore, $\mathbb{C}^*_\circlearrowleft J\pi_{[0,t)}\mathbb{C}_\circlearrowleft x_0 \to \mathbb{C}^*_\circlearrowleft J\mathbb{C}_\circlearrowleft x_0$.

Consequently, $\mathbb{A}^t_\circlearrowleft{}^*\mathcal{P}\mathbb{A}^t_\circlearrowleft \to \mathcal{P} - \mathbb{C}^*_\circlearrowleft J\mathbb{C}_\circlearrowleft$ strongly (where $t \in R$ if the assumption holds on R only), hence (P1)–(P4) are equivalent.

2° *Claim $\mathcal{P}\mathbb{A}^t_\circlearrowleft x_0 \to 0$:* Assume (P1). Then (we use Definition 6.1.1(3.) and Corollary B.3.8)

$$\mathcal{P}\mathbb{A}^t_\circlearrowleft x_0 = \mathbb{C}^*_\circlearrowleft J\mathbb{C}_\circlearrowleft\mathbb{A}^t_\circlearrowleft x_0 = \mathbb{C}^*_\circlearrowleft J\pi_+\tau^t\mathbb{C}_\circlearrowleft x_0 \to \mathbb{C}^*_\circlearrowleft 0 = 0. \tag{9.172}$$

However, (9.172) is not sufficient, by Example 9.13.11, unless $\mathbb{A}_\circlearrowleft$ is stable.

(d2) The proof is analogous to that of (d1) ($\mathbb{C}_\circlearrowleft$ need not be stable here; also here t can be restricted to R).

(e1) This follows from (c3), because $\pi_{[0,t)}\mathbb{D}^*_\circlearrowleft J\mathbb{C}_\circlearrowleft = 0$ for all $t \ge 0$ iff $\pi_+\mathbb{D}^*_\circlearrowleft J\mathbb{C}_\circlearrowleft = 0$.

(e2) Now $(\pi_-\mathbb{N}^*J\mathbb{N}\pi_+)^* = \pi_+\mathbb{N}^*J\mathbb{N}\pi_- = \pi_+\mathbb{N}^*J\mathbb{C}_\circlearrowleft\mathbb{B}_\circlearrowleft = 0$, hence $S' := \mathbb{N}^*J\mathbb{N} \in \mathcal{B}(U)$, by Lemma 2.1.7, and $S' = S'^*$.

(f1) This follows from (c2) and Corollary B.3.8.

(f2) By the proof of (c2), we have $\pi_{[0,t)}\mathbb{N}^*J\mathbb{N}\pi_{[0,t)} = S$ iff $\mathbb{B}^t_\circlearrowleft{}^*\mathcal{P}\mathbb{B}^t_\circlearrowleft = \mathbb{B}^t_\circlearrowleft{}^*\mathbb{C}^*_\circlearrowleft J\mathbb{C}_\circlearrowleft\mathbb{B}^t_\circlearrowleft$. But $\pi_{[0,t)}\mathbb{N}^*J\mathbb{N}\pi_{[0,t)} = S$ for all $t > 0$ iff $\mathbb{N}^*J\mathbb{N} = S$, by Corollary B.3.8, hence we must have $\mathbb{B}^t_\circlearrowleft{}^*\mathcal{P}\mathbb{B}^t_\circlearrowleft = \mathbb{B}^t_\circlearrowleft{}^*\mathbb{C}^*_\circlearrowleft J\mathbb{C}_\circlearrowleft\mathbb{B}^t_\circlearrowleft$ for all $t > 0$, equivalently, $\mathcal{P} = \mathbb{C}^*_\circlearrowleft J\mathbb{C}_\circlearrowleft$ on $H_{\mathbb{B}_\circlearrowleft} = H_{\mathbb{B}}$, by continuity.

But (P1) on $H_{\mathbb{B}}$ implies (P2)–(P4) on $H_{\mathbb{B}}$, as in the proof of (d1). Because each of (P2) and (P3) implies (P4) (on $H_{\mathbb{B}}$) it only remains to assume (P4) and prove that $\mathbb{N}^*J\mathbb{N} = S$.

Assume (P4). Let $T > 0$, and choose $u \in \pi_{[0,T)}\mathrm{L}^2$. Then $\tau(T)u = \pi_-\tau(T)u$, hence

$$\mathbb{B}_\circlearrowleft\tau(t)\pi_+u = \mathbb{B}_\circlearrowleft\tau(t-T)\pi_-\tau(T)u = \mathbb{A}_\circlearrowleft(t-T)x_u, \tag{9.173}$$

where $x_u := \mathbb{B}_\circlearrowleft\pi_-\tau(T)u$. Thus, $\langle\mathbb{B}_\circlearrowleft\tau(t_n)\pi_+u, \mathcal{P}\mathbb{B}_\circlearrowleft\tau(t_n)\pi_+u\rangle \to 0$, as $n \to \infty$, by (P4). By (9.157), it follows that (for $t_n > T$)

$$\langle u, \pi_{[0,t_n)}(S - \mathbb{N}^*J\pi_{[0,t_n)}\mathbb{N})\pi_{[0,t_n)}u\rangle = \langle u, (S - \mathbb{N}^*J\pi_{[0,t_n)}\mathbb{N})u\rangle \to 0, \tag{9.174}$$

as $n \to \infty$. But $(S - \mathbb{N}^* J \pi_{[0,t_n)} \mathbb{N}) u \to (S - \mathbb{N}^* J \mathbb{N}) u$, hence $\langle u, (S - \mathbb{N}^* J \mathbb{N}) u \rangle = 0$. Because $T > 0$ and $u \in \pi_{[0,T)} \mathrm{L}^2$ were arbitrary, we have $S = \mathbb{N}^* J \mathbb{N}$. $\square$

Now we are ready to establish the equivalence between J-critical control and the eIARE, in (c)–(e2) below:

Proposition 9.10.2 *Let Σ, $\Sigma_{\circlearrowleft}$, $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$, J and $\mathcal{P}$ be as in Lemma 9.10.1, and let $\mathbb{C}_{\circlearrowleft}$ be stable. Then the following hold:*

(a1) Conditions (i)–(iv) are equivalent, where

(i) $\langle \mathbb{D}_{\circlearrowleft} \pi_+ u, J \mathbb{C}_{\circlearrowleft} x_0 \rangle = 0$ for all $u \in \mathrm{L}^2_{\mathrm{c}}$ and $x_0 \in H$, and $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$;

(i') $\mathbb{D}^{t\,*} J \mathbb{C}^t_{\circlearrowleft} + \mathbb{B}^{t\,*} \mathcal{P} \mathbb{A}^t_{\circlearrowleft} = 0$ for all $t > 0$ and $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$;

(ii) Equations (9.159)–(9.161) (the eIARE) are satisfied for some $S \in \mathcal{B}(U)$ and some $T := t > 0$, and some of (P1)–(P4) holds for $t \in R := T\mathbf{N}$;

(iii) Equations (9.153)–(9.163) are satisfied for a unique $S = S^* \in \mathcal{B}(U)$ and all $t \geq 0$, and all of (P1)–(P4) hold (for $t \in R := \mathbf{R}_+$);

(iv) We have $\langle y, Jy \rangle = \langle x_0, \mathcal{P} x_0 \rangle + \langle u_{\circlearrowleft}, S u_{\circlearrowleft} \rangle$ for all $x_0 \in H$, $u_{\circlearrowleft} \in \mathrm{L}^2_{\mathrm{c}}$, where $y := \mathbb{C} x_0 + \mathbb{D} u \in \mathrm{L}^2$, $u := \mathbb{K}_{\circlearrowleft} x_0 + \mathbb{M} u_{\circlearrowleft}$.

(Note that $y = \mathbb{C}_{\circlearrowleft} x_0 + \mathbb{D}_{\circlearrowleft} u_{\circlearrowleft} \in \mathrm{L}^2$ in (iv), by Lemma 6.1.11.)

(a2) Let $\Sigma_{\mathrm{ext}}, \Sigma_{\circlearrowleft} \in \mathrm{SOS}$. Then we can replace (P1)–(P4) in (ii) by

(P') $\langle \mathbb{A}^t x_0, \mathcal{P} \mathbb{A}^t x_0 \rangle \to 0$, as $t \to \infty$, for all $x_0 \in H$.

(b1) Assume that $\mathbb{K}_{\circlearrowleft} x_0 \in \mathcal{U}^*_*(x_0)$ for all $x_0 \in H$. Then $\mathbb{M}\mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+; U) \subset \mathcal{U}^*_*(0)$.

(b2) Let $\mathbb{D}_{\circlearrowleft}$ be stable. Then (i) holds iff $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ and $\pi_+ \mathbb{D}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft} = 0$.

(b3) Assume (i). Let $u \in \mathrm{L}^2_\infty(\mathbf{R}_+; U)$ be s.t. $\mathbb{D} u \in \mathrm{L}^2$. Then $\langle \mathbb{B} \tau^t u, \mathcal{P} \mathbb{A}^t_{\circlearrowleft} x_0 \rangle \to -\langle \mathbb{D} u, J \mathbb{C}_{\circlearrowleft} x_0 \rangle$, as $t \to +\infty$, for each $x_0 \in H$.

If, in addition, $\langle \mathbb{D} u, J \mathbb{C}_{\circlearrowleft} x_0 \rangle = 0$ for each $x_0 \in H$, then (9.175) holds.

(b4) If $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ is J-critical, then, for all $u \in \mathcal{U}^*_*(0)$, we have

$$\langle \mathbb{D} u, J \mathbb{D} \mathbb{M} \eta \rangle = \langle \mathbb{M}^{-1} u, S \eta \rangle \quad (\eta \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}; U)). \tag{9.175}$$

*(c) **($\mathcal{U}^*_*$)** The pair $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ is J-critical over $\mathcal{U}^*_*$ and $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ iff (i) holds, $\mathbb{K}_{\circlearrowleft} x_0 \in \mathcal{U}^*_*(x_0)$ for all $x_0 \in H$, and*

$$\langle \mathbb{B}^t u, \mathcal{P} \mathbb{A}^t_{\circlearrowleft} x_0 \rangle \to 0, \quad \text{as } t \to +\infty, \quad (x_0 \in H,\ u \in \mathcal{U}^*_*(0)). \tag{9.176}$$

*(d) **($\mathcal{U}_{\mathrm{exp}}$)** Let $\Sigma_{\circlearrowleft}$ be exponentially stable. Then $\mathcal{U}_{\mathrm{exp}}(0) = \mathbb{M}\mathrm{L}^2(\mathbf{R}_+; U)$. Moreover, (i) holds iff $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ is J-critical over $\mathcal{U}_{\mathrm{exp}}$ and $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$.*

*(e1) **($\mathcal{U}_{\mathrm{out}}$)** The pair $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ is J-critical over $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ iff $\mathbb{K}_{\circlearrowleft}$ is stable and (i) and (9.176) hold.*

(e2) Let $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ be q.r.c.-SOS-stabilizing. Then $\mathcal{U}_{\mathrm{out}}(0) = \mathbb{M}\mathrm{L}^2(\mathbf{R}_+; U)$. Moreover, $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ is J-critical over $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ iff (i) holds.

(f1) **($\boldsymbol{\mathcal{U}}_{\mathbf{str}}$, $\boldsymbol{\mathcal{U}}_{\mathbf{sta}}$)** *Assume that $\mathcal{U}_*^* = \mathcal{U}_{\rm sta}$ (resp. $\mathcal{U}_*^* = \mathcal{U}_{\rm str}$). Then $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ is J-critical over $\mathcal{U}_*^*$ and $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ iff $\left[\ \mathbb{A}_{\circlearrowleft}^{\rm T} \ \middle|\ \mathbb{C}_{\circlearrowleft}^{\rm T} \quad \mathbb{K}_{\circlearrowleft}^{\rm T}\ \right]^{\rm T}$ is (resp. strongly) stable and (i) and (9.176) hold.*

(f2) Assume that $\mathcal{U}_*^* = \mathcal{U}_{\rm sta}$ [or $\mathcal{U}_*^* = \mathcal{U}_{\rm str}$] and that $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ is [strongly] q.r.c.-stabilizing. Then $\mathcal{U}_*^*(0) = \mathbb{M}\mathrm{L}^2(\mathbf{R}_+;U)$. Moreover, $\mathbb{K}_{\circlearrowleft}$ is J-critical over $\mathcal{U}_*^*$ and $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ iff (i) holds.

(g) Let $\left[\ \mathbb{K} \ \middle|\ \mathbb{F}\ \right]$ be J-critical over $\mathcal{U}_*^*$ and $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$. Then $\mathbb{K}_{\circlearrowleft} x_0 + \mathbb{M} u_{\circlearrowleft}$ is J-critical over $\mathcal{U}_*^*$ for x_0 whenever $x_0 \in H$, $u_{\circlearrowleft} \in \mathrm{L}^2_{\rm c}$ and $S u_{\circlearrowleft} \equiv 0$.

(h) We may restrict t to $R := T\mathbf{N}$ or any other unbounded $R \subset \mathbf{N}$ in (9.176), (P') and (P1)–(P4).

By (e2), a J-critical control over $\mathcal{U}_{\rm out}$ can be given in the form of q.r.c.-SOS-stabilizing state feedback iff the eIARE has a q.r.c.-SOS-stabilizing solution. In either case, we obtain $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ from (i). This is written out in Theorem 9.9.10.

By (a1), the solutions of (i) correspond one-to-one to the P-$\mathbb{C}$-stabilizing solutions of the eIARE. (For other $\mathbb{C}$-stabilizing solutions of the eIARE, the corresponding cost $\mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ is finite but $\mathcal{P}$ contains some "phantom cost", hence such solutions are not interesting; cf. Example 9.13.9.)

Proof: (a1) (Here "(b5)", "(c3)", "(c4)", "(d1)" and "(P1)" refer to Lemma 9.10.1.)

"(iii)⇒(ii)": This is trivial. "(i)⇒(i')": This follows from (c3) and (a1). "(i)⇒(iii)": This follows from (c4) with $S := \widetilde{S}$.

"(ii)⇒(i)": By (b5), the equations hold for nt ($n \in \mathbf{N}$); by (d1), equation $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ holds; by (c3) (applied to nt, $n \in \mathbf{N}$), (i) holds.

"(iii)⇒(iv)": This follows from (P1), (9.163) and (9.162).

"(iv)⇒(i)": This is obvious (cf. the proof of Lemma 8.3.7).

(a2) 1° Assume (iii). Then $\mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft} = \mathbb{C}^* J \mathbb{C} - \mathbb{K}^* S \mathbb{K}$, by a direct computation using equations $\mathbb{C}_{\circlearrowleft} = \mathbb{C} + \mathbb{N}\mathbb{K}$, $\mathbb{N}^* J \mathbb{N} = S$ and $\pi_+ \mathbb{N}^* J \mathbb{C}_{\circlearrowleft} = 0$, hence (P') holds, by Lemma 9.9.1(d2).

2° Assume (ii) with (P') in place of (P1)–(P4). By Lemma 9.9.1(d2), we have $\mathcal{P} = \mathbb{C}^* J \mathbb{C} - \mathbb{K}^* S \mathbb{K}$ and $\mathcal{P}\mathbb{A}^{nT} x_0 \to 0$ for each $x_0 \in H$. By the former,

$$\mathcal{P}\mathbb{B}\tau\mathbb{K}_{\circlearrowleft} = \mathbb{C}^* \tau J \tau^* \mathbb{C}\mathbb{B}\tau^t \mathbb{K}_{\circlearrowleft} + \mathbb{K}^* \tau S \tau^* \mathbb{K}\mathbb{B}\tau^t \mathbb{K}_{\circlearrowleft}. \tag{9.177}$$

Fix $x_0 \in H$. Now $\tau^{-nT}\mathbb{C}\mathbb{B}\tau^{nT}\mathbb{K}_{\circlearrowleft} x_0 = \pi_{[nT,+\infty)}\mathbb{D}\pi_{(-\infty,nT)}\mathbb{K}_{\circlearrowleft} x_0 \to 0$, because $\pi_{[nT,+\infty)}\mathbb{D}\mathbb{K}_{\circlearrowleft} x_0 \to 0$ and $\pi_{[nT,+\infty)}\mathbb{K}_{\circlearrowleft} x_0 \to 0$, as $n \to \infty$. Analogously, $\tau^{-nT}\mathbb{K}\mathbb{B}\tau^{nT}\mathbb{K}_{\circlearrowleft} x_0 \to 0$, hence

$$\mathcal{P}\mathbb{A}_{\circlearrowleft}^{nT} x_0 = \mathcal{P}\mathbb{A}^{nT} x_0 + \mathcal{P}\mathbb{B}\tau^{nT}\mathbb{K}_{\circlearrowleft} x_0 \to 0. \tag{9.178}$$

Therefore, (ii) holds.

(b1) Let $t > 0$ and $\eta_{\circlearrowleft} \in \mathrm{L}^2([0,t);U)$. Set $\widetilde{\eta} := \mathbb{M}^t \eta_{\circlearrowleft}$ and define $\eta \in \mathcal{U}_*^*(0)$ as in Lemma 9.7.10. Then

$$\eta = \mathbb{M}^t \eta_{\circlearrowleft} + \tau^{-t}\mathbb{K}_{\circlearrowleft}\mathbb{B}_{\circlearrowleft}\tau^t \eta_{\circlearrowleft} = \tau^{-t}(\pi_-\mathbb{M}\pi_- + \pi_+\mathbb{M}\pi_-)\tau^t \eta_{\circlearrowleft} = \mathbb{M}\eta_{\circlearrowleft}. \tag{9.179}$$

(b2) This follows from the fact that now $\mathbb{D}_{\circlearrowleft}\pi_{[0,t)} u \to \mathbb{D}_{\circlearrowleft}\pi_+ u$, as $t \to +\infty$, by Corollary B.3.8.

(b3) Choose $\omega \in \mathbf{R}$ s.t. $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$. Since (b4) contains no reference to $\mathcal{U}_*^*$, we can (and will) assume that $\mathcal{U}_*^* := \mathcal{U}^\omega_{[0\ 0]}$, so that $u \in \mathcal{U}_*^*(0)$.

1° The convergence claim follows from (iii) and (9.62).

2° For the latter claim, choose $t > 0$ s.t. $\eta = \pi_{(-\infty,t)}\eta$, and set $u_\circlearrowleft := \mathbb{M}^{-1}u$, so that $u_1 := \mathbb{M}\pi_{[0,t)}u_\circlearrowleft \in \mathbb{M}\pi_+\mathrm{L}^2_\mathrm{c}$, hence $\mathbb{D}u_1 \in \mathrm{L}^2$, i.e., $u_1 \in \mathcal{U}_*^*(0)$. Consequently, $u_2 := \mathbb{M}\pi_{[t,\infty)}u_\circlearrowleft = u - \mathbb{M}\pi_{[0,t)}u_\circlearrowleft \in \mathcal{U}_*^*(0)$, hence $\tau^t u_2 \in \mathcal{U}_*^*(0)$ too (because $\pi_{[0,t)}u_2 = 0$). Consequently,

$$\begin{aligned}\langle \mathbb{D}u, J\mathbb{D}\mathbb{M}\eta\rangle &= \langle \mathbb{D}\mathbb{M}\pi_{[0,t)}u_\circlearrowleft, J\mathbb{N}\eta\rangle + \langle \pi_{[t,\infty)}\mathbb{D}\mathbb{M}\pi_{[t,\infty)}u_\circlearrowleft, J\mathbb{N}\eta\rangle \\ &= \langle \pi_{[0,t)}u_\circlearrowleft, S\eta\rangle + 0 = \langle \mathbb{M}^{-1}u, S\eta\rangle,\end{aligned} \tag{9.180}$$

(note that (9.162) holds for all $u, v \in \mathrm{L}^2_\mathrm{c}$, by time-invariance) because $\langle \pi_{[0,t)}u_\circlearrowleft, S\eta\rangle = \langle u_\circlearrowleft, S\eta\rangle$ and

$$\pi_{[t,\infty)}J\mathbb{N}\eta = \tau^{-t}J\pi_+\mathbb{N}\tau^t\eta = \tau^{-t}J\mathbb{C}_\circlearrowleft\mathbb{B}_\circlearrowleft\tau^t\eta, \tag{9.181}$$

hence $\langle \mathbb{D}u_2, \pi_{[t,\infty)}J\mathbb{N}\eta\rangle = \langle \mathbb{D}\tau^t u_2, J\mathbb{C}_\circlearrowleft\mathbb{B}_\circlearrowleft\tau^t\eta\rangle = 0$, by the assumption.

(b4) This follows from (b3). (Note that in (9.175), the (inner product) integral can be taken over over a finite interval only, hence we have allowed $\mathbb{M}^{-1}u \notin \mathrm{L}^2$.)

(c) Since (i) is equivalent to (i'), we obtain from (b3) that (9.176) holds iff $\langle \mathbb{D}u, J\mathbb{C}_\circlearrowleft x_0\rangle = 0$ for all $x_0 \in H$ and all $u \in \mathcal{U}_*^*(0)$. Therefore, (c) holds.

(d) By (8.74), we have $\mathcal{U}_{\exp}(0) = \mathbb{M}\mathrm{L}^2(\mathbf{R}_+;U)$. Therefore, the equivalence follows from (b2).

(e1)&(f1) These follows from (c).

(e2) Let (ii) hold. Then $\mathcal{U}_{\mathrm{out}}(0) = \mathbb{M}\pi_+\mathrm{L}^2$, because $u, \mathbb{D}u \in \mathrm{L}^2 \Leftrightarrow \mathbb{M}^{-1}u \in \mathrm{L}^2$, by Lemma 6.5.6(a1)&(f). Therefore, the equivalence follows from (b2).

(f2) The proof is analogous to that of (e2).

(g) Let $x_0 \in H$, $u_\circlearrowleft \in \mathrm{L}^2_\mathrm{c}$ and $Su_\circlearrowleft \equiv 0$. Set $\widetilde{u} := \mathbb{K}_\circlearrowleft x_0 + \mathbb{M}u_\circlearrowleft$. Then $\langle \mathbb{D}u, J(\mathbb{C}x_0 + \mathbb{D}\widetilde{u})\rangle = 0 + \langle \mathbb{D}u, J\mathbb{D}\mathbb{M}u_\circlearrowleft\rangle = 0$, by (9.175), for all $u \in \mathcal{U}_*^*(0)$. Therefore, $\widetilde{u}$ is J-critical over $\mathcal{U}_*^*$ for x_0.

(h) One observes this from above proofs. □

We have already shown that J-coercivity implies the existence of a unique J-critical control (if the system is stabilizable); here we show that it also implies that the signature operator is invertible:

Lemma 9.10.3 (J-coercive $\Rightarrow S \in \mathcal{GB}(U)$) *Assume that $\mathbb{D}$ is [positively] J-coercive over $\mathcal{U}_*^*$. If $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}\end{array}\right])$ is a $\mathcal{U}_*^*$-stabilizing solution of the eIARE, then $S \in \mathcal{GB}(U)$ [$S \gg 0$].*

See also Lemma 9.9.7(c4)&(c5).

Proof: Choose $\beta \geq \max\{0, \vartheta\}$ s.t. $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\beta$. Set $\varepsilon' := \|\mathbb{M}^{-1}\|^{-1}_{\mathrm{TIC}_\beta}\varepsilon$, $\varepsilon'' := \varepsilon' \int_0^1 e^{-2\beta s}\,ds$. Let ε be as in Definition 8.4.1. Choose $M := \|\pi_{[0,1)}\mathbb{M}^{-1}\pi_{[0,1)}\|_{\mathcal{B}(\mathrm{L}^2)}$.

Let $u_0 \in U$ be given. Set $u_\circlearrowleft := \chi_{[0,1)}u_0$, so that $u := \mathbb{M}u_\circlearrowleft \in \mathcal{U}_*^*(0)$, by Proposition 9.10.2(c)&(b1). Choose $v \in \mathcal{U}_*^*(0)$ s.t. $\|v\|_{\mathcal{U}_*^*} \leq 1$ and $\langle \mathbb{D}v, J\mathbb{D}u\rangle \geq$

$\varepsilon\|u\|_{\mathcal{U}_*^*}$. It follows that $\|v\|_2 \le 1$. The function $\|u\|_{\mathrm{L}^2_\omega}^2 = \int_{\mathbf{R}} \mathrm{e}^{-2\omega t}\|u(t)\|_U^2\,dt$ is decreasing in ω; therefore, by (9.175), we have

$$M\|Su_0\|_U \ge \|\pi_{[0,1)}\mathbb{M}^{-1}\pi_{[0,1)}\|\|v\|_2\|Su_0\|_U \ge |\langle \mathbb{M}^{-1}v, Su_{\circlearrowleft}\rangle| \tag{9.182}$$

$$= |\langle \mathbb{D}v, J\mathbb{D}u\rangle| \ge \varepsilon\|u\|_{\mathcal{U}_*^*} \ge \varepsilon\|u\|_{\mathrm{L}^2_\beta} \ge \varepsilon'\|u_{\circlearrowleft}\|_{\mathrm{L}^2_\beta} = \varepsilon''\|u_0\|_U. \tag{9.183}$$

Because $u_0 \in U$ was arbitrary, S is coercive, hence $S \in \mathcal{GB}(U)$, by Lemma A.3.1(c4) [and necessarily $S \ge 0$, hence $S \gg 0$, Lemma A.3.1(b1)]. $\square$

(See the notes on p. 520.)

9.11 Proofs for Section 9.8: eCARE↔eIARE

If I had only known, I would have been a locksmith.
— Albert Einstein (1879–1955)

Having established the connection between optimal control and a $\mathcal{U}_*^*$-stabilizing solution of the eIARE in the previous section, we now go on to show the eIARE equivalent to the eCARE in the regular case. We start with a technical result:

Lemma 9.11.1 *Let $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix})$ be a solution of the eIARE s.t. $\Sigma_{\text{ext}} \in$ WPLS.*

Let x and y be the state and output of Σ corresponding to initial state $x_0 \in H$ and input $u \in \mathrm{L}^2_\omega(\mathbf{R}_+; U)$. Fix $t \geq 0$, and let x^{d} and u^{d} be the state and output of $\Sigma^{\mathrm{d}}_{\text{ext}}$ corresponding to initial state $x^{\mathrm{d}}_0 := \mathcal{P}x(t)$, and inputs $y^{\mathrm{d}}(s) = Jy(t-s)$, $z^{\mathrm{d}}(s) = S(\mathbb{X}u - \mathbb{K}x_0))(t-s)$ ($s \in [0,t]$). Then, for $s \in [0,t]$, we have $x^{\mathrm{d}}(t-s) = \mathcal{P}x(s)$ and

$$\pi_{[0,t)} u^{\mathrm{d}}(t - \cdot) = -S\pi_{[0,t)} \left(\mathbb{K}x_0 - \mathbb{X}\pi_{[0,t)} u\right). \tag{9.184}$$

Proof: (a) Now $x := \mathbb{A}x_0 + \mathbb{B}\tau u$, $y := \mathbb{C}x_0 + \mathbb{D}u$, $x^{\mathrm{d}}(t-s) := \mathbb{A}(t-s)^* x^{\mathrm{d}}_0 + \mathbb{C}^* \boldsymbol{Я}\tau(t-s) y^{\mathrm{d}} + \mathbb{K}^* \boldsymbol{Я}\tau(t-s) z^{\mathrm{d}}$, and $u^{\mathrm{d}}(t-\cdot) = \mathbb{B}^* x^{\mathrm{d}}_0 + \boldsymbol{Я}\mathbb{D}^* \boldsymbol{Я} y^{\mathrm{d}} + \boldsymbol{Я}\mathbb{F}^* \boldsymbol{Я} z^{\mathrm{d}}$ (see Lemma 6.1.4 and Definition 6.1.5).

Note that $\mathbb{C}^* \boldsymbol{Я} = \mathbb{C}^* \boldsymbol{Я}\pi_-$, $y^{\mathrm{d}}(t-s) = J\pi_+ \boldsymbol{Я}\tau(t) y$ and

$$\boldsymbol{Я}\pi_-\tau(t-s) J\pi_+ \boldsymbol{Я}\tau(t)\pi_+ = \pi_+\tau(-(t-s))\pi_-\tau(t)\pi_+ J = \pi_{[0,t-s)}\tau(s) J. \tag{9.185}$$

Therefore, (recall from Definition 6.1.1, that $\pi_+\tau(s)\mathbb{C} = \mathbb{C}\mathbb{A}(s)$, $\mathbb{B}\tau(t-s)\pi_- = \mathbb{A}(t-s)\mathbb{B}$, and $\pi_+\mathbb{D}\pi_- = \mathbb{C}\mathbb{B}$)

$$\begin{aligned}
x^{\mathrm{d}}(t-s) &= \mathbb{A}(t-s)^* \mathcal{P}\left(\mathbb{A}(t)x_0 + \mathbb{B}\tau(t)u\right) \\
&\quad + \mathbb{C}^* \pi_{[0,t-s)}\tau(s) J\left(\mathbb{C}x_0 + \mathbb{D}u\right) + \mathbb{K}^* \pi_{[0,t-s)}\tau(s) S\left(-\mathbb{K}x_0 + \mathbb{X}u\right) \\
&= \mathbb{A}(t-s)^* \mathcal{P}\mathbb{A}(t-s)\mathbb{A}(s)x_0 + \mathbb{C}^* \pi_{[0,t-s)} J\mathbb{C}\mathbb{A}(s)x_0 - \mathbb{K}^* \pi_{[0,t-s)} S\mathbb{K}\mathbb{A}(s)x_0 \\
&\quad + \left(\mathbb{A}(t-s)^* \mathcal{P}\mathbb{B}\tau(t-s) + \mathbb{C}^* \pi_{[0,t-s)} J\mathbb{D} + \mathbb{K}^* \pi_{[0,t-s)} S\mathbb{X}\right)\left(\pi_{[0,t-s)} + \pi_-\right)\tau(s)u \\
&= \mathcal{P}\mathbb{A}(s)x_0 + 0 + \left(\mathbb{A}(t-s)^* \mathcal{P}\mathbb{B}\tau(t-s) + \mathbb{C}^* \pi_{[0,t-s)} J\mathbb{D} + \mathbb{K}^* \pi_{[0,t-s)} S\mathbb{X}\right)\pi_-\tau(s)u \\
&= \mathcal{P}\mathbb{A}(s)x_0 + \mathcal{P}\mathbb{B}\tau(s)u = \mathcal{P}x(s),
\end{aligned} \tag{9.186}$$

where the last three identities follow from (9.161) (in fact, (9.161)*) and (9.159) (with $\mathbb{B}\tau(t-s)\pi_- = \mathbb{A}(t-s)\mathbb{B}$, $\pi_+\mathbb{D}\pi_- = \mathbb{C}\mathbb{B}$ and $\pi_+\mathbb{X}\pi_- = -\mathbb{K}\mathbb{B}$). Similarly (we omit the details),

$$\begin{aligned}
\pi + \boldsymbol{Я}\tau(t)u^{\mathrm{d}} &= \pi_+\tau(-t)\mathbb{B}^* \mathcal{P}x(t) + \pi_+\tau(-t)\mathbb{D}^* \boldsymbol{Я} J\pi_+ \boldsymbol{Я}\tau(t) y \\
&\quad + \pi_+\tau(-t)\mathbb{F}^* \boldsymbol{Я} S\pi_+ \boldsymbol{Я}\tau(t)(\mathbb{X}u - \mathbb{K}x_0)) \\
&= \pi_{[0,t)} S\mathbb{X}u - S\pi_{[0,t)}\mathbb{K}x_0,
\end{aligned} \tag{9.187}$$

by (9.161) and (9.160), as above. □

In Lemma 9.11.2 and Proposition 9.11.4 we establish the implication eIARE⇒eCARE.

Since the Lyapunov equation does not contain feedthrough operators, it is very handy to move from its differential (instantaneous) form to the integrated one and conversely:

Lemma 9.11.2 (Lyapunov equation) *Let* $S \in \mathcal{B}(U)$ *and* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y\times U)$.

Then $\mathcal{P} \in \mathcal{B}(H)$ *satisfies the* Lyapunov equation $A^*\mathcal{P}+\mathcal{P}A+C^*JC = K^*SK$ *(*$\in \mathcal{B}(H_1,H_{-1}^*)$*) iff (9.159) holds for all* $t \geq 0$. □

(This follows from Lemma 9.7.8 with $\widetilde{J} \mapsto \left[\begin{smallmatrix} -J & 0 \\ 0 & S \end{smallmatrix}\right]$ and $P \mapsto \mathcal{P}$.)

Note that this equation does not require any regularity assumptions unlike the second and third equations of the CARE (which contain explicit feedthrough operators), treated in Proposition 9.11.4.

Now we have obtained the first equation (the Lyapunov equation) of the eCARE. The rest is not as simple. For S, we need different formulae in different occasions. Several such formulae, including the middle equation of the eCARE, can be derived from (9.190) or (9.188), that will be established below.

Proposition 9.11.3 ($X^*SX = D^*JD+\ldots$) *Let the eIARE have a solution* $(\mathcal{P},S,\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$. *Set* $\omega = \max\{0,\omega_A\}$. *Then, for all* $s,z \in \mathbf{C}^+_\omega$ *and* $u_0,v_0 \in U$, *we have*

$$\langle \widehat{\mathbb{X}}(s)u_0, S\widehat{\mathbb{X}}(z)v_0\rangle_U = \langle \widehat{\mathbb{D}}(s)u_0, J\widehat{\mathbb{D}}(z)v_0\rangle_Y + (s+\bar{z})\langle (s-A)^{-1}Bu_0, \mathcal{P}(z-A)^{-1}Bv_0\rangle_Y. \tag{9.188}$$

In particular,

(a) If $\mathbb{D},\mathbb{X} \in \mathrm{WR}$, *then* $X^*SX = \text{w-lim}_{s\to+\infty} D^*JD + B_\mathrm{w}^*\mathcal{P}(s-A)^{-1}B$.

(b) If $\mathbb{D},\mathbb{X} \in \mathrm{SR}$, *then* $X^*SX = D^*JD + \text{s-lim}_{s\to+\infty} 2sB^*(s-A^*)^{-1}\mathcal{P}(s-A)^{-1}B$.

(c) If $\mathbb{D},\mathbb{X},\mathbb{B}\tau \in \mathrm{SVR}$, *then* $D^*JD = X^*SX$.

Note that the conclusions include the convergence of the limits (including the fact that $\mathcal{P}(s-A)^{-1}Bu_0 \in \mathrm{Dom}(B_\mathrm{w}^*)$ for all $u_0 \in U$ in (a), etc.). From (b) we observe that if $\mathbb{D}$ and $\mathbb{X}$ are SR and $\|(s-A)^{-1}B\| \leq s^{-r}$ for some $r > 1/2$ and all real s big enough (this is true for $r = 1/2$, by Theorem 6.2.11(b3)), then $X^*SX = D^*JD$. Thus, the w-lim term can be nonzero only when B is "maximally unbounded". See Remark 9.9.14(b) for some further sufficient assumptions for $X^*SX = D^*JD$.

Proof: (In fact, it suffices that $\mathbb{X}^{t\,*}S\mathbb{X}^t = \mathbb{D}^{t\,*}J\mathbb{D}^t + \mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t$ for all $t > 0$, $\Sigma \in \mathrm{WPLS}(U,H,Y)$, $\mathbb{X} \in \mathrm{TIC}_\infty$, $S,J,\mathcal{P} \in \mathcal{B}$, and that there is $\alpha \geq 0$ s.t. $s,z \in \mathbf{C}^+_\alpha$ and $\mathbb{B}\tau,\mathbb{D},\mathbb{X} \in \mathrm{TIC}_\alpha$.)

Let $u := \mathrm{e}^{st}u_0$, $v := \mathrm{e}^{zt}v_0$, so that $\pi_-u,\pi_-v \in \mathrm{L}^2_\alpha$ for some $\alpha > \omega$, and work as in the proof of Lemma 2.2.4 (note that $\langle \mathbb{B}^t\tau^t u, \mathcal{P}\mathbb{B}^t\tau^t u\rangle_H \to \langle (s-A)^{-1}Bu_0, \mathcal{P}(s-A)^{-1}Bu_0\rangle_H$, by Lemma 6.2.10). Divide the result by $\int_{-\infty}^0 \mathrm{e}^{ts+t\bar{z}}\,dt = (s+\bar{z})^{-1}$ to obtain (9.188).

In the sequel we shall use the facts that $\widehat{\mathbb{B}\tau u_0}(s) = (s-A)^{-1}Bu_0$, by Theorem 6.2.11(b1), and $\mathbb{B}\tau$ is ULR, by Lemma 6.3.16(c).

(a) Let $s \to +\infty$ in (9.188) (recall that "$s \to +\infty$" means "$s \in \mathbf{R}$ and $s \to +\infty$") to obtain

$$\langle Xu_0, S\widehat{\mathbb{X}}(z)v_0\rangle_U = \langle Du_0, J\widehat{\mathbb{D}}(z)v_0\rangle_Y + \langle u_0, B_{\mathrm{w}}^*\mathcal{P}(z-A)^{-1}Bv_0\rangle_Y + \bar{z}\cdot 0 \tag{9.189}$$

(indeed, the limit $\langle Xu_0, S\widehat{\mathbb{X}}(z)v_0\rangle_U - \langle Du_0, J\widehat{\mathbb{D}}(z)v_0\rangle_Y$ exists and u_0 is arbitrary, we have $\mathcal{P}(z-A)^{-1}Bv_0 \in \mathrm{Dom}(B_{\mathrm{w}}^*)$). Then let $z \to +\infty$ to obtain (a).

(b) Substitute $z \mapsto s$ into (9.188), and let $s \to +\infty$ (use Lemma A.3.1(i2)).

(c) Just substitute $z, s \mapsto \beta + iy$ into (9.188), and let $y \to +\infty$. □

When $\mathbb{D}$ and $\mathbb{F}$ are WR, we can derive also the second (at least) and third equations of the eCARE:

Proposition 9.11.4 (WR eIARE⇒eCARE) *Let $\mathbb{D}$ be WR, and let the eIARE have a WR solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$. Then we have the following:*

(a) If $x_0 \in H$, $u_0 \in U$ and $Ax_0 + Bu_0 \in H$, then

$$(B_{\mathrm{w}}^*\mathcal{P} + D^*JC_{\mathrm{w}} + X^*SK_{\mathrm{w}})x_0 = (X^*SX - D^*JD)u_0. \tag{9.190}$$

(b1) For any $u_0 \in U$ we have

$$X^*SXu_0 = D^*JDu_0 + \operatorname*{w-lim}_{\alpha\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(\alpha - A)^{-1}Bu_0. \tag{9.191}$$

(b2) If $\mathbb{N} \in \mathrm{TIC} \cap \mathrm{SHPR}$ and $\mathbb{X} \in \mathrm{SR}$, then $D^*JD = X^*SX$.

(b3) If $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H,U)$, then $X^*SX = D^*JD$.

(c1) If $\mathbb{D}, \mathbb{F} \in \mathrm{SR}$, then $\mathcal{P} \geq 0 \Rightarrow X^*SX \geq D^*JD$.

(c2) If $\mathbb{F} \in \mathrm{SR}$, then $\mathcal{P}, J \geq 0 \Rightarrow X^*SX \geq D^*JD$.

(d) $X^*SKx_0 = -(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)x_0$ for all $x_0 \in \mathrm{Dom}(A)$.

If $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is admissible and $I - F$ is left-invertible (this is the case if $\mathbb{F}$ is SR, by Proposition 6.3.1(a1)), then we can apply (d1) (and (d2) if $F = 0$) of Proposition 6.6.18.

Proof: (a) In the proof of Lemma 9.8.9, we have, by Lemma 9.11.1, that

$$u^{\mathrm{d}} = \mathbb{B}^{\mathrm{d}}x_0^{\mathrm{d}} + \mathbb{D}^{\mathrm{d}}y^{\mathrm{d}} + \mathbb{F}^{\mathrm{d}}z^{\mathrm{d}} = B_{\mathrm{w}}^*x^{\mathrm{d}} + D^*y^{\mathrm{d}} + F^*z^{\mathrm{d}}, \tag{9.192}$$

hence, by (9.184), Theorem 6.2.13(a2), Lemma 6.2.9(b),

$$S\pi_{[0,t)}z = u^{\mathrm{d}}(t-\cdot) = B_{\mathrm{w}}^*\mathcal{P}x + D^*Jy + F^*Sz, \quad \text{equivalently}, \tag{9.193}$$

$$X^*Sz = B_{\mathrm{w}}^*\mathcal{P}x + D^*Jy \quad (\text{on } [0,t)). \tag{9.194}$$

But z is the output of $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K} & \mathbb{X} \end{array}\right]$, hence $z = -K_{\mathrm{w}}x + Xu$. Thus, by setting the argument equal to zero, (9.194) becomes (9.190) when we note that $y(0) = C_{\mathrm{w}}x_0 + Du_0$.

(b1) Take $x_0 := (\alpha - A)^{-1}Bu_0 \in H_B$ and apply (a) (note that, $\begin{bmatrix} C_{\mathrm{w}} \\ K_{\mathrm{w}} \end{bmatrix}(\alpha - A)^{-1}Bu_0 \to 0$ weakly as $\alpha \to +\infty$, by Theorem 6.2.11(d1)).

(b2) We have $\mathbb{D} = \mathbb{N}\mathbb{X}$, hence $\mathbb{D} \in$ SR and $D = NX$, by Lemma 6.2.5. But $\mathbb{N}^* J\mathbb{N} = S$, by Lemma 9.10.1(c2) and continuity, hence $N^*JN = S$, by Lemma 6.3.6(b), hence $D^*JD = X^*SX$.

(b3) By Lemma A.4.4(d3) (with H_{-1} in place of H), we have $(s-A)^{-1}Bu_0 \to 0$ in H, as $s \to +\infty$, for all $u_0 \in U$, hence (b3) follows from (b1). (Note that now also $(B_{\mathrm{w}}^*\mathcal{P})_{\mathrm{w}} = B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H,U)$.)

(c1) This follows from Proposition 9.11.3(b).

(c2) This follows as in (c1), because the term $\langle C_{\mathrm{w}}(s-A)^{-1}Bu_0, JDu_0\rangle$ and its adjoint converge to zero, the term $\langle Du_0, JDu_0\rangle$ is constant, and $\langle C_{\mathrm{w}}(s-A)^{-1}Bu_0, JC_{\mathrm{w}}(s-A)^{-1}Bu_0\rangle \geq 0$.

(d) Take $u_0 = 0$ in (a). □

Having shown the implication eIARE⇒eCARE in Lemma 9.11.2 and in (b1) and (d) of Proposition 9.11.4, we now turn our attention to the converse direction eCARE⇒eIARE. We start by some technical implications of the eCARE:

Lemma 9.11.5 (eCARE⇒) *Let $\mathbb{D}$ be WR, and let $(\mathcal{P}, S, \begin{bmatrix} K & \big| & I-X \end{bmatrix})$ be a solution of the eCARE (in fact, equation $K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC$ need not hold). Then $\mathcal{P} \in \mathcal{B}(H_B, \mathrm{Dom}(B_{\mathrm{w}}^*))$ and*

(a) $(B_{\mathrm{w}}^*\mathcal{P} + D^*JC_{\mathrm{w}} + (X^*SK)_{\mathrm{w}})x_0 = (X^*SX - D^*JD)u_0 = \text{w-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}Bu_0$ whenever $Ax_0 + Bu_0 \in H$.

If, in addition, $X, S \in \mathcal{GB}(U)$, then we have the following:

(b) $K_{\mathrm{w}}, B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H_B, U) \subset \mathcal{B}(H_1, U)$.

(c) $(B_{\mathrm{w}}^*\mathcal{P})_{\mathrm{w}}, B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H_B, U)$, but $(B_{\mathrm{w}}^*\mathcal{P})_{\mathrm{w}} = B_{\mathrm{w}}^*\mathcal{P} - \text{w-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}Bu_0$ although $(B_{\mathrm{w}}^*\mathcal{P})_{\mathrm{w}} = B_{\mathrm{w}}^*\mathcal{P}$ on H_1.

(d) If $\left[\begin{array}{c|c} A & B \\ \hline C & \\ K & \end{array}\right]$ generate a WPLS Σ_{ext}, then Σ_{ext} is WR.

(e) Σ_{ext} in (d) is SR [UR] iff Σ is SR [UR] and the weak limit in the CARE exists as strong [uniform] limit too.

As one observes from 2° of the proof, for any WR $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U,H,Y)$, $J = J^* \in \mathcal{B}(U,Y)$, $\mathcal{P} \in \mathcal{B}(H)$, and $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H_1, U)$, we have $\exists (B_{\mathrm{w}}^*\mathcal{P} + D^*JC)_{\mathrm{w}}x_0$ for all $x_0 \in H_B \Leftrightarrow \exists\,\text{w-lim}_{\alpha\to\infty} B_{\mathrm{w}}^*\mathcal{P}(\alpha - A)^{-1}Bu_0$ for all $u_0 \in U$. Thus, assuming $X, S \in \mathcal{GB}(U)$, the weak regularity of K (i.e., that of $\mathbb{F}$ and $\mathbb{X}$) is contained in (equivalent to) the assumption on convergence of the second equation of eCARE!

Proof: (The first equation of the eCARE is not used in this proof.)

(a) (We prove here also a part of (b).)

1° The inclusion $\mathcal{P}[H_B] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ was noted in Remark 9.1.6. By Lemma A.3.6, it follows that $\mathcal{P} \in \mathcal{B}(H_B, \mathrm{Dom}(B_{\mathrm{w}}^*))$, hence $B_{\mathrm{w}}^*\mathcal{P} \in \mathcal{B}(H_B, U) \subset \mathcal{B}(H_1, U)$.

2° "$H_B \subset \mathrm{Dom}((X^*SK)_{\mathrm{w}})$" & (a): Let $x_0 \in H_B$ and $u_0 \in U$ be s.t. $z_0 := Ax_0 + Bu_0 \in H$ (see Definition 6.1.17). Set $x_s := s(s-A)^{-1}x_0 \in H_1$ so that $(X^*SK)_{\mathrm{w}}x_0 := \text{w-lim}_{s\to+\infty}(X^*SK)x_s$, by Proposition 6.2.8(a1), once the convergence of this limit is established; this will be done below: using the eCARE, we get

$$\begin{aligned} X^*SKx_s &= -D^*JCx_s - B_{\rm w}^*\mathcal{P}x_s \\ &\rightharpoonup -D^*JC_{\rm w}x_0 - B_{\rm w}^*\mathcal{P}x_0 + \operatorname*{w-lim}_{s\to+\infty} B_{\rm w}^*\mathcal{P}(s-A)^{-1}Bu_0, \end{aligned}$$

since

$$\begin{aligned} B_{\rm w}^*\mathcal{P}x_s &= B_{\rm w}^*\mathcal{P}s(s-A)^{-1}x_0 = B_{\rm w}^*\mathcal{P}(I+A(s-A)^{-1})x_0 \\ &= B_{\rm w}^*\mathcal{P}x_0 + B_{\rm w}^*\mathcal{P}(s-A)^{-1}(z_0 - Bu_0) \\ &\rightharpoonup B_{\rm w}^*\mathcal{P}x_0 + 0 - \operatorname*{w-lim}_{s\to+\infty} B_{\rm w}^*\mathcal{P}(s-A)^{-1}Bu_0, \end{aligned}$$

because $(s-A)^{-1}z_0 \underset{H_1}{\to} 0$, by Lemma A.4.4(d3), and $B_{\rm w}^*\mathcal{P} \in \mathcal{B}(H_1, U)$.

(b) Since $X^*S \in \mathcal{B}(U)$, we now have $H_B \subset \operatorname{Dom}(K_{\rm w})$, by 2°. By Lemma A.3.6, this implies that $K_{\rm w} \in \mathcal{B}(H_B, U)$.

(c) If $x_0 \in H_B$ and $u_0 \in U$ are s.t. $Ax_0 + Bu_0 \in H$ (see Definition 6.1.17), then the latter limit in 2° shows that $(B_{\rm w}^*\mathcal{P})_{\rm w}x_0 = B_{\rm w}^*\mathcal{P}x_0 - \text{w-lim}_{s\to+\infty} B_{\rm w}^*\mathcal{P}(s-A)^{-1}Bu_0$ (and that this limit exists). Thus, $B_{\rm w}^*\mathcal{P} \in \mathcal{B}(H_B, U)$ (cf. the proof of (b)). Naturally, both $(B_{\rm w}^*\mathcal{P})_{\rm w}$ and $B_{\rm w}^*\mathcal{P}$ are (continuous) extensions of $B_{\rm w}^*\mathcal{P} \in \mathcal{B}(H_1, U)$.

To get an example where $(B_{\rm w}^*\mathcal{P})_{\rm w}x_0 \neq B_{\rm w}^*\mathcal{P}x_0$, apply (9.117) to any $u_0 \neq 0$ (with, e.g., $x_0 := (s-A)^{-1}Bu_0$).

(d) $\Sigma_{\rm ext}$ is WR iff $H_B \subset \operatorname{Dom}(\left[\begin{smallmatrix} C \\ K \end{smallmatrix}\right]_{\rm w}) = \operatorname{Dom}(C_{\rm w}) \cap \operatorname{Dom}(K_{\rm w})$, by Proposition 6.2.8(a1). If $x_0 \in H_B$, then $x_0 \in \operatorname{Dom}(C_{\rm w})$ by the weak regularity of Σ and $x_0 \in \operatorname{Dom}(K_{\rm w})$ by 2°.

(e) Certainly $\mathbb{D} \in$ SR is necessary, hence we assume that $\mathbb{D}$ is SR, and find out when $\mathbb{F}$ is SR too, i.e., when $K_{\rm w}(s-A)^{-1}B \to 0$ strongly, equivalently, when $X^*SK_{\rm w}(s-A)^{-1}B \to 0$ strongly (because $X^*S \in \mathcal{GB}$).

By substituting $x_0 = s(s-A)^{-1}Bu_0$ into (a), we see that

$$X^*SK_{\rm w}(s-A)^{-1}B - B_{\rm w}^*\mathcal{P}(s-A)^{-1}B = -D^*JC_{\rm w}(s-A)^{-1}B + X^*SX - D^*JD. \tag{9.195}$$

Because $\mathbb{D}$ is SR, the right-hand-side converges strongly, as $s \to +\infty$. Therefore, $X^*SK_{\rm w}(s-A)^{-1}B$ converges strongly iff $B_{\rm w}^*\mathcal{P}(s-A)^{-1}B$ converges strongly.

The same proof applies for uniform regularity (and for any other form of regularity), mutatis mutandis. □

Since the second and third formulae of the eCARE contain feedthrough operators and the term $B_{\rm w}^*\mathcal{P}$, we need to be careful when we "integrate" them to obtain the eIARE, unlike in the simple case of the Lyapunov equation (see Lemma 9.11.2). However, we write the proofs in detail, so that the reader should be able to follow the steps. We start with the slightly simpler one, namely the third equation:

Lemma 9.11.6 ($\mathbb{X}^{t*}S\mathbb{K}^t = -\left(\mathbb{D}^{t*}J\mathbb{C}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{A}^t\right)$) *Let Σ be WR, and let the eCARE have a WR solution $(\mathcal{P}, S, \left[\begin{array}{c|c} K & I-F \end{array}\right])$. Then, for all $t \geq 0$,*

$$\mathbb{X}^{t*}S\mathbb{K}^t = -\left(\mathbb{D}^{t*}J\mathbb{C}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{A}^t\right). \tag{9.196}$$

Proof: Set $\mathbb{T}(t) := \mathbb{X}^{t*}S\mathbb{K}^t + \mathbb{D}^{t*}J\mathbb{C}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{A}^t \in \mathcal{B}(H, \mathrm{L}^2([0,t);U))$ for $t \geq 0$. By density (see Theorem B.3.11), it is enough to show that

$$f(t) := f_{u,x_0}(t) := \langle u, \mathbb{T}(t)x_0 \rangle_{\mathrm{L}^2} = 0 \tag{9.197}$$

for arbitrary $t \geq 0$, $u \in C_c^\infty((0,t);U)$ and $x_0 \in H_1$. Let v, x_0 be as above, and set (see Theorem 6.2.13(b))

$$x := \mathbb{A}(\cdot)x_0 \in C^1([0,\infty);H), \quad z := \mathbb{B}^t u \in C^1(\mathbf{R};H) \cap C(\mathbf{R};H_B), \mathbb{X}u \in \mathrm{W}^{1,2}_{\mathrm{loc}} \subset C \tag{9.198}$$

to obtain $x' = Ax$, $z' = Az + Bu$, $x', z' \in C(\mathbf{R};H)$; $(\mathbb{X}u)(t) = u(t) - K_{\mathrm{w}}z(t)$, $(\mathbb{D}u)(t) = Du(t) + C_{\mathrm{w}}z(t)$, $\mathbb{C}x_0 = Cx(t)$, $\mathbb{K}x_0 = Kx(t)$ (see (6.46)). Thus, (to be brief, we drop here the (t)'s after u, x, x', z, z')

$$f(t) = \int_0^t \langle \mathbb{X}u, S\mathbb{K}x_0 \rangle_U \, dt + \int_0^t \langle \mathbb{D}u, J\mathbb{C}x_0 \rangle_Y \, dt + \langle z, \mathcal{P}x \rangle_H. \tag{9.199}$$

Because $f(0) = 0$ (since $z(0) = 0$), it is enough to show that $f'(t) = 0$ for $t \geq 0$, since, by Lemma B.5.4, $f \in C^1([0,\infty))$ and

$$f'(t) = \langle Xu - K_{\mathrm{w}}z, SKx \rangle_U + \langle Du + C_{\mathrm{w}}z, JCx \rangle_Y + \langle z, \mathcal{P}x \rangle_H'. \tag{9.200}$$

Now $\langle z, \mathcal{P}x \rangle_H' = \langle z, \mathcal{P}x' \rangle_H + \langle z', \mathcal{P}x \rangle = \langle z, \mathcal{P}Ax \rangle_H + \langle Az + Bu, \mathcal{P}x \rangle_H$. Set $z_r := r(r-A)^{-1}z \in H_1$ for $r > \omega_A$, so that $\begin{bmatrix} C_{\mathrm{w}} \\ K_{\mathrm{w}} \end{bmatrix} z = \text{w-lim}_{r \to +\infty} \begin{bmatrix} C \\ K \end{bmatrix} z_r$, by Proposition 6.2.8. Then

$$f'(t) = \lim_{r \to +\infty} \big(\langle u, (X^*SK + D^*JC)x \rangle_U + \langle z_r, (C^*JC - K^*JK)x \rangle_H \tag{9.201}$$

$$+ \langle z, \mathcal{P}Ax \rangle_H + \langle Az_r, \mathcal{P}x \rangle + \langle r(r-A)^{-1}Bu, \mathcal{P}x \rangle \big). \tag{9.202}$$

But $C^*JC - K^*JK = -A^*\mathcal{P} - \mathcal{P}A$, by the eCARE, hence

$$f'(t) = \lim_{r \to +\infty} \big(\langle u, (X^*SK + D^*JC)x \rangle_U + \langle u, B^*r(r-A^*)^{-1}\mathcal{P}x \rangle \big) \tag{9.203}$$

$$= \langle u, (X^*SK + D^*JC + B_{\mathrm{w}}^*\mathcal{P})x \rangle_U = 0, \tag{9.204}$$

by the definition of K. □

Now only the hardest part of eCARE⇒eIARE remains:

Lemma 9.11.7 ($\mathbb{X}^{t*}S\mathbb{X}^t = \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$) *Let Σ be WR, and let the eCARE have a WR solution $(\mathcal{P}, S, \left[\begin{array}{c|c} K & I - X \end{array}\right])$. Then, for all $t \geq 0$, we have*

$$\mathbb{X}^{t*}S\mathbb{X}^t = \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t. \tag{9.205}$$

Proof: The proof below requires even more patience than the one above. One might ask whether the proof could be simplified in most special cases, e.g., when $\mathbb{D} \in \mathrm{MTIC}$. However, it seems that this is not the case; in the SR case, some details would be slightly but not essentially simpler, and even for $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$, we do not know any way to avoid the tricks with the non-commuting limits or similar difficulties in the proof.

Fix $t > 0$ (for $t = 0$ (9.205) is trivial). Let $u \in \mathrm{W}^{1,2}(\mathbf{R};U)$, $\pi_- u = 0$, and set $x := \mathbb{B}\tau u$. Then, by Theorem 6.2.13(b1)(ii), we have $x \in C^1(\mathbf{R};H) \cap C(\mathbf{R};H_B)$,

$x' = Ax + Bu = \mathbb{B}\tau u' \in C(\mathbf{R};H)$, $\mathbb{D}u = C_{\mathrm{w}}x + Du$, $\mathbb{X}u = Xu - K_{\mathrm{w}}x$, $x(0) = \mathbb{B}u = \mathbb{B}\pi_- u = 0$. Because C_{w} and K_{w} are continuous on H_B, by Proposition 6.2.8(a1), functions $\mathbb{D}u$ and $\mathbb{X}u$ are continuous, hence so are functions f, g, h, h_1 and h_2, that we will define later. Let $T \in [0,t)$ be arbitrary. Set (we will write u for $u(T)$, x for $x(T)$ and x' for $x'(T)$ except in "$\mathbb{D}u$" and "$\mathbb{X}u$")

$$\begin{aligned} g(T) &:= \langle (\mathbb{D}u)(T), J(\mathbb{D}u)(T)\rangle_Y = \langle Du + C_{\mathrm{w}}x, JDu + JC_{\mathrm{w}}x\rangle_Y \\ &= \langle u, D^*JDu\rangle_U + \langle u, D^*JC_{\mathrm{w}}x\rangle_U + \langle D^*JC_{\mathrm{w}}x, u\rangle_U + \langle C_{\mathrm{w}}x, JC_{\mathrm{w}}x\rangle_Y. \end{aligned}$$

Set also

$$\begin{aligned} f(T) &:= \langle (\mathbb{X}u)(T), S(\mathbb{X}u)(T)\rangle_U && (9.206)\\ &= \langle Xu, SXu\rangle_U - \langle Xu, SK_{\mathrm{w}}x\rangle_U - \langle K_{\mathrm{w}}x, SXu\rangle_U + \langle K_{\mathrm{w}}x, SK_{\mathrm{w}}x\rangle_U. && (9.207) \end{aligned}$$

Now $x(T) \in H_B$ need not belong to $\mathrm{Dom}(A)$, yet the definition of K and the Riccati equation give us information for $x_0 \in \mathrm{Dom}(A)$ only. To overcome this problem, we define $x_s := s(s-A)^{-1}x(T) \in \mathrm{Dom}(A)$ for $s > \omega_A$, getting (we first use Proposition 6.2.8(a1), then the third and first equation of the eCARE with $S = S^*$)

$$\begin{aligned} f(T) &= \lim_{r\to+\infty}\lim_{s\to+\infty} [\langle u, X^*SXu\rangle_U - \langle u, X^*SKx_s\rangle_U - \langle X^*S^*Kx_r, u\rangle_U + \langle Kx_r, SKx_s\rangle_U] \\ &= \lim_{r\to+\infty}\lim_{s\to+\infty} \big[\langle u, X^*SXu\rangle_U + \langle Xu, (B_{\mathrm{w}}^*\mathcal{P} + D^*JC)x_s\rangle_U \\ &\quad + \langle (B_{\mathrm{w}}^*\mathcal{P} + D^*JC)x_r, Xu\rangle_U + \langle Kx_r, SKx_s\rangle_U\big] \\ &= \lim_{r\to+\infty}\lim_{s\to+\infty} [\langle u, D^*JDu\rangle_U + \langle u, (X^*SX - D^*JD)u\rangle_U \\ &\quad + \langle u, (B_{\mathrm{w}}^*\mathcal{P} + D^*JC)x_s\rangle_U + \langle (B_{\mathrm{w}}^*\mathcal{P} + D^*JC)x_r, u\rangle_U \\ &\quad + \langle Ax_r, \mathcal{P}x_s\rangle_H + \langle \mathcal{P}x_r, Ax_s\rangle_H + \langle Cx_r, JCx_s\rangle_Y] = g(T) + h(T), \end{aligned}$$

where $g(T) := \lim_{r\to+\infty}\lim_{s\to+\infty}[\langle u, D^*JDu\rangle_U + \langle u, D^*JCx_s\rangle_U + \langle D^*JCx_r, u\rangle_U + \langle Cx_r, JCx_s\rangle_Y] = \langle (\mathbb{D}u)(T), (J\mathbb{D}u)(T)\rangle_Y$, and

$$\begin{aligned} h(T) &:= \lim_{r\to+\infty}\lim_{s\to+\infty} \big[\langle u, (X^*SX - D^*JD)u\rangle_U + \langle u, B_{\mathrm{w}}^*\mathcal{P}x_s\rangle_U && (9.208)\\ &\quad + \langle Ax_r, \mathcal{P}x_s\rangle_H + \langle B_{\mathrm{w}}^*\mathcal{P}x_r, u\rangle_U + \langle \mathcal{P}x_r, Ax_s\rangle_H\big]. && (9.209) \end{aligned}$$

On the other hand, $h(T) = h_1(T) + h_2(T)$, where[3]

$$\begin{aligned}
h_2(T) :&= \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle B_{\mathrm{w}}^*\mathcal{P}x_r, u\rangle_U + \langle \mathcal{P}x_r, Ax_s\rangle_H\right]\\
&= \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle B^*s(s-A^*)^{-1}\mathcal{P}x_r, u\rangle_U + \langle \mathcal{P}x_r, Ax_s\rangle_H\right]\\
&= \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle \mathcal{P}x_r, \overline{s(s-A)^{-1}}Bu\rangle_H + \langle \mathcal{P}x_r, \overline{s(s-A)^{-1}}Ax\rangle_H\right]\\
&= \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle \mathcal{P}x_r, s(s-A)^{-1}x'\rangle_H\right]\\
&= \langle \mathcal{P}x, x'\rangle_H = \langle x, \mathcal{P}x'\rangle_H
\end{aligned}$$

(the next to last identity is from Lemma A.4.4(d1)), and

$$h_1(T) := \lim_{r\to\infty}\lim_{s\to\infty}\left[\langle u, (X^*SX - D^*JD)u\rangle_U + \langle u, B_{\mathrm{w}}^*\mathcal{P}x_s\rangle_U + \langle Ax_r, \mathcal{P}x_s\rangle_H\right].$$

Now (we use first the eCARE, then Lemma 9.11.5(a))

$$\lim_{s\to\infty}\langle u, B_{\mathrm{w}}^*\mathcal{P}x_s\rangle_U + \langle u, (X^*SX - D^*JD)u\rangle_U \tag{9.210}$$

$$= \lim_{s\to\infty}\langle u, -(D^*JC + XSK)x_s\rangle_U + \langle u, (X^*SX - D^*JD)u\rangle_U \tag{9.211}$$

$$= \langle u, -(D^*JC_{\mathrm{w}} + XSK_{\mathrm{w}})x + (X^*SX - D^*JD)u\rangle_U = \langle u, B_{\mathrm{w}}^*\mathcal{P}x\rangle_U, \tag{9.212}$$

because $\bar{A}x + Bu = x' \in H$, as required in Lemma 9.11.5. Therefore

$$\begin{aligned}
\langle x', \mathcal{P}x\rangle_H &= \lim_{r\to\infty}\langle r(r-A)^{-1}\left[x'\right], \mathcal{P}x\rangle_H\\
&= \lim_{r\to\infty}\langle \overline{r(r-A)^{-1}}\left[Ax + Bu\right], \mathcal{P}x\rangle_H\\
&= \lim_{r\to\infty}\left[\langle \overline{r(r-A)^{-1}}Ax, \mathcal{P}x\rangle_H + \langle \overline{r(r-A)^{-1}}Bu, \mathcal{P}x\rangle_H\right]\\
&= \lim_{r\to\infty}\langle Ax_r, \mathcal{P}x\rangle_H + \langle u, B_{\mathrm{w}}^*\mathcal{P}x\rangle_U = h_1(T).
\end{aligned}$$

(N.B. $\lim_{r\to\infty}\langle Ax_r, \mathcal{P}x\rangle_H$ exists.)[4] Thus, still for an arbitrary $T \in \mathbf{R}$, we have

$$h(T) = h_1(T) + h_2(T) = \langle x', \mathcal{P}x\rangle_H + \langle x, \mathcal{P}x'\rangle_H = \langle x, \mathcal{P}x\rangle_H'(T). \tag{9.213}$$

Because $T \in \mathbf{R}$ was arbitrary, we get (recall that $\mathbb{D}^t := \pi_{[0,t)}\mathbb{D}\pi_{[0,t)}$)

$$\langle \mathbb{X}^t u, S\mathbb{X}^t u\rangle_{\mathrm{L}^2} - \langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle_{\mathrm{L}^2} = \int_0^t f(T)\,dT - \int_0^t g(T)\,dT \tag{9.214}$$

$$= \int_0^t h(T)\,dT = \int_0^t \langle x, \mathcal{P}x\rangle_H'(T)\,dT \tag{9.215}$$

$$= \langle x(t), \mathcal{P}x(t)\rangle - 0 = \langle \mathbb{B}^t u, \mathcal{P}\mathbb{B}^t u\rangle_H. \tag{9.216}$$

[3]To clarify this part of the proof, we use here bars for (unique continuous) extensions, e.g., by $\overline{(s-A)^{-1}} \in \mathcal{B}(H_{-1}, H)$ we denote the extension of $(s-A)^{-1} \in \mathcal{B}(H, H_1)$. One can easily verify that $\overline{(s-A)^{-1}}$ is the inverse of $s - \bar{A}$ and $\langle x, \overline{(s-A)^{-1}}z\rangle_H = \langle (s-A^*)^{-1}x, z\rangle_{\langle H_1^*, H_{-1}\rangle}$ for all $x \in H$, $z \in H_{-1}$. Consequently, $\langle B_{\mathrm{w}}^* x, u\rangle_U = \lim_{s\to+\infty}\langle sx, \overline{(s-A)^{-1}}Bu\rangle_H$ for any $u \in U$, $x \in \mathrm{Dom}(B_{\mathrm{w}}^*)$ (see Definition 6.1.17). (All this holds for any $\Sigma \in$ WPLS.)

[4]Note that the commutator $(\lim_s\lim_r - \lim_r\lim_s)\left(\langle u, B_{\mathrm{w}}^*\mathcal{P}x_s\rangle_U + \langle Ax_r, \mathcal{P}x_s\rangle_H\right)$ of the expression in $h_1(t)$ is equal to the term $\langle u, (X^*SX - D^*JD)u\rangle_U$ (which is frequently zero, cf. Remark 9.9.14). When this commutator is zero, we can compute $h_1(T)$ in the same way as $h_2(T)$.

Since $\mathcal{C}_c^\infty((0,t);U) \subset \mathrm{W}^{1,2}(\mathbf{R};U)$ is continuous in $\mathrm{L}^2([0,t);U)$, we obtain (9.205). □

Any solution of the CARE with $S \gg 0$ is WR:

Proposition 9.11.8 ($S \gg 0 \Rightarrow \mathcal{P}$ is WR) *Let Σ be WR, and let $(\mathcal{P},S,K)$ be a solution of the eCARE. Then we have $\mathbb{X}^{t*}S\mathbb{X}^t = \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$ on $\mathrm{W}_0^{1,2}(\mathbf{R}_+;U) = \{u \in \mathrm{W}^{1,2}(\mathbf{R};U) \,|\, \pi_- u = 0\}$, for all $t \ge 0$.*

If, in addition, $S \gg 0$ and $X = I$, then $\left[\begin{array}{c|c} A & B \\ \hline K & 0 \end{array}\right]$ generate a WR WPLS $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{K} & \mathbb{F} \end{array}\right]$, and $(\mathcal{P},S,\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is a WR solution of the IARE.

(In the first claim, we have set $(\mathbb{X}u)(t) := -K_{\mathrm{w}}x(t) + Xu(t)$ (a.e.), where $x := \mathbb{B}\tau u$. The claim means that $\langle \mathbb{X}^t u, S\mathbb{X}^t u\rangle = \langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle + \langle \mathbb{B}^t u, \mathcal{P}\mathbb{B}^t u\rangle$ for all $(u \in \mathrm{W}_0^{1,2})$.)

Proof: The original proof of Lemma 9.11.7 will do for the claim on $\mathrm{W}_0^{1,2}$ (see Lemma B.7.9 for $\mathrm{W}_0^{1,2}$). Assume then that $S \gg 0$ and $X = I$.

By Lemma 9.12.2(a1), $\left[\begin{smallmatrix} A \\ \hline K \end{smallmatrix}\right]$ generate a WPLS, hence (1.)–(4.) of Lemma 6.3.13 are satisfied by $\left[\begin{array}{c|c} A & B \\ \hline -K_{\mathrm{w}} & I \end{array}\right]$ (with $C_{\mathrm{c}} \mapsto -K_{\mathrm{w}}$ and $D_{\mathrm{c}} \mapsto I$). From equation $\mathbb{X}^{t*}S\mathbb{X}^t = \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$ we deduce that $\|S^{1/2}\mathbb{X}^t u\|_2 \le M\|u\|_2$ for all $u \in \mathcal{C}_c^\infty((0,t);U)$, so that also (5.) of Lemma 6.3.13 holds. Therefore, $\left[\begin{array}{c|c} A & B \\ \hline -K_{\mathrm{w}} & I \end{array}\right]$ generate a WPLS.

By Lemma 9.11.5(b), we have $K_{\mathrm{w}} \in \mathcal{B}(H_B,U)$, hence this WPLS is WR. □

In the SR case, the above proofs can be modified to cover the case where the [e]CARE is replaced by the corresponding inequality:

Proposition 9.11.9 (Riccati inequality) *Assume that Σ is SR and that some $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $S = S^*, X \in \mathcal{B}(U)$ and $K \in \mathcal{B}(H_1,U)$ satisfy*

$$\begin{cases} K^*SK \le A^*\mathcal{P} + \mathcal{P}A + C^*JC & \in \mathcal{B}(\mathrm{Dom}(A),\mathrm{Dom}(A)^*) \\ X^*SX = D^*JD + \operatorname*{s-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B & \in \mathcal{B}(U) \\ X^*SK = -(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) & \in \mathcal{B}(\mathrm{Dom}(A),U). \end{cases} \tag{9.217}$$

(a) We have $\mathbb{K}^{t*}S\mathbb{K}^t \le \mathbb{A}^{t*}\mathcal{P}\mathbb{A}^t - \mathcal{P} + \mathbb{C}^{t*}J\mathbb{C}^t$ on $\mathrm{Dom}(A)$, where $(\mathbb{K}x_0)(t) := K\mathbb{A}^t x_0$ for all $t \ge 0$, $x_0 \in H_1$.

If $S \gg 0$, then $\left[\begin{smallmatrix} A \\ \hline K \end{smallmatrix}\right]$ generate a WPLS and $\mathbb{K}^{t*}S\mathbb{K}^t \le \mathbb{A}^{t*}\mathcal{P}\mathbb{A}^t - \mathcal{P} + \mathbb{C}^{t*}J\mathbb{C}^t$ (on H) for all $t \ge 0$.

(b) If $H_B \subset \mathrm{Dom}(K_{\mathrm{s}})$ (this holds if $X,S \in \mathcal{GB}(U)$), then we have $\mathbb{X}^{t*}S\mathbb{X}^t \le \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$ on $\mathrm{W}_0^{1,2}(\mathbf{R}_+;U)$, for all $t \ge 0$.

(c) If $\left[\begin{array}{c|c} A & B \\ \hline -K & X \end{array}\right]$ generate a SR WPLS $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K} & \mathbb{X} \end{array}\right]$, then we have $\mathbb{X}^{t*}S\mathbb{X}^t \le \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$ for all $t \ge 0$ and $X^*SX \le D^*JD + \operatorname{s-lim}_{s\to+\infty} 2sB^*(s-A^*)^{-1}\mathcal{P}(s-A)^{-1}B$.

(d) If we can have "$K^*SK+\varepsilon I+$" in place of "K^*SK" in (9.217), then we can have $\mathbb{X}^{t\,*}S\mathbb{X}^t+\varepsilon\mathbb{L}^{t\,*}\mathbb{L}^t$ in place of $\mathbb{X}^{t\,*}S\mathbb{X}^t$ in both (b) and (c).

(e) If $S\gg 0$ and $X=I$, then $\left[\begin{array}{c|c} A & B \\ \hline -K & X\end{array}\right]$ generate a SR WPLS $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K} & \mathbb{X}\end{array}\right]$, hence then (c) applies.

(In (a), we have set $(\mathbb{X}u)(t):=-K_{\mathrm{w}}x(t)+Xu(t)$ (a.e.), where $x:=\mathbb{B}\tau u$. In (b), we have set $\mathbb{L}:=\mathbb{B}\tau\in\mathrm{TIC}_\infty(U,H)$; do not mix up $\mathbb{L}^t:=\pi_{[0,t)}\mathbb{B}\tau\pi_{[0,t)}:\mathrm{L}^2_\omega\to\mathrm{L}^2_\omega$ with $\mathbb{B}^t:=\mathbb{B}\tau^t\pi_+:\mathrm{L}^2_\omega\to H$.)

Thus, for the Riccati inequality (9.217), the first and the third equation of the eIARE become inequalities. However, the third equation seems to be lost in the above case, due to its asymmetry (cf. the proof of Lemma 9.11.6).

By (d), in the case of "$\ll$", we have $\mathbb{X}^{t\,*}S\mathbb{X}^t+\varepsilon\mathbb{L}^{t\,*}\mathbb{L}^t\le\mathbb{D}^{t\,*}J\mathbb{D}^t+\mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t$ $(t\ge 0)$ for some $\varepsilon>0$. To treat the opposite signs, multiply $S,J,\mathcal{P}$ by -1.

See Lemma 9.12.2 for analogous results.

Proof of Proposition 9.11.9: (a) This is contained in Lemma 9.12.2(a1) and (9.220).

(b) 1° We observe from 2° of the proof of Lemma 9.11.5(a) that if $X,S\in\mathcal{GB}(U)$, then $X^*SK_{\mathrm{s}}\in\mathcal{B}(H_B,U)$, hence then $K_{\mathrm{s}}\in\mathcal{B}(H_B,U)$.

2° Set $R:=A^*\mathcal{P}+\mathcal{P}A+C^*JC-K^*SK\in\mathcal{B}(H_1,H_{-1}^*)$, so that $\langle x_0,Rx_0\rangle\ge 0$ for all $x_0\in H_1$. Since $Cx_r\to C_{\mathrm{w}}x$ strongly and $Kx_r\to K_{\mathrm{w}}x$ strongly, as $r\to+\infty$, we may replace "$\lim_{r\to+\infty}\lim_{s\to+\infty}$" by "$\lim_{r=s\to+\infty}$" in the proof of the lemma (by Lemma A.3.1(i2)). (This is the reason for the explicit and implicit "strong regularity assumptions on $\mathbb{D}$ and $\mathbb{X}$").

Then "$f(T)=g(T)+h(T)$" must be replaced by "$f(T)=g(T)+h(T)-k(T)$", where $k(T):=\lim_{r\to+\infty}\langle x_r,Rx_r\rangle\ge 0$ (the limit exists since $g(T)-h(T)-f(T)$ converges, as shown in the proof). Consequently, we have obtained (9.215) with "$\le$" in place of "$=$".

(c) The first claim follows from (b), by density (see Theorem B.3.11(b1)). The second claim follows as in the proof of Proposition 9.11.3(b).

(d) Now, in the proof of (b), there is $\varepsilon>0$ s.t. $\langle x_0,Rx_0\rangle-\varepsilon\langle x_0,x_0\rangle\ge 0$ for all $x_0\in H_1$. Set $\widetilde{C}:=\left[\begin{smallmatrix} C \\ I\end{smallmatrix}\right]$, $\widetilde{D}:=\left[\begin{smallmatrix} D \\ 0\end{smallmatrix}\right]$, $\widetilde{J}:=\left[\begin{smallmatrix} J & 0 \\ 0 & -\varepsilon I\end{smallmatrix}\right]$ to get the setting of (b) with $\widetilde{\Sigma}:=\left(\begin{array}{c|c} A & B \\ \hline \widetilde{C} & \widetilde{D}\end{array}\right)\in\mathrm{WPLS}(U,H,Y\times U)$, $\widetilde{J}$ and $\widetilde{R}:=R-\varepsilon I\ge 0$ in place of Σ, J and R, respectively. Then $\widetilde{\mathbb{D}}^{t\,*}\widetilde{J}\widetilde{\mathbb{D}}^t=\mathbb{D}^{t\,*}J\mathbb{D}^t-\varepsilon\mathbb{L}^{t\,*}\mathbb{L}^t$, where $\mathbb{L}:=\mathbb{B}\tau\in\mathrm{TIC}_\infty(U,H)$, hence this follows from (b). (Analogously, we have $\mathbb{K}^{t\,*}S\mathbb{K}^t+\varepsilon\mathbb{R}^{t\,*}\mathbb{R}^t\le\mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}^t-\mathcal{P}+\mathbb{C}^{t\,*}J\mathbb{C}^t$ on $\mathrm{Dom}(A)$, where $(\mathbb{R}^tx_0)(t):=\mathbb{A}^tx_0$ $x_0\in H$, $t\ge 0$.)

(e) By (a), $\left[\begin{smallmatrix} A \\ K\end{smallmatrix}\right]$ generate a WPLS, hence (1.)–(4.) of Lemma 6.3.13 are satisfied by $\left[\begin{array}{c|c} A & B \\ \hline -K_{\mathrm{w}} & I\end{array}\right]$ (with $C_{\mathrm{c}}\mapsto -K_{\mathrm{w}}$ and $D_{\mathrm{c}}\mapsto I$).

Also (5.) of Lemma 6.3.13 holds, by (b). Therefore $\left[\begin{array}{c|c} A & B \\ \hline -K_{\mathrm{w}} & I\end{array}\right]$ generate a WPLS. Since $K_{\mathrm{s}}\in\mathcal{B}(H_B,U)$ (by (b)), this WPLS is SR. □

Notes for Sections 9.10 and 9.11

In [S97b], Olof Staffans showed that the existence a spectral factorization leads to an optimal state feedback pair. In [S98b, Section 4], this was applied to the stabilized form of a jointly stabilizable and detectable system. In [S98b, Section 5], the state feedback pair was shown to lead to formulae (9.153)–(9.161). The proofs of ("necessity parts" of the) Lemma 9.10.1(a1)–(b4)&(c2) are essentially from there.

In the same sense, Lemma 9.11.1 equals Corollary 5.7 of [S98b], and also Lemma 9.11.2 and formulae (a), (b1), (c1) and (d) of Proposition 9.11.4 are from [S98b, Sections 6–7]. Most of Proposition 9.11.3 is from a preprint of [SW01a] in the same sense.

In the generality of [S98b], most of the stable case of Lemmas 9.11.2, 9.11.5 and 9.11.7 and Proposition 9.11.4 is contained in [Mik97b] and [Mik98].

9.12 Further eIARE and eCARE results

'My feet hath fate, O king,' he said,
'here over the mountains bleeding led,
and what I sought not I have found,
and love it is hat here me bound.
For fairer than are born to Men
A daughter hast thou, Lúthien.'
— J.R.R. Tolkien (1892–1973), "The Lay of Leithian"

In this section, we shall extend some classical results such as the correspondence between open-loop and closed-loop Riccati equations; we shall also study "pseudospectral factorizations" (something close to a spectral factorization).

However, we start by making a remark on "irregular CAREs":

Remark 9.12.1 ("Compatible CARE") *By applying Lemma 6.3.10(b) instead of Theorem 6.2.13 in the proofs, we see that Proposition 9.11.4(a)&(d) hold for any WPLS provided that we make replacements $(C_{\mathrm{w}}, D) \mapsto (C_{\mathrm{c}}, D_{\mathrm{c}})$, $(B_{\mathrm{w}}^*, D^*) \mapsto (B_{\mathrm{c}}^*, D_{\mathrm{d}}^*)$ etc. We have to remind that compatible pairs are not unique in general (the equations hold for any such pairs).*

In particular, the third equation of the eCARE hold in the compatible case too. The first (Lyapunov) equation of the eCARE holds for any WPLS, by Lemma 9.11.2.

Unfortunately, we have no decent formulae for S in the general case, and thus any attempts to define an "eCARE" that would imply the eIARE seem doomed. Therefore, it seems more adviseable to use Section 9.7 with some admissibility condition in the case where the state feedback pair is not known to be regular.

Note that in the above case, formulae such as $\widehat{\mathbb{X}}(s) = X_{\mathrm{c}} - K_{\mathrm{c}}(s-A)^{-1}B$ hold, by Lemma 6.3.10(a), and that $\begin{bmatrix} \mathbb{K} & \mid & \mathbb{F} \end{bmatrix}$ is uniquely determined by K_{c} and X_{c}.

The output stability of Σ is equivalent the solvability of a CARE:

Lemma 9.12.2 ($\mathbb{A}/\mathbb{C}$ is stable $\Leftrightarrow$ CARE) *Let $\widetilde{C} \in \mathcal{B}(H_1, Z)$, $\widetilde{J} \in \mathcal{B}(Z)$. Define $\widetilde{\mathbb{C}} : H_1 \to C(\mathbf{R}_+; Z)$ by $\widetilde{\mathbb{C}}x_0 := \widetilde{C}\mathbb{A}x_0$ ($x_0 \in H_1$). We say that "$\widetilde{\mathbb{C}}$ is stable" if $\left[\frac{A}{\widetilde{C}}\right]$ generate a WPLS $\left[\frac{\mathbb{A}}{\widetilde{\mathbb{C}}}\right]$ with $\widetilde{\mathbb{C}} \in \mathcal{B}(H, \mathrm{L}^2(\mathbf{R}_+; Z))$. The following hold:*

(a1) Assume that $\widetilde{J} \gg 0$. If some $\mathcal{P} \in \mathcal{B}(H)$ satisfies

$$\widetilde{C}^*\widetilde{J}\widetilde{C} \leq A^*\mathcal{P} + \mathcal{P}A + C^*JC \quad \text{on } \mathrm{Dom}(A), \tag{9.218}$$

then $\left[\frac{A}{\widetilde{C}}\right]$ generate a WPLS.

(a2) Assume that $\widetilde{J} \gg 0$. Then $\widetilde{\mathbb{C}}$ is stable iff there is $\mathcal{P} \in \mathcal{B}(H)$ s.t. $\mathcal{P} \geq 0$ and

$$A^*\mathcal{P} + \mathcal{P}A + \widetilde{C}^*\widetilde{J}\widetilde{C} \leq 0 \quad \text{on } \mathrm{Dom}(A). \tag{9.219}$$

(b) Assume that $\widetilde{\mathbb{C}}$ is stable. Then $\mathcal{P} = \widetilde{\mathbb{C}}^*\widetilde{J}\widetilde{\mathbb{C}}$ satisfies $A^*\mathcal{P} + \mathcal{P}A + \widetilde{C}^*\widetilde{J}\widetilde{C} = 0$, and $\widetilde{\mathcal{P}} \geq \mathcal{P}$ for any $\widetilde{\mathcal{P}} \geq 0$ that solves (9.219).

In particular, if $\widetilde{J} \geq 0$, then $\mathcal{P} = \widetilde{\mathbb{C}}^* \widetilde{J} \widetilde{\mathbb{C}}$ is the smallest nonnegative solution of (9.219).

(c) Assume, that $\mathbb{A}$ is strongly stable and $\widetilde{\mathbb{C}}$ stable. Then $\mathcal{P} = \widetilde{\mathbb{C}}^* \widetilde{J} \widetilde{\mathbb{C}}$ is the unique solution (in $\mathcal{B}(H)$) of $A^*\mathcal{P} + \mathcal{P}A + \widetilde{C}^* \widetilde{J} \widetilde{C} = 0$.

(d) The semigroup $\mathbb{A}$ is exponentially stable iff $A^*\mathcal{P} + \mathcal{P}A \ll 0$ for some nonnegative $\mathcal{P} \in \mathcal{B}(H)$ (and any such $\mathcal{P}$ necessarily satisfies $\mathcal{P} > 0$).

Note that we can take $\widetilde{C} := I =: \widetilde{J}$ in (a2) to check the exponential stability of $\mathbb{A}$, by (d). Naturally, we obtain analogous results for $(A, \widetilde{B})$, by duality (recall that strong stability is then mapped to strong-$*$ stability).

Proof: (a1) Fix $t > 0$. By Lemma 9.7.8(b), (9.218) is equivalent to

$$\widetilde{\mathbb{C}}^{t\,*} \widetilde{J} \widetilde{\mathbb{C}}^t \leq \mathbb{A}^{t\,*} \mathcal{P} \mathbb{A}^t - \mathcal{P} + \mathbb{C}^{t\,*} J \mathbb{C}^t \quad (t \geq 0), \tag{9.220}$$

on $\mathrm{Dom}(A)$. Thus, there is $M < \infty$ s.t. $\|\widetilde{J}^{1/2} \widetilde{\mathbb{C}}^t x_0\|^2_{\mathrm{L}^2([0,t);Z)} \leq M \|x_0\|^2_H$ for all $x_0 \in \mathrm{Dom}(A)$. Since $\widetilde{J}^{1/2} \gg 0$ (by Lemma A.3.1(b4)), it follows from Corollary 6.3.14 that $\begin{bmatrix} A \\ \widetilde{C} \end{bmatrix}$ generate a WPLS.

(a2) 1° Let $\mathcal{P}$ be any nonnegative solution of (9.219), and assume that $\widetilde{J} \gg 0$. By Lemma 9.7.8)(b), inequality (9.219) is equivalent to

$$\mathcal{P} \geq \mathbb{A}^{t\,*} \mathcal{P} \mathbb{A}^t + \widetilde{\mathbb{C}}^{t\,*} \widetilde{J} \widetilde{\mathbb{C}}^t \quad (t \geq 0), \tag{9.221}$$

on $\mathrm{Dom}(A)$, hence $\|\widetilde{J}^{1/2} \pi_{[0,t)} \widetilde{\mathbb{C}} x_0\|_2^2 \leq \langle x_0, \mathcal{P} x_0 \rangle \quad (t \geq 0,\ x_0 \in H_1)$. Consequently, $\widetilde{\mathbb{C}}$ has a unique extension $\widetilde{\mathbb{C}} \in \mathcal{B}(H, \mathrm{L}^2(\mathbf{R}_+; Z))$, hence $\begin{pmatrix} A \\ \widetilde{C} \end{pmatrix}$ is an output-stable WPLS.

2° Assume that $\begin{bmatrix} \mathbb{A} \\ \widetilde{\mathbb{C}} \end{bmatrix} \in$ WPLS, $\widetilde{\mathbb{C}}$ is stable and $\widetilde{J} = \widetilde{J}^* \in \mathcal{B}(Z)$. Let Σ' be the system generated by $\left[\begin{array}{c|c} A & 0 \\ \hline C' & D' \end{array}\right] := \left[\begin{array}{c|c} A & 0 \\ \hline \widetilde{C} & 0 \\ 0 & I \end{array}\right]$. Set $J' := \begin{bmatrix} \widetilde{J} & 0 \\ 0 & I \end{bmatrix}$, so that the corresponding cost function becomes

$$\mathcal{J}'(x_0, u) = \langle \widetilde{\mathbb{C}} x_0, \widetilde{J} \widetilde{\mathbb{C}} x_0 \rangle + \|u\|_2^2, \quad \text{and} \tag{9.222}$$

$$\langle \mathbb{C}' x_0 + \mathbb{D}' u, J' \mathbb{D}' \eta \rangle = \langle u, \eta \rangle_{\mathrm{L}^2} \quad (x_0 \in H,\ u, \eta \in \mathcal{U}'_{\mathrm{out}}(x_0) = \mathrm{L}^2(\mathbf{R}_+; U)). \tag{9.223}$$

Thus, $u_{\mathrm{crit}}(x_0) = 0$ is the unique J'-critical control for each $x_0 \in H$, so that $\mathcal{P} = \widetilde{\mathbb{C}}^* \widetilde{J} \widetilde{\mathbb{C}}$. Since $B = 0$ is bounded and $D'^* J' D' = I$, the operator $\mathcal{P}$ corresponds to the unique $\mathcal{U}'_{\mathrm{out}}$-stabilizing solution of the CARE (or B^*_{w}-CARE) $A^*\mathcal{P} + \mathcal{P}A + \widetilde{C}^* \widetilde{J} \widetilde{C} = 0$, $S = I$, $SK = 0$.

(b) Assume that $\widetilde{\mathbb{C}}$ is stable and that $\mathcal{P} = \widetilde{\mathbb{C}}^* \widetilde{J} \widetilde{\mathbb{C}}$, so that $\mathcal{P}$ solves (9.221), by 2°. By (9.221), any other solution $\widetilde{\mathcal{P}} \geq 0$ satisfies $\widetilde{\mathcal{P}} \geq \widetilde{\mathbb{C}}^* \widetilde{J} \widetilde{\mathbb{C}} + \text{s-lim}_{t \to +\infty} \mathbb{A}^{t\,*} \widetilde{\mathcal{P}} \mathbb{A}^t \geq \widetilde{\mathbb{C}}^* \widetilde{J} \widetilde{\mathbb{C}}$.

If $\widetilde{J} \geq 0$, then $\mathcal{P} \geq 0$, hence then $\mathcal{P}$ is the smallest nonnegative solution of (9.219).

(c) Now $\widetilde{\mathcal{P}} = \widetilde{\mathbb{C}}^* \widetilde{J} \widetilde{\mathbb{C}} + \text{s-lim}_{t \to +\infty} \mathbb{A}^{t\,*} \widetilde{\mathcal{P}} \mathbb{A}^t = \widetilde{\mathbb{C}}^* \widetilde{J} \widetilde{\mathbb{C}}$ for any solution $\mathcal{P} \in \mathcal{B}(H)$, as in (b).

(d) Naturally, the inequality "$A^*\mathcal{P} + \mathcal{P}A \ll 0$" (on $\mathrm{Dom}(A)$) means that there is $\varepsilon > 0$ s.t. $\langle A x_0, \mathcal{P} x_0 \rangle_H + \langle x_0, \mathcal{P} A x_0 \rangle_H \leq -\varepsilon \langle x_0, x_0 \rangle_H$ for all $x_0 \in \mathrm{Dom}(A)$ (cf.

Definition A.3.23; it obviously follows that $\mathrm{Ker}(\mathcal{P}) = \{0\}$, i.e., that $\mathcal{P} > 0$ (even $\mathcal{P} \gg 0$ if A is bounded)).

If $A^*\mathcal{P} + \mathcal{P}A \le -\varepsilon I$ and $\mathcal{P} \ge 0$, then $\widetilde{\mathbb{C}}$ is stable, where $\widetilde{\mathbb{C}}^t x_0 := \mathbb{A}^t x_0$ ($t > 0$, $x_0 \in H$), by (a) (set $\widetilde{C} := I$, $J := \varepsilon I \gg 0$), hence then $\mathbb{A}$ is exponentially stable, by Lemma A.4.5(i)&(ii).

Conversely, if $\mathbb{A}$ is exponentially stable, then $A^*\mathcal{P} + \mathcal{P}A + I^*II \le 0$ on $\mathrm{Dom}(A)$ for some $\mathcal{P} \ge 0$, by (a2) (and we can have $\mathcal{P} = \widetilde{\mathbb{C}}^*\widetilde{\mathbb{C}} > 0$, where $\widetilde{\mathbb{C}}$ is as above). $\square$

We now adopt the notation $\mathcal{P} \in \mathrm{eIARE}(\Sigma, J)$ (or $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]) \in \overline{\mathrm{eIARE}}(\Sigma, J)$) for the solutions of the eIARE for Σ and J. In the lemma below, we show how the solutions of a perturbed system correspond to the solutions for the original one:

Lemma 9.12.3 *Let $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ be admissible for Σ with closed-loop system $\Sigma_\flat$. Set $\mathbb{M}' := (I - \mathbb{F}')^{-1}$. Then*

$$(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]) \in \overline{\mathrm{eIARE}}(\Sigma, J) \Leftrightarrow \tag{9.224}$$

$$(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} - \mathbb{X}\mathbb{K}_\flat & I - \mathbb{X}\mathbb{M}' \end{array}\right]) \in \overline{\mathrm{eIARE}}(\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right], J). \tag{9.225}$$

Moreover,

(a) If $\mathbb{F}'$ and $\mathbb{X}\mathbb{M}'$ (resp. and $\mathbb{F}$) are as above and have any strong or uniform regularity property, then so does $\mathbb{F}$ (resp. $\mathbb{X}\mathbb{M}'$).

(b) The two top rows ($\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \end{array}\right]$) of the corresponding closed-loop systems are equal (hence (P) is satisfied for Σ iff it is satisfied for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$) and Lemma 6.7.11(a1)–(a6) apply for the three pairs.

(c) Assume that $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ is [q.]r.c.-SOS-stabilizing and $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$.

Then $\mathcal{P}$ is [q.]r.c.-SOS-stabilizing for Σ iff $\mathcal{P}$ is q.r.c.-SOS-stabilizing (equivalently, stable and [[r.c.-]]SOS-stabilizing) for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$.

(d1) If $\mathcal{U}_*^* = \mathcal{U}_{\exp}$, then the J-critical state feedback pairs (equivalently, exponentially stabilizing solutions of the eIARE) for Σ and $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ correspond to each other through (9.224).

(d2) Assume that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{sta}}$ [$\mathcal{U}_{\mathrm{str}}$/$\mathcal{U}_{\mathrm{out}}$], and that $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ is q.r.c.-SOS-stabilizing

Then the J-critical q.r.c.-[strongly/SOS-]stabilizing state feedback pairs (equivalently, P-q.r.c.-[strongly/SOS-]stabilizing solutions of the eIARE) for Σ and $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ correspond to each other through (9.224).

Proof: Assume that $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]) \in \overline{\mathrm{eIARE}}(\Sigma, J)$. Set $\mathbb{X}_\natural := \mathbb{X}\mathbb{M}'$, $\mathbb{F}_\natural := I - \mathbb{X}_\natural$, $\mathbb{K}_\natural := \mathbb{K} - \mathbb{X}_\natural\mathbb{K}'$. Then it is easy to verify that $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]) \in \overline{\mathrm{eIARE}}(\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right], J)$. Exchange the roles of Σ and $\Sigma_\flat^1$ to obtain the converse.

$$\begin{tikzcd} & \Sigma \arrow[dl, "\left[\,\mathbb{K}' \mid \mathbb{F}'\,\right]"'] \arrow[dr, "\left[\,\mathbb{K} \mid \mathbb{F}\,\right]"] & \\ \Sigma_\flat \arrow[rr, "\left[\,\mathbb{K}-\mathbb{X}\mathbb{K}_\flat \mid I-\mathbb{X}\mathbb{M}'\,\right]"'] & & \Sigma_\circlearrowleft \end{tikzcd}$$

Figure 9.2: The setting of Lemma 9.12.3

$$\begin{tikzcd} & \Sigma \arrow[dl, "K'"'] \arrow[dr, "K=K'+K_{\natural_w}"] & \\ \Sigma_\flat \arrow[rr, "K_\natural"'] & & \Sigma_\circlearrowleft \end{tikzcd}$$

Figure 9.3: The setting of Proposition 9.12.4

(a) Set $\mathbb{X}_\natural := \mathbb{X}\mathbb{M}'$. Since $\mathbb{X}_\natural, \mathbb{X}, \mathbb{M}' \in \mathcal{G}\mathrm{TIC}_\infty(U)$, any strong or uniform regularity property shared by two of these is shared by the third too, since such properties are preserved in compositions, by Lemma 6.2.5.

(b) This is the setting of Lemma 6.7.11(a') with $\left[\,\mathbb{K} \mid \mathbb{F}\,\right]$ in place of $\left[\,\mathbb{K}^2 \mid \mathbb{F}^2\,\right]$ and $\left[\,\mathbb{K}' \mid \mathbb{F}'\,\right]$ in place of $\left[\,\mathbb{K} \mid \mathbb{F}\,\right]$, hence the conclusions (a1)–(a6) apply with these replacements.

(c) This follows from Lemma 6.7.11(a2)&(a1) and Lemma 6.6.17(b).

(d1) This follows from Theorem 9.9.1(a1) and the fact that $\mathbb{A}_\circlearrowleft$ is common for both solutions, by (b) (and Lemma 6.1.10).

(d2) For $\mathcal{U}_{\mathrm{out}}$ this follows from (c) ("q.r.c.-SOS-"), (b) (for "P-") and Theorem 9.9.1(b). Since $\left[\,\mathbb{A}_\circlearrowleft \mid \mathbb{B}_\circlearrowleft\,\right]$ are common for both solutions, by (b), this leads to the claims on $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$. □

Thus, if $K_\natural$ is an optimal feedback for $\Sigma_\flat$, then the preliminary plus the optimal closed-loop feedback $K' + K_\natural$ produce the optimal open-loop feedback K (i.e., the same $\mathbb{A}_\circlearrowleft$ $\mathbb{C}_\circlearrowleft$, hence the same (possibly optimal) state $x_\circlearrowleft$ and output $y_\circlearrowleft$):

Proposition 9.12.4 (Σ-CARE$\cong$ $\Sigma_\flat$-CARE) *Let K' be an admissible SR state feedback operator for Σ with closed-loop system $\Sigma_\flat$.*

The WR solutions of form $(\mathcal{P}, S, \left[\,K \mid 0\,\right])$ of the eCARE for Σ correspond 1-1 to the WR solutions of form $(\mathcal{P}_\natural, S_\natural, \left[\,K_\natural \mid 0\,\right])$ of the eCARE for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ through

$$K = K' + K_{\natural_{\mathrm{w}}}, \quad S = S_\natural, \quad \mathcal{P} = \mathcal{P}_\natural. \tag{9.226}$$

Also (a)–(d2) of Lemma 9.12.3 apply; in particular, if K' and $K_\natural$ (resp. and K) are as above and have any strong or uniform regularity property, then so does K (resp. $K_\natural$).

(To be exact, by $K = K' + K_{\natural_{\mathrm{w}}}$ we mean that $K = K' + K_{\natural_{\mathrm{w}}}|_{\mathrm{Dom}(A)}$.) Thus, all

WR J-critical state feedback operators K for Σ over $\mathcal{U}_{\text{exp}}$ correspond 1-1 to the WR state feedback operators $K_\natural$ for $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat\\\hline \mathbb{C}_\flat & \mathbb{D}_\flat\end{array}\right]$ over $\mathcal{U}_{\text{exp}}^{\Sigma_\flat}$ through $K = K' + K_{\natural_\text{w}}$. (See Theorem 8.4.5(f) for other $\mathcal{U}_*^*$'s.

Conversely, given K, the optimal feedback for $\Sigma_\flat$ is $K_\natural = (K_s - K_w')|_{\text{Dom}(A_\flat)}$, i.e., we must remove the preliminary feedback K' and replace it by K, the optimizing one.

Proof: The correspondence and (a) follow by combining Lemma 9.12.3 and Proposition 6.6.18(f) (interchange the roles of Σ and $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat\\\hline \mathbb{C}_\flat & \mathbb{D}_\flat\end{array}\right]$ for the other direction). The rest follows from Lemma 9.12.3. □

Two different solutions of the eIARE correspond to each other in the following way:

Lemma 9.12.5 *Let* $(\mathcal{P}_1, S_1, \left[\begin{array}{c|c}\mathbb{K}_1 & \mathbb{F}_1\end{array}\right]) \in \overline{\text{eIARE}}(\Sigma, J)$. *Then*

$$(\mathcal{P}_2, S_2, \left[\begin{array}{c|c}\mathbb{K}_2 & \mathbb{F}_2\end{array}\right]) \in \overline{\text{eIARE}}(\Sigma, J) \Leftrightarrow \tag{9.227}$$

$$(\mathcal{P}_2 - \mathcal{P}_1, S_2, \left[\begin{array}{c|c}\mathbb{K}_2 & \mathbb{F}_2\end{array}\right]) \in \overline{\text{eIARE}}(\left(\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\\hline -\mathbb{K}_1 & \mathbb{X}_1\end{array}\right), S_1). \tag{9.228}$$

Equivalently, for any $(\mathcal{P}, S, \left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]) \in \overline{\text{eIARE}}(\Sigma, J)$ we have

$$\text{eIARE}(\Sigma, J) = \mathcal{P} + \text{eIARE}(\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\\hline -\mathbb{K} & \mathbb{X}\end{array}\right], S). \tag{9.229}$$

Naturally, an analogous result holds for $(\mathcal{P}_k, S_k, K_k)$ $(k = 1, 2)$ in the regular case.

Proof: Set $\Sigma^{\mathcal{P}_1} := \left(\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\\hline -\mathbb{K}_1 & \mathbb{X}_1\end{array}\right) \in \text{WPLS}(U, H, U)$.

1° "$\Rightarrow$": Let $(\mathcal{P}_2, S_2, \left[\begin{array}{c|c}\mathbb{K}_2 & \mathbb{F}_2\end{array}\right]) \in \overline{\text{eIARE}}(\Sigma, J)$ and $t > 0$. Set $\mathcal{P} := \mathcal{P}_2 - \mathcal{P}_1$. Substract the two Lyapunov equations to obtain

$$(\mathbb{K}_2^t)^* S_2 \mathbb{K}_2^t = \mathbb{A}^{t*} \mathcal{P} \mathbb{A}^t - \mathcal{P} + (\mathbb{K}_1^t)^* S_1 \mathbb{K}_1^t. \tag{9.230}$$

This is the first (i.e., Lyapunov) equation of the eIARE for $\left(\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\\hline -\mathbb{K}_1 & \mathbb{X}_1\end{array}\right)$ and S_1. Now $(\mathbb{X}_2^t)^* S_2 \mathbb{X}_2^t = \mathbb{D}^{t*} J \mathbb{D}^t + \mathbb{B}^{t*} \mathcal{P}_1 \mathbb{B}^t + \mathbb{B}^{t*} \mathcal{P} \mathbb{B}^t = (\mathbb{X}_1^t)^* S_1 \mathbb{X}_1^t + \mathbb{B}^{t*} \mathcal{P} \mathbb{B}^t$, and

$$-(\mathbb{X}_2^t)^* S_2 \mathbb{K}_2^t = \mathbb{D}^{t*} J \mathbb{C}^t + \mathbb{B}^{t*} \mathcal{P}_2 \mathbb{A}^t = -(\mathbb{X}_1^t)^* S_1 \mathbb{K}_1^t + \mathbb{B}^{t*} \mathcal{P} \mathbb{A}^t. \tag{9.231}$$

Thus, the three equations of the eIARE are satisfied, i.e., $(\mathcal{P}, S_2, \left[\begin{array}{c|c}\mathbb{K}_2 & \mathbb{F}_2\end{array}\right]) \in \overline{\text{eIARE}}(\Sigma^{\mathcal{P}_1}, S_1)$.

2° "$\Leftarrow$": Let $(\mathcal{P}_2 - \mathcal{P}_1, S_2, \left[\begin{array}{c|c}\mathbb{K}_2 & \mathbb{F}_2\end{array}\right]) \in \overline{\text{eIARE}}(\Sigma^{\mathcal{P}_1}, S_1)$. By going 1° backwards, we see that $(\mathcal{P}_2, S_2, \left[\begin{array}{c|c}\mathbb{K}_2 & \mathbb{F}_2\end{array}\right]) \in \overline{\text{eIARE}}(\Sigma, J)$. □

As Example 9.13.9 shows, the CARE may have solutions $(\mathcal{P} + \Delta, S, K)$ for infinitely many $\Delta = \Delta^* \in \mathcal{B}(H)$. If one of these, say $\mathcal{P}$, is $\mathcal{U}_*^*$-stabilizing (or at least P-stabilizing), then Δ corresponds to "fake cost", i.e., $\mathcal{P} + \Delta = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft + \Delta$, where $\mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$ is the corresponding (optimal) closed-loop cost. The following corollary formulates necessary and sufficient conditions:

Corollary 9.12.6 *Let* $(\mathcal{P}, S, \left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]) \in \overline{\text{eIARE}}(\Sigma, J)$ *and* $\Delta = \Delta^* \in \mathcal{B}(H)$. *Then (i)$\Leftrightarrow$(ii) ($\Leftrightarrow$(iii) provided that* $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *is admissible):*

(i) $(\mathcal{P}+\Delta, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]) \in \overline{\mathrm{eIARE}}(\Sigma, J)$*;*

(ii) $\Delta = \mathbb{A}^{t\,*}\Delta\mathbb{A}^t$ *and* $\mathbb{B}^{t\,*}\Delta\mathbb{A}^t = 0 = \mathbb{B}^{t\,*}\Delta\mathbb{B}^t$ *for all* $t \geq 0$*;*

(iii) $\Delta = \mathbb{A}_\circlearrowleft^{t\,*}\Delta\mathbb{A}_\circlearrowleft^t$ *and* $\mathbb{B}_\circlearrowleft^{t\,*}\Delta\mathbb{A}_\circlearrowleft^t = 0 = \mathbb{B}_\circlearrowleft^{t\,*}\Delta\mathbb{B}_\circlearrowleft^t$ *for all* $t \geq 0$.

Proof: By Lemma 9.12.5, we have (i) iff $\Delta \in \mathrm{eIARE}(\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K} & \mathbb{X} \end{array}\right], S)$, i.e., iff (ii) holds.

Assume that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is admissible for Σ. Then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is admissible for $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K} & \mathbb{X} \end{array}\right]$ with closed-loop system $\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline 0 & I \end{array}\right]$, so that $\Delta \in \mathrm{eIARE}(\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline -\mathbb{K} & \mathbb{X} \end{array}\right], S)$ becomes equivalent to (iii), by Lemma 9.10.1(b4)(i)&(iv). $\square$

If, e.g., Σ is exponentially stable, then any stabilizing solution of the IARE leads to the spectral factorization $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$, by Proposition 9.8.11(d1) (see also Corollary 9.9.11). Under weaker assumptions than those in Proposition 9.8.11, we can still obtain a "pseudospectral factorization", a weak form of $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$":

Proposition 9.12.7 *Let* $\Sigma_{\mathrm{ext}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \\ \mathbb{K} & \mathbb{F} \end{array}\right] \in \mathrm{WPLS}(U, H, Y \times U)$*, and let (9.160) hold for all* $t > 0$ *and some* $J, S, \mathcal{P} \in \mathcal{B}$*, where* $\mathbb{X} := I - \mathbb{F}$*. Then*

$$\langle \mathbb{X}v, \pi_{(-\infty,t)} S\mathbb{X}u\rangle = \langle \mathbb{D}v, \pi_{(-\infty,t)} J\mathbb{D}u\rangle + \langle \mathbb{B}\tau^t v, \mathcal{P}\mathbb{B}\tau^t u\rangle \quad (u, v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}; U),\ t \in \mathbf{R}). \tag{9.232}$$

Moreover, we have the following:

(a) If $\mathbb{B}$*,* $\mathbb{D}$ *and* $\mathbb{X}$ *are stable, then*

$$\mathbb{X}^* S\mathbb{X} = \mathbb{D}^* J\mathbb{D} + \operatorname*{s-lim}_{t\to+\infty} \tau^{-t}\mathbb{B}^* \mathcal{P}\mathbb{B}\tau^t. \tag{9.233}$$

If, in addition, $\mathbb{B}$ *is strongly stable, then* $\mathbb{X}^* S\mathbb{X} = \mathbb{D}^* J\mathbb{D}$*.*

(b) If $\mathbb{B}$ *and* $\mathbb{D}$ *are strongly stable and* $u \in \mathrm{L}^2(\mathbf{R}; U)$*, then* $\mathbb{X}^*\pi_{[-T,t)} S\mathbb{X}u \to \mathbb{D}^* J\mathbb{D}u$ *in* $\mathrm{L}^2(\mathbf{R}; U)$*, as* $t, T \to +\infty$ *(independently); in particular, we have (uniformly in* v*)*

$$\lim_{T,t\to+\infty} \langle \pi_{[-T,t)}\mathbb{X}v, \pi_{[-T,t)} S\mathbb{X}u\rangle = \langle \mathbb{D}v, J\mathbb{D}u\rangle \quad (u, v \in \mathrm{L}^2(\mathbf{R}; U)). \tag{9.234}$$

(Naturally, the statements include the convergence of limits presented.)

Recall that the eIARE implies (9.160). If, e.g., $\mathbb{D}$ and $\mathbb{X}$ are stable, $S \in \mathcal{GB}(U)$ and (P) holds on $\mathrm{Ran}(\mathbb{B})$, then we have the spectral factorization $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$, by (9.232) and continuity.

Proof: 1° Let $v_k \in \mathrm{L}^2([-t, 0); U)$ $(k = 1, 2)$, and substitute $\tau^{-t} v_k$ $(\in \mathrm{L}^2([0, t); U))$ into (9.160) to obtain (note that $\pi_J \tau^r = \tau^r \pi_{r+J}$)

$$\langle \mathbb{X}v_2, \pi_- S\mathbb{X}v_1\rangle = \langle \mathbb{D}v_2, \pi_- J\mathbb{D}v_1\rangle + \langle \mathbb{B}v_2, \mathcal{P}\mathbb{B}v_1\rangle. \tag{9.235}$$

This holds for arbitrary $t \in \mathbf{R}$, hence for arbitrary $v_k \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}; U)$ $(k = 1, 2)$, because $\pi_+ v_k$ does not affect the equation.

Let now t, u, v be as in (9.232) and substitute $v_1 := \tau^t u$, $v_2 := \tau^t v$ into (9.235) to obtain (9.232).

(a) Let $\mathbb{B},\mathbb{D},\mathbb{X}$ be stable. By (9.232), we have $\mathbb{X}^*\pi_{(-\infty,t)}S\mathbb{X}u = \mathbb{D}^*\pi_{(-\infty,t)}J\mathbb{D}u + \tau^{-t}\mathbb{B}^*\mathcal{P}\mathbb{B}\tau^t u$ for all $u \in \mathrm{L}^2_{\mathrm{c}}$, hence for all $u \in \mathrm{L}^2$, by continuity. Let $t \to +\infty$ and use Corollary B.3.8 to obtain (9.233). The second claim is obvious (because $\tau^{-t}\mathbb{B}^* = (\mathbb{B}\tau^t)^*$ is bounded $H \to \mathrm{L}^2$).

(b) Now $\mathbb{X}^{\mathrm{d}}\pi_{[-t,T)} \in \mathcal{B}(\mathrm{L}^2)$ for all $t,T > 0$, by Lemma 6.1.11 (since $\mathbb{B}^{\mathrm{d}}$ is stable), hence $\mathbb{X}^*\pi_{[-T,t)}, \pi_{[-T,t)}\mathbb{X} \in \mathcal{B}(\mathrm{L}^2)$. Thus, from (9.232), we obtain

$$\mathbb{X}^* S\pi_{[-T,t)}\mathbb{X}u = \mathbb{D}^*\pi_{[-T,t)}J\mathbb{D}u + \pi_{[-T,t)}\tau^{-t}\mathbb{B}^*\mathcal{P}\mathbb{B}\tau^t\pi_{[-T,t)}u \tag{9.236}$$

(in $\mathrm{L}^2(\mathbf{R};U)$) for all $u \in \mathrm{L}^2_{\mathrm{c}}([-T,+\infty))$ (apply (9.232) to each $v \in \mathrm{L}^2_{\mathrm{c}}$ and recall that $\mathrm{L}^2_{\mathrm{c}}$ is dense in $\mathrm{L}^2 = (\mathrm{L}^2)^*$). By continuity, this holds for all $u \in \mathrm{L}^2$. Since $\pi_{\mathbf{R}\setminus[-T,t)}u \to 0$ (see Corollary B.3.8), we have $\mathbb{B}\tau^t\pi_{[-T,t)}u = \mathbb{B}\tau^t u - \mathbb{B}\tau^t\pi_{\mathbf{R}\setminus[-T,t)}u \to 0$, as $T,t \to +\infty$. Consequently

$$\mathbb{D}^*\pi_{[-T,t)}J\mathbb{D}u + \pi_{[-T,t)}\tau^{-t}\mathbb{B}^*\mathcal{P}\mathbb{B}\tau^t\pi_{[-T,t)}u \to \mathbb{D}^*J\mathbb{D}u, \tag{9.237}$$

i.e., $\mathbb{X}^*\pi_{[-T,t)}S\mathbb{X}u \to \mathbb{D}^*J\mathbb{D}u$ in L^2, as $t,T \to +\infty$ (independently). Thus, (b) holds. □

As indicated in Chapter 5, the factorization "$\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$" corresponding to a J-critical state feedback pair for a stable system need not be stable (that is, a spectral factorization), but we may have $\widehat{\heartsuit}\widehat{\mathbb{X}} \in \mathrm{H}^2 \setminus \mathrm{H}^\infty$ in case $\dim U < \infty$. We state this and more general results below:

Lemma 9.12.8 ($\mathbb{B},\mathbb{D}$ stable $\Rightarrow \mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$ & $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}$) *Assume that $\mathbb{B}$ and $\mathbb{D}$ are stable, $\vartheta = 0$, and $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix})$ is a $\mathcal{U}^*_*$-stabilizing solution of the eIARE. Set $\mathbb{M}^{-1} := \mathbb{X} := I - \mathbb{F}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$, $\widehat{\mathbb{X}}^{\mathrm{d}} := \widehat{\mathbb{X}}(\bar{\cdot})^*$. Let $r > 0$. Then*

(a1) $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\omega(U)$ *for all* $\omega > 0$.

(a2) $(\cdot+1)^{-1}\widehat{\mathbb{N}}, (\cdot+1)^{-1}\widehat{\mathbb{M}}, (\cdot+1)^{-1}\widehat{\mathbb{X}}^{\mathrm{d}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U,*))$*; in particular,* $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}(\mathbf{C}^+;\mathcal{B}(U))$.

(b1) $\mathbb{N},\mathbb{M},\mathbb{X}^*$ *map* $\mathrm{L}^2_{\mathrm{c}}(\mathbf{R};U) \to \mathrm{L}^2$*, and* $\mathbb{X}^*\pi_\pm\mathbb{M}^*, \mathbb{M}\pi_{[-T,t)}, \mathbb{X}^*\pi_{[-T,t)} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U))$ *for all* $T,t > 0$.

(b2) $\mathbb{M}\pi_+\mathbb{X}, \pi_{[-T,t)}\mathbb{X} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U)) \cap \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))$ *for each* $\omega > 0$*, and* $\mathbb{M}\pi_+\mathbb{X}$ *and* $\pi_{[-T,t)}\mathbb{X}$ *have a continuous extensions to* $\mathcal{B}(\mathrm{L}^2(\mathbf{R};U))$.

(c1) $\langle \mathbb{N}u, J\mathbb{N}v\rangle = \langle u, Sv\rangle$ *for all* $u,v \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R};U)$.

(c2) $\mathbb{X}^*\pi_{[-T,t)}S\mathbb{X}u \to \mathbb{D}^*J\mathbb{D}u$ *in* $\mathrm{L}^2(\mathbf{R};U)$*, as* $t,T \to +\infty$*, if* $\mathbb{B}$ *is strongly stable and* $u \in \mathrm{L}^2(\mathbf{R};U)$.

(d) **($\dim U < \infty \Rightarrow \widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$)** *If* $\dim U < \infty$*, then* $(\cdot+1)^{-1}\widehat{\mathbb{X}}^{\pm1} \in \mathrm{H}^2(\mathbf{C}^+;\mathcal{B}(U)) \cap \mathrm{L}^2(i\mathbf{R};\mathcal{B}(U))$*, and* $\widehat{\mathbb{X}} \in \mathcal{G}\mathcal{B}(U)$ *and* $\widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ *a.e. on* $i\mathbf{R}$.

(e) **($(\pi_+\mathbb{D}^*J\mathbb{D}\pi_+)^{-1} = \mathbb{M}\pi_+S^{-1}\mathbb{M}^*$)** *If* $\mathbb{T} := \pi_+\mathbb{D}^*J\mathbb{D}\pi_+$ *is invertible on* $\mathrm{L}^2(\mathbf{R}_+;U)$*.* $\mathbb{B}$ *is strongly stable, then* $S \in \mathcal{G}\mathcal{B}(U)$ *and* $\mathbb{T}^{-1} = \mathbb{M}\pi_+S^{-1}\mathbb{M}^* \in \mathcal{G}\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U))$.

(See Lemma 14.2.8 for additional information.) Recall that $\vartheta = 0$ for $\mathcal{U}_{\text{out}}$, $\mathcal{U}_{\text{sta}}$, $\mathcal{U}_{\text{str}}$ and $\mathcal{U}_{\text{exp}}$.

Proof: (a1)–(c2) These follow as in the proof of Lemma 14.2.8 (alternatively, we can use discretization).

(d) By (a2) and Theorem 3.3.1(c3), we have $(\cdot + 1)^{-1}\widehat{\mathbb{X}}^{\pm 1} \in \mathrm{H}^2(\mathbf{C}^+;\mathcal{B}(U)) \cap \mathrm{L}^2(i\mathbf{R};\mathcal{B}(U))$. By continuity, $\widehat{\mathbb{X}}\widehat{\mathbb{M}} = I = \widehat{\mathbb{M}}\widehat{\mathbb{X}}$ and $\widehat{\mathbb{N}} = \widehat{\mathbb{D}}\widehat{\mathbb{M}}$ a.e. on $i\mathbf{R}$ (since these hold on $\mathbf{C}^+$); in particular, $\widehat{\mathbb{X}} \in \mathcal{GB}(U)$ a.e. on $i\mathbf{R}$.

Since $\langle \mathbb{X}u, S\mathbb{X}v\rangle_{\mathrm{L}^2} = \langle \mathbb{D}u, J\mathbb{D}v\rangle_{\mathrm{L}^2}$, i.e., $\langle \widehat{\mathbb{X}}\widehat{u}, S\widehat{\mathbb{X}}\widehat{v}\rangle_{\mathrm{L}^2(i\mathbf{R};U)} = \langle \widehat{\mathbb{D}}\widehat{u}, J\widehat{\mathbb{D}}\widehat{v}\rangle_{\mathrm{L}^2(i\mathbf{R};U)}$ for all $\widehat{u},\widehat{v} \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+;U)$, we have $\widehat{\mathbb{X}}^* S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}$ a.e. on $i\mathbf{R}$.

(e) This follows from Lemma 14.2.8(e), by discretization. □

Notes

For bounded C, part of Lemma 9.12.2 (with "=" in place of "≤") is known (see [CZ, Section 5]). For general C, Lemma 9.12.2(a2) was essentially given in [Grabowski91, Theorem 3] and (c) in [Sbook, Theorem 9.5.2] (both with "=" in place of "≤"); the latter also contains further necessary and sufficient conditions for $\left[\frac{A}{C}\right]$ to generate an output-stable WPLS.

Formula (9.224) was used in [S98b]. Lemma 14.2.4 is from [Mal00], and Lemma 9.12.5 is its IARE variant. Part of Proposition 9.12.7 was given in [Mik97b].

9.13 Examples of Riccati equations

Though the day of my destiny's over,
And the star of my fate hath declined,
Thy soft heart refused to discover
The faults which so many could find.
— Lord Byron (1788–1824), "Stanzas to Augusta"

In this section, we shall illustrate by examples several "pathological" cases due to which the general Riccati equation theory is rather complex. The part corresponding to different $\mathcal{U}_*^*$'s may be new even for finite-dimensional systems.

The studies on Riccati equations have mainly concentrated on exponentially stabilizing solutions of Riccati equations. The articles on optimization over $\mathcal{U}_{\text{out}}$ or $\mathcal{U}_{\text{sta}}$ have usually either neglected uniqueness or used estimatability (e.g., put some cost on the state) to reduce $\mathcal{U}_{\text{out}}$ to $\mathcal{U}_{\text{exp}}$.

We found the condition (P) for the SOS-stable case in [Mik97b] (cf. Corollary 9.9.11); the condition (PB) is new. In Proposition 9.13.1(a)&(b)&(e) we show that these conditions are necessary. The difference between $\mathcal{U}_{\text{out}}$- and $\mathcal{U}_{\text{exp}}$-stabilizing solutions and the role of coprimeness are illustrated in Example 9.13.2. We also present plenty of examples that illustrate the different aspects of the role of the signature operator S and the insignificance of D^*JD (when the latter does not coincide with the former).

By discretization (see Proposition 9.8.7), all our examples apply also in discrete time (eDAREs). In particular, our CARE "counter-"examples are also eCARE, IARE, eIARE, DARE and eDARE "counter-"examples.

Most examples also have obvious discrete-time analogies (without artificial discretization); in particular, all our finite-dimensional examples have finite-dimensional discrete-time counterparts.

Our examples are mathematically motivated; for physically more motivated examples on WPLSs and Riccati equations, see, e.g., [Sal87].

The following proposition summarizes which conclusions can be drawn from the different examples:

Proposition 9.13.1 (Non-$\mathcal{U}_*^*$-stabilizing solutions)

(a) **((P)$\not\Rightarrow$(PB))** *[Example 9.13.2] There can be several P-SOS-stabilizing solutions of the CARE, only one of which is $\mathcal{U}_{\text{out}}$-stabilizing.*

A $\mathcal{U}_{\text{out}}$-stabilizing solution need not be internally stabilizing, even if there were exponentially stabilizing solutions.

(b) **((PB2)$\not\Rightarrow$(P))** *[Example 9.13.9] There can be several r.c.-stabilizing solutions $\mathcal{P}$ satisfying $\langle \mathbb{B}^t x_0, \mathcal{P}\mathbb{A}_\circlearrowleft^t x_0\rangle \to 0$, of the CARE for a stable minimal system, only one of which is P-stabilizing (hence $\mathcal{U}_{\text{out}}$-stabilizing and equal to the J-critical cost operator over $\mathcal{U}_{\text{out}}$ (and over $\mathcal{U}_{\text{sta}}$)).*

Moreover, our example is exactly reachable and of the standard LQR (minimization) form with $\mathbb{A}^*$ strongly stable.

(c1) **($\mathbb{D} \in$ ULR $\not\Rightarrow \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix} \in$ WR)** *[WW, Example 11.5] There is* $\mathbb{D} \in \mathrm{TIC}(\mathbf{C}) \cap \mathrm{ULR}$, *for which* $D = 0$ *and* $\mathbb{D}^*\mathbb{D} = \mathbb{X}^*\mathbb{X} \gg 0$ *with an irregular (I-spectral factor)* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(\mathbf{C})$ *(in fact,* $\widehat{\mathbb{X}}(2^k) = 2 + (-1)^k$, *hence* $\mathbb{X}, \mathbb{X}^{-1} \notin \mathrm{WR}$*).*

Alternatively, we can choose $\mathbb{D} \in \mathrm{TIC}(\mathbf{C}, \mathbf{C}^2) \cap \mathrm{ULR}$ *s.t.* $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 \\ I \end{bmatrix}$, $D_1 = 0$ *and* $\mathbb{D}^*\mathbb{D} = \mathbb{X}^*\mathbb{X} \gg 0$ *for this same* $\mathbb{X}$ *multiplied by some positive constant.*

In particular, if Σ *is a strongly stable realization of (either)* $\mathbb{D}$ *(as in (6.11)), then the IARE has a strongly PB-r.c.-stabilizing solution, but there are no PB-output-stabilizing solutions of the eCARE.*

(c2) **(Bounded $C \not\Rightarrow S = D^*JD$)** *[Example 9.13.8] There is a stable positively I-coercive WPLS (in standard LQR form) with* $J = I$, *bounded* C *and* $D^*C = 0$ *s.t.* $\mathbb{D}, \mathbb{X} \in \mathrm{ULR}$ *but* $S \neq D^*JD$, K *is unbounded and* $\mathcal{P}[H] \not\subset \mathrm{Dom}(B_{\mathrm{w}}^*)$.

(d1) *Condition* $S = D^*JD \gg 0$ *does not guarantee sufficient coercivity for the existence of a minimizing (J-critical) control over* $\mathcal{U}_{\mathrm{exp}}$*; see Example 9.13.5.*

(d2) *We may have* $D^*JD \gg 0 \gg S$ *for a maximizing solution, hence* D^*JD *does not characterize the signature properties of the problem; see Example 9.13.7.*

(e) *Let* $\Sigma \in \mathrm{SOS}$. *If there is a stable, stabilizing solution* $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid I - \mathbb{X} \end{bmatrix})$ *of the IARE s.t.* $\langle \mathbb{A}^t x_0, \mathcal{P}\mathbb{A}^t x_0 \rangle \to 0$, *as* $t \to +\infty$, *for all* $x_0 \in H_{\mathbb{B}}$, *then* $\mathbb{X}^* S \mathbb{X}$ *is the unique spectral factorization of* $\mathbb{D}^* J \mathbb{D}$, *and the IARE has a (unique) stable PB-r.c.-stabilizing solution, namely* $(\widetilde{\mathcal{P}}, S, \begin{bmatrix} \mathbb{K} \mid I - \mathbb{X} \end{bmatrix})$, *where* $\widetilde{\mathcal{P}} := \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft$.

Thus, the open- and closed-loop systems for $\mathcal{P}$ *and* $\widetilde{\mathcal{P}}$ *are identical; the only difference is that* $\widetilde{\mathcal{P}}$ *is the J-critical cost operator and* $\mathcal{P} = \widetilde{\mathcal{P}} + \mathrm{s\text{-}lim}_{t \to +\infty} \mathbb{A}(t)^* \mathcal{P} \mathbb{A}(t)$, *as in Example 9.13.12.*

(f) *Even if* Σ *is strongly stable and the Popov Toeplitz operator* $\pi_+ \mathbb{D}^* J \mathbb{D} \pi_+$ *is invertible (as in Proposition 8.3.10), so that there is a unique J-critical control for each* $x_0 \in H$ *over* $\mathcal{U}_*^* := \mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$, *1. it may be that there is no J-critical state feedback pair (Example 11.3.7(a)), 2. it may be that the J-critical state feedback pair* $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ *and its closed-loop form* $\begin{bmatrix} \mathbb{K}_\circlearrowleft \mid \mathbb{F}_\circlearrowleft \end{bmatrix}$ *are unstable (Example 11.3.7(b)) though they and* Σ *are UHPR; in particular there is no spectral factorization of* $\mathbb{D}^* J \mathbb{D}$.

Proof: (a)&(b)&(c2)&(d1)&(d2)&(f) See corresponding examples.

(c1) (We do not know whether this can happen for exponentially stable $\mathbb{D}$.) The existence of $\mathbb{X}$ is proved in [WW, Example 11.5] (although any irregular $\mathbb{X}$ would do). We can then simply take $\widehat{\mathbb{D}}(s) := e^{-s}\widehat{\mathbb{X}}(s)$ in the former case; in the latter case we must multiply $\mathbb{X}$ by a positive constant so that $\mathbb{X}^*\mathbb{X} \gg I$. Let $\mathbb{Z}$ be an I-spectral factor of $\mathbb{X}^*\mathbb{X} - I$, and set $\mathbb{D}_1 := e^{-\cdot}\mathbb{Z}$ to guarantee that $\mathbb{D} \in \mathrm{ULR}$ (as in [WW]).

By Corollary 9.9.11(Crit4SOS), we have $\mathbb{X} = I - \mathbb{F}$, where $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$ is the unique $\mathcal{U}_{\mathrm{out}}$-stabilizing solution of the eIARE. Because $\mathbb{F}$ is not WR, the eCARE does not have a $\mathcal{U}_{\mathrm{out}}$-stabilizing solution.

(e) Now $\mathcal{P} = \mathbb{C}^* J\mathbb{C} - \mathbb{K}^* S\mathbb{K}$ on $\mathrm{Ran}(\mathbb{B})$, by (9.159). Consequently, $\mathbb{X}^* S\mathbb{X}u = \mathbb{D}^* J\mathbb{D}u$ for $u \in \mathrm{L}^2$, by (9.160) (since $\mathbb{C}\mathbb{B}^t = \pi_+\mathbb{D}\pi_-\tau^t\pi_+ = \tau^t\pi_{[t,\infty)}\mathbb{D}\pi_{[0,t)} \to 0$, because $\pi_{[t,\infty)}\mathbb{D}\pi_{[0,t)} \to \pi_\emptyset\mathbb{D}\pi_+ = 0$, as $t \to +\infty$, by Corollary B.3.8). Because $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, the operator $\widetilde{\mathcal{P}}$ is q.r.c.-P-stabilizing, by Theorem 9.9.10(b)&(e2), hence it is stable and PB-r.c.-stabilizing (and unique). By (9.159), we have

$$\mathcal{P} - \mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}^t = \mathbb{C}^{t\,*}J\mathbb{C}^t - \mathbb{K}^{t\,*}J\mathbb{K}^t \to \mathbb{C}^* J\mathbb{C} - \mathbb{K}^* S\mathbb{K} = \widetilde{\mathcal{P}} \tag{9.238}$$

strongly, as $t \to +\infty$. □

We continue by presenting the actual examples, starting with a simple (unobservable) system illustrating that the solutions over $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{exp}}$ may be different:

Example 9.13.2 (Std. LQR: $\mathcal{U}_{\mathrm{out}}$-stabilizing $\neq$ $\mathcal{U}_{\mathrm{exp}}$-stabilizing, (P)$\not\Rightarrow$(PB)) Let $U = \mathbf{C} = H = Y$, $A = 1 = B = D = J$, $C = 0$. Then the CARE becomes

$$S = 1, \ \ SK = -\mathcal{P}, \ \ \mathcal{P} + \mathcal{P} + 0 = K^* SK = \mathcal{P}^2. \tag{9.239}$$

Thus, the solutions of the CARE are given by $(0, 1, 0)$ (PB-r.c.-SOS-stabilizing, hence $\mathcal{U}_{\mathrm{out}}$-stabilizing) and $(2, 1, -2)$ ($\mathcal{U}_{\mathrm{exp}}$-stabilizing, hence exponentially P-stabilizing). The corresponding closed-loop systems are

$$\Sigma_\circlearrowleft := \left[\begin{array}{c|c} 1 & 1 \\ \hline 0 & 1 \\ 0 & 0 \end{array}\right] \quad \text{and} \quad \Sigma'_\circlearrowleft := \left[\begin{array}{c|c} -1 & 1 \\ \hline 0 & 1 \\ -2 & 0 \end{array}\right]. \tag{9.240}$$

Now $\mathbb{D} = 1 = \mathbb{D}_\circlearrowleft = \mathbb{D}'_\circlearrowleft$, $\mathbb{C} = 0 = \mathbb{C}_\circlearrowleft = \mathbb{C}'_\circlearrowleft$, $\mathbb{K}_\circlearrowleft = 0 = \mathbb{F}_\circlearrowleft$, $\mathbb{X} = I$, hence $(0, 1, 0)$ is, indeed, the unique $\mathcal{U}_{\mathrm{out}}$-stabilizing solution, (in fact, it is PB-r.c.-SOS-stabilizing and cost-minimizing; the condition (PB) is trivially satisfied for $\mathcal{P} = 0$), whereas $(2, 1, -2)$ is the unique $\mathcal{U}_{\mathrm{exp}}$-stabilizing solution.

Let us see why this happens: One easily verifies that $\widehat{\mathbb{N}'}(s) = (s-1)/(s+1) = \widehat{\mathbb{M}'}(s)$ (indeed, $\mathbb{N}'$ and $\mathbb{M}'$ are not q.r.c., because both have a zero at $s = 1$), and that $\langle \mathbb{N}' u_\flat, J\mathbb{C}'_\circlearrowleft x_0\rangle = 0$ for all $x_0 \in H$ and $u_\flat \in \pi_+\mathrm{L}^2$ (we have $(J\mathbb{C}'_\circlearrowleft x_0)(t) = -2\mathrm{e}^{-t}x_0$ $(t \ge 0)$), but $\langle \mathbb{D}u, J\mathbb{C}'_\circlearrowleft x_0\rangle \neq 0$ for $x_0 \neq 0$, $u \in \pi_+\mathrm{L}^2$ s.t. $\widehat{u}(1) \neq 0$.

Thus, although $\Sigma'_\circlearrowleft$ is J-critical over $\mathcal{U}_{\mathrm{out}}^{\Sigma_\circlearrowleft}$, i.e., J-critical w.r.t. "closed-loop stable" controls ($u_\flat \in \pi_+\mathrm{L}^2$), the system $\Sigma'_\circlearrowleft$ "does not see" signals with $\widehat{u}(1) \neq 0$, i.e., $u_\flat := (\mathbb{M}')^{-1}u \notin \mathrm{L}^2$ for such u, because $\mathbb{N}'$ and $\mathbb{M}'$ are not q.r.c. One can also verify that

$$\langle \mathbb{B}^t u, \mathcal{P}\mathbb{A}'_\circlearrowleft x_0\rangle_H = 2\mathrm{e}^{-t}\overline{x_0}\int_0^t \mathrm{e}^t\mathrm{e}^{-s}u(s)\,ds \to 2\overline{x_0}\widehat{u}(1) \neq 0 \tag{9.241}$$

when $x_0 \neq 0$ and $u \in \mathrm{L}^2$ is s.t. $\widehat{u}(1) \neq 0$, i.e., (PB) does not hold for $\mathcal{P} = 2$ (unless we replace $\mathcal{U}_{\mathrm{out}}$ by $\mathcal{U}_{\mathrm{exp}}$ in (PB), see Theorem 9.9.1(g1)). Trivially, (PB) holds for $\mathcal{P} = 0$. ◁

See also Example 6.6.16

Even for exponentially stable systems, the existence of a unique J-critical control does not guarantee that $\mathbb{D}$ is J-coercive, nor that the CARE has a solution:

Example 9.13.3 (Singular control: unique optimum without CARE and J-coercivity, $\nexists(D^*JD)^{-1}$) Assume that $\mathbb{A}$ is exponentially stable (e.g., $H = \mathbf{C}$ and $A = -1$), $C = 0 = B$, $J = I$, and $\mathbb{D} = D \in \mathcal{B}(U,Y)$ is one-to-one but not coercive (i.e., $S := D^*D \notin \mathcal{GB}(U)$). Let $\mathcal{U}_*^* \in \{\mathcal{U}_{\text{out}}, \mathcal{U}_{\text{sta}}, \mathcal{U}_{\text{str}}, \mathcal{U}_{\text{exp}}\}$.

Then $u = 0$ is the unique minimizing control for x_0, and $\mathcal{U}_*^*(x_0) = \mathrm{L}^2(\mathbf{R}_+;U)$, for each $x_0 \in H$. The corresponding unique $\mathcal{U}_*^*$-stabilizing solution of the eCARE is given by $(0, D^*D, 0)$ (by Theorem 9.9.6, equation $SK = 0$ forces K to be zero).

However, $\mathbb{D}^*J\mathbb{D} = D^*D$ is not J-coercive over $\mathcal{U}_*^*$. $\triangleleft$

Even when Σ is exponentially stable and $S > 0$, there need not exist any PB-stabilizing solutions nor minimizing control over $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$:

Example 9.13.4 (Stable Σ and $S > 0 \not\Rightarrow \exists\,\mathcal{U}_*^*$-stabilizing solution (or optimal control)) Take $U := \ell^2(\mathbf{N}) := Y$, $J := I$. Define $\mathbb{D} := D \in \mathcal{B}(U,Y)$ by $(Du_0)(n) := \mathrm{e}^{-2n}u_0(n)$ $(n \in \mathbf{N},\ u_0 \in U)$ (note that $0 < D^*D \not\gg 0$). Let Σ be the -1-stable exactly observable realization of D, so that $H = \mathrm{L}^2_{-1}(\mathbf{R}_+;Y)$, $\mathbb{C} = \pi_+$. Define $x_0 \in H$ and $u \in \mathrm{L}^2_{\text{loc}}(\mathbf{R}_+;U)$

$$x_0(t) := \mathrm{e}^{-t} \sum_{n\in\mathbf{N}} \pi_{[n,n+1)}(t) e_n \in H, \tag{9.242}$$

$$u_\infty(t) := -\mathrm{e}^{-t} \sum_{n\in\mathbf{N}} \mathrm{e}^{2n} e_n \in H, \tag{9.243}$$

where $\{e_n\}$ is the natural base of U. Then $\mathbb{C}x_0 + Du = 0$, so that u_∞ minimizes $\mathcal{J}(x_0,u) := \|x_0 + Du\|_2^2$ over all $u \in \mathrm{L}^2_{\text{loc}}(\mathbf{R}_+;U)$. Since D is one-to-one, u_∞ is the unique minimum.

However, $\mathcal{J}(x_0, \pi_{[0,T)}u_\infty) = \|\pi_{[0,T)}x_0\|_2^2 = (\mathrm{e}^{-2T}/2)^{1/2} \to 0$ as $T \to \infty$, and $\pi_{[0,T)}u_\infty \in \mathcal{U}_*^*(x_0) = \mathrm{L}^2(\mathbf{R}_+;U)$ (we assume that $\mathcal{U}_*^* \in \{\mathcal{U}_{\text{exp}}, \mathcal{U}_{\text{str}}, \mathcal{U}_{\text{sta}}, \mathcal{U}_{\text{out}}\}$). Since $u_\infty \notin \mathrm{L}^2$, there is no minimum (hence no J-critical control) over $\mathcal{U}_*^*$, hence there is no $\mathcal{U}_*^*$-stabilizing solution of the eIARE (note that $S = D^*D > 0$).

$\triangleleft$

Now we show that for the existence of a minimum, 1. even for an exponentially stable system ($\Sigma_\flat^1$), condition $S \gg 0$ is not sufficient, and 2. a system may be exponentially stabilizable and J-coercive over $\mathcal{U}_{\text{out}}$ without being J-coercive over $\mathcal{U}_{\text{exp}}$:

Example 9.13.5 (Exponentially stabilizable with $S = D^*JD \gg 0$ but no minimum over $\mathcal{U}_{\text{exp}}$) Let $J = 1$.

(a) For $\Sigma := \left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 1 \end{array}\right)$, we have $S = D^*JD = 1 \gg 0$, and $(0,1,0)$ is the unique solution of the CARE $\mathcal{P}^2 = 0$, and corresponding control $u = 0$ is minimizing over $\mathcal{U}_{\text{out}}$ (and $\mathcal{U}_{\text{sta}}$), but there is no minimum over $\mathcal{U}_{\text{exp}}$ (nor $\mathcal{U}_{\text{str}}$), and $\inf_{u\in\mathcal{U}_{\text{exp}}(x_0)} \mathcal{J}(x_0,u) = 0$ for all $x_0 \in H$.

Here $\widehat{x}(s) = s^{-1}(x_0 + \widehat{u}(s))$, so that we must have $\widehat{u}(0) = -x_0$ in some sense (for the boundary function of $\widehat{u} \in \mathrm{H}^2(\mathbf{C}^+)$), but we wish to minimize $\mathcal{J}(x_0,u) =$

$\|\widehat{u}\|_2^2$ under this condition, and the latter expression can be taken arbitrarily small. In time domain, we have $x(t) = x_0 + \int_0^t u(r)\,dr$, so that $\mathcal{J}(x_0, u_n) \to 0+$, as $n \to \infty$, where $u_n := -x_0 n^{-1}\chi_{[0,n)} \in \mathcal{U}_{\text{exp}}(x_0)$.

(b) By using the exponentially stabilizing state feedback operator $K = -1$ (and then removing the state feedback (= third) row), we obtain $\Sigma_\flat^1 := \left(\begin{array}{c|c} -1 & 1 \\ \hline -1 & 1 \end{array}\right)$. The CARE becomes again $\mathcal{P}^2 = 0$ with the unique solution $(0,1,1)$, hence there is no minimizing control over $\mathcal{U}_{\text{exp}} = \mathcal{U}_{\text{out}}$, although the infimum cost over $\mathcal{U}_{\text{exp}}$ is again zero.

Now $y = x' = u - x$, so $\mathcal{J}(x_0, u) = \|u - x\|_2^2$, and it would be optimal to have $u \equiv x \equiv x_0$, but this is not allowed, since we require that $u \in \mathrm{L}^2$.

(c) The system Σ is J-coercive over $\mathcal{U}_{\text{out}}$ (since $\mathbb{D}^* J \mathbb{D} = 1 \gg 0$), hence we knew that there had to be a unique minimizing control over $\mathcal{U}_{\text{out}}$; however, Σ is not J-coercive over $\mathcal{U}_{\text{exp}}$ (an alternative proof for this is that $(s-A)^{-1}B = s^{-1}$ is not majorized by $\widehat{\mathbb{D}} \equiv 1$ on $i\mathbf{R}$; see Proposition 10.3.2(iv)&(i)).

The system $\Sigma_\flat^1$ is not J-coercive over either $\mathcal{U}_{\text{out}}$ or $\mathcal{U}_{\text{exp}}$.

Note that the state feedback $K = -1$ for Σ used in (b) is not r.c.-stabilizing, since $\mathbb{N} = \mathbb{M} = s/(s+1)$ has a common zero at $s = 0$; thus, the minimizing control $u = 0$ over $\mathcal{U}_{\text{out}}$ is lost in this preliminary stabilization. $\triangleleft$

Even if $S = 0$ and the J-critical control is not unique, there might be only one J-critical control in feedback form:

Example 9.13.6 ($\mathcal{U}_{\text{out}}$: Unique $\mathbb{K}_\circlearrowleft$ although $S = 0$) Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right]$ (i.e., $\mathbb{A} = I$ and $\mathbb{B} = 0 = \mathbb{C} = \mathbb{D}$). Then the eCARE becomes $0 = K^*SK$, $X^*SX = 0$, $X^*SK = 0$ (see Theorem 9.9.6(e2) and the remark in its proof). The admissible solutions are the ones with $X \in \mathcal{GB}(U)$; for them we have $S = 0$, so that $\mathcal{P}$, K and X can be arbitrary.

Since $B = 0$, we have $A_\circlearrowleft = A + BK = A$, so that $\mathbb{A}_\circlearrowleft = \mathbb{A} = I$ and hence $K = K_\circlearrowleft = 0$ is the only output-stabilizing state feedback operator (note that $H_1 := \mathrm{Dom}(A) = H$). Condition (P) requires that $\mathcal{P} = 0$. Thus, all P-output-stabilizing solutions are given by $(0, 0, \left[\begin{array}{c|c} 0 & I - X \end{array}\right])$ $(X \in \mathcal{GB}(U))$ (and they all are P-SOS-r.c.-stabilizing, hence $\mathcal{U}_{\text{out}}$-stabilizing, by Theorem 9.9.1(b)).

Obviously, $\mathcal{U}_{\text{out}}(x_0) = \mathrm{L}^2(\mathbf{R}_+; U)$ for all $x_0 \in H$, and each control is J-critical (the cost is zero for each control). Nevertheless, 0 is the only J-critical control in state feedback form. $\triangleleft$

(To obtain the corresponding discrete-time example we must set $A = I$, $B = C = D = 0$ (so that still $\mathbb{A} = I$, $\mathbb{B} = \mathbb{C} = \mathbb{D} = 0$).)

Even for $\mathbb{D}, \mathbb{X} \in \mathrm{MTIC}$, the operator D^*JD need not contain any information on the signature properties of the problem:

Example 9.13.7 [$D^*JD \gg 0 \gg S$] Let $\Sigma \in \mathrm{SOS}$, $\mathbb{D} = \left[\begin{smallmatrix} 2\tau^{-1} \\ I \end{smallmatrix}\right] \in \mathrm{MTIC_d}(U, U^2)$, $J = \left[\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right]$. Then $D^*JD = I \gg 0$ but $\mathbb{D}^* J \mathbb{D} = -3I = I^*SI \in \mathrm{TI}(U)$, where $S = -3I \ll 0$.

It follows that the CARE has a ULR unique $\mathcal{U}_{\text{out}}$-stabilizing solution, and this solution is q.r.c.-SOS-stabilizing and maximizing over $\mathcal{U}_{\text{out}}$ (and $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{str}}$ if $\mathbb{A}$ is strongly stable), by (9.139).

We can set $\mathbb{D} := \begin{bmatrix} 2\tau^{-1} \\ D_2 \end{bmatrix}$, $J = \begin{bmatrix} -I & 0 \\ 0 & \widetilde{J} \end{bmatrix}$ for any $Y_2, D_2 \in \mathcal{B}(U,Y_2)$, $\widetilde{J} = \widetilde{J}^* \in \mathcal{B}(Y_2)$ s.t. $\|D_2^*\widetilde{J}D_2\| < 4$ without affecting the above (except that S is altered but still $S \ll 0$). In particular, $D^*JD = D_2^*\widetilde{J}D_2$ may be uniformly/strictly/nonstrictly positive/negative, zero, or indefinite. $\triangleleft$

G. Weiss and H. Zwart [WZ] have shown that even if C is bounded and $\mathbb{D}, \mathbb{X} \in$ ULR, we may have $S \neq D^*JD$ and K unbounded:

Example 9.13.8 [C bounded, $B_{\mathrm{w}}^*\mathcal{P}, K$ not] Let $\left[\ \mathbb{A} \mid \mathbb{B}\ \right]$ be as in Example 9.8.15, but set

$$\mathcal{J}(x_0,u) := \int_0^\infty (\|\pi_{[0,1)}x(t)\|_H^2 + \|u(t)\|_U^2)\,dt, \tag{9.244}$$

i.e., $\left[\ C \mid D\ \right] := \left[\begin{array}{c|c} \pi_{[0,1)} & 0 \\ 0 & I \end{array}\right]$ and $J := I$. These operators are bounded, and one easily verifies $\mathbb{D}^*J\mathbb{D} = 2I$ and that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_0(\mathbf{C}, H, H \times \mathbf{C})$, where $H := \mathrm{L}^2(\mathbf{R}_+)$ (see (19) of [WZ]). Thus, $S := 2I$, $\mathbb{X} := I$ defines a spectral factorization of $\mathbb{D}^*J\mathbb{D}$ (by Corollary 9.9.11, this corresponds to the stabilizing solution over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$).

However, $D^*JD = I \neq S$, hence $\mathcal{P}[H] \not\subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and $K = -B_{\mathrm{w}}^*\mathcal{P}$ is unbounded (by Proposition 9.11.4(b3)). One can verify from Proposition 8.3.10 that $\mathcal{P} = \pi_{[0,1)} + \frac{1}{2}\pi_{[1,\infty)}$. (See [WZ] for details.) $\triangleleft$

Even for a minimal system (and standard LQR cost function), condition (PB) is not always superfluous:

Example 9.13.9 ((P) is necessary even for minimal weakly stable Σ) We construct here an exactly reachable and approximately observable (weakly, even strong*) stable system Σ with scalar input s.t. the LQR for Σ has a unique solution over $\mathcal{U}_{\mathrm{out}}$, but there are also other (non-PB-)r.c.-stabilizing solutions.

Take $U = \mathbf{C}$, $H = \mathrm{L}^2(\mathbf{R}_-;\mathbf{C})$, $Y = \ell^2(\mathbf{N})$, $\mathbb{A} = \tau\pi_-$, $\mathbb{B} = \pi_-$, $\mathbb{C}_1 = (2^{-k/2}\tau_{-k}\pi_{[-k,0]})_{k\in\mathbf{N}}$, $\mathbb{D}_1 = (2^{-k/2}\tau_{-k})_{k\in\mathbf{N}}$, $\mathbb{C} := \begin{bmatrix} \mathbb{C}_1 \\ 0 \end{bmatrix}$, $\mathbb{D} := \begin{bmatrix} \mathbb{D}_1 \\ I \end{bmatrix}$, $J = I$ to get a standard LQR form minimal (exactly reachable ($\mathbb{B}\mathbb{B}^* = \pi_- \gg 0 \in \mathcal{B}(H)$) and approximately observable ($\mathbb{C}^*\mathbb{C} = \sum_k 2^{-k}\pi_{[-k,0]} = \sum_{k=0}^\infty 2^{-k}\pi_{[-k-1,-k]} > 0$)) stable minimization problem (the minimization of $\mathcal{J}(u,x_0) := \|\mathbb{D}_1 u + \mathbb{C}_1 x_0\|^2 + \|u\|^2$).

The generators of $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ are $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, where $A = \frac{d}{d\theta}$, $B = \delta_0$, $H_1 := \mathrm{Dom}(A) = \mathrm{W}_0^{1,2}(\mathbf{R}_-)$, $(s-A)^{-1}B : u_0 \mapsto e^{s\cdot}u_0 \in H_B = \mathrm{W}^{1,2}(\mathbf{R}_-)$, $A^* = -\frac{d}{d\theta}$, $H_1^* := \mathrm{Dom}(A^*) = \mathrm{W}^{1,2}(\mathbf{R}_-)$, $B_{\mathrm{w}}^* = \delta_{0-}^*$; $C_1 := (2^{-k/2}\delta_{-k}^*)_k : H_1 \to \ell^2(\mathbf{N})$, $C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 0 \\ I \end{bmatrix}$ (here θ is the argument of an element $x_0 \in H$; cf. Example 6.2.14).

Let $\mathcal{P}$ be a multiplication operator, say, $\mathcal{P} \in \mathrm{L}^\infty(\mathbf{R}_-)\cdot$. Then $B_{\mathrm{w}}^*\mathcal{P}x_0 = (\mathcal{P}x_0)(0-) = 0$ for all $x_0 \in H_1$ (we use this only for $\mathcal{P}$ s.t. $\mathcal{P}x_0$ (left-)continuous at 0 for all $x_0 \in H_1$; the general case would follow as at the end of Example 6.2.14), hence then $K = 0$, $\mathbb{K} = 0$, $\mathbb{X} = X \in \mathcal{GB}(U)$ for any admissible multiplication operator solution (so it is optimal to have no feedback, because such would never catch up the τx_0 term moving towards $-\infty$). Thus, all admissible multiplication operator solutions are r.c.-stabilizing. The first equation of the eCARE becomes

$$\langle x_0', \mathcal{P}x_0\rangle_H + \langle x_0, \mathcal{P}x_0'\rangle_H = -\sum_{k=1}^{\infty} 2^{-k}|x_0(-k)|^2 \text{ for all } x_0 \in H_1. \tag{9.245}$$

Setting $\mathcal{P} := \sum_{k=1}^{\infty} r_k\pi_{[-k,-k+1]}$, the left-hand side becomes $\sum_{k=1}^{\infty} r_k[x_0(-k+1) - x_0(k)]$, hence we should have $r_2 - r_1 = -2^{-1}$, $r_3 - r_2 = -2^{-2}$, ..., $r_{n+1} - r_n = -2^{-n}$. Thus, $r_{n+1} = r_1 - \sum_{k=1}^{n} 2^{-k} = 2^{-n} + r_1 - 1$.

Therefore, $\mathcal{P}_r := rI + \sum_{k=0}^{\infty} 2^{-k}\pi_{[-k-1,-k]} \in \mathcal{B}(H)$ is a stabilizing solution of the eCARE for each $r \in \mathbf{R}$. Note that $S = I + \lim_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B = 1 + ((r+1)\mathrm{e}^{s\cdot})(0-) = 2 + r$, hence for $r \neq -2$ and $X = I$, the operator $\mathcal{P}$ becomes a stabilizing solution of the CARE.

Thus, $r = 0$ gives $\mathcal{P}_0 = \mathbb{C}^*\mathbb{C}$, the unique (by Proposition 9.8.11) P-r.c.-stabilizing solution (because it is the J-critical cost). For $r \in \mathbf{R}\setminus\{0\}$, the formula $\mathcal{J}(u_{\min}(x_0), x_0) = \langle x_0, \mathcal{P}_r x_0\rangle$ does not hold and

$$\mathbb{X}_r^* S_r \mathbb{X}_r = S_r = (2+r) \neq 2 = \mathbb{D}^* J\mathbb{D} \tag{9.246}$$

(but $\mathbb{D}^*J\mathbb{D} + \text{w-lim}_{t\to+\infty} \tau(t)^*\mathbb{B}^*\mathcal{P}_r\mathbb{B}\tau(t) = 2 + r$, as in (9.233)).

We conclude that the condition (PB) is, indeed, necessary, even for a minimal weakly stable WPLS. For other values of r, we get a stable, r.c.-stabilizing solution $(\mathcal{P}_r, S_r, K_r)$ s.t. $\mathcal{P}_r$ differs from the critical cost operator $\mathcal{P}_0$, and $\mathbb{X}_r^* S_r \mathbb{X}_r$ is not a spectral factorization of $\mathbb{D}^*J\mathbb{D}$ (since $K_r = 0$ and $\mathbb{X}_r = I$).

Finally, we note that since $\mathbb{A}_{\circlearrowleft} = \mathbb{A} + 0 = \tau\pi_-$ (for any $r \in \mathbf{R}$), we obtain that

$$\langle \mathbb{B}^t u, \mathcal{P}_r\mathbb{A}_{\circlearrowleft}^t x_0\rangle = \langle \pi_-\tau^t u, \mathcal{P}_r\tau^t\pi_- x_0\rangle = 0 \to 0, \quad \text{as } t \to +\infty \tag{9.247}$$

for all $r \in \mathbf{R}$ and $u \in \mathcal{U}_*^*(0)$ (even for all $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+; U)$), since $u = \pi_+ u$. Thus, (9.247) does not imply (P), since the latter is satisfied for $r = 0$ only. $\triangleleft$

Thus, if $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & \mid & \mathbb{F} \end{bmatrix})$ is a $\mathcal{U}_{\mathrm{out}}$-stabilizing (or $\mathcal{U}_{\mathrm{sta}}$-stabilizing) solution of the IARE, CARE or DARE, it can happen that there are solutions of form $(\mathcal{P}', S, \begin{bmatrix} \mathbb{K} & \mid & \mathbb{F} \end{bmatrix})$ (hence necessarily output-stabilizing) s.t. $\mathcal{P}' \gg \mathcal{P} > 0$ or $\mathcal{P} > 0 \gg \mathcal{P}'$, by Example 9.13.9 (discretize it for the DARE example). Thus, even q.r.c.-stabilization does not suffice, but we do have to verify the condition (P) (equivalently, that $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J\mathbb{C}_{\circlearrowleft}$) to avoid "fake cost" (or residual cost at infinity).

Above we added the copy of u to the output to get a standard LQR cost function; we could as well remove it and substract I from S and S_r.

Example 9.13.10 (Unique *J*-critical control though $D^*JD = 0$) In Example 9.13.9, we may remove the second row of $\mathbb{C}$ and $\mathbb{D}$, to obtain exactly same results except that then $D^*JD = 0$ and hence $S = 1$ and $S_r := 1 + r$. Indeed, still $SK = 0$ and $K^*SK = 0$, so that $(\mathcal{P}_r, S_r, 0)$ is again a stable, r.c.-stabilizing solution of the CARE for each $r \neq -1$ ($\mathcal{U}_{\mathrm{out}}$-stabilizing for $r = 0$).

Since S is one-to-one, the J-critical control is unique for each $x_0 \in H$. Note also that $\mathbb{D}^*\mathbb{D} = \sum_k (2^{-k/2})^2 = 2 = S = I^*SI$ is the corresponding I-spectral factorization and that $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{out}}$. $\triangleleft$

Also the system of Example 9.8.15 has $D = 0$ and a unique J-critical control over $\mathcal{U}_{\mathrm{out}}$; however, here we have several output-stabilizing solutions, so that the condition (PB) is needed also in this case.

The next example shows that for unstable $\mathbb{A}_{\circlearrowleft}$ (and non–internally stabilizing $\mathcal{P}$) the condition "$\mathcal{P}\mathbb{A}_{\circlearrowleft}^t x \to 0$ $(x \in H)$" of Lemma 9.10.1(d1) is not sufficient for (P):

Example 9.13.11 ($\mathcal{P}\mathbb{A}_{\circlearrowleft}^t x \to 0 \not\Rightarrow \mathbb{A}_{\circlearrowleft}^{t\,*}\mathcal{P}\mathbb{A}_{\circlearrowleft}^t x \to 0$) Let $H = \mathrm{L}^2(\mathbf{R}_+)$, $\mathbb{A}^t := \mathrm{e}^t\tau^{-t}$, $(\widetilde{\mathcal{P}}x)(s) := \mathrm{e}^{-2s}x(s)$, $D = I = J$, $B = 0$ (so that Σ is ULR). Let $\mathbb{C}$ be stable.

Then, for any solution of the eCARE with $X = I$, we have $\mathbb{A}_{\circlearrowleft} = \mathbb{A}$, $\mathbb{D}_{\circlearrowleft} = \mathbb{D} = D$, $S = I$, $K = -C$, $C_{\circlearrowleft} = C + DK = 0$, $C^*JC = K^*SK$, hence $(0, I, -C)$ is the unique r.c.-SOS-PB-stabilizing solution (hence $\mathcal{U}_{\mathrm{out}}$-stabilizing, by Theorem 9.9.1(b)).

However, $\widetilde{\mathcal{P}} = \mathbb{A}^{t\,*}\widetilde{\mathcal{P}}\mathbb{A}^t \geq 0$, as one easily verifies, hence also $(\widetilde{\mathcal{P}}, I, -C)$ is a r.c.-SOS-stabilizing solution of the eCARE and the eIARE. We have $\|\widetilde{\mathcal{P}}\mathbb{A}^t x\|_H^2 \leq \mathrm{e}^{-2t}\|x\|_H^2 \to 0$ for all $x \in H$, but $\mathbb{A}^{t\,*}\widetilde{\mathcal{P}}\mathbb{A}^t x = \widetilde{\mathcal{P}}x \not\to 0$ for $x \neq 0$. $\triangleleft$

One can add an unobservable and unreachable part to the semigroup and thus alter the properties of the solution without affecting the J-critical cost, nor the J-critical feedback or signature operator:

Example 9.13.12 (Wrong $\mathcal{P}$, fake cost) Let $\mathcal{P}$ be a stable, P-stabilizing solution for Σ and J, and let Σ be stable. Extend Σ to $\Sigma' := \left[\begin{array}{cc|c} \mathbb{A} & 0 & \mathbb{B} \\ 0 & \widetilde{\mathbb{A}} & 0 \\ \hline \mathbb{C} & 0 & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}_0(U, H \times \widetilde{H}, Y)$ with a non–strongly stable $\widetilde{\mathbb{A}}$ (so we have Σ and $\widetilde{\Sigma}$ put together into (stable) Σ').

$\mathcal{P}' := \begin{bmatrix} \mathcal{P} & 0 \\ 0 & \widetilde{\mathcal{P}} \end{bmatrix} = \mathcal{P}'^* \in \mathcal{B}(H \times \widetilde{H})$ is a stable, r.c.-stabilizing solution for Σ' and $J' = J$ iff $\widetilde{A}^*\widetilde{\mathcal{P}} + \widetilde{\mathcal{P}}\widetilde{A} = 0$ (because $(B'^*)_{\mathrm{w}}\mathcal{P}' = \begin{bmatrix} B_{\mathrm{w}}^*\mathcal{P} & 0 \end{bmatrix}$ and $\mathbb{B}'^*\mathcal{P}' = \begin{bmatrix} \mathbb{B}^*\mathcal{P} & 0 \end{bmatrix}$, thus $S' = S$, $\mathbb{K}' = \begin{bmatrix} \mathbb{K} & 0 \end{bmatrix}$, $\mathbb{F}' = \mathbb{F}$, $\mathbb{X}' = \mathbb{X}$). By Lemma 9.8.11(a), such a solution is P-stabilizing iff $\langle \widetilde{\mathbb{A}}\widetilde{x}_0, \widetilde{\mathcal{P}}\widetilde{\mathbb{A}}\widetilde{x}_0 \rangle \to 0$ for all $\widetilde{x}_0 \in \widetilde{H}$, because $\langle \mathbb{A}' \left[\begin{smallmatrix} x_0 \\ \widetilde{x}_0 \end{smallmatrix}\right], \mathcal{P}'\mathbb{A}' \left[\begin{smallmatrix} x_0 \\ \widetilde{x}_0 \end{smallmatrix}\right] \rangle = \langle \mathbb{A}x_0, \mathcal{P}\mathbb{A}x_0 \rangle + \langle \widetilde{\mathbb{A}}\widetilde{x}_0, \widetilde{\mathcal{P}}\widetilde{\mathbb{A}}\widetilde{x}_0 \rangle$, but, by the uniqueness of $\mathcal{P}'$, only $\widetilde{\mathcal{P}} = 0$ can satisfy this condition.

(a) Taking $\widetilde{\mathbb{A}} = \mathrm{e}^{it}$ ($\widetilde{A} = i = -\widetilde{A}^* \in \mathbf{C}^{1\times 1}$) we see that any $\widetilde{\mathcal{P}} \in \mathbf{R}$ defines a stable, r.c.-stabilizing solution $\mathcal{P}'$, but only $\widetilde{\mathcal{P}} = 0$ defines a P-stabilizing solution.

(b) Require, in addition, that Σ is weakly stable. Take $\widetilde{\mathbb{A}} = \tau$ on $\widetilde{H} := \mathrm{L}^2$. Then $\widetilde{\mathbb{A}}$ and hence also $\mathbb{A}'$ is weakly but not strongly stable. Moreover, $\widetilde{A}^* = -\widetilde{A}$, hence $\widetilde{\mathcal{P}} := rI$ defines a stable, r.c.-stabilizing solution for any $r \in \mathbf{R}$. $\triangleleft$

Note that if $\mathbb{A}$ and $\widetilde{\mathbb{A}}$ were strongly stable and $S \gg 0$, then the $\mathcal{U}_{\mathrm{out}}$-stabilizing solution $\left[\begin{smallmatrix} \mathcal{P} & 0 \\ 0 & 0 \end{smallmatrix}\right]$ would be the greatest solution of the eIARE having $S \geq 0$, by Corollary 9.2.11. By (a) (or (b)), the is not the case in the weakly stable case.

Naturally, the minimal cost can be negative:

Example 9.13.13 (Negative minimum, $\mathcal{P} \ll 0$) The system

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \tag{9.248}$$

is minimal and exponentially stable. For $J = \left[\begin{smallmatrix} -1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$, we have $\mathbb{D}^* J \mathbb{D} \gg 0$ (since $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}} = 2 - |s+1|^{-2} \ge 1$ for $s \in i\mathbf{R}$), so that the system is positively J-coercive over $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$.

The [e]CARE has a two solutions, $\mathcal{P} = -2 \pm \sqrt{2}$ (with $S = 2$, $K = -\mathcal{P}/2$). The smaller one is unstable, and the bigger one, $(-2+\sqrt{2}, 2, -1+2^{-1/2})$, minimizing over $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$. Thus, the minimal cost is given by $\langle x_0, \mathcal{P} x_0 \rangle = -(2-\sqrt{2})\|x_0\|^2$. $\triangleleft$

Finally, we give an example of a minimal exponentially stabilizable system, which has a $\mathcal{U}_{\text{str}}$-stabilizing (and $\mathcal{U}_{\text{str}}$-minimizing) solution that is not strongly stabilizing (hence not minimizing over $\mathcal{U}_{\text{exp}}$):

Example 9.13.14 **($\left[\begin{smallmatrix} \mathbb{A} \\ \hline \mathbb{C} \\ \mathbb{K} \end{smallmatrix}\right]$ and $\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right]$ stable but $\mathbb{B}, \mathbb{D}, \mathbb{F}, \mathbb{B}_\circlearrowleft$ unstable)** Let $A = (-k^{-1})_{k \in \mathbf{N}+1}$, $B = I$, $C = (\left[\begin{smallmatrix} k^{-1/2} \\ 0 \end{smallmatrix}\right])_{k \in \mathbf{N}+1}$, $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, $U := H := Y := \ell^2(\mathbf{N}+1)$, as in Example 6.1.14 (we have added a copy of u to the output), and set $J := I$.

By Example 6.1.14(b), $\left[\begin{smallmatrix} \mathbb{A} \\ \hline \mathbb{C} \end{smallmatrix}\right]$ is strongly stable and $\mathbb{B}$ and $\mathbb{D}$ are unstable. Since $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{out}}$, there is a unique $\mathcal{U}_{\text{out}}$-stabilizing solution of the CARE (=B_{w}^*-CARE)

$$S = I,\ K = -\mathcal{P},\ C^*C + A^*\mathcal{P} + \mathcal{P}A = \mathcal{P}^2. \tag{9.249}$$

since B (and A and C and D) is bounded. One can verify that this solution is given by the unique nonnegative diagonal solution, namely

$$\mathcal{P}_k = -k^{-1} + k^{-1}\sqrt{1+k} = -K_k \quad (k \in \mathbf{N}+1). \tag{9.250}$$

Since $\mathcal{P}$ is also $\mathcal{U}_{\text{str}}$-stabilizing (because $A_\circlearrowleft = A + BK = A - \mathcal{P} = (-k^{-1}\sqrt{1+k})$ is strongly stable), it must be $\mathcal{U}_{\text{sta}}$-stabilizing, by Lemma 8.3.3. By Proposition 10.7.3(d), $\mathcal{P}$ is also SOS-stabilizing. One can verify that $\mathbb{K}$ is stable but $\mathbb{F}$ is unstable. The unstability of $\mathbb{B}_\circlearrowleft$ and $\mathbb{B}_\circlearrowleft \tau$ follow as in Example 6.1.14(b).

Since there is no exponentially stabilizing solution (such a solution would be diagonal and nonnegative; an alternative proof follows from Corollary 9.7.2 and the uniqueness of $\mathcal{P}$), we can deduce that there is no J-critical control over $\mathcal{U}_{\text{exp}}$ (for some $x_0 \in H$), equivalently, $\mathbb{D}$ is not J-coercive over $\mathcal{U}_{\text{exp}}$, by Theorem 9.2.16.

(Recall from Example 6.1.14(b) that Σ is minimal and exponentially stabilizable but not detectable.) $\triangleleft$

In the above example, the cost function is coercive enough to provide a minimizing control over $\mathcal{U}_{\text{out}}$, $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{str}}$, but the existence of a minimizing control over $\mathcal{U}_{\text{exp}}$ would require even further coercivity.

Notes

Most of Proposition 9.13.1(c1) is from [WW, Example 11.5] and Example 9.13.8 is from [WZ].

9.14 $(J,*)$-critical factorization ($\mathbb{D}=\mathbb{N}\mathbb{M}^{-1}$)

There comes a critical moment where everything is reversed, after which the point becomes to understand more and more that there is something which cannot be understood.

— Søren Kierkegaard (1813-1855)

In this section, we shall develop an extension of canonical (H^2) factorization theory of $L^\infty(\partial\mathbf{D};\mathbf{C}^{n\times n})$ maps [LS] [GlaGoh] for the case where $\mathbf{C}^{n\times n}$ is replaced by $\mathcal{B}(U)$. Most of our results are given in continuous time.

We start by formulating an additional equivalent condition for a unique J-critical control to be of state feedback form:

Definition 9.14.1 (($J,*$)-critical factorization) *Assume that* $\mathcal{U}_*^*=\mathcal{U}_{\text{out}}$ *or* $\mathcal{U}_*^*=\mathcal{U}_{\text{exp}}$. *Let* $\mathbb{M}\in\mathcal{G}\mathrm{TIC}_\infty(U)$ *be s.t.* $\pi_+\left[\begin{smallmatrix}\mathbb{N}\\ \mathbb{M}\end{smallmatrix}\right]\pi_-\mathbb{M}^{-1}[L^2_\omega]\subset L^2$ *(and* $\pi_+\mathbb{B}\mathbb{M}\tau\pi_-\mathbb{M}^{-1}[L^2_\omega]\subset L^2$ *if* $\mathcal{U}_*^*=\mathcal{U}_{\text{exp}}$*) for some* $\omega\in\mathbf{R}$, *where* $\mathbb{N}:=\mathbb{D}\mathbb{M}$, *and*

$$\langle\mathbb{N}\pi_-\mathbb{M}^{-1}v,J\mathbb{D}\eta\rangle_{L^2(\mathbf{R}_+;Y)}=0\quad(\eta\in\mathcal{U}_*^*(0),\ v\in L^2_\omega(\mathbf{R}_-;U)).\tag{9.251}$$

Then $\mathbb{D}=\mathbb{N}\mathbb{M}^{-1}$ *is called a* $(J,*)$-critical factorization (of $\mathbb{D}$) over $\mathcal{U}_*^*$.

It follows that $\pi_+\left[\begin{smallmatrix}\mathbb{N}\\ \mathbb{M}\end{smallmatrix}\right]\pi_-\mathbb{M}^{-1}\in\mathcal{B}(L^2_\omega,L^2)$ (and $\mathbb{B}\mathbb{M}\tau\pi_-\mathbb{M}^{-1}\in\mathcal{B}(L^2_\omega,\subset L^2)$ for $\mathcal{U}_{\text{exp}}$), by Lemma A.3.6. Note that we may increase ω, since $L^2_{\omega'}(\mathbf{R}_-;U)\subset L^2_\omega$ for $\omega'>\omega$.

Lemma 9.14.2 *Make the assumptions and use the notation of Definition 9.14.1 [and assume that* $\mathcal{U}_*^*=\mathcal{U}_{\text{exp}}$*]. Then*

(a) $\left[\begin{smallmatrix}\mathbb{N}\\ \mathbb{M}\end{smallmatrix}\right][L^2_c(\mathbf{R};U)]\subset L^2$ *[and* $\mathbb{B}\mathbb{M}\tau[L^2_c]\subset L^2$*]; in particular,* $[\mathbb{B}\mathbb{M}\tau,]\mathbb{N},\mathbb{M}\in\mathrm{TIC}_\gamma$ *for all* $\gamma>0$.

(b) Set $\mathbb{T}:=\pi_+\mathbb{M}\pi_-\mathbb{M}^{-1}$, $\mathbb{S}:=\pi_-+\mathbb{T}=\mathbb{M}\pi_-\mathbb{M}^{-1}$. *Then* $\mathbb{T}\in\mathcal{B}(L^2_\omega,L^2)$ *and* $[\pi_+\mathbb{B}\tau\mathbb{S},]\mathbb{T},\pi_+\mathbb{D}\mathbb{S}\in\mathcal{B}(L^2_\omega,L^2)$.

(c) If Σ *is stable,* $\mathcal{U}_*^*=\mathcal{U}_{\text{out}}$, *and* S *is one-to-one, then all choices of* $(\mathbb{N},\mathbb{M})$ *are given by* $(\mathbb{N}E,\mathbb{M}E)$ $(E\in\mathcal{G}\mathcal{B}(U))$.

Proof: (a) Let $v_{\circlearrowleft}\in L^2_c(\mathbf{R}_-;U)$. Then $v:=\pi_-\mathbb{M}v_{\circlearrowleft}\in L^2_c\subset L^2_\omega$, hence $\mathbb{N}v_{\circlearrowleft}=\pi_-\mathbb{N}v_{\circlearrowleft}+\pi_+\mathbb{N}\pi_-\mathbb{M}^{-1}v\in L^2_c+L^2\subset L^2$. Thus, $\mathbb{N}L^2_c\subset L^2$. By Lemma 2.1.13, we have $\mathbb{N}\in\mathrm{TIC}_\gamma$ for all $\gamma>0$. Replace $\mathbb{N}$ by $\mathbb{M}$ [and then by $\mathbb{B}\mathbb{M}\tau$] to obtain the other claims.

(b) Now $\mathbb{S}=\pi_-\mathbb{M}\pi_-\mathbb{M}^{-1}+\pi_+\mathbb{M}\pi_-\mathbb{M}^{-1}=\mathbb{M}\pi_-\mathbb{M}^{-1}$. hence $\mathbb{D}\mathbb{S}=\mathbb{N}\pi_-\mathbb{M}^{-1}$. By assumption, $[\pi_+\mathbb{B}\tau\mathbb{S},]$ $\mathbb{T}$ and $\pi_+\mathbb{D}\mathbb{S}$ map $L^2_\omega\to L^2$, hence $[\pi_+\mathbb{B}\tau\mathbb{S},]\mathbb{T},\pi_+\mathbb{D}\mathbb{S}\in\mathcal{B}(L^2_\omega,L^2)$, by Lemma A.3.6.

(c) This follows from Theorem 9.14.3 and Theorem 8.3.13(f) (since $\mathbb{N}\mathbb{M}^{-1}$ is independent on $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$). □

Now we are ready to show that a strictly optimal control is of state feedback form iff there is a J-critical factorization:

Theorem 9.14.3 (JCF$\Leftrightarrow \exists \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$) *Assume that $\mathbb{D}$ has a unique J-critical control over $\mathcal{U}_*^*$ for each $x_0 \in H$, where $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ or $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$. Then the following are equivalent:*

(i) $\mathbb{D}$ has a $(J,*)$-critical factorization over $\mathcal{U}_*^*$;

(ii) there is a J-critical state feedback pair for Σ over $\mathcal{U}_*^*$;

(iii) there is a $\mathcal{U}_*^*$-stabilizing solution of the eIARE.

Moreover, if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ solves (i), then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] := \left[\begin{array}{c|c} \mathbb{M}^{-1}\mathbb{K}_{\text{crit}} & I - \mathbb{M}^{-1} \end{array}\right]$ is a J-critical state feedback pair for Σ over $\mathcal{U}_*^*$. Conversely, if $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ solves (ii), then $\mathbb{M} := (I - \mathbb{F})^{-1}$ and $\mathbb{N} := \mathbb{D}\mathbb{M}$ form a $(J,*)$-critical factorization over $\mathcal{U}_*^*$.

A sufficient condition for (i)–(iii) (and for the eIARE becoming a CARE) is that any of (1.)–(4.) of Remark 9.9.14 holds and $D^*JD \in \mathcal{GB}(U)$.

In either case, Theorem 8.3.13 applies. See Theorem 9.9.1(a1)&(e) for the correspondence of (ii) to (iii).

Proof: (In fact, it would suffice to assume that there is a J-critical control in WPLS form for Σ, as one observes from the proof of Theorem 8.3.13(c2) and from Proposition 9.3.1.)

1° *(ii)⇒(i):* Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be J-critical (as in (ii)), and set $\mathbb{M} := (I - \mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$. Then $(\pi_+\mathbb{N}\pi_-)\mathbb{M}^{-1} = (\mathbb{C}_{\text{crit}}\mathbb{B}\mathbb{M})\mathbb{M}^{-1} = \mathbb{C}_{\text{crit}}\mathbb{B}$, hence (9.251) holds for any $\omega > \omega_A$. Because $\pi_+\mathbb{M}\pi_-\mathbb{M}^{-1} = \mathbb{K}_{\text{crit}}\mathbb{B} =: \mathbb{T}$, the requirements of Definition 9.14.1 are satisfied in case $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$. For $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, we have on $\mathbf{R}_+$ that

$$\mathbb{B}\mathbb{M}\tau\pi_-\mathbb{M}^{-1} = \mathbb{B}\tau(\pi_- + (\pi_+\mathbb{M}\pi_-)\mathbb{M}^{-1}) \tag{9.252}$$

$$= \mathbb{B}\tau\pi_- + \mathbb{B}\tau(\mathbb{K}_{\text{crit}}\mathbb{B})\mathbb{M}\pi_-\mathbb{M}^{-1} = \mathbb{A}^{\cdot}\mathbb{B} + \mathbb{B}\tau\mathbb{K}_{\text{crit}}\mathbb{B} = \mathbb{A}^{\cdot}_{\text{crit}}\mathbb{B}, \tag{9.253}$$

and $\pi_+\mathbb{A}^{\cdot}_{\text{crit}}\mathbb{B}$ maps $\mathrm{L}^2_\omega \to \pi_+\mathrm{L}^2$, because $\mathbb{A}_{\text{crit}}$ is exponentially stable.

2° *(i)⇒(ii):* With the notation of Lemma 9.14.2(b) (assume, in addition, that $\omega > \omega_A$), $\mathbb{T}$ is the operator of Theorem 8.3.13(a), and $\mathbb{T}\mathbb{M} = \pi_+\mathbb{M}\pi_-$, hence (ii) holds and $\left[\begin{array}{c|c} \mathbb{M}^{-1}\mathbb{K}_{\text{crit}} & I - \mathbb{M}^{-1} \end{array}\right]$ is J-critical, by Theorem 8.3.13(c2).

3° *(ii)⇔(iii):* See Theorem 9.9.1(a1)&(e).

4° The "moreover" claims were established in 1°–2°. The sufficient condition is obtained from Remark 9.9.14(d). □

Next we establish a canonical (H^2) factorization as a special case of J-critical factorization. The main result of this section, Theorem 9.14.6, is given in discrete time, as are the classical results, but we start with two continuous time results.

If a function is H^∞ over the unit circle, then it is also H^2 over the unit circle. The same does not hold in continuous time, hence our continuous time results and assumptions differ somewhat from those of Theorem 9.14.6. We start with the exponentially stable case:

Theorem 9.14.4 (Exponential H^2-SpF) *Assume that $\widehat{\mathbb{D}} - D \in \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_{-\varepsilon};\mathcal{B}(U,Y))$ for some $\varepsilon > 0$, $J = J^* \in \mathcal{B}(Y)$, and $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_+;U))$.*

Then there are $\varepsilon' > 0$ and $\widehat{\mathbb{X}} \in \mathcal{G}(\mathcal{B}(U) + \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_{-\varepsilon'};\mathcal{B}(U)))$, s.t. $\mathbb{X}^*(D^*JD)\mathbb{X} = \mathbb{D}^*J\mathbb{D}$, $X = I$ and $D^*JD \in \mathcal{GB}(U)$. In particular, $\widehat{\mathbb{X}}^*(D^*JD)\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ on $i\mathbf{R}\cup\{\infty\}$.

Moreover, $\mathbb{D} = (\mathbb{D}\mathbb{X}^{-1})\mathbb{X}$ is the unique $(J,*)$-spectral factorization of $\mathbb{D}$ over $\mathcal{U}_{\text{out}}$ having $X = I$.

Note that $\mathrm{H}^2_{\text{strong}}(\mathbf{C}^+_{-\varepsilon};\mathcal{B}(*)) \subset \mathrm{H}^\infty(\mathbf{C}^+_{-\varepsilon/2};\mathcal{B}(*))$, by Lemma F.3.2(a1).

By Lemma 2.2.2(d), we have $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \gg 0$ iff $\mathbb{D}^*J\mathbb{D} \gg 0$, equivalently, iff $\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}} \gg 0$ in $\mathrm{L}^\infty_{\text{strong}}(i\mathbf{R};\mathcal{B}(U))$, by Theorem 3.1.3(a1)&(e1). However, in the indefinite case, the invertibility of the Toeplitz operator is a strictly stronger condition than the invertibility of $\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ (i.e., of $\mathbb{D}^*J\mathbb{D}$). This also applies to Proposition 9.14.5 and Theorem 9.14.6.

Proof of Theorem 9.14.4: See the proof of Theorem 9.2.14(c2) (and (a)2°). The uniqueness follows from Lemma 9.14.2(c). □

In the positive case, we may give up exponential stability:

Proposition 9.14.5 (Positive H^2-SpF) *Assume that $\widehat{\mathbb{D}} - D \in \mathrm{H}^\infty \cap \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+;\mathcal{B}(U,Y))$, $J = J^* \in \mathcal{B}(Y)$, and $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \gg 0$.*

Then there is $\widehat{\mathbb{X}} \in \mathcal{G}(\mathcal{B}(U) + \mathrm{H}^\infty \cap \mathrm{H}^2_{\text{strong}}(\mathbf{C}^+;\mathcal{B}(U)))$ s.t. $\mathbb{X}^*(D^*JD)\mathbb{X} = \mathbb{D}^*J\mathbb{D}$, $X = I$ and $D^*JD \in \mathcal{GB}(U)$. Moreover $\mathbb{D} = (\mathbb{D}\mathbb{X}^{-1})\mathbb{X}$ is the unique $(J,*)$-critical factorization of $\mathbb{D}$ over $\mathcal{U}_{\text{out}}$ having $X = I$.

It follows that, $\langle\widehat{\mathbb{X}}u_0, D^*JD\widehat{\mathbb{X}}u_0\rangle_U = \langle\widehat{\mathbb{D}}u_0, J\widehat{\mathbb{D}}u_0\rangle_Y$ a.e. on $i\mathbf{R}$, for any $u_0 \in U$.

An analogous indefinite result also holds, except that $\widehat{\mathbb{X}}^{\pm 1}$ need not be stable; indeed, the continuous-time equivalent of Theorem 9.14.6 also holds under the assumption that $\widehat{\mathbb{D}} - D \in \mathrm{H}^\infty \cap \mathrm{H}^2_{\text{strong}}$ (use Theorem 9.9.6 instead of Theorem 14.1.6 in the proof, etc.). Note that $\mathbb{D},\mathbb{X} \in \mathrm{ULR}$ in both cases.

Proof of Proposition 9.14.5: Take a strongly stable realization Σ of $\mathbb{D}$ with B bounded (see Theorem 6.9.1(a)&(d2)). Now we obtain our claims from Theorem 10.6.3(a) (see also (d)) (which also contains the converse) combined with Lemma 10.6.2(d)(7.) except for the regularity of $\mathbb{X}$; but, by Theorem 6.9.1(a), $\mathbb{X} - I$ and $\mathbb{X}^{-1} - I$ are in $\mathrm{H}^2_{\text{strong}}(\mathbf{C}^+;\mathcal{B}(U))$. By Theorem 9.14.3, $\mathbb{N}\mathbb{M}^{-1}$ is a $(J,*)$-critical factorization. The uniqueness follows from Lemma 9.14.2(c).

(Note that $\mathbb{X}$ corresponds to a J-critical state feedback pair for Σ over $\mathcal{U}_{\text{out}}$ and to a strongly r.c.-stabilizing solution of the B^*_{w}-CARE.) □

Now we establish the canonical factorization for $\mathrm{L}^\infty_{\text{strong}}(\partial\mathbf{D};\mathcal{B}(U))$ functions of form $\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$, $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(U))$, with an invertible Toeplitz operator:

Theorem 9.14.6 (Discrete H^2-factorization) *Assume that $\mathbb{D} \in \mathrm{tic}(U,Y)$ and $J = J^* \in \mathcal{B}(Y)$ are s.t. $\pi^+\mathbb{D}^*J\mathbb{D}\pi^+ \in \mathcal{GB}(\ell^2(\mathbf{N};U))$.*

Then $\mathbb{D}$ has a unique $(J,*)$-critical factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ over $\mathcal{U}_{\text{out}}$ s.t. $M = I$. Moreover, $\widehat{\mathbb{X}} := \widehat{\mathbb{M}}^{-1} \in \mathcal{G}\mathrm{H}(\mathbf{D};\mathcal{B}(U))$, $\mathbb{N} = \mathbb{D}\mathbb{X}^{-1} \in \mathrm{H}^2_{\text{strong}}(\mathbf{D};\mathcal{B}(U,Y))$ and

$$\langle \mathbb{N}u, J\mathbb{N}v\rangle = \langle u, Sv\rangle \qquad (u,v \in \ell^1(\mathbf{Z};U)) \tag{9.254}$$

for some $S = S^* \in \mathcal{GB}(U)$. Furthermore, $\widehat{\mathbb{M}}, \widehat{\mathbb{X}}(\bar{\cdot})^* \in \mathrm{H}^2_{\text{strong}}(\mathbf{D};\mathcal{B}(U))$, $(\pi^+\mathbb{D}^*J\mathbb{D}\pi^+)^{-1} = \mathbb{M}\pi^+S^{-1}\mathbb{M}^* \in \mathcal{GB}(\ell^2(\mathbf{N};U))$, and all claims in (a1)–(c2) of Lemma 14.2.8 hold.

If $\dim U < \infty$, then $\widehat{\mathbb{X}}, \widehat{\mathbb{M}} \in \mathrm{H}^2(\mathbf{D};\mathcal{B}(U)) \cap \mathrm{L}^2(\partial\mathbf{D};\mathcal{B}(U))$, and $\widehat{\mathbb{X}} \in \mathcal{GB}(U)$ and $\widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ a.e. on $\partial\mathbf{D}$.

If $\mathbb{D}$ is exponentially stable, then so are $\mathbb{X}$ and $\mathbb{X}^{-1}$, and then we have $\mathbb{N}^*J\mathbb{N} = S$ and $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$, i.e.,

$$\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}} = \widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} \quad \text{on } \partial\mathbf{D}. \tag{9.255}$$

(In the above theorem, we have given up Standing Hypotheses on Σ.) By using the Cayley transform, we obtain the corresponding continuous-time "factorization". However, this factorization need not be well-posed, since the "factor" operator $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}(\mathbf{C}^+;\mathcal{B}(U))$ (for $\widehat{\mathbb{D}} \in \mathrm{TIC}(U,Y)$) may satisfy $\mathbb{X}, \mathbb{X}^{-1} \notin \mathrm{TIC}_\infty$ (i.e., $\widehat{\mathbb{X}}, \widehat{\mathbb{X}}^{-1} \notin \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U))$ for all $\omega \in \mathbf{R}$), even when $\dim U < \infty$, as shown in Example 8.4.13 (see also Example 11.3.7). Thus, there is no equivalent result in continuous time (unless we have $\mathbb{D}^*J\mathbb{D} \gg 0$ or $\mathbb{D}$ is assumed to be sufficiently regular; cf. the remark below Proposition 9.14.5).

By Lemma 14.2.8(d), we have

$$\lim_{t\to+\infty} \langle \mathbb{X}u, S\pi_{[0,t)}\mathbb{X}v\rangle = \langle \mathbb{D}u, J\mathbb{D}v\rangle \quad (u,v \in \ell^2(\mathbf{N};U)). \tag{9.256}$$

However, we cannot write "$\widehat{\mathbb{X}}^*S\widehat{\mathbb{X}}$", because it is not even known whether $\widehat{\mathbb{X}}$ has a boundary function. Thus, to obtain $\widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$, we must, e.g., assume that $\mathbb{D}$ is exponentially stable ($\widehat{\mathbb{D}} \in \mathrm{H}^\infty(r\mathbf{D};\mathcal{B}(U))$ for some $r > 1$), in which case the above factorization becomes an exponential spectral factorization).

If $\mathbb{D}^*J\mathbb{D} \gg 0$, then $S \gg 0$, as one observes from the proof and Lemma 9.10.3, hence then it follows from (9.256) that $\mathbb{X}$ is in tic, and from (9.254) that $\mathbb{M}$ is in tic; in particular, then $\mathbb{X} = \mathbb{M}^{-1} \in \mathcal{G}\mathrm{tic}$ (i.e., $\widehat{\mathbb{X}} = \widehat{\mathbb{M}}^{-1} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{D};\mathcal{B}(U))$), so that this would be an alternative (system-theoretic) proof for the positive spectral factorization result Lemma 5.2.1(a) (for operators of form $\mathbb{D}^*J\mathbb{D}$, as is the case with the applications of the lemma in this monograph). (Note that we obtain the corresponding continuous-time result through the Cayley transform in this positive case.)

Proof of Theorem 9.14.6: Choose a strongly stable realization $\Sigma = \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{wpls}(U,H,Y)$ of $\mathbb{D}$. By Proposition 8.3.10, there is a unique J-critical control for each $x_0 \in H$. By Theorem 9.14.3, this corresponds to a $(J,*)$-critical factorization (which is unique, by Lemma 9.14.2(c)) and to a $\mathcal{U}_{\rm out}$-stabilizing solution $(\mathcal{P}, S, K)$ of the DARE (we have $S \in \mathcal{GB}(U)$, by Lemma 9.10.3).

Let $\Sigma_\circlearrowleft$, $\mathbb{X}$, $\mathbb{M}$ and $\mathbb{N}$ be as in Definition 14.1.1, so that $\mathbb{A}_\circlearrowleft$, $\mathbb{C}_\circlearrowleft$ and $\mathbb{K}_\circlearrowleft$ are strongly stable by Theorem 8.3.9(a3). Now we obtain the other claims from Lemma 14.2.8. □

We finish this section by a discussion on spectral and other J-critical factorizations in continuous time.

By Example 8.4.13, J-coercivity over $\mathcal{U}_{\rm out}$ does not imply the existence of a spectral factorization in the indefinite case (see also Example 11.3.7). Instead, we need some additional assumptions, as in Theorems 8.4.9 and 9.2.14.

In discrete time, we always have the "H^2-factor" (the $(J,*)$-critical factor) of Theorem 9.14.6, and it suffices for the purposes of optimal control (by Theorems 9.14.3 and 8.4.3). If the I/O map is exponentially stable, then this J-critical factor is a spectral factor (cf. Theorem 14.3.2).

For a continuous-time I/O map $\mathbb{D}$, the corresponding factor and its inverse may be non-well-posed (i.e., $\mathbb{X}^{\pm 1} \notin \mathrm{TIC}_\infty$, as in Example 11.3.7), and the corresponding "state feedback pair" may thus be non-well-posed.

If $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U,Y))$ is $\mathrm{H}^\infty(\mathbf{K}^c; \mathcal{B}(U,Y))$ for some compact $K \subset \mathbf{C}^-$, then an analogous condition is satisfied by $\widehat{\mathbb{X}}^{\pm 1}$ (since this condition holds iff the map is a Cayley transform of an exponentially stable (discrete time) map), and in this case we do have an exponentially stable spectral factorization (whenever the Popov Toeplitz operator is invertible), but this condition is too strong for our purposes. Unfortunately, it does not seem that continuous time exponential stability would be a sufficient condition (it seems likely that both $\widehat{\mathbb{X}}$ and $\widehat{\mathbb{X}}^{-1}$ might still be unbounded near infinity).

However, if $\widehat{\mathbb{D}} - D \in \mathrm{H}^\infty \cap \mathrm{H}^2_{\rm strong}$ over $\mathbf{C}^+$ (for some $D \in \mathcal{B}(U,Y)$), then the "H^2-spectral factorization" mentioned above is, indeed, well-posed (and ULR) in continuous time too, as noted below Proposition 9.14.5; Then also the continuous time form of Theorem 9.14.6 and its proof are valid.

(In the above setting, the discretized $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is a $\mathcal{U}_{\rm out}$-stabilizing solution of the discretized eIARE, hence the original proof of Theorem 9.14.6 leads to corresponding results on $\Delta^{\rm S}\mathbb{D}$ and $\Delta^{\rm S}\mathbb{X}$, but these are not as strong as the ones obtained from the continuous time version.)

Notes for Sections 9.14 and 9.15

Theorem 9.15.3 is a basic result of classical (generalized) canonical factorization theory, and also variants of Lemmas 9.15.2 and 9.15.5 are well known.

Actually, in the classical theory, one can replace $\mathbb{D}^* J \mathbb{D}$ by any ti_0 operator (i.e., $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$ by any $\mathrm{L}^\infty(\partial\mathbf{D}; \mathbf{C}^{n\times n})$ function), and $\partial\mathbf{D}$ can be replaced by any "standard contour" (and L^2 by L^p). Thus, Theorem 9.14.6 only extends the part of the factorization theory that is needed for standard control theory.

Classical references on the subject in English include [CG81] and [LS], and an up-to-date book [BKS] appeared during the referee process for this monograph. See also the notes on p. 148.

9.15 H^2-factorization when $\dim U < \infty$

Divide et impera!

— Louis XI

We present here some basic facts on a weaker form of finite-dimensional canonical (generalized) factorizations for later use. We mainly work in discrete time (i.e., on $\mathbf{D}$, not $\mathbf{C}^+$); in particular, we relax Standing Hypothesis 9.0.1 (except in Lemma 9.15.4); we still assume that H and Y are Hilbert spaces.

Let $n \in \mathbf{N}+1$, $U := \mathbf{C}^n$. Set $\mathrm{H}^2 := \mathrm{H}^2(\mathbf{D};\mathbf{C}^n)$. Recall that $\mathbf{D} = \{z \in \mathbf{C} \,\big|\, |z| < 1\}$, hence $\partial\mathbf{D} = \{z \in \mathbf{C} \,\big|\, |z| = 1\}$. Note that $\widehat{\pi^+} \sum_{n=-\infty}^{\infty} u_n z^n = \sum_{n=0}^{\infty} L_n u^n$ for each $(u_n)_{n\in\mathbf{Z}} \in \cap_{r>0} \ell^2_r(\mathbf{Z};\mathbf{C}^n)$.

Definition 9.15.1 ($\mathcal{G}\mathrm{H}^2$-factorization) *Let $\mathbb{D} \in \mathrm{tic}(\mathbf{C}^n, Y)$ and $J = J^* \in \mathcal{B}(Y)$. We say that $\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}}$ is a $\mathcal{G}\mathrm{H}^2$-factorization of $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$ if $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}^2(\mathbf{D};\mathbf{C}^{n\times n})$, $S = S^* \in \mathcal{G}\mathcal{B}(\mathbf{C}^{n\times n})$, and $\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}} = \widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$ a.e. on $\partial\mathbf{D}$.*

In standard factorization theory, a $\mathcal{G}\mathrm{H}^2$-factorization with the additional condition "$\widehat{\mathbb{X}}^{-1}\widehat{\pi^+}\widehat{\mathbb{X}}$ is bounded on $\mathrm{L}^2(\partial\mathbf{D};\mathbf{C}^n)$" (cf. Theorem 9.15.3) is called a "generalized canonical (right-)factorization relative to L^2" (see, e.g., pp. 142–143 of [CG81]).

A $\mathcal{G}\mathrm{H}^2$-factorization is unique up to an invertible constant matrix, and one can always redefine $\widehat{\mathbb{X}}$ so that $S = J_1$ (i.e., make S a diagonal matrix with diagonal elements ± 1):

Lemma 9.15.2 (Uniqueness) *Let $\mathbb{D} \in \mathrm{tic}(\mathbf{C}^n, Y)$ and $J = J^* \in \mathcal{B}(Y)$. Let $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$ have a $\mathcal{G}\mathrm{H}^2$-factorization $\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}}$.*

Then all $\mathcal{G}\mathrm{H}^2$-factorizations of $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$ are given by $(E\widehat{\mathbb{X}})^*(E^{-*}SE^{-1})(E\widehat{\mathbb{X}})$ ($E \in \mathcal{G}\mathcal{B}(\mathbf{C}^n)$), and there is $E \in \mathcal{G}\mathcal{B}(\mathbf{C}^n)$ s.t. $(E^{-*}SE^{-1}) = J_1 := \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right] \in \mathcal{B}(\mathbf{C}^k \times \mathbf{C}^{n-k})$ for some $k \in \{0, 1, \ldots, n\}$.

Moreover, then $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}} = (\widehat{\mathbb{X}}^{-1} S^{-1} \widehat{\mathbb{X}}^{-*})^{-1} \in \mathcal{G}\mathrm{L}^1(\partial\mathbf{D};\mathbf{C}^{n\times n})$ and $\widehat{\mathbb{X}}^{\pm 1} \in \mathcal{G}\mathbf{C}^{n\times n}$ a.e. on $\partial\mathbf{D}$.

Proof: 1° Obviously, any $E \in \mathcal{G}\mathcal{B}(\mathbf{C}^n)$ defines a $\mathcal{G}\mathrm{H}^2$-factorization. By Lemmas 2.4.4 and 2.4.1, we have $S = (VE')^* J_1 (VE')$ for some $V, E' \in \mathcal{B}(U)$ (set $k := \dim H_+$).

2° By Theorem 3.3.1(e)&(a4), the functions $\widehat{\mathbb{D}} \in \mathrm{H}^\infty \subset \mathrm{H}^2$, $\widehat{\mathbb{M}} := \widehat{\mathbb{X}}^{-1} \in \mathcal{G}\mathrm{H}^2$, $\widehat{\mathbb{N}} := \widehat{\mathbb{D}}\widehat{\mathbb{X}} \in \mathrm{H}^2$ have L^2 boundary functions on $\partial\mathbf{D}$.

By continuity, $\widehat{\mathbb{M}}\widehat{\mathbb{X}} = I = \widehat{\mathbb{X}}\widehat{\mathbb{M}}$, $\widehat{\mathbb{N}} = \widehat{\mathbb{D}}\widehat{\mathbb{M}}$ and $\widehat{\mathbb{N}}^* J \widehat{\mathbb{N}} = S$ a.e. on $\partial\mathbf{D}$, hence $\widehat{\mathbb{X}}, \widehat{\mathbb{M}}, \widehat{\mathbb{N}}^* J \widehat{\mathbb{N}}, \widehat{\mathbb{D}}^* J \widehat{\mathbb{D}} \in \mathcal{G}\mathcal{B}(U)$ a.e. on $\partial\mathbf{D}$ and $(\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}})^{-1} = \widehat{\mathbb{X}}^{-1} S^{-1} \widehat{\mathbb{X}}^{-*} \in \mathrm{L}^1(\partial\mathbf{D};\mathcal{B}(U))$, by The Hölder Inequality.

3° *All $\mathcal{G}\mathrm{H}^2$-factorizations:* Conversely, let $\widehat{\mathbb{X}}_2^* J_1 \widehat{\mathbb{X}}_2$ be a $\mathcal{G}\mathrm{H}^2$-factorization of $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$. Set $\widehat{\mathbb{X}} := \widehat{\mathbb{X}}_1$, $\widehat{\mathbb{N}}_1 := \widehat{\mathbb{N}}$ (redefined so that $S = J_1$).

By The Hölder Inequality, $E := \widehat{\mathbb{X}}_1 \widehat{\mathbb{X}}_2^{-1} \in \mathcal{G}\mathrm{H}^1(\mathbf{D};\mathcal{B}(U))$. From equations $J_1 = \widehat{\mathbb{N}}_1^* J \widehat{\mathbb{N}}_1$ and $\widehat{\mathbb{N}}_2 = \widehat{\mathbb{N}}_1 E$, we obtain that $E^* J_1 E = J_1$, hence $E = J_1^{-1} E^{-*} J_1$

a.e. on $\partial\mathbf{D}$. By Lemma B.3.6, $(E^{-1})^{\mathrm{d}} := E(\bar z)^{-*} \in \mathcal{G}\mathrm{H}^1(\mathbf{D};\mathcal{B}(U))$. Set $E(z) := J_1^{-1}E(1/\bar z)^{-*}J_1$ for $z \in \overline{\mathbf{D}}^c$, so that $E \in \mathcal{B}(U)$, by Proposition D.1.20. But $\widehat{\mathbb{X}}_1 = E\widehat{\mathbb{X}}_2$, as required. □

If the Popov Toeplitz operator is invertible (i.e., $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{out}}$), then $\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ has a $\mathcal{G}\mathrm{H}^2$-factorization:

Theorem 9.15.3 ($\exists(\pi^+\mathbb{D}^*J\mathbb{D}\pi^+)^{-1} \Rightarrow \exists\mathcal{G}\mathrm{H}^2$-factorization) *Let $\mathbb{D} \in \mathrm{tic}(\mathbf{C}^n,Y)$ and $J = J^* \in \mathcal{B}(Y)$. If $\pi^+\mathbb{D}^*J\mathbb{D}\pi^+ \in \mathcal{G}\mathcal{B}(\ell^2(\mathbf{N};\mathbf{C}^n))$, then $\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ has a $\mathcal{G}\mathrm{H}^2$-factorization $\widehat{\mathbb{X}}^*S\widehat{\mathbb{X}}$.*

Moreover, $\widehat{\mathbb{X}}^{-1}\widehat{\pi^+}\widehat{\mathbb{X}}$ is bounded on $\mathrm{L}^2(\partial\mathbf{D};\mathbf{C}^n)$ and $\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{L}^\infty(\partial\mathbf{D};\mathbf{C}^{n\times n})$.

Naturally, the Toeplitz invertibility condition can be written as $\widehat{\pi^+}\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}\widehat{\pi^+} \in \mathcal{G}\mathcal{B}(H^2(\mathbf{D};\mathbf{C}^n))$,

Proof: The existence of $\widehat{\mathbb{X}}$ and S with $\widehat{\mathbb{X}}^{-1}\widehat{\pi^+}\widehat{\mathbb{X}}$ bounded follows from Theorem 9.14.6. By Lemma 2.2.2(d), $\mathbb{D}^*J\mathbb{D} \in \mathcal{G}\mathrm{ti}(U)$, hence $\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}} \in \mathcal{G}\mathrm{L}^\infty$, by Lemma 13.1.5. □

In fact, the coercivity assumption can be replaced by a weaker condition:

Lemma 9.15.4 (CT: $\left[\begin{array}{c|c}\mathbb{K}_{\mathrm{crit}} & \mathbb{F}_{\mathrm{crit}}\end{array}\right] \Rightarrow \mathcal{G}\mathrm{H}^2$-factorization) *Let $\Sigma = \left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{WPLS}(\mathbf{C}^n,H,Y)$ be s.t. $\mathbb{B}$ and $\mathbb{D}$ are stable, and let $J = J^* \in \mathcal{B}(Y)$. Let $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ be a J-critical state feedback pair for Σ over $\mathcal{U}_{\mathrm{out}}$.*

Then $(\widehat{\heartsuit}\widehat{\mathbb{X}})^*S(\widehat{\heartsuit}\widehat{\mathbb{X}}) = (\widehat{\heartsuit}\widehat{\mathbb{D}})^*J(\widehat{\heartsuit}\widehat{\mathbb{D}})$ is a $\mathcal{G}\mathrm{H}^2$-critical factorization, where $\mathbb{X} := I - \mathbb{F}$ and S is the corresponding signature operator. □

(This follows from Lemma 9.12.8(d) and Lemma 13.2.1(e2). In fact, any $\mathcal{U}_*^*$ with $\vartheta = 0$ would do, with the same proof.)

We shall also need the following local variant of Proposition 5.2.2, which states that if $\mathbb{D}$ is holomorphic around some subarc of $\partial\mathbf{D}$, then so are $\mathbb{X}^{\pm 1}$:

Lemma 9.15.5 (Local holomorphic extension) *Assume that $\widehat{\mathbb{E}} \in \mathrm{L}^\infty(i\mathbf{R};\mathbf{C}^{n\times n})$ and that $\widehat{\mathbb{X}},\widehat{\mathbb{Y}} \in \mathrm{H}^2(\mathbf{D};\mathbf{C}^{n\times n})$. Let $\Omega \subset \mathbf{C}$ be open and $\Gamma := \Omega\cap\partial\mathbf{D} \neq \emptyset$. If $\widehat{\mathbb{E}} = \widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$ a.e. on $\partial\mathbf{D}$ and $\widehat{\mathbb{E}}_{|\Gamma}$ has an holomorphic extension to Ω, then $\widehat{\mathbb{X}}^{\pm 1}$ and $\widehat{\mathbb{Y}}^{\pm 1}$ have holomorphic extensions to $\mathbf{D}\cup\Omega$ and $\mathbf{D}\cup\{1/\bar s \,|\, s \in \Omega\}$, respectively.*

(Note that $\Omega' := \Omega\cap\{1/\bar s \,|\, s \in \Omega\}$ contains Γ. Note also that if $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(\mathbf{C}^n,Y))$ and $\widehat{\mathbb{D}}$ has a holomorphic extension to $\mathbf{D}\cup\Omega$, then $\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}_{|\Gamma}$ has a holomorphic extension to Ω', so that the lemma applies.)

Proof: 1° *Case $-1 \notin \Omega$:* Set $\widehat{\mathbb{Z}}(s) := \widehat{\mathbb{Y}}(1/\bar s)^*$ for $s \in \mathbf{C}$ s.t. $1/\bar s \in \overline{\mathbf{D}}$. We have $s = 1/\bar s$, hence $\widehat{\mathbb{Y}}(s)^* = \widehat{\mathbb{Z}}(s)$ for all $s \in \partial\mathbf{D}$. Since $\widehat{\mathbb{Y}} \in \mathrm{H}(\mathbf{D};\mathbf{C}^{n\times n})$, we have $\widehat{\mathbb{Z}} \in \mathrm{H}(\overline{\mathbf{D}}^c;\mathbf{C}^{n\times n})$.

Consequently, $\widehat{\mathbb{Z}}\widehat{\mathbb{X}} = \widehat{\mathbb{E}}$ a.e. on $\partial\mathbf{D}$, so that $f := \widehat{\mathbb{X}} = \widehat{\mathbb{Z}}^{-1}\widehat{\mathbb{E}}$ a.e. on $\partial\mathbf{D}$ (note from Lemma 9.15.2 that $\widehat{\mathbb{X}},\widehat{\mathbb{Z}} \in \mathcal{G}\mathbf{C}^{n\times n}$ a.e. on $\partial\mathbf{D}$). By Theorem 3.3.1(e)&(a2), $\widehat{\mathbb{X}}(r\cdot) \to \widehat{\mathbb{X}}(\cdot)$ in $\mathrm{L}^2(\partial\mathbf{D};\mathbf{C}^{n\times n})$, as $r \to 1-$. It follows that the (inverse)

Cayley transform $\widehat{\heartsuit}^{-1}\widehat{\mathbb{X}}(r+i\cdot)$ converges locally in L^1 as $r \to 0+$; analogously, $\widehat{\heartsuit}^{-1}\widehat{\mathbb{Z}}(r+i\cdot)$ converges locally in L^1 as $r \to 0-$ (see Lemma 13.2.1(e1)).

We conclude from Proposition D.1.18 that $f : \partial\mathbf{D} \cap \Omega \to \mathbf{C}^{n\times n}$ has an holomorphic extension $f : \Omega \to \mathbf{C}^{n\times n}$ s.t. $f = \widehat{\mathbb{X}}$ on $\mathbf{D} \cap \Omega$ and $f = \widehat{\mathbb{Z}}^{-1}\widehat{\mathbb{E}}$ on $\overline{\mathbf{D}}^c \cap \Omega$. Thus, $\widehat{\mathbb{X}}$ has an holomorphic extension to $\mathbf{D} \cup \Omega$.

Apply the above to $\widehat{\mathbb{E}}^* = \widehat{\mathbb{X}}^*\widehat{\mathbb{Y}}$ in place of $\widehat{\mathbb{E}}$ to obtain the claim on $\widehat{\mathbb{Y}}$ (equivalently, an holomorphic extension of $\widehat{\mathbb{Z}}_{|\overline{\mathbf{D}}^c\cap\Omega}$ to Ω). But $f^{-1} = \widehat{\mathbb{E}}^{-1}\widehat{\mathbb{Z}} \in \mathrm{H}(\Omega;\mathbf{C}^{n\times n})$, and $f^{-1} = \widehat{\mathbb{X}}^{-1}$ on $\mathbf{D} \cap \Omega$, hence $\widehat{\mathbb{X}}^{-1}$ has an holomorphic extension to $\mathbf{D} \cup \Omega$.

2° *Case* $\partial\mathbf{D} \not\subset \Omega$*:* Rotate the functions so that 1° applies.

3° *Case* $\partial\mathbf{D} \subset \Omega$*:* Apply 2° to two subsets of Ω. □

(The notes for this section are given on p. 543.)

Chapter 10

Quadratic Minimization ($\min \mathcal{J}$)

Alas, I am dying beyond my means.

— Oscar Wilde (1856–1900) [as he sipped champagne on his deathbed]

Throughout this chapter we assume that Standing Hypotheses 9.0.1 and 10.6.6 hold (i.e., $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, $\mathcal{U}_*^*$ is reasonable and "$\widetilde{\mathcal{A}}_+$" denotes some ULR class admitting positive spectral factorization). Hypothesis 10.1.1 will be assumed through Section 10.1.

We strongly recommend the reader to start by reading the introduction to this chapter (p. 31), where also the main results (particularly the LQR problem but also the real lemmas and the H^2 problem) are explained.

We shall first present some minimization results for cost functions $\int_0^\infty (\|x\|_H^2 + \|u\|_U^2)\, dm$, $\int_0^\infty (\|y\|_H^2 + \|u\|_U^2)\, dm$ and their variants in Section 10.1. These allow one to simplify significantly the Riccati equation theory, and, for $\int_0^\infty (\|x\|_H^2 + \|u\|_U^2)\, dm$, any nonnegative solution of the LQR-CARE becomes unique, exponentially stabilizing and minimizing (and also a converse holds), so that one only has to find a nonnegative solution without any stabilization requirements.

Section 10.2 contains a more detailed study on minimization for general WPLSs and cost functions. In Section 10.3, we present several conditions that are equivalent to positive J-coercivity over $\mathcal{U}_{\mathrm{out}}$ or over $\mathcal{U}_{\mathrm{exp}}$ and show how they are implied by or equivalent to various classical assumptions for minimization problems in the literature. The H^2 problem is solved in Section 10.4.

In Section 10.5, we present the Bounded Real Lemma and the Positive Real Lemma, which allow one to use the Riccati equation or the Riccati inequality to verify whether $\|\mathbb{D}\|_{\mathrm{TIC}} \le \gamma$ or $\mathrm{Re}\langle \mathbb{D}\cdot, \cdot\rangle \gg 0$, respectively. We give our results for sufficiently regular systems and sketch corresponding more general results (the latter lack necessity unless we accept IAREs in place of CAREs).

In Section 10.6, we present the equivalence between the uniform positivity of the Popov operator ($\mathbb{D}^* J \mathbb{D} \gg 0$), I-spectral factorization ($\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* \mathbb{X}$) and stabilizing solutions of the Riccati equation or of the Riccati inequality under varying assumptions. These are used for the minimization results in other sections. We also give sufficient conditions for different versions of the equivalence.

In Section 10.7, we show how any solutions of positive Riccati equations are WR and admissible, even stabilizing under suitable conditions.

The proofs can most easily be read in the order Section 10.7 $\to$ Section 10.6 $\to$ Section 10.2 $\to$ the other sections.

10.1 Minimizing $\int_0^\infty (\|y\|_H^2 + \|u\|_U^2)\,dm$ (LQR)

In spite of the cost of living, it's still popular.
— Kathleen Norris (1880–1960)

Here we study the Linear Quadratic Regulator (LQR) problem for cost functions $\int_0^\infty (\|x\|_H^2 + \|u\|_U^2)\,dm$ and $\int_0^\infty (\|y\|_H^2 + \|u\|_U^2)\,dm$; or more generally, $\mathcal{J}(x_0,u) := \langle y, Qy\rangle_{\mathrm{L}^2} + \langle u, Ru\rangle_{\mathrm{L}^2}$ for some $Q,R \gg 0$; here $x := \mathbb{A}x_0 + \mathbb{B}\tau u$, $y := \mathbb{C}x_0 + \mathbb{D}u$. More general minimization problems will be studied in Section 10.2, which also provides further definitions, results and explanations. See also the presentation in the introduction (p. 31).

Given an initial state $x_0 \in H$, we minimize the cost function over a set $\mathcal{U}_*^*(x_0)$ of admissible controls. We are mainly interested in the classical cases $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}} := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,|\, x, y \in \mathrm{L}^2\}$ and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}} := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \,|\, y \in \mathrm{L}^2\}$ (see Definition 8.3.2). Note that $\mathcal{J}(x_0,u) = +\infty$ for $u \notin \mathcal{U}_{\mathrm{out}}(x_0)$.

To formulate the system and cost function as before, we augment Σ by the extra row $\left[\begin{array}{c|c} 0 & I \end{array}\right]$ when we wish to apply the results of the other sections:

Standing Hypothesis 10.1.1 (LQR) *Throughout this section, we assume that $\mathbb{D}$ is UR or $\dim U < \infty$ and $\mathbb{D}$ is WR, and that $J = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \in \mathcal{B}(Y \times U)$ and $J \gg 0$.*

By minimization and CAREs (and IAREs), we refer to the augmented system $\Sigma_{\mathrm{aug}} := \left[\begin{array}{c} \Sigma \\ \hline 0 \,|\, I \end{array}\right] \in \mathrm{WPLS}(U,H,Y\times U)$ and to the operator J.

Thus, the maps "C" and "D" in the CARE are replaced by $\begin{bmatrix} C \\ 0 \end{bmatrix}$ and $\begin{bmatrix} D \\ I \end{bmatrix}$, etc., and the cost function becomes $\mathcal{J}(x_0,u) := \langle y, Qy\rangle_{\mathrm{L}^2} + \langle u, Ru\rangle_{\mathrm{L}^2}$.

For general $\mathbb{D}$'s (WR or even irregular), one can apply the results of Section 10.2 to obtain results similar to those in this section. Therefore we omit the most general case and study the (rather general) case that $\mathbb{D}$ and (at least) $\mathbb{X}$ are UR. This allows us to rewrite the CARE as follows and guarantee that any solution is admissible (see Theorem 10.1.4):

Definition 10.1.2 (LQR-CARE) *We call $(\mathcal{P},S,K)$ (or $\mathcal{P}$) a nonnegative solution of the* LQR-CARE *iff $0 \le \mathcal{P} \in \mathcal{B}(H)$, $S \in \mathcal{B}(U)$, $K \in \mathcal{B}(H_1,U)$,*

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*QC, \\ \quad S = R + D^*QD + \lim_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B, \\ \quad K = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*QC), \end{cases} \tag{10.1}$$

and $\lim_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B \ge 0$ or $S \gg 0$. We use prefices and suffices (e.g., "PB-") as in Definition 9.8.1.

If Σ is ULR, then we call $\mathcal{P}$ (or $(\mathcal{P},S,K)$) a nonnegative solution of the LQR-B_{w}^*-CARE *iff $0 \le \mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$ and $\mathcal{P}$ satisfies*

$$K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*QC, \tag{10.2}$$

where $S := R + D^*QD$, and $K = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*QC)$.

(One often has $D = 0$, so that the state feedback becomes $K = -S^{-1}B_{\mathrm{w}}^*\mathcal{P}$.)

Thus, the LQR-CARE is the CARE for $\Sigma_{\rm aug}$ and J with the additional conditions that $S \geq D_{\rm aug}^* J D_{\rm aug}$ or $S \gg 0$ and that the limit in S converges uniformly (i.e., in $\mathcal{B}(U)$; see Lemma A.3.1(h)). The LQR-$B_{\rm w}^*$-CARE is the corresponding $B_{\rm w}^*$-CARE (cf. Definition 9.2.6). If $\dim U < \infty$, then any WR K (or $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$) is UR, by Lemma 6.3.2(a1)&(a2).

The advantage of the LQR-CARE is that any nonnegative solution is SOS-stabilizing (in particular, it is admissible):

Lemma 10.1.3

(a) The nonnegative solutions of the LQR-CARE are exactly the UR nonnegative admissible solutions of the CARE (for $\Sigma_{\rm aug}$).

Moreover, all of them are SOS-stabilizing and have $\mathbb{M}^{\pm 1} \in$ UR*,* $M \in \mathcal{GB}(U)$*,* $\mathbb{N} \in$ WR*,* $S \gg 0$ and $\lim_{s \to +\infty} B_{\rm w}^* \mathcal{P}(s-A)^{-1} B \geq 0$.

(b) Any solution of the LQR-CARE is a solution of the CARE.

(c) $\mathcal{P}$ is a UR $\mathcal{U}_*^*$-stabilizing solution of the eCARE iff $\mathcal{P}$ is a nonnegative $\mathcal{U}_*^*$-stabilizing solution of the LQR-CARE.

(d1) The nonnegative solutions of the LQR-$B_{\rm w}^*$-CARE are exactly the nonnegative solutions of the $B_{\rm w}^*$-CARE (for $\Sigma_{\rm aug}$), hence all of them are nonnegative solutions of the LQR-CARE.

(d2) If Hypothesis 9.2.1 holds (for corresponding $\mathcal{U}_*^*$), then the $\mathcal{U}_*^*$-stabilizing solutions of the LQR-$B_{\rm w}^*$-CARE are exactly the nonnegative $\mathcal{U}_*^*$-stabilizing solutions of the LQR-CARE, hence exactly the UR $\mathcal{U}_*^*$-stabilizing solutions of the eCARE.

Note from Proposition 9.8.10 that the UR nonnegative admissible solutions of the CARE are exactly the UR nonnegative admissible solutions of the IARE modulo (9.114).

By Theorem 9.2.9 and the above lemma, the $\mathcal{U}_*^*$-stabilizing solutions of the LQR-$B_{\rm w}^*$-CARE are exactly the $\mathcal{U}_*^*$-stabilizing solutions of the LQR-CARE if Hypothesis 9.2.1 holds.

Proof: Obviously, any solution of the LQR-CARE is a solution of the CARE (even UR, by Lemma 9.11.5(e)).

(a)&(b) By Proposition 10.7.4, a nonnegative solution of the LQR-CARE is a UR SOS-stabilizing solution of the CARE with $S \gg 0$.

Conversely, any UR admissible nonnegative solution $(\mathcal{P}, S, K)$ of the CARE has a uniform (not merely weak) limit in S, either because $\dim U < \infty$ (so that weak=uniform) or by Lemma 9.11.5(e). Thus, $(\mathcal{P}, S, K)$ satisfies the LQR-CARE (and $\lim_{s \to +\infty} B_{\rm w}^* \mathcal{P}(s-A)^{-1} B = S - (D^* Q D + R) \geq 0$, by Proposition 9.11.4(c2); in particular, $S \geq D^* Q D + R \geq R \gg 0$).

Since $\mathbb{X} =: \mathbb{M}^{-1}$ is necessarily UR for such a solution, we have $\mathbb{M}^{\pm 1} \in$ UR, $M \in \mathcal{GB}(U)$ and $\mathbb{N} = \mathbb{D}\mathbb{M} \in$ WR, by Proposition 6.3.1(b1) and Lemma 6.2.5.

(c) By (a), a nonnegative $\mathcal{U}_*^*$-stabilizing solution of the LQR-CARE is a nonnegative ULR $\mathcal{U}_*^*$-stabilizing solution of the eCARE. Conversely, a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} K & F \end{array}\right])$ of the eCARE has $S \gg 0$, by Lemma

9.10.3, and $\mathcal{P}\ge 0$, since $\mathcal{J}\ge 0$ (see Theorem 9.9.1), and $X\in\mathcal{GB}(U)$, by Proposition 6.3.1(b1), hence it is equivalent to a UR $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P},S,K')$ of the CARE, by Remark 9.8.2, hence of the LQR-CARE, by (a).

(d1) By Proposition 9.2.7, the solutions of the LQR-B_{w}^*-CARE are admissible ULR solutions of the CARE and the IARE, hence of the LQR-CARE too, by (a).

(d2) By (d1) and Proposition 9.2.7(a)&(b), a $\mathcal{U}_*^*$-stabilizing solution of the LQR-B_{w}^*-CARE is a URL $\mathcal{U}_*^*$-stabilizing solution of the CARE; the converse follows from Theorem 9.2.9, and the second equivalence follows from (c). □

Now we state the connection between the LQR-CARE and UR minimizing state feedback operators:

Theorem 10.1.4 ($\min_u \int_0^\infty(\|y\|_Y^2+\|u\|_U^2)$)

(a1) **($\mathcal{U}_*^*$)** *There is a minimizing UR state feedback operator iff there is a [nonnegative] $\mathcal{U}_*^*$-stabilizing solution of the LQR-CARE.*

(a2) **(Uniqueness)** *Any $\mathcal{U}_*^*$-stabilizing solution of the LQR-CARE, minimizing control or minimizing state feedback operator is unique.*

(b1) **($\mathcal{U}_{\mathrm{out}}$)** *There is a UR minimizing state feedback operator over $\mathcal{U}_{\mathrm{out}}$ iff there is a minimal nonnegative solution of the LQR-CARE and this solution satisfies (PB) for $\mathcal{U}_{\mathrm{out}}$.*

(b2) **($\mathcal{U}_{\mathrm{exp}}$)** *There is a UR minimizing state feedback operator over $\mathcal{U}_{\mathrm{exp}}$ iff there is a maximal nonnegative solution of the LQR-CARE and this solution is exponentially stabilizing.*

(b3) **(Smooth Σ)** *If Hypothesis 9.2.1 holds (for the corresponding choice of $\mathcal{U}_*^*$), then we can replace the LQR-CARE by LQR-B_{w}^*-CARE everywhere in this theorem.*

(b4) **(Smooth Σ: $\mathcal{U}_{\mathrm{out}}$)** *Assume that Hypothesis 9.2.1 holds for $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{out}}$. Then the following are equivalent:*

(i) **(Min)** *there is a $\langle y,Qy\rangle+\langle u,Ru\rangle$-minimizing control $u_{\min}(x_0)$ over $u\in\mathrm{L}^2(\mathbf{R}_+;U)$ for each $x_0\in H$;*

(ii) **(FCC)** *for each $x_0\in H$ there is $u\in\mathrm{L}^2(\mathbf{R}_+;U)$ s.t. $y\in\mathrm{L}^2$.*

(iii) The LQR-B_{w}^*-CARE has a nonnegative solution.

(iv) The LQR-B_{w}^*-CARE has a smallest nonnegative solution, and that solution corresponds to a (unique) ULR minimizing state feedback operator over $\mathcal{U}_{\mathrm{out}}$.

(We can replace LQR-B_{w}^*-CARE by LQR-CARE in (iii) and (iv).) If, in addition, $D=0$, then the LQR-B_{w}^*-CARE becomes

$$(B_{\mathrm{w}}^*\mathcal{P})R^{-1}B_{\mathrm{w}}^*\mathcal{P}=A^*\mathcal{P}+\mathcal{P}A+C^*QC \tag{10.3}$$

(in either case, we only require that $0\ll\mathcal{P}\in\mathcal{B}(H,\mathrm{Dom}(B_{\mathrm{w}}^*))$).

(b5) We have above (i)⇔(ii)⇐(iii)⇐(iv)⇐(v) in general.

(b6) **(Smooth Σ: $\mathcal{U}_{\text{exp}}$)** *Assume that $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, and that (1.) Hypothesis 9.2.1 holds, or that (2.) $\mathbb{A}B \in \mathrm{L}^2([0,T);\mathcal{B}(U,H))$, $C_{\text{w}}\mathbb{A} \in \mathrm{L}^1([0,T);\mathcal{B}(H,Y))$ and $C_{\text{w}}\mathbb{A}B \in \mathrm{L}^1([0,T);\mathcal{B}(U,Y))$. Then the following are equivalent:*

(i) There is a [unique] minimizing control for each $x_0 \in H$.

(ii) There is a [unique] exponentially stabilizing solution $(\mathcal{P},S,K)$ of the LQR-CARE.

(iii) $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable, and $\mathbb{D}_{\text{aug}}$ is I-coercive, i.e., there is $\varepsilon > 0$ s.t.

$$(ir-A)x_0 = Bu_0 \Longrightarrow \|C_{\text{w}}x_0 + Du_0\|_Y + \|u_0\|_U \geq \varepsilon\|x_0\|_H \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}). \tag{10.4}$$

Let $(\mathcal{P},S,K)$ be as in (ii). Then $S = R + D^*QD \gg 0$, and K is ULR and the unique minimizing state feedback operator. In case (2.), we have $\mathbb{B}\tau, \mathbb{D}, \mathbb{F} \in \mathrm{MTIC}_\infty^{\mathrm{L}^1}$ and $\mathbb{B}_{\circlearrowleft}\tau, \mathbb{N}, \mathbb{M} \in \mathrm{MTIC}_\omega^{\mathrm{L}^1} \subset \mathrm{UHPR}$ for some $\omega < 0$.

(c1) **($\min_u \int_0^\infty(\|x\|_H^2 + \|u\|_U^2)$: unique $\mathcal{P}$)**
Assume that Σ is estimatable (e.g., $C \in \mathcal{B}(H,Y)$ and $C^*C \gg 0$).

Then there is at most one nonnegative solution of the LQR-CARE. Such a solution (if any) is strictly minimizing over $\mathcal{U}_{\text{out}}$, $\mathcal{U}_{\text{sta}}$, $\mathcal{U}_{\text{str}}$ and $\mathcal{U}_{\text{exp}}$, and exponentially q.r.c.-stabilizing.

Moreover, such a solution defines an exponential normalized q.r.c.f. (even r.c.f. if $C \in \mathcal{B}(H,Y)$) $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, where $\widehat{\mathbb{N}}(s) := D + (C+DK)(s-A-BK)^{-1}B$ and $\widehat{\mathbb{M}}(s) := I + K(s-A-BK)^{-1}B$ are exponentially q.r.c. and stable, $\mathbb{M}^{\pm 1} \in \mathrm{UR}$ and $\mathbb{N}^*Q\mathbb{N} + \mathbb{M}^*R\mathbb{M} = I$.

(c2) Assume that Σ is strongly top row–detectable.

Then there is at most one nonnegative solution of the LQR-CARE. Such a solution (if any) is strictly minimizing over $\mathcal{U}_{\text{out}}$, $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{str}}$.

(d) A CARE solution ($(\mathcal{P},S,K)$) of the form mentioned in any of (a1)–(c2) (except in (b4)(iii)) is unique (of that form) and $\mathcal{U}_*^*$-stabilizing, and the same K is the unique minimizing state feedback operator.

(Note that (2.) of (b6) is implied by the "Parabolic system assumption" Hypothesis 9.5.1, by Lemma 9.5.2.)

Thus, when minimizing over $\mathcal{U}_{\text{out}}$, we only have to find a minimal solution and check the condition (PB) for that solution only, by (b1). Analogously, when minimizing over $\mathcal{U}_{\text{exp}}$, instead of looking for an exponentially stabilizing solution, it suffices to look for a maximal solution and check the exponential stability of $\mathbb{A}_{\circlearrowleft}$ for that solution only. (If none exists or (at least) one exists but does not satisfy (PB), then the minimizing control (if any) cannot be given in the state feedback form.)

By (b3)–(b6), the LQR-CARE can be replaced by the "LQR-B_{w}^*-CARE" if Σ is smooth enough, and in this case any minimizing control is necessarily of state feedback form (and ULR). See Theorems 9.2.10–9.2.12 for further analogous results. See also Corollary 9.5.10 for the case where $\mathbb{A}$ is analytic.

By (c1)&(c2), estimatability or strong detectability implies that a nonnegative solution of the LQR-CARE is unique and minimizing. The same holds when Σ is exponentially q.r.c.-stabilizable or strongly stable, by Theorem 10.1.6.

Proof of Theorem 10.1.4: (a1) This follows from Lemma 10.1.3(c), and Corollary 9.9.2(a2)&(e1)&(e2).

(a2) By Theorem 9.9.1(f2) (Lemma 10.1.3(c)), a $\mathcal{U}_*^*$-stabilizing solution is unique. By Lemma 8.3.8, a minimizing control is unique; consequently, a compatible state feedback operator is unique (on H_B), by Lemma 8.3.17(b).

(b1)&(b2) If we drop the minimality/maximality condition, then the equivalence follows from (a1) and Theorem 9.8.5 (for (b1) we used the fact that $\mathcal{P}$ is SOS-stabilizing, by Lemma 10.1.3(a)).

But the minimality in (b1) (resp. maximality in (b2)) is necessary, by Theorem 9.9.1(a2).

(b3) This follows from Lemma 10.1.3(d2). (Note that in (b1) (resp. (b2)) we must have Hypothesis 9.2.1 for $\mathcal{U}_{\rm out}$ (resp. for $\mathcal{U}_{\rm exp}$) etc.)

(b4) By Theorem 9.2.10(b), (ii) implies (iv); the rest follows from (b5).

(b5) Since $\mathbb{D}_{\rm aug}$ is positively J-coercive over $\mathcal{U}_{\rm out}$, we have (i)$\Leftrightarrow$(ii), by Theorem 8.4.3 and Lemma 10.2.2.

Implication "(iv)$\Rightarrow$(iii)" is trivial. and "(iii)$\Rightarrow$(ii)" follows from the fact that any nonnegative solution is SOS-stabilizing, by Lemma 10.1.3(d1)&(a).

(b6) This follows from Corollary 10.2.9 (case (1.)) or Corollary 10.2.10 (case (2.)), Lemma 10.1.3 and Proposition 10.3.2(ii') (note that $J \gg 0$, hence $\mathbb{D}$ is positively J-coercive iff it is [positively] I-coercive).

(Obviously, $u=0$ is the unique minimizing control for $x_0=0$, hence any minimizing control (for any $\mathcal{U}_*^*$) is unique, by Lemma 8.3.8.)

(c1) This follows from Proposition 10.7.3(d3) except for the last claim, which is from Theorem 10.1.6 (if $C \in \mathcal{B}(H,Y)$, then any exponentially stabilizing state feedback operator is exponentially r.c.-stabilizing, by Lemmas 6.6.25 and 6.6.26, hence then "q.r.c." becomes "r.c.").

(N.B. the last equation is equivalent to $\widehat{\mathbb{N}}(s)^*Q\widehat{\mathbb{N}}(s)+\widehat{\mathbb{M}}(s)^*R\widehat{\mathbb{M}}(s)=I$ on $i\mathbf{R}$.)

(c2) (By top row–detectability we mean that some admissible output injection pair $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ makes $\left[\begin{array}{c|cc}\mathbb{A}_\sharp & \mathbb{H}_\sharp & \mathbb{B}_\sharp\end{array}\right]$ strongly stable. This obviously holds if Σ is strongly detectable. Actually, it suffices to assume that $\Delta^{\rm S}\Sigma$ is top row–detectable, as one observes from the proof.)

1° Let $\mathcal{P}$ be a nonnegative solution of the LQR-CARE, hence SOS-stabilizing, by Lemma 10.1.3(a). By 2°, $\mathbb{A}_\circlearrowleft$ is strongly stable, hence $\mathcal{P}$ is $\mathcal{U}_{\rm str}$-stabilizing, by Theorem 9.8.5, hence unique and strictly minimizing over $\mathcal{U}_{\rm str}$, by (a2).

Let $\mathcal{P}'$ be the J-critical cost operator over $\mathcal{U}_{\rm out}$ (recall that $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\rm out}$). Then $\mathcal{P}'=\mathcal{P}$, by uniqueness (use the discrete version of (c2)). By Lemma 8.3.3, $\mathcal{P}$ is (strictly) minimizing over $\mathcal{U}_{\rm sta}$ too.

2° $\mathbb{A}_\circlearrowleft$ *is strongly stable:* We prove this in discrete time (note that state feedback and output injection pairs can be discretized and also stability is preserved under discretization in both directions).

As at the end of the proof of Lemma 6.7.11(a), we note that also $\Sigma_{\circlearrowleft}$ has the above detectability property (note that $\widetilde{H} := -\mathtt{H}D - B$ implies that $\widetilde{\mathbb{H}}_\sharp = -\mathbb{H}_\sharp D - \mathbb{B}_\sharp$, hence it is stable too; by assumption, so is $\left[\begin{array}{c|cc} \mathbb{A}_\sharp & \mathbb{H}_\sharp & \mathbb{B}_\sharp \end{array}\right]$). Thus, $\mathbb{A}_{\circlearrowleft} + \mathbb{H}'_\sharp \tau \mathbb{C}_{\circlearrowleft}$ and $\mathbb{H}'_\sharp$ become stable for the output injection pair $\left(\frac{H'}{0}\right)$, where $H' := \begin{bmatrix} \mathtt{H} & \widetilde{H} \end{bmatrix}$. Consequently, also $\mathbb{A}_{\circlearrowleft}$ is stable (since $\mathbb{H}'_\sharp \tau^t \mathbb{C}_{\circlearrowleft} x_0 \to 0$, as $t \to \infty$, for all $x_0 \in H$; the proof of this is analogous to that of Lemma 6.6.8(a)).

(d) Since the solutions mentions above are $\mathcal{U}^*_*$-stabilizing, they contain the minimizing K. □

Remark 10.1.5 *The cost is finite for $u \in \mathcal{U}_{\text{out}}(x_0)$ only, hence minimization over all (measurable) $u : \mathbf{R} \to U$ corresponds to minimization over $\mathcal{U}_{\text{out}}$ (this was applied in Theorem 10.1.4(b3)&(b4)).*

In Theorem 10.1.4(b1) we showed that such a minimizing control is generated by an UR state feedback operator iff there is a (necessarily smallest nonnegative) $\mathcal{U}_{\text{out}}$-stabilizing solution of the LQR-CARE. □

In the strongly stable case there is at most one solution of the LQR-CARE (see also Theorem 10.1.4(c1)&(c2)):

Theorem 10.1.6 ($\mathcal{U}_{\text{out}}$: LQR⇔r.c.f.⇔CARE) *Assume that Σ is strongly stable or exponentially q.r.c.-stabilizable. Then the following are equivalent:*

(i) **(*K*)** *There is a [unique] UR minimizing state feedback operator K over $\mathcal{U}_{\text{out}}$.*

(ii) **(CARE)** *The LQR-CARE has a [unique] nonnegative solution $\mathcal{P}$.*

(iii) **(R.c.f.)** *There is a q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ with $\mathbb{M} \in \mathrm{UR}$ and $\mathbb{N}^* Q \mathbb{N} + \mathbb{M}^* R \mathbb{M} = I$.*

Moreover, the following holds:

(a1) The above conditions imply (Crit1+WR)–(Crit4+WR) of Theorem 10.2.14, hence (a1)–(g3) of Theorem 9.9.10 apply (for Σ_{aug} and J).

(a2) The solutions K of (i) and $\mathcal{P}$ of (ii) are unique, UR, strongly q.r.c.-stabilizing, strictly minimizing over $\mathcal{U}_{\text{out}}$, $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{str}}$, and equal to those of Theorem 9.1.7 (so is the solution of (iii) too if we require that $M = I$).

If Σ is exponentially q.r.c.-stabilizable, then K and $\mathcal{P}$ are exponentially q.r.c.-stabilizing and strictly minimizing over $\mathcal{U}_{\text{exp}}$ too.

(b1) **(B^*_{w}-CARE)** *Assume that Hypothesis 9.2.1 holds and $D^*JD \gg 0$.*

Then (i)–(iii) have solutions, we may replace the LQR-CARE by the LQR-B^*_{w}-CARE, and $\mathbb{D}, \mathbb{F}, \mathbb{N}, \mathbb{M}^{\pm 1} \in \mathrm{ULR}$.

(b2) Assume that Σ is strongly stable and satisfies Hypothesis 10.6.1(3.) (resp. (6.)).

Then (i)–(iii) have solutions (resp. and (b1) applies).

(b3) Assume that Σ has a UR (resp. URL) exponentially q.r.c.-stabilizing state feedback operator $\widetilde{K}$ s.t. the resulting closed-loop system $\left[\begin{array}{c|c}\mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_{\mathrm{aug},\flat} & \mathbb{D}_{\mathrm{aug},\flat}\end{array}\right]$ satisfies Hypothesis 10.6.1(3.) (resp. (6.)).

Then (i)–(iii) have solutions (resp. and (b1) applies).

(b4) Assume that $\mathbb{D}\in\widetilde{\mathcal{A}}_+$ and Σ is strongly stable (resp. that $\mathbb{D}$ is exponentially q.r.c.-stabilizable in $\widetilde{\mathcal{A}}_+$).

Then (i)–(iii) have solutions with $\mathbb{N},\mathbb{M}\in\widetilde{\mathcal{A}}_+$.

(See Standing Hypothesis 10.6.6 for $\widetilde{\mathcal{A}}_+$.) Note that (iii) is equivalent to the existence of a (J,I)-inner q.r.c.f. $\mathbb{D}_{\mathrm{aug}}=\mathbb{N}_{\mathrm{aug}}\mathbb{M}^{-1}$ with $\mathbb{M}\in\mathrm{UR}$ (it follows that $\mathbb{N}_{\mathrm{aug}}=\left[\begin{smallmatrix}\mathbb{N}\\ \mathbb{M}\end{smallmatrix}\right]$). For stable (d), this becomes equivalent to the existence of an I-spectral factorization $\mathbb{D}^*Q\mathbb{D}+R=\mathbb{X}^*\mathbb{X}$, by Lemma 6.4.8(a).

If we drop the stability/stabilizability assumptions of the theorem, then K and $\mathcal{P}$ must be assumed to be strongly q.r.c.-stabilizing (or q.r.c.-SOS-stabilizing and q.r.c.-SOS-P-stabilizing) in (i) and (ii), and Σ must be assumed to be strongly q.r.c.-stabilizable in (iii) (or q.r.c.-SOS-stabilizable), by Theorems 9.1.7 and 9.9.10 (or Corollary 10.2.12, which weakens (b3)).

Recall from Theorem 6.7.15(c1) that if Σ is estimatable, then any output-stabilizing state feedback operator is exponentially q.r.c.-stabilizing.

Proof of Theorem 10.1.6: 1° *(i)⇔(ii), (a2):* This follows from Theorem 10.1.4(b1)&(a2) and Proposition 10.7.3(d2)&(d3).

2° *(iii)⇔(ii):* This follows from Lemma 6.5.7(c) and Theorem 9.1.7 (indeed, a solution of (ii) is a q.r.c.-stabilizing solution of the CARE, by (a2) and Lemma 10.1.3; a solution of (iii) can be chosen s.t. $X=I$, by (a2); on the other hand, $\mathbb{M}^{\pm1}\in\mathrm{UR}$ for any solution of (ii) or (iii), by Lemma 10.1.3 and 1°).

(It follows that $\mathbb{N}_{\mathrm{aug}}$ is (J,S)-inner, where $S\gg 0$ is from the (LQR-)CARE; replace $\mathbb{M}$ by $\mathbb{M}S^{-1/2}$ to get $S=I$.)

(a1) This follows from (a2) and Lemma 10.1.3.

(b1)–(b4) These follow from Theorem 10.2.14(b1)–(b4) or Corollary 10.2.15(b1) (resp. (b2)) (since now $\mathbb{X}$ and hence $\mathbb{M}:=\mathbb{X}^{-1}$ (by Proposition 6.3.1(b1)) is necessarily UR (resp. ULR and so is $\mathbb{D}$)), Lemma 10.1.3 and Proposition 9.2.7(c).

For (b4) we note that if $\mathbb{D}\in\widetilde{\mathcal{A}}_+$ (resp. $\mathbb{N},\mathbb{M}\in\widetilde{\mathcal{A}}_+$), then $\mathbb{D}_{\mathrm{aug}}\in\widetilde{\mathcal{A}}_+$ (resp. $\mathbb{D}_{\mathrm{aug},\flat}=\left[\begin{smallmatrix}\mathbb{N}\\ \mathbb{M}\end{smallmatrix}\right]\in\widetilde{\mathcal{A}}_+$). Thus, we could equivalently pose the assumption on Σ_{aug}. □

Notes

One often calls any minimization problems (with a quadratic cost function) *Linear Quadratic Regular* (*LQR*) problems, and the problems of this section (those satisfying Hypothesis 10.1.1) are then called *standard LQR problems* or something similar.

Since the LQR problem is perhaps the most popular subject in infinite-dimensional control theory, we can only try to give a picture of some most general current results. The earlier history of infinite-dimensional Riccati equations is documented in the notes to Section 6 of [CZ].

Most of our results are known for several special cases (see, e.g., Section 6.2 of [CZ] for WPLSs with bounded B and C and Theorems 3.3 and 3.4 of [PS87] for Pritchard–Salamon systems; these treat both $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{exp}}$ to some extent).

For $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$, the case covered by Proposition 9.7.6 was solved in [FLT] (with a "generalized CARE" and possibly non-well-posed feedback; for parabolic systems these issues are well settled and the results very general, see [LT00a]). A similar result for general WPLSs was given in [Zwart], and the regular stable minimization problem was solved independently in [S97b] and [WW] (with roughly the implication "(iii)⇒(i)&(ii)" of Theorem 10.1.6 for stable WPLSs with $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$). Theorem 10.1.4(c2) generalizes Theorem 3.3.2 of [Oostveen] (which assumed bounded B and C); see [Oostveen] for its physical applications. See also the notes on p. 571

10.2 General minimization (LQR)

Incidis in Scyllam, cupens vitare Charybdim.

— Homer

In this section we shall present some applications of the CARE theory to minimization problems, where one wishes to find a minimizing control over some set $\mathcal{U}_*^*$ of allowable controls, in state feedback form (i.e., to find a [regular] state feedback operator K or a pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ that produces a minimizing control). Thus, this is a generalization of Section 10.1.

Since this section is rather technical, lengthy and boring (due to the reasons explained below Corollary 10.2.3), a casual reader might wish to just have a glance at Subsections 10.2.1–10.2.10 (or less) and then proceed to the next section.

Most results become essentially simpler under Hypothesis 9.2.1, as illustrated in Section 9.2 (in particular, in Theorems 9.2.10–9.2.12), or in their discrete-time forms, as illustrated in Section 15.1.

Therefore, in this section we have the emphasis on general WPLS results and on results for MTIC I/O maps. This makes several classical results rather complicated, and almost each piece of simplicity must be obtained at the cost of generality. Consequently, we give certain results under some different assumptions, and leave it to the reader to extra- and interpolate further results under other kind of assumptions, using the results of Sections 8.3–8.4, Chapter 9, and the rest of this chapter.

We use the word "minimizing" in the same way as the word "J-critical":

Definition 10.2.1 (Minimizing $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and K) *We call $u_{\min} \in \mathcal{U}_*^*(x_0)$ a* [strictly] minimizing control *(over $\mathcal{U}_*^*$) for $x_0 \in H$ (and Σ and J) if $\mathcal{J}(x_0, u_{\min}) \le \mathcal{J}(x_0, u)$ for all $u \in \mathcal{U}_*^*(x_0)$ [and $u_{\min}$ is unique].*

Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be an admissible state feedback pair for Σ with closed-loop system $\Sigma_\circlearrowleft$. Then we call $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ minimizing *(over $\mathcal{U}_*^*$ for Σ and J) iff $\mathbb{K}_\circlearrowleft x_0$ is minimizing for each $x_0 \in H$. In this case, we say that the minimizing control is of* state feedback form.

We call a WR admissible state feedback operator $K \in \mathcal{B}(H_1, U)$ minimizing *if $\left[\begin{array}{c|c} K & 0 \end{array}\right]$ generate minimizing pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ for Σ.*

Consequently, $u_{\text{crit}} \in \mathrm{L}^2(\mathbf{R}_+;U)$ is [strictly] minimizing for x_0 over $\mathcal{U}_{\text{out}}$ (resp. $\mathcal{U}_{\exp}$) iff $y := \mathbb{C}x_0 + \mathbb{D}u \in \mathrm{L}^2$ (resp. and $x := \mathbb{A}x_0 + \mathbb{B}\tau u \in \mathrm{L}^2$) and u [strictly] minimizes the *cost function* $\mathcal{J}(x_0,u) := \int_0^\infty \langle y(t), Jy(t)\rangle_Y \, dt$ among such controls. See Definition 9.1.4 (or Definition 6.6.10) for K, $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, and $\Sigma_\circlearrowleft$.

Note that "minimizing over $\mathcal{U}_*^*$" does not affect the other attributes (prefices) and vice versa, i.e., "minimizing exponentially stabilizing state feedback pair over $\mathcal{U}_{\text{out}}$" produces a control that is minimizing over all elements of $\mathcal{U}_{\text{out}}(x_0)$ for each $x_0 \in H$, not just over those produced by exponentially stabilizing state feedback.

Lemma 10.2.2 (Min $\Leftrightarrow$ J-crit. & pos.) *Let $x_0 \in H$. A control $u \in \mathcal{U}_*^*(x_0)$ is [strictly] minimizing for x_0 iff u is J-critical for x_0 and $\langle \mathbb{D}\eta, J\mathbb{D}\eta\rangle \ge 0$ [> 0] for all $\eta \in \mathcal{U}_*^*(0) \setminus \{0\}$.*

Thus, if there is a [strictly] minimizing control for any x_0, then 0 is a [strictly] minimizing control for $x_0 = 0$ (since $\mathcal{J}(0,\eta) = \langle \mathbb{D}\eta, J\mathbb{D}\eta \rangle$).

If there is a minimizing control for all x_0, then we may call the J-critical cost operator $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ (the one for which $\mathcal{J}(x_0, u_{\min}(x_0)) = \langle x_0, \mathcal{P}x_0 \rangle$ ($x_0 \in H$), as in Theorem 8.3.9(b1)) the *minimal cost operator*. Recall that it is equal to the (unique) $\mathcal{U}_*^*$-stabilizing solution of the Riccati equation (if any, i.e., if a minimizing control is of state feedback form).

Proof of Lemma 10.2.2: Proof 1: Corollary 8.1.8 and Remark 8.3.4.

Proof 2: "If": This follows from Lemma 8.3.7(ii). "Only if": This follows from Lemma 8.3.6 and Lemma 8.3.7(ii). □

At their best, our minimization results for general cost functions look like the following one:

Corollary 10.2.3 (Minimization for bounded B) *Assume that $B \in \mathcal{B}(U,H)$, $\dim U < \infty$. Then the following are equivalent:*

(i) there is a unique minimizing control for each $x_0 \in H$;

(ii) $\mathcal{J}(0,\cdot) \geq 0$, $D^*JD \gg 0$, and the "B-CARE"

$$(B^*\mathcal{P} + D^*JC)^*(D^*JD)^{-1}(B^*\mathcal{P} + D^*JC) = A^*\mathcal{P} + \mathcal{P}A + C^*JC \tag{10.5}$$

has a (unique) $\mathcal{U}_*^*$-stabilizing solution $\mathcal{P}$.

(iii) there is a unique minimizing state feedback operator for Σ.

Moreover, if (ii) holds, then minimizing state feedback is given by $u_{\min}(t) = K_{\mathrm{L,s}}x(t)$ a.e., where $K := -(D^*JD)^{-1}(B^*\mathcal{P} + D^*JC)$ is URL, the minimal cost is $\mathcal{J}(x_0, u_{\min}) = \langle x_0, \mathcal{P}x_0 \rangle$ etc. as in Proposition 9.9.1. If $\mathcal{U}_*^* = \mathcal{U}_{\exp}$, then also the following is equivalent to (i):

(iv) Σ is positively J-coercive and optimizable.

(Condition $\mathcal{J}(0,\cdot) \geq 0$ is redundant at least for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$, by Corollary 10.2.6. The operator $K_{\mathrm{L,s}}$ can be replaced by any of its extensions, such as $K_{\mathrm{L,w}}$, K_{s} and K_{w}.)

We recall from Theorem 9.8.5 that "$\mathcal{U}_{\exp}$-stabilizing" means "exponentially stabilizing", whereas "$\mathcal{U}_{\mathrm{out}}$-stabilizing" is rather complicated for general unstable Σ. If Σ is q.r.c.-SOS-stabilizable (e.g., jointly stabilizable and detectable) and J-coercive, then the latter case can be slightly simplified, as in Corollary 10.2.12–Theorem 10.2.14.

Proof of Corollary 10.2.3: By Proposition 10.3.2(e2)&(i)&(v), we have "(ii)⇒(iv)⇒(i)" for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$. The rest follows from Lemma 10.2.2 and Theorem 9.9.6 (see also Theorem 9.9.1(a2)&(e2) etc.). □

Instead of B being bounded, we can assume that Hypothesis 9.2.1 holds and that $D^*JD \gg 0$, by Theorem 9.2.9 (then B^* must be replaced by B_{w}^*, and we must, additionally, require $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ in (ii)). Since the generators of a wpls (a discrete-time system) are bounded, Corollary 10.2.3 always holds in discrete time (we just have to replace equations for $\mathcal{P}$, D^*JD and K by the DARE; this follows from Theorem 15.1.2, as in the above proof). Unfortunately, for general WPLSs, the situation is not as nice (in continuous time):

(1.) If B is unbounded, then $B^*\mathcal{P}$ is not defined in general, and even $B_{\mathrm{w}}^*\mathcal{P}$ need not always be bounded (it is defined at least on $\mathrm{Dom}(A_{\min})$ when $\mathbb{D}$ is WR, by Theorem 9.7.3(b)).

Thus, the "B-CARE" (10.5) must be replaced by the more complicated CARE (for which D^*JD is replaced by the limit S).

(2.) Even worse, the minimizing state feedback need not be regular (even when $\mathbb{D}$ were ULR, by Proposition 9.13.1(c1)), hence the CARE must be replaced by the IARE (or one must use discretization) and the unique minimizing state feedback operator must be replaced by an essentially unique minimizing state feedback pair.

(3.) Still worse: for general WPLSs we do not even know whether (i) implies (ii) and (iii), i.e., whether a minimizing control is of state feedback form (it is of the generalized, non-well-posed sense of Definition 8.3.15, by Theorem 8.3.9); thus, instead of the IARE we must use the "generalized IARE" of Theorem 9.7.1 (or the "CARE on $\mathrm{Dom}(A_{\min})$" of Theorem 9.7.3 if $\mathbb{D}$ is regular).

(4.) If $\dim U = \infty$, then the IARE (resp. CARE) must be replaced by eIARE (eCARE), since we only know that $S > 0$ (even though B were bounded).

We shall present below some results for general WPLS and then use different assumptions to get rid of some of the above problems.

By using Hypothesis 9.2.1, we get rid of (2.), (3.) and most of (1.); some corresponding results are given Section 9.2; see, e.g., Theorems 9.2.10–9.2.12.

A set of weaker assumptions is given in Hypothesis 10.6.1; they allow us to get rid of (2.) and (3.) (and a part of (1.)), as shown in several results below and in Section 10.1. Also Corollaries 10.2.10 and 9.5.10 present similar results under different assumptions.

For stable (or suitably stabilizable) systems, the positive spectral factorization result of Lemma 6.4.7(a) can be used to avoid problem (3.) for general WPLSs; see Corollaries 10.2.6–10.2.13.

Sometimes we use J-coercivity or analogous assumptions to overcome (4.). Under sufficient regularity assumptions we have $S = D^*JD$, hence then we can make S invertible just by assuming that $D^*JD \gg 0$.

In Section 10.1, we give some analogous results for the more specific LQR problem, and in Section 10.6, we give further minimization and [e]IARE and [e]CARE results.

Naturally, one can obtain several analogous theorems by combining the results of this and the previous chapter in different ways; we hope that the following results provide a helpful guideline.

Standard coercivity assumptions guarantee the existence of a unique minimizing control:

Lemma 10.2.4 *Let Z^{s} be reflexive, and let $\mathbb{D}$ be positively J-coercive, i.e., let there be $\varepsilon > 0$ s.t.*

$$\langle \mathbb{D}u, J\mathbb{D}u\rangle \geq \varepsilon \|u\|^2_{\mathcal{U}_*^*} \quad (u \in \mathcal{U}_*^*(0)). \tag{10.6}$$

If $x_0 \in H$ is s.t. $\mathcal{U}_*^*(x_0) \neq \emptyset$, then there is a unique minimizing control for x_0. □

(This follows from Theorem 8.2.5, Lemma 8.2.3 and Lemma 10.2.2 through Remark 8.3.4. Recall that Z^{s} is reflexive for $\mathcal{U}_{\mathrm{exp}}$ and $\mathcal{U}_{\mathrm{out}}$.)

Thus, if $\mathcal{U}_*^*(x_0) \neq \emptyset$ for all $x_0 \in H$ (i.e., the *Finite Cost Condition* holds) and $\mathbb{D}$ is positively J-coercive (see Section 10.3 for several equivalent conditions for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$), then there is a unique minimizing control for each $x_0 \in H$ (provided that Z^{s} is reflexive). The minimizing control and corresponding state, output and cost are then given in WPLS form by Theorem 8.3.9

However, as noted above, without further assumptions we do not know whether the unique minimizing control is of state feedback form. We give below different formulations of sufficient conditions.

We now recall the basic minimization results from Chapter 9 (do not forget our definitions: we require the solutions of Riccati equations to be self-adjoint):

Corollary 10.2.5 (Minimizing control⇔eCARE/eIARE) *Let $\mathcal{J}(0,\cdot) \geq 0$. Then the following hold:*

(a) There is a minimizing state feedback pair $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ for Σ iff the eIARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$.

(b) Let $\mathbb{D}$ be WR. Then there is a minimizing WR state feedback operator K for Σ iff the eCARE

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC, \\ \quad S = D^*JD + \underset{s\to+\infty}{\text{w-lim}} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B, \\ \quad SK = -(B_{\mathrm{w}}^*\mathcal{P} + D^*JC), \end{cases} \tag{10.7}$$

has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \begin{bmatrix} K \mid 0 \end{bmatrix})$.

If such a solution exists and $x_0 \in H$, then the minimizing control $u_{\min}(x_0)$ is given by $u_{\min}(x_0)(t) = K_{\mathrm{w}}x(t)$ a.e., where $x = \mathbb{A}x_0 + \mathbb{B}\tau u_{\min}(x_0)$ is the corresponding state.

(c) Let $\mathcal{P}$ be a $\mathcal{U}_*^*$-stabilizing solution of the eIARE (resp. eCARE, hence of both).

Then $\mathcal{P}$ is unique, $S \geq 0$, and $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is the pair determined by $\mathcal{P}$ (resp. generated by $\begin{bmatrix} K \mid 0 \end{bmatrix}$). The control $\mathbb{K}_\circlearrowleft$ is strictly minimizing iff $S > 0$.

Also parts (f1)–(k) of Theorem 9.9.1 apply to $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix})$; in particular, the minimal cost is given by $\mathcal{J}(x_0, u_{\min}(x_0)) = \langle x_0, \mathcal{P}x_0\rangle$.

If $\mathcal{J}(\cdot,\cdot) \geq 0$ and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$, then $\mathcal{P}$ is the smallest nonnegative output-stabilizing solution of the eIARE (resp. eCARE).

(d) If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then "$\mathcal{J}(0,\cdot) \geq 0$" can be dropped from this corollary if we require the solutions of the eIARE and the eCARE to satisfy $S \geq 0$.

In particular, (b)–(d) apply to the ($\mathcal{U}_*^*$-stabilizing) solutions of the eCARE (for Σ) mentioned in other results of this chapter.

Proof: (a)–(c) This is a combination of Theorem 9.9.1(a2)&(f1)–(k) and Corollary 9.9.2.

(d) If there is a minimizing control for any $x_0 \in H$, then $\mathcal{J}(0,\cdot) \geq 0$, by Lemma 10.2.2, hence then $S \geq 0$, by (c). Conversely, any solution with $S \geq 0$ is minimizing, by Theorem 9.9.1(k). $\square$

By combining (a)&(b)&(d) of Corollary 10.2.5 with Theorem 9.8.5, we obtain:

Corollary 10.2.6 ($\mathcal{U}_{\mathrm{exp}}$-min $\Leftrightarrow$ eIARE) *There is a minimizing state feedback pair over $\mathcal{U}_{\mathrm{exp}}$ iff the eIARE has an exponentially stabilizing solution with $S \geq 0$.*

Let $\mathbb{D}$ be WR. Then there is a minimizing WR state feedback operator over $\mathcal{U}_{\mathrm{exp}}$ iff the eCARE has an exponentially stabilizing solution with $S \geq 0$. $\square$

If $\mathcal{P}$ is such a solution and $S \gg 0$ (equivalently, $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\mathrm{exp}}$, by Proposition 9.9.12), then $\mathcal{P}$ is the greatest admissible solution of the eIARE having $S \geq 0$, by Corollary 15.1.3. Naturally, also Corollary 10.2.5(c) applies.

Next we note some "anomalies" (see Section 9.13 for more):

Example 10.2.7 1. We may have $\mathcal{P} \ll 0$ (this is trivial; see Example 9.13.13).

2. All minimizing controls need not be of the state feedback form even though one of them is (see Example 9.13.6). $\triangleleft$

To start with a simple case, we first generalize Theorem 16.3.3 of [LR] (see Proposition 10.3.1(d) for (10.87) and positive J-coercivity):

Corollary 10.2.8 *Assume that Σ is optimizable and estimatable, $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\mathrm{out}}$, and (1.), (2.) or (4.) of Hypothesis 9.2.2 holds. Then there is a unique exponentially stabilizing solution $(\mathcal{P}, S, K)$ of the B_{w}^*-CARE. Moreover,*

(a) $\mathcal{J}(x_0, u) = \langle x_0, \mathcal{P}x_0\rangle$ for all $x_0 \in H$;

(b) For each $x_0 \in H$, there is a strictly minimizing control over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$, namely that given by the state feedback operator K.

(c) K is ULR and exponentially stabilizing.

(d) If (10.87) holds for some $\varepsilon > 0$, then $\mathcal{P}$ is the unique nonnegative solution of the B_{w}^*-CARE.

(e) $\mathcal{P}$ is the greatest solution of the B_{w}^*-CARE.

See Proposition 10.3.1 for necessary and sufficient conditions for positive J-coercivity over $\mathcal{U}_{\mathrm{out}}$.

Proof: By Lemma 8.3.3, we have $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$. By Lemma 9.2.17, we have $D^*JD \gg 0$. By Corollary 10.2.9, $(\mathcal{P}, S, K)$ exists and claims (b) and (c) hold; we then obtain (a) from Theorem 9.9.1. Claim (d) follows from Proposition 10.7.3(d3) and claim (e) follows from Corollary 9.2.11. $\square$

Under sufficient regularity, minimization over $\mathcal{U}_{\mathrm{exp}}$ is easy:

Corollary 10.2.9 ($\mathcal{U}_{\text{exp}}$: Unique minimum $\Leftrightarrow$ B_{w}^*-CARE$\Leftrightarrow$ J-coercive) *Assume that Hypothesis 9.2.1 holds for $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, and that $D^*JD \gg 0$. Then the following are equivalent:*

(i) There is a unique minimizing control over $\mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$.

(ii) The B_{w}^*-CARE (9.13) has an exponentially stabilizing solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H, \text{Dom}(B_{\text{w}}^*))$.

(iii) Σ is optimizable and $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{exp}}$ (i.e., any (hence all) of Proposition 10.3.2(i)–(iii) holds).

If (ii) holds, then $K := -(D^*JD)^{-1}(B_{\text{w}}^*\mathcal{P} + D^*JC)$ is the unique minimizing state feedback operator over $\mathcal{U}_{\text{exp}}$ and ULR.

If $J \geq 0$, then the word "unique" is redundant in (i). If $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1];\mathcal{B}(U,H))$ then "$D^*JD \in \mathcal{GB}(U)$" is redundant in (iii).

Further results are given in Theorems 9.2.10–9.2.12. See Corollary 9.5.10 for applications of this and the following corollary for parabolic problems.

Recall that if $\mathcal{P}$ is a solution of the B_{w}^*-CARE (which requires that $D^*JD \in \mathcal{GB}(U)$, by Definition 9.2.6), then $A_{\circlearrowleft} := A + BK$ generates a C_0-semigroup $\mathbb{A}_{\circlearrowleft}$; the additional requirement in (ii) is that this semigroup satisfies $\|\mathbb{A}_{\circlearrowleft}(t)\| \leq M\mathrm{e}^{-\varepsilon t}$ $(t \geq 0)$ for some $\varepsilon > 0$ and $M < \infty$.

If $J \gg 0$, then $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{exp}}$ iff there is $\varepsilon > 0$ s.t.

$$(ir - A)x_0 = Bu_0 \Longrightarrow \|C_{\text{w}}x_0 + Du_0\|_Y \geq \varepsilon\|x_0\|_H \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}). \tag{10.8}$$

(Note that $(ir - A)x_0 = Bu_0 \Rightarrow x_0 \in H_B \subset \text{Dom}(C_{\text{w}})$.) The same holds for Corollary 10.2.10.

Proof of Corollary 10.2.9: Set $\mathcal{U}_*^* := \mathcal{U}_{\text{exp}}$.

1° *(i)⇔(ii)⇔(iii):* By Theorem 9.2.16, (i) and (iii) imply (ii). If (ii) holds, then we obtain Theorems 9.2.16 and 9.9.1(k)&(a2) that (i) holds, $\mathbb{D}$ is J-coercive and $\mathcal{J}(0,\cdot) \geq 0$, hence $\mathbb{D}$ is positively J-coercive, hence also (iii) holds.

2° *Proposition 10.3.2(i)–(iii):* By Proposition 10.3.2(c)&(g1), any of Proposition 10.3.2(i)–(iii) implies Proposition 10.3.2(i). Conversely, if (iii) holds, then (ii) holds, hence then Σ has an exponentially stabilizing ULR K, hence Proposition 10.3.2(i)–(iii) are equivalent (hence they all hold), by Proposition 10.3.2(e2) (since $D^*JD \gg 0$).

3° The uniqueness of K follows from Proposition 6.6.18(g). The "$J \geq 0$" claim follows from Lemma 9.3.7(1+). If $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1];\mathcal{B}(U,H))$, then "$D^*JD \in \mathcal{GB}(U)$" is redundant in (iii) by Proposition 10.3.2(e2). □

We now establish Corollary 10.2.9 under "weaker" assumptions (this forces us to replace the B_{w}^*-CARE by the CARE):

Corollary 10.2.10 ($\mathcal{U}_{\text{exp}}$: Unique minimum $\Leftrightarrow$ CARE $\Leftrightarrow$ J-coercive) *Assume that $\mathbb{A}B \in \mathrm{L}^2([0,1);\mathcal{B}(U,H))$, $C_{\text{w}}\mathbb{A} \in \mathrm{L}^1([0,1);\mathcal{B}(H,Y))$, and $C_{\text{w}}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,Y))$.*

Then the following are equivalent:

(i) there is a [unique] minimizing control over $\mathcal{U}_{\mathrm{exp}}(x_0)$ for each $x_0 \in H$, and $D^*JD \gg 0$;

(ii) there is a [unique] exponentially stabilizing solution $(\mathcal{P}, S, K)$ of the CARE, and $S \gg 0$ (or $D^*JD \gg 0$);

(iii) $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is optimizable, and $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\mathrm{exp}}$ (i.e., any (hence all) of Proposition 10.3.2(i)–(iii) holds).

Moreover, any solution of (ii) is as in Theorem 9.2.18 (in particular, $S = D^*JD \gg 0$), and K is ULR and the unique minimizing state feedback operator over $\mathcal{U}_{\mathrm{exp}}$. If $J \geq 0$, then the word "unique" is redundant in (i).

For L^1 in place of L^2 above, we have an analogous corollary of Theorem 9.2.18 (thus, then we must assume positive J-coercivity and we may remove "$D^*JD \gg 0$" from (i) and (ii)).

Proof: 1° *(i)⇒(ii)&(iii):* Assume (i). By Corollary 9.2.19, (ii) and (iii) are satisfied (and $S = D^*JD \gg 0$) except for the word "positively", which is obtained from Proposition 9.9.12(b).

2° *(iii)⇒(i):* This follows from Corollary 9.2.19, since $D^*JD \gg 0$, by Lemma 9.2.17.

3° *(ii)⇒(i):* If $D^*JD \gg 0$, then this is trivial. If $S \gg 0$, then we obtain (iii) from Corollary 9.2.19, and Proposition 9.9.12(b), hence (i) holds, by 2°.

4° *Proposition 10.3.2(i)–(iii):* Use Proposition 10.3.2(e2) in 2° of the proof of Corollary 10.2.9.

5° *Final claims:* By Corollary 9.2.19, any solution of (ii) is as in Theorem 9.2.18 The uniqueness of K follows from Proposition 6.6.18(g). The "$J \geq 0$" claim follows from Lemma 9.3.7(1+). □

For general WPLSs, the above problem becomes rather tricky, and we cannot say much more than in Corollary 10.2.5. To make our result neater, we assume that $J \geq 0$ and start with the case $\dim U < \infty$:

Theorem 10.2.11 ($\mathcal{U}_{\mathrm{exp}}$: Unique $\left[\begin{array}{c|c} \mathbb{K}_{\min} & \mathbb{F}_{\min} \end{array}\right]$ ⇔ IARE ⇔ J-coercive) *Assume that $J \geq 0$, $\dim U < \infty$ and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. Then (i)–(iii) are equivalent.*

(i) There is a unique (modulo (9.114)) minimizing state feedback pair for Σ.

(ii) The IARE has an exponentially stabilizing solution.

(iii) Σ is exponentially stabilizable and [positively] J-coercive.

Moreover, the following hold:

(a) If (ii) holds, then this exponentially stabilizing solution is the greatest nonnegative admissible solution and strictly minimizing.

(b) If Σ has an SR (resp. ULR) exponentially stabilizing state feedback operator with closed-loop system $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ satisfying Hypothesis 10.6.1(1.) (resp. (6.)), then the IARE is equivalent to the CARE (resp. and to the B_{w}^*-CARE), and also (i')–(iii') below become equivalent to (i)–(iii).

Thus, then there is a unique WR (resp. ULR) minimizing state feedback operator iff any (hence all) of these hold.

(c) **($\dim U = \infty$)** *Drop the assumption that* $\dim U < \infty$. *Then all of the above holds if we add to (i) and (i') the requirement "and* $\mathbb{D}$ *is* J*-coercive" (or "and* $\langle \mathbb{D}u, J\mathbb{D}u\rangle \geq \varepsilon\|u\|_2^2$ *for some* $\varepsilon > 0$ *and all* $u \in \mathcal{U}_{\exp}(0)$*").*

The original (i) holds iff the eIARE has an exponentially stabilizing solution with $S > 0$.

(We believe that implication (iii)$\Rightarrow$(i) does not hold in the indefinite case.)

See Proposition 10.3.2 for equivalent conditions for (positive) J-coercivity. For (b) we note that when $\dim U < \infty$ (but not in case (c)), the concept WR (resp. WLR) is equivalent to SR (resp. SLR) and to UR (resp. ULR) (for state feedback operators and pairs and solutions of the AREs), by Lemma 6.3.2(a1)&(a2).

By (a), it suffices to find a maximal nonnegative admissible solution (if any) and check whether it is exponentially stabilizing (if none/not, then (i)–(iii) do not hold).

In addition to the equivalence between (i)–(iii), we have an equivalence between the following, weaker conditions corresponding to a minimizing, possibly non-well-posed "state feedback" (still under the assumptions that $J \geq 0$ and $\dim U < \infty$):

(i') There is a unique minimizing control for Σ over $\mathcal{U}_{\exp}(x_0)$ for each $x_0 \in H$.

(ii') The discretized IARE has an exponentially stabilizing solution.

(iii') Σ is optimizable and [positively] J-coercive over $\mathcal{U}_{\exp}$.

(Any of these is equivalent to the existence of a unique minimizing control in WPLS form, see Theorems 8.3.9.) If $\mathbb{D}$ is SR and $D^*JD > 0$, then (ii') becomes equivalent to the $\mathrm{Dom}(A_{\mathrm{crit}})$-CARE (9.67) having an exponentially stabilizing solution (cf. Remark 9.7.7(b2)).

(Modify the proof of Proposition 9.9.12 to observe that the discrete "S" is $\gg 0$ to obtain (i')$\Leftrightarrow$(iii'). Equivalence (i')$\Leftrightarrow$(ii') follows from the above proposition (in its discrete-time form) and Theorem 14.1.6. If $\mathbb{D}$ is SR and $D^*JD > 0$, then $\mathbb{D}$ (and hence also $\mathbb{D}^{\mathrm{d}}$) is UR and $D^*JD \gg 0$ (since $\dim U < \infty$), so that the last equivalence follows from Remark 9.7.7(b2).)

If Hypothesis 9.2.1 holds for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ and $D^*JD \gg 0$, then any of (i')–(iii') is equivalent to (i)–(iii), as well as to the B_{w}^*-CARE having an exponentially stabilizing solution, by Corollary 10.2.9, even for general J and U. The situation in discrete time is about the same as for bounded B (with the exception that D^*JD must be replaced by $S = D^*JD + B^*\mathcal{P}B$).

Proof of Theorem 10.2.11: 1° *(i)$\Leftrightarrow$(ii):* This follows from Theorem 9.9.1(a2)&(e1)&(f2) (for "(i)$\Rightarrow$(ii)" we note that $S > 0$ implies that $S \gg 0$; for "(ii)$\Rightarrow$(i)" we note that $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft \gg 0$ and $J \geq 0$ imply that $S \geq 0$, hence $S \gg 0$).

2° *(ii)$\Rightarrow$(iii):* This follows from Proposition 9.9.12, since $J \geq 0$ implies that J-coercivity is equivalent to positive J-coercivity (and the invertibility of S implies that of $\mathbb{T}^t$).

3° *(iii)$\Rightarrow$(i):* Assume (iii). Let $\begin{bmatrix} \mathbb{K}' \mid \mathbb{F}' \end{bmatrix}$ be exponentially stabilizing for Σ with closed-loop system $\Sigma_\flat$, so that $\Sigma_\flat$ is positively J-coercive over $\mathcal{U}_{\exp}^{\Sigma_\flat} = \mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}$, by Theorem 8.4.5(d), i.e., $\mathbb{D}_\flat^* J\mathbb{D}_\flat \gg 0$, by Lemma 8.4.11(a2).

By Corollary 10.2.13(b) (whose proof is independent of this), the IARE for $\Sigma_\flat$ has an exponentially stabilizing, minimizing solution, hence so has the IARE for Σ, by Lemma 9.12.3(d1). The uniqueness claim follows from Theorem 9.9.1(f2).

(a) Since $J \geq 0$, we have $S \geq 0$, hence $S \gg 0$ (because $S \in \mathcal{GB}(U)$, by 1°) and the solution is the greatest nonnegative solution, by Theorem 9.9.1(a2)&(e1)&(f2).

(b) Let $\widetilde{K}$ be a SR (resp. ULR) state feedback operator for Σ with closed-loop system $\Sigma_\flat$ having $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}_+$. Let $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ be the corresponding pair.

Assume (iii'). Then $\Sigma_\flat^1 := \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ is (exponentially stable and) positively J-coercive over $\mathcal{U}_{\text{exp}}^{\Sigma_\flat} = \mathcal{U}_{\text{out}}^{\Sigma_\flat}$, by Theorem 8.4.5(d). By Theorem 10.6.3(f1) (resp. (a)), there is a (unique) exponentially stabilizing solution $(\mathcal{P}, S, K_\natural)$ of the CARE (resp. and the B_w^*-CARE) for $\Sigma_\flat^1$. Set $K := \widetilde{K} + K_\natural$. By Proposition 9.12.4(b), $(\mathcal{P}, S, K)$ is an exponentially stabilizing solution of the CARE (resp. and the B_w^*-CARE, by Proposition 9.3.5(a). (By (a), K is strictly minimizing.)

(c) Parts 1°–3° still hold except that now "$S \gg 0$" in 1° must be deduced from Lemma 9.9.7(c3)&(c4) (note that necessarily $\mathcal{P} = \mathbb{C}_\circlearrowleft^* J \mathbb{C}_\circlearrowleft \gg 0$). The last claim follows from Corollary 10.2.5(a)&(c).

The proofs of (a) and (b) do not use the assumption $\dim U < \infty$ (this is why we write "SR" instead of "WR" or "UR" in (b)). □

For most of the rest of this section, we shall present result for $\mathcal{U}_{\text{out}}$ (recall that $\mathcal{U}_{\text{exp}} = \mathcal{U}_{\text{str}} = \mathcal{U}_{\text{sta}} = \mathcal{U}_{\text{out}}$ when Σ is estimatable or exponentially q.r.c.-stabilizable, by Lemma 8.3.3).

When minimizing over $\mathcal{U}_{\text{out}}$, we cannot we cannot reduce the problem to the stable case as in Theorem 10.2.11, (unless we choose to use Theorem 8.4.5(f)), as illustrated in Example 9.13.2. However, if the system is q.r.c.-SOS-stabilizable (e.g., SOS-stable), then the reduction will succeed. This fact will be applied in most results below. We first give two results on the IARE and then two on the CARE.

Corollary 10.2.12 ($\mathcal{U}_{\text{out}}$: Unique $\left[\begin{array}{c|c} \mathbb{K}_{\min} & \mathbb{F}_{\min} \end{array}\right]$ $\Leftrightarrow$ IARE $\Leftrightarrow$ J-coercive$\Leftrightarrow$ r.c.f.) *Then the following conditions are equivalent to each other and stronger than the conditions (Crit1)–(Crit4) of Theorem 9.9.10:*

(Crit1+) **(Minimizing $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$)** *There is a [strictly] minimizing q.r.c.-SOS-stabilizing state feedback pair for Σ over $\mathcal{U}_{\text{out}}$, and $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\text{out}}$.*

(Crit2+) **(IARE)** *The IARE (9.111) has a q.r.c.-SOS-P-stabilizing solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ s.t. $S \gg 0$.*

(Crit3+) **(J-coercivity)** *Σ is positively J-coercive over $\mathcal{U}_{\text{out}}$ and q.r.c.-SOS-stabilizable.*

(Crit4+) **(R.c.f.)** *The map $\mathbb{D}$ has a (J,I)-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, and Σ is q.r.c.-SOS-stabilizable.*

If any of these has a solution, then that solution solves the other conditions and (Crit1)–(Crit4), and (a1)–(g3) of Theorem 9.9.10 apply.

See Theorem 10.2.14 for corresponding CAREs.

We recall that $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{out}}$ iff there is $\varepsilon > 0$ s.t.

$$\langle \mathbb{D}u, J\mathbb{D}u\rangle \geq \varepsilon\left(\|u\|_2^2 + \|\mathbb{D}u\|_2^2\right) \qquad (u \in \mathcal{U}_{\text{out}}(0)). \tag{10.9}$$

Proof: 1° *(Crit1+)⇔(Crit2+)⇔(Crit4+):* Except for (Crit3+), This follows from Theorem 9.9.10(e2) (use normalization $\mathbb{N}' := \mathbb{N}S^{-1/2}$, $\mathbb{M}' := \mathbb{M}S^{-1/2}$ for implication (Crit2+)&(Crit4)⇒(Crit4+)).

2° *(Crit1+)⇔(Crit3+):* Obviously, (Crit1+) implies (Crit3+) (*positive J*-coercivity follows from Lemma 10.2.2). For the converse, assume (Crit3+) and let $\Sigma_\flat$ be the corresponding closed-loop system. By Lemma 8.4.11(c), $\mathbb{D}_\flat$ is J-coercive over $\mathcal{U}_{\text{out}}$, hence there is a minimizing q.r.c.-SOS-stabilizing pair for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$, by (the proof of) Corollary 10.2.13. By Theorem 8.4.5(g1)&(a), there is a minimizing q.r.c.-SOS-stabilizing pair for Σ too.

3° The final claims follow from Theorem 9.9.10(a1)–(a3). □

In the stable case, one more equivalent condition is the uniform positivity of the Popov operator (which is obtained, e.g., by adding $\varepsilon\|u\|^2$ to the nonnegative cost function):

Corollary 10.2.13 ($\mathcal{U}_{\text{out}}$: $\left[\begin{array}{c|c} \mathbb{K}_{\min} & \mathbb{F}_{\min}\end{array}\right]$ — Stable case) *Let* $\Sigma \in$ SOS. *The following conditions are equivalent to and have the same solutions as (Crit1+)–(Crit4+) of Corollary 10.2.12:*

(Crit1SOS+) There is a stable uniformly minimizing SOS-stabilizing feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}\end{array}\right]$ *for* Σ *over* $\mathcal{U}_{\text{out}}$*;*

(Crit2SOS+) the IARE has a stable P-SOS-stabilizing solution having $S \gg 0$*;*

(Crit3SOS+) $\mathbb{D}^* J\mathbb{D} \gg 0$*;*

(Crit4SOS+) $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^*\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\text{TIC}(U)$.

Moreover,

(a) If Σ *satisfies Hypothesis 10.6.1(1.) (resp. (6.)), then we can replace the IARE by the CARE (resp.* B_{w}^**-CARE) above.*

(b) If Σ *is exponentially stable, then solutions of (Crit1SOS+)–(Crit2SOS+) are exactly the minimizing pairs over* $\mathcal{U}_{\exp}$*, equivalently, the exponentially (equivalently, I/O-)stabilizing solutions of the IARE having* $S \gg 0$.

Recall from Lemma 8.4.11(a2) that (Crit3SOS+) holds iff $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{out}}$.

By a "uniformly minimizing" $u_{\min}$ we mean here that $\mathcal{J}(x_0, u_{\min}(x_0) + \eta) - \mathcal{J}(x_0, u_{\min}) \geq \varepsilon\|\eta\|_2^2$ for all $\eta \in \mathrm{L}^2(\mathbf{R}_+; U)$. If $\dim U < \infty$, then we could replace "uniformly minimizing" by "strictly minimizing" in (Crit2SOS+), because then $S \gg 0 \Leftrightarrow S > 0$ (cf. the proof below). The minimizing pair is given by (9.140), i.e., by

$$\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}\end{array}\right] := \left[\begin{array}{c|c} -\pi_+\mathbb{X}^{-*}\mathbb{D}^* J\mathbb{C} & I - \mathbb{X}\end{array}\right], \tag{10.10}$$

where $\mathbb{X}$ is as in (Crit4SOS+).

If Σ is [strongly] stable, then "SOS-" [and "P-"] can be removed from conditions (Crit2SOS+) and (Crit3SOS+) [and "strongly" can be added], by Corollary 6.6.9.

Proof of Theorem 10.2.13: By Theorem 10.6.3(f2)&(b), (Crit2SOS+)–(Crit4SOS+) and (Crit2+) are equivalent (and the (unique) solutions $\mathcal{P}$ of (Crit2+) are exactly the ones of (Crit2SOS+)).

Finally, we obtain (Crit1+)⇔(Crit1SOS+), i.e., the fact that the control $\mathbb{K}_{\circlearrowleft}x_0$ is uniformly minimizing iff $S \gg 0$, by setting $u_{\circlearrowleft} := \mathbb{M}^{-1}\eta$ in Theorem 9.9.10(e1), because $S \gg 0$ iff $\mathbb{M}^{-*}S\mathbb{M}^{-1} \gg 0$ (note that $\mathbb{M} = (I - \mathbb{F})^{-1} \in \mathcal{G}$TIC).

(a) This follows from Theorem 10.6.3(b)&(f1).

(b) This follows from Theorem 9.9.10(c1)&(c2). □

That was the case with IAREs, but we would like to use the CAREs instead of IAREs. Therefore, we shall assume further regularity. One way for this is to use the class $\widetilde{\mathcal{A}}_+$ (see Standing Hypothesis 10.6.6 and (b3) below) $\widetilde{\mathcal{A}}_+$ to formulate conditions under which a unique minimizing control necessarily corresponds to a ($\mathcal{U}_*^*$-stabilizing) solution of the CARE, hence to a WR state feedback operator.

Theorem 10.2.14 ($\mathcal{U}_{\text{out}}$: Unique K_{min} ⇔ CARE ⇔ J-coercive ⇔ r.c.f.) *Let $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$. Assume that $\mathbb{D}$ is WR. Then the following conditions are equivalent to each other and stronger than the conditions (Crit1+)–(Crit4+) of Corollary 10.2.12:*

(Crit1+WR) **(Minimizing K)** *There is a WR [strictly] minimizing q.r.c.-SOS-stabilizing state feedback operator for Σ, and $\mathbb{D}$ is J-coercive.*

(Crit2+WR) **(CARE)** *The CARE has a q.r.c.-SOS-P-stabilizing solution having $S \gg 0$.*

(Crit4+WR) **(WR r.c.f.)** *The map $\mathbb{D}$ has a (J,I)-inner q.r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ with $\mathbb{X} := \mathbb{M}^{-1}$ being WR with $X \in \mathcal{GB}(U)$, and Σ is q.r.c.-SOS-stabilizable.*

Moreover, the following hold:

(a) If any of (Crit1+WR)–(Crit4+WR) has a solution, then that solution solves the other conditions and (Crit1+)–(Crit4+) of Corollary 10.2.12 and (Crit1)–(Crit4), and then (a1)–(g3) of Theorem 9.9.10 apply.

(b1) Assume that Σ has a SR (resp. URL) q.r.c.-SOS-stabilizing state feedback operator $\widetilde{K}$ s.t. the resulting closed-loop system $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ satisfies Hypothesis 10.6.1(1.) (resp. (6.)).

Then (Crit7+WR) (resp. and (Crit6+WR)) is equivalent to (Crit1+WR)–(Crit4+WR):

(Crit6+WR) **(B_{w}^*-CARE)** *The B_{w}^*-CARE has q.r.c.-SOS-P-stabilizing solution.*

(Crit7+WR) $\mathbb{D}$ is positively J-coercive (equivalently, $\mathbb{D}_\flat^* J\mathbb{D}_\flat \gg 0$).

(b2) **(B_w^*-CARE)** *Assume that Hypothesis 9.2.1 holds and $D^*JD \gg 0$.*

Then conditions (Crit1+WR)–(Crit6+WR) are equivalent.

(b3) **(MTIC)** *Assume that Σ is q.r.c.-SOS-stabilizable in $\widetilde{\mathcal{A}}_+$, then (Crit1+WR)–(Crit4+WR) are equivalent to (Crit7+WR), and imply that $\mathbb{N}, \mathbb{M} \in \widetilde{\mathcal{A}}_+$.*

(b4) In (b1)–(b3), conditions (Crit1+WR)–(Crit4+WR) and (Crit1+)–(Crit4+) (and (Crit1)–(Crit4) in (b2)) are all equivalent to each other.

(c) If Σ is exponentially [q.]r.c.-stabilizable, then any I/O-stabilizing or input-stabilizing solution of the CARE having $S \gg 0$ is exponentially [q.]r.c.-stabilizing and minimizing over $\mathcal{U}_{\text{exp}} = \mathcal{U}_{\text{str}} = \mathcal{U}_{\text{sta}} = \mathcal{U}_{\text{out}}$. (See Theorem 6.7.15 for further reductions.)

If Σ is estimatable, then any minimizing pair over $\mathcal{U}_{\text{out}}$ is exponentially q.r.c.-stabilizing and minimizing over $\mathcal{U}_{\text{exp}} = \mathcal{U}_{\text{str}} = \mathcal{U}_{\text{sta}} = \mathcal{U}_{\text{out}}$, by Theorem 9.9.1(d). Therefore, then we get further equivalent conditions from the results for $\mathcal{U}_{\text{exp}}$ in the first part of this section.

Proof: The equivalence and (a) follow from Corollary 10.2.12 and Theorem 9.9.10(d1) (note that the $\mathbb{X}$ of (d1) is replaced by $S^{1/2}\mathbb{X}$).

(a) This follows from the above and Corollary 10.2.12.

(b1) 1° *(1.):* By Proposition 9.12.4(c) the (unique) solutions of (Crit2+WR) correspond 1-1 to the q.r.c.-SOS-P-stabilizing solutions of the CARE for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$. By Corollary 10.2.15(b1) (and (Crit2stable+WR) and (Crit2stable+WR')), such a solution exists iff $\mathbb{D}_\flat^* J \mathbb{D}_\flat \gg 0$, i.e., iff $\mathbb{D}$ is positively J-coercive (by Lemma 8.4.11(b1)).

2° *(5.):* This is contained in 1° except for the B_w^*-CARE claim which can be obtained as in the proof of Proposition 9.3.5.

(Note that Hypothesis 9.2.1 would not necessarily guarantee that the minimizing K is q.r.c.-stabilizing.)

(b2) By the above, we have (Crit1+WR)–(Crit4+WR)$\Rightarrow$(Crit1-4+). By Theorem 9.9.10(d2), (Crit6+WR) is weaker than any of the above, and by Theorem 9.2.9, (Crit6+WR) implies (Crit2+WR).

(b3) This is contained in (b1) except for the fact that $\mathbb{N}, \mathbb{M} \in \widetilde{\mathcal{A}}_+$, which follows from Lemma 10.6.7(b).

(b4) For (b2), this was noted in the proof of (b2). Obviously, (Crit3+) implies (Crit7+WR), hence also the claims on (b1) and (b3) hold.

(c) By Theorem 6.7.15(b1), any I/O-stabilizing solution of the CARE is exponentially [q.]r.c.-stabilizing. The rest follows from (Crit2+WR) (and (a)) and Lemma 8.3.3. □

The stable case is a bit simpler:

Corollary 10.2.15 ($\mathcal{U}_{\text{out}}$: $K_{\min}$ — Stable case) *Let $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$. Assume that $\mathbb{D}$ is WR and Σ is strongly stable. Then the following conditions are equivalent to each other and to (Crit1+WR)–(Crit4+WR).*

(Crit1stable+WR) **(Minimizing** *K***)** *There is a WR [strictly] minimizing stable, stabilizing state feedback operator for Σ, and $\mathbb{D}$ is J-coercive.*

(Crit2stable+WR) **(CARE)** *The CARE has a stable, stabilizing solution having* $S \gg 0$.

(Crit2stable+WR') **(CARE)** *The CARE has a solution having* $\mathbb{M}$ *stable and* $S \gg 0$.

(Crit4stable+WR) **(WR SpF)** *We have* $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* \mathbb{X}$ *for some WR* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$ *having* $X \in \mathcal{GB}(U)$.

Moreover, the following hold:

(a) If any of (Crit1stable+WR)–(Crit4stable+WR) has a solution, then that solution solves the other conditions, (Crit1+WR)–(Crit4+WR) of Theorem 10.2.14, (Crit1+)–(Crit4+) of Corollary 10.2.12 and (i)–(iii) of Theorem 9.1.7, and then (a1)–(g3) of Theorem 9.9.10 apply. In particular, such solutions are stable and strongly r.c.-stabilizing.

(b1) **(MTIC)** *If* Σ *satisfies Hypothesis 10.6.1(1.) (see Lemma 10.6.2), then (Crit1stable+WR)–(Crit4stable+WR) are equivalent to (Crit1+)–(Crit4+) and to*

(Crit7stable+WR) $\mathbb{D}^* J \mathbb{D} \gg 0$.

(b2) **(B^*_{w}-CARE)** *If* Σ *satisfies Hypothesis 10.6.1(6.), then (Crit1stable+WR)–(Crit7stable+WR) are equivalent:*

(Crit6stable+WR) $D^* J D \gg 0$ *and (the* B^*_{w}*-CARE)*

$$(B^*_{\mathrm{w}}\mathcal{P} + D^* JC)^* (D^* JD)^{-1} (B^*_{\mathrm{w}}\mathcal{P} + D^* JC) = A^* \mathcal{P} + \mathcal{P}A + C^* JC \tag{10.11}$$

has a solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H, \mathrm{Dom}(B^*_{\mathrm{w}}))$ *s.t.* $s \mapsto K_{\mathrm{w}}(s - (A + BK_{\mathrm{w}}))^{-1}B$ *is in* $\mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(U))$, *where* $K := -(D^* JD)^{-1}(B^*_{\mathrm{w}}\mathcal{P} + D^* JC)$.

(equivalently, s.t. K is stable and stabilizing).

Proof: (The labels of equivalent conditions follow roughly those of Theorem 9.9.10 and Corollary 10.2.13.)

The above equivalence is that of Corollary 10.2.13 with the additional requirement that $\mathbb{X} = I - \mathbb{F}$ is WR and $X \in \mathcal{GB}(U)$. (Note that we could again remove the J-coercivity assumption from the first condition if we required K to be uniformly minimizing.)

(a) This follows from the above and Corollary 10.2.13(a).

(b1) This follows from Theorem 10.6.3(f1)&(i)&(iv)&(iv').

(b2) This follows from (b1) and Theorem 10.6.3(i)&(iv)&(iv'). □

We extend one more classical result: if Σ is (approximately) observable and the cost function is somewhat standard, then $\mathcal{P} > 0$:

Lemma 10.2.16 ($\mathcal{P} > 0$) *If* Σ *is observable,* $J > 0$, $\mathcal{J}(x_0, u) > 0$ *for all* $x_0 \in H$ *and all nonzero* $u \in \mathcal{U}^*_*(x_0)$, *and there is a minimizing control for each* $x_0 \in H$, *then* $\mathcal{P} > 0$, *where* $\mathcal{P}$ *is the minimal cost operator.*

Proof: Obviously, $u = 0$ is strictly minimizing for $x_0 = 0$, hence the minimizing control is unique for each $x_0 \in H$, by Lemma 8.3.8. By Theorem 8.3.9(b1), we can define the minimal cost operator $\mathcal{P}$.

If $x_0 \in H$ and $0 \geq \langle x_0, \mathcal{P}x_0\rangle = \mathcal{J}(x_0, u_{\text{crit}}(x_0))$, then $u_{\text{crit}}(x_0) = 0$, by the assumption. But then $\langle \mathbb{C}x_0, J\mathbb{C}x_0\rangle = \mathcal{J}(x_0, 0) = 0$, hence then $\|J^{1/2}\mathbb{C}x_0\|_2 = 0$, i.e., $x_0 \in \text{Ker}(\mathbb{C}) = \{0\}$, hence then $x_0 = 0$. We conclude that $\mathcal{P} > 0$. □

Naturally, all of the above theory can be applied to the dual of the LQR problem (which was formulated in [WR00]):

Remark 10.2.17 (Final state estimation problem) *Let* $\Sigma := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \text{WPLS}(U, H, Y)$. *In the* Optimal Final State Estimation Problem (OFSEP) *we wish to find an estimator* $\mathbb{H}_{\circlearrowleft} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U), H)$ *s.t.* $\mathbb{H}_{\circlearrowleft}\mathbb{D}$ *is an optimal estimate for* $\mathbb{C}$, *i.e.,* $\|\mathbb{B} - \mathbb{H}_{\circlearrowleft}\mathbb{D}\|$ *is minimal.*

The dual problem of this is the optimal open-loop L^2-stabilization problem*, where we wish to find a controller* $\mathbb{K}_{\circlearrowleft} \in \mathcal{B}(H, \mathrm{L}^2(\mathbf{R}_+;U))$ *s.t.* $\mathbb{D}\mathbb{K}_{\circlearrowleft}$ *is minimizes* $\mathbb{C}$, *i.e.,* $\|\mathbb{C} + \mathbb{D}\mathbb{K}_{\circlearrowleft}\|$ *is minimal (the solutions of this problem correspond one-to-one to those of the OFSEP for* Σ^{d} *through* $\mathbb{H}_{\circlearrowleft} = -\mathbb{K}_{\circlearrowleft}^{\mathrm{d}}$*); this obviously corresponds to a minimizing control in WPLS form (i.e., the corresponding state feedback need not be well-posed; cf. Theorem 8.3.9).*

If one requires $\mathbb{H}_{\circlearrowleft}$ *to be generated by some weakly regular* $H \in \mathcal{B}(H_{-1}, U)$ *that stabilizes* Σ*, equivalently, if one requires* $\mathbb{K}_{\circlearrowleft}$ *to be generated by some weakly regular* $K \in \mathcal{B}(U, H_1)$ *that stabilizes* Σ*, then one ends up with our LQR problem, hence one can use applicable theorems and corollaries from above. In particular, the OFSEP CARE becomes (here* $J = I$*)*

$$\begin{cases} HSH^* = A\mathcal{P} + \mathcal{P}A^* + BJB^*, \\ \qquad S = DJD^* + \underset{s\to+\infty}{\text{w-lim}}\, C_{\mathrm{w}}\mathcal{P}(s - A^*)^{-1}C_{\mathrm{w}}^*, \\ \qquad H = -S^{-1}(C_{\mathrm{w}}\mathcal{P} + DJB^*). \end{cases} \tag{10.12}$$

In any case, one can find a solution for the OFSEP by using the theory this section. If we minimize $\left\| \begin{bmatrix} \mathbb{B} - \mathbb{H}_{\circlearrowleft}\mathbb{D} \\ \mathbb{H}_{\circlearrowleft} \end{bmatrix} \right\|$ *as in [WR00], the dual cost function becomes that of Section 10.1; in particular, then (the duals of) Theorems 10.1.4 and 10.1.6 apply. (Cf. also Definitions 6.6.10 and 6.6.21.)* □

Section 5 of [WR00] explores the connection between the OFSEP and estimatability (in the open-loop form only, excluding closed-loop systems, factorizations and CAREs). The main idea of this formulation (OFSEP) of the dual problem is to guarantee the existence of a solution regardless whether the solution is of output injection form or not.

For WPLSs with bounded C (e.g., for finite-dimensional systems; see [IOW]), one often defines the system and the cost function without any reference to the output:

Remark 10.2.18 (Bounded C) *Let* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$*, and let* C *be bounded. Set*

$$\widetilde{J} := \begin{bmatrix}C & D\end{bmatrix}^* J \begin{bmatrix}C & D\end{bmatrix} = \begin{bmatrix}C^*JC & C^*JD\\ D^*JC & D^*JD\end{bmatrix} \in \mathcal{B}(H\times U). \tag{10.13}$$

Then $\langle y, Jy\rangle = \langle\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right], \widetilde{J}\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]\rangle$*, where* $x := \mathbb{A}x_0 + \mathbb{B}u$*,* $y := \mathbb{C}x + \mathbb{D}u = Cx + Du$*.*

The converse is trivial: given $\mathbb{A}$*,* $\mathbb{B}$ *and* $\widetilde{J}$*, one may just take* $C := \left[\begin{smallmatrix}I\\0\end{smallmatrix}\right]$*,* $D := \left[\begin{smallmatrix}0\\I\end{smallmatrix}\right]$ *to get* $y = \left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]$*, and set* $J := \widetilde{J}$ *to obtain the same cost function as above.*

If we minimize over $\mathcal{U}_{\mathrm{exp}}$*, then we do not have to know y; knowledge on x, u and* $\mathcal{J}(x_0,u)$ *is enough, hence then it is not a problem that different* $\mathbb{C}$*,* $\mathbb{D}$ *and* J *may result in same* $\widetilde{J}$*.* □

The above remark can be used when translation our results to the language of several of the articles where a bounded output operator is assumed, and conversely. Note that the above remark can be generalized to arbitrary regular WPLSs.

Notes

The special case of a standard cost function (as in Section 10.1) was explained in the notes on p. 555. Implications "(Crit4SOS+)⇒(Crit1SOS+)–(Crit3SOS+)" of Corollary 10.2.13 and the SR case of "(Crit4stable+)⇒(Crit1stable+WR)–(Crit2stable+WR')" are more or less implicitly contained in [S97b] and [WW]. These were extended for jointly stabilizable and detectable systems in [S98b] (cf. Theorems 10.2.11 and 10.2.14). See [WR00] for the first the paragraphs of Remark 10.2.17. All these results treat only the case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$.

Corollary 10.2.9 generalizes Theorem 3.10 of [Keu], which replaces Hypothesis 9.2.1 by the stronger assumption that Σ is a Pritchard–Salamon system; most classical finite-dimensional results (including Theorem 16.3.3 of [LR], Section 14.3 of [ZDG], Section 5.2.2 of [GL] and Corollary 4.5.7 of [IOW]) are special cases of Corollary 10.2.9 or of Corollary 10.2.8, all these results assume positive J-coercivity or something stronger, as one can show by applying Proposition 10.3.2. All these results treat only the case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$.

In a sense, Corollary 10.2.9 uses the weakest possible coercivity assumption (positive J-coercivity; this can be relaxed if eCAREs ($D^*JD \not\gg 0$) are allowed). We do not know analogous results for more general systems (than Pritchard–Salamon systems), but there are some results with stronger coercivity or stabilizability assumptions for several subsets of WPLSs, as explained above and in the notes on p. 555.

10.3 Standard assumptions

Euclid taught me that without assumptions there is no proof. Therefore, in any argument, examine the assumptions.
— Eric Temple Bell (1883–1960)

In this section we study popular assumptions of LQR (minimization) problems; more exactly, we list sufficient (and necessary) conditions for positive J-coercivity over $\mathcal{U}_{\rm exp}$ or $\mathcal{U}_{\rm out}$.

In classical results (in particular, in most minimization results for finite-dimensional [LR] [GL] [ZDG] [IOW] or Pritchard–Salamon systems [Keu] [LW]), one usually assumes some of (i)–(iv) of Proposition 10.3.2 (see also Remark 10.3.3) or something stronger, and shows that the CARE has an exponentially stabilizing solution iff the system is exponentially stabilizable, and that in either case the solution leads to the unique minimizing state feedback over $\mathcal{U}_{\rm exp}$. In those rare results where the assumptions are weaker than positive J-coercivity, the CARE does not need to have a solution, although the eCARE or something similar might have (as in Corollary 10.2.5).

Sometimes the minimization is done over $\mathcal{U}_{\rm out}$ and the assumption is one of (ai)–(biv) of Proposition 10.3.1. The purpose of this section is to show that all these assumptions are equivalent to or stronger than positive J-coercivity. This also leads to a list of different ways to verify the positive J-coercivity of a given WPLS and thus make the minimization results of Section 10.2 more applicable.

Popular classical assumptions for LQR, H^2 and H^∞ problems include conditions "no invariant zeros on $i\mathbf{R}\cup\{\infty\}$" (see (iii) far below) and "no transmission zeros on $i\mathbf{R}\cup\{\infty\}$" ((aiv) below); for minimal finite-dimensional systems these are equivalent, as shown below. We start by showing that the latter condition is equivalent to I-coercivity over $\mathcal{U}_{\rm out}$ (recall that $\mathcal{U}_{\rm out}(0) := \{u\in \mathrm{L}^2(\mathbf{R}_+;U) \mid \mathbb{D}u\in\mathrm{L}^2\}$):

Proposition 10.3.1 ($\mathcal{U}_{\rm out}$: $y\in\mathrm{L}^2 \Rightarrow u\in\mathrm{L}^2$) *Let* $\Sigma := \left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *and* $J = J^*\in\mathcal{B}(Y)$. *Consider the following conditions:*

(ai) $\mathcal{J}(0,u)\ge\varepsilon\left(\|u\|_2^2+\|\mathbb{D}u\|_2^2\right)$ *for some* $\varepsilon>0$ *and all* $u\in\mathcal{U}_{\rm out}(0)$*; i.e.,* $\mathbb{D}$ *is positively J-coercive over* $\mathcal{U}_{\rm out}$.

(aii) $\|\mathbb{D}u\|_2\ge\varepsilon\|u\|_2$ $(u\in\mathrm{L}^2(\mathbf{R}_+;U))$ *for some* $\varepsilon>0$.

(aiii) $\widehat{\mathbb{D}}(s)^*\widehat{\mathbb{D}}(s)\ge\varepsilon I$ *for all* $s\in i\mathbf{R}\cup\{\infty\}$ *and some* $\varepsilon>0$.

(aiv) $\widehat{\mathbb{D}}(s)u_0\ne 0$ *for all* $s\in i\mathbf{R}\cup\{\infty\}$ *and all nonzero* $u_0\in U$.

(bii) $\langle\mathbb{D}u, J\mathbb{D}u\rangle\ge\varepsilon\|u\|_2^2$ $(u\in\mathcal{U}_{\rm out}(0))$ *for some* $\varepsilon>0$.

(biii) $\widehat{\mathbb{D}}(s)^*J\widehat{\mathbb{D}}(s)\ge\varepsilon I$ *for all* $s\in i\mathbf{R}\cup\{\infty\}$ *and some* $\varepsilon>0$.

(biv) $\widehat{\mathbb{D}}(s)^*J\widehat{\mathbb{D}}(s)>0$ *for all* $s\in i\mathbf{R}\cup\{\infty\}$.

(bv) $D^*JC=0$, $D^*JD\gg 0$, $C^*JC\ge 0$.

We have the following implications:

(a) **($J\gg 0$)** *Assume that* $J\gg 0$. *Then (ai)⇔(aii)⇔(bii). If, in addition,* $\dim U\times H\times Y<\infty$, *then (ai)–(biv) are equivalent.*

(b) **(Rational $\widehat{\mathbb{D}}$)** *Assume that* $\dim U \times H \times Y < \infty$. *Then (ai)⇔(bii)⇔(biii)⇔(biv)⇐(bv).*

If, in addition, Σ is exponentially stabilizable and exponentially detectable (e.g., minimal), then also (i)–(vi) of Proposition 10.3.2 are equivalent to (ai) (see Proposition 10.3.2(d)).

(c) **($\mathbb{D} \in$ TIC)** *Assume that $\mathbb{D}$ is stable. Then (ai) ⇔ (bii) ⇔ $\mathbb{D}^* J \mathbb{D} \geq \varepsilon I$ ⇔ $\langle \widehat{\mathbb{D}}(s) u_0, J \widehat{\mathbb{D}}(s) u_0 \rangle_Y \geq \varepsilon \|u_0\|_U^2$ a.e., for all $u_0 \in U$.*

If U is separable or $\widehat{\mathbb{D}}$ is piecewise continuous, then a fifth equivalent condition is that $(\widehat{\mathbb{D}})^* J \widehat{\mathbb{D}} \geq \varepsilon I$ a.e.

(d) Assume that $\widetilde{J} := \begin{bmatrix} C & D \end{bmatrix}^* J \begin{bmatrix} C & D \end{bmatrix} =: \left[\begin{smallmatrix} Q & N^* \\ N & R \end{smallmatrix}\right]$ satisfies $\widetilde{J} \geq 0$, $R \gg 0$, and $\dim \operatorname{Ker}(\widetilde{J}) = \dim \operatorname{Ker}(Q)$, *and* $\dim U \times H \times Y < \infty$.

Then there is $\varepsilon > 0$ s.t. $\langle \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], \widetilde{J} \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \rangle \geq \varepsilon \|u_0\|_U^2$ for all $x_0 \in H$, $u_0 \in U$. Therefore, $\mathcal{J}(x_0, u) \geq \varepsilon \|u\|_2^2$ for all measurable $u : \mathbf{R}_+ \to U$ and all $x_0 \in H$. In particular, then (bii)–(biv) hold.

(Consequently, if we wish to optimize over all measurable controls, we may replace $\begin{bmatrix} C & D \end{bmatrix}$ by $\widetilde{J}^{1/2}$ and J by $I \in \mathcal{B}(H \times Y)$ to make $\mathcal{U}_{\text{out}}(x_0)$ equal to $\{u \mid \mathcal{J}(x_0, u) < \infty\}$; the new system is observable if $\left[\begin{smallmatrix} A \\ Q \end{smallmatrix}\right]$ is.)

Condition (aiv) is the standard assumption that $\widehat{\mathbb{D}}$ has a full column rank on $i\mathbf{R} \cup \{\infty\}$. Equivalently, one can say that $\widehat{\mathbb{D}}$ has no *transmission zeros* on $i\mathbf{R} \cup \{\infty\}$ (see, e.g., Lemma 3.27 of [ZDG]).

Note that, when $J \geq 0$, $\mathbb{D}$ is J-coercive iff $\mathbb{D}$ is positively J-coercive; for stable $\mathbb{D}$ this holds iff $\mathbb{D}^* J \mathbb{D} \gg 0$ (cf. (c)). Note also that (ai) is included in (a) and (c) only (since (bii)–(biv) do not require that $\langle \mathbb{D}u, J\mathbb{D}u \rangle \geq \varepsilon \|\mathbb{D}u\|_2^2$).

When $J \gg 0$, the condition in (bii) holds for all $u \in \mathcal{U}_{\text{out}}(0)$ iff it holds for all $u \in \mathrm{L}^2_{\text{loc}}$.

When $\dim U \times H \times Y < \infty$, the function $\widehat{\mathbb{D}}$ is rational, hence $\widehat{\mathbb{D}}|_{i\mathbf{R}}$ is then well defined also for unstable $\mathbb{D}$. In this case, we define $\langle \widehat{\mathbb{D}} u_0, J \widehat{\mathbb{D}} u_0 \rangle$ as the limit of itself at the poles of A, e.g., if $\widehat{\mathbb{D}}(\cdot) u_0$ has a pole at s_0, it is considered that $\|\widehat{\mathbb{D}}(s_0) u_0\| = \infty$ (in (aiii), (aiv), (biii) and (biv)). Thus, (aiii) holds iff $\|\widehat{\mathbb{D}}(s) u_0\| \geq \varepsilon \|u_0\|$ for $u_0 \in U$ and $s \in i\mathbf{R} \setminus \sigma(A)$. Naturally, $\widehat{\mathbb{D}}(\pm\infty) = D$ (when $\dim H < \infty$). Note also that, in this finite-dimensional case, a minimal system is exponentially stabilizable and detectable (see [LR, p. 91]).

Proof of Proposition 10.3.1: (a) By Lemma 8.4.11(d2), we have (ai)⇔(bii). Trivially, $\langle \mathbb{D}u, J\mathbb{D}u \rangle = \infty \geq \varepsilon \|u\|_2$ whenever $u \in \mathrm{L}^2(\mathbf{R}_+, U) \setminus \mathcal{U}_{\text{out}}(0)$, hence (aii)⇔(bii). The rest of (a) follows from (b).

(b) We assume that $\dim U \times H \times Y < \infty$ and divide the proof in parts.

1° *"If $u, \mathbb{D}u \in \mathrm{L}^2(\mathbf{R}_+; *)$, then $\mathcal{L}\mathbb{D}u = \widehat{\mathbb{D}}\widehat{u}$ a.e. on $i\mathbf{R}$":* (In fact, this holds also whenever, e.g., the set N below is at most countable (this happens whenever A is compact) or $\mathbb{D}$ is stable (then $\mathcal{L}\mathbb{D}u = \widehat{\mathbb{D}}\widehat{u}$ a.e. on $i\mathbf{R}$, by (3.36)).)

Because $\dim H < \infty$, the set $N := \sigma(A)$ is finite, and there is $\omega \in \mathbf{R}$ s.t. $\mathbb{D} \in \mathrm{TIC}_\omega$ and $\widehat{\mathbb{D}} \in \mathrm{H}(\mathbf{C} \setminus N; \mathcal{B}(U, Y))$.

Set $\Omega := \mathbf{C}^+ \setminus N$. The functions $\widehat{\mathbb{D}}\widehat{u} \in \mathrm{H}(\Omega;Y)$ and $G := \widehat{\mathbb{D}u} \in \mathrm{H}^2(\mathbf{C}^+;Y)$ coincide on $\mathbf{C}^+_\omega$, hence they are identical on Ω, by Lemma D.1.2(e). Consequently, $\widehat{\mathbb{D}}\widehat{u}|_{i\mathbf{R}} \in \mathrm{L}^2$ is the boundary function of $G \in \mathrm{H}^2$ a.e., by (a1)(1.) and (a2) of Theorem 3.3.1.)

(Note that if, e.g., $\widehat{\mathbb{D}}(s) = (s-1)^{-1}$, then the map $\widehat{u} \mapsto \widehat{\mathbb{D}}\widehat{u}$ on $i\mathbf{R}$ induces a TI_0 map that is different from $\mathbb{D}$ (which is in $\mathrm{TIC}_\infty \setminus \mathrm{TIC}$). Above we showed that these two maps coincide on the set $\{u \in \mathrm{L}^2 \mid \mathbb{D}u \in \mathrm{L}^2\}$. Another example in this direction is provided below Example 3.3.10.)

2° *(biii)⇒(biv):* This is trivial.

3° *(biii)⇒(bii):* Assume (biii). Let $u \in \mathcal{U}_{\mathrm{out}}(0)$. Then $\mathcal{L}\mathbb{D}u = \widehat{\mathbb{D}}\widehat{u}$ a.e. on $i\mathbf{R}$, by 1°, hence

$$\langle \mathbb{D}u, J\mathbb{D}u\rangle = (2\pi)^{-1/2}\langle \widehat{\mathbb{D}}\widehat{u}, J\widehat{\mathbb{D}}\widehat{u}\rangle \geq (2\pi)^{-1/2}\varepsilon\|\widehat{u}\|_2^2 = \varepsilon\|u\|_2^2, \tag{10.14}$$

by the Plancherel Theorem (see (D.36)). Thus, (bii) holds.

4° *(bii)⇒(biii):* Set $F := \widehat{\mathbb{D}}(-\bar{\cdot})^* J\widehat{\mathbb{D}}$, so that $F \in \mathrm{H}(N^c;\mathcal{B}(U))$, where $N \subset \mathbf{C}$ is finite, and $F = \widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}$ on $i\mathbf{R} \cup \{\infty\}$.

Assume that (biii) is false (we aim to show that then also (bii) is false). Then there are $\varepsilon', \varepsilon'' < \varepsilon$, $ir_0 \in i\mathbf{R} \cup \{\infty\}$, $u_0 \in K := \{u_0 \in U \mid \|u_0\| = 1\}$ s.t. $G(ir_0) < \varepsilon' < \varepsilon''$, where $G(ir) := \langle u_0, F(ir)u_0\rangle$ $(r \in \mathbf{R})$. We may assume that $ir_0 \in i\mathbf{R} \setminus N$ (alter r_0 slightly if necessary).

Note that $G \in \mathrm{H}(N^c)$ and $G(ir) \in \mathbf{R}$ for all $r \in \mathbf{R}$. Let

$$\widehat{g}(s) := \prod_p (s-p)/(s+p+2), \tag{10.15}$$

where p runs over the poles of $\widehat{\mathbb{D}}u_0$ on $\mathbf{C}^+_{-1}$, counting multiplicities.

Then $\widehat{g}\widehat{\mathbb{D}}u_0 \in \mathrm{H}^\infty(\mathbf{C}^+_{-1};Y)$, $\widehat{g} \in \mathrm{H}^\infty(\mathbf{C}^+_{-1})$ and $|\widehat{g}| \leq 1$ on $\mathbf{C}^+_{-1}$. Set $a := |\widehat{g}(ir_0)|$, $M := \|J\|\|\widehat{\mathbb{D}}\widehat{g}u_0\|$, so that $|G||\widehat{g}|^2 \leq M$ on $\overline{\mathbf{C}^+} \cup \{\infty\}$, $M < \infty$ and $a > 0$. Given $\varepsilon_0 \in (0,a)$, there is $\delta > 0$ s.t. $a - \varepsilon_0 < |\widehat{g}(ir)| < a + \varepsilon_0$ on $J_\delta := i(r-\delta, r+\delta)$.

For each $t \in (0,\delta)$, we choose $f_t \in \mathrm{L}^2(\mathbf{R}_+)$ as in Lemma D.1.24, and set $\widehat{h}_t := \widehat{f}_t\widehat{g} \in \mathrm{H}^2(\mathbf{C}^+)$, so that $\widehat{\mathbb{D}}\widehat{h}_t u_0 \in \mathrm{H}^2$; in particular, $u_t := h_t u_0 \in \mathcal{U}_{\mathrm{out}}(0)$. Now

$$R := 2\pi\langle \mathbb{D}h_t u_0, J\mathbb{D}h_t u_0\rangle_{\mathrm{L}^2(\mathbf{R}_+;Y)} = \int_{i\mathbf{R}} \langle \widehat{h}_t u_0, F\widehat{h}_t u_0\rangle_U \, dm = \int_{i\mathbf{R}} G|\widehat{g}\widehat{f}_t|^2 \, dm \tag{10.16}$$

$$\leq \int_{J_\delta} \varepsilon'(a+\varepsilon_0)^2|\widehat{f}_t|^2 \, dm + \int_{i\mathbf{R}\setminus J_\delta} M|\widehat{f}_t|^2 \, dm \leq 2\pi\varepsilon'(a+\varepsilon_0)^2 + M\varepsilon_t, \tag{10.17}$$

where $\varepsilon_t \to 0$ as $t \to 0+$. By choosing t small enough, we get $R < 2\pi\varepsilon''(a+\varepsilon_0)^2$ and $\|\widehat{h}_t\|_2^2 \geq 2\pi(a-\varepsilon_0)^2$. Because $\varepsilon_0 \in (0,a)$ was arbitrary, we have $\langle \widehat{\mathbb{D}}\widehat{h}_t u_0, J\widehat{\mathbb{D}}\widehat{h}_t u_0\rangle \leq \varepsilon''\|h_t u_0\|_2^2$ for ε_0 and t small enough. Therefore, (bii) does not hold.

5° *(ai)⇒(bii):* This follows from Lemma 8.4.11(d1).

6° *(biv)⇒(ai):* Set $F := \widehat{\mathbb{D}}(-\bar{\cdot})^* J\widehat{\mathbb{D}}$, so that $F = \widehat{\mathbb{D}}^* J\widehat{\mathbb{D}}$ on $i\mathbf{R} \cup \{\infty\}$ and $F \in \mathrm{H}(N^c;\mathcal{B}(U))$ (in fact, F is rational), where $N \subset \mathbf{C}$ is finite.

6.1° *"(biv)⇒(biii)" when $\mathbb{D}$ is exponentially stable:* Now $F(i\cdot) \in \mathcal{C}_b(i\mathbf{R} \cup \{\infty\}; \mathcal{B}(U))$. Assume (biv).

Obviously, $(r, u_0) \mapsto \langle u_0, F(ir)u_0 \rangle$ is continuous $i\mathbf{R} \cup \{\infty\} \times U \to (0, +\infty)$. Therefore, $\varepsilon := \min_{r \in i\mathbf{R} \cup \{\infty\}, u_0 \in K} \langle u_0, F(ir)u_0 \rangle$ exists, where $K := \{u_0 \in U \,\big|\, \|u_0\| = 1\}$. By (biv), we have $\varepsilon > 0$, hence (biii) holds (with this ε).

6.2° *(biv)⇒(ai):* Assume (biv). Being rational, $\widehat{\mathbb{D}}$ has an (exponential) r.c.f. $\widehat{\mathbb{D}} = \widehat{\mathbb{N}}\widehat{\mathbb{M}}^{-1}$ (by Lemma 6.5.10(b)). Since

$$0 < \langle \mathbb{D}\mathbb{M}u_{\circlearrowleft}, J\mathbb{D}\mathbb{M}u_{\circlearrowleft} \rangle = \langle \mathbb{N}u_{\circlearrowleft}, J\mathbb{N}u_{\circlearrowleft} \rangle \quad (u_{\circlearrowleft} \in \mathrm{L}^2(\mathbf{R}_+; U)), \tag{10.18}$$

we have $\widehat{\mathbb{N}}^* J \widehat{\mathbb{N}} \geq \varepsilon I$ for some $\varepsilon > 0$, by 5.1°. By Lemma 8.4.11(a2)&(b1), it follows that $\mathbb{D}$ (resp. $\mathbb{N}$) is positively J-coercive over $\mathcal{U}_{\mathrm{out}}$ (resp. $\mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}$), hence (ai) holds.

7° *(bv)⇒(biv):* Assume (bv). Then $\widehat{\mathbb{D}}(s)^* J \widehat{\mathbb{D}}(s) = D^*JD + B^*(s - A)^{-*}C^*JC(s-A)^{-1}B \geq D^*JD \gg 0$ for all $s \in i\mathbf{R} \cup \{\infty\}$, hence then (biv) holds.

8° *(i)–(vi):* Proposition 10.3.2(d) provides the last equivalence.

(c) Now $\mathcal{U}_{\mathrm{out}}(0) = \mathrm{L}^2(\mathbf{R}_+; U)$, hence $\mathbb{D}^* J \mathbb{D} \geq \varepsilon I$ is equivalent to (bii), by Lemma 6.4.6. Trivially, (ai)⇒(bii); the converse also holds since $\|\mathbb{D}u\|_2 \leq \|\mathbb{D}\| \|u\|_2$. By Theorem 3.1.3(e2), also the remaining two equivalences hold.

(d) *Finding $\varepsilon > 0$:* Assume (iv). We have $\widetilde{J} = P^* \begin{bmatrix} \widetilde{Q} & 0 \\ 0 & R \end{bmatrix} P = E^*E$, where $\widetilde{Q} := Q - NR^{-1}N^*$, $P := \begin{bmatrix} I & 0 \\ -R^{-1}N^* & I \end{bmatrix} \in \mathcal{GB}(H \times U)$ (hence $\widetilde{Q} \geq 0$), $E = \begin{bmatrix} \widetilde{Q}^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix} P$. Thus, $0 \leq \langle z, \widetilde{Q}z \rangle \leq \langle z, Qz \rangle$ for all $z \in H \times U$, hence $\mathrm{Ker}(Q) \subset \mathrm{Ker}(\widetilde{Q})$. But $\dim \mathrm{Ker}(\widetilde{Q}) = \dim \mathrm{Ker}(Q) < \infty$, by (iv), hence $\mathrm{Ker}(Q) = \mathrm{Ker}(\widetilde{Q})$. Because $0 \leq \widetilde{Q}$, we must have

$$\mathrm{Ker}(Q) \subset \mathrm{Ker}(NR^{-1}N^*) = \mathrm{Ker}(N^*). \tag{10.19}$$

Therefore, $\mathrm{Ker}(\widetilde{Q}) \subset \mathrm{Ker}(N^*)$, hence $\mathrm{Ker}(\widetilde{Q}^{1/2}) \subset \mathrm{Ker}(N^*)$. By Lemma A.3.1(f), we have $L\widetilde{Q}^{1/2} = -R^{-1}N^*$ for some $L \in \mathcal{B}(U)$. Choose $\varepsilon_R > 0$ s.t. $R \geq \varepsilon_R^2 I$. Given $x_0 \in H$ and $u_0 \in U$, set $v_0 := u_0 - R^{-1}N^*x_0$, so that

$$\kappa(x_0, u_0) := \langle \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \widetilde{J} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \rangle = \langle v_0, Rv_0 \rangle + \langle x_0, \widetilde{Q}x_0 \rangle = \|E \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}\|^2. \tag{10.20}$$

To obtain a contradiction, assume that $x_0 \in H$ and $u_0 \in U$ are s.t. $\|u_0\| = 1$ but $\|E \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\|^2 < \varepsilon := \min\{\varepsilon_R/2, 1/4\|L\|^2\} > 0$. Then $\|v_0\|_U < \varepsilon\varepsilon_R^{-1} \leq 1/2$, hence $\|R^{-1}N^*x_0\| \geq 1/2$, hence $\widetilde{Q}^{1/2}x_0 \geq 1/2\|L\|$, hence $\kappa(x_0, u_0) \geq 1/4\|L\|^2 \geq \varepsilon$, a contradiction, as required. Thus, $\widetilde{J} \geq \varepsilon I$.

2° *The other claims:* Since $y = Cx + Du$, where $x := \mathbb{A}x_0 + \mathbb{B}\tau u$, we have

$$\mathcal{J}(x_0, u) = \int_0^\infty \langle \begin{bmatrix} x \\ u \end{bmatrix}, \widetilde{J} \begin{bmatrix} x \\ u \end{bmatrix} \rangle \, dm \geq \varepsilon \|u\|_2^2 \quad (u \in \mathrm{L}^2(\mathbf{R}_+; U)). \tag{10.21}$$

Thus, the second claim holds. Condition (bii) follows from this, by (b) also (biii) holds (and (biv) if $J \geq 0$).

Let us now study the new system with $\begin{bmatrix} C' & D' \end{bmatrix} := \widetilde{J}^{1/2}$. As shown above, $\mathcal{J}(x_0, u) = \infty$ for all $u \notin \mathrm{L}^2$. With this new setting, $\mathcal{U}'_{\mathrm{out}}(x_0)$ (for Σ') consists of all $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ for which (10.21) is finite; in particular, we may have $\mathbb{C}x_0 + \mathbb{D}u \notin \mathrm{L}^2$ (but $\mathbb{C}'x_0 + \mathbb{D}'u \in \mathrm{L}^2$). Thus, $\mathcal{U}'_{\mathrm{out}}$ includes the original $\mathcal{U}_{\mathrm{out}}$.

Then $(C')^*C' = Q$, hence then the observability of $\left[\begin{smallmatrix} A \\ Q \end{smallmatrix}\right]$ implies that of $\left[\begin{smallmatrix} A \\ C' \end{smallmatrix}\right]$, by the Hautus test (see p. 7 of [IOW]; obviously, $\operatorname{rank} \left[\begin{smallmatrix} \lambda - A \\ (C')^*C' \end{smallmatrix}\right] \le \operatorname{rank} \left[\begin{smallmatrix} \lambda - A \\ C' \end{smallmatrix}\right]$). □

Next we study J-coercivity over $\mathcal{U}_{\mathrm{exp}}(0) := \{u \in \mathrm{L}^2(\mathbf{R}_+;U) \mid \mathbb{B}\tau u \in \mathrm{L}^2\}$ (we shall have $C_{\mathrm{c}} := C_{\mathrm{w}}$, $D_{\mathrm{c}} := D$ for most applications):

Proposition 10.3.2 ($\mathcal{U}_{\mathrm{exp}}$: $y \in \mathrm{L}^2 \Rightarrow u, x \in \mathrm{L}^2$) *Let $\Sigma := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ have compatible generators $(C_{\mathrm{c}}, D_{\mathrm{c}})$. Let $J = J^* \in \mathcal{B}(Y)$, and set*

$$\kappa(x_0,u_0) := \langle (C_{\mathrm{c}}x_0 + D_{\mathrm{c}}u_0), J(C_{\mathrm{c}}x_0 + D_{\mathrm{c}}u_0)\rangle, \qquad \mathcal{J}(0,u) := \langle \mathbb{D}u, J\mathbb{D}u\rangle. \quad (10.22)$$

We have the following implications between the conditions (i)–(iii') given below:

(a1) Condition (i) is invariant under admissible state feedback (in the sense that if $\Sigma_\flat$ is the corresponding closed-loop system, then $\left[\begin{smallmatrix} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \mathbb{C}_\flat & \mathbb{D}_\flat \end{smallmatrix}\right]$ satisfies (i) iff $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ satisfies (i)).

(a2) Conditions (i)–(iii') are invariant under admissible state feedback by a compatible state feedback operator K_{c} (in the above sense, with $(C_\flat)_{\mathrm{c}} := C_{\mathrm{c}} + DK_{\mathrm{c}}$, $(D_\flat)_{\mathrm{c}} := D_{\mathrm{c}}$).

(b) If Σ is estimatable, then $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{exp}}$, hence then (i) becomes equivalent to (ai) of Proposition 10.3.1.

(c) (i)⇔(i")⇐(ii)⇐(iii)⇒(iii') (without further assumptions).

(d) **(dim$<\infty$)** *Assume that $\dim U \times H \times Y < \infty$. Then (iii)⇔(iii')⇐(vii).*

Assume, in addition, that Σ is exponentially stabilizable. Then (i)–(vi) are equivalent to each other (and to (ai) and (bii)–(biv) of Proposition 10.3.1 if Σ is exponentially detectable). Moreover, in (ii), (ii'), (iii) and (iv), we may replace "$r \in \mathbf{R}$" by "$r \in E$", where $E \subset \mathbf{R}$ is dense.

(e1) **(BwCARE)** *Assume that Hypothesis 9.2.1 holds for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, Σ has a compatible exponentially stabilizing state feedback operator, and $D_{\mathrm{c}} = D$.*

Then (v)⇐(i')⇔(ii')⇔(vi)⇒(i)⇔(i")⇔(ii)⇔(iii)⇒(iii'). If, in addition, $D^*JD \gg 0$, then (i)–(iii), (v) and (vi) are equivalent.

(e2) **(Bounded or smoothing B)** *Assume that $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$, $\mathbb{D} \in \mathrm{ULR}$, Σ is optimizable, and $D_{\mathrm{c}} = D$.*

Then (i)–(iii) are equivalent. If, in addition, Hypothesis 9.2.1 holds for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then (i)–(iii), (v) and (vi) are equivalent.

(f) If Σ has an exponentially stabilizing compatible state feedback operator, then (i')⇔(ii'), and (i)⇔(i")⇔(ii)⇔(iii)⇒(iii').

(g1) If $\mathbb{D}$ is ULR and $D_{\mathrm{c}} = D$, then (i')⇒(i), and (ii')⇒(ii).

(g2) If Σ has an exponentially stabilizing compatible state feedback operator s.t. $\mathbb{D}_\flat \in \mathrm{SHPR}$ (or $\mathbb{B}_\flat \tau \in \mathrm{SHPR}$ and $\mathbb{D} \in \mathrm{SLR}$), then (ii)⇒(ii'), and (i)⇒(i') (with $D_\mathrm{c} = D_\flat$).

(i) $\mathcal{J}(0,u) \geq \varepsilon\left(\|u\|_2^2 + \|\mathbb{B}\tau u\|_2^2\right)$ *for some* $\varepsilon > 0$ *and all* $u \in \mathcal{U}_{\exp}(0)$*; i.e.,* $\mathbb{D}$ *is positively J-coercive over* $\mathcal{U}_{\exp}$.

(i') $\mathcal{J}(0,u) \geq \varepsilon\|\mathbb{B}\tau u\|_2^2$ *for some* $\varepsilon > 0$ *and all* $u \in \mathcal{U}_{\exp}(0)$*, and* $D_\mathrm{c}^* J D_\mathrm{c} \gg 0$.

(i") $\mathcal{J}(0,u) \geq \varepsilon\left(\|u\|_2^2 + \|\mathbb{B}\tau u\|_2^2 + \|\mathbb{D}u\|_2^2\right)$ *for some* $\varepsilon > 0$ *and all* $u \in \mathcal{U}_{\exp}(0)$.

(ii) There is $\varepsilon > 0$ *s.t.*

$$(ir - A)x_0 = Bu_0 \Longrightarrow \kappa(x_0,u_0) \geq \varepsilon(\|x_0\|_H^2 + \|u_0\|_U^2) \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}). \tag{10.23}$$

(ii') $D_\mathrm{c}^* J D_\mathrm{c} \gg 0$ *and there is* $\varepsilon > 0$ *s.t.*

$$(ir - A)x_0 = Bu_0 \Longrightarrow \kappa(x_0,u_0) \geq \varepsilon\|x_0\|_H^2 \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}). \tag{10.24}$$

(iii) There is $\varepsilon > 0$ *s.t.* $T_{ir}^* \left[\begin{smallmatrix} I_H & 0 \\ 0 & J \end{smallmatrix}\right] T_{ir} \geq \varepsilon I$ $(r \in \mathbf{R})$ *on* $H \times U$*, where* $T_{ir} := \begin{bmatrix} A - ir & B \\ C_\mathrm{c} & D_\mathrm{c} \end{bmatrix}$.

(iii') $T_{ir}^* \left[\begin{smallmatrix} I_H & 0 \\ 0 & J \end{smallmatrix}\right] T_{ir} > 0$ $(r \in \mathbf{R})$*. Equivalently,*

$$r \in \mathbf{R}\ \&\ \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in H \times U\ \&\ (ir - A)x_0 = Bu_0 \Longrightarrow \kappa(x_0,u_0) > 0. \tag{10.25}$$

(iv) There is $\varepsilon > 0$ *s.t.* $\langle u_0, \widehat{\mathbb{D}}(s)^* J \widehat{\mathbb{D}}(s) u_0\rangle \geq \varepsilon\left(\|u_0\|_U^2 + \|(s-A)^{-1}Bu_0\|_H^2\right)$ *for a.e.* $s \in i\mathbf{R}$.

(v) There is a unique minimizing $u \in \mathcal{U}_{\exp}(x_0)$ *for each* $x_0 \in H$*, and* $D^* J D \gg 0$.

(vi) The B_w^**-CARE has an exponentially stabilizing solution and* $D^* J D \gg 0$.

(vii) (C,A) *has no unobservable nodes on* $i\mathbf{R}$*,* $J = I$*,* $D^* D > 0$ *and* $D^* C = 0$.

In classical LQR results, the assumptions are usually written in some of the following forms:

Remark 10.3.3 *Obviously, any* $J \gg 0$ *is equivalent to* $J = I$ *in all above conditions. Therefore, for* $J \gg 0$*, we get the following equivalent forms of above conditions:*
(i) $\|\mathbb{D}u\|_2 \geq \varepsilon(\|\mathbb{B}\tau u\|_2 + \|u\|_2)$*;*
(ii) $(ir - A)x_0 = Bu_0 \Rightarrow \|C_\mathrm{c}x_0 + D_\mathrm{c}u_0\|_Y \geq \varepsilon(\|x_0\|_H + \|u_0\|_U)$*;*
(ii') $D_\mathrm{c}^* D_\mathrm{c} \gg 0$*, and* $(ir - A)x_0 = Bu_0 \Rightarrow \|C_\mathrm{c}x_0 + D_\mathrm{c}u_0\|_Y \geq \varepsilon\|x_0\|_H$*;*
(iii') $T_{ir} := \begin{bmatrix} A - ir & B \\ C_\mathrm{c} & D_\mathrm{c} \end{bmatrix} : H \times U \to H \times Y$ *has a full column rank (i.e.,* $T_{ir}^* T_{ir} \gg 0$*) for all* $r \in \mathbf{R}$*;*
(iv) $\|\widehat{\mathbb{D}}(s)u_0\|_Y \geq \varepsilon(\|u_0\|_U + \|(s-A)^{-1}Bu_0\|_H)$ *for a.e.* $s \in i\mathbf{R}$ *and all* $u_0 \in U$. □

Recall that we set $\|x\|_X = \infty$ for $x \notin X$, hence $\|T_s \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\|_{H \times Y} = \infty \geq \varepsilon\|\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right]\|_{H \times U}$ for all $x_0 \in H \setminus H_B$, $\varepsilon > 0$. In (iv) and (iv'), the values of $\|(s-A)^{-1}Bu_0\|_H$,

$\|\widehat{\mathbb{D}}(s)u_0\|_U$ and $\langle u_0, \widehat{\mathbb{D}}(s)^* J\widehat{\mathbb{D}}(s)u_0\rangle$ are defined by continuity at ∞ and at the poles of $(s-A)^{-1}$ (in particular, the value $+\infty$ is allowed).

Condition (iii') says that Σ has no *invariant zeros* on $i\mathbf{R}\cup\{\infty\}$ (see, e.g., Definition 3.16 of [ZDG]). If A is bounded (this is the case when $\dim H < \infty$), then $H = H_{-1} = H_1$, hence then necessarily $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ are bounded, $C_c = C$ and $D_c = D$.

When applying the above results, we often assume that $\mathbb{D}$ is WR, so that we can take $C_c = C_w$, $D_c = D$.

Example 9.13.5 shows that positive J-coercivity over $\mathcal{U}_{out}$ does not imply positive J-coercivity over $\mathcal{U}_{exp}$, not even when Σ is one-dimensional and exponentially stabilizable with $D^*JD \gg 0$.

Proof of Proposition 10.3.2: *Remarks:* The numbers $\varepsilon > 0$ in (i)–(iv) need not be equal, but they are always independent of x_0, u_0 and r. Since $\left\langle \left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{y}\end{smallmatrix}\right], \left[\begin{smallmatrix} I_H & 0\\ 0 & J\end{smallmatrix}\right]\left[\begin{smallmatrix}\widetilde{x}\\ \widetilde{y}\end{smallmatrix}\right]\right\rangle_{H\times U} := \|\widetilde{x}\|_H + \langle\widetilde{y}, J\widetilde{y}\rangle_Y = \infty$ for $\widetilde{x}\in H_{-1}\setminus H$, we have $\left\langle T_{ir}\left[\begin{smallmatrix}x_0\\ u_0\end{smallmatrix}\right], \left[\begin{smallmatrix} I_H & 0\\ 0 & J\end{smallmatrix}\right] T_{ir}\left[\begin{smallmatrix}x_0\\ u_0\end{smallmatrix}\right]\right\rangle_{H\times U} = \infty$ when $(A-ir)x_0 + Bu_0 \notin H$; in particular, (iii) need to be checked for $x_0\in H_B$ only.)

About the proof: By Lemma 6.7.8, (i) is equivalent to (i") and $\mathcal{U}_{exp}(0) = \{u\in \mathrm{L}^2(\mathbf{R}_+;U) \mid \mathbb{B}\tau u, \mathbb{D}u \in \mathrm{L}^2\}$. Thus, we may and will neglect (i") in the rest of the proof.

We shall use the notation of Definition 6.6.10 when referring to state feedback.

(a1) This is Theorem 8.4.5(d).

(a2) For (i) this follows from (a1); for (i') this follows from the formula $\mathcal{U}_{exp}(0) = \widetilde{\mathbb{M}}\mathcal{U}_{exp}^{\Sigma_\flat}(0)$ from Theorem 8.4.5(c1) (since $(D_\flat)_c := D_c$, by (6.144)), so we concentrate on (ii)–(iii'). Let K_c be an admissible compatible state feedback operator for Σ, and let $\Sigma_\flat$ be the corresponding closed-loop system. Then

$$\begin{bmatrix} A-s & B\\ C_c & D_c\end{bmatrix}\begin{bmatrix} I & 0\\ K_c & I\end{bmatrix} = \begin{bmatrix} A+BK_c-s & B\\ C_c+D_cK_c & D_c\end{bmatrix} = \begin{bmatrix} A_\flat - s & B_\flat\\ (C_\flat)_c & (D_\flat)_c\end{bmatrix} =: T_s^\flat \quad (10.26)$$

for all $s\in\mathbf{C}$, by Proposition 6.6.18(d2).

In 1°–4°, we assume a claim for $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ (as above), and prove the same claim for $\left[\begin{smallmatrix}\mathbb{A}_\flat & \mathbb{B}_\flat\\ \mathbb{C}_\flat & \mathbb{D}_\flat\end{smallmatrix}\right]$. The converse is always obtained by using $-K_c$ for $\left[\begin{smallmatrix}\mathbb{A}_\flat & \mathbb{B}_\flat\\ \mathbb{C}_\flat & \mathbb{D}_\flat\end{smallmatrix}\right]$ (see Proposition 6.6.18(d2)).

1° *Claim (ii) is K_c-invariant:* Assume (ii). Given $x_0\in H$, $v_0\in U$, set $u_0 := x_0 + K_cv_0$, $y_0 := C_cx_0 + D_cu_0$, so that $\left[\begin{smallmatrix}x_0\\ u_0\end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0\\ K_c & I\end{smallmatrix}\right]\left[\begin{smallmatrix}x_0\\ v_0\end{smallmatrix}\right]$ and $y_0 = (C_\flat)_cx_0 + (D_\flat)_cv_0$, by (10.26).

Now $\kappa(x_0,u_0) = \langle y_0, Jy_0\rangle =: \kappa_\flat(x_0,v_0)$, so that obtain (ii) for $\Sigma_\flat$ too, because $\|x_0\|^2 + \|v_0\|^2 \le (M_0+2)^2(\|x_0\|^2 + \|u_0\|^2)$, where M_0 is as in Lemma 6.3.21 for $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{K} & \mathbb{F}\end{smallmatrix}\right]$ (note that $\|v_0\| \le \|x_0\| + M_0\|u_0\|$, since $v_0 = x_0 - K_cu_0$).

2° *Claim (ii') is K_c-invariant:* Drop the last part from the above proof.

3° *Claim (iii) is K_c-invariant:* Assume (iii). Given $x_0\in H$, $v_0\in U$, $r\in\mathbf{R}$, set $u_0 := x_0 + K_cv_0$ and

$$\begin{bmatrix}x_1\\ y_0\end{bmatrix} := T_{ir}\begin{bmatrix}x_0\\ u_0\end{bmatrix} = T_{ir}^\flat\begin{bmatrix}x_0\\ v_0\end{bmatrix}, \qquad (10.27)$$

as in (10.26).

By (10.29) and (iii), the sum $\|x_1\|_H^2+\langle y_0, Jy_0\rangle_Y$ is at least $\varepsilon(\|x_0\|_H^2+\|u_0\|_U^2)$, hence at least $(\|x_0\|_H^2+\|v_0\|_U^2)\varepsilon/(M_0+1)^2$, as in 1°. Thus, (iii) holds for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$.

4° *Claim (iii') is K_c-invariant:* This is analogous to 3°.

(b) This is given in Lemma 8.3.3.

(c) 1° *(iii)⇒(iii'):* This is trivial. (Moreover, the equivalence in (iii') is obvious from (10.29).)

2° *(ii)⇒(i):* Assume (ii). Let $u \in \mathcal{U}_{\exp}(0)$, so that $u, x, y \in \mathrm{L}^2(\mathbf{R}_+;*)$, where $x := \mathbb{B}\tau u$ and $y := \mathbb{D}u$. By Lemma 6.3.20, we have $(\cdot - A)\widehat{x} = B\widehat{u}$ and $\widehat{y} = C_c\widehat{x} + D_c\widehat{u}$ a.e. on $i\mathbf{R}$. By (3.34), we have

$$\langle \widehat{y}, J\widehat{y}\rangle = \int_{\mathbf{R}} \kappa(\widehat{x}(ir), \widehat{u}(ir))\, dr \geq \varepsilon(\|\widehat{x}\|_2^2 + \|\widehat{u}\|_2^2) \tag{10.28}$$

hence $\mathcal{J}(0,u) = \langle y, Jy\rangle \geq \varepsilon(\|x\|_2^2 + \|u\|_2^2)$, by (3.34). Therefore, (i) holds.

3° *(iii)⇒(ii):* Now

$$\langle \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, T_{ir}^* \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} T_{ir} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \rangle = \|Bu_0 - (ir - A)x_0\|_H^2 + \kappa(x_0, u_0). \tag{10.29}$$

Thus, if $(A - ir)x_0 + Bu_0 = 0$, then (iii) implies that $\kappa(x_0, u_0) \geq \varepsilon(\|x_0\|_H^2 + \|u_0\|_U^2)$ for some $\varepsilon > 0$ independent of x_0 and u_0. Thus, then (ii) holds.

(f) By (c), we only have to show the implications "(i)⇒(ii)⇒(iii)", and "(i')⇔(ii')". This will be done below.

1.1° *(ii)⇒(iii) for exponentially stable Σ:* Let Σ be exponentially stable. Assume (ii) with $\varepsilon \in (0, 1/2)$. By Lemma A.4.4(g1), there is $M < \infty$ s.t. $\|(ir - A)^{-1}z_0\|_H \leq M\|z_0\|_H$ for all $r \in \mathbf{R}$ and $z_0 \in H$ (because $\omega_A < 0$).

Let $\|x_0\|_H^2 + \|u_0\|_U^2 = 1$ and $r \in \mathbf{R}$. Assume that

$$\|z_1\|_H < \delta := \varepsilon/3(1 + M + \|\widehat{\mathbb{C}}\|^2\|J\| + 2\|\widehat{\mathbb{C}}\|\|J\|\|\widehat{\mathbb{D}}\|), \tag{10.30}$$

where $z_1 := (ir - A)x_0 - Bu_0$. Set $x_1 := (ir - A)^{-1}z_1$, $x_2 := x_0 - x_1$, so that $(ir - A)x_2 = Bu_0$, i.e., $x_2 = (ir - A)^{-1}Bu_0$. It follows that

$$y_0 := C_c x_2 + D_c u_0 = \widehat{\mathbb{D}}(ir)u_0 \quad \text{and} \quad C_c x_0 = C_c x_2 + \widehat{\mathbb{C}}(ir)z_1, \tag{10.31}$$

hence $\kappa(x_2, u_0) = \langle y_0, Jy_0\rangle$ and $\kappa(x_0, u_0) = \langle y_0 + \widehat{\mathbb{C}}(ir)z_1, J(y_0 + \widehat{\mathbb{C}}(ir)z_1)\rangle$. Consequently,

$$\kappa(x_0, u_0) - \kappa(x_2, u_0) = \langle \widehat{\mathbb{C}}(ir)z_1, J\widehat{\mathbb{C}}(ir)z_1\rangle + 2\,\mathrm{Re}\langle \widehat{\mathbb{C}}(ir)z_1, Jy_0\rangle \tag{10.32}$$

$$\leq \|\widehat{\mathbb{C}}\|^2\|J\|\delta^2 + 2\|\widehat{\mathbb{C}}\|\|J\|\|\widehat{\mathbb{D}}\|\delta < \varepsilon/3 \tag{10.33}$$

(note that $\delta < \varepsilon/3 < 1$, hence $\delta^2 < \delta$). But $\|x_2\|^2 + \|u_0\|^2 \geq 1 + \|x_2\|^2 - \|x_0\|^2 \geq 1 - 2\|x_1\|\|x_0\| \geq 1 - 2M\|z_1\| \geq 1 - 2\varepsilon/3$, hence $\kappa(x_2, u_0) \geq \varepsilon(1 - 2\varepsilon/3) > 2\varepsilon/3$, by (ii). By (10.32), we obtain that $\kappa(x_0, u_0) > 2\varepsilon/3 - \varepsilon/3 = \varepsilon/3$.

Thus, (10.29)$\geq \min(\delta^2, \varepsilon/6) =: \varepsilon'$, hence (iii) holds with ε' in place of ε.

1.2° *(ii)⇒(iii) for Σ having an exponentially stabilizing K_c:* Apply 1° and twice (a2).

2.1° *(i)⇒(ii) for exponentially stable Σ:* Let Σ be exponentially stable and assume that (ii) does not hold for some $\gamma > 0$, so that there are $x_0 \in H$, $u_0 \in U$ and $r_0 \in \mathbf{R}$ s.t. $(ir_0 - A)x_0 = Bu_0$, $\kappa(x_0, u_0) < \gamma$ and $\|x_0\|^2 + \|u_0\|^2 = 1$. Note that $x_0 = (ir_0 - A)^{-1}Bu_0$.

Let $\varepsilon' \in (0,1)$ be arbitrary. Choose $\delta > 0$ s.t. $\kappa(x_r, u_0) < \gamma$ and $\|x_r\|^2 + \|u_0\|^2 > 1 - \varepsilon'$ for $|r - r_0| < \delta$, where $x_r := (ir - A)^{-1}Bu_0$. Set

$$M := \max\{1, \|\mathbb{D}\|^2\|J\|\}. \tag{10.34}$$

By Lemma D.1.24, there is $f \in \mathrm{L}^2(\mathbf{R}_+)$ s.t. $\int_{|r-r_0|>\delta} |\widehat{f}(ir)|^2\,dr < \varepsilon'/M$ and $\|\widehat{f}\|_2 = 1$. Set $u := fu_0$, $\widehat{y} := \widehat{\mathbb{D}}\widehat{u}$, so that $\|\widehat{y}(ir)\| \le \|\mathbb{D}\|\,\|\widehat{f}(ir)\|$ $(r \in \mathbf{R})$, and hence

$$\int_{|r-r_0|>\delta} \langle \widehat{y}, J\widehat{y}\rangle\,dr \le \int_{|r-r_0|>\delta} M\|\widehat{f}\|^2\,dr < M\varepsilon'/M = \varepsilon', \tag{10.35}$$

$$\int_{|r-r_0|<\delta} \langle \widehat{y}, J\widehat{y}\rangle\,dr = \int_{|r-r_0|<\delta} \kappa(x_r, u_0)|\widehat{f}|^2\,dr < \gamma, \quad \text{and} \tag{10.36}$$

$$\int_{i\mathbf{R}} (\|\widehat{u}\|^2 + \|\widehat{x}\|^2)\,dr > (1-\varepsilon')\int_{|r-r_0|<\delta} \|\widehat{f}\|^2\,dm > (1-\varepsilon')^2 \tag{10.37}$$

In particular, $\mathcal{J}(0,u) < (\gamma + \varepsilon')/2\pi$ and $\|u\|^2 + \|x\|^2 > (1-\varepsilon')^2/2\pi$. Because $\varepsilon' > 0$ was arbitrary, we see that (i) cannot hold for any $\varepsilon > \gamma$.

Thus, if (i) holds for some $\varepsilon > 0$, we must also have (ii) for the same ε (otherwise (ii) would fail for some $\gamma < \varepsilon$ too, hence (i) would be false for ε, a contradiction).

2.2° *(i)⇒(ii) for Σ having an exponentially stabilizing K_c:* Apply 3.1° and twice (a2).

3° *(i')⇔(ii')*: Define Σ' and $\mathcal{J}'$ by setting $J' := \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right]$, $C_\mathrm{c}' := \left[\begin{smallmatrix} C_\mathrm{c} \\ 0 \end{smallmatrix}\right]$, $D_\mathrm{c}' := \left[\begin{smallmatrix} D_\mathrm{c} \\ I \end{smallmatrix}\right]$ (thus, $\mathcal{J}' = \mathcal{J} + \|u\|_2^2$). Then for $\mathcal{J}'$, condition (i) is satisfied, hence so is (ii), by 2°. But $\kappa'(x_0, u_0) = \kappa(x_0, u_0) + \|u_0\|_U^2$, hence (ii') is satisfied by $\mathcal{J}$. The converse is analogous, using "(ii)⇒(i)" (hence it does not even require the stabilizability assumption).

(g1) 1° *(ii')⇒(ii):* To derive a contradiction, assume that $\mathbb{D}$ is ULR, $D_\mathrm{c} = D$ and (ii') holds but (ii) does not hold. Choose $\varepsilon' > 0$ s.t. $D^*JD \gg \varepsilon' I$. Then there are $\{r_n\} \subset \mathbf{R}$, $\{x_n\} \subset H$, $\{u_n\} \subset U$ s.t. $\|u_n\|_U = 1$ $(n \in \mathbf{N})$, $(ir_n - A)x_n = Bu_n$ and $\kappa(x_n, u_n) \to 0$, as $n \to \infty$. Consequently, $x_n \to 0$ (by (ii')).

Given $\delta > 0$, we obtain from Lemma 6.3.22 that there is $M < \infty$ s.t. $\|C_\mathrm{w}x_n\|_Y \le M\|x_n\|_H + \delta\|u_n\|_U/2$ $(n \in \mathbf{N})$. Consequently, there is $N_\delta \in \mathbf{N}$ s.t. $\|C_\mathrm{w}x_n\|_Y < \delta$ for $n \ge N_\delta$. It follows that

$$\kappa(x_n, u_n) = \langle u_n, D^*JDu_n\rangle + 2\,\mathrm{Re}\langle C_\mathrm{w}x, JDu_n\rangle + \langle C_\mathrm{w}x, JC_\mathrm{w}x\rangle \tag{10.38}$$

$$\ge \varepsilon' - 2\|D\|\|J\|\delta - \|J\|\delta^2. \tag{10.39}$$

Because $\delta > 0$ was arbitrary, we have $\liminf_{n\to\infty} \kappa(x_n, u_n) \ge \varepsilon'$, a contradiction, as required.

2° *(i')⇒(i):* To derive a contradiction, assume that $\mathbb{D}$ is ULR, $D_\mathrm{c} = D$ and (i') holds but (i) does not hold. Choose $\varepsilon' > 0$ s.t. $D^*JD \gg \varepsilon' I$. Then there are $\{u_n\} \subset \mathcal{U}_{\exp}(0)$ s.t. $\|u_n\|_2 + \|x_n\|_2 = 1$ $(n \in \mathbf{N})$, and $\langle \mathbb{D}u, J\mathbb{D}u\rangle \to 0$, as $n \to \infty$, where $x_n := \mathbb{B}\tau u_n$. Consequently, $\|x_n\|_2 \to 0$ and $\|u_n\|_2 \to 1$ (by (i')), as $n \to \infty$.

Given $\delta > 0$, choose M as in 1°. Since $(ir - A)\widehat{x}_n(ir) = B\widehat{u}_n(ir)$ for a.e. $r \in \mathbf{R}$, by Lemma 6.3.20, we obtain that

$$\|C_{\mathrm{w}}\widehat{x}_n(ir)\|_Y \leq M\|\widehat{x}_n(ir)\|_H + \frac{\delta}{2}\|\widehat{u}_n(ir)\|_U \quad \text{for a.e. } r \in \mathbf{R} \tag{10.40}$$

and for all $n \in \mathbf{N}$. Consequently, there is $N_\delta > 0$ s.t. $\|C_{\mathrm{w}}\widehat{x}_n(ir)\|_Y \leq \delta\|\widehat{u}_n(ir)\|_U$ for a.e. $r \in \mathbf{R}$ and all $n \geq N_\delta$, hence $\|C_{\mathrm{w}}\widehat{x}_n\|^{\leq}_{\mathrm{L}^2(i\mathbf{R};Y)} 2\pi\delta^2$ for all $n \geq N_\delta$ (because $\|\widehat{u}_n\|_2^2 = 2\pi\|u_n\|_2^2 \leq 2\pi$). But

$$2\pi\mathcal{J}(0,u_n) \geq \langle D\widehat{u}_n, JD\widehat{u}_n\rangle_{\mathrm{L}^2} - \|J\|\|C_{\mathrm{w}}\widehat{x}_n\|_2^2 - 2\|J\|\|D\widehat{u}_n\|_2^2\|C_{\mathrm{w}}\widehat{x}_n\|_2^2, \tag{10.41}$$

by (6.90), hence $\mathcal{J}(0,u_n) \geq \varepsilon'\|u_n\|_2^2 - M'\delta^2 - M'\|u_n\|_2\delta$ for $n \geq N_\delta$, where $M' := \|J\|\max\{\|D\|,1\}$. Consequently, $\liminf \mathcal{J}(0,u_n) \geq \varepsilon' - M'\delta^2$. Because $\delta > 0$ was arbitrary, we have $\liminf \mathcal{J}(0,u_n) \geq \varepsilon'$, a contradiction, as required.

(g2) Let K_{c} be the exponentially stabilizing compatible state feedback operator. Note that $\mathbb{D}_\flat \in$ SHPR in either case, by the assumption or by Lemma 6.3.23.

1° *Case* $K_{\mathrm{c}} = 0$*:* Assume that (i) holds (by (c), this is the case if (ii) holds). By Lemma 6.3.6(c2), we have $D_\flat^* JD_\flat \gg 0$, i.e., $D^*JD \gg 0$. Thus, (i') holds (and (ii') if (ii) holds).

2° *(ii)⇒(ii'), and (i)⇒(i') assuming that* K_{c} *stabilizes* Σ *exponentially,* $\mathbb{B}_\flat\tau \in$ SHPR *and* $\mathbb{D}_\flat \in$ SLR*:* Assume that K_{c} is an exponentially stabilizing compatible state feedback operator for Σ. Assume that the corresponding closed-loop system $\Sigma_\flat$ is s.t. $\mathbb{D}_\flat \in$ SHPR. Assume that (ii) (resp. (i)) holds for $D_{\mathrm{c}} = D_\flat =: D$.

(By applying twice Proposition 6.6.18(d2) (see also Lemma 6.6.14), we see that $-K_{\mathrm{c}}$ is an admissible compatible state feedback operator for $\Sigma_\flat$, and $((C_\flat)_{\mathrm{w}} - D_\flat K_{\mathrm{c}}, D_\flat)$ is a compatible pair for Σ. Thus condition (ii) for $D_{\mathrm{c}} = D_\flat$ is well defined.)

Then (ii) (resp. (i)) holds for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$, by (a2), hence (ii') (resp. (i')) holds for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$, by 1°, hence (ii') (resp. (i')) holds for Σ, by (a2).

(e1) The first chain of implications follows from (f), (g1) and (g2) except for (v) and (vi).

"(ii')⇒(v)": This follows from Lemma 10.2.4 (since (ii') implies (i)). "(v)⇔(vi)⇒(i)": This follows from Theorem 9.2.16 "(vi)⇒(i')": If (vi) holds, then $D^*JD \gg 0$ and (i) holds, by the above, hence then (i') holds.

(e2) 1° *(i)⇒(i')–(iii')&(v):* Assume (i). By Lemma 9.2.17 and its proof, $D^*JD \gg 0$ (hence (i') holds) and there is an exponentially stabilizing ULR $K \in \mathcal{B}(H,U)$, hence (i)–(iii') and (v) hold, by (f).

2° *(i')/(ii)/(ii')/(iii)⇒(i):* This follows from (g1) and (f).

3° *(i)&(i')⇔(vi)⇔(v) under Hypothesis 9.2.1:* Assume Hypothesis 9.2.1 for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. By Theorem 9.2.16, (i)&(i') implies (vi) and (v) implies (vi). Conversely, if (vi) holds, then we obtain from Theorems 9.2.16 and 9.9.1(k)&(a2) that (v) holds, $\mathbb{D}$ is J-coercive and $\mathcal{J}(0,\cdot) \geq 0$, hence $\mathbb{D}$ is positively J-coercive, i.e., (i) holds.

(d) Assume that $\dim U, \dim H, \dim Y < \infty$. (Note that (iv) is not well defined in general, but it is well defined in this finite-dimensional case.)

1° *(iii')⇒(iii):* Then A,B,C,D are bounded. Let $K := \{(x,u) \in H \times U \,|\, \|x\|_H^2 + \|u\|_U^2 = 1\}$. The function

$$F(r,x_0,u_0) := \langle \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right], T_{ir}^* \left[\begin{smallmatrix} I & 0 \\ 0 & J \end{smallmatrix}\right] T_{ir} \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \rangle = \|Ax_0 + Bu_0 - irx_0\|^2 + \kappa(x_0,u_0) \quad (10.42)$$

is continuous on $H \times U$ for a fixed $r \in \mathbf{R}$, hence $M := \max_{x,u \in K} F(0,x,u) < \infty$.

Obviously, there is $R < \infty$ s.t. $F(r,x,u) > M$ for all $|r| > R$ and $(x,u) \in K$. But F is obviously continuous on $[-R,R] \times K$, hence $\varepsilon := \min_{r \in \mathbf{R},\,(x,u) \in K} F(r,x,u)$ exists. By (iii'), $\varepsilon > 0$. Apparently, (iii) holds for this ε.

2° *(vii)⇒(iii'):* The first condition in (vii) means that $\mathrm{Ker}(\left[\begin{smallmatrix} ir-A \\ C \end{smallmatrix}\right]) = \{0\}$ for all $r \in \mathbf{R}$. If $u_0 \neq 0$, then $\kappa(x_0,u_0) \geq \langle u_0, D^*Du_0 \rangle > 0$. If $u_0 = 0$ and $(ir - A)x_0 = Bu_0$ for some $r \in \mathbf{R}$, then we have $Cx_0 \neq 0$, hence then $\kappa(x_0,u_0) = \langle Cx_0, Cx_0 \rangle > 0$.

3° *(ii)⇔(iv):* Let ε be as in (ii), and set $M = \|C\| + \|D\|$. If $ir \in i\mathbf{R} \setminus \sigma(A) =: R$ and $(ir - A)x_0 = Bu_0$, then $Cx_0 + Du_0 = \widehat{\mathbb{D}}(ir)u_0$, hence then $F := \widehat{\mathbb{D}}^* J \widehat{\mathbb{D}}$ satisfies $\langle u_0, F(ir)u_0 \rangle = \kappa(x_0,u_0)$, so that (ii) becomes equivalent to (iv) for $ir \in R$. Since R is dense in $i\mathbf{R} \cup \{\infty\}$, and both sides of the inequalities in (ii) and (iv) are continuous $i\mathbf{R} \cup \{\infty\} \to (0,+\infty]$, any $ir \in i\mathbf{R} \cup \{\infty\}$ will do. (Recall that we interpret the values as their limits as $\mathbf{R} \ni r \to r_0$ for $r_0 = \infty$ and for terms having a pole at ir_0.)

4° *The rest of (d) except E:* If Σ is exponentially stabilizable [and detectable], then we get the other implications from (e) [and (b) and Proposition 10.3.1(b)].

5° *The dense set $E \subset \mathbf{R}$:* In the proof of (ii)⇔(ii')⇔(iii) (including the invariance of (ii), (ii') and (iii) w.r.t. $K_c = K$ in (a2)), we can restrict r to any $E \subset \mathbf{R}$. But $T_\cdot \in \mathcal{C}(i\mathbf{R}; \mathcal{B}(U \times H))$, hence (iii) is invariant under the replacement of $\mathbf{R}$ by its dense subset. The same holds for (iv), because both sides of the inequality in (iv) are continuous $i\mathbf{R} \to [0,+\infty]$. □

In conditions (i)–(i'') of Proposition 10.3.2, we posed requirements on $u \in \mathcal{U}_{\exp}(0)$ only; here we show that even in the finite-dimensional case, this is strictly weaker than requiring the same for all $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ (or for all $u \in \mathcal{U}_{\mathrm{out}}(0)$), and that none of (i)–(iv') implies the same for all $u \in \mathcal{U}_{\mathrm{out}}(0)$ (naturally, this cannot be the case when Σ is exponentially detectable, by (b)):

Example 10.3.4 Let $A = 1 = B = D$, $C = 0$, $\mathbf{C} = U = H = Y$. Then $\mathbb{D}u = u$ for all u, so that $\mathcal{U}_{\mathrm{out}}(0) = \mathrm{L}^2(\mathbf{R}_+;U)$, but $(s-A)^{-1}B = (s-1)^{-1}$, hence $\mathcal{L}\mathbb{B}\tau u = (\cdot - 1)^{-1}\widehat{u}$, so that $\mathcal{U}_{\exp}(0) = \{u \in \mathrm{L}^2 \,|\, \widehat{u}(1) = 0\}$. Moreover, then $|\widehat{\mathbb{D}}| = 1$ and $|(s-A)^{-1}B| \leq 1$ on $i\mathbf{R}$, hence condition (ii') of Proposition 10.3.2 holds for $\varepsilon = 1$, hence (i)–(iv') hold, by (d) (because Σ is exponentially stabilizable).

Nevertheless, we have $\mathcal{J}(0,u) = \|u\|_2^2$ and $\|\mathbb{B}\tau u\|_2 = \infty$ for all $u \in \mathcal{U}_{\mathrm{out}}(0) \setminus \mathcal{U}_{\exp}(0)$, hence we cannot allow for arbitrary $u \in \mathrm{L}^2(\mathbf{R}_+;U)$, not even arbitrary $u \in \mathcal{U}_{\mathrm{out}}(0)$ in (i) (nor in (i') or in (i'')). ◁

Notes

Many of the above results are known for finite-dimensional systems, part of them also more generally. The method of the proof of "(ii')⇔(i')" in Proposition 10.3.2 is partially from [Keu, Section 3], where these two conditions, (v) and (vi) are shown to be equivalent for Pritchard–Salamon systems.

The two propositions show that even in the finite-dimensional case, several classical results contain superfluous or redundant assumptions. See the notes on p. 555 for more on minimization problems and their assumptions.

Proposition 10.3.1 also holds in its discrete-time form, whereas Proposition 10.3.2 needs to be rewritten (since S takes the role of D^*JD); see Proposition 15.2.2.

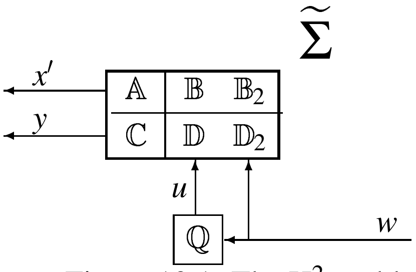

Figure 10.1: The H^2 problem

10.4 The H^2 problem

Grabel's Law:
2 is not equal to 3 – not even for large values of 2.

In this section, we shall show how the solution for the minimization problem of Section 10.2 (for Σ) provides also the solution of the H^2 full information and state feedback problems (for $\widetilde{\Sigma}$) as a corollary.

We first show that given an additional (bounded) input operator $B_2 \in \mathcal{B}(W,H)$, the norm $\|\mathbb{D}u + \mathbb{C}B_2 w_0\|_2$ is minimized by $u(t) = Kx(t)$ (Theorem 10.4.2), i.e., that K solves the H^2 problem. Then we allow for unbounded B_2 but require that K is bounded (Proposition 10.4.3). Finally, we give a corollary, whose regularity, coercivity and stabilizability assumptions guarantee that K necessarily exists and is bounded, so that it also solves the H^2 problem (Corollary 10.4.4).

Standing Hypothesis 10.4.1 ($J = I$, B_2) *Throughout this section, we assume the existence of an additional WR input operator $B_2 \in \mathcal{B}(W, H_{-1})$, such that Σ can be extended to*

$$\widetilde{\Sigma} := \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B} & \mathbb{B}_2 \\ \hline \mathbb{C} & \mathbb{D} & \mathbb{D}_2 \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Y) \quad \textit{with generators} \quad \left[\begin{array}{c|cc} A & B & B_2 \\ \hline C & D & 0 \end{array}\right]. \tag{10.43}$$

We also assume that $J = I$.

(By B_2 being WR we mean that $\mathbb{D}_2$ is WR. The assumption $J = I$ implies that minimization refers to the cost function $\mathcal{J}(x_0, u) := \|\mathbb{C}x_0 + \mathbb{D}u\|_2^2$ (see Section 10.2). This could easily be relaxed to obtain a more general H^2 problem.)

In the H^2 *problem*, we wish to find a "controller" $\mathbb{Q}$ that minimizes the norm

$$\|\widehat{\mathbb{D}}\widehat{\mathbb{Q}} + \widehat{\mathbb{D}}_2\|_{\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(W,Y))} \tag{10.44}$$

where $\widehat{\mathbb{D}}_2(s) = C(s-A)^{-1}B_2$ (see Figure 10.1; this is a generalization of the traditional H^2 problem, by Remark 10.4.6). But $\|\widehat{\mathbb{E}}\|_{\mathrm{H}^2_{\mathrm{strong}}} = \sup_{\|w_0\|_W \le 1} \|\mathbb{E}w_0\delta_0\|_2$ for any $\widehat{\mathbb{E}} \in \mathrm{H}^\infty_\infty$; indeed, the Laplace transform is an isometric isomorphism of $\mathcal{B}(W, \mathrm{L}^2(\mathbf{R}_+;Y))$ onto $\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(W,Y))$, by Lemma F.3.4(d). (Here $\mathbb{E}w_0\delta_0 := \mathcal{L}^{-1}\widehat{\mathbb{E}}w_0$.) Thus, an equivalent definition for the H^2 problem as finding for an arbitrary $w_0 \in W$ a "stabilizing" control (see the proofs of Theorem

10.4.2 and Proposition 10.4.3 for explanations) s.t.

$$\mathcal{J}_{\mathrm{H}^2}(w_0,u) := \frac{1}{2\pi}\|\widehat{\mathbb{D}}\widehat{u}+\widehat{\mathbb{D}}_2 w_0\|^2_{\mathrm{H}^2(\mathbf{C}^+;Y)} \quad (= \|\mathbb{D}u+\mathbb{C}B_2w_0\|_2^2 \text{ if } B_2\in\mathcal{B}(W,H)) \tag{10.45}$$

is minimized. Thus, we are minimizing the energy of the impulse response (the output corresponding to "second input $w_0\delta_0$", where δ_0 is the Dirac delta function) of the system.

This implies that the assumption

$$\widehat{\mathbb{D}}_2 \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(W,Y)) \text{ for some } \omega>0 \tag{10.46}$$

is necessary (because $\widehat{\mathbb{D}}\widehat{u}\in\mathrm{H}^2_\omega$ for all $\widehat{u}\in\mathcal{L}[\mathrm{L}^2_\vartheta(\mathbf{R}_+;U)]$, where $\omega\ge\vartheta$ is s.t. $\widehat{\mathbb{D}}\in\mathrm{H}^\infty_\omega$ (and where ϑ is s.t. the controls are required to be ϑ-stable), it follows that we must have $\widehat{\mathbb{D}}_2 w_0\in\mathrm{H}^2_\omega$, for all $w_0\in W$).

Therefore, $\begin{bmatrix}\mathbb{D} & \mathbb{D}_2\end{bmatrix}$ has a realization with a bounded B_2 (see (6.213), although one is usually more interested in the original system, where B_2 may be unbounded. Usually one also requires the control law $\mathbb{Q}: w_0\mapsto u$ to be of a specific form (e.g., a state feedback controller).

Due to the above, we start with the case $B_2\in\mathcal{B}(W,H)$ (which implies that (10.46) holds). As explained above, at least with such systems we end up with the minimization problem of Section 10.2 except that the initial states are restricted to $B_2[W]$; in particular:

Theorem 10.4.2 ($\mathcal{U}_*^*$: LQR $\Rightarrow$ H^2) *Assume that $B_2\in\mathcal{B}(W,H)$ and that there is a minimizing WR state feedback operator K over $\mathcal{U}_*^*$.*

Then K solves the H^2 *problem (strictly if K is strictly minimizing), i.e., it leads to the minimization of the cost (10.45), for each $w_0\in W$; see Figure 10.2.*

Thus, in this case, the state feedback and full information H^2 problems have a common solution, namely the minimizing K. See Section 10.2 for sufficient and necessary conditions for the existence of K.

Assume for a while that also B is bounded. Then a sufficient condition for the existence of K is that Σ is positively J-coercive (see Section 10.3) and "$\mathcal{U}_*^*$-stabilizable" (i.e., $\mathcal{U}_*^*(x_0)\neq\emptyset$ for all $x_0\in H$) if, e.g., $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{exp}}$.

Moreover, then K is necessarily ULR and the optimal controller $\mathbb{Q}$ becomes the map $\mathbb{Q}=\mathbb{K}_\circlearrowleft B_2: W\to\mathrm{L}^2(\mathbf{R}_+;U)$ (since $\mathbb{C}_\circlearrowleft B_2w_0=\mathbb{C}B_2w_0+\mathbb{D}\mathbb{K}_\circlearrowleft B_2$; note that $\widehat{\mathbb{Q}}=$"$\widehat{\mathbb{F}_{\circlearrowleft 2}}$", where $\mathbb{F}_{\circlearrowleft 2}\in\mathrm{TIC}_\infty$ ($\in$ TIC if $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{exp}}$) is the inverse Laplace transform of $\widehat{\mathbb{Q}}$ as a H^∞_∞ function, whereas $\mathbb{Q}$ is the inverse Laplace transform of $\widehat{\mathbb{Q}}\in\mathcal{B}(W,\mathrm{L}^2(\mathbf{R}_+;U))$ (when, e.g., $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{exp}}$)).

Proof of Theorem 10.4.2: By definition, the control $u:=\mathbb{K}_\circlearrowleft B_2w_0$ minimizes the cost $\|\mathbb{C}B_2w_0+\mathbb{D}u\|_2^2$ over $u\in\mathcal{U}_*^*(B_2w_0)$ (which in case $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{exp}}$ means the controls $u\in\mathrm{L}^2(\mathbf{R}_+;U)$ that make $x:=\mathbb{A}B_2w_0+\mathbb{B}\tau u$ (and hence $y:=\mathbb{C}B_2w_0+\mathbb{D}u$) stable). (If this minimization is strict for all $x_0\in H$ in place of B_2w_0, then it is strict for a particular $w_0\in W$.)

N.B. If $\mathcal{U}_*^*=\mathcal{U}_{\mathrm{exp}}$, then K is an exponentially stabilizing state feedback operator (and it minimizes the H^2 norm over all such operators as well as over

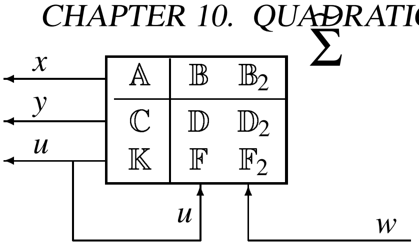

Figure 10.2: The H^2 state feedback problem

all $u \in \mathcal{U}_{\mathrm{exp}}(B_2 w_0)$, for each $w_0 \in W$). □

In classical H^2 problems, one has $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ (and $\dim U \times H \times Y < \infty$), and one assumes that the original system Σ is exponentially stabilizable and positively J-coercive, so that there is a unique (bounded) minimizing state feedback operator. Thus, then Theorem 10.4.2 becomes applicable.

Next we drop the assumption that B_2 is bounded and assume, instead, that the cost-minimizing state feedback operator K is bounded. We again show that K also solves the H^2 problem (assuming that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ and that the two necessary "$\mathrm{L}^2_{\mathrm{strong}}$" assumptions hold):

Proposition 10.4.3 ($\mathcal{U}_{\mathrm{exp}}$: LQR $\Rightarrow$ $\mathbf{H}^2$) *Assume that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. Assume that $\pi_{[0,1)}\mathbb{A}B_2 w_0 \in \mathrm{L}^2([0,1);H)$ and $\pi_{[0,1)}C_{\mathrm{w}}\mathbb{A}B_2 w_0 \in \mathrm{L}^2([0,1);Y)$ for all $w_0 \in W$ (this is necessary for the existence of admissible controls for each $w_0 \in W$).*

Assume also that there is a bounded minimizing state feedback operator K for Σ.

Then K solves the H^2 problem (strictly if K is strictly minimizing); i.e., the minimum of $\|\widehat{\mathbb{D}}\widehat{u} + \widehat{\mathbb{D}}_2 w_0\|_{\mathrm{H}^2(\mathbf{C}^+;Y)}$ over $\mathcal{U}_{\mathrm{exp}}$-stabilizing controls u is equal to

$$\|\widehat{\mathbb{D}}_{\circlearrowleft 2} w_0\|_{\mathrm{H}^2(\mathbf{C}^+;Y)} = \|\mathbb{C}_{\circlearrowleft} B_2 w_0\|_{\mathrm{L}^2(\mathbf{R}_+;Y)} = \|(C_{\circlearrowleft})_{\mathrm{L,s}}\mathbb{A}_{\circlearrowleft} B_2 w_0\|_{\mathrm{L}^2(\mathbf{R}_+;Y)} \qquad (10.47)$$

The above assumptions on B_2 hold iff $(\cdot - A)^{-1}B_2 \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(W,H))$ and $\widehat{\mathbb{D}}_2 \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(W,Y))$ for some $\omega \in \mathbf{R}$, by Lemma 6.8.1(a)&(d1).

Here $\mathcal{U}_{\mathrm{exp}}$-stabilizing controls mean such $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ that the state of Σ with "initial state $B_2 w_0$" is in L^2. Since $\mathbb{B}\tau u \in \mathrm{L}^2_\infty$, the condition $\pi_{[0,1)}\mathbb{A}B_2 w_0 \in \mathrm{L}^2([0,1);H)$ is necessary; it was shown above that $\pi_{[0,1)}C_{\mathrm{w}}\mathbb{A}B_2 w_0 \in \mathrm{L}^2([0,1);Y)$ (and $D_2 = 0$) is also necessary. Thus, the assumptions at the beginning of the proposition do not reduce generality.

Note that we cannot require that $x \in \mathcal{C}$ unless $\mathbb{A}B_2 w_0 \in \mathcal{C}(\mathbf{R}_+;H)$ for all $w_0 \in W$ (e.g., B_2 is bounded), since $\mathbb{B}\tau u \in \mathcal{C}$ (for any $u \in \mathrm{L}^2_\infty(\mathbf{R}_+;U)$); see (10.50).

Proof of Proposition 10.4.3: 1° *$\widetilde{\Sigma}_{\circlearrowleft}$:* Since K is bounded, it extends $\widetilde{\Sigma}$ to

$$\widetilde{\Sigma}_{\mathrm{ext}} := \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B} & \mathbb{B}_2 \\ \hline \mathbb{C} & \mathbb{D} & \mathbb{D}_2 \\ \mathbb{K} & \mathbb{F} & \mathbb{F}_2 \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Y \times U) \quad \text{with generators} \quad \left[\begin{array}{c|cc} A & B & B_2 \\ \hline C & D & 0 \\ K & 0 & 0 \end{array}\right], \qquad (10.48)$$

by Lemma 6.3.16(c). It follows from that $\left[\begin{smallmatrix} K \\ 0 \end{smallmatrix}\right]$ is a bounded ULR exponentially stabilizing state feedback operator for $\widetilde{\Sigma}$, by Lemma 6.6.11 (alternatively, we can use the state feedback operator $\left[\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right]$ for $\widetilde{\Sigma}_{\mathrm{ext}}$). The corresponding closed-loop system is given by

$$\widetilde{\Sigma}_{\circlearrowleft} := \left[\begin{array}{c|cc} \mathbb{A}_{\circlearrowleft} & \mathbb{B}_{\circlearrowleft} & \mathbb{B}_{\circlearrowleft 2} \\ \hline \mathbb{C}_{\circlearrowleft} & \mathbb{D}_{\circlearrowleft} & \mathbb{D}_{\circlearrowleft 2} \\ \mathbb{K}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft 2} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{A} + \mathbb{B}_{\circlearrowleft}\mathbb{K} & \mathbb{B}\mathbb{M} & \mathbb{B}_{\circlearrowleft}\mathbb{F}_2 + \mathbb{B}_2 \\ \hline \mathbb{C} + \mathbb{D}_{\circlearrowleft}\mathbb{K} & \mathbb{D}\mathbb{M} & \mathbb{D}_{\circlearrowleft}\mathbb{F}_2 + \mathbb{D}_2 \\ \mathbb{M}\mathbb{K} & \mathbb{M} - I & \mathbb{M}\mathbb{F}_2 \end{array}\right] \tag{10.49}$$

(use (6.133) with $\mathbb{M} = \left[\begin{smallmatrix} \mathbb{M} & \mathbb{M}\mathbb{F}_2 \\ 0 & I \end{smallmatrix}\right]$ or (6.125) with $I - L\mathbb{D} = \left[\begin{smallmatrix} \mathbb{X} & -\mathbb{F}_2 \\ 0 & I \end{smallmatrix}\right]$; two left columns of $\widetilde{\Sigma}_{\circlearrowleft}$ are exactly $\Sigma_{\circlearrowleft}$, the closed-loop system of Σ corresponding to K).

2° *We have* $\mathbb{A}_{\circlearrowleft} B_2 w_0, (C_{\circlearrowleft})_{\mathrm{w}}\mathbb{A}_{\circlearrowleft} B_2 w_0 \in \mathrm{L}^2$*:* Let $w_0 \in W$ be arbitrary. Since $\mathbb{K} = K\mathbb{A}$, we have

$$\mathbb{A}_{\circlearrowleft} B_2 w_0 = \mathbb{A} B_2 w_0 + \mathbb{B}_{\circlearrowleft} \tau K \mathbb{A} B_2 u_0 \in \mathrm{L}^2([0,1);H) \tag{10.50}$$

since $\mathbb{A} B_2 w_0 \in \mathrm{L}^2([0,1);H)$ and $K \in \mathcal{B}(H,U)$. By Lemma 6.8.1(a), it follows that $\mathbb{A}_{\circlearrowleft} B_2 w_0 \in \mathrm{L}^2(\mathbf{R}_+;U)$, (since $\omega_{\mathbb{A}_{\circlearrowleft}} < 0$). Analogously,

$$\widehat{\mathbb{D}}_{\circlearrowleft 2} w_0 = \widehat{\mathbb{D}_{\circlearrowleft}} \widehat{\mathbb{F}}_2 w_0 + \widehat{\mathbb{D}}_2 w_0 = \widehat{\mathbb{D}_{\circlearrowleft}} K(\cdot - A)^{-1} B_2 w_0 + \widehat{\mathbb{D}}_2 w_0 \in \mathrm{H}^2(\mathbf{C}^+_{\omega};Y) \tag{10.51}$$

for some $\omega > 0$, hence $\widehat{\mathbb{D}}_{\circlearrowleft 2} w_0 \in \mathrm{H}^2(\mathbf{C}^+;Y)$ (i.e., $(C_{\circlearrowleft})_{\mathrm{w}}\mathbb{A}_{\circlearrowleft} B_2 w_0 \in \mathrm{L}^2(\mathbf{R}_+;Y)$), by Lemma 6.8.1(d1).

3° $\mathcal{U}_{\mathrm{exp}}$*-stabilizing controls* u*:* Let $w_0 \in W$. Assume that $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ is $\mathcal{U}_{\mathrm{exp}}$*-stabilizing*, by which we mean that $x \in \mathrm{L}^2$, where

$$x := \mathbb{A} B_2 w_0 + \mathbb{B}\tau u = \mathbb{A}_{\circlearrowleft} B_2 w_0 + \mathbb{B}_{\circlearrowleft} \tau(\mathbb{X} u - \mathbb{K} B_2 w_0) \tag{10.52}$$

(we used here the facts that $\mathbb{A}_{\circlearrowleft} = \mathbb{A} + \mathbb{B}_{\circlearrowleft}\tau\mathbb{K}$ and $\mathbb{B}_{\circlearrowleft} = \mathbb{B}\mathbb{X}^{-1}$). By assumption, $\mathbb{A}_{\circlearrowleft} B_2 w_0 \in \mathrm{L}^2$, hence $\mathbb{B}_{\circlearrowleft}\tau u_{\circlearrowleft} \in \mathrm{L}^2$, where $u_{\circlearrowleft} := \mathbb{X} u - \mathbb{K} B_2 w_0 = u - Kx \in \mathrm{L}^2$. Then (recall that $\widehat{\mathbb{F}}_2(s) = K(s-A)^{-1}B_2 = \widehat{\mathbb{K}}(s) B_2$)

$$\widehat{y} := \widehat{\mathbb{D}}_2 w_0 + \widehat{\mathbb{D}}\widehat{u} = \widehat{\mathbb{D}}_2 w_0 + \widehat{\mathbb{D}_{\circlearrowleft}}\widehat{\mathbb{K}} B_2 w_0 - \widehat{\mathbb{D}_{\circlearrowleft}}\widehat{\mathbb{K}} B_2 w_0 + \widehat{\mathbb{D}_{\circlearrowleft}}\widehat{\mathbb{X}}\widehat{u} = \widehat{\mathbb{D}}_{\circlearrowleft 2} w_0 + \widehat{\mathbb{D}_{\circlearrowleft}} u_{\circlearrowleft}. \tag{10.53}$$

4° *Minimum at* $u_{\circlearrowleft} = 0$*:* Let $w_0 \in W$ be arbitrary, and set $\widehat{z} := \widehat{\mathbb{D}}_{\circlearrowleft 2} w_0$, so that $z = (C_{\circlearrowleft})_{\mathrm{L,s}} \dot{\mathbb{A}}_{\circlearrowleft} B_2 w_0$, by Lemma 6.8.1(d1). For each $n \in \mathbf{N}$, choose $t_n \in (0, 1/n)$ s.t. $x_n := \mathbb{A}^{t_n}_{\circlearrowleft} B w_0 \in H$. Then

$$z_n := (C_{\circlearrowleft})_{\mathrm{L,s}} \dot{\mathbb{A}}_{\circlearrowleft} \mathbb{A}^{t_n}_{\circlearrowleft} B_2 w_0 = \mathbb{C}_{\circlearrowleft} x_n \in \mathcal{X} \quad (n \in \mathbf{N}). \tag{10.54}$$

But $z_n = \tau^{-t_n}\pi_+\tau^{t_n} z$, hence $z_n \to z$ in $\mathrm{L}^2(\mathbf{R}_+;Y)$ as $n \to \infty$, by Corollary B.3.8. Consequently,

$$\langle z, \mathbb{D}_{\circlearrowleft} u_{\circlearrowleft}\rangle_{\mathrm{L}^2} = \lim_n \langle \mathbb{C}_{\circlearrowleft} x_n, \mathbb{D}_{\circlearrowleft} u_{\circlearrowleft}\rangle_{\mathrm{L}^2} = 0 \quad (u_{\circlearrowleft} \in \mathrm{L}^2). \tag{10.55}$$

Thus, if u and $u_{\circlearrowleft}$ are as in 3°, then $\langle \widehat{\mathbb{D}}_{\circlearrowleft 2} w_0, \widehat{\mathbb{D}_{\circlearrowleft}} u_{\circlearrowleft}\rangle_{\mathrm{H}^2} = 0$, so that

$$\|\widehat{\mathbb{D}}_2 w_0 + \widehat{\mathbb{D}}\widehat{u}\|^2_{\mathrm{H}^2} = \|\widehat{\mathbb{D}}_{\circlearrowleft 2} w_0\|^2_{\mathrm{H}^2} + \|\widehat{\mathbb{D}}_{\circlearrowleft} u_{\circlearrowleft}\|^2_{\mathrm{H}^2}. \tag{10.56}$$

Consequently, the minimum is achieved at $u_{\circlearrowleft} = 0$. This minimum is strict iff

$\mathbb{D}$ is injective (on $\mathcal{U}_{\mathrm{exp}}$-stabilizing controls u, e.g., on $\mathrm{L}^2(\mathbf{R}_+;U)$). $\square$

When Σ is very regular (cf. Theorem 9.2.3) and $D^*C = 0$ (or C is bounded), the assumptions of the proposition are satisfied:

Corollary 10.4.4 ($\mathbf{H^2}$ problem when $\mathbb{A}Bu_0 \in \mathrm{L}^2$) *Assume that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, $\mathbb{A}B_2w_0 \in \mathrm{L}^2([0,1);H)$ and $C_{\mathrm{w}}\mathbb{A}B_2w_0 \in \mathrm{L}^2([0,1);Y)$ for all $w_0 \in W$. Assume also that Hypothesis 9.2.1 holds, $D^*C \in \mathcal{B}(H,U)$, and $D^*D \gg 0$.*

Then there is a minimizing control for each $x_0 \in H$ iff Σ is optimizable and $\mathbb{D}$ is I-coercive (see Proposition 10.3.2). Assume that this is the case.

Then there is a bounded strictly minimizing state feedback operator K for Σ (corresponding to the unique exponentially stabilizing solution $(\mathcal{P}, D^*D, K)$ of the B_{w}^*-CARE), and K solves the H^2 problem strictly; i.e., the minimum of $\|\widehat{\mathbb{D}}\widehat{u} + \widehat{\mathbb{D}}_2 w_0\|_{\mathrm{H}^2(\mathbf{C}^+;Y)}$ over $\mathcal{U}_{\mathrm{exp}}$-stabilizing controls u is strict and equal to $\|\widehat{\mathbb{D}}_{\circlearrowleft 2} w_0\|_{\mathrm{H}^2(\mathbf{C}^+;Y)}$.

In particular, the problem can be solved in the parabolic case (when Hypothesis 9.5.1 holds, $\gamma < 1/4$, $\beta < -1/2$, $D^*D \gg 0$ and $D^*C \in \mathcal{B}(H,U)$), by Theorem 9.2.3.

Alternatively, when B and B_2 are bounded, $D^*C \in \mathcal{B}(H,U)$, $D^*D \gg 0$ and $\mathbb{D}$ is J-coercive, all assumptions of the corollary are satisfied. In either case, also the state becomes continuous. Hypothesis 9.2.1 is also satisfied in the case that $\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1);H)$ and $C_{\mathrm{w}}\mathbb{A}Bu_0 \in \mathrm{L}^2([0,1);Y)$ for all $u_0 \in U$, i.e., that also B satisfies the assumptions posed on B_2.

Proof of Corollary 10.4.4: By Corollary 10.2.9, there is a minimizing control for each $x_0 \in H$ iff Σ is optimizable and $\mathbb{D}$ is (necessarily positively, since $I \geq 0$) I-coercive, and such a control is necessarily unique, with $(\mathcal{P}, D^*D, K)$ being the unique exponentially stabilizing solution of the B_{w}^*-CARE. Since $D^*C \in \mathcal{B}(H,U)$, the operator K is bounded. The rest follows from Proposition 10.4.3. $\square$

Remark 10.4.5 ($x_0 \neq 0$) *As one observes from the proofs, the (state feedback) controller $\begin{bmatrix} \mathbb{Q}_0 & \mathbb{Q} \end{bmatrix} : (x_0, w) \mapsto u := \mathbb{K}_{\circlearrowleft} x_0 + \mathbb{K}_{\circlearrowleft} B_2 w_0$ of Theorem 10.4.2, Proposition 10.4.3 and Corollary 10.4.4 actually minimizes the norm*

$$\|\widehat{\mathbb{C}}x_0 + \widehat{\mathbb{D}}\widehat{u} + \widehat{\mathbb{D}}_2 w_0\|_{\mathrm{H}^2(\mathbf{C}^+;Y)} \tag{10.57}$$

for any $x_0 \in H$ (not merely for $x_0 = 0$ as in the H^2 problem). Indeed, the "cost"

$$\|\widehat{\mathbb{C}_{\circlearrowleft}}x_0 + \widehat{\mathbb{D}_{\circlearrowleft}}\widehat{u_{\circlearrowleft}} + \widehat{\mathbb{D}_{\circlearrowleft 2}}w_0\|_{\mathrm{H}^2}^2 = \|\widehat{\mathbb{C}_{\circlearrowleft}}x_0 + \widehat{\mathbb{D}_{\circlearrowleft 2}}w_0\|_{\mathrm{H}^2}^2 + \|\widehat{\mathbb{D}_{\circlearrowleft}}\widehat{u_{\circlearrowleft}}\|_{\mathrm{H}^2}^2 \tag{10.58}$$

induced by K corresponds to the control law $\begin{bmatrix} \mathbb{Q}_0 & \mathbb{Q} \end{bmatrix}$, where $\widehat{u_{\circlearrowleft}}$ is the external input. For $u_{\circlearrowleft} \in \mathrm{L}^2_{\mathrm{c}}(\mathbf{R}_+;U)$ we have $\|\widehat{\mathbb{D}_{\circlearrowleft}}\widehat{u_{\circlearrowleft}}\|_{\mathrm{H}^2}^2 = \langle \widehat{u_{\circlearrowleft}}, S\widehat{u_{\circlearrowleft}} \rangle$ (where $S = D^*D$ if Σ is regular enough; this is the case in Corollary 10.4.4), by (9.162); if $\mathbb{D}_{\circlearrowleft}$ is stable (e.g., $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$), then this is the case for all $u_{\circlearrowleft} \in \mathrm{L}^2(\mathbf{R}_+;U)$.

Thus, not only is there a unique $\mathcal{J}_{\mathrm{H}^2}$-minimizing u for each w_0 (and $x_0 = 0$), there is a unique $\mathcal{J}_{\mathrm{H}^2}$-minimizing u for each w_0 and x_0, and this u is of state *feedback form, namely generated by the ($\mathcal{J}$-)minimizing K (in particular, the (optimal) function from state to u is static).*

As noted below Proposition 10.4.3, conditions $\mathbb{A}B_2w_0, C_{\mathrm{w}}\mathbb{A}B_2w_0 \in \mathrm{L}^2_{\mathrm{loc}}$ ($w_0 \in W$) are necessary for the existence of a solution over $\mathcal{U}_{\mathrm{exp}}$. □

Remark 10.4.6 (Traditional H^2 problem) *Traditionally, in the (finite-dimensional) H^2 problem one minimizes the (squared) norm*

$$\|\widehat{\mathbb{D}}_{\circlearrowleft 2}\|^2 := \sum_{q\in Q} \|\widehat{\mathbb{D}}_{\circlearrowleft 2} w_q\|^2_{\mathrm{H}^2}, \tag{10.59}$$

where $\{w_q\}_{q\in Q}$ is an orthonormal base for W (this form is defined as the H^2 norm of the trace of $\widehat{\mathbb{D}}^*_{\circlearrowleft 2}\widehat{\mathbb{D}}_{\circlearrowleft 2}$, see p. 265 of [IOW] for the equivalence of the two definitions).

Since we minimize $\|\widehat{\mathbb{D}}_{\circlearrowleft 2} w_0\|^2_{\mathrm{H}^2}$ for each $w_0 \in W$, our solution a fortiori *solves the traditional H^2 problem provided that it is solvable, i.e., that the norm can be made finite (e.g., when $\dim W < \infty$ and the conditions of of Theorem 10.4.2, Proposition 10.4.3 or Corollary 10.4.4 are satisfied). Thus, when $\dim W = \infty$, we have to strengthen our optimizability assumption the include the condition that*

$$\sum_{q\in Q} \|\widehat{\mathbb{D}}_2 w_q + \widehat{\mathbb{D}}\widehat{u}_q\|^2_{\mathrm{H}^2} < \infty \tag{10.60}$$

for some functions $u_q \in \mathcal{U}^*_*(B_2w_q)$ ($q \in Q$) to guarantee a solution also for the traditional H^2 problem (if B_2 is unbounded, we must extend $\mathcal{U}^*_*$ to $B_2[\{w_q\}]$ as in 3° of the proof of Proposition 10.4.3). □

We have solved above the state feedback and full information H^2 problems (they have a common solution). In classical literature, one usually also solves other special H^2 problems and then the general H^2 (dynamic partial feedback) control problem for a given (even more extended) system

$$\left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B} & \mathbb{B}_2 \\ \hline \mathbb{C} & \mathbb{D} & \mathbb{D}_2 \\ \mathbb{C}_2 & \mathbb{D}_{21} & \mathbb{D}_{22} \end{array}\right] \in \mathrm{WPLS}(U\times W, H, Y\times Y_2) \quad \text{with generators} \quad \left[\begin{array}{c|cc} A & B & B_2 \\ \hline C & D & 0 \\ C_2 & 0 & D_{22} \end{array}\right]. \tag{10.61}$$

We end this section by a brief overview of the general H^2 problem. One usually poses for $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_2 \\ \hline \mathbb{C}_2 & \mathbb{D}_{22} \end{array}\right]^{\mathrm{d}}$ the same assumptions as on Σ (thus guaranteeing the solution for the LQR problem for that system).

It seems that it is straightforward to extend the classical results for infinite-dimensional systems by using the separation result of Proposition 10.4.3 (and its proof). (An alternative approach would be to formulate the problems in I/O map form and solve them as in Section 12.3.)

Indeed, since $\widehat{\mathbb{D}_{\circlearrowleft}}$ is (I, D^*D)-inner (and it can be made inner by replacing K by $\left[\begin{array}{c|c} K' & F' \end{array}\right]$ with $K' := XK$, $F' := I - X$, $X := (D^*D)^{1/2}$), the cost for a (dynamic partial feedback) controller $\mathbb{Q}_{\circlearrowleft} : w_0 \mapsto u_{\circlearrowleft}$ is equal to $\mathcal{J}_{\mathrm{H}^2} = \|\mathbb{D}_{\circlearrowleft} w_0\|^2_2 + \|\mathbb{Q}_{\circlearrowleft} w_0\|^2_2$. Thus, the solution of the general H^2 problem is obtained by finding the optimal

$\mathbb{Q}_{\circlearrowleft}$, i.e., by solving the H^2 output estimation problem (OEP) for

$$\left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B} & \mathbb{B}_2 \\ \hline -\mathbb{K} & \mathbb{X} & -\mathbb{F}_2 \\ \mathbb{C}_2 & \mathbb{D}_{21} & \mathbb{D}_{22} \end{array}\right] \in \mathrm{WPLS}(U \times W, H, U \times Y_2) \quad \text{with generators} \quad \left[\begin{array}{c|cc} A & B & B_2 \\ \hline -K & I & 0 \\ C_2 & 0 & D_{22} \end{array}\right]. \tag{10.62}$$

Since we cannot use the theory of Section 10.2 directly for the H^2 OEP (unlike for the H^2 full information problem (FIP)), we leave it to the interested reader complete the details under suitable regularity assumptions, such as those of Lemma 6.8.5 for $p = 2 = q$ or those of Hypothesis 9.5.7(3.) (no special assumptions seem to be needed in the discrete-time case). See, e.g., pp. 316 and 395–396 of [ZDG], Chapter 9 of [IOW] or Chapter VIII of [DGKF] for details for finite-dimensional systems.

Notes

See, e.g., [AM90], [AM79], [KS] or [GL, p. 207] for the motivation and the history of the H^2 problem and [IOW] or [ZDG] for complete solutions in the finite-dimensional case. The first state-space solution seems to be given in [DGKF] and a very general one in [IOW].

The classical assumptions for the state feedback and full information H^2 problems are positive J-coercivity, exponential stabilizability and $D^*D \gg 0$, so that corresponding results are contained Corollary 10.4.4 (since B and C are bounded for finite-dimensional systems).

Naturally, by taking causal adjoints we obtain a solution of the dual problems, the H^2 output injection and full control problems (see [GL]). Their stochastic counterpart is called the Kalman filter problem (see, e.g., [GL] or [LR]).

Also the general H^2 problem has a stochastic counterpart, the so called *Linear Quadratic Gaussian (LQG)* problem; it has been studied also for infinite-dimensional systems, see [CP78].

According to [Helton85], p. 17, G. Zames indicated in the late 1970s how the H^∞ problem is usually physically better motivated than the H^2 problem. Since that time the former problem (whose solutions are more complicated) has become much more popular than the latter one.

10.5 Real lemmas

Rocky's Lemma of Innovation Prevention:
Unless the results are known in advance, funding agencies will reject the proposal.

In this section, we present the Bounded Real Lemma (in two forms) and the Strict Positive Real Lemma, which allow one to use the Riccati equation to verify whether $\|\mathbb{D}\|_{\mathrm{TIC}} \le \gamma$ or $\mathrm{Re}\langle \mathbb{D}\cdot, \cdot\rangle \gg 0$, respectively.

We give our results for "L^1-type" systems; see Theorem 10.6.5(e) for alternative regularity assumptions. For general regular systems only the sufficiency part holds (unless we accept IAREs in place of CAREs). See Theorems 15.4.1 and 15.4.3 and Proposition 15.4.2 for the discrete-time counterparts of these results (and to get an overview of this section without any regularity considerations).

Our theorems do not need any verification of stability/stabilizability; this is based on the fact that we may apply the theory of the next section in case "$C^*JC \le 0$", as one observes from the proofs.

Thus, under sufficient regularity, a uniform Riccati inequality has a solution iff Σ is exponentially stable and $\|\mathbb{D}\| < \gamma$:

Theorem 10.5.1 (Generalized Strict Bounded Real Lemma) *Assume that* $\gamma > 0$.

(a) If (1.) or (2.) or (5.) of Hypothesis 9.2.2 holds, or if C is bounded and at least one of the following conditions holds:

1. $\dim Y < \infty$;
2. $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U,H))$;
3. $\pi_{[0,1)}\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1);H)$ *for all* $u_0 \in U$ *and* $D^*C = 0$;
4. $D^*C = 0$ *and* $(\widehat{\mathbb{D}} - D)u_0 \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U,Y))$ *for all* $u_0 \in U$;

then the following are equivalent:

(i) Σ *is exponentially stable and* $\|\mathbb{D}\| < \gamma$;

(ii) There is $\mathcal{P} \le 0$ *s.t.* $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ *and*

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A - C^*C & (B_{\mathrm{w}}^*\mathcal{P} - D^*C)^* \\ B_{\mathrm{w}}^*\mathcal{P} - D^*C & \gamma^2 I - D^*D \end{bmatrix} \gg 0 \quad \text{on} \quad \mathrm{Dom}(A) \times U. \tag{10.63}$$

(iii) There is $\mathcal{P} \le 0$ *s.t.* $S := \gamma^2 I - D^*D + \text{s-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B$ *exists and*

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A - C^*C & (B_{\mathrm{w}}^*\mathcal{P} - D^*C)^* \\ B_{\mathrm{w}}^*\mathcal{P} - D^*C & S \end{bmatrix} \gg 0 \quad \text{on} \quad \mathrm{Dom}(A) \times U. \tag{10.64}$$

(b) If $\pi_{[0,t)}\mathbb{A}B \in \mathrm{L}^1([0,t);\mathcal{B}(U,H))$, $\pi_{[0,t)}C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,t);\mathcal{B}(H,Y))$, *and* $\pi_{[0,t)}C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,t);\mathcal{B}(U,Y))$ *for some* $t > 0$, *then (i)$\Leftrightarrow$(iii)$\Leftarrow$(ii).*

(c1) If $\mathbb{D}$ is ULR, then we have (i)$\Leftarrow$(iii)$\Leftarrow$(ii).

(c2) Any solution of (ii), (iii), (ii') or (iii') is strictly negative ($\mathcal{P} < 0$). Under the assumptions of (a), there is an exponentially stabilizing solution (if (i) holds).

(d) If $\mathbb{D}$ is SR, then we have (ii)$\Leftrightarrow$(ii')$\Rightarrow$(iii)$\Leftrightarrow$(iii'), where

(ii') $\|D\| < \gamma$, and there is $\mathcal{P} \leq 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$(B_{\mathrm{w}}^*\mathcal{P} - D^*C)^*(\gamma^2 I - D^*D)^{-1}(B_{\mathrm{w}}^*\mathcal{P} - D^*C) \ll A^*\mathcal{P} + \mathcal{P}A - C^*C. \tag{10.65}$$

(iii') There is $\mathcal{P} \leq 0$ s.t.

$$S := (\gamma^2 - D^*D) + \operatorname*{s-lim}_{\alpha\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(\alpha - A)^{-1}B \gg 0, \quad \textit{and} \tag{10.66}$$

$$(B_{\mathrm{w}}^*\mathcal{P} - D^*C)^* S^{-1}(B_{\mathrm{w}}^*\mathcal{P} - D^*C) \ll A^*\mathcal{P} + \mathcal{P}A - C^*C. \tag{10.67}$$

We always require that $\mathcal{P} \in \mathcal{B}(H)$. As in Definition 9.1.5, conditions (iii) and (iii') include the requirements that the terms are defined (in particular, that the limits converge strongly; as in Remark 9.1.6, it follows that $\mathcal{P}[H_B] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$). If B is bounded, then $B_{\mathrm{w}}^* = B^*$ and $\mathrm{Dom}(B_{\mathrm{w}}^*) = H$, so that then ((a) applies and) (ii) becomes essentially simpler.

Recall that "$C^*JC \leq 0$" means that $\langle x_0, C^*JCx_0\rangle_{\langle H_1,(H_1)^*\rangle} \leq 0$ for all $x_0 \in H_1 := \mathrm{Dom}(A)$. The phrase "on $\mathrm{Dom}(A) \times U$" refers to the inner product $\langle \left[\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right]\rangle := \langle x_1, x_2\rangle_{H_1,H_{-1}^*} + \langle u_1, u_2\rangle_U$, since H is the pivot space and $H_1 := \mathrm{Dom}(A)$.

Note that (10.65) (resp. (10.66)&(10.67)) is an "inequality form" of the B_{w}^*-CARE (resp. of the (strongly regular) CARE). Hypothesis 9.5.1 is stronger than the assumption of (b), and $\mathbb{D}$ is ULR whenever the assumptions of (a) or (b) hold.

See Theorem 10.6.5(e) (with Σ_{aug} and J_{aug} in place of Σ and J, i.e., with substitutions $\widetilde{C} = \left[\begin{smallmatrix} C \\ 0 \\ I \end{smallmatrix}\right]$, $\widetilde{D} = \left[\begin{smallmatrix} D \\ I \\ 0 \end{smallmatrix}\right]$, $\widetilde{J} = \mathrm{diag}(-I, \gamma^2, \varepsilon)$), for alternative regularity assumptions.

Note that we could also have the real lemmas of this section based on IAREs instead of B_{w}^*-CAREs or CAREs, to make them look like their discrete-time counterparts.

If one replaces J by $-J$ in the proof, the operators $\mathcal{P} \leq 0$ and $S \gg 0$ are replaced by $-\mathcal{P} \geq 0$ and $S \ll 0$, so that the condition (ii) becomes "$\|D\| < \gamma$, and there is $\mathcal{P} \geq 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$-(B_{\mathrm{w}}^*\mathcal{P} + D^*C)^*(\gamma^2 I - D^*D)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*C) \ll A^*\mathcal{P} + \mathcal{P}A + C^*C\text{"}; \tag{10.68}$$

(special cases of) this condition is common in the literature, but we have made the choice that leads to a positive Popov operator and to the setting of Section 10.6. Obviously, analogous changes can be made to the other conditions (multiply the inequalities by -1 and replace $\mathcal{P}$ by $-\mathcal{P}$).

Proof of Theorem 10.5.1: 1° Define $J_{\mathrm{aug}} := \left[\begin{smallmatrix} -I & 0 \\ 0 & \gamma^2 I \end{smallmatrix}\right]$, $\mathbb{C}_{\mathrm{aug}} := \left[\begin{smallmatrix} \mathbb{C} \\ 0 \end{smallmatrix}\right]$, $\mathbb{D}_{\mathrm{aug}} := \left[\begin{smallmatrix} \mathbb{D} \\ I \end{smallmatrix}\right]$ and $\Sigma_{\mathrm{aug}} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C}_{\mathrm{aug}} & \mathbb{D}_{\mathrm{aug}} \end{array}\right]$.

Then $\|\mathbb{D}\|_{\text{TIC}} < \gamma$ iff $\mathbb{D}_{\text{aug}}^* J_{\text{aug}} \mathbb{D}_{\text{aug}} = \gamma^2 I - \mathbb{D}^*\mathbb{D} \gg 0$. Moreover, Σ_{aug} is exponentially stable iff Σ is, by Lemma 6.1.10(a1), and $C_{\text{aug}}^* J_{\text{aug}} C_{\text{aug}} = -C^*C \leq 0$ (and $D_{\text{aug}}^* J_{\text{aug}} C_{\text{aug}} = -D^*C$ and $D_{\text{aug}}^* J_{\text{aug}} D_{\text{aug}} = \gamma^2 I - D^*D$ whenever $\mathbb{D}$ is regular). Consequently, the theorem follows directly from Theorem 10.6.5, by 2°.

2° *The assumptions of (a) imply those of (a) or (e) of Theorem 10.6.5 for* Σ_{aug}*:* Obviously, (1.) or (2.) or (5.) of Hypothesis 9.2.2 for Σ implies that for Σ_{aug}.

Assume then that C is bounded. Case 1. was given in Theorem 10.6.5(e). Cases 2., 3., and 4. (with (i)) imply (4.), (3.) and (6.) of Hypothesis 9.2.2, respectively, because (i) implies that $\|D\| < \gamma$ (hence that $\gamma^2 - D^*D \gg 0$), by Lemma 6.3.2(e) (thus, we need Theorem 10.6.5(e) in case 4.). □

Note that the spectral factorization $\gamma^2 I - \mathbb{D}^*\mathbb{D} = \mathbb{X}^* S \mathbb{X}$ is equivalent to the normalized factorization $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$, $\gamma^2 \mathbb{M}^*\mathbb{M} - \mathbb{N} * \mathbb{N} = S$, $\mathcal{B}(U) \ni S \gg 0$, $\mathbb{M} := \mathbb{X}^{-1} \in \mathcal{G}\text{TIC}(U)$, $\mathbb{N} := \mathbb{D}\mathbb{M} \in \text{TIC}(U,Y)$.

When one wishes to find an estimate for $\mathbb{D}$ without requiring Σ to be exponentially stable, one should use the proposition below instead of the above theorem (and one should know, a priori, that $\mathbb{C}$ is stable).

Proposition 10.5.2 (Nonexp. $\|\mathbb{D}\|_{\text{TIC}} < \gamma$) *Assume that $\mathbb{D}$ is SR and $\gamma > 0$.*

If (ii) or (iii) holds, then $\mathbb{D} \in \text{TIC}$ and $\|\mathbb{D}\| \leq \gamma$.

Conversely, if $\Sigma \in \text{SOS}$, $\|\mathbb{D}\| < \gamma$, and Σ_{aug}, J_{aug} satisfy (2.) (resp. (6.)) of Hypothesis 10.6.1, then (iii) (resp. (iii) and (ii)) holds (also with "$=$" in place of "$\geq$").

Here we have referred to the following conditions:

(ii) $\|D\| < \gamma$, and there is $\mathcal{P} \leq 0$ s.t. $\mathcal{P}[H] \subset \text{Dom}(B_{\text{w}}^*)$ and

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A - C^*C & (B_{\text{w}}^*\mathcal{P} - D^*C)^* \\ B_{\text{w}}^*\mathcal{P} - D^*C & \gamma^2 I - D^*D \end{bmatrix} \geq 0 \quad \text{on} \quad \text{Dom}(A) \times U. \tag{10.69}$$

(iii) There is $\mathcal{P} \leq 0$ s.t. $S := \gamma^2 I - D^*D + \text{s-lim}_{s \to +\infty} B_{\text{w}}^* \mathcal{P}(s-A)^{-1}B \gg 0$, and

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A - C^*C & (B_{\text{w}}^*\mathcal{P} - D^*C)^* \\ B_{\text{w}}^*\mathcal{P} - D^*C & S \end{bmatrix} \geq 0 \quad \text{on} \quad \text{Dom}(A) \times U. \tag{10.70}$$

Moreover, we have (ii)⇔(ii')⇒(iii)⇔(iii'), where (ii') and (iii') are from Theorem 10.5.1(d) with "$\leq$" in place of "$\ll$".

If $\mathbb{B}$ is strongly stable, then we can replace "$\mathcal{P} \leq 0$" by $\mathcal{P} = \mathcal{P}^*$ everywhere in this proposition.

(See the proof of Theorem 10.5.1 for Σ_{aug} and J_{aug}. Note that here we may have $\mathcal{P} = 0$ (take $\Sigma = 0$) whereas $\mathcal{P} < 0$ in the theorems of this section.)

Thus, if $\mathbb{C}$ is stable, $\mathbb{D}$ is SR, and any of (1.)–(10.) (resp. (1.)–(8.)) of Lemma 10.6.2(c) (with $D^*JC \mapsto D^*C$ and $D^*JD \mapsto \gamma^2 - D^*D$) holds, then Hypothesis 10.6.1(2.) (resp. (6.)) holds, and we can estimate $\|\mathbb{D}\|$ as follows:

Take some $\gamma > 0$, and then verify, whether the Riccati inequality condition (iii) (resp. (ii)) has any solutions. If so, then $\|\mathbb{D}\| \leq \gamma$, otherwise $\|\mathbb{D}\| \geq \gamma$. Then vary

γ and find an estimate for $\|\mathbb{D}\|$ by, e.g., a binary search. (Also the corresponding Riccati equation (that is, "$=$" in place of "$\geq$") can be used.)

Proof of Proposition 10.5.2: The proof of Theorem 10.5.1 applies here too, with Proposition 10.6.4 in place of Theorem 10.6.5.

(If $\mathbb{B}^t$ is strongly stable, then, in the proof of Proposition 10.6.4(d), we have $\gamma^2 I - \mathbb{D}^{t\,*}\mathbb{D}^t \geq -\mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t \to 0$, which implies that $\mathbb{D} \in \mathrm{TIC}$ and $\|\mathbb{D}\| \leq \gamma$. Therefore, we do not have to assume $\mathbb{D}$ to be stable for the last claim unlike in Proposition 10.6.4(d).)

Remark: As noted in the proof of Proposition 10.6.4(b), in the converse claim we can choose $\mathcal{P} \leq 0$ so that it is P-SOS-r.c.-stabilizing (also with "$=$" in place of "$\geq$" in (iii) (resp. in (iii) and (ii)). $\square$

In classical literature, an operator $\mathbb{D} \in \mathrm{TIC}(U)$ is called *strictly positive* if $\mathrm{Re}\langle \mathbb{D}\cdot, \cdot\rangle \gg 0$, i.e., if $\mathbb{D} + \mathbb{D}^* \gg 0$ (one can show that $\sigma(\mathbb{D}) \subset \mathbf{C}^+$ is a necessary condition; it is sufficient for normal $\mathbb{D}$). An equivalent condition is that $\widehat{\mathbb{D}} + \widehat{\mathbb{D}}^* \geq \varepsilon I$ in $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R}; \mathcal{B}(U))$ for some $\varepsilon > 0$. We use this definition in the following generalized extension of the classical Strictly Positive Real Lemma:

Theorem 10.5.3 (Generalized Strictly Positive (Real) Lemma)

(a) If C is bounded and $\dim Y < \infty$, or if (1.) or (2.) or (5.) of Hypothesis 9.2.2 holds, then the following are equivalent:

(i) Σ is exponentially stable and $\mathbb{D}$ is strictly positive;

(ii) There is $\mathcal{P} \leq 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A & (B_{\mathrm{w}}^*\mathcal{P} + C)^* \\ B_{\mathrm{w}}^*\mathcal{P} + C & D + D^* \end{bmatrix} \gg 0 \quad \text{on} \quad \mathrm{Dom}(A) \times U. \tag{10.71}$$

(iii) There is $\mathcal{P} \leq 0$ s.t. $S := D + D^* + \text{s-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B$ exists and

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A & (B_{\mathrm{w}}^*\mathcal{P} + C)^* \\ B_{\mathrm{w}}^*\mathcal{P} + C & S \end{bmatrix} \gg 0 \quad \text{on} \quad \mathrm{Dom}(A) \times U. \tag{10.72}$$

(b) If $\pi_{[0,t)}\mathbb{A}B \in \mathrm{L}^1([0,t); \mathcal{B}(U,H))$, $\pi_{[0,t)}C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,t); \mathcal{B}(H,Y))$, and $\pi_{[0,t)}C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,t); \mathcal{B}(U,Y))$ for some $t > 0$, then (i)$\Leftrightarrow$(iii)$\Leftarrow$(ii).

(c1) If $\mathbb{D}$ is ULR, then we have (i)$\Leftarrow$(iii)$\Leftarrow$(ii).

(c2) Any solution of (ii), (iii), (ii') or (iii') is strictly negative ($\mathcal{P} < 0$). Under the assumptions of (a), there is an exponentially stabilizing solution (if (i) holds).

(d) If $\mathbb{D}$ is SR, then we have (ii)$\Leftrightarrow$(ii')$\Rightarrow$(iii)$\Leftrightarrow$(iii'), where

(ii') $D + D^* \gg 0$, and there is $\mathcal{P} \leq 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$(B_{\mathrm{w}}^*\mathcal{P} + C)^*(D + D^*)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + C) \ll A^*\mathcal{P} + \mathcal{P}A \tag{10.73}$$

(iii') There is $\mathcal{P} \le 0$ s.t.

$$S := D + D^* + \operatorname*{s-lim}_{\alpha \to +\infty} B_{\mathrm{w}}^* \mathcal{P}(\alpha - A)^{-1} B \gg 0, \quad \textit{and} \tag{10.74}$$

$$(B_{\mathrm{w}}^* \mathcal{P} + C)^* S^{-1} (B_{\mathrm{w}}^* \mathcal{P} + C) \ll A^* \mathcal{P} + \mathcal{P} A. \tag{10.75}$$

□

(The proof of Theorem 10.5.1 applies mutatis mutandis, with $J_{\mathrm{aug}} := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ (so that $\mathbb{D}_{\mathrm{aug}}^* J_{\mathrm{aug}} \mathbb{D}_{\mathrm{aug}} = \mathbb{D} + \mathbb{D}^*$). Also the comments below Theorem 10.5.1 apply, mutatis mutandis.)

We may also use a binary search for finding an estimate for supremal $\gamma > 0$ s.t. $\operatorname{Re}\langle \mathbb{D}\cdot, \cdot \rangle \ge \gamma^2 I$; then a positive real -variant of Proposition 10.5.2 applies (such a result holds with the same proof, mutatis mutandis).

We can again rewrite the conditions for $\mathcal{P} \mapsto -\mathcal{P} \ge 0$, $S \mapsto -S \ll 0$ and $J \mapsto -J$; e.g., (ii') becomes: $D + D^* \gg 0$, and there is $\mathcal{P} \ge 0$ s.t. $\mathcal{P}[H] \subset \operatorname{Dom}(B_{\mathrm{w}}^*)$ and

$$-(B_{\mathrm{w}}^* \mathcal{P} - C)^* (D + D^*)^{-1} (B_{\mathrm{w}}^* \mathcal{P} - C) \ll A^* \mathcal{P} + \mathcal{P} A. \tag{10.76}$$

Notes for Sections 10.5 and 10.6

In Section 8 of [S98b], it was shown that if "(i)" holds and the corresponding spectral factor is sufficiently regular, then "(iii)" holds; this applies to both "real lemmas".

The results of this section generalize most analogous results, including Theorem 3.7.1 and Problem 3.25 of [GL] (finite-dimensional case), Section 4.5 of [Oostveen] (the strongly stabe case with bounded B and C), and Remark 3.14 of [Keu] (exponentially stable Pritchard–Salamon systems); all these require one to assume, a priori, that Σ is strongly stable, and to check whether the a solution is stabilizing. One obtains such lemmas by using Theorem 10.6.3 (whose conditions (iv) and (v) are popular in the literature) instead of Theorem 10.6.5 in our proofs.

An exception to this is the Strict Bounded Real Lemma given in Section 7.1 of [IOW], which equals Theorem 10.5.1(a) restricted to finite-dimensional systems. Analogously, Theorem 10.6.5 is the generalization of Theorems 4.6.1–4.6.2 of [IOW]. In both cases, the proofs of [IOW] cannot be extended, because $\mathcal{P} < 0$ does not imply that $\mathcal{P} \ll 0$ when $\dim H = \infty$. The strength of our results reflect the power of the integral notation ($\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ and the IARE instead of $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and the CARE), which allows one to observe connections not visible from the generator notation. Similar remarks apply to corresponding discrete-time results (Section 15.4).

Propositions 10.5.2 and 10.6.4 might be new even for finite-dimensional systems. See Chapter 5 for notes on (i)–(ii") of Theorem 10.6.3(a); also the necessity of the existence of a solution to the CARE is well known [WW] [S97b].

10.6 Positive Popov operators ($0 \ll \mathbb{D}^* J \mathbb{D} = \mathbb{X}^* \mathbb{X}$)

Positive*, adj.:*
Mistaken at the top of one's voice.
— Ambrose Bierce (1842–1914), "The Devil's Dictionary"

In this section, we present necessary and sufficient conditions for the uniform positivity of the Popov operator (i.e., for $\mathbb{D}^* J \mathbb{D} \geq \varepsilon I$ for some $\varepsilon > 0$; equivalently, $\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}} \geq \varepsilon I$ in $\mathrm{L}^\infty_{\text{strong}}(i\mathbf{R}; \mathcal{B}(U))$), in terms of [regular] spectral factorizations and Riccati equations or Riccati inequalities. In the finite-dimensional setting, the connection between these concepts is rather simple; the same holds in the infinite-dimensional setting with bounded input and output operators. Therefore, to have an nice overview of the contents of this section, the reader might first wish to read our corresponding discrete-time results, namely Lemma 15.3.1, Proposition 15.3.2 and Theorem 15.3.3.

When working with stable finite-dimensional or Pritchard–Salamon systems, or with stable WPLSs having $\mathbb{D} \in$ MTIC, the uniform positivity of the Popov operator ($\mathbb{D}^* J \mathbb{D} \gg 0$) implies that it has a ULR spectral factorization (by Lemma 10.6.2(b)). For general (or even for ULR) stable WPLSs, this is not the case, by Proposition 9.13.1(c1), and therefore we must sometimes replace the CARE theory by the IARE theory.

To overcome this difficulty, we often assume that $\mathbb{D} \in$ MTIC or that Σ is otherwise sufficiently regular; that is, we assume some of the six alternative hypotheses below:

Hypothesis 10.6.1 ($\mathbb{D}$ admits positive regular SpF) *We have* $\mathbb{D} \in \mathrm{WR} \cap \mathrm{TIC}$*, and if some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ *satisfies* $\mathbb{X}^*\mathbb{X} = \mathbb{D}^* J \mathbb{D}$*, then*

(1.) $\mathbb{X} \in \mathrm{WR}$ *and* $X \in \mathcal{GB}(U)$.

(2.) $\mathbb{X} \in \mathrm{SR}$ *and* $X \in \mathcal{GB}(U)$.

(3.) $\mathbb{X} \in \mathrm{UR}$.

(4.) $\mathbb{X} \in \mathrm{ULR}$.

(5.) $\mathbb{X} \in \mathrm{ULR}$ *and* $X^*X = D^*JD$.

(6.) the B^*_{w}*-CARE has a stable P-I/O-stabilizing solution. Moreover,* $\Sigma \in \mathrm{SOS}$ *and* $\mathbb{D} \in \mathrm{ULR}$.

Note that for any $\mathbb{D} \in \mathrm{TIC}(U,Y)$, we have $\mathbb{X}^*\mathbb{X} = \mathbb{D}^* J \mathbb{D}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ iff $\mathbb{D}^* J \mathbb{D} \gg 0$, by Lemma 6.4.7(a). Conditions (1.)–(5.) of the above hypothesis just require that $\mathbb{D}$ and $\mathbb{X}$ (if any) are regular (by Lemma 6.4.5(a), all possible $\mathbb{X}$'s differ by an unitary constant, hence they are equally regular).

By Theorem 10.6.3(f1)&(i)&(iii)&(d), condition (1.) implies for any $\Sigma \in \mathrm{SOS}$ that the CARE has a unique stable P-I/O-stabilizing solution $\mathcal{P} \geq 0$; condition (6.) just says that this solution must also solve the B^*_{w}-CARE, i.e., that $\mathcal{P}[H] \subset \mathrm{Dom}(B^*_{\mathrm{w}})$.

Next we list sufficient conditions for the above hypotheses (see Standing Hypothesis 10.6.6 for $\widetilde{\mathcal{A}}_+$):

Lemma 10.6.2

(a1) We have (6.)⇒(5.)⇒(4.)⇒(3.)⇒(2.)⇒(1.) in Hypothesis 10.6.1.

(a2) If $\dim U < \infty$*, then (1.)–(3.) of Hypothesis 10.6.1 are equivalent.*

(b) If $\mathbb{D} \in \widetilde{\mathcal{A}}_+$*, then* $\mathbb{D}$ *satisfies (1.)–(4.) of Hypothesis 10.6.1.*

(c) Assume that $\Sigma \in$ SOS *and at least one of conditions (1.)–(10.) below holds. Then* $\mathbb{D}$ *satisfies Hypothesis 10.6.1(1.)–(5.).*

(1.) B is bounded (i.e., $B \in \mathcal{B}(U,H)$*);*

(2.) **(Analytic $\mathbb{A}$)** *Hypotheses 9.5.1 and 9.5.7 hold.*

(3.) $C = \left[\begin{smallmatrix} C_1 \\ 0 \end{smallmatrix}\right] \in \mathcal{B}(H, Y_1 \times Y_2)$*,* $\dim Y_1 < \infty$*, and* $\mathbb{B}$ *is stable;*

(4.) $\mathbb{A}B \in \mathrm{L}^1(\mathbf{R}_+;\mathcal{B}(U,H))$ *and* $C \in \mathcal{B}(H,Y)$*;*

(5.) $\mathbb{A}B \in \mathrm{L}^1([0,1];\mathcal{B}(U,H))$*,* $C \in \mathcal{B}(H,Y)$ *and* $\mathbb{A}$ *is exponentially stable;*

(6.) C is bounded, $D^*JC = 0$*, and* $\mathbb{D} \in \mathcal{B}(U,Y) + \mathcal{B}(U,\mathrm{L}^1(\mathbf{R}_+;Y))*$*;*

(7.) $\mathbb{D} \in \mathcal{B}(U,Y) + \mathcal{B}(U,\mathrm{L}^2(\mathbf{R}_+;Y))*$*;*

(8.) Hypothesis 9.2.1 (or 9.2.2) is satisfied and $D^*JD \gg 0$*;*

(9.) $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$*;*

(10.) **(Analytic $\mathbb{A}$)** *Hypothesis 9.5.1 holds and* $\mathbb{A}$ *is exponentially stable.*

(d) Assume that Σ *is stable and that at least one of conditions (1.)–(8.) above holds. Then* Σ *satisfies Hypothesis 10.6.1(1.)–(6.).*

Assumptions (1.)–(7.) of (c) are roughly the stable versions of (1.)–(7.) of Hypothesis 9.2.2.

Proof: (a1) Trivially, (5.)⇒(4.)⇒(3.), and (2.)⇒(1.) By Theorem 10.6.3(b)(ii)&(iii), (6.) implies (5.). By Proposition 6.3.1(b1), (3.) implies (2.).

(a2) Use (a1) and Lemma 6.3.2(a1)&(a2).

(b) This follows from Standing Hypothesis 10.6.6.

(c) 1° *Cases (9.) and (10.):* We have $\mathbb{D} \in \mathrm{MTIC}^{\mathrm{L}^1}$, by Lemma 9.5.2. Therefore, this follows from Theorem 8.4.9 (and $\mathbb{X} \in \mathrm{MTIC}^{\mathrm{L}^1} \subset$ UHPR).

2° *Cases (1.)–(8.) & (d):* (N.B. We do not know whether Hypothesis 9.2.2 is sufficient without the assumption $D^*JD \gg 0$. As shown below, $\mathbb{D}^*J\mathbb{D} \gg 0 \Rightarrow D^*JD \gg 0$ under any of (1.)–(8.).)

By Proposition 9.2.4, (3.) implies (7.), i.e., that $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U,Y))$. By Lemma 6.8.3(a), (5.) implies (4.). Therefore, any of (1.)–(8.) implies that Hypothesis 9.2.1 holds (see Hypothesis 9.2.2 and Theorem 9.2.3).

By Theorem 6.9.1(d2), we have $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}} \cap \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(U,Y))$ in (1.); the same holds in (7.), hence in (3.) too. In case (2.), we have $\mathbb{D} \in$ UHPR $\subset$ SHPR, by Lemma 9.6.3. In cases (4.), (5.) and (6.), we have $\mathbb{D} \in \mathrm{SMTIC}^{\mathrm{L}^1} \subset$ SHPR, by Theorem 2.6.4(h1).

Assume that $\mathbb{D}^*J\mathbb{D} \gg 0$, i.e., that $\mathbb{D}^*J\mathbb{D} \geq \varepsilon I$ for some $\varepsilon > 0$. Then $D^*JD \geq \varepsilon I$, by Lemma 6.3.5 (cases (1.), (3.) and (7.)) or by Lemma 6.3.6(b) (cases (2.), (4.), (5.) and (6.)) or by assumption (case (8.)). By Lemma 6.4.7(a), we have $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*\mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$, Therefore, there is a J-critical, strictly minimizing, stable, SOS-stabilizing state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ for Σ over $\mathcal{U}_{\rm out}$, by Corollary 9.9.11(Crit1SOS)&(Crit4SOS), and $\mathbb{F} = I - \mathbb{X}$, by (9.140).

Thus, we can apply Theorem 9.2.9(v)&(iii) to obtain that the $B_{\rm w}^*$-CARE has a solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ with $S = D^*JD \gg 0$, $\widetilde{\mathbb{X}} := I - \widetilde{\mathbb{F}} \in$ ULR and $\widetilde{X} = I$. By Theorem 9.9.1(a1)&(f2), we have $\widetilde{\mathbb{X}} = E\mathbb{X}$ for some $E \in \mathcal{GB}(U)$ (hence for $E = X^{-1}$). Consequently, $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}(U)$.

It follows that $\widetilde{\mathbb{N}} := \mathbb{D}\widetilde{\mathbb{X}}^{-1}$ is stable, hence $\widetilde{\mathbb{N}}^* J \widetilde{\mathbb{N}} = S$, by Theorem 9.9.1(g2), so that $\mathbb{D}^* J \mathbb{D} = \widetilde{\mathbb{X}}^* S \widetilde{\mathbb{X}}$. By Lemma 6.4.5(a), it follows that $S = (E^*)^{-1} I E^{-1} = X^*X$. Thus, $D^*JD = S = X^*X$, as required.

3° *Remarks:* The assumption that $\mathbb{C}$ is stable is superfluous (for (c)) in (4.), (5.), (9.) and (10.) and the assumption that $\mathbb{D}$ is stable is redundant in (4.), (5.), (6.), (9.) and (10.).

We note that $\mathbb{X} \in \mathcal{G}\mathrm{MTIC}^{\mathrm{L}^1}$ in cases (4.), (5.), (9.) and (10.) (by 1°), and $\mathbb{X}, \mathbb{X}^{-1} \in \mathcal{B} + \mathrm{H}^\infty \cap \mathrm{H}^2_{\rm strong}(\mathbf{C}^+; \mathcal{B}(U))$ in cases (1.), (3.) and (7.). The latter claim follows from Theorem 4.1.6(j), and from the fact that $\mathbb{D}$ has a stable realization with a bounded B, by Theorem 6.9.1(a)&(d2) and Corollary 6.9.7, so that we can obtain $\mathbb{X}$ for that realization instead of Σ (note that (1.)–(5.) of Hypothesis 10.6.1 depend on $\mathbb{D}$ and J only).

(d) Part "if" from the last claim follows from 2° above; and "only if" from Proposition 9.8.11. □

In fact, the solution of the $B_{\rm w}^*$-CARE in Hypothesis 10.6.1(6.) is actually stable and P-SOS-r.c.-stabilizing, and we have the classical equivalence between positive J-coercivity, I-spectral factorization and Riccati equations in this generality too:

Theorem 10.6.3 ($\mathbb{D}^* J \mathbb{D} \gg 0 \Leftrightarrow$ SpF $\Leftrightarrow$ CARE) *Assume that Σ is SOS and ULR.*

(a) If Hypothesis 10.6.1(6.) holds and Σ is strongly stable, then (i)–(iv') are equivalent (and (v) if Σ is exponentially stable).

(i) $\mathbb{D}^* J \mathbb{D} \gg 0$*;*

(i') $\mathbb{D}^* J \mathbb{D} \gg 0$ *and* $D^*JD \in \mathcal{GB}(U)$*;*

(ii) $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* S \mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $S \gg 0$*;*

(ii') $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* \mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$*;*

(ii") $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* \mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U)$*, and* $\mathbb{D}, \mathbb{X} \in$ ULR *and* $D^*JD = X^*X \gg 0$*;*

(iii) the $B_{\rm w}^*$-CARE (or CARE or IARE) has a stable P-I/O-stabilizing solution with $S \gg 0$;

(iii') the $B_{\rm w}^*$-CARE has a stable P-SOS-r.c.-stabilizing with $S \gg 0$;

(iv) the $B_{\rm w}^*$-CARE (or CARE or IARE) has a solution with $S \gg 0$ and $\mathbb{M} \in$ TIC*;*

(iv') the $B_{\rm w}^*$-CARE has a strongly stabilizing solution with $S \gg 0$;

(iv") the $B_{\rm w}^*$-CARE has a stable strongly r.c.-stabilizing solution with $S \gg 0$;

(v) the B_w^*-CARE has an exponentially stabilizing solution with $S \gg 0$.

(b) If Hypothesis 10.6.1(6.) holds, then (i)–(iii') are equivalent.

(c1) We have (iii')⇒(iii)⇒(ii')⇔(ii)⇔(i), and (iv)⇐(iv')⇐(iv")⇒(iii')⇒(ii")⇒(i')⇒(i).

(c2) If Σ is strongly stable, then (iii')⇔(iv')⇔(iv"), and (iii)⇔(iv).

(d) The solutions of the B_w^*-CARE mentioned in (a)–(b) are unique and equal, and they solve (ii) (and (ii') and (ii")) if we replace $\mathbb{X}$ by $S^{1/2}\mathbb{X}$).

(e) If Σ is strongly stable and the eIARE has a solution with $S \geq 0$, then $\mathbb{D}^* J\mathbb{D} \geq 0$.

(f1) **(CARE)** *Replace "(6.)" by "(1.)", and remove (i') and (ii") and "ULR", and replace "B_w^*-CARE" by "CARE", everywhere in this theorem. Then (a)–(e) still hold.*

(f2) **(General WPLSs)** *Remove (i'), (ii"), "ULR" and Hypothesis 10.6.1(6.), and replace "B_w^*-CARE" by "IARE", everywhere in this theorem. Then (a)–(e) still hold.*

The proposition provides us necessary and sufficient conditions for (i) (i.e., for the positive J-coercivity over $\mathcal{U}_{\text{out}}$) under different stability and regularity assumptions. Such conditions are needed for positive and bounded real lemmas and for minimization problems; the reader can find here additional equivalent conditions for those results under same or different assumptions.

Recall that $S := D^*JD$ for the B_w^*-CARE (but not necessarily for the CARE or IARE), and that any solution of the B_w^*-CARE (and any WR solution of the eCARE) is a solution of the eIARE.

In the unstable case, we have three alternatives for minimization: 1. If Σ is regular enough, we may use the B_w^*-CARE results of Section 9.2. 2. If Σ is stabilizable with closed-loop system $\Sigma_\flat$ as in Hypothesis 10.6.1(6.) (or (1.)), we may combine the above result and Proposition 9.12.4 (cf. Theorem 10.2.14(b1)&(b2)). 3. In the general case, we have to be satisfied with results such as Theorem 10.2.11 and Corollaries 10.2.5(a), 10.2.6 and 10.2.12.

Proof of Theorem 10.6.3: (a) Trivially, (v) implies (iv'); the converse (for exponentially stable Σ) follows from Corollary 6.6.9 (in fact, $\mathcal{P}$ is then exponentially stable and exponentially r.c.-stabilizing). The rest follows from (b) and (c2).

(b) 1° *(i)⇒(ii')&(iii'):* Assume (i), so that $X^*X \gg 0$ and the B_w^*-CARE has a P-I/O-stabilizing solution $(\mathcal{P}, S, K)$ s.t. $\mathbb{X}_K = (X^*X)^{-1/2}\mathbb{X} \in \mathcal{G}\text{TIC}(U)$, where $\widehat{\mathbb{X}}_K := I - K_w(\cdot - A)^{-1}B$, $\mathbb{X} \in \mathcal{G}\text{TIC}(U) \cap \text{ULR}$, $X^*X \gg 0$ and $\mathbb{X}^*\mathbb{X} = \mathbb{D}^*J\mathbb{D}$, by the hypothesis. Consequently, (iii') holds, by Proposition 9.8.11.

Since $\widehat{\mathbb{X}}_K \in \text{ULR}$, we have $\mathbb{X} \in \text{ULR}$ and hence $X \in \mathcal{GB}(U)$. From (9.162), we obtain that $\langle (X^*X)^{1/2}u, (X^*X)^{1/2}v\rangle = \langle u, Sv\rangle$ for all $u, v \in \mathrm{L}^2_c$ (since $\mathbb{D}_\circlearrowleft = \mathbb{D}\mathbb{X}_K^{-1} = \mathbb{D}\mathbb{X}^{-1}(X^*X)^{1/2}$), hence $X^*X = S$; in particular, $S \gg 0$. But $S = D^*JD$, hence $D^*JD = X^*X \gg 0$, so that (ii') holds.

2° The rest of the equivalence follows from (c1).

(c1) Implications "(iii')⇐(iv")⇒(iv')⇒(iv)", "(iii')⇒(iii)", "(ii')⇒(ii)", and "(i')⇒(i)" are trivial, and "(ii')⇒(i')" is obvious. Equivalence "(i)⇔(ii)" follows from Lemma 6.4.7(a), We obtain "(iii)⇒(ii)" from Proposition 9.8.11 (in particular, a stable P-$\mathbb{M}$-stabilizing solution suffices) (and Proposition 9.2.7(b)).

Finally, we obtain "(iii')⇒(ii')" from Proposition 9.2.7(b) and Proposition 9.8.11(c), by replacing $\mathbb{X}$ by $S^{1/2}\mathbb{X}$.

(c2) This follows from Proposition 9.8.11(b).

(d) This follows from the above proofs (especially of that of (c1)).

(e) By (9.160), we have $\langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle \ge -\langle \mathbb{B}^t u, \mathcal{P}\mathbb{B}^t u\rangle \to 0$, as $t \to +\infty$, because $\mathbb{B}$ is strongly stable (by Lemma 6.1.13), hence $\mathbb{D}^* J\mathbb{D} \ge 0$. (Note that whenever $S \ge 0$, $\mathcal{P} \le 0$, we have $\langle \mathbb{D}^t u, J\mathbb{D}^t u\rangle \ge 0$ for all t, so that $\mathbb{D}^* J\mathbb{D} \ge 0$ if $\mathbb{D}$ is stable.)

(f1) This follows as above (or from (f2)) (note from Proposition 10.7.2 that any solution of the CARE with $S \gg 0$ is a WR solution of the IARE and from Proposition 10.7.1 that such a solution is stable if Σ is).

(f2) Also this follows as above. □

For a SOS-stable system $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$ with sufficient regularity, condition (ii) below implies that $\mathbb{D}^* J\mathbb{D} \ge 0$, and, conversely, $\mathbb{D}^* J\mathbb{D} \gg 0$ implies that (ii) holds (by (b) below). For several applications, such as the "strict bounded real lemma" of Proposition 10.5.2, this "almost equivalence" is in practice as good as an equivalence.

Proposition 10.6.4 ($\mathbb{D}^* J\mathbb{D} \ge 0$ ⇔ CARI) *Assume that $\mathbb{D}$ is SR and $C^*JC \le 0$. Then we have (ii)⇒(iii)⇒(i)⇐(iv) for the following conditions:*

(i) $\mathbb{D}^{t\,*} J\mathbb{D}^t \ge 0$ *for all* $t \ge 0$.

(ii) $D^*JD \gg 0$, *and there is* $\mathcal{P} \le 0$ *s.t.* $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ *and*

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A + C^*JC & (B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^* \\ B_{\mathrm{w}}^*\mathcal{P} + D^*JC & D^*JD \end{bmatrix} \ge 0 \quad on \quad \mathrm{Dom}(A) \times U. \tag{10.77}$$

(iii) There is $\mathcal{P} \le 0$ *s.t.* $S := D^*JD + \text{s-lim}_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B \gg 0$, *and*

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A + C^*JC & (B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^* \\ B_{\mathrm{w}}^*\mathcal{P} + D^*JC & S \end{bmatrix} \ge 0 \quad on \quad \mathrm{Dom}(A) \times U. \tag{10.78}$$

(iv) $\mathbb{D} \in \mathrm{TIC}$ *and* $\mathbb{D}^* J\mathbb{D} \ge 0$.

Moreover, the following hold:

(a) If $\mathbb{D} \in \mathrm{TIC}$, *then we have (i)⇔(iv).*

(b) If $\Sigma \in \mathrm{SOS}$, $\mathbb{D}^* J\mathbb{D} \gg 0$, *and (2.) (resp. (6.)) of Hypothesis 10.6.1 holds, then (i), (iii) and (iv) (resp. (i)–(iv)) hold; in fact, we can have equality in (10.81) (resp. in (10.79)).*

(c) We have (ii)⇔(ii')⇒(iii)⇔(iii') (even for a fixed $\mathcal{P}$), where

(ii') $D^*JD \gg 0$, *and there is* $\mathcal{P} \le 0$ *s.t.* $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ *and*

$$(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^*(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) \le A^*\mathcal{P} + \mathcal{P}A + C^*JC. \tag{10.79}$$

(iii') There is $\mathcal{P} \le 0$ *s.t.*

$$S := D^*JD + \operatorname*{s-lim}_{\alpha\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(\alpha - A)^{-1}B \gg 0, \quad \textit{and} \tag{10.80}$$

$$(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^*S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) \le A^*\mathcal{P} + \mathcal{P}A + C^*JC. \tag{10.81}$$

(d) If $\mathbb{D} \in \mathrm{TIC}$ *and* $\mathbb{B}$ *is strongly stable, then we can replace "*$\mathcal{P} \le 0$*" by* $\mathcal{P} = \mathcal{P}^*$ *everywhere in this proposition.*

(The comments below Theorem 10.5.1 apply here too, mutatis mutandis. See Theorem 10.6.3 for more on (b).)

Proof: (Naturally, condition "$C^*JC \le 0$" means that $\langle x_0, C^*JCx_0\rangle \le 0$ for all $x_0 \in \mathrm{Dom}(A)$. In the proof below, the proofs of (c) and (a) come logically first.)

We get "(ii)$\Rightarrow$(iii)" as in the proof of Proposition 9.2.7(a). Implication "(iv)$\Rightarrow$(i)" follows from (a).

To complete the first claim, assume (iii) (i.e., (iii')). By Proposition 9.11.9(e)&(c), there is $\mathbb{X} \in \mathrm{TIC}_\infty(U)$ s.t. $\mathbb{X}^{t\,*}S\mathbb{X}^t \le \mathbb{D}^{t\,*}J\mathbb{D}^t + \mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t$ for all $t \ge 0$, hence $\mathbb{D}^{t\,*}J\mathbb{D}^t \ge 0$. Thus, (iii) implies (i).

(a) Since $C^*JC \le 0$, we have $C_{\mathrm{s}}^*JC_{\mathrm{s}} \le 0$ on $\mathrm{Dom}(C_{\mathrm{s}}) \supset H_B$ (because $\langle C_{\mathrm{s}}x_0, JC_{\mathrm{s}}x_0\rangle_Y = \lim_{s\to+\infty}\langle Cs(s-A)^{-1}x_0, JCs(s-A)^{-1}x_0\rangle_Y \le 0$ for all $x_0 \in \mathrm{Dom}(C_{\mathrm{s}})$). Therefore, for any $t \ge 0$ and $u \in \mathrm{L}^2([0,t);U)$ we have $\int_0^\infty = \int_0^t + \int_t^\infty$, i.e.,

$$\langle \mathbb{D}u, J\mathbb{D}u\rangle_{\mathrm{L}^2} = \langle u, \mathbb{D}^{t\,*}J\mathbb{D}^t u\rangle_{\mathrm{L}^2} + \int_t^\infty \langle C_{\mathrm{s}}\mathbb{B}\tau^r u, JC_{\mathrm{s}}\mathbb{B}\tau^r u\rangle\, dr \le \langle u, \mathbb{D}^{t\,*}J\mathbb{D}^t u\rangle_{\mathrm{L}^2}, \tag{10.82}$$

by Theorem 6.2.13(a2). Consequently, $\mathbb{D}^*J\mathbb{D} \ge 0$ implies that (i) holds. The converse is obvious (use Corollary B.3.8).

(b) Obviously, (iv) holds, hence so does (i), by (a). By (b)&(f1)&(i)&(iii') of Theorem 10.6.3, condition (iii') (resp. (ii')) above holds (with equality in (10.81) (resp. in (10.79))) if we can show that $\mathcal{P} \le 0$. By (c), then the rest of (b) holds too.

Condition $C^*JC \le 0$ implies that $\langle \mathbb{C}x_0, J\mathbb{C}x_0\rangle_{\mathrm{L}^2} = \int_0^\infty \langle \mathbb{A}^t x_0, C^*JC\mathbb{A}^t x_0\rangle\, dt \le 0$ for all $x_0 \in \mathrm{Dom}(A)$, hence $\mathbb{C}^*J\mathbb{C} \le 0$, by density. Since $S \gg 0$, it follows that $\mathbb{C}^*J\mathbb{C} - \mathbb{K}^*S\mathbb{K} \le 0$, hence $\mathcal{P} \le 0$, by (9.142).

(Note that the $\mathcal{P}$ above is P-SOS-r.c.-stabilizing. We could use (1.) instead of (2.) of Hypothesis 10.6.1 if we would have "w-lim" in (10.80).)

(c) Implication "(ii)$\Rightarrow$(iii)" was shown above. If $\mathrm{Dom}(A) = H$ (i.e., if A is bounded), then the equivalences follow from Lemma A.3.1(p2) (with columns and rows interchanged); even general A, the proof of Lemma A.3.1(p2) applies.

(d) The above proofs still apply except that the proof of "(iii)$\Rightarrow$(i)" must be altered as follows: Assume (iii). When $\mathbb{B}^t$ is strongly stable, the inequality $\mathbb{X}^{t\,*}S\mathbb{X}^t \le \mathbb{D}^{t\,*}J\mathbb{D}^t + \mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t$ ($t \ge 0$) implies that $\mathbb{D}^{t\,*}J\mathbb{D}^t \ge -\mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t \to 0$, as $t \to +\infty$, i.e., that $\mathbb{D}^*J\mathbb{D} \ge 0$. By (a), this is equivalent to (i). □

Next we show that, under sufficient regularity, the uniform Riccati inequality has a solution iff Σ is exponentially stable and the Popov operator is uniformly positive:

Theorem 10.6.5 ($\mathbb{D}^* J \mathbb{D} \gg 0 \Leftrightarrow$ CARI) *Assume that $C^* JC \leq 0$.*

(a) If any of Hypothesis 9.2.2(1.)–(6.) holds (the references to Theorem 8.3.9 may be ignored), then the following are equivalent:

(i) Σ is exponentially stable and $\mathbb{D}^* J \mathbb{D} \gg 0$.

(ii) There is $\mathcal{P} \leq 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A + C^*JC & (B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^* \\ B_{\mathrm{w}}^*\mathcal{P} + D^*JC & D^*JD \end{bmatrix} \gg 0 \quad on \quad \mathrm{Dom}(A) \times U. \tag{10.83}$$

(iii) There is $\mathcal{P} \leq 0$ s.t. $S := D^*JD + \text{s-lim}_{s \to +\infty} B_{\mathrm{w}}^* \mathcal{P}(s-A)^{-1}B$ exists, and

$$\begin{bmatrix} A^*\mathcal{P} + \mathcal{P}A + C^*JC & (B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^* \\ B_{\mathrm{w}}^*\mathcal{P} + D^*JC & S \end{bmatrix} \gg 0 \quad on \quad \mathrm{Dom}(A) \times U. \tag{10.84}$$

(b) If $\pi_{[0,t)} \mathbb{A} B \in \mathrm{L}^1([0,t);\mathcal{B}(U,H))$, $\pi_{[0,t)} C_{\mathrm{w}} \mathbb{A} \in \mathrm{L}^1([0,t);\mathcal{B}(H,Y))$, and $\pi_{[0,t)} C_{\mathrm{w}} \mathbb{A} B \in \mathrm{L}^1([0,t);\mathcal{B}(U,Y))$ for some $t > 0$, then (i)$\Leftrightarrow$(iii)$\Leftarrow$(ii).

(c1) If $\mathbb{D}$ is ULR, then we have (i)$\Leftarrow$(iii)$\Leftarrow$(ii).

(c2) Any solution of (ii), (iii), (ii') or (iii') is strictly negative ($\mathcal{P} < 0$). Under the assumptions of (a), there is an exponentially stabilizing solution (if (i) holds).

(d) If $\mathbb{D}$ is SR, then we have (ii)$\Leftrightarrow$(ii')$\Rightarrow$(iii)$\Leftrightarrow$(iii') (even for a fixed $\mathcal{P}$), where

(ii') $D^*JD \gg 0$, and there is $\mathcal{P} \leq 0$ s.t. $\mathcal{P}[H] \subset \mathrm{Dom}(B_{\mathrm{w}}^*)$ and

$$(B_{\mathrm{w}}^*\mathcal{P} + D^*JC)^*(D^*JD)^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) \ll A^*\mathcal{P} + \mathcal{P}A + C^*JC. \tag{10.85}$$

(iii') There is $\mathcal{P} \leq 0$ s.t.

$$\begin{cases} K^*SK \ll A^*\mathcal{P} + \mathcal{P}A + C^*JC & \text{on } \mathrm{Dom}(A)\ , \\ \quad S = D^*JD + \operatorname*{s-lim}_{s \to +\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B & \text{on } U\ , \\ \quad K = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) & \text{on } \mathrm{Dom}(A)\ . \end{cases} \tag{10.86}$$

and $S \gg 0$ (for some S and K).

(e) We can replace the assumptions of (a) (resp. of (b)) by any assumption that together with (i) leads to Hypothesis 10.6.1(6.) (resp. (2.)) (or to $D^*JD \gg 0$ and Hypothesis 9.2.1) for $\widetilde{\Sigma}$ and $\widetilde{J}$ (see the proof), for all sufficiently small $\varepsilon > 0$.

One such assumption is that $C \in \mathcal{B}(H,Y)$ and $\dim Y < \infty$.

See also the comments below Theorem 10.5.1. The case where "$\gg$" is replaced by "$\geq$" is covered by Proposition 10.6.4.

Proof: (Naturally, condition "$C^*JC \leq 0$" means that $\langle x_0, C^*JCx_0 \rangle \leq 0$ for all $x_0 \in \mathrm{Dom}(A)$.) Let $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$.

We shall use (d) tacitly throughout the proof.

1° *"(i)⇒(ii)&(iii)" under Hypothesis 9.2.2:* Assume (i) and Hypothesis 9.2.2. Then $\mathbb{L} := \mathbb{B}\tau \in \mathrm{TIC}(U,H)$, by Lemma 6.1.10, so that $\mathbb{D}^*J\mathbb{D} - \varepsilon\mathbb{L}^*\mathbb{L} \gg 0$ for some $\varepsilon > 0$. Define $\widetilde{\Sigma} := \left(\begin{array}{c|c} A & B \\ \hline \widetilde{C} & \widetilde{D}\end{array}\right) \in \mathrm{WPLS}(U,H,Y\times U)$ by $\widetilde{C} := \left[\begin{smallmatrix} C \\ I \end{smallmatrix}\right]$, $\widetilde{D} := \left[\begin{smallmatrix} D \\ 0 \end{smallmatrix}\right]$, and set $\widetilde{J} := \left[\begin{smallmatrix} J & 0 \\ 0 & -\varepsilon I \end{smallmatrix}\right]$.

Then also $\widetilde{\Sigma}$ satisfies Hypothesis 9.2.2 (including the requirements of Theorem 8.3.9(b2)). Since $\widetilde{\mathbb{D}}^*\widetilde{J}\widetilde{\mathbb{D}} = \mathbb{D}^*J\mathbb{D} - \varepsilon\mathbb{L}^*\mathbb{L} \gg 0$, the B_{w}^*-CARE, the IARE and the CARE for $\widetilde{\Sigma}$ and $\widetilde{J}$ have an exponentially stabilizing solution $(\mathcal{P}, S, K)$ with $S = D^*JD \in \mathcal{GB}(U)$, by Theorem 9.2.9(i) and Proposition 8.3.10.

But the CARE for $\widetilde{\Sigma}$ and $\widetilde{J}$ equals (10.86) with "$= -\varepsilon I +$" in place of "$\ll$". Therefore, we have established (ii') and (iii') once we show that $\mathcal{P} \le 0$ and $S \ge 0$.

By Theorem 9.9.1(a2), $S \ge 0$, hence $S \gg 0$. Condition $C^*JC \le 0$ implies that $\langle \mathbb{C}x_0, J\mathbb{C}x_0\rangle_{\mathrm{L}^2} = \int_0^\infty \langle \mathbb{A}^t x_0, C^*JC\mathbb{A}^t x_0\rangle\, dt \le 0$ for all $x_0 \in \mathrm{Dom}(A)$, hence $\mathbb{C}^*J\mathbb{C} \le 0$, by density. Since $\widetilde{\mathbb{C}}^*\widetilde{J}\widetilde{\mathbb{C}} = \mathbb{C}^*J\mathbb{C} - \varepsilon\mathbb{J}^*\mathbb{J} < 0$, where $(\mathbb{J}x_0)(t) := \mathbb{A}^t x_0$ $(t \ge 0)$, we obtain from (9.142) that $\mathcal{P} = \widetilde{\mathbb{C}}^*\widetilde{J}\widetilde{\mathbb{C}} - \mathbb{K}^*S\mathbb{K} < 0$, as required (recall that $S \gg 0$).

Remark: $\mathcal{P} < 0$ and $\mathcal{P}$ is exponentially stabilizing for Σ when (i) holds (since $\mathcal{P}$ is exponentially stabilizing for $\widetilde{\Sigma}$, hence for (A,B) too). (When the assumptions of (a) are satisfied and (i) holds, there is one such solution; however, the inequalities (ii) and (iii) might have non-stabilizing solutions too.)

2° *"(ii)⇒(iii)" (whenever $\mathbb{D}$ is SR):* This follows from the proof of Proposition 9.2.7(a).

3° *"(iii)⇒(i)" (whenever $\mathbb{D} \in$ ULR):* By Proposition 9.11.9(e), $\left[\begin{array}{c|c} A & B \\ \hline K & 0\end{array}\right]$ generate a SR WPLS. Since $A^*(-\mathcal{P}) + (-\mathcal{P})A \ll 0$ and $-\mathcal{P} \ge 0$, the semigroup $\mathbb{A}$ is exponentially stable (and $-\mathcal{P} > 0$), by Lemma 9.12.2(d).

By Proposition 9.11.9(d), we have $\mathbb{X}^{t*}S\mathbb{X}^t + \varepsilon\mathbb{L}^{t*}\mathbb{L}^t \le \mathbb{D}^{t*}J\mathbb{D}^t + \mathbb{B}^{t*}\mathcal{P}\mathbb{B}^t$ for some $\varepsilon > 0$, where $\mathbb{L} := \mathbb{B}\tau \in \mathrm{TIC}_\infty(H,U)$. Since $S \gg 0$ and $\mathcal{P} \le 0$, we have $\varepsilon\mathbb{L}^{t*}\mathbb{L}^t \le \mathbb{D}^{t*}J\mathbb{D}^t$, hence $\mathbb{D}^*J\mathbb{D} \ge \varepsilon\mathbb{L}^*\mathbb{L}$, hence $\mathbb{D}$ is positively J-coercive, by Proposition 10.3.2(g1) (with $D_{\mathrm{c}} = D$; here we need the assumption that $\mathbb{D}$ is ULR); in particular $\mathbb{D}^*J\mathbb{D} \ge \varepsilon I$.

4° *Remarks:* From 1° we observe that the implication "(i)⇒(ii)" (resp. "(i)⇒(iii)") also holds under a suitable variant of (6.) (resp. of (2.)) or Hypothesis 10.6.1.

(a) Since Hypothesis 9.2.2 implies that $\mathbb{D}$ is ULR, the equivalence follows from 1°–3°.

(b) By Lemma 6.8.5, $\mathbb{D}$ is ULR, hence we have (ii)⇒(iii)⇒(i). If (i) holds, then $\mathbb{D}, \mathbb{L} \in \mathrm{MTIC}^{\mathrm{L}^1}$ (hence $\widetilde{\mathbb{D}} \in \mathrm{MTIC}^{\mathrm{L}^1}$), by Lemma 6.8.5(a), hence then we obtain "(i)⇒(iii)" from 1° with Theorem 9.2.14(c3) in place of Theorem 9.2.9(i).

(c1) This follows from 2° and 3°.

(c2) This was observed in 3°.

(d) Use the proof of Lemma A.3.1(p4) (with rows and columns interchanged) for "(ii)⇔(ii')" and for "(iii)⇔(iii')", and 2° for "(ii)⇒(iii)".

(e) (Note that $\mathbb{D} \in \widetilde{\mathcal{A}}_+$ is not sufficient, since $\widetilde{\mathbb{D}}$ also contains $\mathbb{B}\tau$.)

1° If Σ is exponentially stable, $C \in \mathcal{B}(B,Y)$ and $\dim Y < \infty$, then $\widetilde{\Sigma}$ and $\widetilde{J}$ satisfy Hypothesis 10.6.1(6.) (and hence (2.)) for all $\varepsilon > 0$, by Lemma 10.6.2(d)(6.).

2° If $D^*JD \gg 0$, (i) holds, and Hypothesis 9.2.1 holds for $\widetilde{\Sigma}$ and $\widetilde{J}$, then Hypothesis 10.6.1(6.) is satisfied by $\widetilde{\Sigma}$ and $\widetilde{J}$ for $\varepsilon > 0$ s.t. $D^*JD - \varepsilon I \gg 0$, by Lemma 10.6.2(d)(8.).

3° Under Hypothesis 10.6.1(6.) (resp. (2.)), assumption (i) still implies that the B_{w}^*-CARE (resp. the CARE) for $\widetilde{\Sigma}$ and $\widetilde{J}$ has an exponentially stabilizing solution (cf. 1° (resp. the proof of (b))). Therefore, 1° (resp. the proof of (b)) still applies; the rest follows from (c1). □

Throughout this chapter, $\widetilde{\mathcal{A}}_+$ is assumed to be MTIC or something almost as regular:

Standing Hypothesis 10.6.6 (Class $\widetilde{\mathcal{A}}_+$ is ULR and admits positive SpF)

(1.) $\mathcal{B} \subset \widetilde{\mathcal{A}}_+ \underset{a}{\subset} \mathrm{TIC} \cap \mathrm{ULR}$ *(see Definition 6.2.4), and*

(2.) if $\mathbb{D} \in \widetilde{\mathcal{A}}_+(U,Y)$, $J = J^* \in \mathcal{B}(Y)$, *and* $\mathbb{D}^*J\mathbb{D} \gg 0$, *then* $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_+(U)$.

Thus, we have weakened Hypothesis 8.4.7 to only cover the positive case; consequently, all MTIC classes are now applicable without dimensionality districtions (see (a1) below):

Lemma 10.6.7 ($\widetilde{\mathcal{A}}_+$)

(a1) All classes listed in Theorem 8.4.9 and their exponentially stable versions (see "$\mathcal{A}_{\exp}$" in the Theorem) satisfy Standing Hypothesis 10.6.6.

(a2) Hypothesis 8.4.7 is stronger than Standing Hypothesis 10.6.6.

(a3) The class TIC *satisfies (2.) of Standing Hypothesis 10.6.6.*

(b) Let $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *be a q.r.c.f. with* $\mathbb{N}, \mathbb{M} \in \widetilde{\mathcal{A}}_+$. *Then the following are equivalent*

(i) $\mathbb{N}^*J\mathbb{N} \gg 0$*;*

(ii) $\mathbb{N}^*J\mathbb{N} = \mathbb{X}^*\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_+$*;*

(iii) $\mathbb{D}$ *has a* (J,I)*-inner q.r.c.f.* $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1}$.

Moreover, we have $\mathbb{N}', \mathbb{M}' \in \widetilde{\mathcal{A}}_+$ *and* $M' \in \mathcal{G}\mathcal{B}(U)$ *for any* $(J,*)$*-inner q.r.c.f.* $\mathbb{D} = \mathbb{N}'\mathbb{M}'^{-1}$ *of* $\mathbb{D}$.

Proof: (a1) By Theorem 5.2.8, we may now allow for $\dim U = \infty$ in (β) too; the rest follows from (a2) and Theorem 8.4.9.

(a2) Assume that $\mathbb{D}^*J\mathbb{D} \gg 0$, i.e., that $\pi_+\mathbb{D}^*J\mathbb{D}\pi_+ \gg 0$ (see Lemma 6.4.6), so that $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$ for some $S \in \mathcal{G}\mathcal{B}(U)$. Then $S \geq 0$, hence $S \gg 0$. Replace $\mathbb{X}$ by $S^{1/2}\mathbb{X}$ to satisfy Standing Hypothesis 10.6.6.

(a3) This is Lemma 6.4.7(a).

(b) This follows as in the proof of Corollary 8.4.14 (see also Lemma 8.4.11). $\square$

(See the notes on p. 595.)

10.7 Positive Riccati equations ($\mathcal{J}(0,\cdot) \geq 0$)

I made it a rule to forbear all direct contradictions to the sentiments of others, and all positive assertion of my own. I even forbade myself the use of every word or expression in the language that imported a fixed opinion, such as "certainly", "undoubtedly", etc. I adopted instead of them "I conceive", "I apprehend", or "I imagine" a thing to be so or so; or "so it appears to me at present".

— Autobiography of Benjamin Franklin (1706–1790)

In this section, we shall give additional auxiliary results for Riccati equations in the positive (minimization) case, where $\mathcal{J}(0,\cdot) \geq 0$ (e.g., $J \geq 0$). Recall from Lemma 10.2.2, that $\mathcal{J}(0,\cdot) \geq 0$ implies that a control is minimizing iff it is J-critical.

Such a solution is of state feedback form iff it corresponds to a $\mathcal{U}_*^*$-stabilizing solution of the eIARE. Since such a solution is necessarily nonnegative, we are only interested in nonnegative solutions of the eIARE (or of the eCARE) in this section.

In Proposition 10.7.2 (resp. 10.7.1) we show that a solution of the CARE (resp. the IARE for a stable system) with $S \gg 0$ is WR (resp. well-posed and stable). Analogous results for Riccati inequalities were given in Proposition 9.11.9.

In the two latter propositions we show that for standard LQR cost functions, solutions of the IARE or CARE are well-posed, admissible and stabilizing provided that certain additional assumptions hold.

Proposition 10.7.1 ($S \gg 0$) *Let Σ be [strongly] stable. Assume that the IARE for Σ and J has a solution $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix})$ s.t. $S \gg 0$.*

Then $\mathbb{K}$ and $\mathbb{F}$ are stable, and $\mathbb{X}^* S \mathbb{X} = \mathbb{D}^* J \mathbb{D} + \text{s-lim}_{t\to+\infty} \tau(-t)\mathbb{B}^* \mathcal{P} \mathbb{B} \tau(t)$ [$\mathbb{X}^* S \mathbb{X} = \mathbb{D}^* J \mathbb{D}$].

Proof: By Lemma 9.10.1(b4), the eIARE implies that (9.153)–(9.161) hold. The stability of $\mathbb{X}$ follows from (9.160) as that of $\mathbb{N}$ in Proposition 10.7.3(a); similarly, we obtain the stability of $\mathbb{K}$ from (9.159). The claim on $\mathbb{X}^* S \mathbb{X}$ follows from Proposition 9.12.7(a). $\square$

We recall from Proposition 9.11.8 that a solution of the CARE with $S \gg 0$ is WR:

Proposition 10.7.2 ($S \gg 0$) *Let Σ be WR. If the CARE has a solution $\mathcal{P}$ with $S \gg 0$, then $\mathcal{P}$ is a WR solution of the IARE.* $\square$

For $J \gg 0$, any admissible nonnegative solution is at least $\begin{bmatrix} \mathbb{C} & \mathbb{D} \end{bmatrix}$-stabilizing; for the standard LQR cost function with $C^*C \gg 0$, such a solution is exponentially stabilizing:

Proposition 10.7.3 ($J \gg 0$) *Assume that the eIARE for Σ and $J \gg 0$ has an admissible solution $\mathcal{P} \geq 0$. Then (a)–(c3) hold:*

(a) The maps $\mathbb{C}_\circlearrowleft$ *and* $\mathbb{D}_\circlearrowleft$ *are stable and* $S \ge 0$.

(b) Conversely, any minimizing solution is nonnegative.

(c1) If $\mathbb{C} = \begin{bmatrix} \mathbb{C}_1 \\ 0 \end{bmatrix}$, $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 \\ I \end{bmatrix}$, *and* $J = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \gg 0$, *then (10.87) is satisfied.*

(c2) If $\mathbb{D} \in \mathrm{SR}$ *and* $\begin{bmatrix} C & D \end{bmatrix}^* J \begin{bmatrix} C & D \end{bmatrix} \ge \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ *on* $H_1 \times U$ *for some* $\varepsilon > 0$, *then (10.87) is satisfied.*

(c3) If $C \in \mathcal{B}(H,Y)$, $C^*C \gg 0$ *and* $\begin{bmatrix} C & D \end{bmatrix}^* J \begin{bmatrix} C & D \end{bmatrix} \ge \varepsilon \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$ *on* $H_1 \times U$ *for some* $\varepsilon > 0$, *then (10.87) is satisfied and* Σ *is estimatable, hence then (d3) applies, and* $\mathcal{P}$ *is the unique nonnegative admissible solution of the eIARE and minimizing over* $\mathcal{U}_{\mathrm{exp}}$ *(and* $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ *and* $\mathcal{U}_{\mathrm{str}}$*) and exponentially r.c.-stabilizing.*

Assume, in addition, that there is $\varepsilon > 0$ *s.t.*

$$\langle \mathbb{C}x_0 + \mathbb{D}u, J(\mathbb{C}x_0 + \mathbb{D}u)\rangle_{\mathrm{L}^2(\mathbf{R}_+;Y)} \ge \varepsilon \|u\|_2^2 \quad (x_0 \in H,\ u \in \mathrm{L}^2_{+,\infty}). \tag{10.87}$$

(d) Then $\mathbb{D}$ *is positively* J*-coercive over* $\mathcal{U}_{\mathrm{out}}$, $S \gg 0$, $\mathcal{P}$ *is SOS-stabilizing, and there is a unique minimizing control over* $\mathcal{U}_{\mathrm{out}}$ *for each* $x_0 \in H$. *Moreover, (d1)–(d6) hold:*

(d1) If the minimizing control over $\mathcal{U}_{\mathrm{out}}$ *is given by some state feedback pair* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, *then* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is the pair (modulo* $E \in \mathcal{GB}(U)$*) determined by the smallest nonnegative admissible solution of the eIARE.*

(d2) Assume that Σ *is [strongly [exponentially]] stable.*

Then so are Σ_{ext} *and* $\Sigma_\circlearrowleft$. *In particular, then* $\mathcal{P}$ *is stable, [strongly [exponentially]] r.c.-stabilizing [and strictly minimizing over* $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ *and* $\mathcal{U}_{\mathrm{str}}$ *[and* $\mathcal{U}_{\mathrm{exp}}$*], and* $\mathcal{P}$ *is the unique nonnegative admissible solution].*

(d3) Assume that Σ *is estimatable or exponentially q.r.c.-stabilizable. Then* $\mathcal{P}$ *is the unique nonnegative admissible solution, and it is exponentially q.r.c.-stabilizing and strictly minimizing over* $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$, $\mathcal{U}_{\mathrm{str}}$ *and* $\mathcal{U}_{\mathrm{exp}}$.

Thus, in the case described in (c3), we only have to find one nonnegative solution and check whether it is exponentially stabilizing. If not, then there is no minimizing state feedback pair over any of $\mathcal{U}_{\mathrm{out}}$–$\mathcal{U}_{\mathrm{exp}}$.

The assumption in (c1) is equivalent to having the cost function $\mathcal{J}(x_0,u) = \langle u, Ru\rangle + \langle y_1, Qy_1\rangle$, where $y_1 = \mathbb{C}_1 x_0 + \mathbb{D}_1 u$, $R \gg 0$, $Q \gg 0$ (note that here $y = \begin{bmatrix} y_1 \\ u \end{bmatrix}$). See also Theorem 9.2.10 and Corollary 15.1.6.

Although the assumptions in (d) or (c) (and the existence of $\mathcal{P}$) imply that there is a unique minimizing control over $\mathcal{U}_{\mathrm{out}}$, we do not know in general whether such a control can be given in the feedback form (not even whether there is a minimal control among such controls), neither whether a smallest solution would be cost-minimizing. In discrete time we have no such problems, see, e.g., Corollary 15.1.6 (neither in continuous time when, e.g., B is bounded; see Theorems 9.9.6, 9.2.10 and 10.1.4(b4)&(b6)).

Proof of Proposition 10.7.3: (a) 1° $\mathbb{N} := \mathbb{D}_{\circlearrowleft}$ *is stable and* $S \geq 0$*:* For all $u \in \pi_+ \mathrm{L}^2$, $t > 0$, we have

$$0 \leq \langle \mathbb{N}^t u, J\mathbb{N}^t u\rangle \leq \langle u, Su\rangle \leq \|S\|\|u\|_2^2, \tag{10.88}$$

by (9.157), hence $\|J^{1/2}\mathbb{N}^t u\|_2^2 \leq \|S\|\|u\|_2^2$. But $\|\mathbb{N}^t u\|_2^2 \leq M^2\|J^{1/2}\mathbb{N}^t u\|_2^2$ for some $M < \infty$, hence $\|\mathbb{N}\| \leq M\|S\|^{1/2} < \infty$, i.e., $\mathbb{N} = \mathbb{D}_{\circlearrowleft}$ is stable, by Lemma 6.1.12. From (10.88) we also observe that $S \geq 0$ (take $u = u_0\chi_{[0,1)}$).

2° $\mathbb{C}_{\circlearrowleft}$ *is stable:* By (9.155) and the Monotone Convergence Theorem, $\|J^{1/2}\mathbb{C}_{\circlearrowleft}x_0\|_2 \leq \langle x_0, \mathcal{P}x_0\rangle$ ($x_0 \in H$), hence $\mathbb{C}_{\circlearrowleft}$ is stable.

(b) This follows from equation $\mathcal{P} = \mathbb{C}_{\circlearrowleft}^* J\mathbb{C}_{\circlearrowleft}$ (see Theorem 9.9.1(a2)&(g2)). (In fact, any $\mathbb{C}$-P-stabilizing solution is nonnegative, by Lemma 9.10.1(d1).)

(c1) Now $\mathcal{J}(x_0, u) \geq \langle u, Ru\rangle \geq \varepsilon^2\|u\|_2^2$ for some $\varepsilon^2 > 0$.

(c2) Now $y := \mathbb{C}x_0 + \mathbb{D}u = C_{\mathrm{s}}x + Du$ a.e., hence $\langle y, Jy\rangle \geq \varepsilon\|u\|_2^2$, where $x := \mathbb{A}x_0 + \mathbb{B}\tau u$ (note that $\langle \begin{bmatrix} C_{\mathrm{s}} & D\end{bmatrix}\left[\begin{smallmatrix} x_0 \\ u_0\end{smallmatrix}\right], J\begin{bmatrix} C_{\mathrm{s}} & D\end{bmatrix}\left[\begin{smallmatrix} x_0 \\ u_0\end{smallmatrix}\right]\rangle \geq \varepsilon\|u_0\|_U^2$ can be extended to $\mathrm{Dom}(C_{\mathrm{s}}) \times U$, by first replacing x_0 by $r(r-A)^{-1}x_0$, and then letting $r \to +\infty$).

(c3) Because $C^*C \gg 0$, we have $(C^*C)^{-1}C^*C = I$, hence Σ is estimatable by a bounded $\mathtt{H}$, by Lemma 6.6.25. Consequently, (d3) applies (by (c2)). By Lemma 6.6.26, $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}\end{array}\right]$ is exponentially r.c.-stabilizing (jointly with $\mathtt{H}$).

(d) (Naturally, we allow the value $+\infty$ for the norms.)

1° $\mathcal{P}$ *is SOS-stabilizing:* Let $u_{\circlearrowleft} \in \mathrm{L}^2(\mathbf{R}_+; U)$, $x_0 \in H$. Set

$$u := \mathbb{K}_{\circlearrowleft}x_0 + \mathbb{M}u_{\circlearrowleft}, \quad y := \mathbb{C}x_0 + \mathbb{D}u = \mathbb{C}_{\circlearrowleft}x_0 + \mathbb{D}_{\circlearrowleft}u_{\circlearrowleft}. \tag{10.89}$$

By (a), y is stable, hence so is u. Consequently, $\mathbb{K}_{\circlearrowleft}$ and $\mathbb{M}$ are stable, by Lemma 6.1.12. Thus, $\Sigma_{\circlearrowleft} \in \mathrm{SOS}$.

2° $S \gg 0$ *and* $\mathbb{D}$ *is positively* J*-coercive over* $\mathcal{U}_{\mathrm{out}}$*:* These follow from Proposition 10.3.1(a) and Lemma 9.10.3.

3° *Unique minimizing control over* $\mathcal{U}_{\mathrm{out}}$*:* Because $\mathcal{P}$ is SOS-stabilizing, we have $\mathbb{K}_{\circlearrowleft}x_0 \in \mathcal{U}_{\mathrm{out}}(x_0)$ consequently, $\mathcal{U}_{\mathrm{out}}(x_0) \neq \emptyset$, for each $x_0 \in H$. Thus, by Theorem 8.4.3, there is a unique minimizing control for each x_0.

(d1) This follows from Theorem 9.9.1(a2), since any admissible nonnegative solution is output-stabilizing, by (d).

(d2) If Σ is [strongly [exponentially]] stable, then so is Σ_{ext}, by Proposition 10.7.1, and $\Sigma_{\circlearrowleft}$, by Corollary 6.6.9 (here "$I - L\mathbb{D}$"$= \mathbb{X} \in \mathcal{G}$TIC) and Lemma 6.6.8(c). Now $\mathbb{D}, \mathbb{X}, \mathbb{X}^{-1}$ are [[exponentially]] stable, hence $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F}\end{array}\right]$ is [strongly [exponentially]] r.c.-stabilizing.

[By Theorem 9.9.10(e2)&(c1)&(b), $\mathcal{P}$ is minimizing over $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$ [and $\mathcal{U}_{\mathrm{exp}}$], hence unique.]

(d3) A nonnegative admissible solution $(\mathcal{P}', S', \left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}'\end{array}\right])$ is SOS-stabilizing and has $S' \gg 0$, by (d), hence it is exponentially q.r.c.-stabilizing, by Theorem 6.7.15(c2)&(b1), hence minimizing over $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$, $\mathcal{U}_{\mathrm{str}}$ and $\mathcal{U}_{\mathrm{exp}}$ and unique, by Theorem 9.9.10(e2)&(c1)&(b).

□

In usual quadratic minimization problems, one need not check whether $\mathcal{P}$ is admissible (see Section 10.1 for applications):

Proposition 10.7.4 ($\mathcal{J} = \langle y, Qy \rangle + \langle u, Ru \rangle$) *Let $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ be WR. Let $Y = \mathcal{Y} \times U$, $\mathbb{C} = \left[\begin{smallmatrix} \mathbb{C}_1 \\ 0 \end{smallmatrix}\right]$, $\mathbb{D} = \left[\begin{smallmatrix} \mathbb{D}_1 \\ I \end{smallmatrix}\right]$, $J = \left[\begin{smallmatrix} Q & 0 \\ 0 & R \end{smallmatrix}\right]$, $Q \geq 0$, $R \gg 0$. Assume that $\mathbb{D} \in \mathrm{UR}$ or $\dim U < \infty$.*

Let the CARE

$$\begin{cases} K^*SK = A^*\mathcal{P} + \mathcal{P}A + C^*JC, \\ \quad S = D^*JD + \lim_{s \to +\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B, \\ \quad K = -S^{-1}(B_{\mathrm{w}}^*\mathcal{P} + D^*JC) \end{cases} \tag{10.90}$$

have a solution $\mathcal{P} \in \mathcal{B}(H)$, $\mathcal{P} \geq 0$ s.t. $\lim_{s \to +\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B \geq 0$ or $S \gg 0$.

Then $\mathcal{P}$ is UR and admissible.

If $Q \gg 0$, then $\mathcal{P}$ is SOS-stabilizing and Proposition 10.7.3 applies; in particular, if Σ is estimatable, then $\mathcal{P}$ is the unique nonnegative solution, strictly minimizing over $\mathcal{U}_{\mathrm{out}}$ and exponentially q.r.c-stabilizing.

Note that for $\dim U < \infty$ (resp. $\mathbb{D} \in \mathrm{UR}$), the limit in CARE converges uniformly (as required above) iff the limit in CARE converges weakly (resp. iff $\mathbb{F} \in \mathrm{UR}$, by Lemma 9.11.5(e)).

Proof: Choose some $\omega > \max(0, \omega_A)$. If $\lim_{s \to +\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B \geq 0$, then $S \geq D^*JD = D_1^*QD_1 + R \geq R \gg 0$. Therefore, $S \gg 0$ under either assumption.

By Proposition 10.7.2, $\mathcal{P}$ is a WR solution of the CARE. The map $\mathbb{X}$ is UR, by Lemma 6.3.2(a1)&(a2) or Lemma 9.11.5(e).

Substitute $z = s$ into (9.188) to observe that

$$\widehat{\mathbb{X}}(s)^*S\widehat{\mathbb{X}}(s) \geq \widehat{\mathbb{D}}(s)^*J\widehat{\mathbb{D}}(s) \quad (s \in \mathbf{C}_\omega^+) \tag{10.91}$$

(because $\mathcal{P} \geq 0$). But $\widehat{\mathbb{D}}(s)^*J\widehat{\mathbb{D}}(s) = \widehat{\mathbb{D}}_1(s)^*Q\widehat{\mathbb{D}}_1(s) + R \geq R \gg 0$ ($s \in \mathbf{C}_\omega^+$), hence $\widehat{\mathbb{X}}(s)^*S\widehat{\mathbb{X}}(s) \geq R \gg 0$.

By Proposition 2.2.5, we have $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ (this is why we wanted $\mathbb{X}$ to be UR). Thus, $\mathcal{P}$ is also admissible, and $S \gg 0$. The rest follows now from Proposition 10.7.3. □

If, e.g., Hypothesis 9.5.1 holds, then we have $\|sB^*(s-A)^{-*}\mathcal{P}(s-A)^{-1}B\|_{\mathcal{B}(U)} \to 0$, as $s \in \mathbf{C}_\omega^+$, $|s| \to \infty$, by Lemma 9.4.2(k). Therefore, in that case, self-adjoint solutions are UR and admissible even without the nonnegativity assumption (since then we have, instead of (10.91), that $\widehat{\mathbb{X}}(s)^*S\widehat{\mathbb{X}}(s) \geq \widehat{\mathbb{D}}(s)^*J\widehat{\mathbb{D}}(s) - \varepsilon I \geq R - \varepsilon I$ for some $\varepsilon > 0$, when $\mathrm{Re}\, s$ is big enough, by (9.188), hence also then $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}_\infty^\infty$).

Notes

Part of Proposition 10.7.3 is well known for certain subclasses; see, e.g., Theorem 3.3 of [PS87] or Section 6.2 of [CZ]. In the finite-dimensional case, [LR] is a comprehensive reference for both general and positive Riccati equations and inequalities.

Chapter 11

H^∞ Full-Information Control Problem ($\|w \mapsto z\| < \gamma$)

Of all men's miseries, the bitterest is this: to know so much and have control over nothing.
— Herodotos

In this chapter, we shall solve the H^∞ Full-Information Control Problem (FICP), which is described on p. 33.

Our main results are presented in Section 11.1; applications to parabolic (analytic semigroup) systems are given in Corollary 9.5.11. Further results and proofs are given in Section 11.2 (including the extension of the (J_γ, J_1)-lossless factorization solution of [Green] and [CG97] to an MTIC setting, in Theorem 11.2.7), and the stable case in Section 11.3. Section 11.4 treats minimax J-coercivity, a property of the Popov operator, equivalent to the existence of a nonsingular solution to the H^∞ minimax problem.

The discrete-time H^∞ full-information control problem (ficp) is treated in Section 11.5, and corresponding proofs are given in Section 11.6. The reader might wish to read first these two sections in order to observe the basic characteristics of the H^∞ FICP in a simple setting before going into the technical details required by the unboundedness of the input and output operators in continuous time. A reader interested only in the main results should read only the introduction on p. 33 and Sections 11.5 and 11.1, in that order.

The necessity part of our proofs is based on the solution of the abstract H^∞ FICP (i.e., the FICP in the setting of Section 8.1), which is given in Section 11.7.

The methods used for the stable H^∞ FICP also apply to the (one-block) Nehari problem, where one wishes to estimate $d(\mathbb{D}, \mathrm{TIC}^*)$ or the Hankel norm $\|\pi_+ \mathbb{D} \pi_-\|$ of some $\mathbb{D} \in \mathrm{TIC}$. Therefore, we take a brief look at this problem in Section 11.8.

Standing Hypothesis 11.0.1 *Throughout this chapter (except in Section 11.7; see Hypothesis 11.7.1) we assume that H, U, W, Y, Z are Hilbert spaces, and that the space $\widetilde{\mathcal{A}}(U \times W)$ satisfies Hypothesis 8.4.7.*

(Cf. Theorem 8.4.9(a) and Definition 6.2.4.) Note also that Hypothesis 11.1.1 is assumed through Sections 11.1–11.3, 11.5 and 11.6, Hypothesis 11.8.1 is assumed through that Sections 11.8–11.9, and that Hypotheses 11.2.1, 11.3.1 11.6.1 and 11.7.1 are assumed through corresponding sections.

Thus, we can use the equivalence of Theorem 8.4.12 whenever we assume that $\mathbb{D} \in \widetilde{\mathscr{A}}(U \times W, Y)$, for any Hilbert space Y. (Note that this does not put any restrictions on U and W when one takes $\widetilde{\mathscr{A}}$ to be one of the classes in Theorem 8.4.9(1.). As noted in Lemma 14.3.5, classes $\ell^1_+(U \times W)*$ and $\mathrm{tic}_{\mathrm{exp}}(U \times W)$ satisfy the discrete-time version of Hypothesis 8.4.7.)

As in previous chapters, we denote by $\Sigma_{\circlearrowleft}$ the closed-loop system corresponding to the solution of the Riccati equation. By $\Sigma^{\curvearrowleft}$, we denote the corresponding "semi-closed-loop system" (where only the state feedback loop corresponding to the control u is closed but the second input, the disturbance w is unaffected); see Section 11.1 for details. The sub- or superscripts $\circlearrowleft$ and $\curvearrowleft$ are also used for the components and signals in those systems.

11.1 The H$^\infty$ Full-Info Control Problem (FICP)

The goal of science is to build better mousetraps. The goal of nature is to build better mice.

We strongly recommend the reader to read the introduction to the H$^\infty$ FICP problem (p. 33) and possibly also have a glance at the discrete-time results of Section 11.5 before going into the technical details of this section. The results of this section look like more complicated forms of Theorem 11.5.1, due to the possibly unbounded input and output operators.

In this section, we give necessary and sufficient conditions for the existence of a γ-suboptimal full-information or state-feedback controllers in terms of Riccati equations. For solutions in terms of lossless factorizations, see, e.g., Theorem 11.2.7 or Theorem 11.1.5(b).

In Theorems 11.1.3 and 11.1.4, we treate suboptimal exponentially stabilizing controllers (or "H$^\infty$-FI-pairs") assuming "B_{w}^*-CARE" type regularity or a smoothing semigroup, respectively. In Theorem 11.1.6, we treate suboptimal output-stabilizing controllers assuming stronger nonsingularity (e.g., a copy of input contained in the output) and "B_{w}^*-CARE" type regularity. In Theorem 11.1.5, we treate suboptimal strongly stabilizing controllers assuming that the system is strongly q.r.c.-stabilizable with MTIC closed-loop system (the above three results do not assume any stabilizability). In all above results, we practically give sufficient and necessary conditions for a suboptimal state-feedback (and full-information) controllers to exist, in terms of a Riccati equation and the corresponding signature condition. In Remark 11.1.11 we treate the dual problem (the Full Control Problem).

As described on p. 33, in the H$^\infty$ FICP, we have a system $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{WPLS}(U\times W,H,Z)$, and we wish to find, for each disturbance $w\in \mathrm{L}^2(\mathbf{R}_+;W)$, a "suboptimal" control $u\in\mathrm{L}^2(\mathbf{R}_+;U)$, i.e., one that is stabilizing and makes the (closed-loop) norm $\|w\mapsto z\|$ less than a given constant γ, where z is the output of the system under input $\left[\begin{smallmatrix}u\\ w\end{smallmatrix}\right]$ (and under initial state $x_0=0$). One often also requires that the control is given by some state feedback pair (with or without feedthrough).

We have $\|w\mapsto z\|_{\mathrm{L}^2\to\mathrm{L}^2}<\gamma$ iff the (cost) function $\|z\|_2^2-\gamma^2\|w\|_2^2$ is uniformly negative w.r.t. w, i.e., iff $\|z\|_2^2-\gamma^2\|w\|_2^2\le-\varepsilon\|w\|_2^2$ for some $\varepsilon>0$. Moreover, the J-critical control for this cost function and the corresponding Riccati equation lead to the solution of the H$^\infty$ FICP, as will be shown in following sections.

Therefore, we usually augment Σ by the extra the row $\left[\begin{array}{c|cc}0 & 0 & I\end{array}\right]$ to make the output equal to $y:=\left[\begin{smallmatrix}z\\ w\end{smallmatrix}\right]$ (the input is $\left[\begin{smallmatrix}u\\ w\end{smallmatrix}\right]\in\mathrm{L}^2(\mathbf{R}_+;U\times W)$; see Figure 11.1 without the feedback row and loop), so that we can make the cost $\mathcal{J}:=\langle y,Jy\rangle$ equal to $\|z\|_2^2-\gamma^2\|w\|_2^2$ by setting $J=J_\gamma:=\left[\begin{smallmatrix}I & 0\\ 0 & -\gamma^2 I\end{smallmatrix}\right]$:

Standing Hypothesis 11.1.1 (H$^\infty$ Full-Information Control Problem (FICP)) *Throughout Sections 11.1–11.3, we make the following assumptions:*

$$\Sigma=\left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]=\left[\begin{array}{c|cc}\mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2\\ \hline \mathbb{C}_1 & \mathbb{D}_{11} & \mathbb{D}_{12}\\ 0 & 0 & I\end{array}\right]\in\mathrm{WPLS}(U\times W,H,Z\times W). \tag{11.1}$$

The corresponding discrete-time assumptions (see (13.63)) are made in Sections 11.5 and 11.6.

If the generators of A, B, C and D of Σ are bounded, i.e., $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U \times W, H \times Z \times W)$, then this corresponds to the dynamics

$$\begin{cases} x' = Ax + B_1 u + B_2 w, \\ z = C_1 x + D_{11} u + D_{12} w \end{cases} \tag{11.2}$$

(and $w = Iw$) with initial state $x(0) = x_0 \in H$. In the case of a general weakly regular system Σ, equations (11.2) hold in the strong sense, see Theorem 6.2.13 for details. Note that $C = \left[\begin{smallmatrix} C_1 \\ 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} D_{11} & D_{22} \\ 0 & I \end{smallmatrix}\right]$.

As mentioned above, the suboptimal control is required to be "stabilizing". In the literature, this sometimes means that for any given initial state $x_0 \in H$ and disturbance $w \in \mathrm{L}^2(\mathbf{R}_+; W)$, the state feedback pair (or a more general control law $x_0, w \mapsto u$) must produce a control $u \in \mathrm{L}^2(\mathbf{R}_+; U)$ s.t. the output z (equivalently, $y := \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$) becomes stable, i.e., s.t. $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{out}}(x_0)$.

Often one also requires that $x \in \mathrm{L}^2(\mathbf{R}_+; H)$, i.e., that $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\exp}(x_0)$ (see Lemma 8.3.3). In either case, we denote the set of corresponding controls by $\mathcal{U}_{\mathrm{u}}$ (in this section we set $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ or $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$, depending on the application):

Definition 11.1.2 (Suboptimal $\mathbf{H}^\infty$-FI-pair) *Throughout this section, we use the following notation:*

$$\mathcal{U}_{\exp}(x_0) = \{\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathrm{L}^2(\mathbf{R}_+; U \times W) \,\big|\, \mathbb{A}x_0 + \mathbb{B}\tau \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathrm{L}^2\}; \tag{11.3}$$

$$\mathcal{U}_{\mathrm{out}}(x_0) = \{\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathrm{L}^2(\mathbf{R}_+; U \times W) \,\big|\, \mathbb{C}x_0 + \mathbb{D}\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathrm{L}^2\}; \tag{11.4}$$

$$\mathcal{U}_{\mathrm{u}}(x_0, w) := \{u \in \mathrm{L}^2(\mathbf{R}_+; U) \,\big|\, \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}_*^*(x_0)\} \quad (x_0 \in H,\ w \in \mathrm{L}^2(\mathbf{R}_+; W)), \tag{11.5}$$

$$\gamma_0 := \sup_{w \in \mathrm{L}^2(\mathbf{R}_+; W), \|w\|=1} \ \inf_{u \in \mathcal{U}_{\mathrm{u}}(0,w)} \|\mathbb{D}_{11} u + \mathbb{D}_{12} w\|_2. \tag{11.6}$$

An admissible state feedback pair of form $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ *(resp. admissible WR state feedback operator* $K = \left[\begin{smallmatrix} K_1 \\ 0 \end{smallmatrix}\right]$*) for* Σ *is called a* H^∞-FI-pair *(resp.* H^∞*-SF-operator) if*

$$\left[\begin{array}{c|c} \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft + I \end{array}\right] \begin{bmatrix} x_0 \\ 0 \\ w \end{bmatrix} \in \mathcal{U}_*^*(x_0) \quad \textit{for all } x_0 \in H \textit{ and } w \in \mathrm{L}^2(\mathbf{R}_+; W), \tag{11.7}$$

where $\Sigma_\circlearrowleft$ *is the corresponding closed-loop system (see Figure 11.1 and equation (11.8); we use prefices and suffices as in Definition 6.6.10).*

By γ_{FI} *(resp.* γ_{SF}*) we denote the infimum of the norm* $\|w \mapsto z\|_{\mathrm{L}^2(\mathbf{R}_+; W) \to \mathrm{L}^2(\mathbf{R}_+; Z)}$ *over all* H^∞*-FI-pairs (resp. all* H^∞*-SF-operators). Given* $\gamma > 0$*, a* H^∞*-FI-pair or* H^∞*-SF-operator is called* suboptimal *if* $\|\mathbb{D}_{\circlearrowleft 12}\|_{\mathrm{TIC}} < \gamma$.

By equation (11.8), we have $\|w \mapsto z\| = \|\mathbb{D}_{\circlearrowleft 12}\|_{\mathrm{TIC}}$. Therefore, the H^∞ FICP means finding $\left[\begin{array}{c|c} \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right]$ s.t. $\|\mathbb{D}_{\circlearrowleft 12}\|_{\mathrm{TIC}} < \gamma$. As one observes from Figure 11.1, the map $\left[\begin{array}{c|c} \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right]$ maps $\left[\begin{smallmatrix} x_0 \\ u_\circlearrowleft \\ w \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u - u_\circlearrowleft \\ 0 \end{smallmatrix}\right]$, i.e., from the initial state and

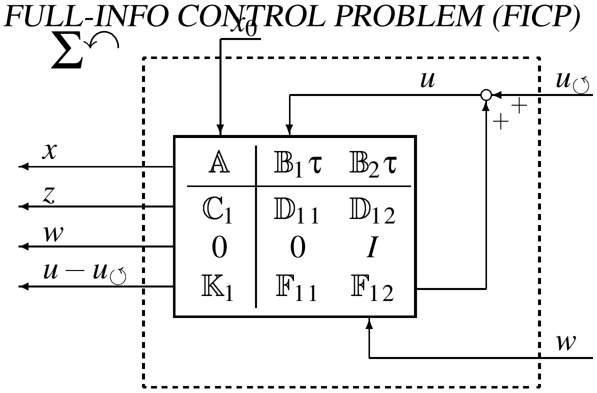

Figure 11.1: A WPLS controlled by a H^∞-FI-pair

external inputs to the feedback signal, so that we must add $\left[\begin{smallmatrix} u_{\circlearrowleft} \\ w \end{smallmatrix}\right]$ to obtain the effective input $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ (the situation is the same as in Definition 6.6.10).

Note also that the controller (feedback) is only allowed to affect u. Therefore, the existence of a H^∞-FI-pair for Σ can be described as "Σ is exponentially stabilizable through B_1" (cf. (11.11) and Figure 11.1). See Lemma 11.1.8 for details.

The standard "state-feedback" setting of Figure 6.3 has become the setting of Figure 11.1, because the feedback loop does not affect w (hence we have omitted the second, zero row of $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$) and because the second row of $\left[\begin{array}{c|c} \mathbb{C} & \mathbb{D} \end{array}\right]$ equals $\left[\begin{array}{c|cc} 0 & 0 & I \end{array}\right]$, so that the lower element of the output "$y = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$" equals w. Recall from Definition 6.6.10, that our state feedback allows a feedforward term, hence it is actually a "full-information feedback". The corresponding closed-loop system $\Sigma^{\curvearrowleft} : \begin{bmatrix} x_0 \\ u_{\circlearrowleft} \\ w \end{bmatrix} \mapsto \begin{bmatrix} x \\ \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} u-u_{\circlearrowleft} \\ 0 \end{smallmatrix}\right] \end{bmatrix}$ is given by (cf. (6.134))

$$\Sigma^{\curvearrowleft} = \left[\begin{array}{c|cc} \mathbb{A}^{\curvearrowleft} & \mathbb{B}_1^{\curvearrowleft} & \mathbb{B}_2^{\curvearrowleft} \\ \hline \mathbb{C}_1^{\curvearrowleft} & \mathbb{D}_{11}^{\curvearrowleft} & \mathbb{D}_{12}^{\curvearrowleft} \\ 0 & 0 & I \\ \mathbb{K}_1^{\curvearrowleft} & \mathbb{F}_{11}^{\curvearrowleft} & \mathbb{F}_{12}^{\curvearrowleft} \\ 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} & \mathbb{D}_{12} \\ 0 & 0 & I \\ \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c|cc} I & 0 & 0 \\ \hline \mathbb{X}_{11}^{-1}\mathbb{K}_1 & \mathbb{X}_{11}^{-1} & -\mathbb{X}_{11}^{-1}\mathbb{X}_{12} \\ 0 & 0 & I \end{array}\right] \tag{11.8}$$

$$= \left[\begin{array}{c|cc} \mathbb{A} + \mathbb{B}_1\tau\mathbb{X}_{11}^{-1}\mathbb{K}_1 & \mathbb{B}_1\mathbb{X}_{11}^{-1} & \mathbb{B}_2 - \mathbb{B}_1\mathbb{X}_{11}^{-1}\mathbb{X}_{12} \\ \hline \mathbb{C}_1 + \mathbb{D}_{11}\mathbb{X}_{11}^{-1}\mathbb{K}_1 & \mathbb{D}_{11}\mathbb{X}_{11}^{-1} & \mathbb{D}_{12} - \mathbb{D}_{11}\mathbb{X}_{11}^{-1}\mathbb{X}_{12} \\ 0 & 0 & I \\ \mathbb{X}_{11}^{-1}\mathbb{K}_1 & \mathbb{X}_{11}^{-1} - I & -\mathbb{X}_{11}^{-1}\mathbb{X}_{12} \\ 0 & 0 & 0 \end{array}\right] \tag{11.9}$$

$$= \left[\begin{array}{c|cc} \mathbb{A}_{\circlearrowleft} - \mathbb{B}_{\circlearrowleft 2}\tau\mathbb{M}_{22}^{-1}\mathbb{K}_{\circlearrowleft 2} & \mathbb{B}_{\circlearrowleft 1} - \mathbb{B}_{\circlearrowleft 2}\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{B}_{\circlearrowleft 2}\mathbb{M}_{22}^{-1} \\ \hline \mathbb{C}_{\circlearrowleft 1} - \mathbb{N}_{12}\mathbb{M}_{22}^{-1}\mathbb{K}_{\circlearrowleft 2} & \mathbb{N}_{11} - \mathbb{N}_{12}\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{N}_{12}\mathbb{M}_{22}^{-1} \\ 0 & 0 & I \\ \mathbb{K}_{\circlearrowleft 1} - \mathbb{M}_{12}\mathbb{M}_{22}^{-1}\mathbb{K}_{\circlearrowleft 2} & \mathbb{M}_{11} - \mathbb{M}_{12}\mathbb{M}_{22}^{-1}\mathbb{M}_{21} - I & \mathbb{M}_{12}\mathbb{M}_{22}^{-1} \\ 0 & 0 & I \end{array}\right] \tag{11.10}$$

$$= \left[\begin{array}{c|c} \mathbb{A} + \mathbb{B}\tau\mathbb{K}^{\curvearrowleft} & \mathbb{B}\overline{\mathbb{X}}^{-1} \\ \hline \mathbb{C} + \mathbb{D}\mathbb{K}^{\curvearrowleft} & \mathbb{D}\overline{\mathbb{X}}^{-1} \\ \mathbb{K}^{\curvearrowleft} & \overline{\mathbb{X}}^{-1} - I \end{array}\right], \tag{11.11}$$

and $\Sigma^{\curvearrowleft} \in \mathrm{WPLS}(U \times W, H, Z \times W \times U \times W)$ (here $\mathbb{X} := I - \mathbb{F}$, $\mathbb{M} := \mathbb{X}^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$, $\overline{\mathbb{X}} := \left[\begin{smallmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} \\ 0 & I \end{smallmatrix}\right] = (I + \mathbb{F}^{\curvearrowleft})^{-1}$; for (11.10) we used the fact that $\mathbb{M} = \overline{\mathbb{X}}^{-1}\underline{\mathbb{M}}$, where $\underline{\mathbb{M}} := \left[\begin{smallmatrix} I & 0 \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{smallmatrix}\right]$, by (A.9)).

If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then condition (11.7) holds iff $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is exponentially stabilizing (see Remark 11.2.5 for more on (11.7)). Most existing literature for finite-dimensional control theory is written for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, whereas also $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ is popular in the infinite-dimensional case.

As obvious from the above definition, we are interested in the infimal value γ_0 of $\|w \mapsto z\|$ over all "stabilizing" controls (the norm of $w \mapsto \|\mathbb{D}_{11}u_{\min}(w) + \mathbb{D}_{12}w\|_2$), in the infimal value γ_{FI} of $\|w \mapsto z\|$ over all "stabilizing" full information controllers, and in the infimal value γ_{SF} of $\|w \mapsto z\|$ over all "stabilizing" pure state feedback controllers (thus, for γ_{SF}, we pose the additional condition that the feedthrough operator F of $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ exists and is equal to zero ($\mathbb{F} \in \mathrm{WR}$ and $F_{11} = 0 = F_{12}$); recall from Definition 6.6.10 that a state feedback pair is allowed to have any admissible feedthrough term). Since $\inf\emptyset = +\infty$, we obviously have

$$0 \le \gamma_0 \le \gamma_{\mathrm{FI}} \le \gamma_{\mathrm{SF}} \le +\infty. \tag{11.12}$$

Thus, $\gamma_0 < \gamma$ corresponds to the existence of a suboptimal control law (for $x_0 = 0$ only), $\gamma_{\mathrm{FI}} < \gamma$ to the existence of a causal control law and $\gamma_{\mathrm{SF}} < \gamma$ to the existence of a strictly causal control law (no feedforward term), with the additional restriction that γ_{FI} and γ_{SF} require the control law to be of state feedback form. If Σ is somewhat regular, then, under the standard assumptions that $\mathbb{D}_{11}$ is coercive and $D_{12} = 0$, we have $\gamma_0 = \gamma_{\mathrm{FI}} = \gamma_{\mathrm{SF}} < \infty$ for systems exponentially stabilizable through B_1, as illustrated in the following theorems. If $\mathbb{D} \in \mathrm{WR}$ and $\|D_{12}\| > \gamma_{\mathrm{FI}}$, then we necessarily have $\gamma_{\mathrm{SF}} > \gamma_{\mathrm{FI}}$ (if for $\mathbb{D} \in \mathrm{SR}$, since then

$$\|\mathbb{D}_{12}^{\frown}\|_{\mathrm{TIC}} = \|\widehat{\mathbb{D}^{\frown}}_{12}\|_{\mathrm{H}^\infty} \geq \|D_{12}^{\frown}\|_{\mathcal{B}(W,Z)} = \|D_{12}\|_{\mathcal{B}(W,Z)} > \gamma_{\mathrm{FI}}, \tag{11.13}$$

by Proposition 6.6.18(d4) (to be exact; this requires that we restrict ourselves to SR H^∞-SF-operators; when $\dim U < \infty$ or Σ is sufficiently regular, this is not a restriction, see, e.g., Theorem 11.1.3(a)). Example 11.1.9 illustrates this.

Now we are ready to present our results. See Proposition 10.3.2 and Theorem 9.2.3 for the two assumptions.

Theorem 11.1.3 ($\mathcal{U}_{\mathrm{exp}}$: H^∞ FICP $\Leftrightarrow$ B_{w}^*-CARE) *Assume that $\gamma > 0$ and that (1.) and (2.) hold.*

(1.) **(Nonsingularity)** *Assume $D_{11}^* D_{11} \gg 0$, and that there is $\varepsilon_1 > 0$ s.t.*

$$(ir - A)x_0 = Bu_0 \Rightarrow \|C_1 x_0 + D_{11} u_0\|_Z \geq \varepsilon_1 \|x_0\|_H \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}). \tag{11.14}$$

(2.) **(Regularity)** *Assume that Σ and J_γ satisfy Hypothesis 9.2.1 and that $\pi_{[0,1)} \mathbb{A} B \in \mathrm{L}^1([0,1); \mathcal{B}(U \times W, H))$ or $D^* J D \in \mathcal{GB}(U \times W)$.*

Then (i)–(iii) are equivalent:

(i) $\gamma > \gamma_0$ and Σ is exponentially stabilizable through B_1;

(ii) $\gamma > \gamma_{\mathrm{FI}}$, i.e., there is a suboptimal H^∞-FI-pair for Σ;

(iii) $D_{12}^* D_{12} - D_{12}^* D_{11} (D_{11}^* D_{11})^{-1} D_{11}^* D_{12} \ll \gamma^2 I$, and (the B_{w}^*-CARE)

$$\left(B_{\mathrm{w}}^* \mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1\right)^* (D^* J_\gamma D)^{-1} \left(B_{\mathrm{w}}^* \mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1\right) = A^* \mathcal{P} + \mathcal{P} A + C_1^* C_1 \tag{11.15}$$

has a solution $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$ s.t. $\mathcal{P} \geq 0$ and $A + BK_{\mathrm{w}}$ generates an exponentially stable semigroup, where $K_{\mathrm{w}} := -(D^* J_\gamma D)^{-1} (B_{\mathrm{w}}^* \mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} (C_{\mathrm{w}})_1)$.

Moreover, the following hold:

(a) Assume that $(\mathcal{P}, S, K)$ satisfies (iii) (here $S := D^* J_\gamma D$). Then

$$\left[\begin{array}{c|cc} -(D_{11}^* D_{11})^{-1} (D_{11}^* C_1 + (B_1^*)_{\mathrm{w}} \mathcal{P}) & 0 & -(D_{11}^* D_{11})^{-1} D_{11}^* D_{12} \\ 0 & 0 & 0 \end{array}\right] \tag{11.16}$$

generate a ULR (exponentially stabilizing) suboptimal H^∞-FI-pair.

There is a suboptimal H^∞-SF-operator iff $\|D_{12}\| < \gamma$; if this is the case, then

$$K_1 := \begin{bmatrix} I & 0 \end{bmatrix} K = -(S_{11} - S_{12} S_{22}^{-1} S_{21})^{-1} (D_{11}^* C_1 + (B_1^*)_{\mathrm{w}} \mathcal{P} - S_{12} S_{22}^{-1} (D_{12}^* C_1 + (B_2^*)_{\mathrm{w}} \mathcal{P})) \tag{11.17}$$

is a ULR (exponentially stabilizing) suboptimal H^∞-SF-operator, where $S := D^* J_\gamma D$.

(b) If (i)–(iii) hold, then the assumptions of Proposition 11.2.8 (including those of (a1)) are satisfied and (FI1)–(FI5) hold.

Applications to parabolic systems of this theorem and the ones to follow are given in Corollary 9.5.11. Recall that $J_\gamma := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$, hence $D^* J_\gamma D = \begin{bmatrix} D_{11}^* D_{11} & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} - \gamma^2 I \end{bmatrix}$.

Under the normalizing conditions

$$D_{12} = 0, \quad D_{11}^* \begin{bmatrix} C_1 & D_{11} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}, \tag{11.18}$$

condition (iii) can be written as follows:

$$((B_1^*)_w \mathcal{P})^* (B_1^*)_w \mathcal{P} - \gamma^{-2} ((B_2^*)_w \mathcal{P})^* (B_2^*)_w \mathcal{P} = A^* \mathcal{P} + \mathcal{P} A + C_1^* C_1 \tag{11.19}$$

with the requirements that $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_w^*))$, $\mathcal{P} \geq 0$, and $A + (\gamma^{-2} B_2 (B_2^*)_w - B_1 (B_1^*)_w) \mathcal{P}$ is exponentially stable. Now $S = J_\gamma := \left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right]$ and $K = \begin{bmatrix} -(B_1^*)_w \mathcal{P} \\ \gamma^{-2} (B_2^*)_w \mathcal{P} \end{bmatrix} \in \mathcal{B}(H, U \times W))$, and if(f) (iii) holds, then $K_1 = -(B_1^*)_w \mathcal{P} \in \mathcal{B}(H, U)$ is a suboptimal H^∞-SF-operator for Σ; this implies that the corresponding closed-loop state is controlled by the equation

$$x' = (A - B_1 (B_1^*)_w \mathcal{P}) x + B_2 w \qquad \text{a.e.} \tag{11.20}$$

Note that equations (6.3.5) of [GL], (20.2.5) of [LR], (16.1) of [ZDG], and (4.11) of [Keu] are special cases of (11.19).

If B is bounded, then (11.19) takes the classical form

$$\mathcal{P}(B_1 B_1^* - \gamma^{-2} B_2 B_2^*) \mathcal{P} = A^* \mathcal{P} + \mathcal{P} A + C_1^* C_1, \tag{11.21}$$

hence $K = \begin{bmatrix} -(B_1^*)_w \mathcal{P} \\ \gamma^{-2} (B_2^*)_w \mathcal{P} \end{bmatrix} \in \mathcal{B}(H, U \times W) \in \mathcal{B}(H, U \times W)$, $K_1 = -B_1^* \mathcal{P} \in \mathcal{B}(H, U)$. Recall from Definition 9.8.1, that the CARE is given on $\mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A^*)^*) =: \mathcal{B}(H_1, H_{-1}^*)$; e.g., (11.19) holds iff

$$\begin{aligned} &\langle (B_1^*)_w \mathcal{P} x_0, (B_1^*)_w \mathcal{P} x_1 \rangle - \gamma^{-2} \langle (B_2^*)_w \mathcal{P} x_0, (B_2^*)_w \mathcal{P} x_1 \rangle \\ &= \langle A x_0, \mathcal{P} x_1 \rangle + \langle \mathcal{P} x_0, A x_1 \rangle + \langle C_1 x_0, C_1 x_1 \rangle \end{aligned} \tag{11.22}$$

for all $x_0, x_1 \in \mathrm{Dom}(A)$ (we can take $x_1 = x_0$ w.l.o.g., by Lemma A.3.5(a)).

All CAREs of this section and the next two sections equal the CARE for Σ and J_γ (under corresponding regularity assumptions, that is, some of the CAREs have been simplified).

Obviously, the K in (iii) is bounded (and hence $K_w = K$) iff $\begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 \in \mathcal{B}(H, U)$; analogously, when $D_{11}^* C_1 \in \mathcal{B}(H, U)$, we may restrict us to bounded H^∞-FI-pairs (with generators $\left[\begin{array}{c|c} K & F \end{array}\right]$, $K = \left[\begin{smallmatrix} K_1 \\ 0 \end{smallmatrix}\right] \in \mathcal{B}(H, U)$, $F = \left[\begin{smallmatrix} F_{11} & F_{12} \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{B}(U \times W)$), i.e., if there is a suboptimal H^∞-FI-pair, then there is a bounded H^∞-FI-pair (by (a)).

Recall from Proposition 9.2.7 that any K of the form in (iii) (with $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_w^*))$) is admissible (in particular, $A + BK_w$ generates a C_0-semigroup). If $B \in \mathcal{B}(U, H)$, then (2.) is satisfied, the stabilizability assumption in (i) holds iff

(A, B_1) is exponentially stabilizable (or optimizable), and $\mathrm{Dom}(B_{\mathrm{w}}^*) = H$.

The equivalence (ii)⇔(iii) is (an extension of) the standard result that there is a suboptimal (exponentially stabilizing) state feedback controller through B_1 iff the Riccati equation has an exponentially stabilizing solution and the signature conditions on $D^* J_\gamma D$ are satisfied. Condition (i) means roughly that there is a suboptimal (exponentially stabilizing) control law $x_0, w \mapsto u$; thus "(i)⇔(ii)" says that such a law can always be realized by a state feedback controller.

If Σ is exponentially stable, then $\mathbb{A}_\circlearrowleft$, $\mathbb{C}_\circlearrowleft$, $\mathbb{K}_\circlearrowleft$ and $\mathcal{P}$ are given by (8.43)–(8.46), by (b) and Proposition 11.3.4(g), whenever (i)–(iii) hold.

One often has $\mathcal{J}(x_0, u, w) = \|x\|_2^2 + \varepsilon\|u\|_2^2 - \gamma^2\|w\|_2^2$ or something similar, so that (1.) is satisfied (see Proposition 10.3.2(e1) and 0.2°; the cost on x should be coercive at least for $x_0 = 0$ to satisfy (1.)); in fact, for this cost the nonsingularity assumptions of all results in this section and the following one are satisfied. The theorem does not hold without this assumption, that is, for "singular H$^\infty$ problems", and such problems are rarely treated in the literature; see [Stoorvogel] for an exception (for finite-dimensional systems).

Proof of Theorem 11.1.3: We have tacitly assumed that $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ (also in (2.)). See Definition 9.2.6 for the B_{w}^*-CARE.

0.1° *Remark: Exponentially stabilizability through B_1:* By this we mean the existence of a H$^\infty$-FI-pair (over $\mathcal{U}_{\exp}$), i.e., of an exponentially stabilizing state feedback pair that does not affect w (i.e., which is of form $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] = \left[\begin{array}{c|cc} \widetilde{\mathbb{K}}_1 & \widetilde{\mathbb{F}}_{11} & \widetilde{\mathbb{F}}_{12} \\ 0 & 0 & 0 \end{array}\right]$, so that the feedback loop goes through B_1 only).

If $\pi_{[0,1)}\mathbb{A}B_1 u_0 \in \mathrm{L}^1([0,1); H)$ for all $u_0 \in U$ (this follows from the middle assumption in (2.)), then this assumption holds iff (A, B_1) is optimizable (equivalently, exponentially stabilizable), by Lemma 11.1.8 (see that lemma for further remarks).

0.2° *Remark: Assumption (1.):* Assumption (1.) implies that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} \end{array}\right]$ is positively I-coercive, by Proposition 10.3.2(g1)&(c). The converse holds if $D_{11}^* D_{11} \gg 0$ or $\mathbb{A}B \in \mathrm{L}^1_{\mathrm{loc}}$, by Proposition 10.3.2(e1)&(e2).

0.3° *Remark: $(D^* J_\gamma D)^{-1}$:* Set $S := D^* J_\gamma D$. Then $S_{11} = D_{11}^* D_{11} \gg 0$, by (1.), and the first condition in (iii) is equal to $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$. By Lemma 11.3.13(i)&(viii), these two together imply that $S \in \mathcal{GB}(U \times W)$, so that (11.15) is well defined.

1° *The equivalence of (i)–(iii):* Now (2.) or (3.) of Remark 9.9.14 is satisfied, hence we obtain the equivalence from Proposition 11.2.6, since (iii) is equivalent to (iii'), by which we denote "(iii)" of the proposition, as shown in 2° and 3° below. (Note that Hypothesis 11.2.1 is satisfied, by Lemma 11.2.2.)

2° *"(iii)⇒(iii')":* Assume (iii). Since $S := D^* J_\gamma D \in \mathcal{GB}(U \times W)$, by 0.3°, $\mathcal{P}$ is a solution of the B_{w}^*-CARE, hence an admissible and ULR solution of the CARE, by Proposition 9.2.7, and $\mathbb{A}_\circlearrowleft$ is generated by $A + BK_{\mathrm{w}}$, by (6.145), so that K is exponentially stabilizing.

Thus, (FI5) of Theorem 11.2.7 is satisfied. By Proposition 11.2.9, it follows that (FI2) holds, i.e., (ii) holds.

3° *"(iii')⇒(iii)":* This follows from Theorem 9.2.9(iii)&(iv) (as noted in 2°, the claim on $A + BK_{\mathrm{w}}$ holds iff K is exponentially stabilizing).

(b) This follows from Proposition 11.2.6 (whose assumptions are now satisfied, as noted in 1° above).

(a) This follows from Proposition 11.2.8(a1). □

We now present a CARE variant under assumptions that are weaker in certain sense:

Theorem 11.1.4 ($\mathcal{U}_{\mathbf{exp}}$: $\mathbf{H}^\infty$ FICP $\Leftrightarrow$ CARE: Case $\mathbb{A}B \in \mathrm{L}^1_{\mathrm{loc}}$) *Assume (1.) and (2.):*

(1.) **(Nonsingularity)** *Assume $D_{11}^* D_{11} \gg 0$, and that there is $\varepsilon_1 > 0$ s.t.*

$$(ir - A)x_0 = Bu_0 \Rightarrow \|C_1 x_0 + D_{11} u_0\|_Z \ge \varepsilon_1 \|x_0\|_H \quad (x_0 \in H,\ u_0 \in U,\ r \in \mathbf{R}). \tag{11.23}$$

(2.) **(Regularity)** *Assume that $\pi_{[0,1)}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U \times W, H))$, $\pi_{[0,1)}(C_1)_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1([0,1);\mathcal{B}(H,Z))$, and $\pi_{[0,1)}(C_1)_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1([0,1);\mathcal{B}(U \times W,Z))$.*

Then (i)–(iii) are equivalent for any $\gamma > 0$:

(i) $\gamma > \gamma_0$, and (A, B_1) is exponentially stabilizable;

(ii) $\gamma > \gamma_{\mathrm{FI}}$, i.e., there is a suboptimal H^∞-FI-pair for Σ;

(iii) $D_{12}^* D_{12} - D_{12}^* D_{11}(D_{11}^* D_{11})^{-1} D_{11}^* D_{12} \ll \gamma^2 I$, and the CARE

$$\left(B_{\mathrm{w}}^* \mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1\right)^* (D^* J_\gamma D)^{-1} \left(B_{\mathrm{w}}^* \mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1\right) = A^* \mathcal{P} + \mathcal{P} A + C_1^* C_1 \tag{11.24}$$

has an exponentially stabilizing solution $\mathcal{P} \in \mathcal{B}(H)$ s.t. $\mathcal{P} \ge 0$ and $\lim_{s \to +\infty} B_{\mathrm{w}}^* \mathcal{P}(s-A)^{-1} B = 0$.

In particular, $\gamma_{\mathrm{FI}} = \gamma_0$ if (A, B_1) is exponentially stabilizable (equivalently, optimizable). Moreover, Theorem 11.1.3(a)&(b) hold.

Most of the remarks made below Theorem 11.1.3 apply here too.

Proof: Condition (2.) implies that $\mathbb{D} \in \mathrm{MTIC}_\infty^{\mathrm{L}^1}(U \times W, Z \times W) \subset \mathrm{ULR}$, by Lemma 6.8.5.

0.1° *Remark: assumption (11.23) $\Leftrightarrow$ $\mathbb{D}_{11}$ is I-coercive:* By Proposition 10.3.2(e2), (11.23) holds iff $\mathbb{D}_{11}$ is I-coercive (and also conditions (i)–(iii) of Proposition 10.3.2 (with $J \mapsto I$, $\Sigma \mapsto \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} \end{array}\right]$) are equivalent to (11.23)).

0.2° *Remark: "(A, B_1) exponentially stabilizable":* By (2.) and Theorem 9.2.12, this holds iff (A, B_1) is optimizable, equivalently, iff Σ has an exponentially stabilizing (bounded) state feedback operator $\widetilde{K} = \begin{bmatrix} \widetilde{K}_1 \\ 0 \end{bmatrix} \in \mathcal{B}(H, U \times W)$.

See also 0.3° of the proof of Theorem 11.1.3.

1° *The equivalence of (i)–(iii):* Denote condition (iii) of Proposition 11.2.6 by "(iii-Prop)". Now Remark 9.9.14(5.) applies, hence we obtain the equivalence from Proposition 11.2.6, because condition (iii) is equivalent to (iii-Prop), as shown in 2° below. (Note that Hypothesis 11.2.1 is satisfied, by Lemma 11.2.2.)

2° *"(iii)⇔(iii-Prop)":* Condition (iii) says that (iii-Prop) holds, $\mathbb{F}$ is UR (by Lemma 9.11.5(e)) and $S = D^* J_\gamma D$. Conversely, if (iii-Prop) holds, then $S = D^* J_\gamma D$ and $\mathbb{F}$ is UR, by Remark 9.9.14(b)&(a).

(a)&(b) The original proofs of Theorem 11.1.3(a)&(b) apply mutatis mutandis. □

The case $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ is much more complicated than the case $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ and hence very rarely treated in the literature. (We have not found any existing research of the H$^\infty$ problems truely over $\mathcal{U}_{\text{out}}$ in the unstable case. Recall, however, that $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$ for estimatable systems, hence many historical results are applicable for such systems, though none before for WPLSs.) We give a more thorough treatment of this case in Section 11.2, under weaker assumptions, but here we show that if Σ is strongly q.r.c.-stabilizable through B_1 with a rather regular closed-loop system, then we have the standard equivalence:

Theorem 11.1.5 ($\widetilde{\mathcal{A}}$, $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{str}}$: H$^\infty$ FICP ⇔ CARE) *Let $\mathcal{U}_*^* := \mathcal{U}_{\text{out}}$. Assume (1.) and (2.):*

(2.) **(Stabilizability)** *There is a strongly q.r.c.-stabilizing UR state feedback operator $\widetilde{K} = \begin{bmatrix} \widetilde{K}_1 \\ 0 \end{bmatrix}$ for Σ, and $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$.*

(1.) **(Nonsingularity)** *There is $\varepsilon_1 > 0$ s.t. $\|\mathbb{D}_{\flat 11} u\|_2 \ge \varepsilon_1 \|u\|_2$ $(u \in \mathrm{L}^2(\mathbf{R}_+; U))$.*

Then (i)–(iii) are equivalent for each $\gamma > 0$:

(i) $\gamma > \gamma_0$;

(ii) $\gamma > \gamma_{\text{FI}}$; i.e., there is a suboptimal H$^\infty$-FI-pair for Σ;

(iii) the CARE

$$\begin{cases} K^* S K = A^* \mathcal{P} + \mathcal{P} A + C_1^* C_1, \\ S = \begin{bmatrix} D_{11}^* D_{11} & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} - \gamma^2 I \end{bmatrix} + \lim_{s \to +\infty} B_{\text{w}}^* \mathcal{P} (s - A)^{-1} B, \\ K = -S^{-1} \left(B_{\text{w}}^* \mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 \right). \end{cases} \tag{11.25}$$

has a q.r.c.-stabilizing solution $(\mathcal{P}, S, K)$, and $\mathcal{P} \ge 0$, $S_{11} \gg 0$ and $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$.

In particular, $\gamma_0 = \gamma_{\text{FI}}$. Moreover, the following hold:

(a) Assume that $(\mathcal{P}, S, K)$ satisfies (iii). Then K is UR and

$$\left[\begin{array}{c|cc} -S_{11}^{-1}(D_{11}^* C_1 + (B_1^*)_{\text{w}} \mathcal{P}) & 0 & -S_{11}^{-1} S_{12} \\ 0 & 0 & 0 \end{array} \right]. \tag{11.26}$$

generate a UR strongly q.r.c.-stabilizing suboptimal H$^\infty$-FI-pair.

There is a suboptimal H$^\infty$-SF-operator iff $S_{22} \ll 0$; if this is the case, then (11.17) is a UR strongly q.r.c.-stabilizing suboptimal H$^\infty$-SF-operator.

(b) Each of conditions (FI1)–(FI7) of Theorem 11.2.7 is equivalent to (i). If any of these holds, then the assumptions of Proposition 11.2.8 and Theorem 11.2.7 are satisfied.

(c) If (1.) of Theorem 11.1.6 holds, and the CARE has a UR solution with $\mathbb{A}_{\circlearrowleft}$, $\mathbb{C}_{\circlearrowleft}$ and $\mathbb{K}_{\circlearrowleft}$ strongly stable, then $\gamma \ge \gamma_{\mathrm{FI}} = \gamma_0$.

See Theorem 11.2.7 and Corollary 11.2.11 for analogous results for $\mathcal{U}_{\mathrm{exp}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{out}}$ under slightly different assumptions.

Some of the remarks made below Theorem 11.1.3 apply here, too, with some minor modifications; in particular, also here we have $\gamma_0 = \gamma_{\mathrm{FI}}$ ($= \gamma_{\mathrm{SF}}$ if $D_{12} = 0$).

An important special case of "(2.)" is the case where Σ is strongly stable and $\mathbb{D} \in \widetilde{\mathcal{A}}$ (take $\widetilde{K} = 0$). By Example 11.3.7(c), even for strongly stable Σ, condition "$\mathbb{D} \in \widetilde{\mathcal{A}}$" cannot be weakened to e.g., "$\mathbb{D} \in$ ULR" without making condition (iii) is strictly stronger than (i)–(ii) (indeed, the IARE corresponding to (iii) has a irregular (i.e., non-WR) q.r.c.-stabilizing solution leading to the minimax control and to a suboptimal irregular H^∞-FI-pair (there are also regular H^∞-FI-pairs, e.g., $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \end{array}\right]$), but the CARE does not have a stabilizing solution).

As noted below Theorem 11.1.6, the "almost equivalence" of (c) suffices for the binary search for γ_{FI}, hence we can use the simpler stabilization condition of (c) (instead of q.r.c.-stabilization) in this strictly nonsingular case.

Since $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ means requiring the (effective) input ($\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$) and output ($y = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$) to be stable ($\in \mathrm{L}^2$), we now must have $u, z \in \mathrm{L}^2$, for all $x_0 \in H$ and $w \in \mathrm{L}^2(\mathbf{R}_+; W)$. Because $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{str}}$ (by Lemma 8.3.3), it follows that then also the state becomes strongly stable ($\|x(t)\|_H \to 0$ as $t \to +\infty$).

Therefore, a H^∞-FI-pair is now an admissible state feedback pair of form $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{smallmatrix} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{smallmatrix}\right]$ s.t. $u, z \in \mathrm{L}^2$ for all $x_0 \in H$ and $w \in \mathrm{L}^2(\mathbf{R}_+; W)$ in Figure 11.1 (see also (11.8)–(11.11) and Remark 11.2.5). By the above, it follows that $\|x(t)\|_H \to 0$, as $t \to +\infty$, for such x_0, w. Nevertheless, for $u_{\circlearrowleft} \neq 0$ the signals u and z (and x) are not required to be stable, hence the middle row of $\mathbb{D}^{\curvearrowleft}$ may be unstable, although we do have $u, z \in \mathrm{L}^2$ and x vanishing for any compactly supported $u_{\circlearrowleft} \in \mathrm{L}^2$, by (6.9).

Thus, for the solutions mentioned in (c), we do not know whether the H^∞-FI-pair defined by (11.26) is stabilizing. Therefore, if one has used (c) to find an estimate on $\gamma_{\mathrm{FI}} = \gamma_0$, one might wish to either 1. verify directly whether the corresponding (strongly internally and output-stabilizing) K is (strongly) stabilizing or 2. increase γ slightly to guarantee that $\gamma > \gamma_{\mathrm{FI}}$ and then find a internally stabilizing solution of the CARE (i.e., one with stable $\mathbb{A}_{\circlearrowleft}$), because then such a solution is necessarily the strongly q.r.c.-stabilizing one, by uniqueness (see Theorem 9.8.12(a)).

Proof of Theorem 11.1.5: (Note from the proof below that we can replace "UR" by "ULR" throughout this theorem.)

0.1° *Remark: Assumptions (1.)–(2.):* Here $\Sigma_\flat$ is the closed-loop system corresponding to $\widetilde{K}$ (cf. Definition 6.6.10). Recall from Theorem 6.6.28 that if $\widetilde{K}$ and some output injection pair are jointly strongly stabilizing and (I/O-)detecting for Σ, then $\widetilde{K}$ is strongly q.r.c.-stabilizing.

If Σ is strongly stable and $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$, then we can take $\widetilde{K} = 0$ in (2.).

0.2° *Remark:* $\mathbb{D}_{11}$ *is I-coercive* Assumption (1.) is equivalent to the (positive) I-coercivity of $\mathbb{D}_{\flat 11}$ (equivalently, of $\mathbb{D}_{11}$; in particular, it is independent of $\widetilde{K}$), by Lemma 8.4.11(b1).

1° *The equivalence of (i)–(iii), (a) and (b):* Except for (c), all claims in the lemma follow from Theorem 11.2.7 and Corollary 11.2.11 (use the fact that a solution of (11.25) is necessarily UR, by Lemma 9.11.5(e); note that Hypothesis 11.2.1 is satisfied, by 0.2°).

(c) By Theorem 9.8.5 and Theorem 9.9.1(c3), $\mathcal{P}$ is $\mathcal{U}_{\mathrm{str}}$-stabilizing, hence $\mathcal{U}_{\mathrm{out}}$-stabilizing. By Lemma 11.2.14(4.)&(b)&(a2) (with $s = +\infty$), it follows that (11.48) is a H^∞-FI-pair satisfying $\|\mathbb{D}_{12}^{\frown}\| \le \gamma$ (in fact, $\|\mathbb{D}_{12}^{\frown} u\|_2 < \gamma \|u\|_2$ for all $u \in \mathrm{L}^2(\mathbf{R}_+;U)$, by the remark in the proof of (a2)). Thus, then $\gamma \ge \gamma_{\mathrm{FI}}$. □

For $\mathcal{U}_{\exp}$, one can obtain the equivalence "(ii)⇔(iii)" directly from Proposition 11.2.19 and Lemma 11.2.13 (or Lemma 11.2.14), provided that Σ is sufficiently regular (e.g., if Hypothesis 9.2.1 holds).

For any other $\mathcal{U}_*^*$ than $\mathcal{U}_{\exp}$, there seems to be a gap between the proposition and the lemma: it seems that $\Sigma^{\frown}$ need not be stable enough. In Theorem 11.1.5, we used the q.r.c.-property to reduce the problem to the stable case, and the (stable) spectral factorization properties of $\widetilde{\mathcal{A}}$ to guarantee the existence of a strongly stabilizing solution in case that $\gamma > \gamma_0$.

In practice, the cost function is often of form $\mathcal{J}(x_0,u,w) := \|z_1\|_2^2 + \|Ru\|_2^2 - \gamma^2 \|w\|_2^2$, where $R^*R \gg 0$. This is the case when $\left[\begin{array}{c|cc} \mathbb{C}_1 & \mathbb{D}_{11} & \mathbb{D}_{12} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{C}_{1a} & \mathbb{D}_{11a} & \mathbb{D}_{12a} \\ 0 & R & 0 \end{array}\right]$ and $\mathcal{B}(U) \ni R^*R \ge \varepsilon_+^2 I$ for some $\varepsilon_+ > 0$; obviously, condition (11.27) is then satisfied. Such a cost function forces u to be stable (for stable outputs, i.e., for a finite cost) and makes it possible to fill the gap mentioned above (as noted in (c) above):

Theorem 11.1.6 ($\mathcal{U}_{\mathbf{out}}$: $\mathbf{H}^\infty$ FICP ⇔ B_{w}^*-CARE) *Assume that* $\gamma > 0$ *and* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$, *and that (1.) and (2.) hold.*

(1.) **(Strict nonsingularity)** *We have* $D_{11}^* D_{11} \gg 0$, *and there is* $\varepsilon_+ > 0$ *s.t.*

$$\|\mathbb{C}_1 x_0 + \mathbb{D}_{11} u + \mathbb{D}_{12} w\|_2 \ge \varepsilon_+ \|u\|_2 \quad (u \in \mathrm{L}^2_{\varepsilon_+}(\mathbf{R}_+;U),\ w \in \mathrm{L}^2(\mathbf{R}_+;W),\ x_0 \in H). \tag{11.27}$$

(2.) **(Regularity)** *Assume that* Σ *and* J_γ *satisfy Hypothesis 9.2.1, and that* $D^* J_\gamma D \in \mathcal{GB}(U \times W)$ *or* $B \in \mathcal{B}(U,H)$.

If condition (iii) below holds, then there is a H^∞*-FI-pair s.t.* $\|w \mapsto z\| \le \gamma$. *Conversely, if there is a* H^∞*-FI-pair s.t.* $\|w \mapsto z\| < \gamma$, *then (iii) holds.*

(iii) $D_{12}^* D_{12} - D_{12}^* D_{11} (D_{11}^* D_{11})^{-1} D_{11}^* D_{12} \ll \gamma^2 I$, *and (the* B_{w}^**-CARE)*

$$\left(B_{\mathrm{w}}^* \mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1\right)^* (D^* J_\gamma D)^{-1} \left(B_{\mathrm{w}}^* \mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1\right) = A^* \mathcal{P} + \mathcal{P} A + C_1^* C_1 \tag{11.28}$$

has a nonnegative $\mathcal{U}_{\mathrm{out}}$*-stabilizing solution* $\mathcal{P} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$.

Moreover, the following hold:

(a1) If $(\mathcal{P}, S, K)$ satisfies (iii), then (11.16) generate a ULR H^∞-FI-pair s.t. $\|w \mapsto z\| = \|\mathbb{D}_{12}^{\curvearrowright}\| \le \gamma$.

(a2) If there is a suboptimal H^∞-SF-operator, then $\|D_{12}\| < \gamma$. Conversely, if $\|D_{12}\| < \gamma$ and (iii) has a solution $(\mathcal{P}, S, K)$, then $K_1 =$(11.17) is a ULR H^∞-SF-operator s.t. $\|w \mapsto z\| = \|\mathbb{D}_{12}^{\curvearrowright}\| \le \gamma$.

The above "almost equivalence" is in practice as good as an equivalence: if we wish to find a state feedback controller s.t. $\|w \mapsto z\|$ is (approximately) minimized, then we can use a binary search over γ (solve (iii) above for different values of γ). See also the remarks below Theorem 11.1.3.

See the remarks below Theorem 11.1.5 for $\mathcal{U}_{\text{out}}$ and for how stable is closed-loop system (in Figure 11.1) corresponding the H^∞-SF-operator defined by (11.17). See Definition 9.8.1 for "$\mathcal{U}_{\text{out}}$-stabilizing" (which is equivalent to "one with stable $\mathbb{A}_\circlearrowleft$, $\mathbb{C}_\circlearrowleft$ and $\mathbb{K}_\circlearrowleft$" if Σ is strongly q.r.c.-stabilizable).

Note from Theorem 6.7.15(c2) that if Σ is estimatable, then (iii) is equivalent to (i)–(iii) of Theorem 11.1.3 (whose (a) and (b) then apply).

Proof of Theorem 11.1.6: (See Definition 9.2.6 for the B_{w}^*-CARE.)

0.1° *Remark on (2.):* Condition $D^* J_\gamma D \in \mathcal{GB}(U \times W)$ can be omitted if $B \in \mathcal{B}(U, H)$ (use Remark 9.9.14(1.)&(b) in 2°) or $D_{12}^* D_{12} - D_{12}^* D_{11} (D_{11}^* D_{11})^{-1} D_{11}^* D_{12} \ll \gamma^2 I$ (cf. 0.3° of the proof of Theorem 11.1.3).

0.2° *Remark: $\mathbb{D}_{11}$ is I-coercive* This follows from (1.). Thus, Hypothesis 11.2.1 is satisfied.

1° *(iii)$\Rightarrow$ "almost H^∞-FI-pair":* Since $S := D^* J_\gamma D \in \mathcal{GB}(U \times W)$, by 0.3° of the proof of Theorem 11.1.3, $\mathcal{P}$ is a solution of the B_{w}^*-CARE, hence an admissible and ULR solution of the CARE, by Proposition 9.2.7.

By Lemma 11.2.14(4.)&(b)&(a2) (with $s = +\infty$), it follows that (11.48) is a H^∞-FI-pair satisfying $\|\mathbb{D}_{12}^{\curvearrowright}\| \le \gamma$ (in fact, $\|\mathbb{D}_{12}^{\curvearrowright} u\|_2 < \gamma \|u\|_2$ for all $u \in \mathrm{L}^2(\mathbf{R}_+; U)$, by the remark in the proof of (a2)).

2° H^∞*-FI-pair$\Rightarrow$(iii):* (We give the proof for the case $D^* J D \in \mathcal{GB}(U \times W)$; use first Remark 9.9.14(1.)&(b) under the alternative assumptions $B \in \mathcal{B}(U, H)$.)

By Proposition 11.2.19(a1) and Theorem 9.2.9(i)&(iv)&(a2), the B_{w}^*-CARE (and hence the CARE and the IARE) has an ULR $\mathcal{U}_{\text{out}}$-stabilizing solution $(\mathcal{P}, S, K)$ with $S = D^* J_\gamma D$. By Proposition 11.2.19(d1), $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$.

(a1) See 1°.

(a2) Drop Lemma 11.2.14(b) from 1° and replace (d1) by (d2) in 2°. □

In [IOW], the signature condition on S is formulated by using the following equivalence:

Lemma 11.1.7 *Instead of Standing Hypothesis 11.1.1, assume only that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in$ WPLS$(U \times W, H, Y)$ is WR, $J = J^* \in \mathcal{B}(Y)$ and $\gamma > 0$.*

(Even) then the CARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ s.t. $S_{11} \gg 0$ and $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$ iff the IARE has a WR $\mathcal{U}_*^*$-stabilizing solution

$(\mathcal{P}, J_\gamma, \begin{bmatrix} \widetilde{\mathbb{K}} & \big| & \widetilde{\mathbb{F}} \end{bmatrix})$ *s.t.* $\widetilde{X}_{11}, \widetilde{X}_{22} \in \mathcal{GB}$, $\widetilde{X}_{21} = 0$, *where* $\widetilde{X} := I - \widetilde{F}$. *All prefices and suffices apply (see Definition 9.8.1).*

Indeed, this latter formulation is equivalent to a Kalman–Popov–Yakubovich system formulation in terms of [IOW]. Note that if U or W is finite-dimensional, then an equivalent condition is that the IARE has WR $\mathcal{U}_*^*$-stabilizing solution with $X_{21} = 0$ (i.e., with no feedforward from u to w).

Proof: (Here Y is an arbitrary Hilbert space.) Let $(\mathcal{P}, S, K)$ be as above. By Lemma 11.3.13(i)&(iii'), there is $\widetilde{X}$ as above s.t. $\widetilde{X}^* J_\gamma \widetilde{X} = S$. By Theorem 9.8.12(s1), the latter condition is satisfied. The converse is obtained analogously. The last claim can be observed from the $\Sigma_{\circlearrowleft E}$ of Theorem 9.8.12(s1). □

The following lemma clarifies our basic concepts:

Lemma 11.1.8 (FCC ⇔ optimizable) *Let* $\mathcal{U}_*^* = \mathcal{U}_{\exp}$. *Then* $\mathcal{U}_{\mathrm{u}}(x_0, w) \neq \emptyset$ *for all* $x_0 \in H$ *and all* $w \in \mathrm{L}^2(\mathbf{R}_+; W)$ *iff* (A, B_1) *is optimizable.*

If there is a H^∞*-FI-pair, then* (A, B_1) *is exponentially stabilizable (hence optimizable). Conversely, if* (A, B_1) *is exponentially stabilizable (or optimizable) and* $\pi_{[0,1)} \mathbb{A} B_1 u_0 \in \mathrm{L}^1([0,1); H)$ *for all* $u_0 \in U$ *then there is a* H^∞*-FI-pair.*

Thus, if there is a "stabilizing u" for each x_0 (and $w = 0$), then, actually, there is a "stabilizing u" for each x_0 and w_0:

Proof: 1° The equivalence follows from "(i)⇔(ii)" of Lemma 11.6.4, by discretization (note that "(i)⇐(ii)" is trivial).

2° If $\begin{bmatrix} \mathbb{K} & \big| & \mathbb{F}_1 & \mathbb{F}_2 \end{bmatrix}$ is a H^∞-FI-pair (equivalently, an exponentially stabilizing pair for Σ with second row equal to zero), then $\begin{bmatrix} \mathbb{K} & \big| & \mathbb{F}_1 \end{bmatrix}$ is obviously an exponentially stabilizing pair for $\begin{bmatrix} \mathbb{A} & \big| & \mathbb{B}_1 \end{bmatrix}$.

3° Assume that (A, B_1) is exponentially stabilizable (or optimizable) and $\pi_{[0,1)} \mathbb{A} B_1 u_0 \in \mathrm{L}^1([0,1); H)$ for all $u_0 \in U$. Then there is an exponentially stabilizing $\widetilde{K}_1 \in \mathcal{B}(H, U)$ for $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} \end{array}\right]$, hence $\begin{bmatrix} \widetilde{K}_1 \\ 0 \end{bmatrix}$ is a H^∞-SF-operator for Σ (by Lemma 6.6.11, it is admissible for Σ; obviously the two closed-loop semigroups $\mathbb{A}_\flat := \mathbb{A} + \mathbb{B}_1 (I - \widetilde{\mathbb{F}}_{11})^{-1} \widetilde{\mathbb{K}}_1$.) are equal, hence exponentially stable).

(This converse holds also under much weaker conditions: The exponential stabilizability of (A, B_1) means the existence of an admissible state feedback pair $\begin{bmatrix} \widetilde{\mathbb{K}}_1 & \big| & \widetilde{\mathbb{F}}_{11} \end{bmatrix}$ for $\begin{bmatrix} \mathbb{A} & \big| & \mathbb{B}_1 \end{bmatrix}$; we just have to know that $\widetilde{K}_1$ and B_2 "fit" to the same WPLS, i.e., that (6.100) is satisfied under substitutions $C \mapsto \widetilde{K}_1$, $B \mapsto B_2$, since then we obtain an admissible state feedback pair with same (exponentially stable) closed-loop semigroup $\mathbb{A}_\flat$, as above.) □

Next we give an example, where the signature condition $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$ is satisfied but the stronger condition $S_{22} \ll 0$ is not, so that there is a suboptimal H^∞-FI-pair but no suboptimal H^∞-SF-operators (for $\gamma < 1$):

Example 11.1.9 **($\gamma_{\mathrm{SF}} > \gamma_{\mathrm{FI}}$)** Let $\mathbb{D} = D = \left[\begin{smallmatrix} I & I \\ 0 & I \end{smallmatrix}\right]$, $J = J_\gamma$, $B = 0 = C$, $A = -I$ and $U = W = Z$. Then $\mathbb{D}^* J_\gamma \mathbb{D} = \begin{bmatrix} I & I \\ I & (1-\gamma^2)I \end{bmatrix} = D^* J_\gamma D =: S \in \mathcal{GB}$, so that the CARE $-2\mathcal{P} = -K^* S K$, $K = 0$ has the unique solution $\mathcal{P} = 0$ (which is exponentially stabilizing).

By Theorem 11.1.3(iii)&(a), the pair (11.16)$= \left(\begin{smallmatrix} 0 & 0 & -I \\ 0 & 0 & 0 \end{smallmatrix}\right)$ is a suboptimal H$^\infty$-FI-pair for Σ (indeed, it leads to $\mathbb{D}_{12}^{\circlearrowleft} = 0$, by (11.8), hence to $\|w \mapsto z\| = \|\mathbb{D}_{12}^{\circlearrowleft}\| = 0 < \gamma$). Since this holds for any $\gamma > 0$, we have $\gamma_{\mathrm{FI}} = 0$.

However, each H$^\infty$-SF-operator $\left(\begin{smallmatrix} K_1 \\ 0 \end{smallmatrix}\right) \in \mathcal{B}(H_1, U \times W)$ leads to $u = 0$ (or $\mathbb{D}_{12}^{\circlearrowleft} = \mathbb{D}_{12} = I$, since $\mathbb{X} = I = \mathbb{M}$, because $B = 0$), hence to the cost

$$\|z\|_2 := \|\mathbb{D}_{11} u + \mathbb{D}_{12} w\|_2 = \|w\|_2, \tag{11.29}$$

so that any H$^\infty$-SF-operator is suboptimal iff $\gamma > 1$. Thus, $\gamma_{\mathrm{SF}} = 1$, whereas $\gamma_{\mathrm{FI}} = 0$ (in accordance to Theorem 11.1.3(iii)&(a), since $S_{22} - S_{21} S_{11}^{-1} S_{12} = -\gamma^2 \ll 0$ for all $\gamma > 0$ but $S_{22} = 1 - \gamma^2 \ll 0$ iff $\gamma > 1$). $\triangleleft$

The proofs of our H$^\infty$ FICP results are based on the "H$^\infty$ minimax game" (11.58) (not "maximin", since traditionally the cost $-\mathcal{J}$ has been used).

This game is often considered as a Stackelberg game where w, the *disturbance* (or evil player, nature, uncertainties, modeling errors, sensor noise, dark side of the force, ...) tries to maximize the cost, whereas u, the *control* (or good player, control engineer, our hero) bravely defends the stability of the system by trying to minimize the cost. Lucky for the good ones, the good player is allowed to act last (although not in a noncausal way, i.e., it has no knowledge on future disturbance, just past and present).

If $\gamma_0 < \gamma$, then, obviously, $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ is the unique solution of the game for $x_0 = 0$; if $\gamma > \gamma_0$, then there can be no solution (i.e., $\max_w \min_u \mathcal{J}(x_0, \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]) = \infty$ for all x_0, since the quadratic term dominates the cost for a fixed x_0).

Naturally, $\mathcal{U}_*^*(x_0) \neq \emptyset$ is a necessary condition for each $x_0 \in H$; by Proposition 11.2.19(a), it is also sufficient for $\gamma_0 < \gamma$. Thus, this game is intimately connected to the H$^\infty$ FICP.

For any $x_0 \in H$, the solution $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$, being a saddle point of the (J-coercive) cost function, constitute the unique J-critical control. This leads to the existence of a unique $\mathcal{U}_*^*$-stabilizing solution of the Riccati equation (11.15), with the saddle point ("minimax", worst disturbance and best control) input $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ being the corresponding state feedback for each initial state $x_0 \in H$, by the results of Section 9.9. See Proposition 11.2.19 for the proofs.

Conversely, given a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ of the Riccati equation (11.15) satisfying the signature condition above the equation, we choose a modified $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, \widetilde{S}, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ of the equation s.t. $\widetilde{S} = \begin{bmatrix} \widetilde{S}_{11} & 0 \\ 0 & \widetilde{S}_{22} \end{bmatrix}$, $\widetilde{S}_{11} \gg 0$, $\widetilde{S}_{22} \ll 0$. It follows that even when we drop the bottom row of $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ (to obtain a H$^\infty$-FI-pair), the state feedback remains suboptimal, because if the disturbance ("the evil player") dares to deviate from the saddle point value, it is punished by the negative cost $\langle w_{\circlearrowleft}, \widetilde{S}_{22} w_{\circlearrowleft} \rangle < 0$ (as in (9.139)), where $w_{\circlearrowleft}$ is the deviation. This leads to the suboptimal H$^\infty$-FI-pair (11.16). See Lemma 11.2.14 for the proofs.

As mentioned above, these facts lead to the proofs of our results. In addition, this shows that for each $x_0 \in H$ and $w \in \mathrm{L}^2(\mathbf{R}_+;H)$, the cost $\|z\|_2^2 - \gamma^2\|w\|_2^2$ is at most (11.58). Thus, although the suboptimality requirement was posed for $x_0 = 0$ only, our suboptimal controller (11.16) (or (11.48) actually makes $\|z\|_2$ small also for $x_0 \neq 0$ (for the "worst w for a given $\|w\|_2$", the controller minimizes the cost, hence also $\|z\|_2$, and for the other w's with same norm, the value of $\|z\|_2$ becomes less though not necessarily minimal). For the same reasons, the same holds to the H$^\infty$-SF-operator (11.17). We emphasize this observation:

Remark 11.1.10 *The controllers (11.16) and (11.17) are "worst-case-optimal" in certain sense also for $x_0 \neq 0$.* □

(By Theorem 9.9.1(h), the reference to (9.139) is allowed also for $\mathcal{U}_*^* \neq \mathcal{U}_{\mathrm{exp}}$ (for $x_0 \in H$, $u_{\circlearrowleft} = 0$, $w_{\circlearrowleft} \in \mathrm{L}^2(\mathbf{R}_+;W)$), because $\mathbb{C}_{\circlearrowleft}$ must be stable for a $\mathcal{U}_*^*$-stabilizing solution, and also $\begin{bmatrix} \mathbb{D}_{12}^{\frown} \\ I \end{bmatrix}$ is stable, by Lemma 11.2.14(a).)

We finish this section by a remark on the dual of the FICP:

Remark 11.1.11 (Full Control Problem = FICP$^{\mathbf{d}}$) *The dual problem of the* H$^\infty$ *FICP is the* H$^\infty$ Full Control Problem, *where one looks for a (suitably) stabilizing output injection pair of form*

$$\left[\begin{array}{c} \mathbb{H} \\ \hline \mathbb{G} \end{array}\right] = \left[\begin{array}{cc} 0 & \mathbb{H}_2 \\ \hline 0 & \mathbb{G}_{12} \\ 0 & \mathbb{G}_{22} \end{array}\right] \tag{11.30}$$

for some $\Sigma = \left[\begin{array}{c|cc} \mathbb{A} & 0 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & I & \mathbb{D}_{12} \\ \mathbb{C}_2 & 0 & \mathbb{D}_{22} \end{array}\right] \in \mathrm{WPLS}(Z \times W, H, Z \times Y)$ *(i.e., we may inject only the lower output, the "measurement"), s.t.* $\|\mathbb{D}_{\sharp\,12}\| < \gamma$ *(see Definition 6.6.21).*

By duality, we obtain a solution for this problem from any of the above solutions for H$^\infty$ *FICP.*

For example, if $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, $\mathbb{D}_{22}^{\mathrm{d}}$ is I-coercive, $\mathbb{A}B_2 \in \mathrm{L}^1_{\mathrm{loc}}$, $C_{\mathrm{w}}\mathbb{A} \in \mathrm{L}^1_{\mathrm{loc}}$, and $C_{\mathrm{w}}\mathbb{A}B_2 \in \mathrm{L}^1_{\mathrm{loc}}$, then there is an exponentially stabilizing output injection pair of form (11.30) for Σ s.t. $\|\mathbb{D}_{\sharp\,12}\| < \gamma$ iff the following hold:

$$D_{22}D_{22}^* \gg 0, \;\; D_{12}D_{12}^* - D_{12}D_{22}^*(D_{22}D_{22}^*)^{-1}D_{22}D_{12}^* \ll \gamma^2 I, \tag{11.31}$$

and there is a nonnegative exponentially stabilizing solution $\mathcal{P} \in \mathcal{B}(H)$ of the Riccati equation

$$\begin{cases} HSH^* = A\mathcal{P} + \mathcal{P}A^* + B_2B_2^*, \\ \quad S = \begin{bmatrix} D_{22}^* & D_{12}^* \\ 0 & I \end{bmatrix}^* J_\gamma \begin{bmatrix} D_{22}^* & D_{12}^* \\ 0 & I \end{bmatrix}, \\ \quad H^* = -S^{-1}\left(\begin{bmatrix} C_2 \\ C_1 \end{bmatrix}_{\mathrm{w}} \mathcal{P} + \begin{bmatrix} D_{22} \\ D_{12} \end{bmatrix} B_2^* \right) \end{cases} \tag{11.32}$$

s.t. $\lim_{s\to+\infty} \begin{bmatrix} C_2 \\ C_1 \end{bmatrix}_{\mathrm{w}} \mathcal{P}(s-A^*)^{-1} \begin{bmatrix} C_2^* & C_1^* \end{bmatrix} = 0.$

(Apply Theorem 11.1.4 to the system Σ_Y :=(12.85) for the proof.) □

The H$^\infty$ FCP is also called the H$^\infty$ filter problem, since it means that $\gamma > \|\mathbb{D}_{\sharp\,12}\|_{\mathrm{TIC}} = \|\mathbb{D}_{12} + \widetilde{\mathbb{M}}_{12}\mathbb{D}_{22}\|_{\mathrm{TIC}}$, where $\widetilde{\mathbb{M}} := (I - \mathbb{G})^{-1}$, i.e., that $-\widetilde{\mathbb{M}}_{12}y$ is an estimate of $z := \mathbb{D}_{12}w$, where $y := \mathbb{D}_{22}w$, $w \in \mathrm{L}^2(\mathbf{R}_+;W)$, with error of norm less than $\gamma\|w\|_2$.

Notes

The H^∞ problems were introduced by G. Zames [Zames]. The first solutions to the problem used frequency-domain methods; their history can be found in [Francis]. Our stable case solution (Section 11.3) is partially based on such methods. The state space solution of this section was given by J. Doyle et al. in [DGKF], for finite-dimensional systems under several simplifying assumptions, and that article also contains the early history of state-space methods. All these works provide solutions to the H^∞ 4BP, see the notes on p. 706 for more on that problem, whose special case the FICP is.

The formulation (11.1) of the H^∞ FICP has been used in several earlier results. The equivalence of (ii) and (iii) in Theorem 11.1.3 is an extension of [DGKF, p. 836], [ZDG, Section 16.4] and [GL, Section 6.3]. [ZDG] and [DGKF] also provide an "all suboptimal controllers" formula, whose extension is contained in Theorem 12.1.8.

The SF-variant of (ii)–(iii) (of, e.g., Theorem 11.1.3(a)) is an extension of [Keu, Theorem 4.4], [IOW, Theorem 10.9.1] and [LR, Theorem 20.2.1]. The results in [Keu] also contain the equivalence with (i).

Except for [Keu], which treats Pritchard–Salamon systems (and hence assumes that $B \in \mathcal{B}(U,H)$), all of the above results assume that U, W, H and Z are finite-dimensional, but otherwise [IOW] has as general assumptions as we do. (Use Proposition 10.3.2 to observe that the assumptions of the above results are stronger than those of ours. Note also that since all results mentioned above assume a bounded B, Hypothesis 9.2.1 is satisfied.)

In the general case (see, e.g., (11.25), (11.17) and (11.26)), the formulae become similar to their discrete-time counterparts (e.g., $S \neq D^*JD$), given in Section 11.5 and in, e.g., [GL, (B.2.31), p. 487] and in [GL, Remark B.2.1, p. 488].

For parabolic (analytic) systems, the equivalence of (i) and (iii) is given in [MT94a] (repeated in [LT00a]), for a setting that allows the input and output operators to be more unbounded than we do (in Theorem 9.5.11; they take $\gamma = 0$ but allow for any $\beta < 1$). The cost function in [MT94a] is rather specific (namely $C = \left[\begin{smallmatrix} R \\ 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, $R \in \mathcal{B}(H,Y)$, so that the signature condition becomes redundant; condition (iii) is also otherwise modified due to different regularity assumptions) and the system is assumed to be estimatable; moreover, condition (ii) is not treated.

Almost the same result is given in [MT94b], for all estimatable WPLSs that have C, D, R as above (in particular, C must be bounded, hence these systems are ULR) with the additional requirements that B_2 is bounded and Σ is exactly reachable in finite time (but $\mathbb{A}$ need not be analytic). In this result, the CARE is treated as in Section 9.7; in particular, non-well-posed solutions are allowed. Thus, Proposition 11.2.19 and Theorem 9.7.3 extend the necessity part of this result to arbitrary regular WPLSs. However, the converse is not true without suitable signature conditions, as illustrated in Example 11.2.17.

The nonsingularity assumptions of all above results are the same or stronger than those of ours. For singular finite-dimensional systems, the H^∞ problems have been solved in [Stoorvogel].

In Theorems 11.1.3 and Theorem 11.1.6, we have not stated that $\gamma_0 = \gamma_{\mathrm{FI}}$; however, this follows from the theorems if (2.) holds independently of γ, e.g., if Hypothesis 9.2.2 is satisfied.

The state-space results mentioned above treat the case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. In the frequency-domain setting, one usually works with stable rational H$^\infty$ transfer functions, in which case there is no difference between "$\mathcal{U}_{\mathrm{exp}}$ and $\mathcal{U}_{\mathrm{out}}$", but there are also more general solutions, involving only the I/O maps (or transfer functions), such as in, e.g., [FF].

See the notes on p. 652 for solutions of the H$^\infty$ FICP in terms of spectral and "J-lossless" factorizations. Historical remarks on the stable H$^\infty$ FICP are given on p. 669.

11.2 The H^∞ FICP: proofs

Thu chanted a song of wizardry,
of piercing, opening, of treachery,
revealing, uncovering, betraying.
Then sudden Felagund there swaying
sang in answer a song of staying,
resisting, battling against power,
of secrets kept, strength like a tower,
and trust unbroken, freedom, escape;
of changing and of shifting shape,
of snares eluded, broken traps,
the prison opening, the chain that snaps.
— J.R.R. Tolkien (1892–1973), "The Lay of Leithian"

In this section, we shall prove the results of the previous section and present some new, more technical ones. In Theorem 11.2.7 we solve the H^∞ FICP in terms of J-lossless factorizations, assuming q.r.c.-stabilizability with MTIC. The assumptions are then weakened in Proposition 11.2.8. Most other results of this section are rather technical generalizations, parts of proofs, or counter-examples against further reduction of assumptions.

In addition to Standing Hypotheses 11.0.1 and 11.1.1, we assume the following:

Standing Hypothesis 11.2.1 (H^∞ Full-Information Control Problem (FICP)) *Throughout this section, we make the following assumptions: Hypothesis 9.0.1 is satisfied (with $U \mapsto U \times W$ and $Y \mapsto Z \times W$), $\gamma > 0$ and there is $\varepsilon_+ > 0$ s.t. $\|\mathbb{D}_{11} u\|_2 \geq \varepsilon_+ \| \left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right] \|_{\mathcal{U}_*^*}$ for all $u \in \mathcal{U}_{\mathrm{u}}(0,0)$.*

The first assumption says that $\mathcal{U}_*^*$ is something reasonable (and it is satisfied if, e.g., $\mathcal{U}_*^* \in \{\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}, \mathcal{U}_{\mathrm{exp}}\}$). The third assumption is the standard nonsingularity assumption:

Lemma 11.2.2 ($\mathbb{D}_{11}$ I-coercive) *(Drop Standing Hypothesis 11.2.1 for the moment).*

If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ (resp. $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{str}}$, $\mathcal{U}_{\mathrm{sta}}$) and $\gamma > 0$, then Hypothesis 11.2.1 holds iff $\mathbb{D}_{11}$ is I-coercive over $\mathcal{U}_{\mathrm{exp}}$ (resp. $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{str}}$, $\mathcal{U}_{\mathrm{sta}}$).

Thus, we obtain several equivalent assumptions from Proposition 10.3.2 (resp. Proposition 10.3.1, the two propositions and Lemma 8.3.3, -"-). Recall from Definition 8.4.1 that I-coercivity is equivalent to positive I-coercivity. It is up to the reader to choose $\mathcal{U}_*^*$, i.e., to decide which controls shall be allowed (cf. (11.6) and (11.7)).

Proof of Lemma 11.2.2: (By I-coercivity, we refer to realization $\Sigma_{11} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} \end{array}\right]$ of $\mathbb{D}_{11}$.)

The lemma follows directly from the definitions. Indeed, we obviously have

$$\| \left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right] \|_{\mathcal{U}_{\mathrm{exp}}^{\Sigma}} = \|u\|_{\mathcal{U}_{\mathrm{exp}}^{\Sigma_{11}}} \quad \text{for all} \quad u \in \mathcal{U}_{\mathrm{u}}(0,0) = \mathcal{U}_{\mathrm{exp}}^{\Sigma_{11}}(0); \tag{11.33}$$

the same holds with $\mathcal{U}_{\text{out}}$, $\mathcal{U}_{\text{str}}$ or $\mathcal{U}_{\text{sta}}$ in place of $\mathcal{U}_{\text{exp}}$. □

In addition to Definition 11.1.2, we need some extra notation:

Definition 11.2.3 *Throughout this chapter, we use also the following notation:*

$$\mathbb{D}_1 := \begin{bmatrix}\mathbb{D}_{11}\\ 0\end{bmatrix} \quad \mathbb{D}_2 := \begin{bmatrix}\mathbb{D}_{12}\\ I\end{bmatrix}, \quad Y := Z\times W, \quad J := J_\gamma := \begin{bmatrix} I & 0\\ 0 & -\gamma^2 I\end{bmatrix} \tag{11.34}$$

The cost function is given by

$$\mathcal{J}(x_0,u,w) := \mathcal{J}(x_0,[\begin{smallmatrix}u\\ w\end{smallmatrix}]) := \langle y, J_\gamma y\rangle_{\mathrm{L}^2(\mathbf{R}_+;Y)}, \text{ where } y := \mathbb{C}x_0 + \mathbb{D}[\begin{smallmatrix}u\\ w\end{smallmatrix}] \tag{11.35}$$

(for $x_0\in H$, $[\begin{smallmatrix}u\\ w\end{smallmatrix}]\in \mathcal{U}_*^*(x_0)$).

Thus, $\mathbb{D} = \begin{bmatrix}\mathbb{D}_1 & \mathbb{D}_2\end{bmatrix} \in \mathcal{B}(U\times W, Y)$; this short-hand-notation makes many formulae simpler.

Lemma 11.2.4 *We have $\gamma > \gamma_0$ iff there is $\varepsilon > 0$ s.t. $\inf_{u\in\mathcal{U}_{\mathrm{u}}(0,w)} \mathcal{J}(0,u,w) \le -\varepsilon\|w\|_2^2$ for all $w\in \mathrm{L}^2(\mathbf{R}_+;W)$.*

A H^∞-FI-pair (or H^∞-SF-operator) is suboptimal for Σ iff $\mathbb{D}_2^{\circlearrowleft *} J_\gamma \mathbb{D}_2^{\circlearrowleft} \ll 0$.

Thus, a H^∞-FI-pair is suboptimal if it makes $\mathcal{J}$ uniformly negative w.r.t. w. H^∞-FI-pairs and H^∞-SF-operators and $\Sigma^{\circlearrowleft}$ are defined as in Definition 11.1.2.

Proof: 1° *Case $\gamma>\gamma_0$:* Given $[\begin{smallmatrix}u\\ w\end{smallmatrix}]\in \mathrm{L}^2(\mathbf{R}_+;U\times W)$, we have $\langle y, J_\gamma y\rangle = \|z\|_2^2 - \gamma^2\|w\|_2^2$, where $[\begin{smallmatrix}z\\ w\end{smallmatrix}] := y := \mathbb{D}[\begin{smallmatrix}u\\ w\end{smallmatrix}]$, hence the cost function $\mathcal{J}$ becomes uniformly negative w.r.t. w ($\mathcal{J}(0,u_w,w)\le -\varepsilon\|w\|_2^2$ for all w and some $\varepsilon>0$) iff the control law $w\mapsto u_w$ makes the norm $\|w\mapsto z\|$ less than γ, i.e., iff $\|z\|_2^2-\gamma^2\|w\|_2^2 \le \varepsilon\|w\|_2^2$ for some $\varepsilon>0$.

2° *Suboptimality:* Since $\mathbb{D}_2^{\circlearrowleft} = \begin{bmatrix}\mathbb{D}_{12}^{\circlearrowleft}\\ I\end{bmatrix}$ (see (11.8)), we have $\mathbb{D}_2^{\circlearrowleft *}J_\gamma\mathbb{D}_2^{\circlearrowleft} = (\mathbb{D}_{12}^{\circlearrowleft})^*\mathbb{D}_{12}^{\circlearrowleft} - \gamma^2 I \ll 0$, iff $\|\mathbb{D}_{12}^{\circlearrowleft}\|<\gamma$, by Lemma A.3.1(e2). □

A H^∞-FI-pair is a state feedback (through u only) pair for which the (controlled) input $[\begin{smallmatrix}u\\ w\end{smallmatrix}]$ is in $\mathcal{U}_*^*$ for all $x_0\in H$ and $w\in \mathrm{L}^2(\mathbf{R}_+;W)$:

Remark 11.2.5 *Let $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right] = \left[\begin{array}{c|cc}\mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12}\\ 0 & 0 & 0\end{array}\right]$ be an admissible state feedback pair for Σ. Then $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is a H^∞-FI-pair iff the closed-loop control $u := \mathbb{K}_1^{\circlearrowleft}x_0 + \mathbb{F}_{12}^{\circlearrowleft}w$ produced by $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is in $\mathcal{U}_{\mathrm{u}}(x_0,w)$ for each $x_0\in H$ and $w\in\mathrm{L}^2(\mathbf{R}_+;W)$. In particular, $\mathbb{C}^{\circlearrowleft}$ and $\mathbb{D}_2^{\circlearrowleft}$ must be stable.*

Thus, if $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ (resp. $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$), then $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is a H^∞-FI-pair iff the controller makes the control, state and output (resp. control and output) stable for all x_0 and w; equivalently, iff $\Sigma^{\circlearrowleft}$ is exponentially stable (resp. iff $\mathbb{C}_1^{\circlearrowleft}$, $\mathbb{K}_1^{\circlearrowleft}$, $\mathbb{D}_{12}^{\circlearrowleft}$ and $\mathbb{F}_{12}^{\circlearrowleft}$ are stable).

Proof: The first equivalence is trivial. Assume that $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ (the case $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ is analogous and hence omitted). By definition, $[\begin{smallmatrix}u\\ w\end{smallmatrix}]\in\mathcal{U}_{\text{exp}}(x_0)$ iff $[\begin{smallmatrix}u\\ w\end{smallmatrix}], x, y\in \mathrm{L}^2$. By Lemma A.4.5, the closed-loop system is exponentially stable iff $x := \mathbb{A}^{\circlearrowleft}x_0\in\mathrm{L}^2$ for all $x_0\in H$; conversely, if this is the case, then also the output $\begin{bmatrix}y\\ u\\ w\end{bmatrix}$ is stable, so that $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ is a H^∞-FI-pair. □

The $\mathcal{U}_{\mathrm{exp}}$-results of Section 11.1 are based on the following:

Proposition 11.2.6 ($\mathcal{U}_{\mathrm{exp}}$: (i)–(iii)) *Suppose that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ and that some of (1.)–(6.) of Remark 9.9.14 hold. Then (i)–(iii) are equivalent:*

(i) $\gamma > \gamma_0$, and there is an exponentially stabilizing H^∞-FI-pair for Σ;

(ii) $\gamma > \gamma_{\mathrm{FI}}$, i.e., there is a suboptimal H^∞-FI-pair for Σ;

(iii) The CARE has a UR exponentially stabilizing solution $(\mathcal{P}, S, K)$ s.t. $\mathcal{P} \geq 0$, $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{11} \ll 0$.

Moreover, if (ii) holds, then the assumptions of Proposition 11.2.8 (also those of (a1) and (a2)) are satisfied, (FI1)–(FI5) hold, and the solution of (iii) is unique and ULR.

Note that "there is a H^∞-FI-pair for Σ" means that Σ is exponentially stabilizable through B_1. Cf. also Lemma 11.1.8.

Proof: 0° *Weakening the assumptions:* In fact, it suffices that $(\Sigma, J_\gamma) \in$ coerciveCARE over $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ except that we have to require an UR state feedback operator instead of a SR one (implications (ii)⇒(i)⇒(iii) are true even without this extra requirement); in particular, also (7.) and (8.) or Remark 9.9.14 with "UR" in place of "SR" are sufficient.

1° *(iii)⇒(ii):* This follows from Proposition 11.2.9 (implication (FI5)⇒(FI2)).

2° *(ii)⇒(i):* This follows from (11.12).

3° *(i)⇒(iii) and Proposition 11.2.8:* Assume (i). By Proposition 11.2.19(a1) and Remark 9.9.14(a), the CARE has an ULR $\mathcal{U}_{\mathrm{exp}}$-stabilizing solution $(\mathcal{P}, S, K)$ (and $\mathbb{D} \in$ ULR). Since $\mathbb{D}_\circlearrowleft^* J_\gamma \mathbb{D}_\circlearrowleft = I^* SI$, by Theorem 9.9.1(g2), also the closed-loop I/O map $\mathbb{D}_\flat$ corresponding to any exponentially stabilizing H^∞-FI-pair has a spectral factorization, by Lemma 6.7.13. Therefore, we can apply Proposition 11.2.8(a1), to observe that $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, so that (iii) and (FI1)–(FI5) hold. □

If Σ is smoothly exponentially stabilizable through u, then we have the classical equivalence (for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$; we also give here a variant of this result for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$):

Theorem 11.2.7 ($\widetilde{\mathcal{A}}$: FICP) *Assume that $\widetilde{K} = \begin{bmatrix} \widetilde{K}_1 \\ 0 \end{bmatrix}$ is a UR state feedback operator for Σ with closed-loop system $\Sigma_\flat$ s.t. $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$. Assume also that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ and $\widetilde{K}$ is exponentially stabilizing (resp. $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ and $\widetilde{K}$ is [q.]r.c.-SOS-stabilizing). Then (FI1)–(FI5) are equivalent:*

(FI1) $\gamma > \gamma_0$; i.e., $\inf_{u \in \mathcal{U}_{\mathrm{u}}(0,\cdot)} \mathcal{J}(0, u, \cdot) \ll 0$;

(FI2) $\gamma > \gamma_{\mathrm{FI}}$; i.e., there is a suboptimal H^∞-FI-pair for Σ;

(FI3) $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat = \mathbb{X}_\natural^* J_1 \mathbb{X}_\natural$, where $\mathbb{X}_\natural, \mathbb{X}_{\natural 11} \in \mathcal{G}\mathrm{TIC}$;

(FI4) the IARE has an exponentially stabilizing (resp. P-[q.]r.c.-SOS-stabilizing) solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$*, and* $\mathcal{P} \geq 0$*, and* $\widetilde{S} := (\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})(s_0)$ *satisfies* $\widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$ *for some (equivalently, all)* $s_0 \in \mathbf{C}$ *with* $\operatorname{Re} s > \max\{0, \omega_A\}$.

(FI5) the CARE (11.36) has a UR exponentially stabilizing (resp. a UR P-[q.]r.c.-SOS-stabilizing) solution $(\mathcal{P}, S, K)$*, and* $\mathcal{P} \geq 0$*,* $S_{11} \gg 0$ *and* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$.

Moreover, the following hold:

(a) There is a suboptimal H^∞*-SF-operator iff (FI5) has a solution with* $S_{22} \ll 0$*; if this is the case, then* K_1 *=(11.40) is a UR exponentially (resp. [q.]r.c.-SOS-) stabilizing suboptimal* H^∞*-SF-operator.*

(b) For any solutions of (FI3)–(FI5) (resp. (FI3)–(FI7)) we have $\mathbb{X}_\natural \in \widetilde{\mathcal{A}}$*,* $\mathbb{X}_{\natural 11} \in \mathcal{G}\widetilde{\mathcal{A}}(U)$*,* $\mathbb{M}_{22} \in \mathcal{G}\widetilde{\mathcal{A}}(W)$*,* $\mathbb{D}, \mathbb{F}, \mathbb{D}_\circlearrowleft, \mathbb{F}_\circlearrowleft, \mathbb{N}, \mathbb{M}^{\pm 1}, \mathbb{X}_\natural^{\pm 1}, \mathbb{D}_\flat \in \mathrm{UR}$.

Moreover, if there is a suboptimal H^∞*-FI-pair, then (11.39) generate a UR suboptimal exponentially (resp. [q.]r.c.-SOS-) stabilizing* H^∞*-FI-pair.*

(c) If any of (FI1)–(FI5) have solutions, then the assumptions of Proposition 11.2.8 are satisfied (also those of (a1)&(a2); in particular, $(\mathcal{P}, S, K)$ *is unique).*

(d) If $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is q.r.c.-SOS-stabilizing, then (FI1)–(FI7) (and (FI8) if* $\widetilde{K}$ *is ULR; and (FI9) if* $\dim U < \infty$ *or* $\dim W < \infty$*) are equivalent:*

(FI6) $\mathbb{D}$ *has a* (J_γ, J_1)*-inner [q.]r.c.f.* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *with* $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}(W)$*;*
(FI7) $\mathbb{D}$ *has a* (J_γ, J_1)*-lossless [q.]r.c.f.* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *s.t.* $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$*;*
(FI8) $\mathbb{D}$ *has a* (J_γ, J_1)*-lossless [q.]r.c.f.* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *s.t.* $M_{22} \in \mathcal{G}\mathcal{B}(W)$*;*
(FI9) $\mathbb{D}$ *has a* (J_γ, J_1)*-lossless [q.]r.c.f.* $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$.

(e) This theorem also holds with "$\mathrm{TIC} \cap \mathrm{ULR}$*" in place of "*$\widetilde{\mathcal{A}}$*" if any of Remark 9.9.14(1.)–(6.) holds. Moreover, this theorem always holds with "ULR" in place of "UR".*

Naturally, $\mathbb{X} := I - \mathbb{F}$ in (FI4). Note from (11.58) that the state feedback K of (FI5) (or $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ of (FI4)) produces the unique "minimax" control. In this theorem the CARE can also be written as (11.25), by Lemma 9.11.5(e).

Recall from Definition 6.4.4 that $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is a (J_γ, J_1)-inner [q.]r.c.f. iff $\mathbb{N}$ and $\mathbb{M}$ are [q.]r.c., $\mathbb{M} \in \mathcal{G}\mathrm{TIC}_\infty$, and $\mathbb{N}^* J_\gamma \mathbb{N} = J_1$ (and $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$).

By Example 11.2.16, condition (FI9) is not sufficient in general (by 11.2.15, 11.2.17 and 11.1.9, also the conditions on $\mathbb{X}_{11}$ and S are not superfluous).

Proof of Theorem 11.2.7: 1° $\mathbb{D}, \widetilde{\mathbb{X}}^{\pm 1} \in \mathrm{UR}$, $\mathbb{D}_\flat \in \mathrm{ULR}$*:* By the assumptions, $\mathbb{D}, \widetilde{\mathbb{X}} := I - \widetilde{\mathbb{F}} \in \mathrm{UR}$. By Proposition 6.3.1(b1), also $\widetilde{\mathbb{M}} := \widetilde{\mathbb{X}}^{-1}$ is UR. Since $\mathbb{D}_\flat \in \widetilde{\mathcal{A}} \subset \mathrm{ULR}$, it follows that $\mathbb{D} = \mathbb{D}_\flat \widetilde{\mathbb{X}}$ is UR.

2° *(FI1)–(FI7)⇒SpF:* By Proposition 11.2.9 and Lemma 11.2.10, any of (FI3)–(FI5) (resp. (FI3)–(FI7)) implies that (FI3) holds, in particular, that $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat$ has a spectral factorization $\mathbb{X}_\diamond^* S_\diamond \mathbb{X}_\diamond$ (with $S_\diamond = J_1$).

If (FI1) or (FI2) holds, then (FI1) holds, by (11.12), hence then $\mathbb{D}$ is J_γ-coercive, by Proposition 11.2.19, hence so is $\mathbb{D}_\flat$, by Theorem 8.4.5(d)&(g1), hence $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat$ has a spectral factorization $\mathbb{X}_\diamond^* S_\diamond \mathbb{X}_\diamond$ with $\mathbb{X}_\diamond \in \mathcal{G}\widetilde{\mathcal{A}}(U \times W)$ (since $\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$).

3° *The equivalence of (FI1)–(FI7):* By 1°&2°, the assumptions of Proposition 11.2.8(a2) (and (a1)) are satisfied (since $\mathbb{D}_\flat, \mathbb{X}_\diamond \in \widetilde{\mathcal{A}} \subset \text{ULR} \subset \text{UR}$) whenever any of (FI1)–(FI5) (resp. (FI1)–(FI7)) is satisfied. Thus, these conditions are equivalent, by Proposition 11.2.8(d).

(c) This was established above.

(d) This was shown in 3° above for (FI6) and (FI7). The rest follows from Lemma 11.2.10, whose proof shows that $\mathbb{M}_{22} = (\mathbb{X}_\natural^{-1})_{22}$ (hence $\mathbb{M}_{22} \in \text{ULR}$, because necessarily $\mathbb{X}_\natural^{-1} \in \widetilde{\mathcal{A}} \subset \text{ULR}$).

(a) This follows from Proposition 11.2.8(a2) (see 3° above).

(b) b.1° *(FI3)&(FI6)–(FI9):* By 1°&3° above, $\mathbb{D}, \widetilde{\mathbb{X}}, \widetilde{\mathbb{M}} \in \text{UR}$ and $\mathbb{D}_\flat, \mathbb{X}_\natural, \mathbb{X}_\diamond \in \widetilde{\mathcal{A}} \subset \text{ULR}$ (from the proof of Lemma 11.2.10, we observe that this holds for solutions of (FI8) and (FI9), except that $\mathbb{X}_{\natural 11}$ and $\mathbb{M}_{22}$ might be noninvertible if the additional assumptions in (d) are not met).

Therefore, $\mathbb{X}_{\natural 11} \in \mathcal{G}\widetilde{\mathcal{A}}$, hence $\mathbb{M}_{22} = (\mathbb{X}_\natural^{-1})_{22} \in \mathcal{G}\widetilde{\mathcal{A}}$ (and $\mathbb{M}_{21} = (\mathbb{X}_\natural^{-1})_{21} \in \widetilde{\mathcal{A}}(U, W)$) if $\mathbb{M}$ is as in (FI6), by the proof of Proposition 11.2.8(d); note that then also $\mathbb{M} = \widetilde{\mathbb{M}}\mathbb{X}_\natural^{-1}$ is UR.

b.2° *(FI4)&(FI5):* (Here $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ refers to a pair solving (FI4) or (FI5), $\mathbb{X} := I - \mathbb{F}$, $\mathbb{M} := \mathbb{X}^{-1} = \mathbb{F}_\circlearrowleft + I$, $\mathbb{N} := \mathbb{D}_\circlearrowleft := \mathbb{D}\mathbb{M}$, as in Definition 9.8.1.)

By Proposition 11.2.8 (and b.1°), $\mathbb{F} = I - E'\mathbb{X}_\diamond\widetilde{\mathbb{X}} \in \text{UR}$, hence $\mathbb{X}, \mathbb{M}, \mathbb{F}_\circlearrowleft, \mathbb{N}, \mathbb{D}_\circlearrowleft \in \text{UR}$.

b.3° *Suboptimal* H^∞*-FI-pairs:* This follows from Proposition 11.2.8(a1) (its assumptions hold by (c), since now we have assumed (FI2)).

(e) In fact, this theorem also holds with "TIC$\cap$UR" in place of "$\widetilde{\mathcal{A}}$" if either $\mathbb{D}_\flat$ is not J-coercive or $\mathbb{D}_\flat$ has an UR spectral factorization (except that (FI8) might become strictly stronger than the other conditions; this is not the case when the factor is ULR), as shown below.

1° $\widetilde{\mathcal{A}}$*:* Indeed, the proof only uses from "$\mathbb{D}_\flat \in \widetilde{\mathcal{A}}$" the facts that $\mathbb{D}_\flat$ is UR and that if $\mathbb{D}_\flat$ is J_γ-coercive, then $\mathbb{D}_\flat$ has a UR spectral factorization (if we replace $\widetilde{\mathcal{A}}$ by TIC$\cap$UR in (a)). The only exception is that the necessity of (FI8) was shown above assuming that $\mathbb{X}_\natural \in \text{ULR}$ (so that even it is true in 2° below).

2° *ULR:* Indeed, if $\widetilde{K}$ is ULR, then so are all the other operators claimed to be UR in this theorem; cf. the proof of (b).

(Even "SR" would be otherwise acceptable but it might lead to problems with (FI5) unless we make some additional assumption in (FI5) (cf. Proposition 2.2.5) or we assume that $\dim W < \infty$.) □

In most classical results, one assumes that $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ (or exponential detectability, which implies that $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$), that Σ is exponentially stabilizable and that B is bounded (this includes [Keu]). By Theorem 9.2.12, this implies that we can take a bounded exponentially stabilizing $\widetilde{K}$ and choose the $\widetilde{\mathcal{A}}$ of Theorem 8.4.9(γ), so that the assumptions of Theorem 11.2.7 are satisfied.

However, without sufficient regularity assumptions, the above equivalence does not hold, at least in case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$: By Example 11.3.7(b), (FI1) does not imply any of (FI3)–(FI5) in general (not even for strongly stable uniformly half-plane-regular systems (so that one can take $\widetilde{K} = 0$); in this example, the $\mathcal{U}_*^*$-stabilizing solution the CARE is not I/O-stabilizing; in Example 11.3.7(a), there is no $\mathcal{U}_*^*$-stabilizing solution of the CARE, nor of the IARE). Although this counter-example only treats the cases $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$, it is believed that an example similar to Example 11.3.7(a) could be constructed for $\mathcal{U}_{\mathrm{exp}}$ too; see the comments below the example.

To avoid this problem, we made the $\widetilde{\mathcal{A}}$-assumption above, and in the general WPLS result below we have to make a weaker spectral factorization assumption (which is necessary for (FI3), hence for (FI4) and (FI5) too, by Proposition 11.2.9 and Lemma 11.2.10):

Proposition 11.2.8 (FICP) *Assume that* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] = \left[\begin{array}{c|cc} \widetilde{\mathbb{K}}_1 & \widetilde{\mathbb{F}}_{11} & \widetilde{\mathbb{F}}_{12} \\ 0 & 0 & 0 \end{array}\right]$ *is a state feedback pair for* Σ *with closed-loop system* $\Sigma_\flat$, *and that* $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat = \mathbb{X}_\diamond^* S_\diamond \mathbb{X}_\diamond$ *for some* $\mathbb{X}_\diamond \in \mathcal{G}\mathrm{TIC}(U \times W)$ *and* $S_\diamond \in \mathcal{GB}(U \times W)$. *Assume also that* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ *and* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is exponentially stabilizing (resp.* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ *and* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is [q.]r.c.-SOS-stabilizing).*

Then (FI1)–(FI4) of Theorem 11.2.7 are equivalent to each other and implied by (FI5). Also the following hold:

(a1) **(CARE)** *Assume that* $\mathbb{D}$ *is WR and the CARE*

$$\begin{cases} K^* S K = A^* \mathcal{P} + \mathcal{P} A + C_1^* C_1, \\ S = \begin{bmatrix} D_{11}^* D_{11} & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} - \gamma^2 I \end{bmatrix} + \operatorname*{w-lim}_{s \to +\infty} B_{\mathrm{w}}^* \mathcal{P} (s - A)^{-1} B, \\ K = -S^{-1} (B_{\mathrm{w}}^* \mathcal{P} + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1). \end{cases} \tag{11.36}$$

has a UR exponentially (resp. P-[q.]r.c.-SOS-)stabilizing solution $(\mathcal{P}, S, K)$. *Then* $\mathcal{P}$, S *and* K *are unique.*

Moreover, (FI1) holds iff $S_{11} \gg 0$ *and* $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$; *if this is the case, then (11.39) generate a UR exponentially (resp. [q.]r.c.-SOS-) stabilizing suboptimal* H^∞*-FI-pair and (FI5) holds.*

There is a suboptimal H^∞*-SF-operator iff* $S_{11} \gg 0$ *and* $S_{22} \ll 0$; *if this is the case, then* $K_1 =$*(11.40) is a UR exponentially (resp. [q.]r.c.-SOS-) stabilizing suboptimal* H^∞*-SF-operator.*

(a2) Assume that $\mathbb{D}$ *is WR,* $\widetilde{\mathbb{F}}$ *and* $\mathbb{X}_\diamond$ *are UR and* $\widetilde{F} = 0$. *Then (FI1)–(FI5) are equivalent, and the CARE has a UR exponentially (resp. P-[q.]r.c.-SOS-)stabilizing solution* $(\mathcal{P}, S, K)$.

(b1) The condition on $\widetilde{S}$ *in (FI4) is independent on the choice of* S *and* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *(and* $s_0 \in \mathbf{C}^+$*), and* $\mathcal{P}$ *is unique. Condition* $\mathbb{X}_{\natural 11} \in \mathcal{G}\mathrm{TIC}(U)$ *in (FI3) is independent on* $\mathbb{X}_\natural$ *(by (c1)).*

(b2) An exponentially (resp. P-[q.]r.c.-SOS-)stabilizing solution of the CARE is unique.

(b3) A solution of (FI3) or (FI4) is unique modulo an invertible constant.

(c1) If (FI1) holds, $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat = \mathbb{X}_\natural^* J_1 \mathbb{X}_\natural$ and $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, then $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ and $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TIC}} < 1$.

(c2) Any solution $\mathcal{P}$ of (FI5) is unique and a solution of (FI4).

(c3) If $\mathbb{X}_\natural$ and $\mathbb{F}$ are as in (FI3) and (FI4), respectively, then $\mathbb{X}_\natural := E\mathbb{X}_\diamond$ and $(I - \mathbb{F}) = E'\mathbb{X}_\diamond(I - \widetilde{\mathbb{F}})$ for some $E, E' \in \mathcal{GB}(U \times W)$.

(d) If $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ is q.r.c.-SOS-stabilizing, then (FI1)–(FI4) are equivalent to (FI6) and to (FI7) (and to (FI9) if $\dim U < \infty$ or $\dim W < \infty$).

(e) Any UR solution of (FI3) can be redefined s.t. $X = \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}$, $X_{11}, X_{22} \in \mathcal{GB}$.

(f) Even without the above spectral factorization assumption (that of $\mathbb{X}_\diamond$ and $S_\diamond$), we have (FI5)⇒(FI4)⇔(FI3)⇒(FI2)⇒(FI1), part (d) is true with (FI3) in place of (FI1), and part (a1) is still true (whenever $(\mathcal{P}, S, K)$ exists).

By Proposition 11.2.9, (FI5) and (FI4) are sufficient for (a1) and for (FI1)–(FI4) without any further assumptions. Unfortunately, (FI1) implies neither of (FI4) and (FI5) in general. By Example 11.3.7(c), condition (FI5) is strictly stronger than (FI1)–(FI4).

Proof: (Note from Lemma 6.7.13 that the existence of $\mathbb{X}_\diamond$ and $S_\diamond$ (if any) is independent on $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$.) We set $\widetilde{\mathbb{X}} := I - \widetilde{\mathbb{F}}$, $\widetilde{\mathbb{M}} := \widetilde{\mathbb{X}}^{-1}$. By (FI1s)–(FI5s), we refer to the conditions of Theorem 11.3.3 for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$.

1° *(FI1)⇔(FI2)⇔(FI3)⇔(FI1s)–(FI4s):* By Lemma 11.2.22, condition (FI1s) of Proposition 11.3.4 for $\Sigma_\flat^1 := \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ is equivalent to (FI1), and (FI2s) for $\Sigma_\flat^1$ to (FI2). By Proposition 11.3.4, (FI1s)–(FI4s) are equivalent (for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ this follows from the fact that $\mathcal{U}_{\exp}^{\Sigma_\flat} = \mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}$, by Theorem 8.4.5(e)). Condition (FI3) is trivially (FI3s) for $\Sigma_\flat^1$.

2° *(FI4)⇔(FI4s):* By Lemma 9.12.3(d1)&(d2) (and uniqueness), the solutions of (FI4) and (FI4s) correspond to each other as in (9.224).

Indeed, then $\mathbb{X}_\natural = \mathbb{X}\widetilde{\mathbb{M}}$, by (9.224), hence $S' := \widehat{\mathbb{X}_\natural}^* S \widehat{\mathbb{X}_\natural} = \widehat{\widetilde{\mathbb{M}}}^* \widetilde{S} \widehat{\widetilde{\mathbb{M}}}$ where $\widetilde{S} := \widehat{\mathbb{X}}^* S \widehat{\mathbb{X}}$ (at some $s_0 \in \mathbf{C}^+$). Thus, the claim follows from Lemma 11.3.13(i)&(ii") (the "(hence all)" claim), because $\widetilde{\mathbb{M}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$ (note that if $\omega > \omega_A$, then $\widehat{\mathbb{X}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right] \in \mathrm{H}^\infty(\mathbf{C}_\omega^+; \mathcal{B}(U \times W))$, by Lemma 6.1.10 and Theorem 6.2.1).

3° *(FI5)⇒(FI1):* This is given in Proposition 11.2.9.

(a1) If (FI1) holds, then so does (FI2), by 1°. Since $\mathcal{P}$ is $\mathcal{U}_*^*$-stabilizing (see 3° above), the signature conditions $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$ are necessary, by Proposition 11.2.19(d1)&(d2). The rest follows from Lemma 11.2.13.

(a2) a2.1° $(\mathcal{P}, S, K)$ *exists:* Now $\widetilde{\mathbb{M}}$ is UR and $\widetilde{M} = \widetilde{X}^{-1} = I$, by Proposition 6.3.1(a3), hence $\mathbb{D}_\flat = \mathbb{D}\widetilde{\mathbb{M}}$ is WR. By Corollary 9.9.11, the IARE for $\Sigma_\flat$ has a UR exponentially (resp. stable and P-[q.]r.c.-SOS-)stabilizing solution with zero feedthrough (use Lemma 6.7.15(c2) in case $\mathcal{U}_*^* = \mathcal{U}_{\exp}$), hence so does that for Σ, by Lemma 9.12.3(d1)&(d2) (and uniqueness).

a2.2° *(FI1)–(FI5) are equivalent:* We have already shown above that (FI1)–(FI4) are equivalent and that (FI5) implies (FI4). By a2.1° and (a1), a solution of (FI4) is necessarily a solution of (FI5).

Remark: In this case (FI1)–(FI5) are equivalent to (FI1s)–(FI5s) (see Proposition 11.3.4(a)). Now $\mathcal{P}$ and S are the same in (FI5) and (FI5s) (if either (hence both) holds), and $K = \widetilde{K} + K'$, where K corresponds to (FI5) and K' to (FI5s), by (9.226).

(b1)–(c3)&(e) Most of these follow easily from the above and Proposition 11.3.4; the rest follow as in its proof.

(d) This follows form Lemma 11.2.10.

Remark: We necessarily have $\mathbb{M}(I-\mathbb{F}) \in \mathcal{G}\mathrm{TIC}(U \times W)$ for solutions $\mathbb{M}$ of (FI6) and $\mathbb{F}$ of (FI4), by the proof of Lemma 11.2.10.

(f) 1° *(FI5)⇒(FI4)⇔(FI3)⇒(FI2)⇒(FI1):* If (FI3) holds, then so do (FI1)–(FI4) (since we can take $\mathbb{X}_\diamond := \mathbb{X}$), by this proposition. Implications (FI5)⇒(FI4)⇒(FI3) follow from Lemma 11.2.9.

2° *(a1)&(d):* The assumption that (FI5) holds is more than sufficient for the proof of (a1). The modified part (d) still follows from Lemma 11.2.10. □

As explained above, the equivalence of (FI1)–(FI4) does not hold under a mere stabilizability assumption. However, we still have the following sufficiency results of proposition and lemma below:

Proposition 11.2.9 (FICP: CARE ⇒ H^∞-SF-operator) *Assume that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. If (FI4) or (FI5) holds, then (FI1)–(FI4) hold and the assumptions of Lemma 11.2.10 and Proposition 11.2.8 are satisfied.*

Thus, if (FI5) holds, then we can apply Proposition 11.2.8(a1).

Proof: 1° *(FI4)⇒(FI1)–(FI3):* Assume (FI4). Redefine S and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ by (11.52), so that $S_{11} \gg 0 \gg S_{22}$. By Lemma 11.2.14(a), $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right] :=$ is a suboptimal H^∞-FI-pair and $\mathbb{M}_{22}^{-1} \in \mathrm{TIC}$.

If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is exponentially stabilizing, by Remark 11.2.5. If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$, then we observe from (11.10) that $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is stable and SOS-stabilizing, and from (11.11) that also $\overline{\mathbb{X}}^{-1} = \mathbb{M}\underline{\mathbb{M}}$ and $\mathbb{D}^{\curvearrowleft} = \mathbb{D}_{\circlearrowleft}\underline{\mathbb{M}}^{-1}$ are [q].r.c., where $\underline{\mathbb{M}} := \left[\begin{smallmatrix} I & 0 \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{smallmatrix}\right] \in \mathcal{G}\mathrm{TIC}$, by Lemma 6.4.5(c), hence also $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is [q].r.c.-SOS-stabilizing,

In either case, we have shown that the assumptions of Lemma 11.2.10 are satisfied, hence (FI1)–(FI4) hold and the assumptions of Proposition 11.2.8 are satisfied.

2° *(FI5)⇒(FI1)–(FI4):* The proof is analogous to that of 1° (see Lemma 11.2.13 and 3° of the proof of Lemma 11.2.10). □

Lemma 11.2.10 (FICP: SpF/IARE $\Rightarrow$ H$^\infty$-FI-pair) *Assume that* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] = \left[\begin{array}{c|cc} \widetilde{\mathbb{K}}_1 & \widetilde{\mathbb{F}}_{11} & \widetilde{\mathbb{F}}_{12} \\ 0 & 0 & 0 \end{array}\right]$ *is a state feedback pair for* Σ *with closed-loop system* $\Sigma_\flat$. *Assume also that* $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$ *and* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is exponentially stabilizing (resp.* $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$ *and* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is [q.]r.c.-SOS-stabilizing).*

Either of (FI3) and (FI4) implies that (FI1)–(FI4) and the assumptions of Proposition 11.2.8 are satisfied. If $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is q.r.c.-SOS-stabilizing, then (FI3) is equivalent to (FI6) and to (FI7) (and to (FI8) if* $\mathbb{M}_{22} \in \text{ULR}$ *and to (FI9) if* $\dim U < \infty$ *or* $\dim W < \infty$*).*

Proof: 1° *(FI3):* For (FI3) this is obvious: take $\mathbb{X}_\diamond := \mathbb{X}_\natural$, $S_\diamond := J_1$.

2° *(FI6):* By Lemma 6.4.8(b), the solutions of (FI3) and (FI6) correspond to each other through $\mathbb{M} = \widetilde{\mathbb{M}}\mathbb{X}_\natural^{-1}$ (if we neglect $\mathbb{X}_{\natural 11}$ and $\mathbb{M}_{22}$), hence $\mathbb{M}_{22} = (\mathbb{X}_\natural^{-1})_{22}$ (because $\widetilde{\mathbb{M}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$). But $(\mathbb{X}_\natural^{-1})_{22} \in \mathcal{G}\text{TIC}(W)$ iff $\mathbb{X}_{\natural 11} \in \mathcal{G}\text{TIC}(U)$, by Lemma A.1.1(c1), hence (FI3) and (FI6) are equivalent.

3° *(FI7), (FI8) and (FI9):* We obtain "(FI6)$\Leftrightarrow$(FI7)" from Corollary 2.5.5 (since $\mathbb{N} = \mathbb{D}\mathbb{M}$, so that $\mathbb{N}_{22} = \mathbb{M}_{22}$, because $\mathbb{D} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$), "(FI6)$\Leftrightarrow$(FI9)" from Proposition 2.5.4 (if $\dim U < \infty$ or $\dim W < \infty$; not in general!), and "(FI7)$\Leftrightarrow$(FI8)" from Proposition 6.3.1(c) (if $\mathbb{M}_{22} \in \text{ULR}$).

4° *(FI4):* Assume (FI4). Let $\Sigma_\circlearrowleft$ be the closed-loop system corresponding to $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$. Then $\mathbb{D}_\circlearrowleft^* J_\gamma \mathbb{D}_\circlearrowleft = S = I^* S I$, by Theorem 9.9.10(a2)&(a1). Thus, also $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat$ has a spectral factorization, by Lemma 6.7.13 (in case of $\mathcal{U}_{\text{exp}}$) or Lemma 6.4.5(c) (in case of $\mathcal{U}_{\text{out}}$, since then $\mathbb{D}_\flat = \mathbb{D}_\circlearrowleft \mathbb{U}$, hence $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat = \mathbb{U}^* S \mathbb{U}$, for some $\mathbb{U} \in \mathcal{G}\text{TIC}$). Therefore, the (preliminary) assumptions of Proposition 11.2.8 are satisfied, hence also (FI1)–(FI3) hold. $\square$

The above results also hold for $\mathcal{U}_{\text{str}}$ or $\mathcal{U}_{\text{sta}}$ in place of $\mathcal{U}_{\text{out}}$, mutatis mutandis:

Corollary 11.2.11 (FICP for $\mathcal{U}_{\text{str}}$ and $\mathcal{U}_{\text{sta}}$) *Assume that* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ *is [strongly] q.r.c.-stabilizing. Then the following hold for Theorem 11.2.7, Propositions 11.2.8 and 11.2.9 and Lemma 11.2.10:*

We have $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{sta}}[= \mathcal{U}_{\text{str}}]$*, and "SOS-" can be omitted [or replaced by strongly; moreover, "P-SOS-" can be replaced by strongly].*

Proof: By Theorem 8.4.5(g2), $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{sta}}[= \mathcal{U}_{\text{str}}]$. By Theorem 6.7.15(a1)[(a2)], prefices "q.r.c.-SOS-stabilizing" and "q.r.c.-stabilizing", [and "strongly q.r.c.-stabilizing"] are equivalent for admissible pairs for Σ or $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$. [Obviously, that "strongly" implies "P-".] For $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$, "q.r.c.-SOS-stabilizing" and "stable and SOS-stabilizing" are equivalent, by Lemma 6.6.17(b). $\square$

In the setting of Proposition 11.2.8, all suboptimal (i.e., s.t. $\|w \mapsto z\| < \gamma$) stable causal time-invariant control laws $\mathbb{U} : w \mapsto u$ can be formulated as follows:

Lemma 11.2.12 (All suboptimal TIC controllers) *Assume that* $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] = \left[\begin{array}{c|cc} \widetilde{\mathbb{K}}_1 & \widetilde{\mathbb{F}}_{11} & \widetilde{\mathbb{F}}_{12} \\ 0 & 0 & 0 \end{array}\right]$ *is a q.r.c.-SOS-stabilizing state feedback pair for* Σ *with closed-loop system* $\Sigma_\flat$.

Then all $\mathbb{U} \in \mathrm{TIC}(W,U)$ *s.t.* $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathrm{TIC}} < \gamma$ *are given by*

$$\{\widetilde{\mathbb{M}}_{11}\mathbb{U}_{\mathrm{st}} + \widetilde{\mathbb{M}}_{12} \mid \mathbb{U}_{\mathrm{st}} \in \mathrm{TIC}(W,U) \text{ is s.t. } \|\widetilde{\mathbb{N}}_{11}\mathbb{U}_{\mathrm{st}} + \widetilde{\mathbb{N}}_{12}\| < \gamma\}. \tag{11.37}$$

Given a solution for (FI3), we obtain the parametrization of all such (closed-loop suboptimal TIC control laws) $\mathbb{U}_{\mathrm{st}}$ from Theorem 11.3.6.

Recall from Theorem 6.7.15(c2), that if Σ is estimatable, then any exponentially stabilizing state feedback pair is exponentially q.r.c.-stabilizing (hence q.r.c.-SOS-stabilizing).

If we drop the q.r.c.-condition, then "$\mathbb{U}_{\mathrm{st}} \in \mathrm{TIC}$" must be replaced by "$\mathbb{U}_{\mathrm{st}} \in \mathrm{TIC}$ & $\widetilde{\mathbb{M}}_{11}\mathbb{U}_{\mathrm{st}} + \widetilde{\mathbb{M}}_{12} \in \mathrm{TIC}$".

The existence of a solution $\mathbb{U}$ to the I/O map problem (or frequency-domain problem) formulated above obviously implies that $\gamma > \gamma_0$. Recall that if the assumptions of, e.g., Theorem 11.3.6 hold, then $\gamma_0 = \gamma_{\mathrm{FI}}$, hence then also the converse holds (since this problem obviously lies between those corresponding to γ_0 and γ_{FI}), so that (FI3) is applicable for the above parametrization whenever a solution exists.

Proof of Lemma 11.2.12: (N.B. We observe from the proof that the theorem also holds with "$\le \gamma$" in place of "$< \gamma$".) As elsewhere, we have set

$$\widetilde{\mathbb{M}} := (I - \widetilde{\mathbb{F}})^{-1} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right] \in \mathrm{TIC} \quad \widetilde{\mathbb{N}} := \mathbb{D}\widetilde{\mathbb{M}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right] \in \mathrm{TIC}. \tag{11.38}$$

1° $\mathbb{U}_{\mathrm{st}} \implies \mathbb{U}$ *(sufficiency):* Let $\mathbb{U}_{\mathrm{st}} \in \mathrm{TIC}(W,U)$. Set $\mathbb{U} := \widetilde{\mathbb{M}}_{11}\mathbb{U}_{\mathrm{st}} + \widetilde{\mathbb{M}}_{12} \in \mathrm{TIC}(W,U)$, so that $\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] = \widetilde{\mathbb{M}}\left[\begin{smallmatrix} \mathbb{U}_{\mathrm{st}} \\ I \end{smallmatrix}\right]$. Then $\mathbb{N}\left[\begin{smallmatrix} \mathbb{U}_{\mathrm{st}} \\ I \end{smallmatrix}\right] = \mathbb{D}\mathbb{M}\left[\begin{smallmatrix} \mathbb{U}_{\mathrm{st}} \\ I \end{smallmatrix}\right] = \mathbb{D}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right]$, hence $\mathbb{N}_{11}\mathbb{U}_{\mathrm{st}} + \mathbb{N}_{12} = \mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}$, so that $\mathbb{U}$ is suboptimal for $\mathbb{D}$ iff $\mathbb{U}_{\mathrm{st}}$ is suboptimal for $\mathbb{N}$.

2° $\mathbb{U} \implies \mathbb{U}_{\mathrm{st}}$ *(necessity):* Let $\mathbb{U} \in \mathrm{TIC}(W,U)$ be s.t. $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathrm{TIC}} < \gamma$. Set $\mathbb{U}_{\mathrm{st}} := \begin{bmatrix} I & 0 \end{bmatrix}\mathbb{M}^{-1}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right]$. Since $\mathbb{D}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] \in \mathrm{TIC}$ and $\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] \in \mathrm{TIC}$, we have $\mathbb{M}^{-1}\mathbb{M}\left[\begin{smallmatrix} \mathbb{U}_{\mathrm{st}} \\ I \end{smallmatrix}\right] \in \mathrm{TIC}$, by Lemma 6.5.6(b), i.e., $\mathbb{U}_{\mathrm{st}} \in \mathrm{TIC}$. Thus, we have the setting of 1°. □

In the next two lemmas we list some implications of the Riccati equation and formulate a sufficient condition for suboptimality (see Lemma 11.2.14(a)):

Lemma 11.2.13 (General $\mathcal{U}_*^*$: CARE ⇒ FICP) *Assume that CARE has a UR* $\mathcal{U}_*^*$*-stabilizing solution* $(\mathcal{P}, S, K)$ *s.t.* $\mathcal{P} \ge 0$, $S_{11} \gg 0$ *and* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$. *Then the assumptions of Lemma 11.2.14 are satisfied (including (4.)).*

In particular, if $\mathcal{U}_*^* = \mathcal{U}_{\exp}$ *(or* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ *and* $(I - \mathbb{F})^{-1} \in \mathrm{TIC}$*), then (11.48) is a suboptimal* H^∞*-FI-pair for* Σ*, with generators*

$$\left[\begin{array}{c|cc} -S_{11}^{-1}(D_{11}^* C_1 + (B_1^*)_{\mathrm{w}}\mathcal{P}) & 0 & -S_{11}^{-1}S_{12} \\ 0 & 0 & 0 \end{array}\right]; \tag{11.39}$$

if, in addition, $S_{22} \ll 0$*, then*

$$K_1 := \begin{bmatrix} I & 0 \end{bmatrix} K = -(S_{11} - S_{12}S_{22}^{-1}S_{21})^{-1}(D_{11}^*C_1 + (B_1^*)_{\mathrm{w}}\mathcal{P} - S_{12}S_{22}^{-1}(D_{12}^*C_1 + (B_2^*)_{\mathrm{w}}\mathcal{P})) \tag{11.40}$$

is a UR suboptimal H^∞*-SF-operator for* Σ*.*

Proof: (By Theorem 9.8.12(s4)&(s3), $\mathcal{P}$, S and K are unique.)

1° *All claims except the formulae (11.39) and (11.40):* By Proposition 9.8.10, $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix})$ is a UR $\mathcal{U}_*^*$-stabilizing solution of the IARE. Obviously, (4.) is satisfied for $s_0 = +\infty$. The claim on K_1 follows from (a) of Lemma 11.2.14, that on (11.48) from (b) and (a) (with $\widetilde{S} = S$).

Thus, it only remains to establish (11.39) and (11.40).

2° *The generators of (11.48) are given by (11.39):* If $(\mathcal{P}, S, K)$ is a UR solution of the CARE (i.e., $s_0 = +\infty$ and $\widetilde{S} = S$), then $F = 0$, hence $\bar{F} = \begin{bmatrix} 0 & -S_{11}^{-1}S_{12} \\ 0 & 0 \end{bmatrix}$, and

$$(\bar{K})_1 = K_1 + S_{11}^{-1}S_{12}K_2 = -\begin{bmatrix} I & S_{11}^{-1}S_{12} \end{bmatrix} S^{-1}(D^*J_\gamma C + B_{\mathrm{w}}^*\mathcal{P}) \tag{11.41}$$

$$= -\begin{bmatrix} I & S_{11}^{-1}S_{12} \end{bmatrix} \begin{bmatrix} I & -S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & (S_{22}')^{-1} \end{bmatrix} \begin{bmatrix} I & -S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix}^* (D^*J_\gamma C + B_{\mathrm{w}}^*\mathcal{P}) \tag{11.42}$$

$$= -\begin{bmatrix} S_{11}^{-1} & 0 \end{bmatrix} (D^*J_\gamma C + B_{\mathrm{w}}^*\mathcal{P}) = -S_{11}^{-1}(D_{11}^*C_1 + (B_1^*)_{\mathrm{w}}\mathcal{P}). \tag{11.43}$$

3° *We have* $K_1 =$*(11.40):* By Lemma A.1.1(c1) (substitute $A \mapsto S$),

$$\mathcal{GB}(U) \ni (S^{-1})_{11}^{-1} = S_{11} - S_{12}S_{22}^{-1}S_{21} =: S_{11}'. \tag{11.44}$$

Set $L := D^*J_\gamma C + B_{\mathrm{w}}^*\mathcal{P}$. Then

$$K_1 = -\begin{bmatrix} I & 0 \end{bmatrix} S^{-1}L = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -S_{22}^{-1}S_{21} & I \end{bmatrix} \begin{bmatrix} (S_{11}')^{-1} & 0 \\ 0 & S_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & -S_{12}S_{22}^{-1} \\ 0 & I \end{bmatrix} L \tag{11.45}$$

$$= -(S_{11}')^{-1} \begin{bmatrix} I & -S_{12}S_{22}^{-1} \end{bmatrix} L = -(S_{11}')^{-1}(L_1 - S_{12}S_{22}^{-1}L_2) \tag{11.46}$$

=(11.17). □

Our sufficiency results are based in the following lemma (all three cases of (a2) are used in Section 11.1):

Lemma 11.2.14 (General $\mathcal{U}_*^*$: IARE ⇒ FICP) *Assume that IARE has a* $\mathcal{U}_*^*$*-stabilizing solution* $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix})$ *s.t.* $\mathcal{P} \geq 0$*. Assume, in addition, that at least one of (1.)–(4.) holds:*

(1.) $\min(\dim U, \dim W) < \infty$*;*

(2.) $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$*;*

(3.) $\widehat{\mathbb{X}}_{11}(s_0) \in \mathcal{GB}(U)$ *for some* $s_0 \in \mathbf{C}_\alpha^+$*;*

(4.) $\widetilde{S} := \widehat{\mathbb{X}}(s_0)^*S\widehat{\mathbb{X}}(s_0)$ *safisfies* $\widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$ *for some* $s_0 \in \mathbf{C}_\alpha^+$*.*

(In (3.) and (4.), we allow for any $\alpha \ge \max\{0,\vartheta\}$ s.t. $\mathbb{F} \in \mathrm{TIC}_\alpha$; for $\mathbb{F} \in \mathrm{UR}$ we also allow for $s_0 = +\infty$. In (2.) and (3.), we can allow right-invertibility (being onto) instead of invertibility.)

Then the following is true:

(a1) If $S_{22} \ll 0$, then $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ satisfies $\mathbb{X}_{11}, \mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$, $\mathbb{D}_{12}^{\frown}, \mathbb{M}_{22}^{-1}, \mathbb{X}_{21}\mathbb{X}_{11}^{-1}, \mathbb{M}_{22}^{-1}\mathbb{M}_{21} \in \mathrm{TIC}$ and $\|\mathbb{D}_{12}^{\frown}\|_{\mathrm{TIC}} \le \gamma$.

(a2) Assume that 1. $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, or that 2. $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ and $\mathbb{M} \in \mathrm{TIC}$, or that 3. $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ and there is $\varepsilon > 0$ s.t.

$$\|\mathbb{C}_1 x_0 + \mathbb{D}_{11}u + \mathbb{D}_{12}w\|_2 \ge \varepsilon\|u\|_2 \quad (u \in \mathrm{L}^2_\varepsilon(\mathbf{R}_+;U),\ w \in \mathrm{L}^2(\mathbf{R}_+;W),\ x_0 \in H). \tag{11.47}$$

If $S_{22} \ll 0$, then $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ is a H^∞-FI-pair; in cases 1. and 2. it is suboptimal (and $\|\mathbb{D}_{12}^{\frown}\| \le \gamma$ also in case 3.).

(b) If (4.) holds, then the $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S', \left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right])$ defined by (11.52) satisfies the assumptions of (a) (i.e., $\mathcal{P} \ge 0$, $S'_{22} \ll 0$ and (4.) holds for $\widehat{\mathbb{X}}'(s_0)S'\widehat{\mathbb{X}}'(s_0)$); the corresponding pair "$\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$" is given by

$$\left[\begin{array}{c|cc} \mathbb{K}_1 + \widetilde{S}_{11}^{-1}\widetilde{S}_{12}\mathbb{K}_2 & \mathbb{F}_{11} + \widetilde{S}_{11}^{-1}\widetilde{S}_{12}\mathbb{F}_{21} & \mathbb{F}_{12} + \widetilde{S}_{11}^{-1}\widetilde{S}_{12}(\mathbb{F}_{22} - I) \\ 0 & 0 & 0 \end{array}\right] \tag{11.48}$$

In (a), we have used the standard notation $\mathbb{X} := I - \mathbb{F}$, $\mathbb{M} := \mathbb{X}^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$, $\mathbb{D}_{12}^{\frown} := \mathbb{N}_{12}\mathbb{M}_{22}^{-1}$; cf. Definition 9.8.4 and (11.8).

Note that we can choose, e.g., $\alpha = \max\{0, \vartheta, \omega_A + 1\}$. Recall that for $\mathcal{U}_{\mathrm{out}}$ and $\mathcal{U}_{\mathrm{exp}}$ one requires that the control is stable, i.e., $\vartheta = 0$.

Proof of Lemma 11.2.14: *Remark 1:* This lemma also holds without the assumption on the existence of ε_+ (see Standing Hypothesis 11.2.1); indeed, this proof does not use it even implicitly. The same remark applies also to Lemma 11.2.13.

Remark 2: By Theorem 9.8.12(s4), $\widetilde{S}$ (and $\mathcal{P}$) is independent on the choice of a $\mathcal{U}_*^*$-stabilizing $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$. However, conditions (2.) and (3.) may depend on $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$: if $\mathbb{D}^* J_\gamma \mathbb{D} = J_1$, $S = -J_1$ and $X = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$, then $\widetilde{S} = J_1$ and (4.) (and (1.) if we choose so) hold although (2.) and (3.) are false for this solution (but not for all solutions, by (b)); this problem did not occur with the CARE (see Lemma 11.2.13).

(a1) By Theorem 9.9.1(a1)&(e1), $\Sigma_\circlearrowleft$ is a J_γ-critical control in state feedback form. By Theorem 8.3.9(a1'), $\mathbb{C}_\circlearrowleft$ is stable and $\mathbb{K}_\circlearrowleft$ is ϑ-stable. By Lemma 6.1.11 (and Remark 6.1.9), $\mathbb{M} = \mathbb{F}_\circlearrowleft + I \in \mathrm{TIC}_\omega(U \times W)$ for all $\omega > \vartheta$.

1° $\mathbb{M}_{22}$ *satisfies some of (1)–(5) of Proposition 2.2.5:* If any of (1.)–(3.) holds, then $\mathbb{X}_{11}$ satisfies some of Proposition 2.2.5(1)–(5), hence so does $\mathbb{M}_{22}$, by Lemma A.1.1(c1) (if $\dim U < \infty$, then $\mathbb{M}_{22}$ satisfies (4), which is observed by exchanging the columns and rows of $\mathbb{M}$).

If (4.) holds, then we can apply Lemma 11.3.13(b2) to $\widetilde{S} := \widehat{\mathbb{X}}(s_0)^* S \widehat{\mathbb{X}}(s_0)$ to observe that $\widehat{\mathbb{X}}(s_0)_{11} \in \mathcal{GB}(U)$, equivalently, that $\widehat{\mathbb{M}}(s_0)_{22} \in \mathcal{GB}(W)$, so that (5) holds.

2° $\exists \mathbb{M}_{22}^{-1} \in \mathrm{TIC}$*:* Since $\mathcal{P} \ge 0$ and $\mathbb{N}_{2*} = \mathbb{M}_{2*}$ (because $\mathbb{D} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$), we obtain from the (2,2)-block of (9.157) that

$$\gamma^2 (\mathbb{M}_{22}^t)^* \mathbb{M}_{22}^t - (\mathbb{N}_{12}^t)^* \mathbb{N}_{12}^t \ge -\pi_{[0,t)} S_{22} \ge \varepsilon^2 I \quad (t \ge 0). \tag{11.49}$$

It follows from Corollary 2.2.6, 1° and (11.49) that $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$ and $\|\mathbb{M}_{22}\|_{\mathrm{TIC}} \le \gamma/\varepsilon$.

3° $\|\mathbb{D}_{12}^{\frown}\|_{\mathrm{TIC}} \le \gamma$*:* From $(\mathbb{M}_{22}^{-t})^* \cdot (11.49) \cdot \mathbb{M}_{22}^{-t}$ we obtain that

$$\gamma^2 I \ge (\mathbb{D}_{12}^{\frown t})^* \mathbb{D}_{12}^{\frown t} + \varepsilon^2 (\mathbb{M}_{22}^{-t})^* \mathbb{M}_{22}^{-t}, \tag{11.50}$$

hence $\|\mathbb{D}_{12}^{\frown}\|_{\mathrm{TIC}} \le \gamma$, by Lemma 2.1.14.

Remark: Given $w \in \mathrm{L}^2(\mathbf{R}_+; W)$, we have $\|\mathbb{D}_{12}^{\frown} w\|_2^2 \le \gamma^2 - \varepsilon^2 \|\mathbb{M}_{22}^{-1} w\|_2^2$, hence $\|\mathbb{D}_{12}^{\frown} w\|_2 < \gamma$. If $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$, then $\Sigma_{\circlearrowleft}$ is exponentially stable, hence then $\mathbb{M}_{22} \in \mathrm{TIC}$, hence then $\|\mathbb{D}_{12}^{\frown}\| \le (\gamma^2 - \varepsilon^2/\|\mathbb{M}_{22}\|^2)^{1/2} < \gamma$.

4° $\mathbb{X}_{21}\mathbb{X}_{11}^{-1} \in \mathrm{TIC}$*:* Since $(\mathbb{D}_1^t)^* J_\gamma \mathbb{D}_1^t = (\mathbb{D}_{11}^t)^* \mathbb{D}_{11}^t \ge 0$ and $\mathcal{P} \ge 0$, we obtain from the (1,1)-block of (9.160) that

$$\begin{bmatrix} \mathbb{X}_{11}^t \\ \mathbb{X}_{21}^t \end{bmatrix}^* S \begin{bmatrix} \mathbb{X}_{11}^t \\ \mathbb{X}_{21}^t \end{bmatrix} \ge 0 \quad (t \ge 0). \tag{11.51}$$

Apply Lemma A.3.1(q) to obtain that there is $\delta > 0$ s.t. $(\mathbb{X}_{11}^t)^* \mathbb{X}_{11}^t \ge \delta (\mathbb{X}_{21}^t)^* \mathbb{X}_{21}^t$, i.e., $\delta^{-1} \ge (\mathbb{V}^t)^* \mathbb{V}^t$, for all $t > 0$, hence $\mathbb{V} \in \mathrm{TIC}$, by Lemma 2.1.14, where $\mathbb{V} := \mathbb{X}_{21}\mathbb{X}_{11}^{-1} = -\mathbb{M}_{22}^{-1}\mathbb{M}_{21}$ (by the (2,1)-block of $\mathbb{M}\mathbb{X} = I$).

(a2) 1° $\mathcal{U}_{\mathrm{exp}}$*: Suboptimality:* Since $\mathcal{P}$ is $\mathcal{U}_{\mathrm{exp}}$-stabilizing, $\Sigma_{\circlearrowleft}$ (and $\mathbb{B}_{\circlearrowleft}\tau$, by Lemma 6.1.10) is exponentially stable. From (11.10) we observe that $\mathbb{A}^{\frown} x_0 \in \mathrm{L}^2$ for all $x_0 \in H$, hence $\Sigma^{\frown}$ is exponentially stable, by Lemma A.4.5. Thus, $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is a H$^\infty$-FI-pair, by Remark 11.2.5. By the remark in 3°, $\|\mathbb{D}_{12}^{\frown}\|_{\mathrm{TIC}} < \gamma$ (since $\mathbb{M} = \mathbb{F}_{\circlearrowleft} + I \in \mathrm{TIC}$), i.e., $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is suboptimal.

2° $\mathcal{U}_{\mathrm{out}}$*: Suboptimality when* $\mathbb{M}_{12}, \mathbb{M}_{22} \in \mathrm{TIC}$*:* Because $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is $\mathcal{U}_{\mathrm{out}}$-stabilizing, it is output-stabilizing, by Theorem 9.8.5. This and the stability of $\mathbb{M}_{12}$ (by the assumption), $\mathbb{D}_{12}^{\frown} = \mathbb{N}_{12}\mathbb{M}_{22}^{-1}$ and $\mathbb{M}_{22}^{-1}$ (by 1°&2°), imply that $\mathbb{C}^{\frown}$, $\mathbb{K}^{\frown}$, $\mathbb{D}_{12}^{\frown}$ and $\mathbb{F}_{12}^{\frown}$ are stable (see (11.10)). Thus, $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is a H$^\infty$-FI-pair, by Remark 11.2.5. By the remark in 2°, $\|\mathbb{D}_{12}^{\frown}\|_{\mathrm{TIC}} < \gamma$ (since $\mathbb{M}_{22} \in \mathrm{TIC}$), i.e., $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is suboptimal.

3° *Case* $\|\mathbb{C}_1 x_0 + \mathbb{D}_{11} u + \mathbb{D}_{12} w\|_2 \ge \varepsilon \|u\|_2$*:* (Recall from (a1) that $\|\mathbb{D}_{12}^{\frown}\| \le \gamma$, hence $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is also here "almost suboptimal".) Since $\mathbb{C}_{\circlearrowleft}$ and $\mathbb{K}_{\circlearrowleft}$ are output stable (by Theorem 9.8.5), we have $\mathbb{D}_{\circlearrowleft}, \mathbb{F}_{\circlearrowleft} \in \mathrm{TIC}_\omega$ for all $\omega > 0$, by Lemma 6.1.11. This, (a1)2° and (11.10) imply that $\mathbb{C}^{\frown}$, $\mathbb{D}^{\frown}$, $\mathbb{K}^{\frown}$ and $\mathbb{F}^{\frown}$ are ω-stable for all $\omega > 0$.

By (a1), $\mathbb{D}_{12}^{\frown}$ is stable. Consequently, $\mathrm{L}^2 \ni \mathbb{D}_{12}^{\frown} w = \mathbb{D}_{12} w + \mathbb{D}_{11}\mathbb{F}_{12}^{\frown} w$, by (11.9), hence $\mathbb{F}_{12}^{\frown} w \in \mathrm{L}^2$, by (11.47), for all $w \in \mathrm{L}^2(\mathbf{R}_+; W)$. Thus, $\mathbb{F}_{12}^{\frown} \in \mathrm{TIC}$, by Lemma 6.1.12.

By (11.10), $\mathbb{C}_1^{\frown} = \mathbb{C}_{\circlearrowleft 1} - \mathbb{D}_{12}^{\frown}\mathbb{K}_{\circlearrowleft 2}$ is stable, hence $\mathrm{L}^2 \ni \mathbb{C}_1^{\frown} x_0 = \mathbb{C}_1 x_0 + \mathbb{D}_{11}\mathbb{K}_1^{\frown} x_0$, hence $\mathbb{K}_1^{\frown} x_0 \in \mathrm{L}^2$, for all $x_0 \in H$, so that $\mathbb{K}_1^{\frown}$ is stable.

The stability of $\mathbb{C}_1^{\frown}$, $\mathbb{D}_{12}^{\frown}$, $\mathbb{K}_1^{\frown}$ and $\mathbb{F}_{12}^{\frown}$ imply that $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is a H$^\infty$-FI-pair, by (11.8) and Remark 11.2.5.

(b) By Lemma 9.8.12(s1), also $(\mathcal{P}, S', \left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right])$ is a $\mathcal{U}_*^*$-stabilizing solution of the IARE, where

$$\begin{bmatrix} \mathbb{K}' & | & \mathbb{F}' \end{bmatrix} := \begin{bmatrix} \widetilde{E}\mathbb{K} & | & \widetilde{E}\mathbb{F} + I - \widetilde{E} \end{bmatrix}, \ S' := \begin{bmatrix} \widetilde{S}_{11} & 0 \\ 0 & S'_{22} \end{bmatrix}, \tag{11.52}$$

$E := \left[\begin{smallmatrix} I & \widetilde{S}_{11}^{-1}\widetilde{S}_{12} \\ 0 & I \end{smallmatrix}\right] \in \mathcal{GB}(U \times W)$, $\widetilde{E} := E\widehat{\mathbb{X}}(s_0)^{-1} \in \mathcal{GB}$, and $S'_{22} := \widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$, because $S' = E^{-*}\widetilde{S}E^{-1} = \widetilde{E}^{-*}S\widetilde{E}^{-1}$. Since $S'_{22} \ll 0$ (and $\widetilde{S}$ remains invariant), the assumptions of (a) are satisfied by $(\mathcal{P}, S', \begin{bmatrix} \mathbb{K}' & | & \mathbb{F}' \end{bmatrix})$.

Remark: We could replace $\widetilde{E}$ by $\begin{bmatrix} \widetilde{S}_{11}^{1/2} & 0 \\ 0 & (-\widetilde{S}_{22})^{1/2} \end{bmatrix}$ to obtain $S' = J_1$ above (this would slightly alter (11.48)). □

Next we note that $\mathbb{X}_{\natural 11} \in \mathcal{G}\mathrm{TIC}(U)$ is not superfluous in (FI3) (even $\mathbb{X}_{\natural 11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ is not sufficient):

Example 11.2.15 ($\mathbb{X}_{\natural 11} \in \mathcal{G}\mathrm{TIC}(U)$ is not superfluous) Let $\mathbb{D} := \left[\begin{smallmatrix} R & 2I \\ 0 & I \end{smallmatrix}\right] \in \mathrm{MTIC}_\mathrm{d}(U \times U) \subset \mathrm{TIC}(U \times U)$, where $R := \tau^{-1}$ (here U may be any Hilbert space; we have taken $Z = W = U$ and $\mathcal{U}_*^* = \mathcal{U}_\mathrm{out}$). Then $\mathbb{D}_{11}^*\mathbb{D}_{11} = I \gg 0$, hence Standing Hypotheses 11.1.1 and 11.2.1 are satisfied. Moreover, $\mathbb{D}^* J_1 \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$, where

$$\mathbb{X} = \begin{bmatrix} \frac{4}{3}R + \frac{1}{3} & 2 \\ \frac{2}{3}R + \frac{2}{3} & 1 \end{bmatrix} \in \mathcal{G}\mathrm{MTIC}_\mathrm{d} \subset \mathcal{G}\mathrm{TIC}. \tag{11.53}$$

However, (11.106)$= 3\pi_{[0,1)} - \pi_{[1,\infty)} \not\ll 0$, hence $\mathbb{D}$ is not minimax J-coercive (alternatively, this follows from Lemma 11.4.3(b), since $\mathbb{X}_{11} \notin \mathcal{G}\mathrm{TIC}(U)$, because $\widehat{\mathbb{X}}(\log 4 + \pi i) = 0$).

This also shows that condition $\mathcal{P} \ge 0$ is not superfluous in Lemmas 11.2.14 and 11.3.9(a). (Note that $\mathcal{P}$ can be computed from (8.46), once we choose a realization $\Sigma \in$ SOS of $\mathbb{D}$ (e.g., the shift realization (6.11); since $\mathbb{D}$ is exponentially stable, this realization can be chosen to be exponentially stable, so that $\mathcal{P}$ becomes exponentially stabilizing and $\mathcal{U}_\mathrm{out} = \mathcal{U}_\mathrm{exp}$).) ◁

If both U and W are infinite-dimensional, then the dimensions of positive and negative eigenspaces of $J_1 \in \mathcal{B}(U \times W)$ do not determine those of $\mathbb{D}^* J \mathbb{D}$ unless we require $\widehat{\mathbb{M}}_{22}$ to be invertible somewhere:

Example 11.2.16 ($\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$ is not superfluous in (FI7)) Let R be the right shift τ^{-1} on $\ell^2(\mathbf{N}) =: U =: W =: Z$, $L := R^*$, $P_0 := I - RL$, $Q := RL$. Set $\mathbb{D} := D := \left[\begin{smallmatrix} R & \sqrt{2}P_0 \\ 0 & I \end{smallmatrix}\right]$ to obtain $D^* J_1 D = \left[\begin{smallmatrix} I & 0 \\ 0 & 2P_0 - I \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & P_0 - Q \end{smallmatrix}\right] = X^* J_1 X$, where $\mathbb{X} := X := \left[\begin{smallmatrix} R & P_0 \\ 0 & L \end{smallmatrix}\right] \in \mathcal{GB}(U \times W)$.

Then $\mathbb{D}_{11}^*\mathbb{D}_{11} = I \gg 0$, so that Standing Hypotheses 11.1.1 and 11.2.1 are satisfied, and so are the assumptions of Propositions 11.2.8 and 11.3.4 (set $\begin{bmatrix} \widetilde{\mathbb{K}} & | & \widetilde{\mathbb{F}} \end{bmatrix} = \begin{bmatrix} 0 & | & 0 \end{bmatrix}$).

Set $M := X^{-1} = \left[\begin{smallmatrix} L & 0 \\ P_0 & R \end{smallmatrix}\right]$, $N := DM = \left[\begin{smallmatrix} Q + \sqrt{2}P_0 & 0 \\ P_0 & R \end{smallmatrix}\right]$ to obtain a (J_1, J_1)-lossless r.c.f. of $\mathbb{D}$, so that (FI9) is satisfied. Then $N_{22} = R \notin \mathcal{GB}(W)$, hence (FI6)–(FI8) do not hold (hence none of (FI1)–(FI8) holds). ◁

The existence of a $\mathcal{U}_*^*$-stabilizing solution is not a sufficient condition; we have to know signature properties that guarantee that the solution is really a minimax control:

Example 11.2.17 ($S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$ is necessary) To observe that the conditions on S are not superfluous in (FI5) (or in Theorem 11.1.3), set

$$A := -1,\ B := 0,\ C := 0,\ D := \begin{bmatrix} \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] & \left[\begin{smallmatrix}0\\2\end{smallmatrix}\right] \\ 0 & 1 \end{bmatrix} \in \mathcal{B}(\mathbf{C} \times \mathbf{C}, \mathbf{C}^2 \times \mathbf{C}). \tag{11.54}$$

Then the CARE $-2\mathcal{P} = K^*SK$, $S = \left[\begin{smallmatrix}1 & 0\\ 0 & 4-\gamma^2\end{smallmatrix}\right]$, $SK = 0$ has a unique solution $\mathcal{P} = 0$, which is exponentially stabilizing (unless $\gamma = 2$, in which case there is no solution of the CARE, and the solutions of the eCARE are given by $(\mathcal{P}, \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\ K_2\end{smallmatrix}\right])$). Since

$$\|\mathbb{D}_{11}u + \mathbb{D}_{12}w\|_2^2 = \|u\|_2^2 + 4\|w\|_2^2, \tag{11.55}$$

the optimal H^∞-FI-pair for Σ is given by $\left[\ \mathbb{K} \mid \mathbb{F}\ \right] = \left[\ 0 \mid 0\ \right]$, and $\gamma_0 = \gamma_{\rm FI} = \gamma_{\rm SF} = 2$. Indeed, Theorem 11.1.3(a) confirms this, since $S_{22} - S_{21}S_{11}^{-1}S_{12} = S_{22} = 4 - \gamma^2 \ll 0$ iff $\gamma > 2$. For $\gamma < 2$, the J-critical control is obviously a "min-min" control. ◁

The following lemma will be needed for the Riccati equation form of the solution of the H^∞ 4BP:

Lemma 11.2.18 ($\mathbb{M}_{22} \in \mathcal{G}$TIC $\Longrightarrow$ H^∞-FI-pair) *Assume that $(\mathcal{P}, S, \left[\ \mathbb{K} \mid \mathbb{F}\ \right])$ is an exponentially stabilizing solution of the IARE and $\mathcal{U}_*^* = \mathcal{U}_{\rm exp}$ (resp. that $(\mathcal{P}, S, \left[\ \mathbb{K} \mid \mathbb{F}\ \right])$ is a q.r.c.-SOS-stabilizing solution of the IARE and $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$).*

If $S_{22} \ll 0$ and $\mathbb{M}_{22} \in \mathcal{G}$TIC, then $\left[\ \overline{\mathbb{K}} \mid \overline{\mathbb{F}}\ \right] := \left[\begin{smallmatrix}\mathbb{K}_1\\0\end{smallmatrix} \middle| \begin{smallmatrix}\mathbb{F}_{11} & \mathbb{F}_{12}\\ 0 & 0\end{smallmatrix}\right]$ is a suboptimal H^∞-FI-pair and $\mathcal{P} \ge 0$.

If $\mathbb{F}$ is UR, then (11.48) generates another suboptimal pair (use the lemma and Proposition 11.2.19(d1)), but these pairs need not be equal in general (the operator $\mathbb{M}$ above need not be equal to I).

Proof: (Here, as elsewhere, $\mathbb{X} := I - \mathbb{F}$, $\mathbb{M} := \mathbb{X}^{-1}$.)

1° $\left[\ \overline{\mathbb{K}} \mid \overline{\mathbb{F}}\ \right] := \left[\begin{smallmatrix}\mathbb{K}_1\\0\end{smallmatrix} \middle| \begin{smallmatrix}\mathbb{F}_{11} & \mathbb{F}_{12}\\ 0 & 0\end{smallmatrix}\right]$ *is a H^∞-FI-pair:* Since $\mathcal{P}$ is $\mathcal{U}_{\rm exp}$-stabilizing, the closed-loop system $\Sigma_\circlearrowleft$ (and $\mathbb{B}_\circlearrowleft\tau$, by Lemma 6.1.10) is exponentially stable. From (11.10) we observe that $\mathbb{A}^\frown x_0 \in L^2$ for all $x_0 \in H$, hence $\Sigma^\frown$ is exponentially stable, by Lemma A.4.5. (For $\mathcal{U}_*^* = \mathcal{U}_{\rm out}$, we observe from (11.10) that $\Sigma^\frown \in$ SOS.) Thus, $\left[\ \overline{\mathbb{K}} \mid \overline{\mathbb{F}}\ \right]$ is a H^∞-FI-pair, by Remark 11.2.5.

2° $\left[\ \overline{\mathbb{K}} \mid \overline{\mathbb{F}}\ \right]$ *is suboptimal:* By (11.11), we have

$$\mathbb{D}^\frown := \mathbb{D}\begin{bmatrix}\mathbb{X}_{11} & \mathbb{X}_{12}\\ 0 & I\end{bmatrix}^{-1} = \mathbb{D}\mathbb{X}^{-1}\underline{\mathbb{M}}^{-1}, \tag{11.56}$$

where $\underline{\mathbb{M}} = \left[\begin{smallmatrix}I & 0\\ \mathbb{M}_{21} & \mathbb{M}_{22}\end{smallmatrix}\right]$, $\mathbb{M} := \mathbb{X}^{-1}$, by Schur decomposition (Lemma A.1.1(d1)(A.9)). Therefore, from $(\mathbb{D}\mathbb{X}^{-1})^*J_\gamma\mathbb{D}\mathbb{X}^{-1} = S$, we obtain that

$$(\mathbb{D}_2^\frown)^*J_\gamma\mathbb{D}_2^\frown = \begin{bmatrix}0 & I\end{bmatrix}\mathbb{D}^{\frown *}J_\gamma\mathbb{D}^\frown\begin{bmatrix}0\\ I\end{bmatrix} = \begin{bmatrix}0 & I\end{bmatrix}\underline{\mathbb{M}}^*S\underline{\mathbb{M}}\begin{bmatrix}0\\ I\end{bmatrix} = \mathbb{M}_{22}^*S_{22}\mathbb{M}_{22}, \tag{11.57}$$

hence $(\mathbb{D}_2^{\frown})^* J_\gamma \mathbb{D}_2^{\frown} \ll 0$, because $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}$, by Lemma A.1.1(c1). By Lemma 11.2.4, it follows that $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is suboptimal.

3° $\mathcal{P} \ge 0$*:* This follows from Proposition 11.2.19(b2) (since $\gamma > \gamma_{\mathrm{FI}} \ge \gamma_0$, by 2°).

Remark: Parts 1°–2° hold also for the eFICP of Theorem 11.3.6 (i.e., for any $\Sigma \in \mathrm{WPLS}(U \times W, H, Y)$; we do not need Standing Hypothesis 11.1.1 nor 11.2.1).

If we do assume Standing Hypothesis 11.1.1, then Standing Hypothesis 11.2.1 becomes redundant: Since Standing Hypothesis 11.2.1 is not used in 1°–2°, as noted above, we obtain Standing Hypothesis 11.2.1 from Lemma 11.2.22(a) (with $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] := \left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$). Indeed, by (d1) (resp. (c)) Lemma 9.12.3, $\mathcal{P}$ is an exponentially (resp. q.r.c.-SOS-)stabilizing solution for the CARE for $\Sigma_\flat$ and J_γ, hence $\mathbb{D}_\flat$ has a spectral factorization, by Corollary 9.9.11. □

Next we present a rather general necessity result that was used in the proofs of our main theorems:

Proposition 11.2.19 (General $\mathcal{U}_*^*$: Necessary conditions) *Assume that $\gamma > \gamma_0$ and that Z^{s} is reflexive. Then $\mathbb{D}$ is J_γ-coercive. Assume in addition that $\mathcal{U}_*^*(x_0) \ne \emptyset$ for each $x_0 \in H$. Then the following hold:*

(a1) There is a unique J_γ-critical input $\begin{bmatrix} u_{\mathrm{crit}}(x_0) \\ w_{\mathrm{crit}}(x_0) \end{bmatrix}$ for each $x_0 \in H$, and this input corresponds to the (unique) arguments of

$$\max_{w \in \mathrm{L}^2(\mathbf{R}_+;W)} \min_{u \in \mathcal{U}_{\mathrm{u}}(x_0,w)} \mathcal{J}(x_0,u,w). \tag{11.58}$$

(a2) If $(\Sigma, J) \in$ coerciveCARE *(see Remark 9.9.14), then there is a (SR) unique $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ of the CARE; the corresponding closed-loop control equals $\begin{bmatrix} u_{\mathrm{crit}}(x_0) \\ w_{\mathrm{crit}}(x_0) \end{bmatrix}$ for each $x_0 \in H$ (with no external input).*

(b1) The J_γ-critical input (called the minimax control*) can be given in state feedback form iff the [e]IARE has a $\mathcal{U}_*^*$-stabilizing solution.*

(b2) Assume that a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ of the (e)IARE exists. Then $\mathcal{P} \ge 0$, $S \in \mathcal{GB}(U \times W)$ and $(\mathbb{X}^{t*} S \mathbb{X}^t)_{11} \ge \epsilon_+^2 e^{-2t\max\{0,\vartheta\}} I$ for all $t \ge 0$.

If, in addition, $\vartheta \le 0$, then $(\mathbb{X}^{t*} S \mathbb{X}^t)_{11} \ge \epsilon_+^2 I$ for all $t \ge 0$, and we can choose S and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ so that $S = \begin{bmatrix} I & 0 \\ 0 & J_W \end{bmatrix}$, where $J_W = J_W^* = J_W^{-1} \in \mathcal{GB}(W)$.

(b3) If a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ exists s.t. $\mathbb{F} \in \mathrm{MTIC}_\infty$, then the CARE has a unique $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$, and $S_{11} \ge \epsilon_+^2 I$.

If, in addition, there is a suboptimal MTIC_∞ H^∞*-FI-pair (resp.* H^∞*-SF-operator), then $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$ (resp. $S_{22} \ll 0$).*

(c) Assume that a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ exists, there is a suboptimal H^∞*-FI-pair, and $\vartheta \le 0$. Then $(\mathbb{X}^{t*} S \mathbb{X}^t)_{11} \ge \epsilon_+^2 I$ for all $t \ge 0$,*

and $\widehat{\mathbb{S}} := \widehat{\mathbb{X}}^* S \widehat{\mathbb{X}}$ *satisfies*

$$\widehat{\mathbb{S}}_{11} \geq \varepsilon_+^2 I \quad \text{and} \quad \widehat{\mathbb{S}}_{22} - \widehat{\mathbb{S}}_{21}\widehat{\mathbb{S}}_{11}^{-1}\widehat{\mathbb{S}}_{12} \ll 0 \quad \text{on } \mathbf{C}_\omega^+, \tag{11.59}$$

for any $\omega \geq 0$ *s.t.* $\mathbb{X} \in \mathrm{TIC}_\omega$. *Moreover, there there is a* $\mathcal{U}_*^*$*-stabilizing solution* $(\mathcal{P}, J_1, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ *of the IARE s.t.* $\widetilde{\mathbb{X}}_{11} \in \mathcal{G}\mathrm{TIC}_\infty$, $\|\widetilde{\mathbb{X}}_{11}^{-1}\|_{\mathrm{TIC}} \leq \varepsilon_+^{-1}$, *and* $\|\widetilde{\mathbb{X}}_{21}\widetilde{\mathbb{X}}_{11}^{-1}\|_{\mathrm{TIC}} \leq 1$.

Assume that the CARE has a SR $\mathcal{U}_*^*$*-stabilizing solution* $(\mathcal{P}, S, K)$ *and that* $\vartheta \leq 0$. *Then* $S_{11} \geq \varepsilon_+^2 I$ *and the following hold:*

(d1) $\mathbf{(S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0)}$ *If there is a suboptimal SR* H^∞*-FI-pair, then* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$.

(d2) $\mathbf{(S_{22} \ll 0)}$ *If there is a suboptimal SR* H^∞*-SF-operator, then* $S_{22} \ll 0$.

(e) If $\|\pi_{[0,t)}\mathbb{D}_{11}u\|_2 \geq \varepsilon_+ \|\pi_{[0,t)}u\|_2$ *for all* $u \in \mathrm{L}^2(\mathbf{R}_+; U)$ *and* $t > 0$, *then the assumption "*$\vartheta \leq 0$*" can be removed everywhere in this theorem.*

(f) In (b3), (c), (d1) and (d2), the existence of a suboptimal H^∞*-FI-pair (resp.* H^∞*-SF-operator) is not needed if there is* $\mathbb{U} \in \mathrm{TIC}_\infty(W,U)$ *(resp. SR* $\mathbb{U} \in \mathrm{TIC}_\infty(W,U)$ *having* $\widehat{\mathbb{U}}(+\infty) = 0$*) s.t.* $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathrm{TIC}} < \gamma$ *(in (b3) we must also require that* $\mathbb{U} \in \mathrm{MTIC}_\infty(W,U)$*; in (c) that* $\mathbb{U} \in \mathrm{TIC}_\omega$*, and in (d1) that* $\mathbb{U}$ *is SR).*

See, e.g., Lemma 11.2.14 for the converses. Note that $\mathcal{U}_{\mathrm{exp}}$ and $\mathcal{U}_{\mathrm{out}}$ have Z^{s} reflexive (and $\vartheta = 0$, hence $\mathrm{e}^{-2t\vartheta} = 1$ in (b2)). If $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is as in (b2), then $\mathcal{P}$ is unique and S and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ are unique modulo an invertible constant as in (9.114), by Theorem 9.9.1(a1)&(f1)&(f2). See Example 11.1.9 for the difference between (d1) and (d2).

Under the assumptions of Theorem 9.9.1(k), we can make J_W equal to $-I$ in (b2); this fact was used in Proposition 11.2.8. However, we do not know whether S reflects all signature properties of the problem for general $\mathcal{U}_*^*$, hence we cannot improve (b2) in the general case; see the notes on p. 481 for this problem.

Proof: By Theorem 11.7.2(b) (and Remark 8.3.4; note that Hypothesis 11.7.1 requires the reflexivity assumption), $\mathbb{D}$ is J_γ-coercive.

(a1) By Theorem 11.7.2(c), (11.58) exists and equals the J_γ-critical input.

(a2) Combine (a1) with the definition of coerciveCARE. (Note that under (4.) or (5.) of Remark 9.9.14, we have $\mathbb{F} \in \mathrm{MTIC}_\infty^{\mathrm{L}^1}$, so that (b3) applies. Under any of (1.)–(6.) of Remark 9.9.14, the solution is ULR, hence SR, so that then (d) applies.)

(b1) This follows from by Theorem 9.9.1(a1) (the eIARE is equivalent to the IARE, by (b2)).

(b2) 1° By Lemma 9.10.3, we have $S \in \mathcal{GB}(U \times W)$. By Theorem 9.9.1(f1), we have $\mathcal{P} = \mathbb{C}_{\mathrm{crit}}^* J_\gamma \mathbb{C}_{\mathrm{crit}}$. But $\mathcal{J}(x_0, u, 0) \geq 0$ for each x_0 and u, hence (11.58)≥ 0, i.e., $\mathbb{C}_{\mathrm{crit}}^* J_\gamma \mathbb{C}_{\mathrm{crit}} \geq 0$. Thus, $\mathcal{P} \geq 0$.

Let $t > 0$ and $u' \in \mathrm{L}^2([0,t); U)$ be given. Set $x_t := \mathbb{B}_1 \tau^t u$, $\widetilde{u} := u' + \tau^{-t} u$, where $u \in \mathcal{U}_*^*(x_t)$ is to be defined later. By Lemma 9.7.9, $\widetilde{u} \in \mathcal{U}_*^*(0)$. We first compute that

$$y_u := \mathbb{C}\mathbb{B}_1^t \widetilde{u} + \mathbb{D}_1 u = \pi_+ \mathbb{D}_1 (\pi_+ + \pi_-) \tau^t \widetilde{u} = \pi_+ \mathbb{D}_1 \tau^t \widetilde{u} \tag{11.60}$$
$$= \mathbb{D}_1 \tau^t \widetilde{u} - \pi_{[-t,0)} \mathbb{D} \tau^t \widetilde{u} = \mathbb{D}_1 \tau^t \widetilde{u} - \tau^t \mathbb{D}_1^t u'. \tag{11.61}$$

Consequently, (11.58) implies that

$$\langle \mathbb{B}_1^t u', \mathcal{P} \mathbb{B}_1^t u' \rangle = \max_{w \in \mathrm{L}^2(\mathbf{R}_+;U)} \min_{u \in \mathcal{U}_{\mathrm{u}}(x,w)} \mathcal{J}(\mathbb{B}_1^t u', u, w) \geq \min_{u \in \mathcal{U}_{\mathrm{u}}(x,0)} \mathcal{J}(\mathbb{B}_1^t u', u, 0) \tag{11.62}$$
$$= \min_{u \in \mathcal{U}_{\mathrm{u}}(x,0)} \langle y_u, J_\gamma y_u \rangle_{\mathrm{L}^2} = \min_{u \in \mathcal{U}_{\mathrm{u}}(x,0)} \langle \tau^{-t} y_u, \tau^{-t} J_\gamma y_u \rangle_{\mathrm{L}^2} \tag{11.63}$$
$$\overset{\dagger}{=} \min_{u \in \mathcal{U}_{\mathrm{u}}(x,0)} \big(\langle \mathbb{D}_1 \widetilde{u}, J_\gamma \mathbb{D}_1 \widetilde{u} \rangle - \langle \mathbb{D}_1^t u', J_\gamma \mathbb{D}_1^t u' \rangle \big) \tag{11.64}$$
$$= \min_{u \in \mathcal{U}_{\mathrm{u}}(x,0)} \langle \mathbb{D}_1 \widetilde{u}, J_\gamma \mathbb{D}_1 \widetilde{u} \rangle - \langle u', (\mathbb{D}_1^t)^* J_\gamma \mathbb{D}_1^t u' \rangle \tag{11.65}$$

($\dagger$: here both crossterms are negative, hence the minus sign though positive quadratic terms). But $\mathbb{X}^{t*} S \mathbb{X}^t = \mathbb{D}^{t*} J_\gamma \mathbb{D}^t + \mathbb{B}^{t*} \mathcal{P} \mathbb{B}^t$, by the eIARE, hence it follows from the above and Standing Hypothesis 11.2.1 that

$$\langle u', (\mathbb{X}^{t*} S \mathbb{X}^t)_{11} u' \rangle = \min_{u \in \mathcal{U}_{\mathrm{u}}(x,0)} \langle \mathbb{D}_1 \widetilde{u}, J_\gamma \mathbb{D}_1 \widetilde{u} \rangle \geq \varepsilon_+^2 \| \left[\begin{smallmatrix} \widetilde{u} \\ 0 \end{smallmatrix}\right] \|_{\mathcal{U}_*^*}^2 \tag{11.66}$$
$$\geq \varepsilon_+^2 \|\widetilde{u}\|_{\mathrm{L}^2_\vartheta}^2 \geq \varepsilon_+^2 \|u'\|_{\mathrm{L}^2_\vartheta}^2 \geq \varepsilon_+^2 \mathrm{e}^{-2t \max\{0,\vartheta\}} \|u'\|_2^2. \tag{11.67}$$

Thus, $(\mathbb{X}^{t*} S \mathbb{X}^t)_{11} \geq \varepsilon_+^2 \mathrm{e}^{-2t \max\{0,\vartheta\}}$ for all $t \geq 0$.

2° *Case* $\vartheta \leq 0$*:* Since $\mathrm{e}^{-2t \max\{0,\vartheta\}} = 1$, we now have $(\mathbb{X}^{t*} S \mathbb{X}^t)_{11} \geq \varepsilon_+^2 I$ for all $t \geq 0$.

For any $\omega > \omega_A$, we have $\mathbb{X} \in \mathrm{TIC}_\omega$; if also $\omega \geq 0$, then $(\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})_{11} \geq \varepsilon_+^2 I$ on $\mathbf{C}_\omega^+$, by Lemma 2.2.4. Choose $s_0 \in \mathbf{C}_\omega^+$. Then, $T := (\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})_{11}(s_0) \geq \varepsilon_+^2 I$.

By Lemma 11.3.14, there is $E_0 \in \mathcal{GB}(U \times W)$ s.t. $T = E_0^* \widetilde{S} E_0$, where $\widetilde{S} = \left[\begin{smallmatrix} I & 0 \\ 0 & J_W \end{smallmatrix}\right]$, $J_W = J_W^* = J_W^{-1}$. Set $E := E_0 \widehat{\mathbb{X}}(s_0)^{-1} \in \mathcal{GB}(U \times W)$. By Theorem 9.9.1(f2), also $(\mathcal{P}, \widetilde{S}, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ is a $\mathcal{U}_*^*$-stabilizing solution of the IARE, where $\widetilde{S} := E^{-*} S E^{-1} = E_0^{-*} T E_0^{-1}$, $\widetilde{\mathbb{K}} = E\mathbb{K}$ and $(I - \widetilde{\mathbb{F}}) = E(I - \mathbb{F})$.

Remark: It seems that we could get $S = \left[\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right]$ (i.e., $J_W = I$) at least when both U and W are unseparable, by using Lemma B.3.16; however, we have not been able to establish this for general U and W (and hence obtain "(FI1)⇒(FIn)" for $n \in \{2,3,4,5\}$) without reduction to the stable case (under the assumptions of Theorem 11.2.7 or Proposition 11.2.8), where it is ultimately reduced to Lemma 11.4.3(a)&(c).

(b3) (Note in particular that the MTIC_∞ assumption allows us to avoid the "$\vartheta \leq 0$" assumption.)

1° *The CARE:* We have $\mathrm{MTIC}_\infty \subset \mathrm{ULR}$, by Theorem 2.6.4(f), hence we can redefine X to I so that we obtain a solution of the CARE, by Corollary 9.9.8 (still with $\mathbb{F} \in \mathrm{MTIC}_\infty$).

2° *Claim* $S_{11} \geq \varepsilon_+^2$*:* Let $u_0 \in U$. Set $u' := t^{-1/2} \pi_{[0,t)} u_0$. Let $t \to 0+$ in

(11.66) to obtain (by simple computations using Theorem 2.6.4(i3)) that

$$\langle X \left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right], SX \left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right]\rangle_{U\times W} \geq \varepsilon_+^2 \mathrm{e}^0 \|u_0\|_U^2. \tag{11.68}$$

Since $u_0 \in U$ was arbitrary, this says that $(X^*SX)_{11} \geq \varepsilon_+^2 I$. By Lemma 11.3.14, we can redefine $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ and S so that $S = \left[\begin{smallmatrix} I & 0 \\ 0 & J_W \end{smallmatrix}\right]$ (cf. (b2)2°).

3° *The suboptimal case:* We obtain this from (11.78) as in 2° above (cf. the proofs of (d1)–(d2)).

Remark: In fact, (b3) holds even if we replace MTIC_∞ by $\mathrm{SMTIC}_\infty \cap \mathrm{SR}^{\mathrm{d}}$ or by $\mathcal{B}(U, \mathrm{L}^p_\omega(\mathbf{R}_+;Y)) \cap \mathrm{SR}^{\mathrm{d}}$ for some $\omega \in \mathbf{R}$ (since we only need that $\lim_{t\to 0+}(\mathbb{F}\chi_{\mathbf{R}_+} u_0)(t)$ exists and $\mathbb{F}^{\mathrm{d}} \in \mathrm{SR}$, and that $\mathbb{X}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right]$ also has same properties, where (the suboptimal) $\mathbb{U}$ (if any) is as in the proof of (c)).

(Indeed, then $\mathbb{F}, \mathbb{F}^{\mathrm{d}} \in \mathrm{SR}$, hence $X \in \mathcal{GB}$, by Proposition 6.2.8(a2), and we can work as above.)

(c) 1° Let $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ be a H^∞-FI-pair for Σ with closed-loop system $\Sigma^{\curvearrowleft}$ (we still denote by $\Sigma_{\circlearrowleft}$ the closed-loop system corresponding to K). Set $\mathbb{U} := \widetilde{\mathbb{F}}^{\curvearrowleft}_{12}$, so that $\mathbb{D}_2^{\curvearrowleft} = \mathbb{D}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] \in \mathrm{TIC}(W,Y)$ (see Remark 11.2.5).

Let $t > 0$. Let $\pi_{[0,t)} w \in \mathrm{L}^2([0,t);W)$ be arbitrary and set $x_t := \mathbb{B}\tau^t \left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w$. Set $\pi_{[t,\infty)} w := \tau^{-t}\mathbb{K}_{\circlearrowleft 2} x_t \in \mathrm{L}^2([t,\infty);W)$, so that (use the fact that $\left[\begin{smallmatrix} u_{\mathrm{crit}}(x_t) \\ w_{\mathrm{crit}}(x_t) \end{smallmatrix}\right] = \mathbb{K}_{\circlearrowleft} x_t$)

$$\mathcal{J}(x_t, \pi_+\tau^t \mathbb{U} w, \pi_+\tau^t w) \geq \min_{u\in\mathcal{U}_{\mathrm{u}}} \mathcal{J}(x_t, u, \mathbb{K}_{\circlearrowleft 2} x_t) = \max_{\widetilde{w}} \min_{u\in\mathcal{U}_{\mathrm{u}}} \mathcal{J}(x_t, u, \widetilde{w}) = \langle x_t, \mathcal{P} x_t\rangle. \tag{11.69}$$

We have

$$\tau^t \pi_{[t,\infty)} \mathbb{D}_2^{\curvearrowleft} w = \pi_+ \tau^t \mathbb{D}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w = \pi_+ \mathbb{D}(\pi_+ + \pi_-)\tau^t \left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w = \mathbb{D}\pi_+\tau^t \left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w + \mathbb{C} x_t, \tag{11.70}$$

and $\mathbb{D}_2^{\curvearrowleft} = \pi_{[t,\infty)}\mathbb{D}_2^{\curvearrowleft} + \mathbb{D}_2^{\curvearrowleft t}$, hence

$$\langle \mathbb{D}_2^{\curvearrowleft} w, J_\gamma \mathbb{D}_2^{\curvearrowleft} w\rangle - \langle \mathbb{D}_2^{\curvearrowleft t} w, J_\gamma \mathbb{D}_2^{\curvearrowleft t} w\rangle = \langle \pi_{[t,\infty)}\mathbb{D}_2^{\curvearrowleft} w, J_\gamma \pi_{[t,\infty)}\mathbb{D}_2^{\curvearrowleft} w\rangle \tag{11.71}$$
$$= \langle \tau^t \pi_{[t,\infty)}\mathbb{D}_2^{\curvearrowleft} w, J_\gamma \tau^t \pi_{[t,\infty)} J_\gamma \mathbb{D}_2^{\curvearrowleft} w\rangle \tag{11.72}$$
$$= \langle \mathbb{C} x_t + \mathbb{D}\pi_+\tau^t \left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w, J_\gamma(\mathbb{C} x_t + \mathbb{D}\pi_+\tau^t \left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w)\rangle \tag{11.73}$$
$$= \mathcal{J}(x_t, \pi_+\tau^t \mathbb{U} w, \pi_+\tau^t w) \tag{11.74}$$
$$\geq \langle x_t, \mathcal{P} x_t\rangle = \langle \left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w, \mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t \left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w\rangle, \tag{11.75}$$

by (11.69). From this and (9.160) we obtain that

$$\langle \mathbb{D}_2^{\curvearrowleft} w, J_\gamma \mathbb{D}_2^{\curvearrowleft} w\rangle \geq \langle \mathbb{X}^t \left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w, S\mathbb{X}^t \left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] w\rangle. \tag{11.76}$$

If $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ is suboptimal, then there is $\varepsilon > 0$ s.t. $\langle \mathbb{D}_2^{\curvearrowleft} w, J_\gamma \mathbb{D}_2^{\curvearrowleft} w\rangle \leq -\varepsilon\|w\|_2^2$ for all $w \in \mathrm{L}^2(\mathbf{R}_+;W)$. Since $t > 0$ and $\pi_{[0,t)} w \in \mathrm{L}^2$ were arbitrary, we obtain from (11.76) that

$$\langle \mathbb{X}^t \left[\begin{smallmatrix} \mathbb{U}^t \\ I \end{smallmatrix}\right] w, S\mathbb{X}^t \left[\begin{smallmatrix} \mathbb{U}^t \\ I \end{smallmatrix}\right] w\rangle \leq -\varepsilon\|w\|_2^2 \quad (t \geq 0,\ w \in \mathrm{L}^2([0,t);W)), \tag{11.77}$$

i.e., that $((\mathbb{X}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right])^t)^* S (\mathbb{X}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right])^t \leq -\varepsilon I$ for all $t \geq 0$. By Lemma 2.2.4(a), this implies that

$$(\widehat{\mathbb{X}}\begin{bmatrix}\widehat{\mathbb{U}}\\ I\end{bmatrix})^* S\widehat{\mathbb{X}}\begin{bmatrix}\widehat{\mathbb{U}}\\ I\end{bmatrix} \leq -\varepsilon I \tag{11.78}$$

on $\mathbf{C}^+_\omega$, for any $\omega \geq 0$ s.t. $\mathbb{X}, \mathbb{U} \in \mathrm{TIC}_\omega$.

2° *Implication "$\vartheta \leq 0 \Rightarrow S' = J_1$" etc.:* Assume that $\vartheta \leq 0$. Choose $\omega \geq 0$ s.t. $\mathbb{X} \in \mathrm{TIC}_\omega$ (note that $\mathbb{U} := \widetilde{\mathbb{F}}^{\circlearrowleft}_{12} \in \mathrm{TIC} \subset \mathrm{TIC}_\omega$, because $\mathbb{U}w \in \mathrm{L}^2_\vartheta \subset \mathrm{L}^2$ for all $w \in \mathrm{L}^2(\mathbf{R}_+;W)$, by (11.7)). Since we assumed that $\vartheta \leq 0$, we have $\widehat{\mathbb{S}}_{11} \geq \varepsilon_+^2 I$ on $\mathbf{C}^+_\omega$, by (b2). This and (11.78) imply that $\widetilde{S} := \widehat{\mathbb{S}}(s_0)$ satisfies (i') (hence (i)–(vi)) of Lemma 11.3.13 for any $s_0 \in \mathbf{C}^+_\omega$. Consequently, (11.59) holds and $\widetilde{S} = \widetilde{X}^* J_1 \widetilde{X}$ for some $\widetilde{X} = \left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right] \in \mathcal{GB}$ (with $X_{11}, X_{22} \gg 0$), hence $S = E^* J_1 E$, where $E := \widetilde{X}\widehat{\mathbb{X}}(s_0)^{-1} \in \mathcal{GB}$.

By using this $\widetilde{E}$, we obtain $(\mathcal{P}, J_1, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ as in (b2)2° above. Moreover, then $\widehat{\widetilde{\mathbb{X}}}(s_0) = E\widehat{\mathbb{X}}(s_0) = \widetilde{X}$. By (b2), we have

$$\varepsilon_+^2 I \leq (\widehat{\mathbb{X}}^* J_1 \widehat{\mathbb{X}})_{11} = \widehat{\mathbb{X}}_{11}^* \widehat{\mathbb{X}}_{11} - \widehat{\mathbb{X}}_{21}^* \widehat{\mathbb{X}}_{21} \tag{11.79}$$

on $\mathbf{C}^+_\omega$. This and the invertibility of $\widehat{\widetilde{\mathbb{X}}}(s_0)_{11} = \widetilde{X}_{11}$, imply that $\widetilde{\mathbb{X}}_{11} \in \mathcal{G}\mathrm{TIC}_\omega(U)$, by Proposition 2.2.5(5). The rest follows from Lemma 11.2.21.

(d) By (b2), we have $((\mathbb{X}^t)^* S \mathbb{X}^t)_{11} \geq \varepsilon_+^2 I_U$, hence $(\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})_{11} \geq \varepsilon_+^2 I_U$ on $\mathbf{C}^+_\omega$ for ω big enough, by Lemma 2.2.4. Therefore, for any $u_0 \in U$, we have

$$\varepsilon_+^2 \|u_0\|^2 \leq \langle \widehat{\mathbb{X}}(s) \left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right], S\widehat{\mathbb{X}}(s) \left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right] \rangle_{U \times W} \to \langle \left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right], S \left[\begin{smallmatrix} u_0 \\ 0 \end{smallmatrix}\right] \rangle_{U \times W}, \tag{11.80}$$

as $s \to +\infty$, by strong regularity. Thus, $S_{11} \geq \varepsilon_+^2 I$.

(d1) In (b2)2° we showed that $(\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})_{11}(r) \geq \varepsilon_+^2 I$ for big r; let $r \to +\infty$ to obtain that $S_{11} \geq \varepsilon_+^2 I$ (since $\widehat{\mathbb{X}}(+\infty) = I$). Analogously, from (11.78) we obtain that $(I\left[\begin{smallmatrix} \widetilde{U} \\ I \end{smallmatrix}\right])^* S (I\left[\begin{smallmatrix} \widetilde{U} \\ I \end{smallmatrix}\right]) \leq -\varepsilon I$, where $\widetilde{U} := \widehat{\mathbb{U}}(+\infty)$. By Lemma 11.3.13(i')&(i), this means that $S_{22} - S_{21} S_{11}^{-1} S_{12} \leq -\varepsilon I$.

(d2) Now we have $\widetilde{U} = 0$ in the proof of (d1), hence $S_{22} \leq -\varepsilon I$.

Remark: In (d1) and (d2), we can use any $\varepsilon > 0$ s.t. $\langle \mathbb{D}^{\circlearrowleft}_{\circlearrowleft 2} w, J_\gamma \mathbb{D}^{\circlearrowleft}_{\circlearrowleft 2} w \rangle \leq -\varepsilon \|w\|_2^2$ for all $w \in \mathrm{L}^2(\mathbf{R}_+;W)$ (where $\Sigma^{\circlearrowleft}$ is the closed-loop system corresponding to the suboptimal pair or operator, as in the proof of (c)).

Remark: In (d)–(d2), the assumption on the CARE may be replaced by the weaker assumption that the IARE has a SR solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ s.t. $F = 0$.

(By Proposition 9.8.10, the only difference is that $\mathbb{D}$ need not be WR; obviously, the regularity of $\mathbb{D}$ is not needed in the above proofs.)

(e) (The assumption of (e) is rather common, since one usually has "$\mathcal{J} = \|u\|_2^2 + \|y\|_2^2$" or something similar.)

The assumption that $\vartheta \leq 0$ was only used in (b2)2° (and later to justify referations to (b2)), to show that $(\mathbb{X}^{t*} S \mathbb{X}^t)_{11} \geq \varepsilon_+^2 I$; under the assumption of (e) this inequality follows directly from (11.66).

(f) In the proofs of (b3), (c), (d1) and (d2), we did not need $\mathbb{K}$ or $\mathbb{F}$, just a

$\mathbb{U}$ with the properties mentioned in (f). (Note that (f) and (e) are compatible.) $\square$

From the above proposition, one can conclude that the existence of a well-posed solution of the FICP in the I/O sense (i.e., of a suboptimal well-posed control law) implies a solution in the usual sense if the finite cost condition ($\mathcal{U}_*^* \neq 0$) is satisfied and Σ is smooth enough to guarantee that unique J_γ-critical control (of a J_γ-coercive system) corresponds to a WR state feedback operator (i.e., to a CARE):

Lemma 11.2.20 (I/O FICP $\Rightarrow$ FICP) *Assume that there is* $\mathbb{U} \in \mathrm{TIC}_\infty$ *s.t.* $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\| < \gamma$ *and* $\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right][\mathrm{L}^2(\mathbf{R}_+;W)] \subset \mathcal{U}_{\exp}(0)$. *Assume also that* Σ *is optimizable and* $(\Sigma, J) \in \mathrm{coerciveCARE}$ *(cf. Remark 9.9.14).*

Then there is a unique exponentially stabilizing solution $(\mathcal{P}, S, K)$ *of the CARE. Moreover,* $\mathcal{P} \geq 0$, $S_{11} \geq \varepsilon_+ I$, $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, *and (11.39) generate a SR suboptimal* H^∞*-FI-pair (here* $K_1 := \begin{bmatrix} I & 0 \end{bmatrix} K$*).*

(This lemma will be used for the H^∞ 4BP.)

Proof: (As one observes from the proof, we could allow for $\mathcal{U}_{\mathrm{out}}$ instead of $\mathcal{U}_{\exp}$ if we would require (11.47) for some $\varepsilon > 0$ and replace "exponentially stabilizing" by "$\mathcal{U}_{\mathrm{out}}$-stabilizing".)

The assumptions $\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right][\mathrm{L}^2(\mathbf{R}_+;W)] \subset \mathcal{U}_*^*(0)$ and $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\| < \gamma$ imply that $\gamma > \|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\| \geq \gamma_0$ (we could replace "$\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right][\mathrm{L}^2(\mathbf{R}_+;W)] \subset \mathcal{U}_*^*(0)$" by "$\gamma > \gamma_0$" in the assumptions).

By Proposition 11.2.19(a2) the CARE for Σ and J_γ has a (unique) SR $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$.

By Proposition 11.2.19(b2)&(f)&(c), we have that $\mathcal{P} \geq 0$ and (11.59) holds (the condition "$\mathbb{U} \in \mathrm{TIC}_\omega$" is redundant since $\mathbb{U} \in \mathrm{TIC}(W, U)$ (because $\mathbb{U}[\mathrm{L}^2(\mathbf{R}_+;W)] \subset \mathcal{U}_{\exp}(0) \subset \mathrm{L}^2$; use also (2.13))).

Consequently, Lemma 11.2.14(4.)&(b)&(a2) can be applied to obtain a SR (since $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is SR) suboptimal state feedback pair $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ as above. The existence of such a pair implies that $S_{11} \geq \varepsilon_+ I$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, by Proposition 11.2.19(d1).

$\square$

The following lemma was used above:

Lemma 11.2.21 *Let* $\omega \in \mathbf{R}$, $\mathbb{X} \in \mathcal{G}\mathrm{TIC}_\omega(U \times W)$, $\widehat{\mathbb{X}}_{11}(s_0) \in \mathcal{GB}(U)$ *for some* $s_0 \in \mathbf{C}_\omega^+$ *and* $(\mathbb{X}^{t*} J_1 \mathbb{X}^t)_{11} \geq \varepsilon I$ *(on* $\mathrm{L}^2([0,t);U)$*) for all* $t > 0$. *Then* $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}_\omega(U)$, $\|\mathbb{X}_{11}^{-1}\|_{\mathrm{TIC}} \leq \varepsilon^{-1/2}$ *and* $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TIC}} \leq 1$.

Proof: By Lemma 2.2.4(a) we have

$$\varepsilon I \leq (\widehat{\mathbb{X}}^* J_1 \widehat{\mathbb{X}})_{11} = \widehat{\mathbb{X}}_{11}^* \widehat{\mathbb{X}}_{11} - \widehat{\mathbb{X}}_{21}^* \widehat{\mathbb{X}}_{21} \tag{11.81}$$

on $\mathbf{C}_\omega^+$, hence $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}_\omega$, by Proposition 2.2.5(5). But

$$\varepsilon I \leq (\mathbb{X}^{t*} J_1 \mathbb{X}^t)_{11} = \mathbb{X}^t{}_{11}^* \mathbb{X}^t{}_{11} - \mathbb{X}^t{}_{21}^* \mathbb{X}^t{}_{21} \tag{11.82}$$

implies that

$$\varepsilon \mathbb{X}^{t\,-*}_{11}\mathbb{X}^{t\,-1}_{11} + \mathbb{X}^{t\,-*}_{11}\mathbb{X}^{t\,*}_{21}\mathbb{X}^{t}_{21}\mathbb{X}^{t\,-1}_{11} \leq I. \tag{11.83}$$

By Lemma 2.1.14, it follows that $\mathbb{X}_{11}^{-1}, \mathbb{X}_{21}\mathbb{X}_{11}^{-1} \in \mathrm{TIC}$, $\|\mathbb{X}_{11}^{-1}\| \leq \varepsilon^{-1/2}$, and $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\| \leq 1$ $\square$

The proofs of Theorem 11.2.7 and Proposition 11.2.8 were based on the fact that under their assumptions, we are able to reduce the problem to the (SOS-)stable case:

Lemma 11.2.22 ($\boldsymbol{\Sigma \Leftrightarrow \Sigma_\flat}$) *Assume that $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] = \left[\begin{array}{c|cc} \widetilde{\mathbb{K}}_1 & \widetilde{\mathbb{F}}_{11} & \widetilde{\mathbb{F}}_{12} \\ 0 & 0 & 0 \end{array}\right]$ is an admissible state feedback pair for Σ with closed-loop system $\Sigma_\flat$, and that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ (or that $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ and $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ is q.r.c.-SOS-stabilizing).*

(a) (Drop Standing Hypothesis 11.2.1 for part (a).)

Then $\mathbb{D}_{\flat 11}^*\mathbb{D}_{\flat 11} \gg 0$ iff Hypothesis 11.2.1 holds. A sufficient condition for this is that $\mathbb{D}_\flat^* J_\gamma \mathbb{D}_\flat$ has a spectral factorization.

(b) We have, $\gamma > \gamma_0$ iff

$$\gamma > \sup_{w \in \mathrm{L}^2(\mathbf{R}_+;W)\setminus\{0\}} \inf_{u \in \mathrm{L}^2(\mathbf{R}_+;U)} \|\mathbb{D}_\flat \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]\|_2/\|w\|_2. \tag{11.84}$$

(c) There is a suboptimal H^∞-FI-pair for Σ iff there is a suboptimal H^∞-FI-pair for $\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{array}\right]$ (the same holds for WR suboptimal H^∞-SF-operators if $\widetilde{\mathbb{F}}$ is SR and $\widetilde{F} = 0$).

Recall from Theorem 8.4.5(g2) that if $\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right]$ is [[exponentially] strongly] q.r.c.-stabilizing, then $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}}[= \mathcal{U}_{\mathrm{str}}[= \mathcal{U}_{\mathrm{exp}}]]$.

Proof: (a) 1° *The equivalence, case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ (resp. $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$):* From (11.10) (and (6.133)), we easily observe that $\left[\begin{array}{c|c} \widetilde{\mathbb{K}}_1 & \mathbb{F}_{11} \end{array}\right]$ is an admissible state feedback pair for $\Sigma_{11} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C} & \mathbb{D}_1 \end{array}\right]$, with closed loop system $\Sigma_{\flat 11} := \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_{\flat 1} \\ \hline \mathbb{C}_\flat & \mathbb{D}_{\flat 1} \end{array}\right]$, which is exponentially (resp. SOS-)stable, because so is $\Sigma_\flat$.

By Lemma 11.2.2, Hypothesis 11.2.1 is satisfied iff Σ_{11} is I-coercive. By Theorem 8.4.5(d) (resp. and (g1)), Σ_{11} is I-coercive iff $\Sigma_{\flat 11}$ is I-coercive, i.e., iff $\mathbb{D}_{\flat 11} \gg 0$ (by Lemma 8.4.11(b1)).

2° The latter claim follows from Lemma 11.3.12.

(b)&(c) *I Case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$:*

*I.*1° *"$\gamma_0 = \gamma_{\flat,0}$":* Set $\widetilde{\mathbb{M}} := (I - \widetilde{\mathbb{F}})^{-1} =: \left[\begin{smallmatrix} \widetilde{\mathbb{M}}_{11} & \widetilde{\mathbb{M}}_{12} \\ 0 & I \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(U \times W)$. By Theorem 8.4.5(e)&(c1), $\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(x_0) = \mathrm{L}^2(\mathbf{R}_+; U \times W)$ for all $x_0 \in H$ and $\mathcal{U}_{\mathrm{exp}}(0) = \widetilde{\mathbb{M}}[\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(0)]$. Thus, if $\left[\begin{smallmatrix} u_\flat \\ w \end{smallmatrix}\right] \in \mathrm{L}^2(\mathbf{R}_+; U \times W)\,\mathcal{U}_{\mathrm{exp}}^{\Sigma_\flat}(0))$, then $u := \left[\begin{matrix} \widetilde{\mathbb{M}}_{11} & \widetilde{\mathbb{M}}_{12} \end{matrix}\right] \left[\begin{smallmatrix} u_\flat \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{u}}(0,w)$ and $\|\mathbb{D}\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]\| = \|\mathbb{D}_\flat \left[\begin{smallmatrix} u_\flat \\ w \end{smallmatrix}\right]\|$.

Therefore, $f(w) := \min_{u\in\mathcal{U}_{\mathrm{u}}(0,w)} \mathcal{J}(0,u,w) \le \min_{u_\flat\in\mathcal{U}_{\mathrm{u}}^{\Sigma_\flat}(0,w)} \mathcal{J}_\flat(0,u,w) =: f_\flat(w)$. Since w was arbitrary, $f \ll 0$ if $f_\flat \ll 0$. Exchange the roles of Σ and $\Sigma_\flat$ to obtain the converse.

I.2° *Suboptimal* H^∞*-FI-pairs:* If $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is a suboptimal H^∞-FI-pair for $\Sigma_\flat$ (i.e., it is exponentially stabilizing and its second row equals zero, by Remark 11.2.5), then also (6.193) is exponentially stabilizing and has a zero second row. Since $\mathbb{D}^\frown$ is the same for these two pairs, (6.193) is a suboptimal H^∞-FI-pair for Σ. Exchange the roles of Σ and $\Sigma_\flat$ to obtain the converse.

I.3° *Suboptimal* H^∞*-SF-operators:* The proof of 2° applies except that we have to use Proposition 6.6.18(f) (which shows that any WR $\begin{bmatrix} (K_\natural)_1 \\ 0 \end{bmatrix}$ for $\Sigma_\flat$ corresponds to the WR state feedback operator $\begin{bmatrix} K_1 \\ 0 \end{bmatrix} := \begin{bmatrix} \widetilde{K}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} (K_\natural)_1 \\ 0 \end{bmatrix}$ for Σ).

II Case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$*:* The proofs is analogous to that above. In particular, *I*.1° applies since again $\mathcal{U}_{\mathrm{out}}(0) = \widetilde{\mathbb{M}}[\mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}(0)] = \mathrm{L}^2(\mathbf{R}_+;U\times W)$ (see Theorem 8.4.5(g1) and its proof).

II.1° *Suboptimal* H^∞*-FI-pairs —* $\Sigma_\flat \Rightarrow \Sigma$*:* Assume then that $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is a suboptimal H^∞-FI-pair for $\Sigma_\flat$. Define $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ and $\Sigma'_\circlearrowleft$ by (6.193). Let $x_0 \in H$ and $w \in \mathrm{L}^2(\mathbf{R}_+;W)$.

By Theorem 8.4.5(g1)&(e), $\mathbb{M}_\natural\mathbb{K}_\natural x_0 + \mathbb{M}_\natural \left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}(x_0) = \mathrm{L}^2$, hence $\mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}}\left[\begin{smallmatrix} u_\flat \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{out}}(x_0)$, where $\mathbb{M}_\natural := I - \mathbb{F}_\natural$ and $\widetilde{\mathbb{M}} := I - \widetilde{\mathbb{F}}$. But $\mathbb{K}_\flat x_0 + \widetilde{\mathbb{M}}\left[\begin{smallmatrix} u_\flat \\ w \end{smallmatrix}\right] = \mathbb{K}'_\circlearrowleft x_0 + \mathbb{M}'\left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right]$; because x_0 and w were arbitrary, $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ is a H^∞-FI-pair for Σ. Since y is the same for both pairs, by Theorem 8.4.5(g1)&(c1), the pair $\left[\begin{array}{c|c} \mathbb{K}' & \mathbb{F}' \end{array}\right]$ is suboptimal for $\Sigma_\flat$.

II.2° *Suboptimal* H^∞*-FI-pair —* $\Sigma \Rightarrow \Sigma_\flat$*:* For the converse, assume that $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is a suboptimal H^∞-FI-pair for Σ with closed-loop system $\Sigma_\circlearrowleft$. Define $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ by (6.180) and set $\mathbb{M}_\natural := (I-\mathbb{F}_\natural)^{-1}$, $\mathbb{M} := (I-\mathbb{F})^{-1}$. By (6.183), we have for any $x_0 \in H$ and $w \in \mathrm{L}^2(\mathbf{R}_+;W)$ that

$$\begin{bmatrix} u_{\flat\circlearrowleft} \\ w \end{bmatrix} := \begin{bmatrix} \mathbb{M}_\natural\mathbb{K}_\natural & \mathbb{M}_\natural \end{bmatrix} \begin{bmatrix} x_0 \\ 0 \\ w \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{X}}\mathbb{K}_\circlearrowleft - \widetilde{\mathbb{K}} & \widetilde{\mathbb{X}}\mathbb{M} \end{bmatrix} \begin{bmatrix} x_0 \\ 0 \\ w \end{bmatrix} = -\widetilde{\mathbb{K}}x_0 + \widetilde{\mathbb{X}}\left[\begin{smallmatrix} u_\circlearrowleft \\ w \end{smallmatrix}\right], \tag{11.85}$$

where $\left[\begin{smallmatrix} u_\circlearrowleft \\ w \end{smallmatrix}\right] := \mathbb{K}_\circlearrowleft x_0 + \mathbb{M}\left[\begin{smallmatrix} 0 \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{out}}(x_0)$, so that $\left[\begin{smallmatrix} u_{\flat\circlearrowleft} \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{out}}^{\Sigma_\flat}(x_0) = \mathrm{L}^2$, by Theorem 8.4.5(g1)&(c1). As in *II*.1°, we see that $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right]$ is suboptimal for $\Sigma_\flat$.

II.3° *Suboptimal* $\mathcal{U}_{\mathrm{out}}$*-SF pairs:* This goes as in *I*.3°. □

Notes

The "short-hand-notation" or "extended FICP" formulation of Definition 11.2.3 and Lemma 11.2.4 was given in [S98d] for the stable case. The principle to reduce the $\mathcal{U}_{\mathrm{exp}}$ problem to the stable case (Lemma 11.2.22) is old. Lemma 11.2.18 and its proof are close to [S98d, Theorem 7.2].

Condition (FI9) was used by Michael Green [Green] to solve a problem close to the FICP; the results of [Green] were extended to $\mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}(\mathbf{C}^n,\mathbf{C}^m)$ I/O maps in [CG97]. Hidenori Kimura and others have produced analogous results for (possibly nonlinear) finite-dimensional systems by using the conjugation method [KK]. See also the notes on pp. 628 and 669.

Naturally, if we strengthen (FI1) by requiring that the "minimax" control can be given in [regular] state feedback form, then (FI1) becomes equivalent to the IARE [and CARE] having a $\mathcal{U}_*^*$-stabilizing solution (see Proposition 11.2.19 and Theorem 9.9.1).

The reduction the stable case and the coverage of several equivalent conditions has made the proofs "unnecessarily complex". The main results can be obtained much more directly, as explained below.

The sufficiency of (FI5) (for (FI2)) is shown in Lemma 11.2.13 (and in Lemma 11.2.14; see Proposition 11.2.9 for final details). This proof is direct and constructive: we open the disturbance feedback from the closed-loop system corresponding to the CARE, and it appears that the resulting pair is a H^∞-FI-pair (i.e., it stabilizes the system in the desired way (depending on $\mathcal{U}_*^*$)) and suboptimal.

The necessity of (FI5) is shown in Proposition 11.2.19(a2)&(d1) (assuming that $(\Sigma, J) \in$ coerciveCARE; note that this is always the case in discrete time for $\mathcal{U}_{\text{exp}}$; this is the case in continuous time too if we make the assumptions of Theorem 11.2.7 or Proposition 11.2.8(a2), so that any unique J_γ-critical control is necessarily of SR state feedback form).

Also this necessity proof is direct and constructive: we first show that if $\gamma > \gamma_0$, then $\mathbb{D}$ is J_γ-coercive over $\mathcal{U}_*^*$, so that the CARE has a $\mathcal{U}_*^*$-stabilizing solution $\mathcal{P}$; we also show that the corresponding closed-loop input is the minimax input, from which we deduce that $\mathcal{P} \geq 0$. Then we assume the existence of a causal suboptimal control law $w \mapsto u$ (e.g., of a suboptimal H^∞-FI-pair) and use the minimax property to show that this implies the signature condition $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$ of (FI5).

However, in the proofs of our main results (see the proof of Proposition 11.2.8), we have reduced our solution to the stable case for two reasons:

1. this way we were able to extend the equivalence to (FI1), i.e., we were able to show that if there are any suboptimal controls for each x_0 and w, then there is a (causal!) suboptimal state feedback controller (H^∞-FI-pair) (cf. also the remark in the proof of Proposition 11.2.19(b2));
2. this allowed us to use the techniques of [S98d] to provide the formula for all solutions; this formula will be needed for the H^∞ 4BP.

The results 1. and 2. were established in [S98d] for stable WPLSs over separable Hilbert spaces (for $\mathcal{U}_*^* = \mathcal{U}_{\text{out}}$), and we use essentially the same methods except for the treatment of the unseparable case; see the next section for details. Theorem 11.1.5 illustrates a case where it was not possible to use this reduction to the stable case; this explains the missing "(i)" (or "(FI1)").

The results of this section and Section 11.3 also contain [partial] sufficiency results under further alternative conditions, some of which are needed for the H^∞ 4BP.

11.3 The H^∞ FICP: stable case

All the world's a stage, And all the men and women merely players. They have their exits and their entrances, And one man in his time plays many parts, His acts being seven ages.

— William Shakespeare (1564–1616), "As You Like It"

In this section, we shall solve the H^∞ FICP in the stable case, parameterize all solutions, and present some additional results. The stable case is interested in its own right, but it is also useful for the H^∞ 4BP and for the proofs of the results in previous sections, although we were able to prove part of them (e.g., the equivalence of (FI2)–(FI5)) directly in the unstable case.

In addition to Standing Hypotheses 11.0.1 and 11.1.1, we shall assume the following:

Standing Hypothesis 11.3.1 (Stable case) *Throughout this section, we assume that* $\gamma > 0$, $\Sigma \in \mathrm{SOS}$, $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ *and* $\mathbb{D}_{11}^*\mathbb{D}_{11} \gg 0$.

(It follows that $\vartheta = 0$ and Z^{s} is reflexive, by Definition 8.3.2.)

Recall from Lemma 8.3.3, that if Σ is [[exponentially] strongly] stable, then $[[\mathcal{U}_{\mathrm{exp}} =] \mathcal{U}_{\mathrm{str}} =] \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{out}}$.

Since $\mathbb{D}$ is stable, the norm $\|\cdot\|_{\mathcal{U}_{\mathrm{out}}}$ is equivalent to $\|\cdot\|_2$. Consequently, Standing Hypothesis 11.3.1 is equivalent to Hypothesis 11.2.1 with the additional conditions $\Sigma \in \mathrm{SOS}$ and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$. In particular, Hypothesis 11.2.1 holds.

Lemma 11.3.2 *An admissible state feedback pair* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ *is a* H^∞*-FI-pair iff* $\mathbb{K}^{\curvearrowleft}$ *and* $\mathbb{F}_{12}^{\curvearrowleft}$ *are stable.*

However, in the theorem and proposition below, we shall make such assumptions that the existence of a H^∞-FI-pair implies the existence of stable H^∞-FI-pair.

Proof: This follows from Definition 11.1.2, because now $\mathcal{U}_{\mathrm{out}}(x_0) = \mathrm{L}^2(\mathbf{R}_+; U \times W)$ for all $x_0 \in H$ (note that $\mathbb{F}^{\curvearrowleft} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$). □

Now we state the main result of this section, the stable counterpart of Theorem 11.2.7:

Theorem 11.3.3 ($\widetilde{\mathcal{A}}$: Stable FICP) *Assume that* $\mathbb{D} \in \widetilde{\mathcal{A}}$. *Then the following are equivalent:*

(FI1s) $\gamma > \gamma_0$; *i.e.,* $\inf_{u \in \mathcal{U}_{\mathrm{u}}(0,\cdot)} \mathcal{J}(0,u,\cdot) \ll 0$;

(FI$1\frac{1}{2}$s) $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\| < \gamma$ *for some* $\mathbb{U} \in \mathrm{TIC}(W,U)$;

(FI2s) $\gamma > \gamma_{\mathrm{FI}}$; *i.e., there is a suboptimal* H^∞*-FI-pair for* Σ;

(FI3s) $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$ *where* $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$;

(FI4s) the IARE has a stable, P-SOS-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$, *and* $\mathcal{P} \geq 0$, *and* $\widetilde{S} := (\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})(s_0)$ *safisfies* $\widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$ *for some (equivalently, all)* $s_0 \in \mathbf{C}^+$.

(FI5s) the CARE has a UR stable, P-SOS-stabilizing solution $(\mathcal{P}, S, K)$, and $\mathcal{P} \geq 0$, $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$.

Moreover, (a)–(f) of Proposition 11.3.4 apply, and any solutions ($\mathbb{X}$ or $\mathbb{F}$) of (FI3)–(FI5) belong to $\widetilde{\mathcal{A}}$.

In particular, then there is a suboptimal H^∞-FI-pair (resp. H^∞-SF-operator) iff the CARE has a (necessarily unique) stable, P-SOS-stabilizing solution $(\mathcal{P}, S, K)$, and $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$ (resp. $S_{11} \gg 0$ and $S_{22} \ll 0$).

If this is the case, then (11.39) generates (resp. K_1 =(11.40) is) a ULR stable, r.c.-SOS-stabilizing suboptimal H^∞-FI-pair (resp. H^∞-SF-operator).

Note from (d1) that "stable, P-SOS-stabilizing" means "exponentially stabilizing" when Σ is exponentially stable. Naturally, $\mathbb{X} := I - \mathbb{F}$ in (FI4s).

Condition (FI1$\frac{1}{2}$s) can be considered as the "frequency-domain stable H^∞ FICP". Without the requirement $\mathbb{U} \in \text{TIC}$, (FI1$\frac{1}{2}$s) would always be equivalent to (FI1s), by Lemma 11.3.10.

We conclude from the theorem that $\mathbb{D} \in \widetilde{\mathcal{A}}$ implies that $\gamma_0 = \gamma_{\text{FI}}$.

Proof of Theorem 11.3.3: Assume that some of (FI1s)–(FI5s) holds. As obvious from the proof of Proposition 11.3.4, this implies that (FI1s) holds, hence $\mathbb{D}$ is J_γ-coercive, by Lemma 11.3.10, hence $\mathbb{D}^* J_\gamma \mathbb{D}$ has a spectral factorization $\mathbb{X}_\diamond^* S_\diamond \mathbb{X}_\diamond$ with $\mathbb{X}_\diamond \in \widetilde{\mathcal{A}}$ (because $\mathbb{D} \in \widetilde{\mathcal{A}}$). Thus, the assumptions of the proposition are satisfied.

Since $\mathbb{X}_\diamond \in \widetilde{\mathcal{A}} \subset \text{ULR}$, the assumptions of (a) are satisfied and thus we obtain the other claims (with $\mathbb{F} = I - E\mathbb{X}_\diamond \in \widetilde{\mathcal{A}}$ for some $E \in \mathcal{GB}(U \times W)$); in particular, K and $\mathbb{F}$ are necessarily ULR. □

Since (FI1s) does not imply any of (FI3s)–(FI5s) for general strongly stable WPLSs, by Example 11.3.7(a), we made the $\widetilde{\mathcal{A}}$ assumption above, and we shall make a weaker spectral factorization assumption in the following stable variant of Proposition 11.2.8 (this assumption is necessary for (FI3s)–(FI5s)):

Proposition 11.3.4 (Stable FICP) *Assume that $\mathbb{D} \in \text{TIC}$ and $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}_\diamond^* S_\diamond \mathbb{X}_\diamond$ for some $\mathbb{X}_\diamond \in \mathcal{G}\text{TIC}(U \times W)$ and $S_\diamond \in \mathcal{GB}(U \times W)$.*

Then (FI1s)–(FI4s) are equivalent and implied by (FI5s). Also the following hold:

(a) Assume that $\mathbb{D}$ is WR and $\mathbb{X}_\diamond$ is UR. Then (FI1s)–(FI5s) are equivalent, and the CARE has a unique stable, P-SOS-stabilizing solution $(\mathcal{P}, S, K)$.

There is a suboptimal H^∞-FI-pair (resp. H^∞-SF-operator) iff $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$ (resp. $S_{11} \gg 0$ and $S_{22} \ll 0$); if this is the case, then (11.39) generate (resp. K_1 =(11.40) is) a UR stable, r.c.-SOS-stabilizing suboptimal H^∞-FI-pair (resp. H^∞-SF-operator).

(b1) The condition on $\widetilde{S}$ in (FI4s) is independent on the choice of S and $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ (and $s_0 \in \mathbf{C}^+$), and $\mathcal{P}$ is unique. Condition $\mathbb{X}_{\natural 11} \in \mathcal{G}\text{TIC}(U)$ in (FI3) is independent on $\mathbb{X}_\natural$ (by (c1)).

(b2) A stable, P-SOS-stabilizing solution of the CARE is unique.

(b3) A solution of (FI3s) or (FI4s) is unique modulo an invertible constant.

(c1) If (FI1s) holds, $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$ *and* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$*, then* $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ *and* $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TIC}} < 1$.

(c2) Any solution of (FI5s) is unique and a solution of (FI4s).

(c3) If $\mathbb{X}$ *and* $\mathbb{F}$ *are as in (FI3s) and (FI4s), respectively, then* $\mathbb{X} := E(I - \mathbb{F}) = E'\mathbb{X}_\diamond$ *for some* $E, E' \in \mathcal{GB}(U \times W)$.

(d1) Assume that Σ *is exponentially stable. Then "stable, P-SOS-stabilizing" has the following equivalent forms: "exponentially stabilizing" and "I/O-, input-, output- or internally stabilizing".*

(d2) Assume that Σ *is strongly stable. Then "stable, P-SOS-stabilizing" has the following equivalent forms: "stable, strongly stabilizing", "internally stabilizing (i.e.,* $\mathbb{A}$*-stabilizing)".*

(e) Any UR solution of (FI3s) can be redefined s.t. $X = \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}$, $X_{11}, X_{22} \in \mathcal{GB}$.

(f) Even without the above spectral factorization assumption (on $\mathbb{X}_\diamond$ *and* $S_\diamond$*), we have (FI5s)⇒(FI4s)⇔(FI3s)⇒(FI2s)⇒(FI1*$\frac{1}{2}$*s)⇒(FI1s), and (FIns)⇔(FIn) for* $n = 1, 2, 3, 4, 5$.

(g) If (FI1s) holds, then $\mathbb{A}_\circlearrowleft$, $\mathbb{C}_\circlearrowleft$, $\mathbb{K}_\circlearrowleft$ *and* $\mathcal{P}$ *are given by (8.43)–(8.46).*

By Example 11.2.16, condition $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ is not redundant in general; i.e., $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$ and $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W)$ do not imply any of (FI1s)–(FI5s)).

Recall from Lemma 5.2.1(d) that the spectral factorization assumption could be formulated by "$\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{Y}^* \mathbb{X}_\diamond$ for some $\mathbb{Y}, \mathbb{X}_\diamond \in \mathcal{G}\mathrm{TIC}(U \times W)$" (since necessarily $\mathbb{Y} = S\mathbb{X}_\diamond$ for some $S = S^* \in \mathcal{GB}(U \times W)$). By Proposition 11.2.8(d), under this assumption also conditions (FI1)–(FI4), (FI6) and (FI7) are equivalent to any of (FI1s)–(FI4s) (naturally, this refers to the $\mathcal{U}_{\mathrm{out}}$ forms of (FI1)–(FI7) (not to $\mathcal{U}_{\mathrm{exp}}$ forms unless Σ is exponentially stable)).

Even if the above spectral factorization assumption (on $\mathbb{X}_\diamond$ and $S_\diamond$) does not hold, we have (FI*n*s)⇔(FI*n*) for $n = 1, 2, 3, 4, 5$, as one observes from Corollary 9.9.11 (and Lemma 11.3.9(b)).

Proof of Proposition 11.3.4: 1° *(FI3s)⇒(FI2s)&(FI4s):* By Corollary 9.9.11, the pair (9.140) defines a stable and P-r.c.-SOS-stabilizing solution of the IARE. By Lemma 11.2.18, $\mathcal{P} \geq 0$ and $\left[\; \overline{\mathbb{K}} \;\middle|\; \overline{\mathbb{F}} \;\right]$ is a suboptimal H$^\infty$-FI-pair, hence (FI2s) holds; from Proposition 11.2.19(c) we obtain (FI4s).

2° *(FI2s)⇒(FI1*$\frac{1}{2}$*s):* By (11.8)–(11.9) we can take $\mathbb{U} := \mathbb{F}_{12}^{\frown}$.

3° *(FI1*$\frac{1}{2}$*s)⇒(FI1s):* This follows from Lemma 11.3.10.

4° *(FI1s)⇔(FI3s):* By Lemma 11.3.10, (FI1s) holds iff $\mathbb{D}$ is minimax J_γ-coercive. By Lemma 11.4.3(a)&(c), the latter is equivalent to (FI3s) (and any J_1-spectral factor $\mathbb{X}$ of $\mathbb{D}^* J_\gamma \mathbb{D}$ satisfies $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ and $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}} < 1$).

5° *(FI4s)⇒(FI2s):* This is contained in Lemma 11.3.9.

6° *(FI5s)⇒(FI4s):* This is given in the proof of Lemma 11.3.8.

(a) 1° *Uniqueness:* A solution of (FI5s) (if any) is $\mathcal{U}_*^*$-stabilizing (by Proposition 9.8.11(iii)&(ii)), hence unique.

2° *The CARE:* By Corollary 9.9.11 (and Theorem 9.9.10(a1)–(b)), the spectral factorization $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}_\diamond^* S_\diamond \mathbb{X}_\diamond$ defines a UR J_γ-critical stable, r.c.-P-SOS-stabilizing and $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S_\diamond, \begin{bmatrix} \mathbb{K}_\diamond & | & \mathbb{F}_\diamond \end{bmatrix})$ of the IARE.

All stable, P-SOS-stabilizing solutions of the IARE are given by (9.114). One of them, say $(\mathcal{P}, S, K)$, is a solution of the CARE, since $I - F_\diamond \in \mathcal{GB}$, by Proposition 6.3.1(b1) (note that $\mathbb{F}_\diamond = I - \mathbb{X}_\diamond$, $\mathbb{F} = I - E\mathbb{X}_\diamond$ and $\mathbb{K} = E\mathbb{K}_\diamond$ for some $E \in \mathcal{GB}$, hence K inherits the uniform regularity of $\mathbb{X}_\diamond$).

3° *(FI2s)⇒(FI5s):* This follows from Proposition 11.2.19(d1).

4° *"iff $S_{11} \gg 0 \gg S_{22}$:* "Only if" follows from Proposition 11.2.19(d2); "if" follows from Lemma 11.3.8.

(b1) This follows from (b3).

(b2) This is Theorem 9.8.12(d)&(s3) (with Proposition 9.8.11).

(b3) For (FI4s), this follows from Theorem 9.8.12(d)&(s1) (with Proposition 9.8.11) (i.e., all triples are given by (9.114)).

If $\mathbb{X}^* J_1 \mathbb{X} = \widetilde{\mathbb{X}}^* J_1 \widetilde{\mathbb{X}}$ and $\mathbb{X}, \widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}$, then $\widetilde{\mathbb{X}} = E\mathbb{X}$ for some $E \in \mathcal{GB}$ s.t. $E^* J_1 E = J_1$ (so not all invertible E's will do).

(c1) This follows from Lemma 11.4.3(b) (note also that $\mathbb{X}_{21}\mathbb{X}_{11}^{-1} = -\mathbb{M}_{22}^{-1}\mathbb{M}_{21}$).

(c2) (Here $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ refers to the pair generated by K.) Uniqueness follows from Theorem 9.8.12(e)&(s1), the rest follows from (11.59).

(c3) The claim on $\mathbb{X}_\diamond$ follows from Lemma 6.4.5(a). If $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ is as in (FI4s), then $\widetilde{\mathbb{X}} := E(I - \mathbb{F}) \in \mathcal{G}\mathrm{TIC}$ satisfies $\widetilde{\mathbb{X}}^* J_1 \widetilde{\mathbb{X}} = \mathbb{D}^* J_\gamma \mathbb{D}$ for some $E \in \mathcal{GB}$, by Lemma 11.3.13(i)&(iii), hence $\widetilde{\mathbb{X}}$ satisfies (FI3s), by (c1). Therefore, $E := \mathbb{X}(I - \mathbb{F})^{-1} \in \mathcal{GB}$, by (b3). (Remark: the pair $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ is necessarily stable and r.c.-SOS-stabilizing.)

(d1)&(d2) These follow from Proposition 9.8.11(d2)&(d3) (and (a)2°). (Note that "$\mathcal{U}_{\mathrm{out}}$-stabilizing", "$\mathcal{U}_{\mathrm{sta}}$-stabilizing" and "stable and strongly r.c.-stabilizing" are also equivalent forms in (d1) and (d2); so is also "exponentially stable and exponentially r.c.-stabilizing" in (d1).)

(e) Set $S := X^* J_1 X$. By Proposition 6.3.1(b1), we have $X, X_{11} \in \mathcal{GB}$. It follows from (c1) and Lemma 11.3.13(vi)&(iii') that $S = \widetilde{X}^* J_1 \widetilde{X}$, where $\widetilde{X}$ is as in (e). Replace $\mathbb{X}$ by $\widetilde{X} X^{-1} \mathbb{X}$ to complete the proof.

(f) In the above proofs of implications (FI5s)⇒(FI4s)⇔(FI3s)⇒(FI2s)⇒(FI1$\frac{1}{2}$s)⇒(FI1s) we did not use the spectral factorization assumption.

The claim "(FIns)⇔(FIn)" is trivial for $n = 1, 2, 3$; for $n = 4, 5$ we obtain this from Corollary 9.9.11 (and Lemma 11.3.9(b)). (Here we referred to $\mathcal{U}_{\mathrm{out}}$ forms of (FI1)–(FI5); they are equivalent to $\mathcal{U}_{\mathrm{exp}}$ forms if Σ is exponentially stable).

(g) By Lemma 11.3.10, (FI1s) implies that $\mathbb{D}$ is minimax J_γ-coercive, hence J_γ-coercive, i.e., the Toeplitz operator $\pi_+ \mathbb{D}^* J_\gamma \mathbb{D} \pi_+$ is invertible. By (a)1°, a pair $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$ corresponding to (FI4s) or (FI5s) is J_γ-critical. Thus, Proposition 8.3.10 applies to $\Sigma_\circlearrowleft$ and $\mathcal{P}$. □

We have shown above that the spectral factorization condition (FI3s) is sufficient (and necessary when $\mathbb{D} \in \widetilde{\mathcal{A}}$) for the existence of a suboptimal control

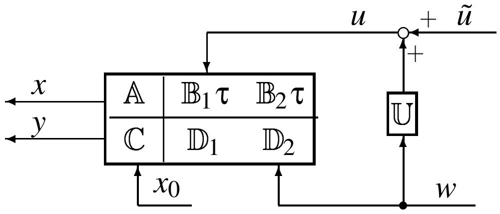

Figure 11.2: Dynamic feedforward compensator

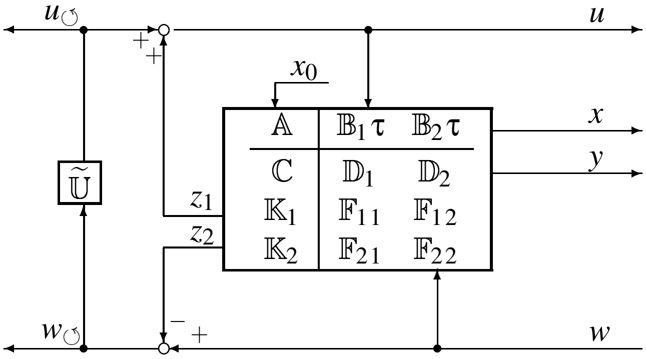

Figure 11.3: Parameterization of all suboptimal compensators

law, and that such a law can be realized as a state feedback controller. For the solution of the H$^\infty$ four-block problem (see Chapter 12), we shall need the following standard parametrization of all suboptimal TIC controllers:

Corollary 11.3.5 (All suboptimal controllers) *If $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$ for some $\mathbb{X} \in \mathrm{TIC}(U \times W)$ s.t. $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$, then all solutions $\mathbb{U} \in \mathrm{TIC}(W,U)$ for $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathrm{TIC}} < \gamma$ are given by Theorem 11.3.6 below.*

The problem "$\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\| < \gamma$" is illustrated in Figure 11.2 and its solution in Figure 11.3 (recall that $y = \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$). Indeed, from Figure 11.3 we observe that $\begin{bmatrix} \widetilde{\mathbb{U}} \\ I \end{bmatrix} w_\circlearrowleft = \mathbb{X}\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$, i.e., $\begin{bmatrix} \mathbb{U}_1 \\ \mathbb{U}_2 \end{bmatrix} w_\circlearrowleft = \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$, hence $u = \mathbb{U}w$ (when $\mathbb{U}_2$ is invertible, and this is the case then $\|\widetilde{\mathbb{U}}\| \le 1$). Thus, each suboptimal controller can be realized as a state feedback plus a subunitary dynamic parameter.

Proof of Corollary 11.3.5: (Recall our hypothesis that $\mathbb{D}_{11}^*\mathbb{D}_{11} \gg 0$, hence $\mathbb{D}_1^* J_\gamma \mathbb{D}_1 \gg 0$, as required in Theorem 11.3.6.)

Indeed, we have

$$(\mathbb{D}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right])^* J_\gamma (\mathbb{D}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right]) = (\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12})^*(\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}) - \gamma^2 I \ll 0 \tag{11.86}$$

iff $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\| < \gamma$, by Lemma A.3.1(d). □

The formulae in the actual proof become simpler for the more general extended FICP below:

Theorem 11.3.6 (All suboptimal controllers) *Assume that $J = J^* \in \mathcal{B}(Y)$, and that $\mathbb{D} := \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \end{bmatrix} \in \mathrm{TIC}(U \times W, Y)$ has a (spectral) factorization $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$ s.t. $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$, and $\mathbb{D}_1^* J \mathbb{D}_1 \gg 0$. Set $\mathbb{M} := \mathbb{X}^{-1}$.*

Then all solutions $\mathbb{U} \in \mathrm{TIC}(W,U)$ to $(\mathbb{D}\left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right])^* J \mathbb{D}\left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right] \ll 0$ (equivalently, to $(w \mapsto \mathcal{J}(0,\mathbb{U}w,w)) \ll 0$) are given by

$$\mathbb{U} := \mathbb{U}_1\mathbb{U}_2^{-1}, \quad \begin{bmatrix}\mathbb{U}_1\\ \mathbb{U}_2\end{bmatrix} = \mathbb{M}\begin{bmatrix}\widetilde{\mathbb{U}}\\ I\end{bmatrix}, \quad (\widetilde{\mathbb{U}} \in \mathrm{TIC}(W,U), \ \|\widetilde{\mathbb{U}}\|_{\mathrm{TIC}} < 1) \tag{11.87}$$

(by $\|\widetilde{\mathbb{U}}\|_{\mathrm{TIC}} \le 1$ we get all solutions to $(\mathbb{D}\left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right])^* J \mathbb{D}\left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right] \le 0$).

Two alternative formulations of (11.87) are given by (all these three formulae produce the same $\mathbb{U}$ for any given $\widetilde{\mathbb{U}} \in \mathrm{TIC}(W,U)$ s.t. $\|\widetilde{\mathbb{U}}\| < 1$ (or $\|\widetilde{\mathbb{U}}\| \le 1$)):

(2.) $\mathbb{U} = \mathbb{Q}_2^{-1}\mathbb{Q}_1$, $\begin{bmatrix}\mathbb{Q}_2 & -\mathbb{Q}_1\end{bmatrix} = \begin{bmatrix}I & -\widetilde{\mathbb{U}}\end{bmatrix}\mathbb{X}$.

(3.) $\mathbb{U} = \mathcal{F}_\ell(\mathbb{T},\widetilde{\mathbb{U}})$*, where*

$$\mathbb{T} := \begin{bmatrix}I & 0\\ \mathbb{X}_{21} & \mathbb{X}_{22}\end{bmatrix}\begin{bmatrix}\mathbb{X}_{11} & \mathbb{X}_{12}\\ 0 & I\end{bmatrix}^{-1} = \begin{bmatrix}\mathbb{M}_{11} & \mathbb{M}_{12}\\ 0 & I\end{bmatrix}\begin{bmatrix}I & 0\\ \mathbb{M}_{21} & \mathbb{M}_{22}\end{bmatrix}^{-1}. \tag{11.88}$$

Moreover, we have the following:

$$\mathbb{U} = (\mathbb{M}_{11}\widetilde{\mathbb{U}} + \mathbb{M}_{12})(\mathbb{M}_{21}\widetilde{\mathbb{U}} + \mathbb{M}_{22})^{-1} = (\mathbb{X}_{11} - \widetilde{\mathbb{U}}\mathbb{X}_{21})^{-1}(-\mathbb{X}_{12} + \widetilde{\mathbb{U}}\mathbb{X}_{22}), \tag{11.89}$$

$$\widetilde{\mathbb{U}} = (\mathbb{M}_{11} - \mathbb{U}\mathbb{M}_{21})^{-1}(-\mathbb{M}_{12} + \mathbb{U}\mathbb{M}_{22}) = (\mathbb{X}_{11}\mathbb{U} + \mathbb{X}_{12})(\mathbb{X}_{21}\mathbb{U} + \mathbb{X}_{22})^{-1}, \tag{11.90}$$

$$\widetilde{\mathbb{U}}_2^{-1} := (\mathbb{X}_{21}\mathbb{U} + \mathbb{X}_{22})^{-1} = \mathbb{U}_2 = \mathbb{M}_{21}\widetilde{\mathbb{U}} + \mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}, \tag{11.91}$$

$$\mathbb{Q}_2^{-1} := (\mathbb{X}_{11} - \widetilde{\mathbb{U}}\mathbb{X}_{21})^{-1} = \mathbb{M}_{11} - \mathbb{U}\mathbb{M}_{21} \in \mathcal{G}\mathrm{TIC}, \tag{11.92}$$

$$\mathbb{U} = \begin{bmatrix}I & 0\end{bmatrix}\begin{bmatrix}\mathbb{X}_{11} & \mathbb{X}_{12}\\ 0 & I\end{bmatrix}^{-1}\begin{bmatrix}\widetilde{\mathbb{U}}\mathbb{U}_2^{-1}\\ I\end{bmatrix}. \tag{11.93}$$

If $\mathbb{D} \in \widetilde{\mathcal{A}}$, then $\mathbb{X} \in \widetilde{\mathcal{A}}$, hence then $\mathbb{U} \in \widetilde{\mathcal{A}} \Leftrightarrow \widetilde{\mathbb{U}} \in \widetilde{\mathcal{A}}$ (this also holds with $\mathrm{TIC}_{\exp}$ in place of $\widetilde{\mathcal{A}}$).

As shown in [S98d], much (but not all) of Proposition 11.3.4 also holds for the above more general "extended" H^∞ FICP ("eFICP") in the stable case.

We shall meet the above more general "extended" H^∞ FICP ("eFICP") again in connection with the H^∞ four-block problem (this corresponds to the (dual of) (Factor2Z) part of Theorem 12.3.7). However, we shall then reduce that problem to a FICP (see Lemma 12.4.8). It seems that this is the easiest way to show that the hypotheses of eFICP theory are satisfied, hence we have no use for a direct eFICP theory and will not treat it further.

Proof of Theorem 11.3.6: (The theorem and this proof hold even without Standing Hypotheses 11.1.1 and 11.2.1.)

We prove the parametrization of all controllers and obtain the other formulae on the way.

1° *Sufficiency:* Let $\widetilde{\mathbb{U}}, \mathbb{U}_1, \mathbb{U}_2, \mathbb{U}$ be as in the statement of the theorem. The $(2,1)$-block of equation $\mathbb{M}\mathbb{X} = I$ implies that $\mathbb{M}_{21}\mathbb{X}_{11} + \mathbb{M}_{22}\mathbb{X}_{21} = 0$, i.e., that $\mathbb{M}_{22}^{-1}\mathbb{M}_{21} = -\mathbb{X}_{21}\mathbb{X}_{11}^{-1}$ (we have $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}$, by Lemma A.1.1(c1)).

But $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\| < 1$, by Lemma 11.4.3(c), hence $\|\mathbb{M}_{22}^{-1}\mathbb{M}_{21}\widetilde{\mathbb{U}}\| \le \|\mathbb{M}_{22}^{-1}\mathbb{M}_{21}\| < 1$, hence $\mathbb{M}_{22}^{-1}\mathbb{M}_{21}\widetilde{\mathbb{U}} + I \in \mathcal{G}\mathrm{TIC}(W)$, equivalently, $\mathcal{G}\mathrm{TIC}(W) \ni \mathbb{M}_{21}\widetilde{\mathbb{U}} + \mathbb{M}_{22} =: \mathbb{U}_2$.

From $\begin{bmatrix} 0 & I \end{bmatrix} \mathbb{X} \left[\begin{smallmatrix} \mathbb{U}_1 \\ \mathbb{U}_2 \end{smallmatrix}\right] = \begin{bmatrix} 0 & I \end{bmatrix} \mathbb{X}\mathbb{M} \left[\begin{smallmatrix} \widetilde{\mathbb{U}} \\ I \end{smallmatrix}\right]$ we obtain that $I = \mathbb{X}_{21}\mathbb{U}_1 + \mathbb{X}_{22}\mathbb{U}_2 = (\mathbb{X}_{21}\mathbb{U} + \mathbb{X}_{22})\mathbb{U}_2$, hence (11.91) holds.

Now $\mathbb{D}_1\mathbb{U} + \mathbb{D}_2 = \mathbb{D}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right] = \mathbb{D}\left[\begin{smallmatrix} \mathbb{U}_1 \\ \mathbb{U}_2 \end{smallmatrix}\right]\mathbb{U}_2^{-1} = \mathbb{D}\mathbb{M}\left[\begin{smallmatrix} \widetilde{\mathbb{U}} \\ I \end{smallmatrix}\right]\mathbb{U}_2^{-1}$, hence

$$(\mathbb{D}_1\mathbb{U} + \mathbb{D}_2)^* J (\mathbb{D}_1\mathbb{U} + \mathbb{D}_2) = \mathbb{U}_2^{-*} \begin{bmatrix} \widetilde{\mathbb{U}} \\ I \end{bmatrix}^* J_1 \begin{bmatrix} \widetilde{\mathbb{U}} \\ I \end{bmatrix} \mathbb{U}_2^{-1} = \mathbb{U}_2^{-*}[\widetilde{\mathbb{U}}^*\widetilde{\mathbb{U}} - I]\mathbb{U}_2^{-1}$$

is $\ll 0$ [≤ 0] iff $\|\widetilde{\mathbb{U}}\| < 1$ [≤ 1] (see Lemma A.3.1(b)&(d)).

2° *Rest of the formulae:* Assume (11.87).

(3.) The formula $\mathbb{U} = \mathcal{F}_\ell(\mathbb{T}, \widetilde{\mathbb{U}})$ can now be verified by a direct computation using the formula $\mathbb{M}\mathbb{X} = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$. We note from Lemma A.3.1(d1) that

$$\mathbb{T} = \begin{bmatrix} \mathbb{X}_{11}^{-1} & -\mathbb{X}_{11}^{-1}\mathbb{X}_{12} \\ \mathbb{X}_{21}\mathbb{X}_{11}^{-1} & \mathbb{X}_{22} - \mathbb{X}_{21}\mathbb{X}_{11}^{-1}\mathbb{X}_{12} \end{bmatrix} = \begin{bmatrix} \mathbb{M}_{11} - \mathbb{M}_{12}\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{M}_{12}\mathbb{M}_{22}^{-1} \\ -\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{M}_{22}^{-1} \end{bmatrix}; \tag{11.94}$$

in particular, $\mathbb{T}_{11}, \mathbb{T}_{22} \in \mathcal{G}\mathrm{TIC}$.

(2.) From $\begin{bmatrix} I & -\widetilde{\mathbb{U}} \end{bmatrix} \mathbb{X}\mathbb{M} \left[\begin{smallmatrix} \widetilde{\mathbb{U}} \\ I \end{smallmatrix}\right] = 0$ we get that $(\mathbb{X}_{11} - \widetilde{\mathbb{U}}\mathbb{X}_{21})^{-1}(\mathbb{X}_{12} - \widetilde{\mathbb{U}}\mathbb{X}_{22}) = -(\mathbb{M}_{11}\widetilde{\mathbb{U}} + \mathbb{M}_{12})(\mathbb{M}_{21}\widetilde{\mathbb{U}} + \mathbb{M}_{22})^{-1}$ (clearly $\mathbb{X}_{11} - \widetilde{\mathbb{U}}\mathbb{X}_{21} = (I - \widetilde{\mathbb{U}}\mathbb{X}_{21}\mathbb{X}_{11}^{-1})\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$), which is equal to $-\mathbb{U}$. Formulation (11.89) follows from this and (11.87), and from (11.89) we get (2.). But (2.) implies that

$$\mathbb{Q}_2 \begin{bmatrix} I & -\mathbb{U} \end{bmatrix} \mathbb{M} = \begin{bmatrix} \mathbb{Q}_2 & -\mathbb{Q}_1 \end{bmatrix} \mathbb{M} = \begin{bmatrix} I & -\mathbb{U} \end{bmatrix}, \tag{11.95}$$

hence $\mathbb{Q}_2(\mathbb{M}_{11} - \mathbb{U}\mathbb{M}_{21}) = I$, so that (11.92) holds. The formula (11.90) can be obtained in a similar way.

By applying $\mathbb{M} = \left[\begin{smallmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} \\ 0 & I \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} I & 0 \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{smallmatrix}\right]$ (from (A.9)) to $\mathbb{U} = \mathbb{U}_1\mathbb{U}_2^{-1}$ one obtains (11.93).

The claim "$\mathbb{U} \in \widetilde{\mathcal{A}} \Leftrightarrow \widetilde{\mathbb{U}} \in \widetilde{\mathcal{A}}$" follows from (11.89), (11.90), and Lemma 8.4.10.

3° *Necessity assuming that* $\mathbb{X}_{21}\mathbb{U} + \mathbb{X}_{22} \in \mathcal{G}\mathrm{TIC}$*:* Let $(\mathbb{D}_1\mathbb{U} + \mathbb{D}_2)^* J (\mathbb{D}_1\mathbb{U} + \mathbb{D}_2) \ll 0$ [≤ 0]. Define $\left[\begin{smallmatrix} \widetilde{\mathbb{U}}_1 \\ \widetilde{\mathbb{U}}_2 \end{smallmatrix}\right] := \mathbb{X}\left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right]$, so that $\widetilde{\mathbb{U}}_2 = \mathbb{X}_{21}\mathbb{U} + \mathbb{X}_{22}$. Assume that $\widetilde{\mathbb{U}}_2 \in \mathcal{G}\mathrm{TIC}$.

Then

$$\widetilde{\mathbb{U}}_1^*\widetilde{\mathbb{U}}_1 - \widetilde{\mathbb{U}}_2^*\widetilde{\mathbb{U}}_2 = \begin{bmatrix} \widetilde{\mathbb{U}}_1 \\ \widetilde{\mathbb{U}}_2 \end{bmatrix}^* J_1 \begin{bmatrix} \widetilde{\mathbb{U}}_1 \\ \widetilde{\mathbb{U}}_2 \end{bmatrix} = \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}^* \mathbb{X}^* J_1 \mathbb{X} \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} \tag{11.96}$$

$$= \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix}^* \mathbb{D}^* J \mathbb{D} \begin{bmatrix} \mathbb{U} \\ I \end{bmatrix} = (\mathbb{D}_1\mathbb{U} + \mathbb{D}_2)^* J (\mathbb{D}_1\mathbb{U} + \mathbb{D}_2) \ll 0 \;\; [\le 0], \tag{11.97}$$

hence the norm of $\widetilde{\mathbb{U}} := \widetilde{\mathbb{U}}_1\widetilde{\mathbb{U}}_2^{-1}$ (a r.c.f.) is < 1 [≤ 1].

Moreover, $\left[\begin{smallmatrix} \mathbb{U}_1 \\ \mathbb{U}_2 \end{smallmatrix}\right] := \mathbb{M}\left[\begin{smallmatrix} \widetilde{\mathbb{U}} \\ I \end{smallmatrix}\right] = \mathbb{M}\left[\begin{smallmatrix} \widetilde{\mathbb{U}}_1 \\ \widetilde{\mathbb{U}}_2 \end{smallmatrix}\right]\widetilde{\mathbb{U}}_2^{-1} = \left[\begin{smallmatrix} \mathbb{U} \\ I \end{smallmatrix}\right]\widetilde{\mathbb{U}}_2^{-1}$, hence $\mathbb{U}_1\mathbb{U}_2^{-1} = \mathbb{U}$, so $\mathbb{U}$ is of the form claimed above.

4° *Necessity completed:* Let Υ be the set of all solutions $\mathbb{U} \in \mathrm{TIC}(W,U)$ for which $(w \mapsto \mathcal{J}(0,\mathbb{U}w,w)) \le 0$, and Υ_0 those of $\mathbb{U} \in \Upsilon$ for which $\mathbb{X}_{21}\mathbb{U} + \mathbb{X}_{22} \in \mathcal{G}\mathrm{TIC}$ (as in 3°). We shall show that $\Upsilon_0 = \Upsilon$; this together with 3° implies the necessity of (11.87).

Let $\mathbb{U} \in \Upsilon$ be arbitrary. Set $\mathbb{U}_0 := \mathbb{M}_{12}\mathbb{M}_{22}^{-1} \in \Upsilon_0$, $s := \inf[0,1] \setminus E_0$, where $E_0 := \{t \in [0,1] \,|\, \mathbb{U}_t := \mathbb{U}_0 + t(\mathbb{U} - \mathbb{U}_0) \in \Upsilon_0\}$. Assuming $s < \infty$, we will derive a contradiction, thus showing that $E_0 = [0,1]$, hence $(\mathbb{U} =)\mathbb{U}_1 \in \Upsilon_0$.

If $\mathcal{J}(0,u,w) \le 0$ and $\mathcal{J}(0,\widetilde{u},w) \le 0$, and $f(r) := \mathcal{J}(0,u + r(\widetilde{u} - u),w)$, then $f''(r) = 2\langle \widetilde{u} - u, (\mathbb{D}_1^* J \mathbb{D}_1)\widetilde{u} - u\rangle \ge 0$ for all $r \in [0,1]$, hence f has no maximum on $(0,1)$, so that $f \le 0$. This convexity leads us to conclude that $\mathbb{U}_t \in \Upsilon$ for all $t \in [0,1]$.

For each $t \in E_0$, define $\widetilde{\mathbb{U}}_t$ as in 3°, so that $\|\widetilde{\mathbb{U}}_t\| \le 1$. We have $(\mathbb{X}_{21}\mathbb{U}_t + \mathbb{X}_{22})^{-1} = \mathbb{M}_{21}\widetilde{\mathbb{U}}_t + \mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}(W)$ for all $t \in E_0$, by 1°, hence for all $t < s$. Since $\|\mathbb{M}_{21}\widetilde{\mathbb{U}}_t + \mathbb{M}_{22}\| \le \|\mathbb{M}_{21}\| + \|\mathbb{M}_{22}\|$ for all $t \in E_0$, we have $\mathbb{X}_{21}\mathbb{U}_t + \mathbb{X}_{22} \in \mathcal{G}\mathrm{TIC}(W)$ for $t = s$ too, by Lemma A.3.3(A3). On the other hand, $\mathcal{G}\mathrm{TIC}$ is open, hence s cannot be the infimum of E_0, QED. □

Minimax J-coercivity does not guarantee that the J-critical (i.e., minimax) control would be given by a state feedback controller:

Example 11.3.7 (Minimax J-coercive $\not\Rightarrow \exists \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, $\exists \mathbb{X}^* S \mathbb{X}$)

(a) **($\nexists \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, $\nexists$ CARE)** Let $\gamma > 0$. There is a strongly stable system $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(\mathbf{C}^2 \times \mathbf{C}^2, \mathrm{L}^2(\mathbf{R}_+;\mathbf{C}^6), \mathbf{C}^4 \times \mathbf{C}^2)$ s.t. $\mathbb{D}_{11}^*\mathbb{D}_{11} \gg 0$, $\mathbb{D} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, $\gamma > \gamma_0$ (i.e., $\mathbb{D}$ is minimax J_γ-coercive, hence J_γ-coercive) but the unique J_γ-critical ("minimax") control for Σ over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$ is not of (well-posed) state feedback form; it is ill-posed in both open-loop and closed-loop forms.

Condition (FI1s) of Theorem 11.3.3 holds but (FI3s)–(FI5s) do not; in fact, $\mathbb{D}^* J_\gamma \mathbb{D}$ does not have a spectral factorization. Analogously, condition (FI1) of Theorem 11.2.7 holds but (FI3)–(FI5) do not (this corresponds to case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$).

(b) **(Unstable $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$)** Let $\gamma > 0$. There is a strongly stable UHPR system $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(\mathbf{C}^2 \times \mathbf{C}^2, \mathrm{L}^2(\mathbf{R}_+;\mathbf{C}^6), \mathbf{C}^4 \times \mathbf{C}^2)$ s.t. $\mathbb{D}_{11}^*\mathbb{D}_{11} \gg 0$, $\mathbb{D} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, $\gamma > \gamma_0$ (i.e., $\mathbb{D}$ is minimax J_γ-coercive, hence J_γ-coercive) but the unique J_γ-critical control for Σ over $\mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$ corresponds to an unstable state feedback pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, whose closed-loop form $\left[\begin{array}{c|c} \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right]$ is also unstable, since $\widehat{\mathbb{X}}, \widehat{\mathbb{X}}^{-1}$ are unbounded at $\pm i$.

As in (a), conditions (FI1) and (FI1s) hold but (FI3)–(FI5) and (FI3s)–(FI5s) do not, since $\mathbb{D}^* J_\gamma \mathbb{D}$ does not have a spectral factorization. However, in this case the CARE and the IARE have a (unique and UHPR) $\mathcal{U}_{\mathrm{out}}$-stabilizing solution $\mathcal{P} \ge 0$ (which is neither stable nor SOS-stabilizing).

Nevertheless, $\widehat{\mathbb{D}}$ and $\widehat{\mathbb{X}}^{\pm 1}$ are holomorhic at infinity, hence uniformly half-plane-regular.

(c) **($\exists \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$, $\nexists$ CARE, although $\mathbb{D} \in$ ULR)** Let $\gamma > 0$. There is a strongly stable ULR system $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(\mathbf{C} \times \mathbf{C}, \mathrm{L}^2(\mathbf{R}_+; \mathbf{C}^2), \mathbf{C} \times \mathbf{C})$ s.t. $\mathbb{D}_{11}^* \mathbb{D}_{11} \gg 0$, $\mathbb{D} = \begin{bmatrix} \mathbb{D}_{11} & 0 \\ 0 & I \end{bmatrix}$, $\gamma > \gamma_{\mathrm{FI}}$ (hence, $\mathbb{D}$ is minimax J_γ-coercive, hence J_γ-coercive).

Moreover, conditions (FI1s)–(FI4s) of Theorem 11.3.3 hold but (FI5s) does not, since the spectral factor of $\mathbb{D}^* J_\gamma \mathbb{D}$ is not WR; in particular, the CARE does not have a stabilizing solution. Analogously, conditions (FI1)–(FI4) of Theorem 11.2.7 hold but (FI5) does not. (This corresponds to case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{str}}$.)

$\triangleleft$

(Note that Σ satisfies (Standing) Hypotheses 11.1.1, 11.2.1 and 11.3.1 (for $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$).)

The anomalies in (a) and (c) cannot happen in discrete time, since in discrete time a unique J_γ-critical control is always of state feedback form and corresponds to a DARE, by Theorem 14.1.6 (recall that discrete-time maps are always "regular" due to bounded input and output operators). However, the anomaly in (b) happens in discrete time too, mutatis mutandis (with $\widehat{\mathbb{X}}, \widehat{\mathbb{X}}^{-1} \in \mathrm{H}^2(\mathbf{D}; \mathcal{B}) \setminus \mathrm{H}^\infty(\mathbf{D}; \mathcal{B})$) unless, e.g., $\widehat{\mathbb{D}}$ is exponentially stable.

In the above examples, we have $\mathcal{U}_*^* \in \{\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}\}$. An expert on the area considers it almost sure that the I/O map of Example 8.4.13 can be modified so that it is exponentially stable (so that we can let Σ be an exponentially stable realization of $\mathbb{D}$); if this is the case, then the anomalities of (a) and (c) (but not that in (b)) also exist in case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$. As mentioned above with $\mathcal{U}_{\mathrm{exp}}$ in discrete time (but the Cayley transform of $\mathrm{TIC}_{\mathrm{exp}}$ covers much more than $\mathrm{tic}_{\mathrm{exp}}$).

Proof of Example 11.3.7: (a) (We assume here that $\gamma = \sqrt{2}$ as in Example 8.4.13. For general $\gamma > 0$, one has to replace $\mathbb{D}$ by $\begin{bmatrix} (\gamma/\sqrt{2})I & 0 \\ 0 & I \end{bmatrix} \mathbb{D}$ and $\mathbb{X}$ by $(\gamma/\sqrt{2})\mathbb{X}$.)

Let $\heartsuit\mathbb{D} := \mathbb{D}_0$ and $J_\gamma := \widetilde{J} = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$, where $\mathbb{D}_0$ and $\widetilde{J}$ are the maps from of Example 8.4.13(b), so that $\mathbb{D} \in \mathrm{TIC}(\mathbf{C}^2 \times \mathbf{C}^2, \mathbf{C}^4 \times \mathbf{C}^2)$ is minimax J_γ-coercive (equivalently, $\gamma > \gamma_0$, by Lemma 11.3.10), and $\widehat{\heartsuit}\widehat{\mathbb{D}}$ has a $\mathcal{G}\mathrm{H}^2$-factorization (see Definition 9.15.1) $(\widehat{\heartsuit}\widehat{\mathbb{X}})^* J_1 (\widehat{\heartsuit}\widehat{\mathbb{X}})$ s.t. $\widehat{\mathbb{X}}^{\pm 1} \in \mathrm{H}(\mathbf{C}^+; \mathcal{B}(U)) \setminus \mathrm{H}^\infty(\mathbf{C}_\omega^+; \mathcal{B}(U))$ for all $\omega \in \mathbf{R}$, as noted in Example 8.4.13.

Let Σ be the strongly stable (shift) realization (13.46) of $\mathbb{D}$. Due to minimax J_γ-coercivity, there is a unique J_γ-critical (i.e., minimax) input for each $x_0 \in H$ over $\mathcal{U}_{\mathrm{out}}$. By Lemma 8.3.3, $\mathcal{U}_{\mathrm{str}} = \mathcal{U}_{\mathrm{sta}} = \mathcal{U}_{\mathrm{out}}$.

If $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is any J_γ-critical state feedback pair for Σ, then $I - \mathbb{F} = E\mathbb{X}$ for some $E \in \mathcal{GB}(U))$, by Lemma 9.15.4 (and Lemma 9.15.2); but then $\mathbb{F}, \mathbb{F}_\circlearrowleft \notin \mathrm{TIC}_\infty$, i.e., the corresponding "controller" is non-well-posed in both its open-loop and closed-loop forms!

By Corollary 9.9.11, $\mathbb{D}^* J_\gamma \mathbb{D}$ does not have a spectral factorization (although it is J_γ-coercive), since otherwise the J_γ-critical control could be given state feedback form. We do not know whether (FI2s) (equivalently, (FI2)) holds.

However, by Theorem 8.3.9 (see also Section 9.7; Remark 9.7.7 in particular), this J_γ-critical control can be written in WPLS form, i.e., as non-well-posed state feedback (where there are no well-posed maps between an external (closed-loop) input "$u_{\circlearrowleft}$" and the internal (open-loop) control "u").

(By Theorem 11.4.11(i) of [Sbook], one can apply Cayley transform to Σ to obtain a strongly stable wpls "$\heartsuit\Sigma$" (whose semigroup is a contraction). The above $\mathcal{G}\mathrm{H}^2$-factorization defines the unique (modulo E) J_γ-critical (well-posed and admissible) state feedback pair "$\heartsuit \begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix}$" for $\heartsuit\Sigma$, but, as noted above, its continuous-time equivalent is not well-posed.)

(b) The proof of (a) applies mutatis mutandis, just use (c) instead of (b) of Example 8.4.13.

(c) 1° *The proof:* (We shall assume that $\gamma = 1$. For general $\gamma > 0$, one has to replace $\mathbb{D}$ by $\left[\begin{smallmatrix} \gamma I & 0 \\ 0 & I \end{smallmatrix}\right] \mathbb{D}$ and $\mathbb{X}$ by $\gamma\mathbb{X}$.)

Let $\mathbb{D}_{11}$ (resp. $\mathbb{X}_{11}$) be the element $\mathbb{D}$ (resp. $\mathbb{X}_{11}$) of Proposition 9.13.1(c1). Then $\mathbb{X} := \left[\begin{smallmatrix} \mathbb{X}_{11} & 0 \\ 0 & I \end{smallmatrix}\right]$ satisfies $\mathbb{X}^* J_1 \mathbb{X} = \mathbb{D}^* J_\gamma \mathbb{D}$ and $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ (in particular, (FI3s) holds), but $\mathbb{D} \in \mathrm{ULR}$ (since $\mathbb{D}_{11} \in \mathrm{ULR}$) and $\mathbb{X}^{\pm 1} \notin \mathrm{WR}$.

Let Σ be the shift realization (6.11). Then (c) is satisfied, by Proposition 11.3.4 (indeed, by Theorem 9.9.1(c)&(e2)&(f1), the eIARE has a unique output-stabilizing solution and this solution is not WR (because $\mathbb{F} = I - \mathbb{X}$ is not WR), hence it is not a solution of the [e]CARE).

2° *Remark:* If the reader is not happy with the fact that $\mathbb{D}_{12} \neq 0$ (which means that $\begin{bmatrix} \mathbb{K} & | & \mathbb{F} \end{bmatrix} = \begin{bmatrix} 0 & | & 0 \end{bmatrix}$ is a suboptimal H^∞-FI-pair, so that the FICP is trivial), (s)he may take first some rational (or $\mathrm{MTIC}^{\mathrm{L}^1}$) $\widehat{\mathbb{D}}_0$ s.t. $\mathbb{D}_0$ is minimax J_γ-coercive, so that $\mathbb{X}_0^* J_1 \mathbb{X}_0 = \mathbb{D}_0^* J_\gamma \mathbb{D}_0$ for some rational (or $\mathrm{MTIC}^{\mathrm{L}^1}$) $\widehat{\mathbb{X}}_0$, and then use $\begin{bmatrix} \mathbb{D}_{11} & 0 \\ 0 & \mathbb{D}_0 \end{bmatrix}$ in place of $\mathbb{D}$, so that $\mathbb{X}$ is replaced by $\begin{bmatrix} \mathbb{X}_{11} & 0 \\ 0 & \mathbb{X}_0 \end{bmatrix}$ (when J_γ is replaced by $\begin{bmatrix} I & 0 \\ 0 & J_\gamma \end{bmatrix}$); here $\mathbb{D}_{11}$ refers to the original $\mathbb{D}_{11}$, so that the dimension of Z is increased by one.

This way the FICP becomes nontrivial but (c) is unchanged except for the dimensions and the fact that $\mathbb{D}_{12} \neq 0$. □

The main part of this section ends here; the rest consists on auxiliary lemmas that were used above; some of the lemmas also have further use in Chapter 12.

From a solution of the CARE we obtain a suboptimal pair or operator as follows:

Lemma 11.3.8 (CARE ⇒ H^∞-FI-pair) *Assume that the CARE has a UR stable, P-SOS-stabilizing solution $(\mathcal{P}, S, K)$ s.t. $\mathcal{P} \geq 0$, $S_{11} \gg 0$ and $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$ (resp. and $S_{22} \ll 0$).*

Then (11.39) generate a UR suboptimal H^∞*-FI-pair (resp. K_1 =(11.40) is a UR suboptimal* H^∞*-SF-operator), which is stable and r.c.-SOS-stabilizing.*

(By Proposition 9.8.11(iii)&(ii), a stable, P-SOS-stabilizing solution of the CARE is $\mathcal{U}_*^*$-stabilizing, hence $\mathcal{P}$, S and K are unique.)

Proof: By Proposition 9.8.10, the assumptions of Lemma 11.3.9(a) (resp. and (b)) are satisfied for $s_0 = +\infty$. □

Lemma 11.3.8 can be extended to cover the IARE (thus we need not assume any regularity):

Lemma 11.3.9 (IARE $\Rightarrow$ H^∞-FI-pair) *Assume that the IARE has a stable, P-SOS-stabilizing solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ s.t. $\mathcal{P} \geq 0$. Then*

(a) If $S_{22} \ll 0$ and any of (1.)–(4.) of Lemma 11.2.14 holds (for $\alpha = 0$), then $\left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ is a stable and r.c.-SOS-stabilizing suboptimal H^∞-FI-pair, and $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$.

(b) If $\widetilde{S} := (\widehat{\mathbb{X}}^* S \widehat{\mathbb{X}})(s_0)$ safisfies $\widetilde{S}_{11} \gg 0$ and $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$ for some (equivalently, all) $s_0 \in \mathbf{C}^+$ ($s_0 = +\infty$ can be allowed for $\mathbb{X} \in \mathrm{UR}$), then (11.48) is a stable and r.c.-SOS-stabilizing suboptimal H^∞-FI-pair.

Here, as elsewhere, $\mathbb{X} := I - \mathbb{F}$.

Proof: (By Theorem 9.8.12(s4) and Proposition 9.8.11(iii)&(ii), $\widetilde{S}$ (and $\mathcal{P}$) is independent on the choice of a stable, P-SOS-stabilizing solution, and such a solution is $\mathcal{U}_*^*$-stabilizing and r.c.-SOS-stabilizing.)

(a) By Lemma 11.3.15, $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ is a stable and r.c.-SOS-stabilizing H^∞-FI-pair. By Proposition 9.8.11(iii)&(ii), $\mathcal{P}$ is $\mathcal{U}_{\mathrm{out}}$-stabilizing. By Lemma 11.2.14(a), $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is suboptimal (recall that $\mathbb{M} = \mathbb{F}_{\circlearrowleft} + I \in \mathrm{TIC}$) and $\mathbb{M}_{22}^{-1} \in \mathrm{TIC}(W)$; since $\vartheta = 0$ and $\mathbb{F} \in \mathrm{TIC}$, we can take $\alpha = 0$. By Lemma A.1.1(c1), it follows that $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$.

(b) 1° *"Some $s_0 \in \mathbf{C}^+$" suffices:* This follows from Lemma 11.2.14(b)&(a) (see the proof of (a) above).

2° *"Equivalently, all $s_0 \in \mathbf{C}^+$":* This follows from (11.59), since $\gamma > \gamma_{\mathrm{FI}} \geq \gamma_0$, and $\mathcal{P}$ is $\mathcal{U}_*^*$-stabilizing (and $\vartheta = 0$). □

Lemma 11.3.10 *The following are equivalent:*

(i) $\gamma > \gamma_0$.

(ii) $\mathbb{D}$ is minimax J_γ-coercive.

(iii) $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;W),\mathrm{L}^2(\mathbf{R}_+;Z))} < \gamma$ for some $\mathbb{U} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;W), \mathrm{L}^2(\mathbf{R}_+;U))$.

See Definition 11.4.1 for minimax J_γ-coercivity. The above result allows us to use this property in the proof of implication (FI1s)$\Rightarrow$(FI3s) (see Lemma 11.4.3(a)).

Proof: 1° *(i)$\Leftrightarrow$(ii):* Set $\mathbb{T} := \pi_+\mathbb{D}_1^* J_\gamma \mathbb{D}_1 \pi_+ \gg 0$ (on $\mathrm{L}^2(\mathbf{R}_+;U)$; see Standing Hypothesis 11.3.1). By Fréchet differentiation (or by completing the square or by applying (8.19) suitably), we see that $u_{\min} := -\mathbb{T}^{-1}\pi_+\mathbb{D}_1^* J_\gamma \mathbb{D}_2 \pi_+ w =: \mathbb{U}w$ minimizes $\mathcal{J}(0, \left[\begin{smallmatrix} \cdot \\ w \end{smallmatrix}\right])$, for any $w \in \mathrm{L}^2(\mathbf{R}_+;W)$ (we have added redundant π_+'s above and below to make the computations easier). Combine this with Lemma 11.2.4 to observe that $\gamma > \gamma_0$ iff

$$\min_{u\in\mathcal{U}_{\mathrm{u}}(0,w)} \mathcal{J}(0,u,w) = \langle \mathbb{D}\pi_+ \left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right] w, J_\gamma \mathbb{D}\pi_+ \left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right] w\rangle \le -\varepsilon\|w\|_2^2 \tag{11.98}$$

for some $\varepsilon > 0$, i.e., iff

$$0 \gg \begin{bmatrix}\mathbb{U}\\ I\end{bmatrix}^* \pi_+\mathbb{D}^* J_\gamma \mathbb{D}\pi_+ \begin{bmatrix}\mathbb{U}\\ I\end{bmatrix} = \cdots = \pi_+\mathbb{D}_2^* J_\gamma (I-P_1)\mathbb{D}_2\pi_+ \tag{11.99}$$

(the equality follows by a straightforward computation (as in Lemma 2.6 of [S98d])), where $P_1 := \pi_+\mathbb{D}_1\mathbb{T}^{-1}\mathbb{D}_1^* J_\gamma \pi_+ = P_1^2$. But (11.99) holds iff $\mathbb{D}$ is minimax J_γ-coercive, by (11.106) and Definition 11.4.1.

2° *(i)⇔(iii):* For a given $\mathbb{U}\in\mathcal{B}$, condition (iii) holds iff $(\mathbb{D}_{11}\mathbb{U}+\mathbb{D}_{12})^*(\mathbb{D}_{11}\mathbb{U}+\mathbb{D}_{12}) \ll 0$ (by Lemma A.3.1(d)), i.e., iff (11.98) holds for this $\mathbb{U}$. This implies that (11.98) holds for the minimizing $\mathbb{U}$; take the minimizing $\mathbb{U}$ to obtain the converse. □

Lemma 11.3.11 ((FI3s) ⇒ $X_{21}=0$) *Assume that (FI3s) is satisfied by some SR $\mathbb{X}$ with $X\in\mathcal{GB}(U\times W)$. Then we can choose $\mathbb{X}$ s.t. $X_{11}, X_{22}\in\mathcal{GB}$ and $X_{21}=0$.*

Proof: By Proposition 11.3.4(f), (FI4s) has a solution $(\mathcal{P}, \widetilde{S}, \left[\begin{array}{c|c}\widetilde{\mathbb{K}} & \widetilde{\mathbb{F}}\end{array}\right])$ and (FI2s) holds. By Theorem 9.8.12(s1), we can have $\widetilde{F}=0$. But then $\widetilde{S}_{11}\gg 0$ and $\widetilde{S}_{22}-\widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12}\ll 0$, by Proposition 11.2.19(d1) (it was remarked in the proof how we may use the IARE instead of the CARE).

By Lemma 11.3.13(i)&(iii'), we can make redefine $\left[\begin{array}{c|c}\widetilde{\mathbb{K}} & \widetilde{\mathbb{F}}\end{array}\right]$ s.t. $X_{11}, X_{22}\in\mathcal{GB}$, $X_{21}=0$ and $S=J_1$, where $X := I-\widetilde{F}$. Set $\mathbb{X} := I-\widetilde{\mathbb{F}}\in\mathcal{G}\mathrm{TIC}(U\times W)$. By the proof of Proposition 11.3.4(f), we have $\mathbb{X}^* J_1\mathbb{X} = \mathbb{D}^* J\mathbb{D}$. By Proposition 11.3.4(c1), $\mathbb{X}_{11}\in\mathcal{G}\mathrm{TIC}(U)$, hence also this new $\mathbb{X}$ satisfies (FI3s). □

Lemma 11.3.12 (SpF ⇒ $\mathbb{D}_{11}$ *I*-coercive) *(Drop Standing hypothesis 11.2.1 for the moment). If $\mathbb{D}^* J_\gamma\mathbb{D}$ has a spectral factorization, then $\mathbb{D}_{11}$ is I-coercive (as required in Hypotheses 11.2.1 and 11.2.1).*

Proof: Let $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X}^* S\mathbb{X}$, $\mathbb{X}\in\mathcal{G}\mathrm{TIC}(U\times W)$, $S\in\mathcal{GB}(U\times W)$. Set $\varepsilon := \|\mathbb{X}^{-1}S^{-1}\mathbb{X}^{-*}\mathbb{D}^* J_\gamma\|^{-1} > 0$. Then

$$\|v\|_2 = \|\mathbb{X}^{-1}S^{-1}\mathbb{X}^{-*}\mathbb{D}^* J_\gamma\mathbb{D}v\|_2 \le \varepsilon_+^{-1}\|\mathbb{D}v\|_2, \tag{11.100}$$

i.e., $\|\mathbb{D}v\|_2\ge\varepsilon_+\|v\|_2$, for all $v\in\mathrm{L}^2(\mathbf{R}_+;\mathcal{B}(U\times W))$. Consequently,, $\|\mathbb{D}_{11}u\|_2 = \|\mathbb{D}\left[\begin{smallmatrix}u\\ 0\end{smallmatrix}\right]\|_2 \ge \varepsilon_+\|u\|_2$ for all $u\in\mathrm{L}^2(\mathbf{R}_+;\mathcal{B}(U))$. □

We have already used the following lemma several times:

Lemma 11.3.13 ($\mathbf{S_{11}\gg 0}$ and $\mathbf{S_{22}-S_{21}S_{11}^{-1}S_{12}\ll 0}$) *Let $S=S^*\in\mathcal{B}(U\times W)$ and $\gamma>0$. Then the following are equivalent:*

(i) $S_{11} \gg 0$ *and* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$.

(i') $S_{11} \gg 0$ *and* $\left[\begin{smallmatrix} F \\ I \end{smallmatrix}\right]^* S \left[\begin{smallmatrix} F \\ I \end{smallmatrix}\right] \ll 0$ *for some* $F \in \mathcal{B}(W,U)$.

(ii) $\widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$ *for some (hence all)* $F \in \mathcal{B}(W,U)$, *where* $\widetilde{S} := \left[\begin{smallmatrix} I & F \\ 0 & I \end{smallmatrix}\right]^* S \left[\begin{smallmatrix} I & F \\ 0 & I \end{smallmatrix}\right]$.

(ii') $S_{11} = \widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} \ll 0$ *for some* $F \in \mathcal{B}(W,U)$, *where* $\widetilde{S} := \left[\begin{smallmatrix} I & F \\ 0 & I \end{smallmatrix}\right]^* S \left[\begin{smallmatrix} I & F \\ 0 & I \end{smallmatrix}\right]$.

(ii") $\widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$ *for some (hence all)* $Z := \left[\begin{smallmatrix} Z_{11} & Z_{12} \\ 0 & Z_{22} \end{smallmatrix}\right] \in \mathcal{GB}(U \times W)$ *s.t.* $Z_{11} \in \mathcal{GB}(U)$, *where* $\widetilde{S} := Z^*SZ$.

(iii) $S = X^*J_\gamma X$ *for some* $X = \left[\begin{smallmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{smallmatrix}\right] \in \mathcal{GB}(U \times W)$ *s.t.* $X_{11} \in \mathcal{GB}$.

(iii') $S = \left[\begin{smallmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{smallmatrix}\right]^* J_\gamma \left[\begin{smallmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{smallmatrix}\right]$, *where* $X_{11} \gg 0$, $X_{22} \gg 0$.

(iv) $S = X^*J_\gamma X$ *for some* $X \in \mathcal{GB}(U \times W)$ *s.t.* $X_{11} \in \mathcal{GB}(U)$ *and* $\begin{bmatrix} H & I \end{bmatrix} X \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] = 0$ *for some* $H \in \mathcal{B}(U,W)$ *s.t.* $\|H\| < \gamma^{-1}$.

(v) $S_{11} \gg 0$ *and* $S = X^*J_\gamma X$ *for some* $X \in \mathcal{GB}(U \times W)$ *s.t.* $X_{11} \in \mathcal{GB}(U)$.

(vi) $S = X^*J_\gamma X$, *where* $X, X_{11} \in \mathcal{GB}$ *and* $\|X_{21}X_{11}^{-1}\| < \gamma^{-1}$.

(vi') $M^*SM = J_\gamma$, *where* $M, M_{22} \in \mathcal{GB}$ *and* $\|M_{22}^{-1}M_{21}\| < \gamma^{-1}$.

(vii) $\widetilde{S}_{11} \gg 0$ *and* $\widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \ll 0$, *where* $\widetilde{S} := \operatorname{diag}(I_{U'}, S, -I_{W'}) \in \mathcal{B}((U' \times U) \times (W \times W'))$, *for some (hence all) Hilbert spaces* W' *and* U'.

(viii) $S_{11} \gg 0$, $S \in \mathcal{GB}(U \times W)$ *and* $(S^{-1})_{22} \ll 0$.

Moreover,

(a) If $\dim U < \infty$, *then one more equivalent condition is that* $S_{11} \gg 0$ *and* $S = X^*J_\gamma X$ *for some* $X \in \mathcal{GB}$.

(b1) If (i) holds and $S = X^*J_\gamma X$ *for some* $X \in \mathcal{GB}(U \times W)$, *then* X *is as in (vi).*

(b2) If (i) holds, $S = X^*TX$, $X \in \mathcal{GB}(U \times W)$, $T_{11} \gg 0$ *and* $T_{22} \ll 0$, *then* $X_{11} \in \mathcal{GB}(U)$.

(b3) If (i) holds and $S = X^*J_\gamma X$ *for some* $X \in \mathcal{GB}(U \times W)$ *s.t.* $X_{21} = 0$, *then* $X_{11}, X_{22} \in \mathcal{GB}$.

(c) If (i) holds, then any $F \in \mathcal{B}(W,U)$ *is as in (ii).*

(d) If $S_{11} \geq \varepsilon^2 I$, $\varepsilon > 0$, $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, $S = X^*J_\gamma X$ *and* $X \in \mathcal{GB}(U \times W)$, *then* $X_{11} \in \mathcal{GB}(U)$ *and* $\|X_{11}^{-1}\| \leq \varepsilon^{-1}$.

(e) If $S_{11} \gg 0$, $S_{22} \ll 0$, $X \in \mathcal{GB}(U \times W)$ *and* $S = X^*J_\gamma X$, *then* $X_{11}, X_{22} \in \mathcal{GB}$, $\|X_{21}X_{11}^{-1}\| < \gamma^{-1}$ *and* $\|M_{11}^{-1}M_{12}\| = \|X_{12}X_{22}^{-1}\| < \gamma$, *where* $M := X^{-1}$.

(f) If X *and* S *are as in (iii), then* $X_{22}^*X_{22} = \gamma^{-2}(S_{12}^*S_{11}^{-1}S_{12} - S_{22})$.

We note that "$S = X^*J_\gamma X$, $X, X_{11} \in \mathcal{GB}$" is not sufficient for (i)–(vi): take $X := \left[\begin{smallmatrix} \gamma I & I \\ I & 0 \end{smallmatrix}\right]$, so that $X, X_{11} \in \mathcal{GB}(U \times W)$ but $S_{11} := (X^*J_\gamma X)_{11} = 0 \not\gg 0$. By Example 11.2.16 (take $S = X^*J_1X$), condition "$\dim U < \infty$" is not superfluous in (a) and condition "$X_{11} \in \mathcal{GB}$" is not superfluous in (iii), nor in (v).

Proof: 1° *(iii')⇒(iii)⇒(i):* Obviously, (iii')⇒(iii). If (iii) holds, then $S_{11} = \widetilde{X}_{11}^*\widetilde{X}_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} = -\gamma^2\widetilde{X}_{22}^*\widetilde{X}_{22} \ll 0$.

2° *(i)⇒(ii')&(iii'):* Assume (i). Set

$$X_{11} := S_{11}^{1/2} \gg 0,\ X_{12} := X_{11}^{-1}S_{12},\ X_{22} := \gamma^{-1}(S_{21}S_{11}^{-1}S_{12} - S_{22})^{1/2} \gg 0. \tag{11.101}$$

Then

$$S = \begin{bmatrix} I & S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22}' \end{bmatrix} \begin{bmatrix} I & S_{11}^{-1}S_{12} \\ 0 & I \end{bmatrix} = X^*J_\gamma X, \tag{11.102}$$

where $S_{22}' := S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, $X = \left[\begin{smallmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{smallmatrix}\right] \in \mathcal{GB}(U \times W)$, $X_{11} \gg 0$, $X_{22} \gg 0$. Thus, (ii') and (iii') hold.

3° *(ii')⇔(i'):* Condition (i') is a reformulation of (ii'), because, obviously, $\widetilde{S}_{11} = S_{11}$ and $\widetilde{S}_{22} = \left[\begin{smallmatrix} F \\ I \end{smallmatrix}\right]^* S \left[\begin{smallmatrix} F \\ I \end{smallmatrix}\right]$ in (ii').

4° *(ii')⇒(ii):* Assume (ii'). Since $\widetilde{S}_{21} = \widetilde{S}_{12}^*$ and $\widetilde{S}_{11}^{-1} \gg 0$, we have $-\widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} \leq 0$, hence (ii) holds.

5° *(ii)⇒(iii):* Assume (ii). By 1°, we have $\widetilde{S} = \widetilde{X}^*J_\gamma\widetilde{X}$ for some $\widetilde{X} = \left[\begin{smallmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ 0 & \widetilde{X}_{22} \end{smallmatrix}\right] \in \mathcal{GB}$. Take $X := \widetilde{X}\left[\begin{smallmatrix} I & F \\ 0 & I \end{smallmatrix}\right]^{-1}$ to obtain (iii).

6° *(iii)⇔(iv):* Trivially (iii)⇒(iv) (take $H = 0$). Assume (iv). Set $X' := \left[\begin{smallmatrix} I & 0 \\ 0 & \gamma \end{smallmatrix}\right]X$, $H' := \gamma H$, so that $X'^*J_1X' = S$, $\|H'\| < 1$ and $\left[\begin{smallmatrix} H' & I \end{smallmatrix}\right]X'\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] = \gamma 0 = 0$. Apply Lemma 12.4.11 to obtain $Z' = \left[\begin{smallmatrix} Z_{11}' & Z_{12}' \\ 0 & Z_{22}' \end{smallmatrix}\right] \in \mathcal{GB}$ s.t. $S = Z'^*J_1Z' = Z^*J_\gamma Z$, where $Z := \left[\begin{smallmatrix} I & 0 \\ 0 & \gamma \end{smallmatrix}\right]Z'$.

7° *(iii)⇒(vi)⇒(iv):* Trivially, (iii) implies (vi). If (vi) holds, then $H := -X_{21}X_{11}^{-1}$ satisfies (iv).

8° *(v)⇔(vi):* Since $S_{11} = X_{11}^*(I - \gamma^2X_{11}^{-*}X_{21}^*X_{21}X_{11}^{-1})X_{11}$, (v) is a reformulation of (vi), by Lemma A.3.1(e2).

9° *(vi)⇔(vi'):* Condition (vi') is a reformulation of (vi) (through $M = X^{-1}$, note that $X_{11} \in \mathcal{GB}(U) \Leftrightarrow M_{22} \in \mathcal{GB}(W)$, and that in either case we have $X_{21}X_{11}^{-1} = -M_{22}^{-1}M_{21}$, by Lemma A.3.1(c1)).

10° *(ii): "(hence all)"* Assume (ii) for some F. Let $S' := \left[\begin{smallmatrix} I & F' \\ 0 & I \end{smallmatrix}\right]^* S \left[\begin{smallmatrix} I & F' \\ 0 & I \end{smallmatrix}\right]$. Then $\widetilde{S} = \left[\begin{smallmatrix} I & F-F' \\ 0 & I \end{smallmatrix}\right]^* S' \left[\begin{smallmatrix} I & F-F' \\ 0 & I \end{smallmatrix}\right]$, hence $S_{11}' \gg 0$ and $S_{22}' - S_{21}'(S_{11}')^{-1}S_{12} \ll 0$, by "(i)⇔(ii)".

11° *(ii")⇔(iii) and "(hence all)"* By "(i)⇔(iii)", we have $\widetilde{S} = X^*J_1X$ for some $X = \left[\begin{smallmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{smallmatrix}\right] \in \mathcal{GB}$ s.t. $X_{11} \in \mathcal{GB}$. Consequently, $S = (XZ^{-1})^*J_1XZ^{-1}$, hence also S satisfies the condition in (iii) (since also XZ^{-1} is of the required form, by Lemma A.1.1(b)). The converse is analogous.

12° *(vii)⇔(i):* Obviously, $\widetilde{S}_{11} \gg 0 \Leftrightarrow S_{11} \gg 0$ (we have made the partition so that $\widetilde{S}_{11} \in \mathcal{B}(U' \times U)$). But

$$\widetilde{S}_{22}' := \widetilde{S}_{22} - \widetilde{S}_{21}\widetilde{S}_{11}^{-1}\widetilde{S}_{12} = \operatorname{diag}(S_{22}', -I_{W'}), \tag{11.103}$$

where $S_{22}' = S_{22} - S_{21}S_{11}^{-1}S_{12}$, hence $\widetilde{S}_{22}' \ll 0 \Leftrightarrow S_{22}' \ll 0$.

13° *(i)⇔(viii):* If (i) holds, then $S \in \mathcal{GB}$ and $(S^{-1})_{22}^{-1} = S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, by Lemma A.1.1(d1). If (viii) holds, then $(S_{22} - S_{21}S_{11}^{-1}S_{12})^{-1} \ll 0$, hence then $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, by Lemma A.1.1(i) holds, by Lemma A.3.1(b1).

(a) By (i) and (iii), the condition of (a) is necessary (regardless of U). Conversely, if S and X are as in (a), then $0 \ll S_{11} = X_{11}^*X_{11} - X_{21}^*X_{21}$, hence $X_{11}^*X_{11} \gg 0$, hence $X_{11} \in \mathcal{GB}(U)$ (if $\dim U < \infty$), so that (v) holds.

(b1) By (b2), $X_{11} \in \mathcal{GB}(U)$. Since $0 \ll S_{11} = X_{11}^*X_{11} - \gamma^2 X_{21}^*X_{21}$, we obtain $\|X_{21}X_{11}^{-1}\| < \gamma^{-1}$ as in 8°.

(b2) (Remark: We cannot replace $T_{22} \ll 0$ by $T_{22} - T_{21}T_{11}^{-1}T_{12} = 1 - 2 \ll 0$; set $T = \left[\begin{smallmatrix} 2 & 2 \\ 2 & 1 \end{smallmatrix}\right]$, $X = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ to obtain a counter-example. On the other hand, if $\dim U < \infty$ or $\dim W < \infty$, then $T_{22} \le 0$ would suffice, by Lemma A.1.1(c1) (and the fact that $X_{11}^*X_{11} \gg 0$ as shown below).)

Apply Lemma A.3.1(q) to the (1,1)-block of $S = X^*TX$ to obtain that $X_{11}^*X_{11} \gg 0$.

By Lemma A.1.1(c1), $S, T \in \mathcal{GB}(U \times W)$ and $(S^{-1})_{22} = (S_{22} - S_{21}S_{11}^{-1}S_{12})^{-1}$, hence $(S^{-1})_{22} \ll 0$, by Lemma A.3.1(b1); similarly, $(T^{-1})_{11} = (T_{11} - T_{12}T_{22}^{-1}T_{21})^{-1} \gg 0$. Set $M := X^{-1}$ to obtain that $S^{-1} = MT^{-1}M^*$ and hence

$$0 \ll -(S^{-1})_{22} \le \left[\begin{smallmatrix} 0 & I \end{smallmatrix}\right] MT^{-1}M^* \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right] = \left[\begin{smallmatrix} M_{21} & M_{22} \end{smallmatrix}\right](-T^{-1})\begin{bmatrix} M_{21}^* \\ M_{22}^* \end{bmatrix}. \tag{11.104}$$

By Lemma A.3.1(q) (interchange the rows and also the columns and recall that $-(T^{-1})_{11}) \ll 0$), we have $M_{22}M_{22}^* \gg 0$. By Lemma A.1.1(c2), this implies that also X_{11} is right-invertible, i.e., $X_{11}X_{11}^* \gg 0$, hence X_{11} is invertible.

(b3) By (b2), $X_{11} \in \mathcal{GB}(U)$, hence $X_{22} \in \mathcal{GB}(W)$, by Lemma A.1.1(b2).

(c) Now $\widetilde{S}_{11} = S_{11} \gg 0$ and $\widetilde{S} = E^*X^*J_\gamma XE$, where X is as in (v) and $E = \left[\begin{smallmatrix} I & F \\ 0 & I \end{smallmatrix}\right]$, so that $(XE)_{11} = X_{11} \in \mathcal{GB}(U)$, Therefore, also $\widetilde{S}$ satisfies (v), hence it satisfies (i) too (in the place of S), so that F is as in (ii).

(d) By (v) and (b1), $X_{11} \in \mathcal{GB}(U)$. Because $\varepsilon^2 I \le S_{11} = X_{11}^*X_{11} - \gamma^2 X_{21}^*X_{21}$, we have $\|X_{11}^{-1}\| < \varepsilon$, by Lemma A.3.1(c1)(ii)&(1').

(e) By (ii') and (b), X is as in (vi). Assume, w.l.o.g., that $\gamma = 1$ (cf. 6°). By duality, $X_{22} \in \mathcal{GB}(W)$ and $\|X_{12}X_{22}^{-1}\| < 1$ (apply (b) to $-S_{\mathrm{d}} = X_{\mathrm{d}}^*J_1X_{\mathrm{d}}$, where $S_{\mathrm{d}} := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]S\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$, $X_{\mathrm{d}} := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]X\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ (note that $J_1 = -\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]J_1\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$)). From the $(1,2)$-block of $MX = I$ we obtain that $M_{11}^{-1}M_{12} = -X_{12}X_{22}^{-1}$.

(f) From $X^*J_\gamma X = S$ we obtain that $X_{11}^*X_{11} = S_{11}$,

$$X_{12} = X_{11}^{-*}S_{12}, \text{ and } \gamma^2 X_{22}^*X_{22} = X_{12}^*X_{12} - S_{22}, \tag{11.105}$$

from which we obtain (f). □

We have also used the following:

Lemma 11.3.14 *Let $X, S \in \mathcal{GB}(U \times W)$, $S = S^*$ and $(X^*SX)_{11} \gg 0$. Then $X^*SX = \widetilde{X}^*\widetilde{S}\widetilde{X}$ for some $\widetilde{X}, \widetilde{S} \in \mathcal{GB}(U \times W)$ s.t. $\widetilde{S} = \widetilde{S}^* = \left[\begin{smallmatrix} I & 0 \\ 0 & J_W \end{smallmatrix}\right]$, where $J_W = J_W^* = J_W^{-1} \in \mathcal{GB}(W)$.*

Proof: Set $T := X^*SX$. By (A.5), $T = E_a^*T'E_a$, where $T' := \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22}' \end{bmatrix}$, $T_{22}' := T_{22} - T_{21}T_{11}^{-1}T_{12}$, $E_a := \begin{bmatrix} I & T_{11}^{-1}T_{12} \\ 0 & I \end{bmatrix}$. By Lemma 2.4.4, $T_{22}' = G^*J_WG$, where $J_W = J_W^* = J_W^{-1} \in \mathcal{GB}(W)$ and $G \in \mathcal{GB}(W)$. Thus, $T' = E_b^* \widetilde{S} E_b$, where $E_b := \begin{bmatrix} T_{11}^{-1/2} & 0 \\ 0 & G^{-1} \end{bmatrix}$. Consequently, $X^*SX = \widetilde{X}^*\widetilde{S}\widetilde{X}$, where $\widetilde{X} := E_b E_a X \in \mathcal{GB}(U \times W)$. $\square$

Lemma 11.3.15 **(H^∞-FI-pair)** *Assume that* $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ *is a stable and SOS-stabilizing state-feedback pair for* Σ*, and* $I - \mathbb{F}_{11} \in \mathcal{G}\mathrm{TIC}(U)$*. Then* $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{K}_1 & \mathbb{F}_{11} & \mathbb{F}_{12} \\ 0 & 0 & 0 \end{array}\right]$ *is a stable and r.c.-SOS-stabilizing* H^∞*-FI-pair.*

Proof: From (11.9) we observe that $\Sigma^\frown \in \mathrm{SOS}$, i.e., that $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is SOS-stabilizing; since $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is also stable, it is a r.c.-SOS-stabilizing state feedback pair, by Lemma 6.6.17(b). By Lemma 11.3.2, it follows that $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}} \end{array}\right]$ is a H^∞-FI-pair. $\square$

Notes

The stable H^∞ FICP was solved by Olof Staffans in [S98d], which proves the implications "(FI3s)⇒(FI2s)⇒(FI1$\frac{1}{2}$s)⇒(FI1s)" and "(FI3s)⇒(FI5s)" (the latter in the case of a regular WPLSs and spectral factors), thus establishing the equivalence of (FI1s)–(FI3s) (and the equality $\gamma_0 = \gamma_{\mathrm{FI}}$) for Σ s.t. $(\mathbb{D}, J_\gamma) \in \mathrm{SpF}$ for all $\gamma > 0$. Also Lemma 11.3.10, much of Proposition 11.3.4 and most of Corollary 11.3.5 and Theorem 11.3.6 are from [S98d], and so are most of the corresponding proofs. Cf. the notes on p. 673.

This spectral factorization approach is rather old, see, e.g., [Francis], [Green] and [CG97], which all contain the equivalence of (FI1s)–(FI3s) in some sense for rational transfer functions ([CG97] for $\mathrm{MTIC}^{\mathrm{L}^1}_{\exp}(\mathbf{C}^n, \mathbf{C}^m)$ I/O maps).

It seems that many of the results of this and previous sections also hold for the extended FICP described in Theorem 11.3.6 (part of this is shown in [S98d]), but, e.g., the implication "(FI5s)⇒(FI2s)" would need additional assumptions.

11.4 Minimax J-coercivity

In some ways we are more confused than ever, but we feel that we are confused on a higher level and about more important things.

Here we state some minimax results that are needed for the solution of H^∞ and Nehari problems. Here $\mathbb{D}\in \mathrm{TI}(U\times W,Y)$, $J=J^*\in\mathcal{B}(Y)$, and $J_1 := \left[\begin{smallmatrix} I & 0\\ 0 & -I\end{smallmatrix}\right]\in \mathcal{GB}(U\times W)$ (in practical applications we will have $\mathbb{D}\in\mathrm{TIC}$, but TI formulations allow us to obtain the dual results more neatly). Recall that $\|\cdot\|_{\mathrm{TI}} := \|\cdot\|_{\mathcal{B}(\mathrm{L}^2,\mathrm{L}^2)}$.

Definition 11.4.1 (Minimax J-coercivity, $\mathbb{F}$) *A map* $\mathbb{D}=\begin{bmatrix}\mathbb{D}_1 & \mathbb{D}_2\end{bmatrix}\in\mathrm{TI}$ *is* minimax J-coercive *iff* $\mathbb{F}:=\pi_+\mathbb{D}^*J\mathbb{D}\pi_+\in\mathcal{B}(\pi_+\mathrm{L}^2)$ *satisfies* $\mathbb{F}_{11}\gg 0$ *and* $\mathbb{F}_{22}-\mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12}\ll 0$ *(on* $\pi_+\mathrm{L}^2$*).*

$\mathbb{D}\in\mathrm{TI}$ *is* co-minimax J-coercive *iff* $\mathbb{F}:=\pi_-\mathbb{D}^*J\mathbb{D}\pi_-\in\mathcal{B}(\pi_-\mathrm{L}^2)$ *satisfies (equivalently,* $\widetilde{\mathbb{F}}:=\pi_+\mathbb{D}^{\mathrm{d}}J(\mathbb{D}^{\mathrm{d}})^*\pi_+ = \mathbf{Я}\mathbb{F}\mathbf{Я}$ *satisfies)* $\mathbb{F}_{11}\gg 0$ *and* $\mathbb{F}_{22}-\mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12}\ll 0$.

As noted in [S98d, Lemma 3.2], the minimax J-coercivity of $\mathbb{D}\in\mathrm{TIC}$ means that $\mathcal{J}(x_0,\left[\begin{smallmatrix}u\\ w\end{smallmatrix}\right])$ is uniformly convex w.r.t. u, and $\min_{u\in\mathrm{L}^2(\mathbf{R}_+;U)}\mathcal{J}(x_0,\left[\begin{smallmatrix}u\\ w\end{smallmatrix}\right])$ is uniformly concave w.r.t. w. By Lemma 11.3.10, an equivalent condition is that $\gamma>\gamma_0$ when the assumptions of previous section are satisfied. Similarly, cominimax J-coercivity is "roughly equivalent" to the solvability of the Nehari problem, see Theorem 11.8.3. We write the negative term out for later use:

$$0\gg\mathbb{F}_{22}-\mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12}=\pi_+\mathbb{D}_2^*J\mathbb{D}_2\pi_+-\pi_+\mathbb{D}_2^*J\mathbb{D}_1\pi_+(\pi_+\mathbb{D}_1^*J\mathbb{D}_1\pi_+)^{-1}\pi_+\mathbb{D}_1^*J\mathbb{D}_2\pi_+. \tag{11.106}$$

By combining (A.11) and Lemma A.3.1(b1) we obtain

Lemma 11.4.2 *If* $\mathbb{D}=\begin{bmatrix}\mathbb{D}_1 & \mathbb{D}_2\end{bmatrix}\in\mathrm{TI}$ *is minimax J-coercive, then* $\mathbb{F}\in\mathcal{GB}(\pi_+\mathrm{L}^2)$ *and* $(\mathbb{F}^{-1})_{22}=(\mathbb{F}_{22}-\mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12})^{-1}\ll 0$ *(in particular,* $\mathbb{D}$ *is J-coercive).*

Similarly, if $\mathbb{D}=\begin{bmatrix}\mathbb{D}_1 & \mathbb{D}_2\end{bmatrix}\in\mathrm{TI}$ *is co-minimax J-coercive, then* $\mathbb{F}\in\mathcal{GB}(\pi_-\mathrm{L}^2)$ *and* $(\mathbb{F}^{-1})_{22}=(\mathbb{F}_{22}-\mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12})^{-1}\ll 0$. □

Note that $\mathbb{E}:=\mathbb{D}^*J\mathbb{D}$ may satisfy $\mathbb{E}_{11}\gg 0$ and $\mathbb{E}_{22}-\mathbb{E}_{21}\mathbb{E}_{11}^{-1}\mathbb{E}_{12}\ll 0$ and still $\mathbb{F}$ may be noninvertible (take $\mathbb{D}=\begin{bmatrix} I & 0\\ \tau(1) & I\end{bmatrix}$ and $J=J_1$; similarly $\mathbb{D}^{\mathrm{d}}$ (which is of Nehari form with $\mathcal{G}=\tau(1)$) is not co-minimax J-coercive).

Minimax J-coercivity can also be formulated in terms of a spectral factorization (if $\mathbb{D}^*J\mathbb{D}$ is regular enough to have one):

Lemma 11.4.3 (SpF) *Let* $\mathbb{D}=\begin{bmatrix}\mathbb{D}_1 & \mathbb{D}_2\end{bmatrix}\in\mathrm{TI}(U\times W,Y)$, $J=J^*\in\mathcal{B}(Y)$, *and* $\gamma>0$.

(a) Assume that $\mathbb{D}^*J\mathbb{D}=\mathbb{Y}^*\mathbb{Z}$, *where* $\mathbb{Y},\mathbb{Z}\in\mathcal{G}\mathrm{TIC}$.

Then $\mathbb{D}$ *is minimax J-coercive iff* $\mathbb{D}^*J\mathbb{D}=\mathbb{X}^*J_\gamma\mathbb{X}$ *with* $\mathbb{X},\mathbb{X}_{11}\in\mathcal{G}\mathrm{TIC}$ *and* $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}}<\gamma$.

(b) Assume that $\mathbb{D}^*J\mathbb{D}=\mathbb{X}^*J_\gamma\mathbb{X}$, *where* $\mathbb{X}\in\mathcal{G}\mathrm{TIC}$. *Then* $\mathbb{D}$ *is minimax J-coercive iff* $\mathbb{X}_{11}\in\mathcal{G}\mathrm{TIC}$ *and* $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}}<\gamma$.

(c) Assume that $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* J_\gamma \mathbb{X}$*, where* $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$.

Then $\mathbb{D}$ *is minimax J-coercive iff* $\mathbb{D}_1^* J\mathbb{D}_1 \gg 0$ *iff* $\mathbb{F}_{11} \gg 0$ *iff* $\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} \ll 0$ *iff* $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}} < \gamma$ *iff* $\|(\mathbb{X}^{-1})_{22}^{-1}(\mathbb{X}^{-1})_{21}\|_{\mathrm{TI}} < \gamma$.

(d) Assume that $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$*,* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W) \cap \mathrm{UR}$*,* $X = I$ *and* $S \in \mathcal{G}\mathcal{B}(U \times W)$*. If* $\mathbb{D}$ *is minimax J-coercive, then* $S_{11} \gg 0$ *and* $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$.

We used (a) and Lemma 11.3.10 to show that (FI1s) implies (FI3s) in Proposition 11.3.4.

Proof: (We take $\gamma = 1$ by using $\left[\begin{smallmatrix} I & 0 \\ 0 & \gamma \end{smallmatrix}\right] \mathbb{X}$ instead of $\mathbb{X}$.)

Let $\mathbb{F} := \pi_+ \mathbb{D}^* J\mathbb{D}\pi_+ \in \mathcal{B}(\pi_+ \mathrm{L}^2)$, so that $\mathbb{F}_{11} \gg 0$ and $(\mathbb{F}^{-1})_{22} \ll 0$ iff $\mathbb{D}$ is minimax J-coercive, by Lemma 11.4.2.

(a) 1° *"If":* Since now $\mathbb{F} = \pi_+ \mathbb{X}^* \pi_+ J_1 \pi_+ \mathbb{X}\pi_+$, and $\pi_+ \mathbb{X}\pi_+ \in \mathcal{G}\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+; U \times W))$, we obtain "If" from Lemma 11.3.13(vi)&(i).

2° *"Only if":* Let $\mathbb{D}$ be minimax J-coercive and $\mathbb{D}^* J\mathbb{D} = \mathbb{Y}^*\mathbb{Z}$, $\mathbb{Y}, \mathbb{Z}$. Then we have $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, by Lemma 11.4.8. The $(1,1)$-block of $\pi_+ \mathbb{D}^* J\mathbb{D}\pi_+ = \pi_+ \mathbb{X}^* J_1 \mathbb{X}\pi_+$ implies that

$$0 \ll \mathbb{F}_{11} = \pi_+ \mathbb{D}_1^* J\mathbb{D}_1 \pi_+ = \pi_+(\mathbb{X}_{11}^*\mathbb{X}_{11} - \mathbb{X}_{21}^*\mathbb{X}_{21})\pi_+, \tag{11.107}$$

hence $\pi_+ \mathbb{X}_{11}^* \mathbb{X}_{11} \pi_+ \gg 0$, i.e., $\mathbb{X}_{11}^*\mathbb{X}_{11} \gg 0$, by Lemma 6.4.6.

By Lemma A.1.1(c), the [left-]invertibility of $\mathbb{X}_{11}$ is equivalent to that of $\mathbb{M}_{22}$, where $\mathbb{M} := \mathbb{X}^{-1}$. Therefore, $\mathbb{M}_{22}^*\mathbb{M}_{22} \gg 0$, so by Lemma 2.2.3, it is enough to prove that $\mathbb{M}_{22}\pi_+\mathbb{M}_{22}^* \gg 0$ on $\mathrm{L}^2(\mathbf{R}_+; W)$ to obtain $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}$ (hence $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$). We shall do it.

Clearly $\mathbb{F}^{-1} = \pi_+ \mathbb{M}\pi_+ J_1 \mathbb{M}^* \pi_+$, hence $0 \gg (\mathbb{F}^{-1})_{22} = \mathbb{M}_{21}\pi_+\mathbb{M}_{21}^* - \mathbb{M}_{22}\pi_+\mathbb{M}_{22}^*$ on $\pi_+\mathrm{L}^2$, by Lemma 11.4.2, which implies that $\mathbb{M}_{22}\pi_+\mathbb{M}_{22}^* \gg 0$ on $\pi_+\mathrm{L}^2$, therefore $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}$, i.e., $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$.

From (11.107) and Lemma 6.4.6 we get $0 \ll \mathbb{X}_{11}^*\mathbb{X}_{11} - \mathbb{X}_{21}^*\mathbb{X}_{21}$, equivalently (by Lemma A.3.1(e2)), $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}} < 1$.

(b) "If" follows from (a) ($\mathbb{Y} := J_1\mathbb{X}$, $\mathbb{Z} := \mathbb{Z}$). Conversely, if $\mathbb{D}$ is minimax J-coercive, then the proof of (a) shows that an arbitrary J_1-spectral factor of $\mathbb{D}^* J\mathbb{D}$ is as in (B).

(c) From $(\pi_+\mathbb{X}_{11}^{-1}\pi_+)(\pi_+\mathbb{X}_{11}\pi_+) = \pi_+ = (\pi_+\mathbb{X}_{11}\pi_+)(\pi_+\mathbb{X}_{11}^{-1}\pi_+)$ we see that
$(\pi_+\mathbb{X}_{11}\pi_+), (\pi_+\mathbb{X}_{11}^{-1}\pi_+) \in \mathcal{G}\mathcal{B}(\pi_+\mathrm{L}^2)$, hence $0 \ll \mathbb{F}_{11}$ is equivalent to

$$0 \ll (\pi_+\mathbb{X}_{11}^{-*}\pi_+)\mathbb{F}_{11}(\pi_+\mathbb{X}_{11}^{-1}\pi_+) = \pi_+ - \mathbb{X}_{11}^{-*}\mathbb{X}_{21}^*\mathbb{X}_{21}\mathbb{X}_{11}^{-1})\pi_+ \tag{11.108}$$

(because $\mathbb{F}_{11} = \pi_+(\mathbb{X}_{11}^*\mathbb{X}_{11} - \mathbb{X}_{21}^*\mathbb{X}_{21})\pi_+$), i.e., $1 > \|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}}$ (cf. 1°). By (b), this is equivalent to the minimax J-coercivity of $\mathbb{D}$.

On the other hand, $0 \ll \mathbb{F}_{11} = \pi_+\mathbb{D}_1^* J\mathbb{D}_1\pi_+ \Leftrightarrow 0 \ll \mathbb{D}_1^* J\mathbb{D}_1$. The last "iff" follows from $\mathbb{M}_{22}^{-1}\mathbb{M}_{21} = -\mathbb{X}_{12}\mathbb{X}_{11}^{-1}$ and the third from Lemma A.1.1(c1).

(d) By (a), $\mathbb{D}^* J\mathbb{D} = \mathbb{Z}^* J_1 \mathbb{Z}$ for some $\mathbb{Z} \in \mathcal{G}\mathrm{TIC}(U \times W)$ s.t. $\mathbb{Z}_{11} \in \mathcal{G}\mathrm{TIC}(U)$ and $\|\mathbb{Z}_{21}\mathbb{Z}_{11}^{-1}\|_{\mathrm{TI}} < 1$. By Lemma 6.4.5(a), $\mathbb{Z} = Z\mathbb{X}$ and $S = Z^* J_1 Z$ for some $Z \in \mathcal{G}\mathcal{B}$, hence $\mathbb{Z}$ is UR. By Proposition 6.3.1(b1), $Z_{11} \in \mathcal{G}\mathcal{B}(U)$. Thus, $\|Z_{21}Z_{11}^{-1}\|_{\mathrm{TI}} < 1$. By Lemma 11.3.13(vi)&(i), $S_{11} \gg 0$ and $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$.

$\square$

Corollary 11.4.4 (MTI SpF) *Let $\mathbb{D} \in \widetilde{\mathcal{A}}(U \times W, Y)$ and $J = J^* \in \mathcal{B}(Y)$.*
Then $\mathbb{D}$ is minimax J-coercive iff $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* J_1 \mathbb{X}$, where $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ and $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\|_{\mathrm{TI}} < 1$.

Naturally, the weaker assumption $\mathbb{D} \in \mathrm{TIC}(U \times W, Y)$, $(\mathbb{D}, J) \in \mathrm{SpF}$ would be sufficient, with essentially the same proof.

Proof: If $\mathbb{D}$ is minimax J-coercive, then $\pi_+ \mathbb{D}^* J \mathbb{D} \pi_+$ is invertible, by Lemma 11.4.2, so the existence of a factorization follows from Theorem 8.4.12(ii). The rest of the claims and the converse is obtained from Lemma 11.4.3. $\square$

Corollary 11.4.5 *Let $\mathbb{D} = \begin{bmatrix} \mathbb{D}_1 & \mathbb{D}_2 \end{bmatrix} \in \mathrm{TI}$, $\mathbb{X}, \mathbb{M} \in \mathrm{TIC}$, $\mathbb{M} = \mathbb{X}^{-1}$, $J = J^* \in \mathcal{B}$ and $\mathbb{D}^* J \mathbb{D} = \mathbb{X} J_1 \mathbb{X}^*$.*
The operator $\mathbb{D}$ is co-minimax J-coercive iff $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ and $\|\mathbb{X}_{11}^{-1}\mathbb{X}_{12}\| < 1$. If $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$, then $\mathbb{D}$ is minimax J-coercive iff $\|\mathbb{X}_{11}^{-1}\mathbb{X}_{12}\|_{\mathrm{TI}} < 1$ iff $\mathbb{F}_{11} \gg 0$ iff $\mathbb{D}_1^* J \mathbb{D}_1 \gg 0$ iff $\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} \ll 0$ (here $\mathbb{F} := \pi_- \mathbb{D}^* J \mathbb{D} \pi_-$).

Proof: Apply Lemma 11.4.3 to $((\mathbb{D}^{\mathrm{d}})^*)^* J ((\mathbb{D}^{\mathrm{d}})^*) = (\mathbb{X}^{\mathrm{d}})^* J_1 \mathbb{X}^{\mathrm{d}}$. $\square$

For $\mathbb{D}$ of Nehari type, the condition $\|\mathbb{X}_{11}^{-1}\mathbb{X}_{12}\| < 1$ is redundant:

Corollary 11.4.6 (Nehari-form) *Let $\mathbb{D} = \left[\begin{smallmatrix} I & \mathbb{G} \\ 0 & I \end{smallmatrix}\right] \in \mathrm{TIC}$, $\mathbb{X} \in \mathcal{G}\mathrm{TIC}$, $J = J^* \in \mathcal{B}$ and $\mathbb{D}^* J \mathbb{D} = \mathbb{X} J_1 \mathbb{X}^*$. The operator $\mathbb{D}$ is co-minimax J-coercive iff $\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} \ll 0$ iff $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ (here $\mathbb{F} := \pi_- \mathbb{D}^* J \mathbb{D} \pi_-$). Moreover, in that case always $\|\mathbb{X}_{11}^{-1}\mathbb{X}_{12}\| < 1$.*

Note that $\|\mathbb{X}_{11}^{-1}\mathbb{X}_{12}\| = \|\mathbb{M}_{12}\mathbb{M}_{22}^{-1}\|$, where $\mathbb{M} := \mathbb{X}^{-1}$, because the $(1,2)$-block of equation $\mathbb{X}\mathbb{M} = I$ is $\mathbb{X}_{11}\mathbb{M}_{12} + \mathbb{X}_{12}\mathbb{M}_{22} = 0$.

Proof: The first "iff" follows from Definition 11.4.1 & Lemma 11.4.2, because always $\mathbb{F}_{11} = \pi_- \gg 0$. Thus the latter "iff" and the inequality follow from Corollary 11.4.5. $\square$

Corollary 11.4.7 (Nehari-form) *Let $\mathbb{D} = \left[\begin{smallmatrix} I & \mathbb{G} \\ 0 & I \end{smallmatrix}\right] \in \mathrm{TIC}(U \times W)$, $\mathbb{Y}, \mathbb{Z} \in \mathcal{G}\mathrm{TIC}(U \times W)$, $J = J_\gamma \in \mathcal{B}(U \times W)$ and $\mathbb{D}^* J \mathbb{D} = \mathbb{Y}\mathbb{Z}^*$.*
Then the operator $\mathbb{D}$ is co-minimax J-coercive iff $\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{12} \ll 0$ iff $\mathbb{D}^* J \mathbb{D} = \mathbb{X} J_1 \mathbb{X}^*$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W)$ having $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$. Moreover, in that case always $\|\mathbb{X}_{11}^{-1}\mathbb{X}_{12}\| < 1$.

Proof: By Lemma 11.4.2 and Corollary 11.4.6, we only have to obtain the third condition assuming the first.

Let $\mathbb{D}$ be co-minimax J-coercive. Set $\mathbb{E} := \boldsymbol{Я}\mathbb{D}^* J \mathbb{D}\boldsymbol{Я} = (\mathbb{Y}^{\mathrm{d}})^* \mathbb{Z}^{\mathrm{d}}$, so that $\mathbb{F} := \pi_- \boldsymbol{Я}\mathbb{E}\boldsymbol{Я}\pi_-$ is as in Definition 11.4.1, hence $\mathbb{F}_{11} \gg 0$ and $\mathbb{F}_{22} -$

$\mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} \ll 0$ on $\mathrm{L}^2(\mathbf{R}_-;U\times W)$. We obviously have $\mathbb{E}_{11} = I \gg 0$. Moreover, $\mathbb{G}$ of Lemma 11.4.8 satisfies

$$\boldsymbol{Я}\pi_+\mathbb{G}\pi_+\boldsymbol{Я} = \boldsymbol{Я}\pi_+\mathbb{E}_{22}\pi_+\boldsymbol{Я} - \boldsymbol{Я}\pi_+\mathbb{E}_{21}\mathbb{E}_{12}\pi_+\boldsymbol{Я} = \boldsymbol{Я}\pi_+\mathbb{E}_{22}\pi_+\boldsymbol{Я} - \boldsymbol{Я}\pi_+\mathbb{E}_{21}\pi_+\mathbb{E}_{12}\pi_+\boldsymbol{Я} \tag{11.109}$$

$$= \boldsymbol{Я}\pi_+\mathbb{E}_{22}\pi_+\boldsymbol{Я} - \pi_-\boldsymbol{Я}\mathbb{E}_{21}\boldsymbol{Я}\pi_-\boldsymbol{Я}\mathbb{E}_{12}\boldsymbol{Я}\pi_- = \mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{12} \ll 0 \tag{11.110}$$

on $\mathrm{L}^2(\mathbf{R}_-;U\times W)$, hence the assumptions of Lemma 11.4.8 are satisfied, by Lemma 6.4.6. We conclude that $\mathbb{E} = \widetilde{\mathbb{X}}^* J_1 \widetilde{\mathbb{X}}$ for some $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}(U\times W)$, hence $\mathbb{D}^* J \mathbb{D} = \boldsymbol{Я}\widetilde{\mathbb{X}}^* J_1 \widetilde{\mathbb{X}}\boldsymbol{Я} = \mathbb{X} J_1 \mathbb{X}^*$, where $\mathbb{X} := \widetilde{\mathbb{X}}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}$. The rest follows from Corollary 11.4.6. □

In the proof of Lemma 11.4.3(a), we used the following:

Lemma 11.4.8 *Assume that* $\mathbb{E} = \mathbb{E}^* \in \mathrm{TI}(U\times W)$, $\mathbb{E}_{11} \gg 0$, $\mathbb{G} := \mathbb{E}_{22} - \mathbb{E}_{21}\mathbb{E}_{11}^{-1}\mathbb{E}_{12} \ll 0$, $\mathbb{E} = \mathbb{Y}^*\mathbb{Z}$ *for some* $\mathbb{Z}, \mathbb{Y} \in \mathcal{G}\mathrm{TIC}(U\times W)$ *and* $\gamma > 0$.
Then $\mathbb{E} = \mathbb{X}^* J_\gamma \mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U\times W)$.

(One could go on to obtain further results as in Lemma 11.4.3.)

Proof: By Lemma 5.2.1(d), $\mathbb{E} = \mathbb{X}^* S \mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U\times W)$ and $S = S^* \in \mathcal{G}\mathcal{B}(U\times W)$. By Lemma 6.4.7(a), we have $\mathbb{E}_{11} = \mathbb{R}^*\mathbb{R}$ and $-\mathbb{G} = \mathbb{T}^*\mathbb{T}$ for some $\mathbb{R} \in \mathcal{G}\mathrm{TIC}(U)$ and $\mathbb{T} \in \mathcal{G}\mathrm{TIC}(W)$. By the Schur decomposition (A.5), we have

$$S = \mathbb{X}^{-*}\mathbb{E}\mathbb{X}^{-1} = \mathbb{X}^{-*}\begin{bmatrix} I & 0 \\ \mathbb{E}_{21}\mathbb{E}_{11}^{-1} & I \end{bmatrix}^* \begin{bmatrix} \mathbb{E}_{11} & 0 \\ 0 & \mathbb{G} \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbb{E}_{21}\mathbb{E}_{11}^{-1} & I \end{bmatrix} \mathbb{X}^{-1} \tag{11.111}$$

$$= \mathbb{X}^{-*}\begin{bmatrix} I & 0 \\ \mathbb{E}_{21}\mathbb{E}_{11}^{-1} & I \end{bmatrix}^* \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{T} \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{T} \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbb{E}_{21}\mathbb{E}_{11}^{-1} & I \end{bmatrix} \mathbb{X}^{-1}, \tag{11.112}$$

i.e., $S = \mathbb{U}^* J_1 \mathbb{U}$, where $\mathbb{U} := \begin{bmatrix} \mathbb{R} & 0 \\ 0 & \mathbb{T} \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbb{E}_{21} & \mathbb{E}_{11}^{-1} \quad I \end{bmatrix} \mathbb{X}^{-1} \in \mathcal{G}\mathrm{TI}(U\times W)$. Consequently, $S = E^* J_\gamma E$ for some $E \in \mathcal{G}\mathcal{B}(U\times W)$, by Theorem 2.4.5. □

We have thus generalized Lemma 5.4 of [S98d] to Hilbert spaces of arbitrary dimensions (see Lemma 11.4.3(a)). Because the assumed separability of the Hilbert spaces used in [S98d] was needed only in the proof of [S98d, Lemma 5.4] (as mentioned at the end of [S98d, Section 1]), the above lemma shows that the separability assumptions are not needed in [S98d]:

Corollary 11.4.9 *All separability assumptions in [S98d] can be removed.* □

Notes

Lemma 11.4.3(a) is roughly Lemma 5.4 of [S98d], and the proofs (including Lemma 11.4.8) use same methods; the difference is that we do not need any separability assumptions, due to Theorem 2.4.5.

11.5 The discrete-time H^∞ ficp

A walking shadow, a poor player,
that struts and frets his hour upon the stage,
And then is heard no more.
— William Shakespeare (1564–1616), "Macbeth"

As in other discrete-time sections, our references to continuous-time results, definitions and hypotheses refer to their discrete-time forms (cf. Theorems 13.3.13 and 11.5.2).

Recall that throughout this section and Section 11.6, we assume (in addition to Standing Hypothesis 11.0.1) that Standing Hypothesis 11.1.1 holds, i.e., we consider the system

$$\begin{cases} x_{n+1} = Ax_n + B_1 u_n + B_2 w_n, \\ \quad z_n = C_1 x_n + D_{11} u_n + D_{12} w_n \end{cases} \qquad (n \in \mathbf{N}) \tag{11.113}$$

(and $w_n = Iw_n$) with initial state $x_0 \in H$, disturbance input $w \in \ell^2(\mathbf{N};W)$, control input $u \in \ell^2(\mathbf{N};U)$ and objective output $z \in \ell^2(\mathbf{N};Z)$ (and second output equal to the disturbance input w_n; cf. (12.31)); here $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H\times U\times W, H\times Z\times W)$ are the generators of $\Sigma \in \mathrm{wpls}(U\times W, H, Z\times W)$ (see Lemma 13.3.3) and $C_2 = 0$, $D_{21} = 0$, $D_{22} = I$. Condition (i) below says that

$$\gamma_0 := \sup_{\|w\|_{\ell^2}=1} \inf\{\|z\|_{\ell^2} \mid u \in \ell^2 \text{ is s.t. } x \text{ is stable (i.e., } x \in \ell^2)\}; \tag{11.114}$$

thus, γ_0 equals $\inf\|w\mapsto z\|_{\mathcal{B}(\ell^2)}$, the infimal disturbance-to-output norm $\|w\mapsto z\|$, over all control laws that make the system exponentially stable ($u \in \ell^2(\mathbf{N};U)$ s.t. $x \in \ell^2$). By (ii), infimum of over (causal) state-feedback (plus feedthrough) controllers

$$u(t) = K_1 x(t) + F_{12} w(t) \tag{11.115}$$

is as low. By (iii) and (a), a control law achieving a performance below γ_0 is found iff the DARE (11.117) has a nonnegative exponentially (or power) stabilizing solution satisfying the signature condition (for S). Moreover, such a solution determines one possible choice of K_1 and F_{12}, by (11.118). Finally, if $S_{22} \ll 0$, we can take $F_{12} = 0$ to obtain the pure state-feedback controller (11.119).

Theorem 11.5.1 ($\mathcal{U}_{\mathrm{exp}}$: H^∞ ficp) *Assume that $\gamma > 0$ and that there is $\varepsilon > 0$ s.t.*

$$(z-A)x_0 = Bu_0 \implies \|C_1x_0 + D_{11}u_0\|_Y \geq \varepsilon(\|x_0\|_H + \|u_0\|_U) \quad (x_0 \in H,\ u_0 \in U,\ z \in \partial\mathbf{D}). \tag{11.116}$$

Then (i)–(iii) are equivalent:

(i) $\gamma > \gamma_0 := \sup_{w:\mathbf{N}\to W,\ \|w\|_{\ell^2}=1} \inf_{u\in\mathcal{U}_{\mathrm{u}}(0,w)} \|\mathbb{D}_{11}u + \mathbb{D}_{12}w\|_{\ell^2}$, *and* (A, B_1) *is exponentially stabilizable;*

(ii) $\gamma > \gamma_{\mathrm{FI}}$, *i.e., there is a suboptimal* H^∞*-FI-pair for* Σ*;*

(iii) the DARE

$$\begin{cases} \mathcal{P} = A^*\mathcal{P}A + C_1^*C_1 - K^*SK, \\ S = \begin{bmatrix} D_{11}^*D_{11} & D_{11}^*D_{12} \\ D_{12}^*D_{11} & D_{12}^*D_{12}-\gamma^2 I \end{bmatrix} + B^*\mathcal{P}B, \\ K = -S^{-1}\left(\begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 + B^*\mathcal{P}A \right), \end{cases} \tag{11.117}$$

has a solution $\mathcal{P} \in \mathcal{B}(H)$ *s.t.* $\mathcal{P} \geq 0$, $S_{11} \gg 0$, $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$ *and* $\rho(A+BK) < 1$.

Moreover, the following hold:

(a) Assume that $(\mathcal{P}, S, K)$ *satisfies (iii). Then*

$$\left(\begin{array}{c|cc} -S_{11}^{-1}(D_{11}^*C_1 + B_1^*\mathcal{P}A) & 0 & -S_{11}^{-1}S_{12} \\ 0 & 0 & 0 \end{array} \right); \tag{11.118}$$

is a suboptimal (exponentially stabilizing) H^∞*-FI-pair for* Σ.

There is a suboptimal H^∞*-SF-operator iff* $S_{22} \ll 0$*; if this is the case, then*

$$K_1 := \begin{bmatrix} I & 0 \end{bmatrix} K = -(S_{11} - S_{12}S_{22}^{-1}S_{21})^{-1}(D_{11}^*C_1 + B_1^*\mathcal{P}A - S_{12}S_{22}^{-1}(D_{12}^*C_1 + B_2^*\mathcal{P}A)) \tag{11.119}$$

is a suboptimal (exponentially stabilizing) H^∞*-SF-operator for* Σ.

(b) If (i)–(iii) hold, then the assumptions of Proposition 11.2.8 (also those of (a1)) are satisfied and (FI1)–(FI5) hold.

One more equivalent condition is that $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathcal{B}(\ell^2(\mathbf{N};W),\ell^2(\mathbf{N};Z))} < \gamma$ for some $\mathbb{U} : \ell^2(\mathbf{N};W) \to \ell^2(\mathbf{N};U)$, and (A, B_1) is exponentially stabilizable (obviously this is stronger than (i) and weaker than (ii)).

We recall from Section 11.1 that $\mathcal{U}_\mathrm{u}$ refers to the controls that make the state and output "$\mathcal{U}_*^*$-stable", i.e.,

$$\mathcal{U}_\mathrm{u}(x_0, w) := \{u \in \ell^2(\mathbf{N};U) \mid \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}_*^*(x_0)\}, \tag{11.120}$$

(in Theorem 11.5.1, $\mathcal{U}_*^* := \mathcal{U}_\mathrm{exp}$, so that "$\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}_*^*(x_0)$" can be replaced by "$\mathbb{B}\tau\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \ell^2$", i.e., by the assumption that u makes the state trajectory belong to ℓ^2), and that by γ_FI (resp. γ_SF) we denote the infimum of the norm $\|w \mapsto z\|_{\ell^2(\mathbf{N};W)\to\ell^2(\mathbf{N};Z)}$ (i.e., of $\|\mathbb{D}_{12}^{\circlearrowleft}\|_\mathrm{TIC}$; see (11.8)) over all $\mathcal{U}_*^*$-stabilizing state feedback pairs (resp. $\mathcal{U}_*^*$-stabilizing state feedback operators) for Σ of form

$$\left(\begin{array}{c|cc} K_1 & F_{11} & F_{12} \\ 0 & 0 & 0 \end{array} \right) \tag{11.121}$$

(i.e., we allow state feedback through the first input (the control u) only). Thus, γ_SF requires that $F_{11} = 0 = F_{12}$. Trivially, $\infty \geq \gamma_\mathrm{SF} \geq \gamma_\mathrm{FI} \geq \gamma_0 \geq 0$ (cf. (11.132))

Proof of Theorem 11.5.1: 0.1° *Remark: Assumption (11.116):* By Proposition 15.2.2(c), this implies that $\mathbb{D}_{11}$ (with realization $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B}_1 \\ \mathbb{C}_1 & \mathbb{D}_{11} \end{smallmatrix}\right]$) is I-coercive. If (A, B_1) is exponentially stabilizable, then (11.116) is equivalent to the I-coercivity of $\mathbb{D}_{11}$, by Proposition 15.2.2(e) (see the proposition for several sufficient and some equivalent conditions).

1° We get "(i)⇒(iii)" from Proposition 11.6.2, "(iii)⇒(ii)" from Lemma 11.6.3, and "(ii)⇒(i)" from (11.12) and Lemma 11.6.4. (See also remark 0.3° of the roof of Theorem 11.1.3. Recall from Lemma 13.3.7(iv) that $\rho(A+BK) < 1$ iff $\mathcal{P}$ (that is, K) is exponentially stabilizing.)

Also (a) follows from Proposition 11.6.2, and Lemma 11.6.3. By Proposition 11.2.9, (b) holds. □

Practically all our H^∞ FICP results hold also in their discrete-time forms:

Theorem 11.5.2 (Discrete form of H^∞ FICP results) *All results of Sections 11.1–11.4 and 11.8–11.9 hold also in their discrete-time forms (i.e., after the changes listed in Theorem 13.3.13), except that in Theorems 11.1.3, 11.1.4 and 11.1.6, assumption (2.) can be removed but equation (11.39) must be replaced by (11.118) and equation (11.17) by (11.119).*

When applying the above results, do not forget (the discrete-time forms of) Standing Hypotheses 11.0.1, 11.1.1, 11.2.1 and 11.3.1 (the last two of which are only assumed for the results of corresponding sections). Note that we have written explicitly the simplified discrete-time forms of some major results in this section and Section 11.6.

Recall from Lemma 14.3.5 that we can have $\widetilde{\mathcal{A}} = \mathrm{tic}_{\mathrm{exp}}$ or $\widetilde{\mathcal{A}} = \ell^1*$; this is particularly useful in Theorems 11.2.7 and 11.3.3. Therefore, in discrete time, we may allow for general (exponentially stabilizable) WPLSs in the $\mathcal{U}_{\mathrm{exp}}$ case of Theorem 11.2.7, while Example 11.3.7(b) (its discrete-time variant) shows that (FI1) does not imply any of (FI3)–(FI5) when $\mathcal{U}_*^* \in \{\mathcal{U}_{\mathrm{out}}, \mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}\}$.

Proof of Theorem 11.5.2: Recall that these changes include having the DARE (11.117) in place of the $[B_{\mathrm{w}}$-]CARE.

This follows roughly by applying (13.63) also to the proofs (recall from Lemma 14.3.5 that $\widetilde{\mathcal{A}} := \mathrm{tic}_{\mathrm{exp}}$ satisfies Standing Hypothesis 11.0.1).

(An alternative proof of the I/O part of Sections 11.8–11.9 is obtained by using the Cayley transform (Theorem 13.2.3).)

Note that (11.17) and (11.17) used the fact that "$S = D^*J_\gamma D$", which is not necessarily true under our discrete-time assumptions (or any reasonable counterparts, cf. Example 14.2.9). However, assumption (2.) can be removed from Theorems 11.1.3, 11.1.4 and 11.1.6, since we can use Theorem 14.1.6 (and Lemma 9.9.7(c2)) instead of Theorems 9.9.6 and Theorem 9.2.9 in the proofs. □

From the above theorem and Theorem 11.2.8(f)&(d), we deduce that if the system is q.r.c.-SOS-stabilizable through u (as in Theorem 11.2.8), then any of the factorization conditions (FI6)–(FI8) on p. 633 are sufficient for (ii), i.e., for the existence of a γ-suboptimal controller (over $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$). If, in addition, the q.r.c.-SOS-stabilized I/O map has, e.g., ℓ^1 impulse response, then these conditions are also necessary. If (A, B_1) is exponentially stabilizable and (A, C) is detectable, then (FI1)–(FI8) become equivalent (and (FI9) if $\dim U < \infty$ or $\dim W < \infty$; use Theorem 6.7.15(c3) and $\widetilde{\mathcal{A}} = \mathrm{tic}_{\mathrm{exp}}$ in Theorem 11.2.7).

Notes

Much of the notes of previous sections also apply to the discrete-time H^∞ ficp. For example, the equivalence of (ii) and (iii) (and part (a)) in Theorem 11.5.1 is given in [IOW] under the same and in [GL] under stronger simplifying assumptions, although both books assume that Σ is finite-dimensional. However, discrete-time H^∞ problems are more rarely treated than their continuous-time counterparts, partially because they are more complicated (as long as the input and output operators are bounded; in our generality the continuous-time results are much more involved).

If we delete "(i)" from Theorem 11.5.1, then its proof can be obtained rather directly from Proposition 11.6.2(a1)&(d1) and Lemma 11.2.13, as noted in the notes on p. 652.

11.6 H^∞ ficp: proofs

By the time the fool has learned the game, the players have dispersed.
— Ghanaian Proverb

In addition to Standing Hypotheses 11.0.1 and 11.1.1, we assume the following:

Standing Hypothesis 11.6.1 ($\mathbf{H}^\infty$ full-information control problem (ficp)) *Throughout this section, we make the following assumptions: Hypothesis 14.0.1 is satisfied (with $U \mapsto U \times W$ and $Y \mapsto Z \times W$), $\gamma > 0$, and there is $\varepsilon_+ > 0$ s.t. $\|\mathbb{D}_{11}u\|_{\ell^2} \geq \varepsilon_+ \|\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]\|_{\mathcal{U}_*^*}$ for all $u \in \mathcal{U}_{\mathrm{u}}(0,0)$.*

(See the remarks below Hypothesis 11.2.1, which is the continuous-time counterpart of this hypothesis.)

Due to bounded generators, the discrete-time variant of Proposition 11.2.19 becomes simpler:

Proposition 11.6.2 (Necessary conditions) *Assume that $\gamma > \gamma_0$ and that Z^{s} is reflexive. Then $\mathbb{D}$ is J_γ-coercive. Assume in addition that $\mathcal{U}_*^*(x_0) \neq \emptyset$ for each $x_0 \in H$. Then the following hold:*

- *(a1) The DARE has a unique $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$, $\mathcal{P} \geq 0$, $S \in \mathcal{GB}(U \times W)$ and $S_{11} \geq \varepsilon_+^2 I$.*
- *(a2) For each $x_0 \in H$, the corresponding closed-loop second output $\begin{bmatrix} u_{\mathrm{crit}}(x_0) \\ w_{\mathrm{crit}}(x_0) \end{bmatrix} := \mathbb{K}_\circlearrowleft x_0$ is the unique J_γ-critical input (called the* minimax control*), and this input corresponds to the (unique) arguments of*

$$\max_{w \in \ell^2(\mathbf{N};W)} \min_{u \in \mathcal{U}_{\mathrm{u}}(x_0,w)} \mathcal{J}(x_0,u,w). \tag{11.122}$$

- *(b2) The IARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, \widetilde{S}, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ s.t. $S = \left[\begin{smallmatrix} I & 0 \\ 0 & J_W \end{smallmatrix}\right]$, where $J_W = J_W^* = J_W^{-1} \in \mathcal{GB}(W)$.*
- *(d1)* **($\mathbf{S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0}$)** *If there is a suboptimal H^∞-FI-pair, then $S_{22} - S_{21}S_{11}^{-1}S_{12} \ll 0$, and the IARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, J_1, \left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right])$ s.t. $\widetilde{\mathbb{X}}_{11}, \widetilde{\mathbb{X}}_{22} \in \mathcal{G}\mathrm{tic}_\infty$, $\widetilde{X}_{21} = 0$, $\|\widetilde{\mathbb{X}}_{11}^{-1}\|_{\mathrm{tic}} \leq \varepsilon_+^{-1}$, and $\|\widetilde{\mathbb{X}}_{21}\widetilde{\mathbb{X}}_{11}^{-1}\|_{\mathrm{tic}} \leq 1$.*
- *(d2)* **($\mathbf{S_{22} \ll 0}$)** *If there is a suboptimal H^∞-SF-operator, then $S_{22} \ll 0$.*
- *(f) In (d1) (resp. (d2)), the existence of a suboptimal H^∞-FI-pair (resp. H^∞-SF-operator) is not needed if there is $\mathbb{U} \in \mathrm{tic}_\infty(W,U)$ s.t. $\|\mathbb{D}_{11}\mathbb{U} + \mathbb{D}_{12}\|_{\mathrm{tic}} < \gamma$ (resp. and $\widehat{\mathbb{U}}(0) = 0$).*

Proof: Most of this follows as in the proof of Proposition 11.2.19. We explain the least obvious changes below.

(a1)&(a2) The unique J_γ-critical control corresponds to the unique $\mathcal{U}_*^*$-stabilizing solution of the DARE, by Theorem 14.1.6. Substitute $t = 1$ into (11.66) (whose continuous-time proof applies, mutatis mutandis) to obtain

$$\langle u_0', S_{11} u_0' \rangle_U \geq \varepsilon_+^2 \|u'\|_{\ell^2_\vartheta}^2 = \varepsilon_+^2 \|u_0'\|_U^2, \tag{11.123}$$

where $u_0' \in U$ is arbitrary. Thus, $S_{11} \geq \varepsilon_+^2 I$.

(d1) Substitute $t = 1$ into (11.77) to observe that

$$\left\langle \begin{bmatrix} \widetilde{U} \\ I \end{bmatrix} w_0, S \begin{bmatrix} \widetilde{U} \\ I \end{bmatrix} w_0 \right\rangle_{U \times W} \leq -\varepsilon \|w_0\|_W^2 \quad (w \in W), \tag{11.124}$$

where $\widetilde{U} \in \mathcal{B}(W, U)$ is the feedthrough operator of $\mathbb{U} := \widetilde{\mathbb{F}}_{12}^{\circlearrowleft}$ (the suboptimal closed-loop $w \mapsto u$ map). By Lemma 11.3.13(i')&(i)&(iii'), this means that $S_{22} - S_{21} S_{11}^{-1} S_{12} \leq -\varepsilon I$ and that

$$S = \widetilde{X}^* J_1 \widetilde{X}, \quad \text{where } \widetilde{X} = \begin{bmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ 0 & \widetilde{X}_{22} \end{bmatrix}, \ \widetilde{X}_{11} \gg 0 \gg \widetilde{X}_{22}. \tag{11.125}$$

By Theorem 9.9.1(f1), also $(\mathcal{P}, J_1, \left(\ \widetilde{X} K, I - \widetilde{X} \ \right))$ is a $\mathcal{U}_*^*$-stabilizing solution of the IARE. Since $\widetilde{X}_{11}, \widetilde{X}_{22} \in \mathcal{GB}$, we have $\widetilde{\mathbb{X}}_{11}, \widetilde{\mathbb{X}}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$. The rest of (d1) follows from (11.77) as in the proofs of Proposition 11.2.19(c) and Lemma 11.2.21 (we do not need to study $\widehat{\widetilde{\mathbb{X}}}$, since we already know that $\widetilde{\mathbb{X}}_{11} \in \mathcal{G}\mathrm{tic}_\infty$).

(d2) Now $\widetilde{U} = 0$ in (11.124), hence $S_{22} \leq -\varepsilon I$.

Remark: We did not have to assume that $\vartheta \leq 0$. This follows from the inequality $\|\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]\|_{\ell^2_\vartheta} \geq \|u_0\|_U \ \ (u \in \ell^2_\vartheta)$, whose continuous-time analogy does not hold.

To obtain an analogous result in continuous time, we have to let $t \to 0+$, which requires additional regularity, as in Proposition 11.2.19(b3). □

Also the sufficiency part becomes simpler:

Lemma 11.6.3 (General $\mathcal{U}_*^*$: DARE $\Rightarrow$ ficp) *Assume that DARE has a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ s.t. $\mathcal{P} \geq 0$, $S_{11} \gg 0$ and $S_{22} - S_{21} S_{11}^{-1} S_{12} \ll 0$.*

Then the assumptions of Lemma 11.2.14 are satisfied (including (4.)). In particular, if $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$ (or $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ and $(I - \mathbb{F})^{-1} \in \mathrm{tic}$), then (11.48) is a suboptimal H^∞-FI-pair for Σ, with generators (11.118). If, in addition, $S_{22} \ll 0$, then (11.119) is a suboptimal H^∞-SF-operator for Σ. □

(The proofs of Lemmas 11.2.13 and 11.2.14 apply mutatis mutandis; note that this lemma also holds without the assumption on the existence of ε_+ (see Standing Hypothesis 11.6.1).)

If we wish that some kind of controller $x_0, w \mapsto u$ stabilizes Σ exponentially (cf. (ii)), then so does a strict state feedback controller (i.e., a H^∞-SF-operator); conversely, such a controller is sufficient for our "H^∞ Finite Cost Condition" $\mathcal{U}_{\mathrm{u}}(x_0, w) \neq \emptyset$ for all x_0, w:

Lemma 11.6.4 (FCC $\Leftrightarrow$ $(\mathbf{A}, \mathbf{B_1})$ exp. stab.) *Let $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$. Then the following are equivalent:*

(i) $\left(\begin{array}{c|c} A & B_1 \end{array}\right)$ *is exponentially stabilizable (or optimizable);*

(ii) $\mathcal{U}_{\text{u}}(x_0, w) \neq \emptyset$ *for all* $x_0 \in H$ *and* $w \in \ell^2(\mathbf{N}; W)$*;*

(iii) $\mathcal{U}_{\text{u}}(x_0, 0) \neq \emptyset$ *for all* $x_0 \in H$*;*

(iv) there is a H^∞*-FI-pair for* Σ*;*

(v) there is a H^∞*-SF-operator for* Σ*.*

Proof: 1° *"(v)⇒(iv)" and "(ii)⇒(iii)":* These are trivial.

2° *"(iv)⇒(ii)":* Given $x_0 \in H$ and $w \in \ell^2(\mathbf{N}; W)$, we have $\mathbb{A}x_0 + \mathbb{B}_1\tau u + \mathbb{B}_2\tau w \in \ell^2$, where $u := \mathbb{X}_{11}^{-1}\mathbb{K}_1 x_0 - \mathbb{B}_1\mathbb{X}_{11}^{-1}\mathbb{X}_{12}w \in \ell^2(\mathbf{N}; U)$, by (11.9).

3° *"(iii)⇔(i)⇒(v)":* Obviously, (iii) says that $\left(\begin{array}{c|c} A & B_1 \end{array}\right)$ is optimizable. By Proposition 13.3.14, $\left(\begin{array}{c|c} A & B_1 \end{array}\right)$ is optimizable iff it is estimatable (hence we have (i)⇔(iii)), and in either case it has an exponentially stabilizing state feedback operator K_1 (i.e., $\rho(A + B_1K_1) < 1$). But then $\left[\begin{smallmatrix} K_1 \\ 0 \end{smallmatrix}\right]$ is obviously exponentially stabilizing for Σ (because the corresponding closed-loop state-to-state map is also given by $A + B_1K_1$); hence it is a H^∞-SF-operator, by Remark 11.2.5. □

Thus, if $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, then we must assume that $\left(\begin{array}{c|c} A & B_1 \end{array}\right)$ is exponentially stabilizable, so that the problem can be reduced to the exponentially stable case, by Lemma 11.2.22 (which obviously holds in discrete time too).

(See the notes on p. 677.)

11.7 The abstract H^∞ FICP

You can not win the game, and you are not allowed to stop playing.

— The Third Law Of Thermodynamics (this is what we wish to make for the destiny of the second player (nature/disturbance))

In this section, we solve the H^∞ FICP in the abstract setting of Section 8.1. These results were applied to WPLSs and wpls's in previous sections.

For simplicity of notation, we replace the space U of inputs by $U \times W$, where $u \in U$ is considered as "control" and $w \in W$ is considered as "disturbance"; otherwise we share the notation of Section 8.1:

Standing Hypothesis 11.7.1 *Throughout this section, we shall assume that U, W, X, Y^{s} and Z^{s} are reflexive Banach spaces, that Y and Z are TVSs, and that the embeddings $Y^{\mathrm{s}} \subset Y$ and $Z^{\mathrm{s}} \subset Z$ are continuous. We also assume that $\left[\begin{smallmatrix} A & B_1 & B_2 \\ C & D_1 & D_2 \end{smallmatrix}\right] \in \mathcal{B}(X \times U \times W, Z \times Y)$ and $J = J^* \in \mathcal{B}(Y^{\mathrm{s}}, Y^{\mathrm{s}*})$.*

Note that $U \times W$ has now taken the role of U; we also set

$$\mathcal{J}(x,u,w) := \mathcal{J}(x, \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]) := \langle D\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right], JD\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]\rangle_{Y^{\mathrm{s}}} \quad (\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}(x)). \tag{11.126}$$

In applications one usually takes $U = \mathrm{L}^2(\mathbf{R}_+; U_0)$, $W = \mathrm{L}^2(\mathbf{R}_+; W_0)$ for some Hilbert spaces U_0, W_0 (cf. Remark 8.3.4).

As in previous sections, in the H^∞ FICP one wishes to find a control law $X \times W \ni (x,w) \mapsto u_{x,w} \in U$ s.t. $\left[\begin{smallmatrix} u_{x,w} \\ w \end{smallmatrix}\right]$ is a "stabilizing" input for the given "initial state" x, for each "disturbance" w, i.e., $\left[\begin{smallmatrix} u_{x,w} \\ w \end{smallmatrix}\right] \in \mathcal{U}(x)$ for all $x \in X$ and all $w \in W$. We denote the set of these "admissible controls" by

$$\mathcal{U}_{\mathrm{u}}(x,w) := \{u \in U \,\big|\, \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}(x)\} = \{u \in U \,\big|\, Cx + Du \in Y^{\mathrm{s}} \;\&\; Ax + Bu \in Z^{\mathrm{s}}\}. \tag{11.127}$$

(We set $A = 0 = B$ or $Z^{\mathrm{s}} = Z$ if we only wish to require the output to be stable.)

Moreover, this control law should be "suboptimal" (see p. 613)., However, to keep the notation simple, we study here the more general *extended Full-Information* H^∞ *Control Problem (*H^∞ *eFICP)*, where the suboptimality condition is replaced by the condition that there is $\varepsilon > 0$ s.t. $\mathcal{J}(0, u_{x,w}, w) \leq -\varepsilon\|w\|^2$. As noted on p. 613, this condition is equivalent to suboptimality in the (special case) setting of (11.1) (with $\mathbb{D} := D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$).

In the following theorem, we first show that the existence of a suboptimal control law implies J-coercivity, and then we show how the unique J-critical control is of maximin form. In Propositions 11.6.2 and 11.2.19, these facts were used to show to necessity of the signature condition and the existence of a unique stabilizing solution to the Riccati equation.

Theorem 11.7.2 ($\mathbf{H}^\infty$ eFICP) *Set $\mathcal{U}_{\mathrm{u}}(x,w) := \{u \in U \mid \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}(x)\}$. Assume that there is $\varepsilon_{\mathrm{u}} > 0$ s.t.*

$$\langle D_1 u, J D_1 u\rangle \geq \varepsilon_{\mathrm{u}} \| \left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right] \|_D^2 \qquad (u \in \mathcal{U}_{\mathrm{u}}(0,0)). \tag{11.128}$$

Then the following hold:

(a) We have (i)⇔(ii), where

(i) There is $\varepsilon_{\mathrm{w}} > 0$ s.t. $\inf_{u \in \mathcal{U}_{\mathrm{u}}(0,w)} \mathcal{J}(0,u,w) \leq -\varepsilon_{\mathrm{w}} \|w\|^2$ for all $w \in W$.
(ii) There is $F^{\curvearrowleft} \in \mathcal{B}(W,U)$ s.t. $F^{\curvearrowleft} w \in \mathcal{U}_{\mathrm{u}}(0,w)$ for all $w \in W$, $D^{\curvearrowleft} := D \left[\begin{smallmatrix} F^{\curvearrowleft} \\ I \end{smallmatrix}\right] \in \mathcal{B}(W, Y^{\mathrm{s}})$, and $D^{\curvearrowleft *} J D^{\curvearrowleft} \ll 0$ (on W).

(b) If (i) (or (ii)) holds, then D is J-coercive.

(c) Assume that (i) (or (ii)) holds and $\mathcal{U}(x) \neq \emptyset$ for all $x \in X$. Then there are $F^{\curvearrowleft} \in \mathcal{B}(W,U)$, $K_u^{\curvearrowleft} \in \mathcal{B}(X,U)$, $K_{\mathrm{crit}}^{\mathrm{w}} \in \mathcal{B}(X,W)$ s.t.

$$\begin{bmatrix} K_u^{\curvearrowleft} & F^{\curvearrowleft} \end{bmatrix} \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right] \in \mathcal{U}_{\mathrm{u}}(x,w) \qquad (x \in X,\ w \in W), \tag{11.129}$$

$$\mathcal{J}(x, K_u^{\curvearrowleft} x + F^{\curvearrowleft} w, w) = \min_{u \in \mathcal{U}_{\mathrm{u}}(x,w)} \mathcal{J}(x,u,w) \quad (x \in X,\ w \in W), \quad \text{and} \tag{11.130}$$

$$\min_{u \in \mathcal{U}_{\mathrm{u}}(x, K_{\mathrm{crit}}^{\mathrm{w}} x)} \mathcal{J}(x,u,K_{\mathrm{crit}}^{\mathrm{w}} x) = \max_{w \in W} \min_{u \in \mathcal{U}_{\mathrm{u}}(x,w)} \mathcal{J}(x,u,w) \quad (x \in X). \tag{11.131}$$

Moreover, then for any $x \in X$, the unique J-critical input is given by Kx, where $K := \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} := \begin{bmatrix} K_u^{\curvearrowleft} + F^{\curvearrowleft} K_{\mathrm{crit}}^{\mathrm{w}} \\ K_{\mathrm{crit}}^{\mathrm{w}} \end{bmatrix}$, and the corresponding cost is

$$\mathcal{J}(x, K_1 x, K_2 x) = \max_{w \in W} \min_{u \in \mathcal{U}_{\mathrm{u}}(x,w)} \mathcal{J}(x,u,w) \geq \min_{u \in \mathcal{U}_{\mathrm{u}}(x,0)} \langle Cx + D_1 u, J(Cx + D_1 u)\rangle. \tag{11.132}$$

In the FICP (see (11.1), we have $J \geq 0$ on $\mathrm{Ran}(\begin{bmatrix} C & D_1 \end{bmatrix})$, so that $\mathcal{J}(x, K_1 x, K_2 x) \geq 0$ for all $x \in X$, by (11.132); this leads to a nonnegative J-critical cost operator $\mathcal{P}$ (see Proposition 11.2.19). Condition (11.128) is the standard nonsingularity assumption, and it is necessary for the J-coercivity stated in the theorem.

Here $F^{\curvearrowleft}$ and $K_u^{\curvearrowleft}$ refer to a system where only u is controlled ("the optimal state feedback through the control input u"), whereas the J-critical "maximin state feedback" K controls both u and w. By (11.132), (i) implies the J-critical control corresponds to "the best control u under the worst disturbance w" (or "the maximin control $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$").

By slightly modifying the proof of (a), one can show that conditions (i) and (ii) hold iff D is J-coercive and $\min_{u \in \mathcal{U}_{\mathrm{u}}(0,w)} \mathcal{J}(0,u,w) \leq 0$ for all $w \in W$. Mere J-coercivity is not sufficient (take, e.g., $D = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right]^{\mathrm{T}}$, $J = \mathrm{diag}(1,1,-\gamma)$, $\gamma < 1$, so that $D^* J D = \begin{bmatrix} 1 & 0 \\ 0 & (1-\gamma) \end{bmatrix} \gg 0$), we must also know that the cost function is concave in w (it is convex in u, by (11.128)). If $\mathcal{U}(x) \neq \emptyset$ for all $x \in H$ ("the Finite Cost

Condition is satisfied"), then a fourth equivalent condition is the existence of a J-critical K (as above) satisfying a suitable signature condition.

However, such additional conditions can be obtained in more useful forms when working with causal systems, therefore we have placed a further treatment in Sections 11.2 and 11.6.

Proof of Theorem 11.7.2: (a) 1° *(ii)⇒(i):* Assume (ii). Then there is $\varepsilon_{\mathrm{w}} > 0$ s.t. $\mathcal{J}(0, F^{\curvearrowleft}w, w) = \langle D^{\curvearrowleft}w, JD^{\curvearrowleft}w\rangle \le -\varepsilon_{\mathrm{w}}\|w\|^2$ for all $w \in W$, hence (i) holds.

2° *(i)⇒(ii):* Assume (i). Then $\mathcal{U}_{\mathrm{u}}(0,w) \neq \emptyset$ for all $w \in W$ (since $\inf\emptyset = +\infty$). The map D_1 with the system $\Sigma_{\mathrm{u}} := \left[\begin{array}{c|c} B_2 & B_1 \\ \hline D_2 & D_1 \end{array}\right]$ is J-coercive (this corresponds to substitutions $X \mapsto W$, $\mathcal{U}_*^* \mapsto \mathcal{U}_{\mathrm{u}}(0,\cdot)$ and $\mathcal{J}(w,u) \mapsto \mathcal{J}(0,u,w)$), by (11.128); indeed, we obviously have $\|u\|_{D_1} = \|\left[\begin{smallmatrix} u \\ 0 \end{smallmatrix}\right]\|_D$. Therefore, is a unique J-critical control $F^{\curvearrowleft}w$ for Σ_{u} for each $w \in W$, where

$$\begin{bmatrix} B^{\curvearrowleft} \\ D^{\curvearrowleft} \\ F^{\curvearrowleft} \end{bmatrix} \in \mathcal{B}(W, Z^{\mathrm{s}} \times Y^{\mathrm{s}} \times U) \tag{11.133}$$

is as in Theorem 8.1.10; in particular, $F^{\curvearrowleft}w \in \mathcal{U}_{\mathrm{u}}(0,w)$ and $\langle D_2 w + D_1 F^{\curvearrowleft}w, Du\rangle = 0$ for all $u \in \mathcal{U}_{\mathrm{u}}(0,w)$ and $w \in W$, and

$$\mathcal{J}(0, F^{\curvearrowleft}w, w) = \min_{u \in \mathcal{U}_{\mathrm{u}}(0,w)} \mathcal{J}(0,u,w) \le -\varepsilon_{\mathrm{w}}\|w\|^2 \text{ for each } w \in W. \tag{11.134}$$

Since $D^{\curvearrowleft} := D_2 + D_1 F^{\curvearrowleft} \in \mathcal{B}(W, Y^{\mathrm{s}})$, we obtain (ii) from (11.134) (because $\mathcal{J}(0, F^{\curvearrowleft}w, u) = \langle D^{\curvearrowleft}w, JD^{\curvearrowleft}w\rangle$). (Intuitively, $F^{\curvearrowleft}$ is the "optimal stabilizing controller" $w \mapsto u$.)

(b) Assume (i). We have $\begin{bmatrix} D_1 & D^{\curvearrowleft} \end{bmatrix} = DE$, where $E := \left[\begin{smallmatrix} I & F^{\curvearrowleft} \\ 0 & I \end{smallmatrix}\right]$, where $F^{\curvearrowleft}$ is as in (a)2°. Because $\left[\begin{smallmatrix} F^{\curvearrowleft}w \\ w \end{smallmatrix}\right] \in \mathcal{U}(0)$ for all $w \in W$, we have $E\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \in \mathcal{U}(0) \Leftrightarrow u \in \mathcal{U}_{\mathrm{u}}(0,0)$ & $w \in W$, by Lemma 8.1.4; thus, $\mathcal{U}(0) = E[\mathcal{U}_{\mathrm{u}}(0,0) \times W]$. From this and (a)2° we see that the assumptions of Lemma 11.7.3 are satisfied for $\left[\begin{array}{c|c} A & BE \\ \hline C & DE \end{array}\right]$, so that DE is J-coercive. By Lemma 8.2.4(c), it follows that also D is J-coercive.

(c) 1° $\mathcal{U}_{\mathrm{u}}(x,w) \neq \emptyset$ *for all* x *and* w*:* Given $x \in X$ and $w \in W$, there are $\begin{bmatrix} u' \\ w' \end{bmatrix} \in \mathcal{U}(x)$ and $u'' \in \mathcal{U}_{\mathrm{u}}(0, w - w')$, by the assumptions, so that $\left[\begin{smallmatrix} u' + u'' \\ w \end{smallmatrix}\right] \in \mathcal{U}(x)$, i.e., $u' + u'' \in \mathcal{U}_{\mathrm{u}}(x,w)$. Thus, $\mathcal{U}_{\mathrm{u}}(x,w) \neq \emptyset$ for all x and w.

2° *Equations (11.129) and (11.130):* These are obtained as in (a)2° (with substitutions $X \mapsto \left[\begin{smallmatrix} X \\ W \end{smallmatrix}\right]$, $x \mapsto \left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]$, $\mathcal{U}_*^*(\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right]) \mapsto \mathcal{U}_{\mathrm{u}}(x,w)$ and $\mathcal{J}(\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right], u) \mapsto \mathcal{J}_{\mathrm{u}}(\left[\begin{smallmatrix} x \\ w \end{smallmatrix}\right], u) := \mathcal{J}(x,u,w)$). (Obviously, $F^{\curvearrowleft}$ is the same as in (a)2°.)

3° *Equation (11.131):* We also obtain in 2° that

$$\Sigma_{\mathrm{ext}}^{\curvearrowleft} := \left[\begin{array}{c|c} A^{\curvearrowleft} & B^{\curvearrowleft} \\ \hline C^{\curvearrowleft} & D^{\curvearrowleft} \end{array}\right] := \left[\begin{array}{c|c} A + B_1 K_u^{\curvearrowleft} & B_2 + B_1 F^{\curvearrowleft} \\ \hline C + D_1 K_u^{\curvearrowleft} & D_2 + D_1 F^{\curvearrowleft} \end{array}\right] \in \mathcal{B}(X \times W, Z \times Y^{\mathrm{s}}) \tag{11.135}$$

and $D^{\curvearrowleft *} J D^{\curvearrowleft} \ll 0$.

Apply Corollary 8.2.7 and Theorem 8.1.10 to $\Sigma_{\mathrm{ext}}^{\curvearrowleft}$ and $-J$ (with cost function $=: -\mathcal{J}^{\curvearrowleft}$) to obtain that there is $K_{\mathrm{crit}}^{\mathrm{w}} \in \mathcal{B}(X, W)$ s.t.

$$\mathcal{J}^{\curvearrowleft}(x, K_{\mathrm{crit}}^{\mathrm{w}} x) = \max_{w \in W} \mathcal{J}^{\curvearrowleft}(x,w) = \max_{w \in W} \min_{u \in \mathcal{U}(x,w)} \mathcal{J}(x,u,w) \quad (x \in X). \tag{11.136}$$

4° "$\begin{bmatrix} K_u^{\frown} + F^{\frown} K_{\mathrm{crit}}^{\mathrm{w}} \\ K_{\mathrm{crit}}^{\mathrm{w}} \end{bmatrix} x$ *is the unique J-critical control for any* $x \in X$"*:* Let $x \in X$. By (b), D is J-coercive, so that there is a unique J-critical control for x, by Theorem 8.2.5. Thus, we only have to show that this control is given by $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] := \begin{bmatrix} K_u^{\frown} + F^{\frown} K_{\mathrm{crit}}^{\mathrm{w}} \\ K_{\mathrm{crit}}^{\mathrm{w}} \end{bmatrix} x$.

Let $\begin{bmatrix} \eta \\ \eta' \end{bmatrix} \in \mathcal{U}(0)$ be arbitrary. Set $\eta'' := \eta - F^{\frown}\eta' = \eta - u_{\mathrm{crit}}(0, \eta')$. Apply Lemma 8.1.7(ii) to $\mathcal{J}_{\mathrm{u}}$, to $\mathcal{J}_{\mathrm{w}}$ and then again to $\mathcal{J}_{\mathrm{u}}$ to obtain (here $u_{\mathrm{crit}}(x', w') := \begin{bmatrix} K_u^{\frown} & F^{\frown} \end{bmatrix} \begin{bmatrix} x' \\ w' \end{bmatrix}$ for all x', w') that $\mathcal{J}(x, u + \eta, w + \eta')$

$$= \mathcal{J}(x, K_u^{\frown} x + F^{\frown}(w + \eta') + \eta'', w + \eta') \tag{11.137}$$
$$= \mathcal{J}_{\mathrm{u}}(\left[\begin{smallmatrix} x \\ w+\eta' \end{smallmatrix}\right], u_{\mathrm{crit}}(x, w + \eta')) + \mathcal{J}_{\mathrm{u}}(0, \eta'') \tag{11.138}$$
$$= \mathcal{J}^{\frown}(x, w + \eta') + \mathcal{J}_{\mathrm{u}}(0, \eta'') = \mathcal{J}^{\frown}(x, w) + \mathcal{J}^{\frown}(0, \eta') + \mathcal{J}_{\mathrm{u}}(0, \eta'') \tag{11.139}$$
$$= \mathcal{J}(x, u, w) + \mathcal{J}_{\mathrm{u}}(\begin{bmatrix} 0 \\ \eta' \end{bmatrix}, u_{\mathrm{crit}}(0, \eta')) + \mathcal{J}_{\mathrm{u}}(0, \eta'') \tag{11.140}$$
$$= \mathcal{J}(x, u, w) + \mathcal{J}_{\mathrm{u}}(\begin{bmatrix} 0 \\ \eta' \end{bmatrix}, u_{\mathrm{crit}}(0, \eta') + \eta'') = \mathcal{J}(x, u, w) + \mathcal{J}(0, \eta, \eta'). \tag{11.141}$$

By "(ii)⇒(i)" of Lemma 8.1.7, $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ is J-critical for x.

5° *Equation (11.132):* This follows from (11.130) and (11.131) (the inequality follows by taking $w = 0$). □

The following result for "uncoupled" H$^\infty$ control was used above:

Lemma 11.7.3 *Assume that* $\mathcal{U}(0) = U_1 \times W$ *for some* $U_1 \subset U$, $D_2^* J D_2 \le 0$, $\langle D_1 u, D_2 w \rangle = 0$ *for all* $u \in U_1$, $w \in W$, $\varepsilon_{\mathrm{u}} > 0$, *and* $\langle D_1 u, J D_1 u \rangle \ge \varepsilon_{\mathrm{u}}(\|u\|_U^2 + \|B_1 u\|_{Z^{\mathrm{s}}}^2 + \|D_1 u\|_{Y^{\mathrm{s}}}^2)$ *for all* $u \in U$. *Then* D *is J-coercive iff* $D_2^* J D_2 \ll 0$.

Proof: Since $\mathcal{U}(0)$ is a subspace of $U \times W$, U_1 is a subspace of U. Since $B_2[W] \subset Z^{\mathrm{s}}$ and $D_2[W] \subset Y^{\mathrm{s}}$, we have $B_2 \in \mathcal{B}(W, Z^{\mathrm{s}})$ and $D_2 \in \mathcal{B}(W, Y^{\mathrm{s}})$, by Lemma A.3.6 (so that $D_2^* J D_2$ is well defined on W).

1° *"Only if":* If D is J-coercive, then, for each nonzero $w \in W$, there is a nonzero $w' \in W$ s.t. $\langle D_2 w', J D_2 w \rangle \ge \varepsilon' \|w\| \|w'\|$, where ε' is as in (8.12) (take $u = 0$). Consequently, then $D_2^* J D_2 \ll 0$. by Lemma A.3.4(c1)(xi)&(c4)&(b1).

2° *"If":* Assume that $D_2^* J D_2 \ll -\varepsilon_{\mathrm{w}} I$. Given $u \in U_1$ and $w \in W$, we have

$$\langle D\left[\begin{smallmatrix} u \\ -w \end{smallmatrix}\right], JD\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \rangle = \varepsilon_{\mathrm{u}}(\|u\|^2 + \|B_1 u\|^2 + \|D_1 u\|^2) + \varepsilon_{\mathrm{w}} \|w\|^2 \tag{11.142}$$
$$\ge \varepsilon'(\|u\|^2 + \|w\|^2 + \|\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]\|^2 + \|B_1 u\|^2 + \|B_2\|^2 \|w\|^2 + \|D_1 u\|^2 + \|D_2\|^2 \|w\|^2), \tag{11.143}$$

where $\varepsilon' := \min\{\varepsilon_{\mathrm{u}}, \varepsilon_{\mathrm{w}}\}$, from which one easily obtains (8.12). □

Notes

For finite-dimensional WPLSs, the idea to first minimize w.r.t. u and then maximize w.r.t. w is very old, although we do not know any references to the exact technique used in this section. See the previous sections for applications to WPLSs.

11.8 The Nehari problem

If you think the problem is bad now, just wait until we've solved it.
— Arthur Kasspe

In the Nehari problem, one wishes to determine the norm of the Hankel operator $\pi_+\mathbb{G}\pi_-$ of a given map $\mathbb{G} \in \mathrm{TIC}(W,U)$, or alternatively, to find a *suboptimal approximation* $-\mathbb{U}^* \in \mathrm{TIC}^*(W,U)$ of $\mathbb{G}$.

Given $\gamma > 0$, one has $\|\pi_+\mathbb{G}\pi_-\| < \gamma$ iff $\inf_{\mathbb{U}\in\mathrm{TIC}(U,W)} \|\mathbb{G} + \mathbb{U}^*\| < \gamma$, i.e., iff $d(\mathbb{G}, \mathrm{TIC}^*) < \gamma$ (or $d(\widehat{\mathbb{G}}, \mathrm{H}^\infty(\mathbf{C}^-; \mathcal{B}(W,U))) < \gamma$), by the Nehari Theorem 11.8.3. As the theorem shows, one can reduce the problem of finding such an operator $\mathbb{U}$ to a co-spectral factorization problem, and such a factorization leads to a parameterization of all solutions $\mathbb{U}$ of $\|\mathbb{G} + \mathbb{U}^*\| < \gamma$.

(Sometimes the problem is formulated so that $\mathbb{G}$ belongs to MTI or some other noncausal decomposing class. Then one may reduce the problem to the above one by replacing $\mathbb{G}$ with its causal part $\mathbb{G}_+$, where $\mathbb{G} = \mathbb{G}_+ + \mathbb{G}_-$, $\mathbb{G}_+, \mathbb{G}_-^* \in \mathrm{TIC}$.)

In this section, we take a very brief look at the aspects of the Nehari problem that closely resemble those of the stable H^∞ FICP.

Standing Hypothesis 11.8.1 *Throughout Sections 11.8–11.9, we assume the following:* $\Sigma := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{G} \end{array}\right] \in \mathrm{WPLS}(W,H,U)$ *is stable and* $\gamma > 0$.

Recall from Definition 6.1.6 that any $\mathbb{G} \in \mathrm{TIC}(W,U)$ has a realization Σ (for some H).

We shall show that the following conditions are equivalent (under some additional assumptions):

Definition 11.8.2 *We define* $\mathbb{D} := \begin{bmatrix} I & \mathbb{G} \\ 0 & I \end{bmatrix} \in \mathrm{TIC}(U \times W)$, *and we define the following conditions:*

(i) There is $\mathbb{U} \in \mathrm{TIC}(U,W)$ *s.t.* $\|\mathbb{G} + \mathbb{U}^*\|_{\mathcal{B}(\mathrm{L}^2)} < \gamma$ *(i.e.,* $d(\mathbb{G}, \mathrm{TIC}^*) < \gamma$*).*

(ii) The Hankel norm $\|\pi_+\mathbb{G}\pi_-\| = \rho(\mathbb{B}\mathbb{B}^*\mathbb{C}^*\mathbb{C})^{1/2}$ *of* $\mathbb{G}$ *is less than* γ.

(iii) The I/O map $\mathbb{D}$ *is co-minimax* J_γ*-coercive [Def11.4.1].*

(iv) The Toeplitz operator $\mathbb{F} := \pi_-\mathbb{D}^* J \mathbb{D}\pi_-$ *satisfies* $\mathbb{F}_{22} - \mathbb{F}_{21}\mathbb{F}_{11}^{-1}\mathbb{F}_{12} \ll 0$.

(v) The I/O map $\mathbb{D} := \begin{bmatrix} I & \mathbb{G} \\ 0 & I \end{bmatrix} \in \mathrm{TIC}$ *has a (co-spectral) factorization* $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X} J_1 \mathbb{X}^*$ *with* $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$.

We have $\pi_+\mathbb{G}\pi_- = \mathbb{C}\mathbb{B}$, by Definition 6.1.1, hence

$$\rho(\mathbb{B}\mathbb{B}^*\mathbb{C}^*\mathbb{C})^{1/2} = \rho(\mathbb{B}^*\mathbb{C}^*\mathbb{C}\mathbb{B})^{1/2} = \|\mathbb{C}\mathbb{B}\| = \|\pi_+\mathbb{G}\pi_-\|, \tag{11.144}$$

by Lemma A.3.3(s2), Lemma A.3.1(c6) and Definition 6.1.1. Note that $\mathbb{B}\mathbb{B}^*$ is the reachability Gramian and $\mathbb{C}^*\mathbb{C}$ is the observability Gramian of Σ. Except for this observation (and the equality in (ii)), our claims only concern the I/O map $\mathbb{G}$, not the realization Σ.

If $\mathbb{D} \in \mathrm{MTIC}$ and $\dim U \times W < \infty$, or if $\mathbb{D} \in \mathrm{MTIC}_{T\mathbf{Z}}$, then (i)–(v) are equivalent:

Theorem 11.8.3 (Nehari) *We have (v)⇒(i)⇒(ii)⇔(iii)⇔(iv). If U and W are separable, then (i)–(iv) are equivalent.*

If $\mathcal{A}^* = \text{Я}\mathcal{A}\text{Я} = \mathcal{A} \underset{a}{\subset} \mathrm{TI}$, $\mathbb{G} \in \mathcal{A}$, and $\mathcal{A}(U \times W)$ admits spectral factorization (e.g., (α) or (β) of Theorem 8.4.9 holds), then (i)–(v) are equivalent.

If (v) holds, then all solutions to (i) are the ones given by Theorem 11.9.4 (if $\mathbb{G} \in \mathcal{A}$, then $\mathbb{X} \in \mathcal{A}$, by Theorem 11.9.3, hence then we can take $\mathbb{U} \in \mathcal{A}$, by Theorem 11.9.4, and all solutions $\mathbb{U} \in \mathcal{A}$ correspond to all parameters $\widetilde{\mathbb{U}} \in \mathcal{A}$, as in Theorem 11.3.6).

By "$\mathcal{A}(U \times W)$ admits spectral factorization" we mean that if $\mathbb{E} \in \mathcal{A}(U \times W)$ and $\pi_+ \mathbb{E} \pi_+$ is (boundedly) invertible on $\mathrm{L}^2(\mathbf{R}_+; U \times W)$ (this condition is necessary for any $\mathcal{A} \underset{a}{\subset} \mathrm{TI}$), then $\mathbb{E} = \mathbb{X}^* S \mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W)$, $S \in \mathcal{G}\mathcal{B}(U \times W)$.

Proof: 1° The first chain of implications follows from Theorems 11.9.4, 11.9.2 and 11.9.1 and Definition 11.4.1.

2° Implication (ii)⇒(i) follows from Theorem 11.9.2.

3° By 1° and Theorem 11.9.3, (i)–(v) are equivalent. (From the proof of Theorem 8.4.9 we observe that each of assumptions (α) and (β) is sufficient for the admissibility assumption; obviously the former condition is satisfied, i.e., $\mathcal{A} \underset{a}{\subset} \mathrm{TI}$ and $\mathbb{E}^*, \text{Я}\mathbb{E}\text{Я} \in \mathcal{A}$ for each $\mathbb{E} \in \mathcal{A}$.) □

One observes directly from the definition that $\mathbb{D}$ is minimax J_γ-coercive. However, we have no use for this fact.

Notes

Above, we have primarily given frequency-domain results. One traditionally also establishes the connection to two particular Riccati equations having nonnegative solutions; see, e.g., [IOW] for the finite-dimensional case (both continuous and discrete time), [CZ] for the case with bounded B and C, and [CZ94] for the case of smooth Pritchard–Salamon systems. The above references only treat the exponentially stable case; see [CO98] for WPLSs with $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ stable and B, C bounded (but $\mathbb{A}$ possibly unstable; note also that $\mathbb{A}$ could be allowed to be unstable through Sections 11.8–11.9).

For $\dim U \times Y < \infty$, the Nehari problem has an extended version, the *Hankel norm approximation problem*, whose I/O form was presented and solved in [AAK] (the *Adamjan–Arov–Krein theorem*). An up-to-date state-space solution for exponentially stable analytic and PS-systems is given in [Sasane] (partially also in [SC]).

Since the Nehari Riccati equation theory does not follow from the theory of Chapter 9, we omit the state-space part corresponding to the above results; it might be worth of a separate study (where results such as Theorem 8.3.9 should be rewritten in this "noncausal" setting (e.g., we have $\mathbb{N}^* J \mathbb{N} = S$, where $\mathbb{N} := \mathbb{D}\mathbb{X}^{-*} \notin \mathrm{TIC} \cup \mathrm{TIC}^*$ in general)). Part of the state-space theory (including a realization of the spectral factor) is given in [SM], due to Olof Staffans.

The results of this section are well known in the generality treated in the above references (excluding [SM]), and the implication "(i)⇔(ii)" is contained in [Treil85] in the separable case.

11.9 The proofs for Section 11.8

HOW TO PROVE IT, PARTS 1–2:
Proof by intimidation: 'Trivial'.
Proof by vigorous handwaving: Works well in a classroom or seminar setting.
Proof by cumbersome notation: Best done with access to at least four alphabets and special symbols.

FURTHER PROOF TECHNIQUES:
Blatant assertion.
Changing all the 2's to n's.
Mutual consent.
Lack of a counterexample.

Now we can prove the implications compiled into Theorem 11.8.3:

Theorem 11.9.1 *(iii)⇔(ii).*

Proof: We have

$$\mathbb{F} := \pi_+ \mathbb{D}^* J \mathbb{D} \pi_+ = \begin{bmatrix} \pi_- & \pi_- \mathbb{G} \\ \mathbb{G}^* \pi_- & \pi_- \mathbb{G}^* \mathbb{G} \pi_- - \gamma^2 \pi_- \end{bmatrix}, \tag{11.145}$$

hence the co-minimax J-coercivity of $\mathbb{D}$ is equivalent to $0 \gg \pi_- \mathbb{G}^* \mathbb{G} \pi_- - \gamma^2 \pi_- - \mathbb{G}^* \pi_- \mathbb{G} = -(\gamma^2 \pi_- - \pi_- \mathbb{G}^* \pi_+ \mathbb{G} \pi_-)$, hence to $\gamma^2 \pi_- \gg \pi_- \mathbb{G}^* \pi_+ \mathbb{G} \pi_-)$, hence to $\|\pi_+ \mathbb{G} \pi_-\| < \gamma$, by Lemma A.1.1(d). □

Theorem 11.9.2 *We have (i)⇒(ii). The converse is true for separable U and W.*

Proof: 1° "(i)⇒(ii)": Because $\pi_+ \mathbb{U}^* \pi_- = 0$, we have

$$\|\pi_+ \mathbb{G} \pi_-\| = \|\pi_+ (\mathbb{G} + \mathbb{U}^*) \pi_-\| \leq \|\mathbb{G} + \mathbb{U}^*\| \tag{11.146}$$

for $\mathbb{U} \in \mathrm{TIC}$, hence (i) implies (ii).

2° "(ii)⇒(i)": By, e.g., the Theorem on p. 57 of [Treil85] (and Theorem 13.2.3), this is true at least in the separable case, and (in the sepable case) we also have that

$$\|\pi_+ \mathbb{G} \pi_-\| = \inf_{\widehat{\mathbb{U}}(-\cdot) \in \mathrm{H}^\infty} \|\widehat{\mathbb{G}} + \widehat{\mathbb{U}}\|_\infty = \inf_{\mathbb{U} \in \mathrm{TIC}} \|\mathbb{G} + \mathbb{U}^*\|_{\mathcal{B}(\mathrm{L}^2)} = \min_{\mathbb{U} \in \mathrm{TIC}} \|\mathbb{G} + \mathbb{U}^*\|_{\mathcal{B}(\mathrm{L}^2)}. \tag{11.147}$$

□

Theorem 11.9.3 *Assume that $\mathcal{A}^* = \text{Я}\mathcal{A}\text{Я} = \mathcal{A} \underset{a}{\subset} \mathrm{TI}$, that $\mathbb{G} \in \mathcal{A}$, and that $\mathcal{A}(U \times W)$ admits spectral factorization. Then (iii) implies (v) (with $\mathbb{X} \in \mathcal{G}\mathcal{A}$).*

Proof: Assume (iii). Then, by Lemma 11.4.2, $\pi_- \mathbb{D}^* J \mathbb{D} \pi_- \in \mathcal{G}\mathcal{B}(\mathrm{L}^2(\mathbf{R}_-; U \times W))$, i.e., $\pi_+ \mathbb{D}^{\mathrm{d}} J (\mathbb{D}^{\mathrm{d}})^* \pi_+ \in \mathcal{G}\mathcal{B}(\mathrm{L}^2(\mathbf{R}_+; U \times W))$. Consequently, there are $\mathbb{X}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}(U \times W) \cap \mathcal{A}(U \times W)$ and $S \in \mathcal{G}\mathcal{B}(U \times W)$ s.t. $(\mathbb{X}^{\mathrm{d}})^* S \mathbb{X}^{\mathrm{d}} = \mathbb{D}^{\mathrm{d}} J (\mathbb{D}^{\mathrm{d}})^*$, i.e., $\mathbb{X} S \mathbb{X}^* = \mathbb{D}^* J \mathbb{D}$.

By Corollary 11.4.7, we can have $S = J_1$ and $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$. $\square$

All solutions to $\|\mathbb{G} + \mathbb{U}^*\| < \gamma$ are given by the standard formula (cf. Theorem 3.2 of [CO98]):

Theorem 11.9.4 (All solutions) *Let the I/O map* $\mathbb{D} := \left[\begin{smallmatrix} I & \mathbb{G} \\ 0 & I \end{smallmatrix}\right] \in \mathrm{TIC}$ *have a co-spectral factorization* $\mathbb{D}^* J_\gamma \mathbb{D} = \mathbb{X} J_1 \mathbb{X}^*$ *with* $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ *(i.e., let (v) hold). Then* $\mathbb{U}^* := \mathbb{M}_{21}^* \mathbb{M}_{22}^{-*} = -\mathbb{X}_{11}^{-*} \mathbb{X}_{21}^*$ *satisfies* $\|\mathbb{G} + \mathbb{U}^*\| < \gamma$*, where* $\mathbb{M} := \mathbb{X}^{-1}$*; in particular, (i) holds.*

Moreover, in that case, all solutions to $\|\mathbb{G} + \mathbb{U}^*\| < \gamma$ *[*$\le \gamma$*] are given by* $\mathbb{U}^* := \mathbb{U}_1^* \mathbb{U}_2^{-*}$, $\left[\begin{smallmatrix} \mathbb{U}_1^* \\ \mathbb{U}_2^* \end{smallmatrix}\right] = \mathbb{M}^* \left[\begin{smallmatrix} \widetilde{\mathbb{U}}^* \\ I \end{smallmatrix}\right]$, $\widetilde{\mathbb{U}} \in \mathrm{TIC}$ *and* $\|\widetilde{\mathbb{U}}\| < 1$ *[*≤ 1*].*

Thus, all solutions are given by $\mathbb{U} = \mathbb{U}_2^{-1} \mathbb{U}_1$ (clearly a l.c.f.), $\begin{bmatrix} \mathbb{U}_1 & \mathbb{U}_2 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{U}} & I \end{bmatrix} \mathbb{M}$, $\widetilde{\mathbb{U}} \in \mathrm{TIC}$ and $\|\widetilde{\mathbb{U}}\| < 1$ [≤ 1].

Proof: By Corollary 11.4.6, we have $\|\mathbb{X}_{11}^{-1} \mathbb{X}_{12}\| < 1$, therefore, the proof of Theorem 11.3.6 applies, with substitutions $\mathbb{U} \mapsto \mathbb{U}^*$, $\widetilde{\mathbb{U}} \mapsto \widetilde{\mathbb{U}}^*$, $\mathbb{X} \mapsto \mathbb{X}^*$, $\mathbb{M} \mapsto \mathbb{M}^*$, $\left[\begin{smallmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} \\ \mathbb{X}_{21} & \mathbb{X}_{22} \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} \mathbb{X}_{11}^* & \mathbb{X}_{21}^* \\ \mathbb{X}_{12}^* & \mathbb{X}_{22}^* \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} \mathbb{M}_{11}^* & \mathbb{M}_{21}^* \\ \mathbb{M}_{12}^* & \mathbb{M}_{22}^* \end{smallmatrix}\right]$, $\mathbb{U}_k \mapsto \mathbb{U}_k^*$, $\widetilde{\mathbb{U}}_k \mapsto \widetilde{\mathbb{U}}_k^*$, $\mathbb{Q}_k \mapsto \mathbb{Q}_k^*$ $(k = 1, 2)$, $(W, U) \mapsto (U, W)$ and $\mathrm{TIC} \mapsto \mathrm{TIC}^*$ (or $\mathrm{TIC} \mapsto \mathrm{TI}$, since causality is not needed in the proof). $\square$

(See the notes on p. 686.)

Chapter 12

H^∞ Four-Block Problem ($\|\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})\| < \gamma$)

What is now proved was once only imagin'd.
— William Blake (1757–1827)

The H^∞ Four-Block Problem (H^∞ 4BP) is presented on p. 36. In Section 12.1 we solve the 4BP, in Section 12.3 we solve the corresponding frequency-domain problem, and in Section 12.2 we treat the discrete-time counterparts of these problems. The rest of this chapter consists of proofs and minor results.

Our main contributions are the solution of the continuous-time H^∞ problem in terms of two independent Riccati equations and a coupling condition (Theorems 12.1.4 and 12.1.5, under different assumptions) and the parametrization of all sub-optimal controllers (Theorem 12.1.8), as well as the corresponding discrete-time results (including Theorem 12.2.1). Another important result is the factorization solution of the same problem (Theorem 12.3.6 solves the frequency domain problem for MTIC systems, Theorem 12.3.7 (partially) for more general ones; Theorem 12.3.5 connects these solutions to the state-space solution).

In this chapter, $\widetilde{\mathcal{A}}$ stands for $\mathrm{MTIC}_{T\mathbf{Z}}$ or for some other suitable class:

Standing Hypothesis 12.0.1 *Throughout this chapter we assume that H,U,W,Y,Z are Hilbert spaces and the spaces $\widetilde{\mathcal{A}}(U \times W)$, $\widetilde{\mathcal{A}}(Y \times U)$ and $\widetilde{\mathcal{A}}(Y \times Z)$ satisfy Hypothesis 8.4.7 and that $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}^{\mathrm{d}}$.*

(Cf. Theorem 8.4.9(c), Lemma 14.3.5 and Definition 6.2.4.)

Note also that Hypothesis 12.3.1 is assumed through Sections 12.3–12.4 and Hypothesis 12.1.1 through Sections 12.1, 12.2, 12.5 and 12.6.

In this chapter, we allow the controllers to have internal loops unless we use the term "well-posed controller"; cf. Figures 7.8 and 7.10 (or Figures 7.9 and 7.11)). We also often drop the prefix "DPF-", since we study no other controllers than DPF-controllers in this chapter (see Section 7.3).

12.1 The standard H^∞ problem (H^∞ 4BP)

Not every problem is necessarily due to the capitalist mode of production.
— Herbert Marcuse

We assume the dynamics of form (1.25) (see p. 37):

Standing Hypothesis 12.1.1 *Throughout Sections 12.1, 12.2, 12.5 and 12.6, we assume that* $\gamma > 0$ *and*

$$\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{C}_2 & \mathbb{D}_{21} & \mathbb{D}_{22} \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Z \times Y). \qquad (12.1)$$

Definition 12.1.2 (Suboptimal controller) *A stabilizing DPF-controller for* Σ *is called* suboptimal *if it makes the norm of the map* $w \mapsto z$ *less that* γ*.*

See Definition 7.3.1 for stabilizing Dynamic Partial Feedback (DPF) controllers for Σ (recall that in this chapter the words "with internal loop" are usually omitted unlike in Chapter 7).

The map $w \mapsto z$ is usually denoted by $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})$, where $\mathbb{Q}$ is the I/O-map of the controller; see Corollary 7.3.20 (or Lemma 12.3.2) for $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})$. As noted on p. 321, we have $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q}) = \mathbb{D}_{12} + \mathbb{D}_{11}\mathbb{Q}(I - \mathbb{D}_{21}\mathbb{Q})\mathbb{D}_{22}$ for well-posed $\mathbb{Q}$. In the literature, often "w comes before u", hence the latter subindices should be interchanged in the formula for $\mathcal{F}_\ell$ for comparison.

We shall show in Theorems 12.1.4 and 12.1.5 that, under standard coercivity assumptions and certain regularity assumptions, the existence of such a controller is equivalent to the standard CARE, signature and coupling conditions. Moreover, in Theorem 12.1.8 we parametrize all such controllers and show that all of them are well-posed when $D_{21} = 0$ (as one usually assumes). Thus, for most readers it suffices to consider well-posed controllers only. We also make some less important remarks on the problem under different assumptions.

Recall from Definition 7.3.1 that $\widetilde{\Sigma}$ being an exponentially stabilizing DPF-controller means that the closed-loop system in Figure 7.9 (or 7.11) is exponentially stable, i.e., that all maps between the signals in this figure are exponentially stable. By Lemma 6.1.10, this is the case iff the corresponding closed-loop semigroup is exponentially stable.

Thus, our concept of a suboptimal exponentially stabilizing DPF-controller is a direct generalization of the standard concept (the "suboptimal admissible controller" of [ZDG], "γ-admissible controller" of [Keu], or "stabilizing γ-contracting controller" of [IOW]).

In most applications of Definition 12.1.2, we can show that Hypothesis 12.3.1 is satisfied. See Proposition 7.3.4 and Lemma 12.5.7 for $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})$ in the general case.

We first note that the problem of this section contains that of Section 12.3:

Lemma 12.1.3 (exp-4BP ⇒ I/O-4BP) *The I/O map of any suboptimal [exponentially] stabilizing DPF-controller for* Σ *is a suboptimal [exponentially] stabilizing DPF-controller for* $\mathbb{D}$. □

(This is trivial; see Definitions 12.3.3 and 7.3.1 (cf. Theorem 7.3.11(c1)).)

In the first form of our solution we shall assume "B_{w}^*-CARE" type regularity to obtain simpler Riccati equations and have the solution look exactly like its finite-dimensional counterparts (with the exception of weak Weiss extensions B_{w}^* and C_{w} in place of B^* and C):

Theorem 12.1.4 ($\mathbf{H}^\infty$ 4BP ⇔ B_{w}^*-CAREs)

(A1) **(Regularity)** *Assume that at least one of (I)–(V) holds, where*

(I) **(Analytic $\mathbb{A}$)** *1. A generates an analytic semigroup on H, $B_1 \in \mathcal{B}(U, H_{\beta_1})$, $B_2 \in \mathcal{B}(W, H_{\beta_2})$, $C_1 \in \mathcal{B}(H_{\gamma_1}, Z)$, $C_2 \in \mathcal{B}(H_{\gamma_2}, Y)$, $D \in \mathcal{B}(U \times W, Z \times Y)$, $\beta_k, \gamma_k \in (-1/2, 1/2)$ $(k = 1,2)$; 2. $\gamma_1 < 1/4$ or $\gamma_1 - \min\{\beta_1, \beta_2\} < 1/2$; and 3. $\beta_2 > -1/4$ or $\max\{\gamma_1, \gamma_2\} - \beta_2 < 1/2$;*

(II) B is bounded (i.e., $B \in \mathcal{B}(U \times W, H)$) and $\pi_{[0,1)} C_{\mathrm{w}} \mathbb{A} \in \mathrm{L}^1([0,1); \mathcal{B}(H, Z \times Y))$;

(III) $\pi_{[0,1)} \mathbb{A} B \in \mathrm{L}^1([0,1]; \mathcal{B}(U \times W, H))$ and $C \in \mathcal{B}(H, Z \times Y)$;

(IV) $\pi_{[0,1)} \mathbb{A} B v_0 \in \mathrm{L}^2([0,1]; H)$, $\pi_{[0,1)} \mathbb{A}^* C^* t_0 \in \mathrm{L}^2([0,1]; H)$, $\pi_{[0,1)} C_{\mathrm{w}} \mathbb{A} B v_0 \in \mathrm{L}^2([0,1]; Z \times Y)$, $\pi_{[0,1)} B_{\mathrm{w}}^* \mathbb{A}^* C^* t_0 \in \mathrm{L}^2([0,1]; U \times W)$ for all $v_0 \in U \times W$, $t_0 \in Z \times Y$ (equivalently, $(\cdot - A)^{-1} B, (\cdot - A^*)^{-1} C^*$, $C_{\mathrm{w}} (\cdot - A)^{-1} B, B_{\mathrm{w}}^* (\cdot - A^*)^{-1} C^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(*,*))$ for some $\omega \in \mathbf{R}$);

(V) $\mathbb{A}$ is exponentially stable and $\widehat{\mathbb{D}} - D, \widehat{\mathbb{D}}(\bar{\cdot})^* - D^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(*,*))$ for some $\omega < 0$.

(A2) **(Nonsingularity)** *Assume that $D_{11}^* D_{11} \gg 0$ and $D_{22} D_{22}^* \gg 0$, and that there is $\varepsilon > 0$ s.t.*

$$(ir - A)x_0 = B_1 u_0 \Longrightarrow \|C_{1\mathrm{w}} x_0 + D_{11} u_0\|_Z \geq \varepsilon \|x_0\|_H \quad \text{and} \tag{12.2}$$

$$(ir - A^*)x_0 = C_2^* y_0 \Longrightarrow \|B_{2\mathrm{w}}^* x_0 + D_{22}^* y_0\|_W \geq \varepsilon \|x_0\|_H \tag{12.3}$$

for all $x_0 \in H$, $u_0 \in U$, $y_0 \in Y$, $r \in \mathbf{R}$.

Then there is a suboptimal exponentially stabilizing DPF-controller for Σ (possibly with internal loop) iff (1.)–(3.) hold:

(1.) **($\mathcal{P}_{\mathrm{X}}$-CARE)** *$D_{12}^* D_{12} - D_{12}^* D_{11} (D_{11}^* D_{11})^{-1} D_{11}^* D_{12} \ll \gamma^2 I$, and the ($B_{\mathrm{w}}^*$-)CARE*

$$\begin{cases} K_X^* S_X K_X = A^* \mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}} A + C_1^* C_1, \\ S_X = \begin{bmatrix} D_{11}^* D_{11} & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} - \gamma^2 I \end{bmatrix}, \\ K_X = -S_X^{-1} \left(\begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 + B_{\mathrm{w}}^* \mathcal{P}_{\mathrm{X}} \right), \end{cases} \tag{12.4}$$

has a solution $(\mathcal{P}_{\mathrm{X}}, S_X, K_X) \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*)) \times \mathcal{B}(U \times W) \times \mathcal{B}(H_1, U \times W)$ s.t. $\mathcal{P}_{\mathrm{X}} \geq 0$, and the semigroup generated by $A + BK_X$ is exponentially stable.

(2.) **($\mathcal{P}_{\mathrm{Y}}$-CARE)** $D_{12}D_{12}^* - D_{12}D_{22}^*(D_{22}D_{22}^*)^{-1}D_{22}D_{12}^* \ll \gamma^2 I$, *and the ($B_{\mathrm{w}}^*$-)CARE*

$$\begin{cases} K_Y^* S_Y K_Y = A\mathcal{P}_{\mathrm{Y}} + \mathcal{P}_{\mathrm{Y}}A^* + B_2B_2^*, \\ \quad S_Y = \begin{bmatrix} D_{22}D_{22}^* & D_{22}D_{12}^* \\ D_{12}D_{22}^* & D_{12}D_{12}^* - \gamma^2 I \end{bmatrix}, \\ \quad K_Y = -S_Y^{-1}\left(\begin{bmatrix} D_{22} \\ D_{12} \end{bmatrix} B_2^* + \begin{bmatrix} C_{2\mathrm{w}} \\ C_{1\mathrm{w}} \end{bmatrix} \mathcal{P}_{\mathrm{Y}} \right), \end{cases} \tag{12.5}$$

has a solution $(\mathcal{P}_{\mathrm{Y}}, S_Y, K_Y) \in \mathcal{B}(H, \mathrm{Dom}(\begin{bmatrix} C_{2\mathrm{w}} \\ C_{1\mathrm{w}} \end{bmatrix})) \times \mathcal{B}(Y \times Z) \times \mathcal{B}(H_1^*, Y \times Z)$ *s.t.* $\mathcal{P}_{\mathrm{Y}} \geq 0$, *and the semigroup generated by* $A^* + \begin{bmatrix} C_2^* & C_1^* \end{bmatrix} K_Y$ *is exponentially stable.*

(3.) **(Coupling condition)** $\rho(\mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}}) < \gamma^2$.

(We recall from Theorem 9.8.12(a) that any exponentially stabilizing solutions of Riccati equations are unique.)

Assume that (1.)–(3.) are satisfied. Then the following hold:

(a) All suboptimal exponentially stabilizing DPF-controllers for Σ *are the ones parametrized in Theorem 12.1.8 (for* $\widetilde{\mathcal{A}} = \mathrm{MTIC}_{\mathrm{exp}}^{\mathrm{L}^1}$ *in cases (I)–(IV) and* $\widetilde{\mathcal{A}}$*=Theorem 8.4.9(*γ'*) in case (V)); note that if* $D_{21} = 0$*, then all of them are well-posed. Moreover, also condition (4.) of Theorem 12.1.8 can be written as a* B_{w}^**-CARE, i.e., as follows:*

(4.) **($\mathcal{P}_{\mathrm{Z}}$-CARE)** *For some (equivalently, all)* $X \in \mathcal{GB}(U \times W)$ *s.t.* $X_{21} = 0$ *and* $S = X^* J_1 X$*, the CARE*

$$\begin{cases} K_Z^* S_Z K_Z = A_Z^*\mathcal{P}_{\mathrm{Z}} + \mathcal{P}_{\mathrm{Z}}A_Z + B_2 X_{22}^{-1} X_{22}^{-*} B_2^*, \\ \quad S_Z = D_Z^* J_1 D_Z, \\ \quad K_Z = -S_Z^{-1}\left(\begin{bmatrix} D_{22}X_{22}^{-1} \\ X_{12}X_{22}^{-1} \end{bmatrix} X_{22}^{-*} B_2^* + (B_Z^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{Z}} \right), \end{cases} \tag{12.6}$$

has a solution $(\mathcal{P}_{\mathrm{Z}}, S_Z, K_Z) \in \mathcal{B}(H, \mathrm{Dom}((B_Z^*)_{\mathrm{w}})) \times \mathcal{B}(Y \times Z) \times \mathcal{B}(\mathrm{Dom}(A_Z), Y \times U)$ *s.t.* $\mathcal{P}_{\mathrm{Z}} \geq 0$, $S_{Z11} \gg 0$, $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$ *and the semigroup generated by* $A_Z + B_Z K_Z$ *is exponentially stable.*

(b) The systems Σ_{X}, Σ_{Y}, Σ_{Z}, $\Sigma_{\mathbb{E}^{\mathrm{d}}}$, $\Sigma_{\mathbb{T}}$ *and* Σ_{alt} *satisfy the assumptions of Lemma 6.8.5 for* $p = 1 = q$ *if any of (I)–(IV) holds (and for* $p = 2 = q$ *if (IV) holds). If (I) holds, then they also satisfy Hypothesis 9.5.1.*

(See Corollary 9.5.12(b) for additional smoothness for case (I). By Corollary 9.5.12(a), conditions "2." and "3." may be omitted from (I) if we write (1.)–(2.) to the form of Theorem 12.1.5.)

Under the normalizing conditions

$$D_{12} = 0, \qquad D_{11}^* \begin{bmatrix} C_1 & D_{11} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}, \tag{12.7}$$

condition (1.) can be written as follows:

$$((B_1^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}})^*(B_1^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} - \gamma^{-2}((B_2^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}})^*(B_2^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} = A^*\mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}}A + C_1^*C_1 \tag{12.8}$$

with the requirements that $\mathcal{P}_{\mathrm{X}} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$, $\mathcal{P}_{\mathrm{X}} \geq 0$, and $A + (\gamma^{-2}B_2(B_2^*)_{\mathrm{w}} - B_1(B_1^*)_{\mathrm{w}})\mathcal{P}_{\mathrm{X}}$ is exponentially stable. (Note that now $S_X = J_\gamma := \left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right]$ and $K_X = \left[\begin{smallmatrix} -(B_1^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} \\ \gamma^{-2}(B_2^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}} \end{smallmatrix}\right] \in \mathcal{B}(H, U \times W)$.) If B is bounded, then (12.8) takes the classical form

$$\mathcal{P}_{\mathrm{X}}(B_1B_1^* - \gamma^{-2}B_2B_2^*)\mathcal{P}_{\mathrm{X}} = A^*\mathcal{P}_{\mathrm{X}} + \mathcal{P}_{\mathrm{X}}A + C_1^*C_1. \tag{12.9}$$

See p. 618 for further simplification and remarks. Analogous remarks apply to (2.) and (4.). We thus observe that the classical H^∞ CAREs become special cases of (1.) and (2.) (cf. p. 618).

Recall from Definition 9.8.1, that each of these $\mathcal{P}_{\mathrm{X}}$-CAREs is given on $\mathcal{B}(H_1, H_{-1}^*) := \mathcal{B}(\mathrm{Dom}(A), \mathrm{Dom}(A^*)^*)$; e.g., (12.8) holds iff

$$\begin{aligned} &\langle (B_1^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}}x_0, (B_1^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}}x_1\rangle - \gamma^{-2}\langle (B_2^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}}x_0, (B_2^*)_{\mathrm{w}}\mathcal{P}_{\mathrm{X}}x_1\rangle \\ &\qquad = \langle Ax_0, \mathcal{P}_{\mathrm{X}}x_1\rangle + \langle \mathcal{P}_{\mathrm{X}}x_0, Ax_1\rangle + \langle C_1x_0, C_1x_1\rangle \end{aligned} \tag{12.10}$$

for all $x_0, x_1 \in \mathrm{Dom}(A)$ (we can take $x_1 = x_0$ w.l.o.g., by Lemma A.3.1(g1)). Analogously, the $\mathcal{P}_{\mathrm{Y}}$-CAREs are given on $\mathrm{Dom}(A^*)$ and the $\mathcal{P}_{\mathrm{Z}}$-CAREs (see Theorem 12.1.8) on $\mathrm{Dom}(A_Z)$.

See the remark in the proof for weakening (A1) (e.g., by assuming that Hypothesis 9.2.1 is satisfied by certain systems with J_γ or J_1). See Remark 12.1.7 for several equivalent conditions for (A2).

One usually assumes that (A, B_1) is exponentially stabilizable and (A, C_2) is exponentially detectable. By the above, such assumptions are necessary but redundant under (A1)–(A2) (or somewhat weaker analogous assumptions, see Theorem 12.1.5 or Theorem 12.2.1) .

Note that in "$(\mathcal{P}_{\mathrm{X}}, S_X, K_X) \in$ ..." only "$\mathcal{P}_{\mathrm{X}} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$" is a requirement; the other two conditions are automatically satisfied whenever $\mathcal{P}_{\mathrm{X}} \in \mathcal{B}(H, \mathrm{Dom}(B_{\mathrm{w}}^*))$ and S_X and K_X are determined by the second and third equations of (12.4). An analogous remark applies to (2.) and (4.) (and to any other B_{w}^*-CARE).

Proof of Theorem 12.1.4: 0.1° *Remarks on (I)–(V):* Conditions (I) says that Hypothesis 9.5.1 holds for Σ (hence for Σ_{X} and Σ_{Y} too) and that Hypothesis 9.5.7(2.) (without the D^*JD condition) is satisfied by Σ_{X} and Σ_{Y} (but not necessarily by Σ). The assumptions in (II)–(V) could be weakened analogously.

Even without the standing hypotheses, (I) implies that $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ generate an ULR WPLS, by Lemma 6.3.15. For (II) (resp. (III)), we must add the conditions (4.) (resp. (2.)) and (1.) of Lemma 6.3.15. Conditions (IV) and (V) are far from sufficient to guarantee the axioms of WPLSs.

The $\mathrm{H}^2_{\mathrm{strong}}$-condition in (V) could be rephrased as "$\mathbb{D} - D \in \mathcal{B}(U \times W, \mathrm{L}^2(\mathbf{R}_+; Z \times Y))*$ and $\mathbb{D}^{\mathrm{d}} - D \in \mathcal{B}(Z \times Y, \mathrm{L}^2(\mathbf{R}_+; U \times W))*$", i.e., as "$C_{\mathrm{w}}\mathbb{A}Bv_0 \in \mathrm{L}^2(\mathbf{R}_+; Z \times Y)$ and $B_{\mathrm{w}}^*\mathbb{A}^*C^*\widetilde{y}_0 \in \mathrm{L}^2(\mathbf{R}_+; U \times W))*$ for all $v_0 \in U \times W$ and $\widetilde{y}_0 \in Z \times Y$".

0.2° *Remark: alternatives for (A1):* In Theorem 12.1.4, an alternative for (A1) would be to assume that $(\Sigma_{\mathrm{X}}, J_\gamma)$ and $(\Sigma_{\mathrm{Y}}, J_\gamma)$ have the property of Remark 9.9.14(c).

Then (1.)–(3.) become necessary, by Lemma 12.1.13, but sufficiency would require a slightly stronger assumption to guarantee that also $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ or Σ_{Z} is as smooth (whenever (1.) holds).

An even weaker assumption can also suffice: if $(\Sigma_\mathrm{X},J_\gamma),(\Sigma_\mathrm{Y},J_\gamma)\in$ coerciveCARE, then (1.)–(3.) are still necessary (possibly in a modified form, see Lemma 12.1.13 and Remark 12.1.6; also (A2) must possibly be altered if $\mathbb{D}$ is not ULR), but for sufficiency one needs several additional assumptions, cf. the remark in the proof of Lemma 12.1.12

1° *Case (V):* This is contained in Theorem 12.1.11 (let $\widetilde{\mathcal{A}}$ be the class of Theorem 8.4.9(γ'); note that Hypothesis 12.5.1 holds, by Remark 12.1.7(a)), since the two forms of (1.)–(2.) are equivalent, by Remark 12.1.6 (because Σ_X and Σ_Y satisfy Hypothesis 9.2.2(7.)). Consequently, we do not treat case (V) in 2°–3° below.

2° *Condition (A1) of Theorem 12.1.5 is satisfied:* For (II)–(IV) this is obvious, since $\mathbb{A}\in\mathcal{C}(\mathbf{R}_+;\mathcal{B}(H))\subset\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+;\mathcal{B}(H))$; for (I) this follows from Lemma 9.5.2.

3° *The equivalence and all controllers:* We deduce from Theorem 12.1.5 that (1.)–(3.) of that theorem are equivalent to the existence of an exponentially stabilizing suboptimal DPF-controller for Σ (possibly with internal loop), and that if either holds, then all such controllers are parametrized by Theorem 12.1.8 (and that in this case the assumptions of Theorem 12.1.8 are satisfied).

It only remains to be shown that (1.)–(2.) can be written as in this theorem.

4° *(1.)–(3.) are equivalent to (1.)–(3.) of Theorem 12.1.5:* Note first that if either conditions hold, then $D_X^*J_\gamma D_X, D_Y^*J_\gamma D_Y\in\mathcal{GB}$, by (A2) and the signature conditions, hence the condition "$D^*JD\in\mathcal{GB}$" of Hypothesis 9.2.2(4.) (or of Hypothesis 9.5.7(2.)) is satisfied by both Σ_X and Σ_Y.

Condition (A1) implies that Hypothesis 9.2.2 is satisfied by both Σ_X and Σ_Y; indeed, (I) implies (2.) or (3.) (resp. (2.) or (3.)) of Hypothesis 9.2.2 for Σ_X (resp. for Σ_Y), (II) implies (1.) (resp. (4.)), (III) implies (4.) (resp. (1.)), (IV) implies (5.) (resp. (5.)), (V) implies (7.) (resp. (7.)).

By Theorem 9.2.3, it follows that Hypothesis 9.2.1 is satisfied by $(\Sigma_\mathrm{X},J_\gamma)$ and by $(\Sigma_\mathrm{Y},J_\gamma)$, hence the two CAREs become equivalent to BwCAREs, by Theorem 9.2.9.

(a) The claim (a) is contained in Theorem 12.1.5 except for the fact that the $\mathcal{P}_\mathrm{Z}$-CARE can be written as (4.) above; equivalently, for the fact that Σ_Z and J_1 satisfy Hypothesis 9.2.1. This will be established below.

(a)&(b) We shall establish (b) and show that Σ_Z satisfies Hypothesis 9.2.2, so that Σ_Z and J_1 satisfy Hypothesis 9.2.1. This completes the proof of (a). It also implies that (4.) is satisfied (since "(4.)" of Theorem 12.1.8 is necessarily satisfied, by the theorem).

a.1° The systems Σ_X, Σ_Y, Σ_Z and $\Sigma_{\mathbb{E}^\mathrm{d}}$ satisfy the assumptions of Lemma 6.8.5 for $p=1=q$ if any of (I)–(IV) holds (and for $p=2=q$ if (IV) holds):

For Σ_X and Σ_Y, this was noted in 1°. Since $B_\mathrm{w}^*\mathcal{P}_\mathrm{X}\in\mathcal{B}(H,U\times W)$, the operator K_X is of the form described in Lemma 6.8.5 (for Σ_X, hence also for Σ), hence also $\Sigma_\circlearrowleft$ and $\Sigma_\circlearrowleft^\mathrm{d}$ satisfy the assumptions of the lemma (by the lemma). Consequently, so does $\Sigma_{\mathbb{E}^\mathrm{d}}$ (since it is a part of $\Sigma_\circlearrowleft^\mathrm{d}$, by Lemma 12.5.15), hence so does Σ_Z, by Lemma 6.8.5(b).

a.2° The systems $\Sigma_\mathbb{T}$ and Σ_alt satisfy the assumptions of Lemma 6.8.5 for

$p = 1 = q$ if (4.) and any of (I)–(IV) hold (and for $p = 2 = q$ if (4.) and (IV) hold): Since K_Z in (4.) is of the form described in Lemma 6.8.5 (by $a.1°$) for Σ_Z, it follows that Σ_{alt} satisfies the assumptions of Lemma 6.8.5, hence so does $\Sigma_{\mathbb{T}}$ (because it is a part of Σ_{alt}).

$a.3°$ The rest of (a) and (b): We show this separately under each assumption. Note that except for claim on Hypothesis 9.5.1, it suffices to be shown that Hypothesis 9.2.2 is satisfied (so that (4.) holds).

We shall use the fact that "(4.)" of Theorem 12.1.8 is satisfied and $S_Z = D_Z^* J_1 D_Z \in \mathcal{GB}$, as noted in the proof of Theorem 12.1.5. (N.B., even if we had not assumed (1.)–(3.), conditions (1.) and (4.) above would imply that also the corresponding CAREs were satisfied (with $S_Z = D_Z^* J D_Z$), by Theorem 9.2.9(iii)&(iv), hence then (1.)–(3.) would again hold, by Theorem 12.1.8 and the above.)

(II): Since C_Z is bounded, by (II) and (12.94), Hypothesis 9.2.2(4.) is satisfied by Σ_Z.

(III): Now C and K_X are bounded (since $C_X = \left[\begin{smallmatrix} C_1 \\ 0 \end{smallmatrix}\right]$ is), hence B_Z is is bounded, by (12.94). Thus, Hypothesis 9.2.2(1.) is satisfied by Σ_Z.

(IV): By $a.1°$, Hypothesis 9.2.2(5.) is satisfied by Σ_Z.

(I): Since $C_X = \left[\begin{smallmatrix} C_1 \\ 0 \end{smallmatrix}\right] \in \mathcal{B}(H_{\gamma_1}, Z \times W)$, we have $K_X \in \mathcal{B}(H_{\gamma_1}, U \times W)$, hence also Σ_Z is analytic, by Lemma 9.5.4.

It follows that $B_Z \in \mathcal{B}(Y \times U, (H_{\beta_Z})^*)$, where $\beta_Z := -\max\{\gamma_1, \gamma_2\}$, by (12.94). Moreover, $C_Z \in \mathcal{B}(H_{\gamma_Z}, W \times U)$, where $\gamma_Z = -\beta_2$, by (12.94).

We conclude from (I) that ($\beta_Z, \gamma_Z \in (-1/2, 1/2)$ and) $\gamma_Z < 1/4$ or $\gamma_Z - \beta_Z < 1/2$, hence (2.) or (3.) of Hypothesis 9.5.7 is satisfied by Σ_Z (recall that $D_Z^* J_1 D_Z \in \mathcal{GB}$), hence so is Hypothesis 9.2.2(2.). Consequently, the $\mathcal{P}_Z$-CARE becomes a B_{w}^*-CARE.

It follows that $K_Z \in \mathcal{B}(H_{\gamma_Z}, Y \times U)$. We conclude from Proposition 12.5.19 and Lemma 9.5.4 that Σ_{alt} satisfies Hypothesis 9.5.1, hence so does $\Sigma_{\mathbb{T}}$. (Note that Hypothesis 9.5.1 is stronger than the assumptions of Lemma 6.8.5, by Lemma 9.5.2.) □

Next we shall allow for any "L^1 systems" instead of the "B_{w}^*-CARE" type regularity above. Thus, assumptions (I)–(III) above become special cases of those in (A1) below. This leads to more general (read: more complicated) Riccati equations:

Theorem 12.1.5 (H$^\infty$ 4BP $\Leftrightarrow$ CAREs)

(A1) **(Regularity)** *Assume that* $\pi_{[0,1)} \mathbb{A} B \in \mathrm{L}^1([0,1); \mathcal{B}(U \times W, H))$, $\pi_{[0,1)} C_{\mathrm{w}} \mathbb{A} \in \mathrm{L}^1([0,1); \mathcal{B}(H, Z \times Y))$, *and* $\pi_{[0,1)} C_{\mathrm{w}} \mathbb{A} B \in \mathrm{L}^1([0,1); \mathcal{B}(U \times W, Z \times Y))$, *or that (IV) of Theorem 12.1.4 holds.*

(A2) **(Nonsingularity)** *Assume that* $D_{11}^* D_{11} \gg 0$ *and* $D_{22} D_{22}^* \gg 0$, *and that there is* $\varepsilon > 0$ *s.t.*

$$(ir - A)x_0 = B_1 u_0 \implies \|C_{1w} x_0 + D_{11} u_0\|_Z \geq \varepsilon \|x_0\|_H \quad \textit{and} \tag{12.11}$$

$$(ir - A^*)x_0 = C_2^* y_0 \implies \|B_{2\mathrm{w}}^* x_0 + D_{22}^* y_0\|_W \geq \varepsilon \|x_0\|_H \tag{12.12}$$

for all $x_0 \in H,\ u_0 \in U,\ y_0 \in Y,\ r \in \mathbf{R}$.

Then there is a suboptimal exponentially stabilizing DPF-controller (possibly with internal loop) for Σ *iff (1.)–(3.) hold:*

(1.) **($\mathcal{P}_X$-CARE)** $D_{12}^*D_{12} - D_{12}^*D_{11}(D_{11}^*D_{11})^{-1}D_{11}^*D_{12} \ll \gamma^2 I_W$, *and the CARE*

$$\begin{cases} K_X^* S_X K_X = A^*\mathcal{P}_X + \mathcal{P}_X A + C_1^* C_1, \\ S_X = \begin{bmatrix} D_{11}^*D_{11} & D_{11}^*D_{12} \\ D_{12}^*D_{11} & D_{12}^*D_{12} - \gamma^2 I \end{bmatrix}, \\ K_X = -S_X^{-1}(\begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 + B_{\mathrm{w}}^*\mathcal{P}_X), \end{cases} \tag{12.13}$$

has a solution $(\mathcal{P}_X, S_X, K_X) \in \mathcal{B}(H) \times \mathcal{B}(U \times W) \times \mathcal{B}(H_1, U \times W)$ *s.t.* $\mathcal{P}_X \geq 0$, K_X *is exponentially stabilizing for* $\left(\begin{array}{c|c} A & B \end{array} \right)$, *and* $\lim_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}_X(s-A)^{-1}B = 0$.

(2.) **($\mathcal{P}_Y$-CARE)** $D_{12}D_{12}^* - D_{12}D_{22}^*(D_{22}D_{22}^*)^{-1}D_{22}D_{12}^* \ll \gamma^2 I_Z$, *and the CARE*

$$\begin{cases} K_Y^* S_Y K_Y = A\mathcal{P}_Y + \mathcal{P}_Y A^* + B_2 B_2^*, \\ S_Y = \begin{bmatrix} D_{22}D_{22}^* & D_{22}D_{12}^* \\ D_{12}D_{22}^* & D_{12}D_{12}^* - \gamma^2 I \end{bmatrix}, \\ K_Y = -S_Y^{-1}(\begin{bmatrix} D_{22} \\ D_{12} \end{bmatrix} B_2^* + \begin{bmatrix} C_{2\mathrm{w}} \\ C_{1\mathrm{w}} \end{bmatrix} \mathcal{P}_Y), \end{cases} \tag{12.14}$$

has a solution $(\mathcal{P}_Y, S_Y, K_Y) \in \mathcal{B}(H) \times \mathcal{B}(Y \times Z) \times \mathcal{B}(H_1^*, Y \times Z)$ *s.t.* $\mathcal{P}_Y \geq 0$, K_Y *is exponentially stabilizing for* $\left(\begin{array}{c|cc} A^* & C_2^* & C_1^* \end{array} \right)$, *and* $\lim_{s\to+\infty} \begin{bmatrix} C_2 \\ C_1 \end{bmatrix}_{\mathrm{w}} \mathcal{P}_Y(s-A)^{-1}\begin{bmatrix} C_2^* & C_1^* \end{bmatrix} = 0$.

(3.) **(Coupling condition)** $\rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2$.

If (1.)–(3.) are satisfied, then all suboptimal exponentially stabilizing DPF-controllers for Σ *are the ones parametrized in Theorem 12.1.8 (for* $\widetilde{\mathcal{A}} = \mathrm{MTIC}_{\mathrm{exp}}^{\mathrm{L}^1}$, *or for class ($\gamma'$) of Theorem 8.4.9 under the alternative assumption "(IV)" in (A1)); note that if* $D_{21} = 0$, *then all of them are well-posed.*

We remark from Theorem 11.1.4(iii)&(i) that (1.) implies that the FICP for Σ_X has a solution, and (2.) implies that the (dual) filter problem for Σ_Y has a solution.

A sufficient condition for (A1) is condition (A1)(I)1. of Theorem 12.1.4, by Corollary 9.5.12. See Theorem 12.1.11 for alternatives for (A1) and (A2).

Proof of Theorem 12.1.5: If there is a suboptimal exponentially stabilizing DPF-controller (possibly with internal loop), then (1.)–(3.) hold, by Lemma 12.5.22. (Thus, a fortiori, this is the case when there is a *well-posed* suboptimal exponentially stabilizing DPF-controller.)

Conversely, if (1.)–(3.) hold, then Hypothesis 12.5.1 and conditions (1.) and (4.) of Theorem 12.1.8 are satisfied, by Lemmas 12.5.21 and 12.5.20. Thus, then there are suboptimal exponentially stabilizing DPF-controllers for Σ, by Theorem 12.1.8 (whose assumptions are satisfied).

We remark from 3° of the proof of Lemma 12.5.20 that $S_Z = D_Z^* J D_Z$ when (1.)–(3.) hold. □

Remark 12.1.6 (Different forms of (1.)–(2.)) *We first recall from Theorem 9.8.12(b) that any solution of any (1.), (2.) or (4.) of this section is unique.*

We have used three forms of the $\mathcal{P}_X$-CARE ("(1.)") and the $\mathcal{P}_Y$-CARE ("(2.)"). The "weakest" forms (the standard forms) are given in Lemma 12.1.12 (they apply to any UR settings).

Those given in Theorem 12.1.5 are equivalent to those given in Lemma 12.1.12 combined with assumptions $S_X = D_X^* J_\gamma D_X$ and $S_Y = D_Y^* J_\gamma D_Y$; these assumptions are redundant when $\mathbb{N}_u$ and $\widetilde{\mathbb{N}_y}^{d}$ satisfy Hypothesis 8.4.8, as noted in the proof of Lemma 12.5.21.

The strongest forms of (1.)–(2.) (the B_w^*-CARE forms) are given in Theorem 12.1.4; they are equivalent to either of the weaker ones whenever Hypothesis 9.2.1 is satisfied by (Σ_X, J_γ) and (Σ_Y, J_γ) (see (12.84)–(12.85)).

In all our results referring to any of these CAREs, one can always replace the CAREs by any weaker ones (of the three forms listed above).

Moreover, under the assumptions of Theorem 12.1.4, 12.1.5 or 12.1.11, we can use the middle form for the $\mathcal{P}_Z$-CARE (4.) or (4.') too (i.e., require that $S_Z = D_Z^* J D_Z$). □

(This is straightforward; use Theorem 9.2.9(iii)&(iv) for the B_w^*-CARE forms.)

Condition (A2) is virtually equivalent to the I-coercivity of $\mathbb{D}_{11}$ and $\mathbb{D}_{22}^{d}$:

Remark 12.1.7 (Different forms of (A2)) *(a) Assume that (A1) of Theorem 12.1.4 holds (or that (A1) of Theorem 12.1.5 holds). Assume that (A, B_1) is optimizable and (A, C_2) is estimatable (this is a necessary condition, by Theorem 7.3.12(a)) or that $D_{11}^* D_{11} \gg 0$ and $D_{22} D_{22}^* \gg 0$.*

Then condition (A2) of Theorem 12.1.4 holds iff $\mathbb{D}_{11}$ and $\mathbb{D}_{22}^{d}$ are I-coercive over $\mathcal{U}_{\exp}$; in fact, then any of (i)–(iii) of Proposition 10.3.2 or Remark 10.3.3 (for $\Sigma_{\mathbb{D}_{11}}$ and $\Sigma_{\mathbb{D}_{22}^{d}}$) are equivalent. One more equivalent condition is that Hypothesis 12.5.1 holds.

(b) If Σ is exponentially stable, then $\mathbb{D}_{11}$ and $\mathbb{D}_{22}^{d}$ are I-coercive over $\mathcal{U}_{\exp}$ iff $\mathbb{D}_{11}^* \mathbb{D}_{11} \gg 0$ and $\mathbb{D}_{22} \mathbb{D}_{22}^* \gg 0$ (equivalently, iff Hypothesis 12.5.1 holds).

(c) The I-coercivity of $\mathbb{D}_{11}$ and $\mathbb{D}_{22}^{d}$ over $\mathcal{U}_{\exp}$ is equivalent to (12.78) if the rest of Hypothesis 12.5.1 holds and (A, B) is optimizable.

In this chapter, the I-coercivity of $\mathbb{D}_{11}$ (resp. of $\mathbb{D}_{22}$) refers to the realization $\Sigma_{\mathbb{D}_{11}} := \left(\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right)$ (resp. $\Sigma_{\mathbb{D}_{22}^{d}} := \left(\begin{array}{c|c} A^* & C_2^* \\ \hline B_2^* & D_{22}^* \end{array}\right)$).

Proof: (a) By Proposition 10.3.2(e2) (use (e1) and (g2) for (V)), conditions (i)–(iii) are equivalent (note that this include I-coercivity and (A2)). The rest is given in Lemma 12.5.4 (use (b) for (V)) (since any of (I)–(IV) of Theorem 12.1.4(A1) implies (A1) of Theorem 12.1.5, by 2° of the proof of Theorem 12.1.4).

(b) This follows from (c) (take $\begin{bmatrix} \mathbb{K}_u & \mathbb{F}_u \end{bmatrix} = 0 = \begin{bmatrix} \mathbb{H}_y \\ \mathbb{G}_y \end{bmatrix}$).

(c) By Lemma 12.5.2(i)&(vi), $\begin{bmatrix} \mathbb{K}_u & \mathbb{F}_u \end{bmatrix}$ is exponentially stabilizing for Σ, hence $\left[\begin{array}{c|c} \mathbb{K}_{u1} & \mathbb{F}_{u11} \\ 0 & 0 \end{array}\right]$ is exponentially stabilizing for $\Sigma_{\mathbb{D}_{11}}$ (since the two closed-loop semigroups are equal).

Therefore $\mathbb{D}_{11}$ is I-coercive over $\mathcal{U}_{\exp}$ iff $\mathbb{N}_{u11}$ is (positively) I-coercive over $\mathcal{U}_{\rm out}$, by Theorem 8.4.5(d). By Lemma 8.4.11(a2), this is the case iff $\mathbb{N}_{u11}^*\mathbb{N}_{u11} \gg 0$. By dual arguments we obtain that $\mathbb{D}_{22}^{\rm d}$ is I-coercive over $\mathcal{U}_{\exp}$ iff $\widetilde{\mathbb{N}}_{y22}\widetilde{\mathbb{N}}_{y22}^* \gg 0$. □

As in, e.g., [IOW], the $\mathcal{P}_Z$-CARE is not determined by the exponentially stabilizing solution of the $\mathcal{P}_X$-CARE but by a(ny) exponentially stabilizing solution of the $\mathcal{P}_X$-IARE having $X_{11}, X_{22} \in \mathcal{GB}$ and $X_{21} = 0$ (see (9.114)). Analogously, for the parametrization of all controllers we use a modified solution of the $\mathcal{P}_Z$-IARE(see (12.94)):

Theorem 12.1.8 (All solutions to the 4BP) *Assume that Hypothesis 12.5.1 is satisfied with $\mathbb{N}_{\rm u}, \mathbb{M}_{\rm u} \in \widetilde{\mathcal{A}}$. Then there is a suboptimal exponentially stabilizing DPF-controller for Σ iff conditions (1.) and (4.) below hold:*

(1.) **($\mathcal{P}_X$-CARE)** *The CARE*

$$\begin{cases} K_X^* S_X K_X = A^*\mathcal{P}_X + \mathcal{P}_X A + C_1^* C_1, \\ S_X = \begin{bmatrix} D_{11}^*D_{11} & D_{11}^*D_{12} \\ D_{12}^*D_{11} & D_{12}^*D_{12}-\gamma^2 I\end{bmatrix} + \lim_{s\to+\infty} B_{\rm w}^*\mathcal{P}_X(s-A)^{-1}B, \\ K_X = -S_X^{-1}\left(\begin{bmatrix} D_{11}^* \\ D_{12}^*\end{bmatrix} C_1 + B_{\rm w}^*\mathcal{P}_X\right), \end{cases} \tag{12.15}$$

has a solution $(\mathcal{P}_X, S_X, K_X) \in \mathcal{B}(H)\times\mathcal{B}(U\times W)\times\mathcal{B}(H_1, U\times W)$ s.t. $\mathcal{P}_X \geq 0$, $S_{X11} \gg 0$, $S_{X22} - S_{X21}S_{X11}^{-1}S_{X12} \ll 0$ and K_X is exponentially stabilizing for $\left(\begin{array}{c|c} A & B \end{array}\right)$.

(4.) **($\mathcal{P}_Z$-CARE)** *The CARE*

$$\begin{cases} K_Z^* S_Z K_Z = A_Z^*\mathcal{P}_Z + \mathcal{P}_Z A_Z + B_2 X_{22}^{-1} X_{22}^{-*} B_2^*, \\ S_Z = D_Z^* J D_Z + \lim_{s\to+\infty} (B_Z^*)_{\rm w}\mathcal{P}_Z(s-A_Z)^{-1}B_Z, \\ K_Z = -S_Z^{-1}\left(\begin{bmatrix} D_{22}X_{22}^{-1} \\ X_{12}X_{22}^{-1}\end{bmatrix} X_{22}^{-*}B_2^* + (B_Z^*)_{\rm w}\mathcal{P}_Z\right), \end{cases} \tag{12.16}$$

has a solution $(\mathcal{P}_Z, S_Z, K_Z) \in \mathcal{B}(H)\times\mathcal{B}(Y\times Z)\times\mathcal{B}(\mathrm{Dom}(A_Z), Y\times U)$ s.t. $\mathcal{P}_Z \geq 0$, $S_{Z11} \gg 0$, $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$ and K_Z is exponentially stabilizing for $\left(\begin{array}{c|c} A_Z & B_Z \end{array}\right)$.

Given the solution $(\mathcal{P}_X, S_X, K_X)$ of (1.), choose any $X \in \mathcal{GB}(U\times W)$ s.t. $X^*J_1X = S_X$ and $X_{21} = 0$. Then by the operators A_Z, B_Z, C_Z, D_Z appearing in the $\mathcal{P}_Z$-CARE we mean the following (here $K_X = \begin{bmatrix} K_{X1} \\ K_{X2}\end{bmatrix} \in \mathcal{B}(H_1, U\times W)$):

$$\left[\begin{array}{c|c} A_Z & B_Z \\ \hline C_Z & D_Z \end{array}\right] = \left[\begin{array}{c|cc} A^* + K_{X2}^*(B_2^*)_{\rm w} & C_2^* + K_{X2}^*D_{22}^* & -K_{X1}^* \\ \hline X_{22}^{-*}(B_2^*)_{\rm w} & X_{22}^{-*}D_{22}^* & X_{22}^{-*}X_{12}^* \\ 0 & 0 & I \end{array}\right] \tag{12.17}$$

$$\in \mathcal{B}(\mathrm{Dom}(A_Z)\times Y\times U, H\times W\times U); \tag{12.18}$$

$$\mathrm{Dom}(A_Z) := \{x \in H \,|\, A_Z x \in H\} = \{x \in H_{C,K}^* \,|\, A_Z x \in H\}. \tag{12.19}$$

Moreover, any solutions of (1.) or (4.) are unique.

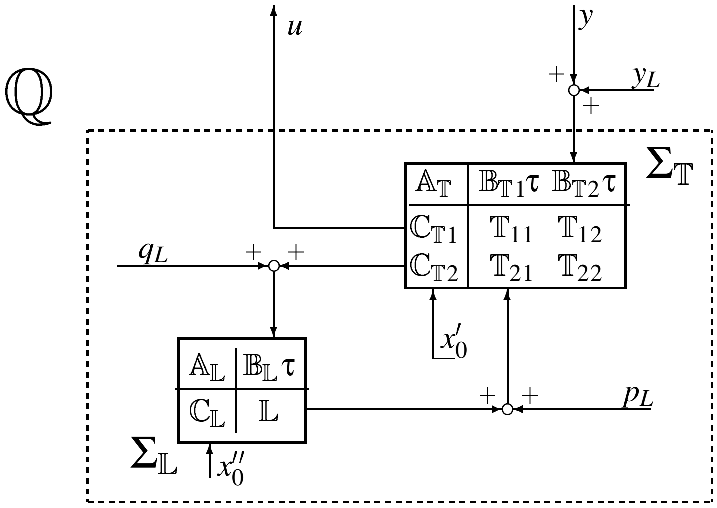

Figure 12.1: The suboptimal controller $\mathbb{Q} := \mathcal{F}_\ell(\mathbb{T},\mathbb{L}) : y \mapsto u$

Assume (1.) and (4.). Then we can construct all suboptimal exponentially stabilizing DPF-controllers for Σ as follows:

Choose $G \in \mathcal{GB}(Y \times U)$ s.t. $G^* J_1 G = S_Z$, $G_{21} = 0$ and $G_{11}, G_{22} \in \mathcal{GB}$, and then set $R := G \begin{bmatrix} I & 0 \\ 0 & M_{11}^* \end{bmatrix}$. If $D_{21} = 0$, then the assumptions of Proposition 12.5.19 are satisfied, hence then $\Sigma_{\mathbb{T}_{12}}$ is exponentially stabilizing for Σ, and all well-posed exponentially stabilizing suboptimal controllers for Σ are given by the connection "$\Sigma_{\mathbb{Q}}$" of $\Sigma_{\mathbb{T}}$ and $\Sigma_{\mathbb{L}}$ in Figure 12.1, where $\Sigma_{\mathbb{L}}$ is any exponentially stable realization of any $\mathbb{L} \in \mathrm{TIC}_{\exp}(Y,U)$ s.t. $\|\mathbb{L}\|_{\mathrm{TIC}} < 1$ (note that the I/O-map of controller $\Sigma_{\mathbb{Q}}$ is $\mathcal{F}_\ell(\mathbb{T},\mathbb{L})$; cf. (12.26)). Moreover, any exponentially stabilizing suboptimal controller with internal loop for Σ is equivalent to a well-posed one.

Here $\mathbb{T} \in \mathrm{TIC}_\infty(U \times Y)$ is a map that has a realization (denoted by $\Sigma_{\mathbb{T}} \in \mathrm{WPLS}(U \times Y, H, U \times Y)$) with the following generators:

$$A_{\mathbb{T}} := A + BK_I + B_{\mathbb{T}2} C_{\circlearrowleft 2} \tag{12.20}$$

$$B_{\mathbb{T}1} := -B_{\circlearrowleft 2} X_{22}^{-*} (D_{22}^* R_{11}^{-1} R_{12} + X_{12}^*) R_{22}^{-1} - B_{\circlearrowleft 1} R_{22}^{-1} \tag{12.21}$$

$$B_{\mathbb{T}2} := -B_{\circlearrowleft 2} X_{22}^{-*} D_{22}^* (R_{11}^* R_{11})^{-1} \tag{12.22}$$

$$C_{\mathbb{T}1} := -K_{I1} - R_{12}^* R_{11}^{-*} C_{\circlearrowleft 2} \tag{12.23}$$

$$C_{\mathbb{T}2} := R_{11}^{-*} C_{\circlearrowleft 2} \tag{12.24}$$

$$D_{\mathbb{T}} := \begin{bmatrix} R_{22}^* & -R_{12}^* R_{11}^{-*} \\ 0 & R_{11}^{-*} \end{bmatrix}, \tag{12.25}$$

where $B_{\circlearrowleft} := BX^{-1}$ and $C_{\circlearrowleft} := C + DK_X$.

For general $D_{21} \in \mathcal{B}(U,Y)$, the above formulae still parametrize all suboptimal exponentially stabilizing DPF-controllers for Σ modulo equivalence (see Definition 7.3.1) except that we have to add to $\mathbb{Q}$ the output feedback through $-D_{21} =: -E$, as in Figure 7.12 (and in Lemma 7.3.23), and that this internal loop need not be well-posed (but the combined system of Figure 7.12 is well-posed).

Note that the $\mathcal{P}_Z$-CARE is not uniquely defined (since X may vary slightly); this is not important, because condition (4.) (and $\mathcal{P}_Z$) is independent on the choice of X (under the above restrictions). An analogous formulation is given in [IOW] (see pp. 281 and 308–309 of [IOW]); the parametrizations on pp. 136–137 of [Keu] or on p. 297 of [GL] are simpler due to simplifying assumptions on the generators of Σ.

The condition $\mathbb{N}_u, \mathbb{M}_u \in \widetilde{\mathcal{A}}$ can be weakened significantly if we replace "iff" by "if" at the beginning of the theorem, as one observes from the proof (see Theorem 12.3.7(a) for the corresponding frequency-space claim).

We repeat that all exponentially stabilizing suboptimal controllers for $\mathbb{D}$ are given by

$$\mathbb{Q} = \mathcal{F}_\ell(\mathbb{T},\mathbb{L}) \quad (\mathbb{L} \in \mathrm{TIC}_{\mathrm{exp}}(Y,U) \text{ is s.t. } \|\mathbb{L}\|_{\mathrm{TIC}} < 1) \tag{12.26}$$

(when $D_{21} = 0$; cf. Lemma 7.3.23). As noted below Theorem 12.3.7, the controller $\mathbb{Q}$ in (12.47) is different for different parameters $\mathbb{L}$.

The maps $\mathbb{T}$ and $\mathbb{L}$ are the same as in Theorem 12.3.7(c) and Proposition 12.5.19, as one observes from the proof below, hence parts (a)–(g) of the theorem apply for Σ and $\Sigma_{\mathbb{T}}$.

In particular, $\mathbb{T} \in \mathrm{ULR}$, since $\mathbb{T}$ is given by (12.48) and $\widetilde{\mathbb{N}}_+, \widetilde{\mathbb{M}}_+ \in \widetilde{\mathcal{A}}$. Analogously, $\widetilde{\mathbb{Q}}_2, \widetilde{\mathbb{Q}}_1 \in \widetilde{\mathcal{A}}(Y,*)$ and $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}(Y)$, where $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ is the l.c.f. of $\mathbb{Q}$ parametrized by (12.47) (here $\widetilde{\mathbb{Q}}_*$, $\widetilde{\mathbb{N}}_+$, $\widetilde{\mathbb{M}}_+$ and $\mathbb{T}$ correspond to Σ with 0 in place of D_{21}; note that the actual controller has the additional loop through $-D_{21}$).

Remark 12.1.9 (Figure 12.1) *Figure 12.1 illustrates the map* $\mathbb{Q} := \mathcal{F}(\mathbb{T},\mathbb{L})$. *The combined closed-loop connection of* $\Sigma_{\mathbb{T}}$ *and* $\Sigma_{\mathbb{L}}$ *is a system having* $\begin{bmatrix} * & \mathbb{Q} \\ * & * \end{bmatrix}$ *as its I/O map (see Lemma 7.3.2 with* $\Sigma \mapsto \Sigma_{\mathbb{T}}$ *and* $\widetilde{\Sigma} \mapsto \Sigma_{\mathbb{L}}$*). The signals* p_L *and* q_L *refer to external inputs and* x_0' *and* x_0'' *to the initial states.*

Thus, this combined system is not a realization of $\mathbb{Q}$ *in the sense of Definition 6.1.6; however, there exists also a realization of* $\mathbb{Q}$ *in this strict sense, by Theorem 12.3.5.* □

(Note that the parametrization of Figure 12.1 was given in (the Main) Theorem 5.4 of [Keu] too.)

Naturally, any $\mathbb{Q}$ given by (12.26) has infinitely many realizations (and so do $\mathbb{T}_{12}$ and $\mathbb{T}$), and not all of them are stabilizing for Σ. (Recall from Definition 7.3.1 that for a realization of $\mathbb{Q}$ to be stabilizing for Σ, this realization must be also internally stabilized by the connection.) The term "all controllers" has traditionally been used as above, meaning all (suboptimal [exponentially] stabilizing) $\mathbb{Q}$'s (with some realizations), not all realizations of such $\mathbb{Q}$'s.

Our choice to use pairs (J_γ, J_1) for the $\mathcal{P}_X$-CARE (or (Factor1X) of Theorem 12.3.7) and (J_1, J_1) for $\mathcal{P}_Z$-CARE (or (Factor1Z)) has slightly scaled the rows and columns of Σ_Z compared to those in [Keu] or [ZDG] (if $\gamma \neq 1$), but there is no essential difference (except that we have no simplifying assumptions such as "$D_{12} = 0 = D_{21}$"). This results in the formula $\mathcal{P}_Z = \mathcal{P}_Y(\gamma^2 I - \mathcal{P}_X\mathcal{P}_Y)^{-1}$ instead of $\mathcal{P}_Z = \mathcal{P}_Y(I - \gamma^{-2}\mathcal{P}_X\mathcal{P}_Y)^{-1}$ in Lemma 12.6.4.

Proposition 12.1.10 (Non-exponentially stabilizing H^∞ 4BP) *Theorem 12.1.8 holds even iff we remove the word "exponentially" everywhere from the theo-*

rem and replace (1.) by (1.'), (4.) by (4.') and "$\mathrm{TIC}_{\mathrm{exp}}$*" by "*TIC*". Here we have referred to the following conditions:*

(1.') **($\mathcal{P}_{\mathbf{X}}$-CARE)** *Condition (1.) holds except that on K_X we only require that K_X is P-stabilizing for Σ, and that $\mathbb{D}(I-\mathbb{F}_X)^{-1}$ and $(I-\mathbb{F}_X)^{-1}$ are r.c.*

(4.') **($\mathcal{P}_{\mathbf{Z}}$-CARE)** *Condition (1.) holds except that on K_Z we only require that K_Z is P-stabilizing for $\Sigma_Z := \left(\begin{array}{c|c} A_Z & B_Z \\ \hline C_Z & D_Z \end{array}\right)$, and that $(I-\mathbb{F}_Z)(I-\widetilde{\mathbb{F}}) \in \mathcal{G}\mathrm{TIC}(Y \times U)$ and $\mathbb{K}_Z + (I-\mathbb{F}_Z)\widetilde{\mathbb{K}}$ is stable, where $\left[\begin{array}{c|c} \mathbb{K}_Z & \mathbb{F}_Z \end{array}\right]$ is the pair generated by K_Z.*

Moreover, (1.) has a solution iff (1.') has a solution and Σ is exponentially stabilizable. If(f) Σ is exponentially stabilizable, then (1.) and (1.') have same solutions (if any), and so do (4.) and (4.'); thus, then any suboptimal stabilizing controller parametrized by the modified theorem is exponentially stabilizing iff $\Sigma_{\mathbb{L}}$ is exponentially stable.

Note that the assumptions of the theorem (by Lemma 12.5.9) and formulae (12.20)–(12.25) do not depend on D_{21}. However, for $\mathbb{Q}$ to be suboptimal and stabilizing for Σ, we must add the output feedback through $-D_{21}$ as in Figure 7.12 (with $E := D_{21}$); this combined connection is always well-posed (and stable). See the comments below Lemma 12.5.17 for why the r.c. condition in (1.') has a more complicated counterpart in (4.').

We conclude from Proposition 12.1.10 that if Hypothesis 12.5.1 is satisfied with $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$, then the H^∞ 4BP can be solved as follows:

0. Choose some initial estimate $\gamma > 0$.
1. If (1.') does not have a solution, then there are no suboptimal controllers.
2. Otherwise, choose X as in (4.'). If (4.') does not have a solution, then there are no suboptimal controllers.
3. Iterate 1.–2. for different values of γ (by using, e.g., a binary search) so as to find a good approximate of the infimal (optimal) γ (but above it, so that (1.') and (4.') have solutions).
4. Choose G and construct $\mathbb{T}$ (and $\Sigma_{\mathbb{Q}}$) as in Theorem 12.1.8 to obtain all suboptimal controllers.

Usually we are looking for an exponentially stabilizing solution; if this is the case, then we must replace (1.') by (1.) and (4.') by (4.) above (see Theorem 12.1.8). If (A,B) is exponentially stabilizable, then the two pairs of conditions become equivalent (by, e.g., Lemma 12.5.2(ii)&(i) and Theorem 6.7.15(c1)).

Under the alternative, more easily verifiable assumptions (A1) and (A2) of Theorem 12.1.5 (or of Theorem 12.1.4, or under the assumptions of Theorem 12.1.11), we can use the following alternative procedure:

0. Choose some initial estimate $\gamma > 0$.
1. There are suboptimal solutions iff (1.)–(3.) of Theorem 12.1.5 hold.

2. Iterate 1. for different values of γ (by using, e.g., a binary search) so as to find a good approximate of the infimal (optimal) γ (but above it, so that (1.)–(3.) have solutions).
3. Set $\mathcal{P}_Z = \mathcal{P}_Y(\gamma^2 I - \mathcal{P}_X\mathcal{P}_Y)^{-1}$ to obtain a solution of (4.).
4. Choose G and construct $\mathbb{T}$ (and $\mathbb{Q}$) as in Theorem 12.1.8 to obtain all suboptimal controllers.

Thus, in this case it suffices to study the $\mathcal{P}_X$-CARE and the $\mathcal{P}_Y$-CARE to find an estimate of infimal γ, but we still need the $\mathcal{P}_Z$-CARE for finding a suboptimal controller (note that $\mathcal{P}_Z$-CARE depends on $\mathcal{P}_X$-CARE, whereas $\mathcal{P}_X$-CARE and $\mathcal{P}_Y$-CARE depend on the original system Σ and γ only and are therefore more suitable for computations). This latter procedure is the classical one.

For the nonexponentially stabilizable case covered by the former procedure, conditions analogous to (1.)–(3.) are necessary, by Lemma 12.1.12, but we do not know whether they are sufficient.

Proof of Theorem 12.1.8 and Proposition 12.1.10: (We prove here both Theorem 12.1.8 and Proposition 12.1.10 at once.)

(We note that if (1.) holds, then $(\mathcal{P}_X, J_1, \begin{pmatrix} XK_X & \big| & I-X \end{pmatrix})$ is an exponentially stabilizing solution of the $\mathcal{P}_X$-IARE; cf. Lemmas 11.1.7 and 12.5.12. Analogously, if also (4.) holds, then $(\mathcal{P}_Z, J_1, \begin{pmatrix} GK_Z & \big| & I-G \end{pmatrix})$ is an exponentially stabilizing solution of the $\mathcal{P}_Z$-IARE, as in Proposition 12.5.19, as one observes from II.2° below. Same "upper triangular" IARE (or "KPYS") solutions were used also on pp. 280–281 of [IOW] (in the corresponding finite-dimensional result).)

0° We first note that any P-stabilizing solution of a CARE is unique, by Theorem 9.8.12(b)&(s1).

Part I: There is a suboptimal stabilizing DPF-controller for Σ iff (1.') and (4.') hold:

I.1° *We only have to show that (4BP3) holds iff (1.') and (4.') hold:* By Lemma 12.5.3, Hypothesis 12.3.1 is satisfied. By Theorem 12.3.5(b) (and Theorem 6.6.28), (4BP1) is equivalent to the existence of a stabilizing DPF-controller *for* Σ. By Lemma 12.3.10, (4BP1)–(4BP3) of Theorem 12.3.7 are equivalent.

I.2° *(Factor1)$\Leftrightarrow$(1.'):* This follows from Lemma 12.5.12 (since a solution has necessarily $\mathbb{X},\mathbb{N},\mathbb{M} \in \widetilde{\mathcal{A}} \subset$ ULR $\subset$ UR, by Lemma 12.3.10(a)).

(A detailed verification that the condition on $\mathcal{P}_X$-CARE in Lemma 12.5.12 is equivalent to (1.') is given in 6° below. An analogous comment applies to (4.'), (1.) and (4.) too.)

I.3° *Consequences of (Factor1):* If (Factor1) holds, then Hypothesis 12.5.13 holds, by Lemma 12.5.12, and $\mathbb{X},\mathbb{N},\mathbb{M} \in \widetilde{\mathcal{A}} \subset$ ULR, as noted above; in particular, then $\mathbb{M}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, by Proposition 6.3.1(c).

I.4° *(4BP3)$\Rightarrow$(1.')&(4.'):* Assume (4BP3). By 2°, (1.') holds. But (Factor2Z) has an ULR solution, by Lemma 12.3.10(a) (and Theorem 12.3.7(e1)), hence also (4.')=Lemma 12.5.17(ii) holds, by Lemma 12.5.17(c2).

I.5° *(1.')&(4.')$\Rightarrow$(4BP3):* If $D_{21} = 0$ and (1.') and (4.') hold, so that the facts listed in 3° hold, then we obtain (Factor2) from Lemma 12.5.17(a) (and Theorem 12.3.7(e1)), thus, then (4BP3) and hence also (4BP1) hold. But (4BP1) is independent of D_{21}, by Lemma 12.5.9, hence so is (4BP3); on the other hand, (1.') and (4.') are also independent of D_{21}, by Lemma 12.5.9. Thus, (1.') and (4.') imply (4BP3) regardless of D_{21}.

I.6° *The CAREs (1.') and (4.'):* Equation (12.17) follows from (12.94), Proposition 6.6.18(d4) and (6.145) (note that the $\mathcal{P}_Z$-CARE corresponds to $(\mathcal{P}, S_X, \begin{pmatrix} XK_X & \big| & I-X \end{pmatrix})$, where X is chosen as in (4.') (i.e., as in (4.)), by the last claim of Lemma 12.5.12; note also that we prove the equivalence regardless of the choice of X (within these restrictions)).

By Lemma 9.11.5(e), the "lim" (instead of w-lim) in (1.') (or (1.)) and (4.') (or (4.)) is equivalent to the solution being UR (since Σ_X and Σ_Z (if any) are necessarily UR). Actually, since any spectral factorization is necessarily in $\widetilde{\mathcal{A}}$, hence ULR, we could as well write w-lim.

I.7° *The set* $\mathrm{Dom}(A_Z)$*:* By Lemma A.4.6,

$$\mathrm{Dom}(A_Z) := \{x \in H \,\big|\, A_Z x \in H\}. \tag{12.27}$$

By Proposition 6.6.18(a1), the space "H_C" for $\Sigma_{\circlearrowleft}$ equals that for Σ_{ext} (of (6.132), which we denote by

$$H^*_{C,K} := (r-A^*)^{-1}[H + C^*[Z\times Y] + K^*[U\times W]] \tag{12.28}$$

$$= \{x_0 \in H \,\big|\, A^*x_0 + C^*\left[\begin{smallmatrix} z_0 \\ y_0\end{smallmatrix}\right] + K^*\left[\begin{smallmatrix} u_0 \\ w_0\end{smallmatrix}\right] \in H \text{ for some } (z_0, y_0, u_0, w_0) \in Z\times Y\times U\times W\} \tag{12.29}$$

(for any $r > \omega_A$); cf. Definition 6.1.17. Naturally, this is the space "H_B" for $\Sigma^{\mathrm{d}}_{\circlearrowleft}$, hence contained by the space "H_B" for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$. By Proposition 6.6.18(a1), the space H_{B_Z} (for Σ_Z) equals "H_B" for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$, and $\mathrm{Dom}(A_Z) \subset H_{B_Z}$, hence we can replace H in (12.27) by H_{B_Z}, hence by its superset $H^*_{C,K}$ (on which B^*_{w} is defined, by the regularity of Σ_{ext} and Proposition 6.2.8(a2), so that the formula for $\mathrm{Dom}(A_Z)$ below (12.17) is well defined).

By Proposition 6.6.18(d3), equation (12.17) actually holds on $H_{B_Z}\times U$, but the values of A_Z on $H_{B_Z}\setminus \mathrm{Dom}(A_Z)$ lie outside H.

Part II: All controllers: Within Part II we assume that (1.') and (4.') hold.

II.1° By I.3°, I.5° and Lemma 12.3.10(a), Hypothesis 12.5.13 holds and (Factor1X) and (Factor2Z) have solutions with $\mathbb{X}, \mathbb{N}, \mathbb{M}, \mathbb{E}, \mathbb{Z} \in \widetilde{\mathcal{A}} \subset \mathrm{ULR}$ and $\mathbb{M}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

II.2° $R = Z^*$ *and* $\mathbb{Z}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(Y)$ *when* $D_{21} = 0$*:* (Note that G can be obtained from (11.101).) By (a)2° and (d)1° of the proof of Lemma 12.5.17(a), the solution $\mathbb{Z}$ of (Factor2Z) obtained in I.5° satisfies $Z = \widetilde{X}^*G^*$ (where $G^* = \widetilde{Z}$). Since $\widetilde{X} = \begin{bmatrix} I & 0 \\ -D^*_{21} & M^*_{11}\end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & M^*_{11}\end{bmatrix}$, we have $R = Z^*$, where R is as in Theorem 12.1.8. Since $Z_{11} = G^*_{11} \in \mathcal{G}\mathcal{B}$, we have $\mathbb{Z}_{11} \in \mathcal{G}\mathrm{TIC}_\infty$, by Proposition 6.3.1(c).

II.3° *All controllers and their well-posedness when* $D_{21} = 0$*:* The assumptions of Proposition 12.5.19 are satisfied, by Part I and II.2°, hence we thus obtain the parametrization of all suboptimal stabilizing controllers.

To obtain (12.20)–(12.25), first write out the generators of $\Sigma_{\circlearrowleft}$, (12.102) (12.103) and (12.104) using (13.57) (see II.3° for $R = Z^*$), and then use these to obtain (12.20)–(12.25) (recall that $X^{-1}K = K_X$).

Since $D_{\mathbb{T}21} = 0$, by (12.25), the condition $I - \mathbb{L}\mathbb{T}_{21} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ in (12.105) always holds (by Proposition 6.3.1(c))), hence all controllers parametrized by (12.26) are well posed.

II.4° *Case* $D_{21} \neq 0$*:* By Lemma 12.5.9, (1.') and (4.') are independent of D_{21}. Therefore, the above shows that all suboptimal stabilizing DPF-controllers for Σ' are given by (12.26), where Σ' is equal to Σ with 0 in place of D_{21}.

Thus, the claim on D_{21} at the end of Theorem 12.1.8 follows from Proposition 12.5.19(g) (alternatively, directly from Lemma 7.3.23).

Remarks on case $D_{21} \neq 0$*:* By Lemma 7.3.23, this output feedback connection of $\mathbb{Q}$ and $-D_{21}$ may be non-well-posed (and it corresponds to a controller with d.c. internal loop, as in the lemma), but when we add the connection with $\mathbb{D}$ (as in Figure 7.12), all signals in this final connection become well-posed. If $I + D_{21}\mathbb{Q} \in \mathcal{G}\mathrm{TIC}_\infty(Y)$, then the final controller becomes well-posed, an alternative formula for it is given by $\mathbb{Q}' = (I + \mathbb{Q}D_{21})^{-1}\mathbb{Q}$ $(= \mathbb{Q}(I + D_{21}\mathbb{Q})^{-1})$, as noted below Lemma 7.2.18.

The factorization (Factor1X) is independent of D_{21}, but the solution $\mathbb{Z}$ of (Factor2Z) used in II.1°–II.4° (for the application of Proposition 12.5.19) solves the condition (Factor2Z) corresponding to Σ', not that corresponding to Σ, i.e., we use the parametrization for the wrong (but equivalent) problem and correct it by the additional output feedback through $-D_{21}$ (since we do not know whether the solution of (Factor2Z) corresponding to Σ' has $\mathbb{Z}_{11}$ invertible, as required by Proposition 12.5.19; moreover, in this case we do not know whether $\mathbb{Q}$ is well-posed).

Part III: Conditions (1.) and (4.) compared to (1.') and (4.'):

If any of the equivalent conditions of Lemma 12.5.2 hold, e.g., Σ is exponentially stabilizable, then Parts I–II hold with the primes removed if we require the controller to be exponentially stabilizing (and use Theorem 12.3.5(e) instead of Theorem 12.3.5(b)). We observe from this and Part I that then (1.) and (1.') have same solutions, and so do (4.) and (4.'). Conversely, (1.) obviously implies that Σ (equivalently, (A, B)) is exponentially stabilizable.

Moreover, if Σ is exponentially stabilizable (i.e., $\left[\begin{array}{c|c} \mathbb{K}_\mathrm{u} & \mathbb{F}_\mathrm{u} \end{array}\right]$ is exponentially stabilizing), then the above formulation of all suboptimal stabilizing controllers equals that of all suboptimal exponentially stabilizing controllers modulo the fact that $\Sigma_\mathbb{L}$ is required to be exponentially stable, as noted in Proposition 12.5.19(e). (Cf. the comments below Theorem 12.1.8.)

Note also that $\Sigma_\mathbb{L}$ must be optimizable and estimatable for the closed-loop system to be exponentially stable, by, e.g., Theorem 6.7.10(d)(viii), Lemma 6.7.11(c) and (twice) Lemma 6.7.18; by Theorem 6.7.10(d)(viii), this is the case iff $\Sigma_\mathbb{L}$ is exponentially stable. □

We give here an alternative set of conditions under which the conditions (1.)–(3.) are necessary and sufficient:

Theorem 12.1.11 (MTIC$_{TZ}$: H^∞ 4BP $\Leftrightarrow$ CAREs) *Assume that Hypothesis 12.5.1 is satisfied with* $\mathbb{N}_u, \mathbb{M}_u, \widetilde{\mathbb{M}_y}, \widetilde{\mathbb{N}_y} \in \widetilde{\mathcal{A}}$ *(e.g., that* Σ *is exponentially stable,* $\mathbb{D} \in \widetilde{\mathcal{A}}$, $\mathbb{D}_{11}^*\mathbb{D}_{11} \gg 0$ *and* $\mathbb{D}_{22}\mathbb{D}_{22}^* \gg 0$*) and that* $\widetilde{\mathcal{A}}$ *satisfies Hypothesis 8.4.8.*

Then there is a suboptimal exponentially stabilizing DPF-controller for Σ *iff conditions (1.)–(3.) of Theorem 12.1.5 hold. If (1.)–(3.) hold, then all suboptimal exponentially stabilizing DPF-controllers for* Σ *are given by Theorem 12.1.8 (in particular, (1.) and (4.) of Theorem 12.1.8 hold).*

Obviously, this is particularly useful for exponentially stable systems.

Proof: (If Σ is exponentially stable and $\mathbb{D} \in \widetilde{\mathcal{A}}$, then we can take $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right] = 0$, $\left[\begin{array}{c} \mathbb{H}_y \\ \hline \mathbb{G}_y \end{array}\right] = 0$ in Hypothesis 12.5.1 to obtain that $\mathbb{N}_u = \mathbb{D} = \widetilde{\mathbb{N}_y} \in \widetilde{\mathcal{A}}$ and $\mathbb{M}_u = I = \widetilde{\mathbb{M}_y} \in \mathcal{B} \subset \widetilde{\mathcal{A}}$.)

By Lemma 12.5.20 and Theorem 12.1.8, conditions (1.)–(3.) are sufficient (and imply (1.)–(4.)); by Lemma 12.1.12, they are also necessary. $\square$

The necessity of (1.)–(3.) can be shown under more general conditions:

Lemma 12.1.12 ($\widetilde{\mathcal{A}}$: H^∞ 4BP $\Rightarrow$ CAREs) *Assume that Hypothesis 12.5.1 is satisfied with* $\mathbb{N}_u, \mathbb{M}_u, \widetilde{\mathbb{M}_y}, \widetilde{\mathbb{N}_y} \in \widetilde{\mathcal{A}}$.

If there is a suboptimal exponentially stabilizing DPF-controller for Σ*, then (1.)–(3.) below hold.*

If there is a suboptimal stabilizing DPF-controller for Σ*, then (1.'), (2.') and (3.) hold.*

(1.) **($\mathcal{P}_X$-CARE)** *Condition (1.) of Theorem 12.1.8 holds.*

(2.) **($\mathcal{P}_Y$-CARE)** *The dual of (1.) holds, i.e., the CARE*

$$\begin{cases} K_Y^* S_Y K_Y = A\mathcal{P}_Y + \mathcal{P}_Y A^* + B_2 B_2^*, \\ S_Y = \begin{bmatrix} D_{22}D_{22}^* & D_{22}D_{12}^* \\ D_{12}D_{22}^* & D_{12}D_{12}^* - \gamma^2 I \end{bmatrix} + \lim_{s\to+\infty} \begin{bmatrix} C_{2w} \\ C_{1w} \end{bmatrix} \mathcal{P}_Y (s-A)^{-1} \begin{bmatrix} C_2^* & C_1^* \end{bmatrix}, \\ K_Y = -S_Y^{-1}\left(\begin{bmatrix} D_{22} \\ D_{12} \end{bmatrix} B_2^* + \begin{bmatrix} C_{2w} \\ C_{1w} \end{bmatrix} \mathcal{P}_Y \right), \end{cases} \tag{12.30}$$

has a solution $(\mathcal{P}_Y, S_Y, K_Y) \in \mathcal{B}(H) \times \mathcal{B}(Y \times Z) \times \mathcal{B}(H_1^*, Y \times Z)$ *s.t.* $\mathcal{P}_Y \geq 0$, $S_{Y11} \gg 0$, $S_{Y22} - S_{Y21}S_{Y11}^{-1}S_{Y12} \ll 0$ *and* K_Y *is exponentially stabilizing for* $\left(\begin{array}{c|cc} A^* & C_2^* & C_1^* \end{array}\right)$.

(3.) **(Coupling condition)** $\rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2$.

Here "(1.')" is the condition of Proposition 12.1.10 and "(2.')" is its dual condition (i.e., equal to (2.) with corresponding modifications).

However, we do not know whether the converse claims hold in general.

Proof: Assume that there is a suboptimal exponentially stabilizing DPF-controller for Σ (the proof below applies to the latter claim too, mutatis mutandis).

By Theorem 12.1.8, conditions (1.) and (4.) of Theorem 12.1.8 hold. Apply Lemma 12.5.6 to obtain that "(1.)" holds for Σ_d too (see p. 740), i.e., that (2.) holds.

It was noted in I.2°–I.3° the proof of Theorem 12.1.8 that (Factor1) is satisfied with $\mathbb{M}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$. We conclude from Lemma 12.5.18 and Lemma 12.6.4(a) that (3.) holds.

Remark — Why the converse is open: For the exponential claim, the problem is that since we have here given up the condition "$S = D^*JD$" of Hypothesis 8.4.8, we can no longer use part 3° of Lemma 12.5.20 to show that S_Z is as required in (4.), so that Theorem 12.1.8 would be applicable.

For the latter claim, the problem is that Lemma 12.6.4(b) does not say anything of the preservation of I/O-stabilization (from $\mathcal{P}_\mathrm{Y}$-DARE to $\mathcal{P}_\mathrm{Z}$-DARE), hence we would only obtain an internally P-stabilizing solution, which is not enough for derivation of (4.') (even if we would assume Hypothesis 8.4.8 to be able to establish the requirements on S_Z); in discrete-time, we face the same problem (although there the problem on S_Z disappears, by Lemma 12.6.4(c)). □

In fact, in the exponential case it suffices that Σ is somewhat smooth:

Lemma 12.1.13 ($\mathbf{H}^\infty$ 4BP $\Rightarrow$ (1.)–(3.)) *Assume that* $(\Sigma_\mathrm{X}, J_\gamma), (\Sigma_\mathrm{Y}, J_\gamma) \in$ coerciveCARE *over* $\mathcal{U}_\mathrm{exp}$ *and that* $\mathbb{D}_{11}$ *and* $\mathbb{D}_{22}^\mathrm{d}$ *are I-coercive over* $\mathcal{U}_\mathrm{exp}$.

If there is an exponentially stabilizing DPF-controller for Σ*, then (1.)–(3.) of Lemma 12.1.12 hold (with* s-lim *in place of* lim*).*

See Remark 12.1.6 for different (equivalent) forms of (1.)–(3.), and Remark 12.1.7 for the above coercivity conditions. See the remark in the proof of Lemma 12.1.12 for why the converse is open.

Proof: (Note that if any of (1.)–(6.) of Remark 9.9.14 holds, then we need not replace lim by s-lim, by Lemma 9.11.5(e).)

We observe from Lemma 12.5.7 that the assumptions of Lemma 11.2.20 are satisfied for Σ_X (we need the coercivity assumption on $\mathbb{D}_{11}$ to satisfy Hypothesis 11.2.1). Consequently, (1.) is satisfied. By dual arguments (see Lemma 12.5.6), we obtain (2.).

Let $\mathbb{O}$ be an exponentially stabilizing DPF-controller for Σ. By discretization (see Theorem 13.4.4(e1)), we observe that $\Delta^\mathrm{S}\mathbb{O}$ is an exponentially stabilizing DPF-controller for $\Delta^\mathrm{S}\Sigma$, hence conditions (1.)–(3.) of Theorem 12.2.1 are satisfied (use Theorem 13.4.4(g) for its coercivity conditions (12.32) and (12.33)), even by same $\mathcal{P}_\mathrm{X}$ and $\mathcal{P}_\mathrm{Y}$, by Proposition 9.8.7(a) (and uniqueness, see Theorem 14.1.4(a)). Thus, (3.) holds. □

Notes

Our result, Theorem 12.1.4 (and Theorem 12.1.8), is of standard form and extends and generalizes at least most nonsingular state-space solutions to the H^∞ 4BP. In the finite-dimensional case, such earlier results include Theorem 1 of [GD88] Theorem 3 of [DGKF] Theorem 5.1 of [GGLD], Theorem 8.3.2 of [GL], Theorem 16.4 of [ZDG] and Theorem 10.3.1 of [IOW].

(Note that most of them interchange the subindices corresponding to u and w, as compared to our formulae. As explained on p. 317, we have used the "u comes before w" practice, which is more popular in the FICP literature, See also Lemma 12.6.2 and the rest of Section 12.6 for how the notation of [IOW] corresponds to that of ours.)

The early history of the problem is explained on p. 328 of [IOW], where it is said that the assumptions and formulae of [IOW] are more general than any earlier (nonsingular) ones; those assumptions and formulae are essentially the same as ours, except that they assume that $\gamma = 1$ and $D_{21} = 0$. In addition, we also treat non-well-posed controllers.

In some of the above results, one only speaks of $\mathbb{Q}$ stabilizing $\mathbb{D}$ (cf. Section 12.3), but in them it is assumed that $\mathbb{D}$ has an exponentially stabilizable and detectable realization and that such a realization is chosen for $\mathbb{Q}$ too, so that one ends up with our setting (see Theorem 7.3.11(c1)).

These results were extended to smooth Pritchard–Salamon systems in Theorem 5.4 of [Keu], by Bert van Keulen. Although our results allow for approximately twice as much unboundedness as the Pritchard–Salamon class does, the result in [Keu] is not exactly contained in ours: in [Keu], the Riccati equation "(2.)" is given on a space $\mathcal{W}$ embedded in $H(:= \mathcal{V})$. The two Riccati equations in [Keu] correspond to the "bounded B" case of the FICP (for Σ and for its "dual" Σ_{d}). (Note also that any exponentially stable (not necessarily smooth) Pritchard–Salamon system satisfies (A1)(V) of Theorem 12.1.4, by Theorem 6.9.6.)

All results mentioned above also make the coercivity assumptions (A2). The *singular* case, where (A2) is replaced by something weaker, is treated in, e.g., [Stoorvogel] for the finite-dimensional case; in that case the proofs and solutions become more complicated than in the standard setting. See Section 17.3 of [ZDG] or Section 5.4.2 of [Keu] for a discussion on how to circumvent (A2) by using "ε-perturbations" of the system (that satisfy (A2)) and then letting $\varepsilon \to 0+$.

The notes on pp. 446–447 of [ZDG] describe the historical development of solutions to the finite-dimensional H$^\infty$ 4BP through several computationally difficult formulations to the simple "(1.)–(3.)" formulation of [GD88] and this section. Also the first few paragraphs of the notes on p. 628 are relevant to the H$^\infty$ 4BP.

Outside this monograph, we do not know any research on nonexponentially stabilizing suboptimal controllers (cf. Proposition 12.1.10 and Lemma 12.1.12).

In Section 12.3, we shall solve the frequency-domain 4BP (the I/O map 4BP), see Theorems 12.3.6 and 12.3.7. (This is close to the Youla parameterization approach of [Doyle84] and [Francis87].) We obtain that the 4BP is equivalent to two nested spectral factorizations "(Factor1X)" and "(Factor2Z)" (equivalently, to two nested (J_1, J_1)-lossless coprime factorizations).

The CAREs (1.) and (4.) of Theorem 12.1.8 correspond to these two factorizations. The CARE (2.) is the dual of (1.) (and the 4BP is invariant under duality; see also Lemma 12.5.6).

In Lemma 12.5.18, we shall show that the $\mathcal{P}_Z$-CARE (4.) is equivalent to (2.)&(3.), thus completing the proof that (1.)–(3.) are equivalent to the solvability of the 4BP. Since a direct continuous-time proof seems almost impossible (unless, e.g., the plant Σ has bounded generators), we have reduced the proof to the discrete time, where the standard computations can be extended to the infinite-dimensional case (see Lemma 12.6.4).

As in other chapters, here and in Section 12.5, we encounter the fact that the equivalence between the CAREs and the factorizations require certain regularity assumptions, and so does also the equivalence between the factorizations and the J-coercivity properties connected to the solvability of the 4BP (cf. Example 11.3.7); this is why most of our results have some kind of $\widetilde{\mathcal{A}}$ regularity assumption.

In discrete time, we have no such problems (since $\mathrm{tic}_{\mathrm{exp}} = \widetilde{\mathcal{A}}$ is a valid choice, by Theorem 14.3.2); see Theorem 12.2.1.

The parametrization (12.26) of all suboptimal controllers will be obtained in the frequency-domain theory (see Theorem 12.3.7), by reducing the problem to a frequency-domain FICP (see Theorem 12.3.7(c) and Proposition 12.5.19). To get this parametrization satisfactory, we must have a realization of the I/O map $\mathbb{T}$ (e.g., the one given by (12.20)–(12.25)); this will be done in Proposition 12.5.19, simply by following the steps guided by (12.49)–(12.50).

We also note that the solution in Theorem 12.1.8 is not symmetric; by replacing Σ by Σ_{d} (see p. 740) in the proofs, one would obtain (2.) in place of (1.) and a fourth Riccati equation ("(5.)") in place of (4.).

12.2 The discrete-time H$^\infty$ problem (H$^\infty$ 4bp)

If you only have a hammer, you tend to see every problem as a nail.
— Abraham Maslow (1908–1970)

As mentioned above, we assume that the discrete-time form of Standing Hypothesis 12.1.1 holds i.e., we consider the system

$$\begin{cases} x_{n+1} = Ax_n + B_1u_n + B_2w_n, \\ \quad z_n = C_1x_n + D_{11}u_n + D_{12}w_n, \qquad (n \in \mathbf{N}) \\ \quad y_n = C_2x_n + D_{21}u_n + D_{22}w_n \end{cases} \tag{12.31}$$

with initial state $x_0 \in H$, disturbance input $w \in \ell^2(\mathbf{N};W)$, control input $u \in \ell^2(\mathbf{N};U)$, objective output $z \in \ell^2(\mathbf{N};Z)$ and measurement output $y \in \ell^2(\mathbf{N};Y)$ (the controller input); here $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U \times W, H \times Z \times Y)$ are the generators of Σ (see Lemma 13.3.3). (As noted in Lemma 14.3.5, we can have, e.g., $\widetilde{\mathcal{A}} = \ell^1_+*$ or $\widetilde{\mathcal{A}} = \mathrm{tic}_{\exp}$.)

We now present the discrete-time counterpart of the theory of Section 12.1, i.e., we try to find a controller $\mathbb{Q} : y \mapsto u$ (possibly with internal loop; as in the continuous-time case, internal loop is unnecessary at least when $D_{21} = 0$) s.t. it stabilizes the above system exponentially and makes the norm $\|w \mapsto z\|_{\mathcal{B}(\ell^2)}$ less than the given number $\gamma > 0$. (See the explanation on p. 36 for the H$^\infty$ four-block problem.) We also record the discrete-time forms of all other results in this chapter (Theorem 12.2.2).

We first present the discrete-time counterpart of Theorem 12.1.4, and then we briefly list the other results of Section 12.1 that can be converted to discrete time (the most important of which is the parametrization of all suboptimal controllers, Theorem 12.1.8).

Theorem 12.2.1 (H$^\infty$ 4BP $\Leftrightarrow$ DAREs) *Assume that there is $\varepsilon > 0$ s.t.*

$$(z - A)x_0 = B_1u_0 \implies \|C_1x_0 + D_{11}u_0\|_Z \geq \varepsilon(\|x_0\|_H + \|u_0\|_U) \quad \textit{and} \tag{12.32}$$
$$(z - A^*)x_0 = C_2^*y_0 \implies \|B_2^*x_0 + D_{22}^*y_0\|_W \geq \varepsilon(\|x_0\|_H + \|y_0\|_Y) \tag{12.33}$$

for all $x_0 \in H$, $u_0 \in U$, $y_0 \in Y$, $z \in \partial\mathbf{D}$. (Alternatively, we can assume that Hypothesis 12.5.1 is satisfied.)

Then there is a suboptimal exponentially stabilizing DPF-controller for Σ iff (1.)–(3.) hold:

(1.) **($\mathcal{P}_X$-DARE)** *the DARE*

$$\begin{cases} \mathcal{P}_X = A^*\mathcal{P}_XA + C_1^*C_1 - K_X^*S_XK_X, \\ S_X = \begin{bmatrix} D_{11}^*D_{11} & D_{11}^*D_{12} \\ D_{12}^*D_{11} & D_{12}^*D_{12}-\gamma^2 I \end{bmatrix} + B^*\mathcal{P}_XB, \\ K_X = -S_X^{-1}(\begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C_1 + B^*\mathcal{P}_XA), \end{cases} \tag{12.34}$$

has a solution $(\mathcal{P}_X, S_X, K_X) \in \mathcal{B}(H) \times \mathcal{B}(U \times W) \times \mathcal{B}(H, U \times W)$ s.t. $\mathcal{P}_X \geq 0$, $S_{X11} \gg 0$, $S_{X22} - S_{X21}S_{X11}^{-1}S_{X12} \ll 0$ and $\rho(A + BK_X) < 1$;

(2.) **($\mathcal{P}_Y$-DARE)** *the DARE*

$$\begin{cases} \mathcal{P}_Y = A\mathcal{P}_Y A^* + B_2 B_2^* - K_Y^* S_Y K_Y, \\ S_Y = \begin{bmatrix} D_{22}D_{22}^* & D_{22}D_{12}^* \\ D_{12}D_{22}^* & D_{22}D_{22}^* - \gamma^2 I \end{bmatrix} + \begin{bmatrix} C_2 \\ C_1 \end{bmatrix} \mathcal{P}_Y \begin{bmatrix} C_2^* & C_1^* \end{bmatrix}, \\ K_Y = -S_Y^{-1}\left(\begin{bmatrix} D_{22} \\ D_{12} \end{bmatrix} B_2^* + \begin{bmatrix} C_2 \\ C_1 \end{bmatrix} \mathcal{P}_Y A^* \right), \end{cases} \tag{12.35}$$

has a solution $(\mathcal{P}_Y, S_Y, K_Y) \in \mathcal{B}(H) \times \mathcal{B}(Y \times Z) \times \mathcal{B}(H, Y \times Z)$ *s.t.* $\mathcal{P}_Y \geq 0$, $S_{Y11} \gg 0$, $S_{Y22} - S_{Y21}S_{Y11}^{-1}S_{Y12} \ll 0$ *and* $\rho(A^* + \begin{bmatrix} C_2^* & C_1^* \end{bmatrix} K_Y) < 1$*;*

(3.) **(Coupling condition)** $\rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2$.

If (1.)–(3.) hold, then $\Sigma_{\mathbb{T}_{12}}$ is a suboptimal DPF-controller for Σ (the central controller*), and all suboptimal DPF-controllers are parametrized in Theorem 12.1.8 (see Figure 12.1).*

Note that (12.34) is the DARE for Σ_X and J_γ (exactly as in Theorem 11.5.1), and (12.35) is the DARE for Σ_Y and J_γ. See also the remarks in Section 12.1.

Proof: (We shall again refer to continuous-time results, thereby meaning their discrete-time forms; cf. Theorem 13.3.13 and Theorem 12.2.2.)

Set $\widetilde{\mathcal{A}} := \mathrm{tic}_{\mathrm{exp}}$ (recall from Lemma 14.3.5 that $\mathrm{tic}_{\mathrm{exp}}$ satisfies Standing Hypothesis 12.0.1).

1° *Necessity of (1.)–(3.):* Necessity follows from, e.g., Lemma 12.1.13 (recall that "$\in$ coerciveCARE" is redundant in discrete time, as noted below Remark 9.9.14).

2° *Sufficiency under Hypothesis 12.5.1:* Even though we would assume no more than Standing Hypothesis 12.1.1, we would obtain from Lemma 12.6.4(b)&(c), that conditions (1.) and (4.) hold iff (1.)–(3.) hold.

Consequently, under (1.)–(3.) and Hypothesis 12.5.1, we obtain the existence of an exponentially stabilizing suboptimal DPF-controller from Theorem 12.1.8.

3° *Sufficiency under assumptions (12.32)–(12.33):* Assume (1.)–(3.) and (12.32)–(12.33). Then Hypothesis 12.5.1 is satisfied (even with "exponentially jointly" in place of "jointly") by Lemmas 12.6.7 and 12.6.6. Consequently, we can apply 2°.

4° *Remarks:* As noted in 2°, (1.) and (4.) hold iff (1.)–(3.) hold. However, the equivalence to the existence of an exponentially stabilizing DPF-controller requires further conditions (e.g., if $B = 0 = D$, then necessarily $S_X = 0$, so that (1.) cannot hold), such as the ones used in 2° or 3°.

Note from Proposition 15.2.2(c) that (12.32) and (12.33) say that $\left(\left(\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right), I\right)$ and $\left(\left(\begin{array}{c|c} A^* & C_2^* \\ \hline B_2^* & D_{22}^* \end{array}\right), I\right)$ are I-coercive over $\mathcal{U}_{\mathrm{exp}}$. as one can verify from the proof below, even weaker assumptions would suffice. □

Practically all our H^∞ 4BP results hold also in their discrete-time forms:

Theorem 12.2.2 (Discrete form of H^∞ 4BP results) *The following results hold also in their discrete-time forms (i.e., after the changes listed in Theorem 13.3.13):*

Lemmas 12.1.3 and 12.1.12, Theorems 12.1.11 and 12.1.8, and everything in Sections 12.3, 12.4 and 12.5.

Moreover, in the exponential case of Lemma 12.1.12 also the converse holds (this is Theorem 12.2.1 with the alternative assumption).

The $\mathcal{P}_Z$-DARE means the DARE for Σ_Z =(12.123) and J_1, hence the condition (4.) can now be written as

(4.) **($\mathcal{P}_Z$-DARE)** *The DARE (see (12.94))*

$$\begin{cases} K_Z^* S_Z K_Z = A_Z^* \mathcal{P}_Z A_Z - \mathcal{P}_Z + B_2 X_{22}^{-1} X_{22}^{-*} B_2^*, \\ S_Z = D_Z^* J D_Z + B_Z^* \mathcal{P}_Z B_Z, \\ K_Z = -S_Z^{-1}\left(\begin{bmatrix} D_{22}X_{22}^{-1} \\ X_{12}X_{22}^{-1} \end{bmatrix} X_{22}^{-*} B_2^* + B_Z^* \mathcal{P}_Z A_Z \right), \end{cases} \tag{12.36}$$

has a solution $(\mathcal{P}_Z, S_Z, K_Z) \in \mathcal{B}(H) \times \mathcal{B}(Y \times Z) \times \mathcal{B}(H, Y \times U)$ s.t. $\mathcal{P}_Z \geq 0$, $S_{Z11} \gg 0$, $S_{Z22} - S_{Z21} S_{Z11}^{-1} S_{Z12} \ll 0$ and $\rho(A_Z + B_Z K_Z) < 1$.

As noted around Example 14.2.9, we almost never have "$S = D^* J D$" in discrete time (thus, in practice we only meet the DARE equivalent of the "weakest" of the CAREs in Remark 12.1.6, and Theorem 12.1.11 becomes rather unnecessary).

See the remarks below Lemma 12.6.4 for the three DAREs; in particular, X must be chosen as in Lemma 12.6.1 (equivalently, as in Theorem 12.1.8).

Proof of Theorem 12.2.2: *Remark:* Recall that these changes include CARE$\mapsto$DARE, i.e., $A_*^* \mathcal{P}_* + \mathcal{P}_* A_* \mapsto A_*^* \mathcal{P}_* A_* - \mathcal{P}_*$ etc., as above.

The proof: This follows roughly by applying (13.63) also to the proofs (recall from Lemma 14.3.5 that $\widetilde{\mathcal{A}} := \mathrm{tic}_{\exp}$ satisfies Standing Hypothesis 12.0.1).

Alternatively, one could use Lemma 12.6.7, Lemma 12.6.6, Lemma 13.1.7, Lemma 6.6.11 and Lemma 13.3.12 to make the proof slightly shorter (and more "discrete-time self-contained"). □

Notes

For finite-dimensional systems, the discrete-time H$^\infty$ 4BP is more complicated than the continuous-time H$^\infty$ 4BP — the same holds for infinite-dimensional systems if we require the input and output operators to be bounded — but in general the continuous-time setting becomes very complicated. This is why part of the proof of our continuous-time results (in particular, the equivalence of (1.)–(3.) and (1.)&(4.)) has been reduced to discrete time (in the last two sections of this chapter).

Theorem 12.2.1 extends the classical nonsingular results to the infinite-dimensional case. Theorem 10.12.1 of [IOW] is possibly the most general of all the nonsingular finite-dimensional results; it is essentially Theorem 12.2.1 (and Theorem 12.1.8) with the assumptions that $\gamma = 1$ and $D_{21} = 0$. Section B.4.2 of [GL] contains a result close to (the discrete-time form of) Theorem 12.1.8. See also the notes on p. 706. The history of the solutions for the discrete-time H$^\infty$ 4BP is explained on p. 501 of [GL].

The nonsingular finite-dimensional case (where (12.32)–(12.33) have been replaced by weaker assumptions) has been treated in [Stoorvogel].

12.3 The frequency-space (I/O) H$^\infty$ 4BP

From a certain point onward there is no longer any turning back. That is the point that must be reached.

— Franz Kafka (1883–1924)

In this section, we solve the *frequency-domain (or I/O)* H$^\infty$ *Four-Block Problem (I/O* H$^\infty$ *4BP)*. This means that, given a plant $\mathbb{D} : \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$ and $\gamma > 0$, we determine whether there is a DPF-controller $\mathbb{Q} : y \mapsto u$ for $\mathbb{D}$ that makes the norm $\|w \mapsto z\| = \|\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q})\|$ less than γ.

In Theorem 12.3.6, we extend to $\mathrm{MTIC}_{T\mathbf{Z}}$ (and beyond) the fact that this problem has a solution iff certain two nested lossless coprime factorizations exist (for rational maps this was established in [Green]), and we parameterize all solutions in terms of these factorizations. The exact conditions on the factorizations depend on whether we require $\mathbb{Q}$ to be well-posed (i.e., without internal loop) or not. In Theorem 12.3.7(a)&(d), the sufficiency part of the above equivalence is extended to general WPLSs (we also extend the necessity part under the assumption that certain maps admit spectral factorization). Theorem 12.3.5 connects these frequency domain solutions to the state-space problem.

First we list the standard I/O H$^\infty$ 4BP assumptions and define the problem.

Standing Hypothesis 12.3.1 (I/O 4BP assumptions) *Throughout this section and Section 12.4, we assume that* $\mathbb{D} = \left[\begin{smallmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{smallmatrix}\right] \in \mathrm{TIC}_\infty(U \times W, Z \times Y)$, *that* $\gamma > 0$, *and that* $\mathbb{D}$ *has a d.c.f.* $\mathbb{D} = \mathbb{N}_{\mathrm{u}} \mathbb{M}_{\mathrm{u}}^{-1} = \widetilde{\mathbb{M}}_{\mathrm{y}}^{-1} \widetilde{\mathbb{N}}_{\mathrm{y}}$ *of the form*

$$\mathbb{D} = \begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ \mathbb{N}_{\mathrm{u}\,21} & \mathbb{N}_{\mathrm{u}\,22} \end{bmatrix} \begin{bmatrix} \mathbb{M}_{\mathrm{u}\,11} & \mathbb{M}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}^{-1} = \begin{bmatrix} I & \widetilde{\mathbb{M}}_{\mathrm{y}\,12} \\ 0 & \widetilde{\mathbb{M}}_{\mathrm{y}\,22} \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{\mathbb{N}}_{\mathrm{y}\,11} & \widetilde{\mathbb{N}}_{\mathrm{y}\,12} \\ \widetilde{\mathbb{N}}_{\mathrm{y}\,21} & \widetilde{\mathbb{N}}_{\mathrm{y}\,22} \end{bmatrix} \tag{12.37}$$

s.t. $\mathbb{N}_{\mathrm{u}\,21}$ *and* $\mathbb{M}_{\mathrm{u}\,11}$ *are r.c. and* $\widetilde{\mathbb{N}}_{\mathrm{y}\,21}$ *and* $\widetilde{\mathbb{M}}_{\mathrm{y}\,22}$ *are l.c. We also make the nonsingularity assumptions*

$$\mathbb{N}_{\mathrm{u}\,11}^{*} \mathbb{N}_{\mathrm{u}\,11} \gg 0, \quad \widetilde{\mathbb{N}}_{\mathrm{y}\,22} \widetilde{\mathbb{N}}_{\mathrm{y}\,22}^{*} \gg 0. \tag{12.38}$$

By Proposition 7.3.14 and Lemma 7.3.16, any d.c.f. of $\mathbb{D}$ of form (12.37) satisfies also the rest of the above hypothesis (when the hypothesis holds).

Obviously, the hypothesis is a generalization of the assumptions of Theorem 4.4 of [Green] and of those of Theorem 5.6 of [CG97]. Therefore, Theorem 12.3.6 generalizes those (frequency-domain) results. The d.c.f. assumption roughly says that $\mathbb{D}$ can be stabilized through y and u, and (12.38) is the standard nonsingularity assumption.

The results of this section will also be used in the proof of the results of Section 12.1; indeed, under assumptions (A1) and (A2) of Theorem 12.1.5 (or (A1) and (A2) of Theorem 12.1.4), the existence of an exponentially stabilizing controller for Σ (alternatively, conditions (1.)–(3.)) implies the above hypothesis, as noted in the proof of the theorem. Thus, the results of this section also apply to any classical (nonsingular) state-space H$^\infty$ 4BPs.

The DPF-stabilizing controllers for $\mathbb{D}$ are Youla parametrized in Corollary 7.3.20, in which the map $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q}) : w \mapsto z$ was defined; we repeat that definition here:

Lemma 12.3.2 ($\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})$) *Let $\mathbb{Q}$ DPF-stabilize $\mathbb{D}$ (with internal loop). Then $\mathbb{Q}$ is a map with d.c. internal loop and $\mathbb{Q} = \mathbb{Q}_1\mathbb{Q}_2^{-1} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$, where $\mathbb{Q}_1, \mathbb{Q}_2, \widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in$* TIC, *$\widetilde{\mathbb{M}_{\mathrm{y}}}_{22}\mathbb{Q}_2 - \widetilde{\mathbb{N}_{\mathrm{y}}}_{21}\mathbb{Q}_1 = I$ and $\widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}\,11} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}\,21} = I$.*

The corresponding closed-loop $w \mapsto z$ map is given by

$$\mathcal{F}_\ell(\mathbb{D},\mathbb{Q}) := \mathbb{N}_{\mathrm{u}\,11}\mathbb{U} + \mathbb{N}_{\mathrm{u}\,12} = \widetilde{\mathbb{N}_{\mathrm{y}}}_{12} + \overline{\mathbb{U}}\widetilde{\mathbb{N}_{\mathrm{y}}}_{22}, \tag{12.39}$$

where $\mathbb{U} := \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}\,22} - \widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}\,12}$, $\overline{\mathbb{U}} := \widetilde{\mathbb{N}}_{11}\mathbb{Q}_1 - \widetilde{\mathbb{M}}_{12}\mathbb{Q}_2$. □

For well-posed $\mathbb{Q}$, we have $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q}) := \mathbb{D}_{12} + \mathbb{D}_{11}\mathbb{Q}(I - \mathbb{D}_{21}\mathbb{Q})\mathbb{D}_{22}$, by (7.65). (Note that, in the literature, the latter subindices are often interchanged, i.e., w comes before u.) As noted below Corollary 7.3.20, the map $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})$ depends on $\mathbb{D}$ and $\mathbb{Q}$ only.

We remind the reader that, in this chapter, we often drop "DPF-" and we allow the (DPF-)controllers to be non-well-posed (i.e., to have an internal loop).

We repeat here the definition of a solution of the I/O H^∞ 4BP:

Definition 12.3.3 (I/O H^∞ 4BP) *A map $\mathbb{Q}$ (with internal loop) is a* suboptimal stabilizing DPF-controller (for $\mathbb{D}$) *if $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ and $\|\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})\| < \gamma$.*

A solution of the I/O (H^∞) 4BP (for $\mathbb{D}$) *means a suboptimal stabilizing DPF-controller for $\mathbb{D}$.*

Recall from Section 7.2, that any well-posed (i.e., $\mathrm{TIC}_\infty(Y,U)$) map is a map with internal loop, hence the above definition covers all well-posed controllers too. In the results we shall also specify when there exists a well-posed solution.

The purpose of this suboptimal problem is that its solution can be used for a binary search over γ's to find an estimate of the optimal γ and an "almost optimal" $\mathbb{Q}$.

The solution of the 4BP is interplay between the original problem and its dual, hence we record the following (recall that $\mathbb{E}^{\mathrm{d}} := \boldsymbol{Я}\mathbb{E}^*\boldsymbol{Я}$):

Lemma 12.3.4 (Dual problem) *The map $\mathbb{D}_{\mathrm{d}} := \begin{bmatrix} \mathbb{D}_{22}^{\mathrm{d}} & \mathbb{D}_{12}^{\mathrm{d}} \\ \mathbb{D}_{21}^{\mathrm{d}} & \mathbb{D}_{11}^{\mathrm{d}} \end{bmatrix}$ also satisfies Standing Hypothesis 12.3.1. Moreover, $\mathbb{Q}$ is suboptimal for $\mathbb{D}$ iff $\mathbb{Q}^{\mathrm{d}}$ is suboptimal for $\mathbb{D}_{\mathrm{d}}$.* □

(This follows from Proposition 7.3.4(d).

Next we note how a solution of the frequency-domain problem of this section leads to a solution of the corresponding state-space problem.

If Σ and $\widetilde{\Sigma}$ are realizations of $\mathbb{D}$ and $\mathbb{Q}$, respectively (if $\mathbb{Q}$ is considered as a map with internal loop, then $\widetilde{\Sigma}$ may be any realization of a representative of $\mathbb{Q}$), then, obviously, $\widetilde{\Sigma}$ I/O-DPF-stabilizes Σ iff $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$. If Σ is SOS-stabilizable, then $\widetilde{\Sigma}$ can be chosen to be SOS-DPF-stabilizing, i.e., such that the resulting closed-loop system is SOS-stable; similar claims hold also for stronger stabilizability properties of Σ:

Theorem 12.3.5 (I/O 4BP ⇒ 4BP) *Let $\mathbb{Q}$ be a stabilizing DPF-controller for $\mathbb{D}$ (with internal loop). Let Σ be a realization of $\mathbb{D}$. Given a realization $\widetilde{\Sigma}$ or $\mathbb{Q}$, the resulting closed-loop connection system will be denoted by Σ_I^o (see (7.60) and (6.125) for Σ_I^o; cf. Figure 7.11).*

Then Theorem 7.2.3 applies, in particular, the following holds:

(a) If Σ is SOS-stabilizable, then $\widetilde{\Sigma}$ can be chosen s.t. $\Sigma_I^o \in \mathrm{SOS}$.

(b) If Σ is [strongly] r.c.-stabilizable, then $\widetilde{\Sigma}$ can be chosen s.t. Σ_I^o is [strongly] stable.

(c) If Σ is stabilizable and [strongly] detectable, then $\widetilde{\Sigma}$ can be chosen s.t. Σ_I^o is [strongly] stable.

(d) $\mathbb{D}$ and $\mathbb{Q}$ have such realizations that their closed-loop connection system Σ_I^o becomes strongly stable.

(e) If Σ_{21} is exponentially jointly stabilizable and detectable, then $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ exponentially with internal loop iff it has a realization that stabilizes Σ exponentially with an internal loop.

Recall from Theorem 6.6.28 that $\mathbb{D}$ has a strongly jointly r.c.-stabilizable and l.c.-detectable realization, because it has a d.c.f., by Standing Hypothesis 12.3.1. Moreover, one can choose the realization so that it satisfies also Hypothesis 12.5.1, by Lemma 12.5.23.

Proof: (a)–(d) Because $\mathbb{Q}$ has a d.c.f., by Corollary 7.3.20, $\mathbb{Q}$ has a strongly jointly r.c.-stabilizable and l.c.-detectable realization $\widetilde{\Sigma}$, by Theorem 6.6.28 (if $\mathbb{Q}$ is a well-posed controller; in the general (non-well-posed) case we may take $\widetilde{\Sigma}$ to be a strongly stable realization (as in Definition 6.1.6) of a stable representative of $\mathbb{Q}$ (cf. Definition 7.2.11)).

By Theorem 7.2.3, $\widetilde{\Sigma}$ stabilizes Σ as in (a)–(d) (for (d) we use the fact that, by Theorem 6.6.28, Σ can be chosen to be as in (b)).

(e) This is contained in Lemma 7.3.6(b1) (even without any standing assumptions). □

Michael Green showed in [Green] (Theorem 4.4) that the frequency-domain H^∞ 4BP has a solution iff certain two nested spectral factorizations exist (in the rational finite-dimensional case). This result was extended to $\mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}$ with $\dim U \times W \times Y \times Z < \infty$ by Green and Ruth Curtain [CG97] (Theorem 5.6). The following (see (c)) is a direct generalization of these results:

Theorem 12.3.6 ($\widetilde{\mathcal{A}}$: $\mathbf{H}^\infty$ 4BP (I/O)) *Assume that $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$. Then conditions (4BP1$\widetilde{\mathcal{A}}$)–(4BP3$\widetilde{\mathcal{A}}$) are equivalent:*

(4BP1$\widetilde{\mathcal{A}}$) The I/O 4BP has a well-posed solution "in $\widetilde{\mathcal{A}}$"; i.e., there are $\widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}(U) \cap \mathcal{G}\mathrm{TIC}_\infty$ and $\widetilde{\mathbb{Q}}_1 \in \widetilde{\mathcal{A}}(Y,U)$ s.t. $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ (DPF-)stabilizes $\mathbb{D}$ and makes $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q}) < \gamma$.

(4BP2$\widetilde{\mathcal{A}}$) (Factor1$\widetilde{\mathcal{A}}$) holds, and there is a solution $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}$ to the ASP

$$\begin{bmatrix} \widetilde{\mathbb{U}} & I \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \begin{bmatrix} \mathbb{N}_{22} & -\mathbb{N}_{21} \\ -\mathbb{M}_{12} & \mathbb{M}_{11} \end{bmatrix} \textit{ for some } \widetilde{\mathbb{U}} \in \mathrm{TIC} \textit{ with } \|\widetilde{\mathbb{U}}\| < 1 \tag{12.40}$$

s.t. $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

(4BP3$\widetilde{\mathcal{A}}$) (Factor1$\widetilde{\mathcal{A}}$) and (Factor2$\widetilde{\mathcal{A}}$) hold.

Here we refer to the following:

(Factor1$\widetilde{\mathcal{A}}$) There is a r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *s.t.* $\widehat{\mathbb{M}}_{11}(+\infty), \widehat{\mathbb{M}}_{22}(+\infty) \in \mathcal{G}\mathcal{B}$.

$$\begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix} \in \mathrm{TIC}(U \times W, Z \times W) \ \ \textit{is } (J_\gamma, J_1)\textit{-lossless.} \tag{12.41}$$

(Factor2$\widetilde{\mathcal{A}}$) The map $\mathbb{D}_+ := \left[\begin{smallmatrix} \mathbb{N}_{21} & \mathbb{N}_{22} \\ I & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ 0 & I \end{smallmatrix}\right]^{-1} \in \mathrm{TIC}_\infty(U \times W, Y \times U)$ *has a l.c.f.* $\mathbb{D}_+ = \widetilde{\mathbb{M}}_+^{-1}\widetilde{\mathbb{N}}_+$ *s.t.* $\mathbb{W}^{\mathrm{d}}$ *is* (J_1, J_1)*-lossless and* $\widehat{\widetilde{\mathbb{N}}}_{+21}(+\infty), \widehat{\widetilde{\mathbb{M}}}_{+22}(+\infty) \in \mathcal{G}\mathcal{B}(U)$*, where*

$$\mathbb{W} := \begin{bmatrix} \widetilde{\mathbb{N}}_{+12} & \widetilde{\mathbb{M}}_{+12} \\ \widetilde{\mathbb{N}}_{+22} & \widetilde{\mathbb{M}}_{+22} \end{bmatrix} \in \mathrm{TIC}(W \times U, Y \times U). \tag{12.42}$$

(a) Any r.c.f. of $\mathbb{D}$ *or l.c.f. of* $\mathbb{D}_+$ *is in* $\widetilde{\mathcal{A}}$*. If (Factor1$\widetilde{\mathcal{A}}$) and (Factor2$\widetilde{\mathcal{A}}$) hold, then* $\mathbb{N}, \mathbb{M}, \widetilde{\mathbb{N}}_+, \widetilde{\mathbb{M}}_+ \in \widetilde{\mathcal{A}}$*, and all well-posed suboptimal DPF-stabilizing controllers are given by*

$$\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1, \ \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} := \begin{bmatrix} \mathbb{L} & I \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{M}}_{+11} & \widetilde{\mathbb{N}}_{+11} \\ \widetilde{\mathbb{M}}_{+21} & \widetilde{\mathbb{N}}_{+21} \end{bmatrix}, \ \mathbb{L} \in \mathrm{TIC}(Y, U), \ \|\mathbb{L}\| < 1 \tag{12.43}$$

with the additional condition that $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty(U)$ *(e.g., take* $\mathbb{L} \in \widetilde{\mathcal{A}}$*,* $\widehat{\mathbb{L}}(+\infty) = 0$*).*

We have $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}$ *iff* $\mathbb{L} \in \widetilde{\mathcal{A}}$*.*

(b) If $\widetilde{\mathbb{M}_{\mathrm{y}}}, \widetilde{\mathbb{N}_{\mathrm{y}}} \in \widetilde{\mathcal{A}}$*, then (4BP1$\widetilde{\mathcal{A}}$) holds iff there is a well-posed solution* $\mathbb{Q} = \mathbb{Q}_1\mathbb{Q}_2^{-1}$ *s.t.* $\mathbb{Q}_1, \mathbb{Q}_2 \in \widetilde{\mathcal{A}}$*.*

(c) Conditions "$\widehat{\mathbb{M}}_{22}(+\infty), \widehat{\widetilde{\mathbb{M}}}_{+22}(+\infty) \in \mathcal{G}\mathcal{B}(U)$*" are redundant if* $\dim U < \infty$*.*

(d) If $D_{21} = 0$*, then (4BP1$\widetilde{\mathcal{A}}$) is equivalent to (4BP1), i.e., if there is any suboptimal stabilizing DPF-controller for* $\mathbb{D}$*, then there is a well-posed suboptimal stabilizing DPF-controller for* $\mathbb{D}$ *"in* $\widetilde{\mathcal{A}}$*".* □

(This will be proved in Lemma 12.4.16.)

By $\widetilde{\mathcal{A}}(U)^{-1}\widetilde{\mathcal{A}}(Y,U)$ we mean maps of form $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ s.t. $\widetilde{\mathbb{Q}}_1 \in \widetilde{\mathcal{A}}(Y,U)$, $\widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}(U)$ and $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty$.

See Definition 6.4.4 for lossless factorizations. Note that we could replace "(J_1,J_1)-lossless" by "frequency-domain (J_1,J_1)-lossless" in (Factor1$\widetilde{\mathcal{A}}$) and (Factor2$\widetilde{\mathcal{A}}$), by Corollary 2.5.5.

The factorization $\mathbb{D}_+ := \left[\begin{smallmatrix} \mathbb{N}_{21} & \mathbb{N}_{22} \\ I & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \mathbb{M}_{11} & \mathbb{M}_{12} \\ 0 & I \end{smallmatrix}\right]^{-1}$ is a r.c.f. when (Factor1$\widetilde{\mathcal{A}}$) holds, by Remark 12.4.5.

The above *ASP* (analytic system problem) formulations are due to [Green].

Above we assumed that $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$. In the general case, we cannot guarantee the sufficient factorization conditions to be necessary, and the conditions become slightly more complicated unless we assume some regularity. This is stated below; we also give weaker sufficient conditions for the equivalence.

Theorem 12.3.7 (I/O H$^\infty$ 4BP) *We consider the following conditions:*

(4BP1) The I/O 4BP has a solution, i.e., some $\mathbb{Q}$ stabilizes $\mathbb{D}$ and makes $\mathcal{F}_\ell(\mathbb{D},\mathbb{Q}) < \gamma$.

(4BP2) (Factor1) holds, and there is a solution $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}$ to the ASP

$$\begin{bmatrix}\widetilde{\mathbb{U}} & I\end{bmatrix} = \begin{bmatrix}\widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2\end{bmatrix}\begin{bmatrix}\mathbb{N}_{22} & -\mathbb{N}_{21}\\ -\mathbb{M}_{12} & \mathbb{M}_{11}\end{bmatrix} \text{ for some } \widetilde{\mathbb{U}} \in \mathrm{TIC} \text{ with } \|\widetilde{\mathbb{U}}\| < 1. \tag{12.44}$$

(4BP3) (Factor1) and (Factor2) hold.

(Factor1) There is a r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ s.t. $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$ and (12.41) holds.

(Factor1X) There is $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U\times W)$, s.t. $\mathbb{X}^* J_1 \mathbb{X} = \left[\begin{smallmatrix}\mathbb{N}_{u11} & \mathbb{N}_{u12}\\ 0 & I\end{smallmatrix}\right]^* J_\gamma \left[\begin{smallmatrix}\mathbb{N}_{u11} & \mathbb{N}_{u12}\\ 0 & I\end{smallmatrix}\right]$ and $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}(U)$.

(Factor2) The map $\mathbb{D}_+ := \left[\begin{smallmatrix}\mathbb{N}_{21} & \mathbb{N}_{22}\\ I & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}\mathbb{M}_{11} & \mathbb{M}_{12}\\ 0 & I\end{smallmatrix}\right]^{-1}$ with d.c. internal loop can be written as $\mathbb{D}_+ = \widetilde{\mathbb{M}}_+^{-1}\widetilde{\mathbb{N}}_+$ (cf. Remark 12.4.5) so that $\mathbb{W}^{\mathrm{d}}$ is (J_1,J_1)-lossless and $\mathbb{W}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, where

$$\mathbb{W} := \begin{bmatrix}\widetilde{\mathbb{N}}_{+12} & \widetilde{\mathbb{M}}_{+12}\\ \widetilde{\mathbb{N}}_{+22} & \widetilde{\mathbb{M}}_{+22}\end{bmatrix} \in \mathrm{TIC}(W\times U, Y\times U). \tag{12.45}$$

(Factor2Z) There is $\mathbb{Z} \in \mathcal{G}\mathrm{TIC}(Y\times U)$ s.t. $\mathbb{E}J_1\mathbb{E}^* = \mathbb{Z}J_1\mathbb{Z}^*$, and $(\mathbb{Z}^{-1}\mathbb{E})_{22} \in \mathcal{G}\mathrm{TIC}$, where

$$\mathbb{E} := \begin{bmatrix}\mathbb{N}_{22} & -\mathbb{N}_{21}\\ -\mathbb{M}_{12} & \mathbb{M}_{11}\end{bmatrix} \in \mathrm{TIC}(W\times U, Y\times U). \tag{12.46}$$

(Note that we define (Factor2) only if (Factor1) holds.) The following holds:

(a) We have (4BP3)⇒(4BP2)⇒(4BP1). If $(\left[\begin{smallmatrix}\mathbb{N}_{u11} & \mathbb{N}_{u12}\\ 0 & I\end{smallmatrix}\right], J_\gamma) \in \mathrm{SpF}$, then (4BP2)⇔(4BP1). If (Factor1) holds and $(\mathbb{E}^{\mathrm{d}}, J_1) \in \mathrm{SpF}$, then (4BP3)⇔(4BP2)⇔(4BP1).

(b) Assume (Factor1). Then (4BP1)⇔(4BP2).

If $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2$ satisfy (4BP2), then a solution of the I/O 4BP is given by $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ (which is obviously a l.c.f.). Conversely, any solution $\mathbb{Q}$ of (4BP1) is of form $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$, where $(\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2)$ solves (4BP2).

(c) If (Factor1) and (Factor2) hold, then all suboptimal DPF-controllers (all solutions to the I/O 4BP) are given by the l.c.f.

$$\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1,\ \begin{bmatrix}\widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2\end{bmatrix} := \begin{bmatrix}\mathbb{L} & I\end{bmatrix}\begin{bmatrix}\widetilde{\mathbb{M}}_{+11} & \widetilde{\mathbb{N}}_{+11}\\ \widetilde{\mathbb{M}}_{+21} & \widetilde{\mathbb{N}}_{+21}\end{bmatrix},\ \mathbb{L} \in \mathrm{TIC}(Y,U),\ \|\mathbb{L}\| < 1 \tag{12.47}$$

(by $\|\mathbb{L}\| \le 1$ we get all $\mathbb{Q}$'s s.t. $\|\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})\| \le \gamma$); the well-posed solutions are parametrized by (12.47) with the additional condition that $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

If $\widetilde{\mathbb{N}}_{+21} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ *(cf. (d)), then (12.47) can be written as* $\mathbb{Q} = \mathcal{F}_\ell(\mathbb{T},\mathbb{L}) := \mathbb{T}_{12} + \mathbb{T}_{11}\mathbb{L}(I-\mathbb{T}_{21}\mathbb{L})^{-1}\mathbb{T}_{22} \in \mathrm{TIC}_\infty(Y,U)$ *(for same* $\mathbb{L}$*'s s.t.* $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty(U)$*; this parametrizes the well-posed suboptimal stabilizing controllers for* $\mathbb{D}$*), where*

$$\mathbb{T} := \begin{bmatrix} \widetilde{\mathbb{N}}_{+11} & I \\ \widetilde{\mathbb{N}}_{+21} & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & \widetilde{\mathbb{M}}_{+11} \\ I & \widetilde{\mathbb{M}}_{+21} \end{bmatrix} \in \mathrm{TIC}_\infty(U\times Y). \tag{12.48}$$

(d) Assume that (Factor1) and (Factor2Z) are satisfied with $\mathbb{Z} \in \mathrm{ULR}$. *Then (Factor2) has a solution having* $\widetilde{\mathbb{N}}_{+21} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ *iff there is a well-posed solution* $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ *of the I/O 4BP s.t.* $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}\cap\mathrm{ULR}$.

(e1) We have (Factor1)⇔(Factor1X), and (Factor2)⇔(Factor2Z).

(e2) The solutions of (Factor1) and (Factor1X) correspond 1-1 to each other through formulae

$$\mathbb{N} = \mathbb{N}_{\mathrm{u}}\mathbb{X}^{-1},\ \ \mathbb{M} = \mathbb{M}_{\mathrm{u}}\mathbb{X}^{-1};\ \ \mathbb{X} = \mathbb{M}^{-1}\mathbb{M}_{\mathrm{u}};\ \ \begin{bmatrix} \mathbb{N}_{\mathrm{u}11} & \mathbb{N}_{\mathrm{u}12} \\ 0 & I \end{bmatrix}\mathbb{X}^{-1} = \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix}. \tag{12.49}$$

The solutions of (Factor2) and (Factor2Z) correspond 1-1 to each other through formulae

$$\widetilde{\mathbb{N}}_+ = \mathbb{Z}^{-1}\begin{bmatrix} 0 & \mathbb{N}_{22} \\ I & -\mathbb{M}_{12} \end{bmatrix},\ \ \widetilde{\mathbb{M}}_+ = \mathbb{Z}^{-1}\begin{bmatrix} I & -\mathbb{N}_{21} \\ 0 & \mathbb{M}_{11} \end{bmatrix};\ \ \mathbb{Z}^{-1} = \begin{bmatrix} \widetilde{\mathbb{M}}_{+11} & \widetilde{\mathbb{N}}_{+11} \\ \widetilde{\mathbb{M}}_{+21} & \widetilde{\mathbb{N}}_{+21} \end{bmatrix} \tag{12.50}$$

(hence $\mathbb{W} := \begin{bmatrix} \widetilde{\mathbb{N}}_{+*2} & \widetilde{\mathbb{M}}_{+*2} \end{bmatrix} = \mathbb{Z}^{-1}\mathbb{E}$*).* □

(This will be proved in Lemma 12.4.14.)

The well-posedness and independence of the above conditions, as well as several additional facts are presented in the lemmas below.

Note that in $\begin{bmatrix} \mathbb{N}_{21} & \mathbb{N}_{22} \\ I & 0 \end{bmatrix} \in \mathrm{TIC}(U\times W, Y\times U)$ in (Factor2) does not have its identity operator on the diagonal. This choice (an analogous was done in [Green] and [CG97]) was done to avoid having to interchange the rows of $\mathbb{E},\mathbb{Z},\mathbb{W}$ etc., which would end up with $\mathbb{W}J_1\mathbb{W}^* = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$.

Obviously, the controller $\mathbb{Q}$ in (12.47) is different for different parameters $\mathbb{L}$. Note that this parametrizes all well-posed solutions one-to-one as well as all non-well-posed solutions one-to-one modulo equivalence (see Definition 7.2.11).

However, the solutions $\begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix}$ of (12.44) are usually different from those of (12.47) (although both parametrize all solutions $\mathbb{Q}$ of the I/O 4BP, by Lemma 12.4.3(c)).

In [Green], the signature operator "$S = J_\gamma$" was used in (Factor1) and (Factor2) instead of this simplest choice "$S = J_1$" (this corresponds to the (J_γ, J_γ)-lossless r.c.f.'s used in [Green] and [CG97] instead of our (J_*, J_1)-lossless r.c.f.'s). Their choice would introduce several additional γ's in the proofs, and the second columns of certain maps would have to be multiplied by $\gamma^{\pm 1}$, but there is no essential difference. See also the corresponding remark below Theorem 12.1.8.

By Theorem 7.3.19, the standard assumptions (see Standing Hypothesis 12.3.1) imply that any stabilizing DPF-controller of the plant $\mathbb{D}$ has a d.c. internal loop (hence it has a d.c.f. if it is well-posed).

Though we use internal loop techniques to handle the temporary plant $\mathbb{D}_+$, the proofs could be written in the well-posed sense whenever $\mathbb{D}_+$ is well-posed, in particular, for the $\widetilde{\mathcal{A}}$ case treated in Theorem 12.3.6. We would mainly just have to refer to Sections 7.1 and 6.4 instead of Section 7.2 (the same change would be needed in Section 7.3 too).

Lemma 12.3.8 ((4BP1)–(4BP3) are independent on $\mathbb{N}_{\rm u}, \mathbb{M}_{\rm u}, \widetilde{\mathbb{N}_{\rm y}}, \widetilde{\mathbb{M}_{\rm y}}$) *Conditions (Factor1)–(Factor2Z) are independent of the preliminary factorizations $\mathbb{N}_{\rm u}\mathbb{M}_{\rm u}{}^{-1}$ and $\widetilde{\mathbb{M}_{\rm y}}{}^{-1}\widetilde{\mathbb{N}_{\rm y}}$ (of $\mathbb{D}$) satisfying Standing Hypothesis 12.3.1, as well as of factors $\mathbb{N}, \mathbb{M}, \mathbb{X}$ (if any) satisfying (Factor1) or (Factor1X) and of factors $\widetilde{\mathbb{N}}_+, \widetilde{\mathbb{M}}_+, \mathbb{Z}$ (if any) satisfying (Factor2[Z]) (i.e., they depend on $\mathbb{D}$ only).*

Moreover, the sets of allowable $(\mathbb{N}, \mathbb{M})$'s, $\mathbb{X}$'s, $\mathbb{Z}$'s and $\mathbb{Q}$'s in above conditions as well as the map $\mathbb{E}^* J_1 \mathbb{E}$ are independent of factors $\mathbb{N}_{\rm u}, \mathbb{M}_{\rm u}, \widetilde{\mathbb{N}_{\rm y}}, \widetilde{\mathbb{M}_{\rm y}}$. The solutions $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2$ of (4BP2) are independent in the sense that they always define the same set of $\mathbb{Q}$'s (which is the set of $\mathbb{Q}$'s solving (4BP1)) through $\mathbb{Q} := \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$.

Maps $\mathbb{D}_+$ and $\mathbb{E}$ and the sets of allowable $\widetilde{\mathbb{M}}_+$'s and $\widetilde{\mathbb{N}}_+$'s depend on $\mathbb{D}, \mathbb{M}, \mathbb{N}$ only. □

(This will be proved in Lemma 12.4.6.)

If $\dim U < \infty$, then the invertibility of $\mathbb{M}_{22}$, $\mathbb{X}_{11}$, $\widetilde{\mathbb{M}}_{+22}$ and $(\mathbb{Z}^{-1}\mathbb{E})_{22}$ becomes redundant:

Lemma 12.3.9 (Case $\dim U < \infty$) *If $\dim U < \infty$, then condition $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$ is redundant in (Factor1) and $\mathbb{W}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ is redundant in (Factor2).* □

(This follows from Proposition 2.5.4(1).)

Lemma 12.3.10 (Case $\mathbb{N}_{\rm u}, \mathbb{M}_{\rm u} \in \widetilde{\mathcal{A}}$) *Assume that $\mathbb{N}_{\rm u}, \mathbb{M}_{\rm u} \in \widetilde{\mathcal{A}}$. Then (4BP1)–(4BP3) are equivalent, and claim (a) below holds. If (4BP1) holds, then we have the following:*

- *(a) All possible choices of $\mathbb{N}, \mathbb{M}, \mathbb{X}, \mathbb{E}, \mathbb{Z}, \mathbb{W}, \widetilde{\mathbb{M}}_+, \widetilde{\mathbb{N}}_+$ are in $\widetilde{\mathcal{A}}$.*
- *(b) The solutions $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ of the I/O 4BP are given by (12.47), and we have $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}} \Leftrightarrow \mathbb{L} \in \widetilde{\mathcal{A}}$.*
- *(c) We can choose $\mathbb{X} \in \widetilde{\mathcal{A}}$ so that $\mathbb{M}_{11}, \mathbb{X}_{22}, \widetilde{\mathbb{M}}_+ \in \mathcal{G}\mathrm{TIC}_\infty$, $X = \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}$, $M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & X_{22}^{-1} \end{bmatrix}$, $X_{11}, X_{22}, M_{11} \in \mathcal{GB}$, and $\mathbb{D}_+$ becomes well-posed. The d.c.f. (12.61) of $\mathbb{D}_+$ is over $\widetilde{\mathcal{A}}$.*
- *(d) There is a well-posed solution $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ of the I/O 4BP with $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}$ iff we can choose $\mathbb{Z}$ so that $(\mathbb{Z}^{-1})_{22} = \widetilde{\mathbb{N}}_{+21} \in \mathcal{G}\mathrm{TIC}_\infty$.*
- *(e) We have $\widetilde{\mathbb{N}_{\rm y}}, \widetilde{\mathbb{M}_{\rm y}} \in \widetilde{\mathcal{A}}$ iff the d.c.f. $\mathbb{D} = \mathbb{N}_{\rm u}\mathbb{M}_{\rm u}{}^{-1} = \widetilde{\mathbb{M}_{\rm y}}{}^{-1}\widetilde{\mathbb{N}_{\rm y}}$ is over $\widetilde{\mathcal{A}}$. If this is the case, then also the d.c.f. $\mathbb{D}_{21} = \mathbb{N}_{{\rm u}\,21}\mathbb{M}_{{\rm u}\,11}^{-1} = \widetilde{\mathbb{M}_{\rm y}}_{22}^{-1}\widetilde{\mathbb{N}_{\rm y}}_{21}$ is over $\widetilde{\mathcal{A}}$.*

(f) We have the following implications: (4BP1$\widetilde{\mathcal{A}}$)$\Rightarrow$(4BP1), (4BP2$\widetilde{\mathcal{A}}$)$\Rightarrow$(4BP2), (4BP3$\widetilde{\mathcal{A}}$)$\Rightarrow$(4BP3), (Factor1$\widetilde{\mathcal{A}}$)$\Rightarrow$(Factor1), (Factor2$\widetilde{\mathcal{A}}$)$\Rightarrow$(Factor2) *(and any solutions of former ones solve the latter ones too).*

(g) The claims (a)–(e) also apply to any solutions of (Factor1$\widetilde{\mathcal{A}}$) *and* (Factor2$\widetilde{\mathcal{A}}$) *(and for the corresponding* $\mathbb{X}$ *and* $\mathbb{Z}$*).* □

(This will be proved in Lemma 12.4.15.)

Lemma 12.3.11 ((Factor1&2))

(a) If (Factor1) holds, then $\mathbb{M}_{22}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$ *and* $\|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\| = \|(\mathbb{X}^{-1})_{22}^{-1}(\mathbb{X}^{-1})_{21}\| < 1$. *If, in addition, (Factor2) holds, then* $\mathbb{W}_{22} \in \mathcal{G}\mathrm{TIC}$ *and* $\|\mathbb{W}_{12}\mathbb{W}_{22}^{-1}\| < 1$.

(b) If $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *solves (Factor1) (resp.* $\mathbb{X}$ *solves (Factor1X)), then all solutions of (Factor1) (resp. of (Factor1X) for a fixed* $\mathbb{N}_{\mathrm{u}}$*) are given by*

$$\mathbb{D} = (\mathbb{N}E)(\mathbb{M}E)^{-1} \text{ (resp. } E^{-1}\mathbb{X}) \ , \ E^* J_1 E = J_1, \ E \in \mathcal{G}\mathcal{B}(U \times W). \quad (12.51)$$

If $\mathbb{Z}$ *solves (Factor2Z), then all solutions are given by* $\mathbb{Z}' = \mathbb{Z}F^{-1}$, $F^* J_1 F = J_1$, $F \in \mathcal{G}\mathcal{B}(Y \times U)$. *If* $\widetilde{\mathbb{M}}_+^{-1}\widetilde{\mathbb{N}}_+$ *solves (Factor2), then all solutions of (Factor2) for a fixed pair* $(\mathbb{N},\mathbb{M})$ *are given by* $\mathbb{D}_+ = (F\widetilde{\mathbb{M}}_+)^{-1}(F\widetilde{\mathbb{N}}_+)$ *(hence* $\mathbb{W}' = F\mathbb{W}$*).*

(c) If (Factor1) is satisfied, then every $\mathbb{X} \in \mathcal{G}\mathrm{TIC}(U \times W)$ *s.t.* $\mathbb{X}^* J_1 \mathbb{X} = \begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}^* J_\gamma \begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}$ *is a solution of (Factor1X). If (Factor2) is satisfied, then every* $\mathbb{Z} \in \mathcal{G}\mathrm{TIC}(Y \times U)$ *s.t.* $\mathbb{Z} J_1 \mathbb{Z}^* = \mathbb{E} J_1 \mathbb{E}^*$ *is a solution of (Factor2Z).*

(d) If (Factor1X) holds and $\mathbb{X} \in \mathrm{UR}$, *then we can have* $X_{21} = 0$, $X_{11}, X_{22} \in \mathcal{G}\mathcal{B}$. *If, in addition,* $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \mathrm{UR}$, *then it follows that* $\mathbb{D}, \mathbb{N}, \mathbb{M}, \mathbb{E} \in \mathrm{UR}$, $M_{21} = 0$, $M_{11}, M_{22} \in \mathcal{G}\mathcal{B}$.

(e) Assume that $D_{21} = 0$, *(Factor1X) holds,* $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \mathbb{X} \in \mathrm{UR}$, *and* X *is chosen as in (d). Then* $E = \begin{bmatrix} N_{22} & 0 \\ -M_{12} & M_{11} \end{bmatrix}$, *and any UR solution of (Factor2) has* $Z_{11}, \widetilde{\mathbb{N}}_{+21} \in \mathcal{G}\mathcal{B}$. □

(This will be proved in Lemma 12.4.6.)

We finish this section by an intuitive explanation of the role of $\mathbb{D}_+$ of (Factor2$\widetilde{\mathcal{A}}$). This map obviously maps $\left[\begin{smallmatrix} u \\ w_\circlearrowleft \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} y \\ u_\circlearrowleft \end{smallmatrix}\right]$ (because $\mathbb{M}$ maps $\left[\begin{smallmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$ and $\mathbb{N}$ maps $\left[\begin{smallmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$).

When (Factor1$\widetilde{\mathcal{A}}$) is satisfied, the map $\mathbb{D}$ equals the connection of $\left[\begin{smallmatrix} \mathbb{D}_{11}^{\curvearrowleft} & \mathbb{D}_{12}^{\curvearrowleft} \\ \underline{\mathbb{X}}_{21} & \underline{\mathbb{X}}_{22}^{-1} \end{smallmatrix}\right] : \left[\begin{smallmatrix} u_\circlearrowleft \\ w \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} y \\ w_\circlearrowleft \end{smallmatrix}\right]$ and $\mathbb{D}_+$, as illustrated in Figure 12.2. Here $\mathbb{D}^{\curvearrowleft}$ is defined by (11.10) and $\underline{\mathbb{X}} := \left[\begin{smallmatrix} I & 0 \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} I & 0 \\ -\mathbb{M}_{22}^{-1}\mathbb{M}_{21} & \mathbb{M}_{22}^{-1} \end{smallmatrix}\right] : \left[\begin{smallmatrix} u_\circlearrowleft \\ w \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{smallmatrix}\right]$.

Since (12.41):$\left[\begin{smallmatrix} u_\circlearrowleft \\ w_\circlearrowleft \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$ is (J_γ, J_1)-lossless, one can deduce that $\|w_\circlearrowleft \mapsto u_\circlearrowleft\| < 1$ iff $\|w \mapsto z\| < \gamma$. Thus, $\mathbb{Q}$ is suboptimal for $\mathbb{D}$ and γ iff $\mathbb{Q}$ is suboptimal for $\mathbb{D}_+$ and 1. It follows that we have reduced the solution to the case where $\mathbb{D}$ is of the special form (of $\mathbb{D}_+$) described in (Factor2$\widetilde{\mathcal{A}}$). These facts are often

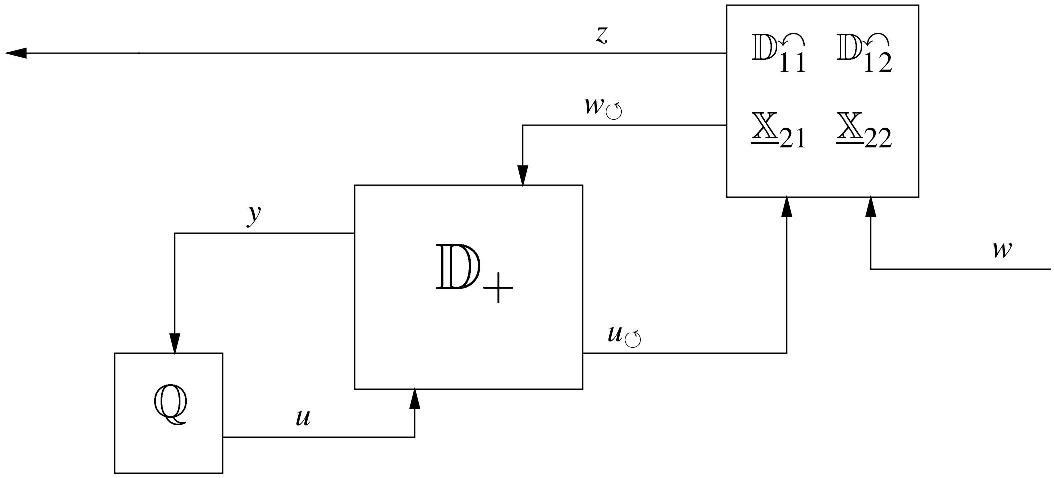

Figure 12.2: The map $\mathbb{D}_+$

used in the proof of the 4BP (see, e.g., pp. 421– of [ZDG]), but we use them only implicitly (in Section 12.4). In fact, in the more general setting of Theorem 12.3.7, the system $\mathbb{D}_+$ need not be well-posed.

Notes

Theorem 12.3.6 is a direct generalization of Theorem 4.4 of [Green] and of those of Theorem 5.6 of [CG97], as explained before the theorem.

We had essentially written this section during early 1998 (but at that stage, we needed improvements to our Riccati equation theory in order to solve the state-space H$^\infty$ 4BP of Section 12.1). At the end of year 2000, Hans Zwart showed us the report [IZ00], which contains something like Theorems 12.3.6 and 12.3.7 with $\widetilde{\mathcal{A}} = \mathrm{MTIC}^{\mathrm{L}^1}$ and $\dim U \times W \times Z \times Y < \infty$; thus our assumptions are also more general than the ones in [IZ00]. Also [IZ00] uses pure frequency-domain methods, and its methods seem essentially finite-dimensional (see, e.g., Lemma 2.2.1(c1), Proposition 2.5.4, Lemma 7.1.4 and the remark in the proof of Lemma 2.2.2(c2)); due to the same reasons, the methods of [Green] and [CG97] do not apply to this general case.

Our formulae (and proofs) become mathematically more complete due to the fact that we allow for controllers with internal loop (hence we do not need any additional invertibility conditions); see Theorem 12.3.6 or Theorem 12.3.7(c)&(d) for well-posed solutions.

The frequency-domain results have their own merits, as explained in [CG97], but they can also be used to derive the solution to the corresponding state-space problem, and that is what we will do in the last two sections of this chapter.

Under sufficient regularity, one can formulate the two lossless factorizations as Riccati equations; this leads to Theorem 12.1.8 and Proposition 12.1.10, i.e., to CAREs "(1.)" and "(4.)". Because "(4.)" is formulated in terms of a perturbed equation, one usually wishes to replace it with "(2.)" and "(3.)" of Theorem 12.1.5. This connection is established in Lemma 12.6.4 for the discrete-time setting and in Section 12.5 for the continuous-time setting (by discretization); the proofs require some extra regularity (as compared to Theorem 12.1.8 or to

Theorem 12.3.6) in the continuous-time case.

One might ask whether something similar could be done to the frequency-domain solution, i.e., whether (Factor1) and its dual condition with some coupling condition ("(3.)") would suffice, so that the perturbed factorization condition (Factor2) would not be needed. A positive answer to this question is given in Remark 12.5.25 (in the most popular setting where $\mathbb{D}$ has an exponential d.c.f.), but we still lack a simple formulation of the frequency-domain coupling condition.

Although the problem is symmetric, our solution is not. If we replace $\mathbb{D}$ by $\mathbb{D}_{\mathrm{d}}$ in the proofs, we obtain Theorems 12.3.6 and 12.3.7 with the dual of (Factor1) (the factorization corresponding to "(2.)") in place of (Factor1) and, analogously, a fourth factorization condition in place of (Factor2) (the situation is the same with the state-space problem and the Riccati equations, as noted on p. 708).

12.4 Proofs for Section 12.3

When confronted by a difficult problem, you can solve it more easily by reducing it to the question, "How would the Lone Ranger have handled this?"

The proofs of the results of Section 12.3 are more complicated than in the rational case (cf. [Green]) or in the case of the Callier–Desoer class (cf. [CG97]), because we lack several algebraic properties satisfied by the rational transfer functions, and because the (possibly) infinite dimensions of input and output spaces cause additional problems.

We start by proving some results concerning (Factor1) (Lemma 12.4.2). Then we list additional conditions (Lemma 12.4.3) equivalent to (4BP1)–(4BP3) (under suitably assumptions). Thereafter, we go on to study (Factor2) (Lemma 12.4.4–) until we are ready to prove the implications and additional results of Section 12.3 (Lemma 12.4.7–).

The proofs will refer to the general WPLS case of Theorem 12.3.7, where we do not require $\mathbb{Q}$ to be well-posed (i.e., $\mathrm{TIC}_\infty(Y,U)$) but we allow $\mathbb{Q}$ to be any map with internal loop. However, the readers interested only in the cases illustrated in Theorem 12.3.6 and analogous results may consider well-posed $\mathbb{Q}$'s only, i.e., the case where $\mathbb{Q}_2 \in \mathcal{G}\mathrm{TIC}_\infty(Y)$ and $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty(U)$.

We shall need the following notion:

Lemma 12.4.1 (4BP $\Rightarrow$ FICP & FCP) *Let the 4BP for $\mathbb{D}$ have a solution. Then $\begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} \widetilde{\mathbb{N}}^{\mathrm{d}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}^{\mathrm{d}}_{\mathrm{y}\,12} \\ 0 & I \end{bmatrix}$ are minimax J_γ-coercive.*

See Lemma 12.5.7 for the analogous and further state-space results under weaker assumptions.

Proof: From (12.39) we observe that there are solutions $\mathbb{U} \in \mathrm{TIC}$ of $\|\mathbb{N}_{\mathrm{u}\,11}\mathbb{U} + \mathbb{N}_{\mathrm{u}\,12}\| < \gamma$ and $\overline{\mathbb{U}} \in \mathrm{TIC}$ of $\|\widetilde{\mathbb{N}}^{\mathrm{d}}_{\mathrm{y}\,22}\overline{\mathbb{U}} + \widetilde{\mathbb{N}}^{\mathrm{d}}_{\mathrm{y}\,12}\| < \gamma$. It follows from Lemma 11.3.10 that $\begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} \widetilde{\mathbb{N}}^{\mathrm{d}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}^{\mathrm{d}}_{\mathrm{y}\,12} \\ 0 & I \end{bmatrix}$ are minimax J_γ-coercive, hence J_γ-coercive, by Lemma 11.4.2. □

Lemma 12.4.2 *We give here partial proofs of Theorem 12.3.7 and Lemmas 12.3.8 and 12.3.11(a)–(c); these proofs will be completed in lemmas below.*

Proof: *I Theorem 12.3.7(e1)&(e2) (partially):*

1° *"(Factor1X)⇒(Factor1)":* Assume (Factor1X). Set $\mathbb{V} := \mathbb{X}^{-1}$, $\mathbb{Y} := \begin{bmatrix} \mathbb{N}_{\mathrm{u}\,11} & \mathbb{N}_{\mathrm{u}\,12} \\ 0 & I \end{bmatrix}\mathbb{V}$ to obtain $\mathbb{Y}^* J_\gamma \mathbb{Y} = J_1$. Set $\mathbb{N} := \mathbb{N}_{\mathrm{u}}\mathbb{V}$, $\mathbb{M} := \mathbb{M}_{\mathrm{u}}\mathbb{V}$ to obtain a r.c.f. (by Lemma 6.4.5(c)) with

$$\mathbb{N} = \begin{bmatrix} \mathbb{Y}_{11} & \mathbb{Y}_{12} \\ * & * \end{bmatrix}, \quad \mathbb{M} = \begin{bmatrix} * & * \\ \mathbb{V}_{21} & \mathbb{V}_{22} \end{bmatrix} = \begin{bmatrix} * & * \\ \mathbb{Y}_{21} & \mathbb{Y}_{22} \end{bmatrix} \tag{12.52}$$

(because $\mathbb{M}_{\mathrm{u}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$), satisfying (Factor1), because $\mathbb{Y}$ is (J_γ, J_1)-lossless, by Corollary 2.5.5 (since $\mathbb{Y}_{22} = \mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$).

2° *"(Factor1)⇒(Factor1X)":* (This is the above proof backwards.) Assume (Factor1). By Lemma 6.4.5(c), we have $\mathbb{V} := \mathbb{M}_u{}^{-1}\mathbb{M} \in \mathcal{G}\mathrm{TIC}(U \times W)$. Set $\mathbb{N} := \mathbb{N}_u\mathbb{V}$, $\mathbb{M} := \mathbb{M}_u\mathbb{V}$ to obtain that $\mathbb{Y} := \begin{bmatrix} \mathbb{N}_{u11} & \mathbb{N}_{u12} \\ 0 & I \end{bmatrix} \mathbb{V}$ satisfies $\mathbb{Y} = \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ \mathbb{M}_{21} & \mathbb{M}_{22} \end{bmatrix}$. Thus, $\mathbb{Y}^*J_\gamma\mathbb{Y} = J_1$ implies that $\begin{bmatrix} \mathbb{N}_{u11} & \mathbb{N}_{u12} \\ 0 & I \end{bmatrix}^* J_\gamma \begin{bmatrix} \mathbb{N}_{u11} & \mathbb{N}_{u12} \\ 0 & I \end{bmatrix} = \mathbb{V}^{-*}J_1\mathbb{V}^{-1}$. Therefore, $\mathbb{X} := \mathbb{V}^{-1}$ is as in (Factor1X) (use again Corollary 2.5.5).

3° *Equations (12.49) of (e2):* This follows from 1°–2°.

II Lemma 12.3.8 — The first claim on $\mathbb{N}_u, \mathbb{M}_u, \widetilde{\mathbb{M}_y}, \widetilde{\mathbb{N}_y}$ *(partially):* This is obvious for (4BP2), (Factor1) and (Factor2), hence for (4BP3) too; for (Factor1X) this follows from the above equivalence, for (Factor2Z) this will follow from the equivalence that will be established in Lemma 12.4.4.

III Lemma 12.3.11: (a), (c) and first half of (b): (a) For (Factor1X), this follows from Lemma 11.4.3(a)&(c) (with substitutions $\mathbb{D} \mapsto \begin{bmatrix} \mathbb{N}_{u11} & \mathbb{N}_{u12} \\ 0 & I \end{bmatrix}$, $J_\gamma \mapsto J_1$, $J \mapsto J_\gamma$; recall from Standing Hypothesis 12.3.1 that $\mathbb{N}_u{}_{11}^*\mathbb{N}_{u11} \gg 0$). The claims on (Factor2Z) follow analogously.

(c) This follows directly from Lemma 11.4.3(b).

(b) (the (Factor1[X]) part only) By ((c) and) Lemma 6.4.5(a), all factorizations of form (Factor1X) are given by $E^{-1}\mathbb{X}$, where $J_1 = E^*J_1E$, $E \in \mathcal{GB}(U \times W)$. (Recall that $\mathbb{N}_u$ was taken fixed here.)

If $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is as in (Factor1), then the above proof of "(Factor1)⇒(Factor1X)" shows that $\mathbb{N} = \mathbb{N}_u\mathbb{X}^{-1}$, where $\mathbb{X}$ is as above, hence such factorizations again differ by E only.

(Remark — $\mathbb{X}$ depends on $\mathbb{N}_u$ as follows: If an r.c.f. $\mathbb{D} = \mathbb{N}_u{}'(\mathbb{M}_u{}')^{-1}$ is also of form $\mathbb{M}_u{}' = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix}$, then $\mathbb{N}_u{}' = \mathbb{N}_u\mathbb{U}$ and $\mathbb{M}_u{}' = \mathbb{M}_u\mathbb{U}$ for some $\mathbb{U} \in \mathcal{G}\mathrm{TIC}$ with $\mathbb{U} = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix}$, by Lemma 7.3.16. It follows that $\begin{bmatrix} \mathbb{N}_u{}'_{11} & \mathbb{N}_u{}'_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} \mathbb{N}_{u11} & \mathbb{N}_{u12} \\ 0 & I \end{bmatrix}\mathbb{U}$, hence $\mathbb{X}\mathbb{U}$ is a spectral factor of $\begin{bmatrix} \mathbb{N}_u{}'_{11} & \mathbb{N}_u{}'_{12} \\ 0 & I \end{bmatrix}$ iff $\mathbb{X}$ is a spectral factor of $\begin{bmatrix} \mathbb{N}_{u11} & \mathbb{N}_{u12} \\ 0 & I \end{bmatrix}$. Thus, for $\mathbb{N}_u{}' = \mathbb{N}_u\mathbb{U}$, $\mathbb{U} = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$, (Factor1X) is satisfied by $\mathbb{X}' = E\mathbb{X}\mathbb{U}$, where E is as above.) □

To simplify the formulae in the proof, we show that the preliminary factorization $\mathbb{D} = \mathbb{N}_u\mathbb{M}_u{}^{-1}$ can be replaced by "a semioptimal preliminary factorization" $\mathbb{D} = \mathbb{N}_u^\frown\mathbb{M}_u^{\frown -1}$. Then we establish further equivalent conditions for (4BP1):

Lemma 12.4.3 (Equivalent conditions (4BP1)–(4BP7)) *Assume (Factor1X) and set* $\mathbb{V} := \mathbb{X}^{-1}$, $\underline{\mathbb{V}} := \begin{bmatrix} I & 0 \\ \mathbb{V}_{21} & \mathbb{V}_{22} \end{bmatrix}$, $\underline{\mathbb{X}} := \underline{\mathbb{V}}^{-1}$, $\overline{\mathbb{X}} := \begin{bmatrix} \mathbb{X}_{11} & \mathbb{X}_{12} \\ 0 & I \end{bmatrix}$, $\overline{\mathbb{V}} := \overline{\mathbb{X}}^{-1}$, $\mathbb{M}_u^\frown := \mathbb{M}_u\overline{\mathbb{V}} = \begin{bmatrix} \mathbb{M}_u^\frown{}_{11} & \mathbb{M}_u^\frown{}_{12} \\ 0 & I \end{bmatrix}$, $\mathbb{N}_u^\frown := \mathbb{N}_u\overline{\mathbb{V}}$, $\mathbb{M} = \mathbb{M}_u\mathbb{X}^{-1} = \mathbb{M}_u^\frown\underline{\mathbb{V}}$ *and* $\mathbb{N} := \mathbb{N}_u\mathbb{X}^{-1} = \mathbb{N}_u^\frown\underline{\mathbb{V}}$. *Then* $\mathbb{X} = \underline{\mathbb{X}}\overline{\mathbb{X}}$, $\mathbb{V} = \overline{\mathbb{V}}\underline{\mathbb{V}}$, *and the following holds.*

(a) The r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ *is as in (Factor1), and the r.c.f.* $\mathbb{D} = \mathbb{N}_u^\frown\mathbb{M}_u^{\frown -1}$ *is as in Standing Hypothesis 12.3.1.*

(b) The conditions (4BP1), (4BP2) and (4BP4)–(4BP7) are equivalent and independent of $\mathbb{N}$, $\mathbb{M}$ *and* $\mathbb{X}$.

(c1) A map $\mathbb{Q}$ *with internal loop solves the 4BP iff it can be written as* $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$, *where* $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2$ *satisfy (4BP2). Note that any maps* $\widetilde{\mathbb{Q}}_1$ *and* $\widetilde{\mathbb{Q}}_2$ *satisfying (4BP2) are necessarily l.c.*

The above claims hold also with (4BP4), (4BP5), (4BP6) and (4BP7) in place of (4BP2).

(c2) (For a fixed pair $(\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2)$ we have "(4BP4)⇔(4BP5)" and "(4BP2)⇒(4BP6)⇔(4BP7)", whereas for $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ (as a map with l.c. internal loop) all these are equivalent.)

(c3) Assume that $\mathbb{N}_{\rm u},\mathbb{M}_{\rm u}\in \mathrm{TIC}_{\rm exp}$. If any of (4BP1), (4BP2) and (4BP4)–(4BP7) has a solution with $\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2\in\mathrm{TIC}_{\rm exp}$, then so do all of them.

(This also holds with $\mathcal{A}$ in place of $\mathrm{TIC}_{\rm exp}$ if $\mathcal{A}\underset{a}{\subset}\mathrm{TIC}$ is closed under spectral factorization.)

(d) If (4BP5) (and hence (4BP1)–(4BP7) except possibly (4BP3)) holds, then the l.c.f. $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1=\mathbb{Q}$ given in (4BP5) is unique, and the following equations hold:

$$\widetilde{\mathbb{U}}_o=(I-\mathbb{U}_o\mathbb{V}_{21})^{-1}\mathbb{U}_o\mathbb{V}_{22}, \tag{12.53}$$

$$\mathbb{U}_o=(I-\widetilde{\mathbb{U}}_o\underline{\mathbb{X}}_{21})^{-1}\widetilde{\mathbb{U}}_o\underline{\mathbb{X}}_{22}=\widetilde{\mathbb{U}}_o(\mathbb{V}_{21}\widetilde{\mathbb{U}}_o+\mathbb{V}_{22})^{-1}, \tag{12.54}$$

$$(I-\widetilde{\mathbb{U}}_o\underline{\mathbb{X}}_{21})^{-1}=I-\mathbb{U}_o\mathbb{V}_{21}\in\mathcal{G}\mathrm{TIC}, \tag{12.55}$$

$$(\mathbb{V}_{21}\widetilde{\mathbb{U}}_o+\mathbb{V}_{22})^{-1}=\underline{\mathbb{X}}_{22}+\underline{\mathbb{X}}_{21}\mathbb{U}_o\in\mathcal{G}\mathrm{TIC}. \tag{12.56}$$

(e) This lemma holds even if we replace "$<$" signs of the lemma by "$\leq$" signs (this includes requiring that $\|\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})\|\leq\gamma$ instead of $<\gamma$).

We have referred to the following conditions:

(4BP4) There are $\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2\in\mathrm{TIC}$ s.t. $\widetilde{\mathbb{Q}}_2\mathbb{M}^{\frown}_{\rm u\,11}-\widetilde{\mathbb{Q}}_1\mathbb{N}^{\frown}_{\rm u\,21}=I$ and $\mathbb{U}_o:=\widetilde{\mathbb{Q}}_1\mathbb{N}^{\frown}_{\rm u\,22}-\widetilde{\mathbb{Q}}_2\mathbb{M}^{\frown}_{\rm u\,12}\in\mathrm{TIC}$ solves (the FICP) $\|\mathbb{N}^{\frown}_{\rm u\,11}\mathbb{U}_o+\mathbb{N}^{\frown}_{\rm u\,12}\|<\gamma$.

(4BP5) There are $\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2\in\mathrm{TIC}$ s.t. $-\widetilde{\mathbb{Q}}_1\mathbb{N}^{\frown}_{\rm u\,21}+\widetilde{\mathbb{Q}}_2\mathbb{M}^{\frown}_{\rm u\,11}=I$ and $\mathbb{U}_o:=\widetilde{\mathbb{Q}}_1\mathbb{N}^{\frown}_{\rm u\,22}-\widetilde{\mathbb{Q}}_2\mathbb{M}^{\frown}_{\rm u\,12}\in\mathrm{TIC}$ is of the form $\mathbb{U}_o=\mathbb{U}_1\mathbb{U}_2^{-1}$, where $\begin{bmatrix}\mathbb{U}_1\\ \mathbb{U}_2\end{bmatrix}=\underline{\mathbb{V}}\begin{bmatrix}\widetilde{\mathbb{U}}_o\\ I\end{bmatrix}$, $\widetilde{\mathbb{U}}_o\in\mathrm{TIC}$ and $\|\widetilde{\mathbb{U}}_o\|<1$.

(4BP6) There are $\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2\in\mathrm{TIC}$ s.t. $\widetilde{\mathbb{Q}}_2\mathbb{M}_{11}-\widetilde{\mathbb{Q}}_1\mathbb{N}_{21}\in\mathcal{G}\mathrm{TIC}(U)$, and $\|(\widetilde{\mathbb{Q}}_2\mathbb{M}_{11}-\widetilde{\mathbb{Q}}_1\mathbb{N}_{21})^{-1}(-\widetilde{\mathbb{Q}}_2\mathbb{M}_{12}+\widetilde{\mathbb{Q}}_1\mathbb{N}_{22})\|_{\mathrm{TIC}(W,U)}<1$.

(4BP7) **(ASP)** *There are $\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2\in\mathrm{TIC}$ s.t. the operators*

$$\begin{bmatrix}\widetilde{\mathbb{U}}_1 & \widetilde{\mathbb{U}}_2\end{bmatrix}:=\begin{bmatrix}\widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2\end{bmatrix}\begin{bmatrix}\mathbb{N}_{22} & -\mathbb{N}_{21}\\ -\mathbb{M}_{12} & \mathbb{M}_{11}\end{bmatrix}\in\mathrm{TIC}(W\times U,U) \tag{12.57}$$

satisfy $\widetilde{\mathbb{U}}_2\in\mathcal{G}\mathrm{TIC}$ and $\|\widetilde{\mathbb{U}}_2^{-1}\widetilde{\mathbb{U}}_1\|<1$.

Thus, condition "(Factor1X) and (4BPn) hold" is equivalent to (4BP1) (and to (4BP2)) whenever $\left(\begin{bmatrix}\mathbb{N}_{\rm u\,11} & \mathbb{N}_{\rm u\,12}\\ 0 & I\end{bmatrix},J_\gamma\right)\in\mathrm{SpF}$, by Theorem 12.3.7(a).

Note that, for each $\mathbb{Q}$ solving the 4BP (cf. (c)), the pairs $(\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2)$ in (4BP2) and (4BP5) need not be the same (in fact this happens very seldom, when

$\widetilde{\mathbb{U}}\mathbb{V}_{22}^{-1}\mathbb{V}_{21} = 0$, as shown in the proof of (4BP2)⇒(4BP5)); we only know that they differ by an element of $\mathcal{G}$TIC (see Definition 7.2.11).

The equivalence "(4BP1)⇔(4BP4)" holds for the original r.c.f. $\mathbb{D} = \mathbb{N}_{\mathrm{u}}\mathbb{M}_{\mathrm{u}}{}^{-1}$ too, as one observes from the proof.

If one l.c.f. of $\mathbb{Q}$ satisfies the ASP given in (v) (and (iv) as well), then clearly every l.c.f. of $\mathbb{Q}$ does (cf. Lemma 6.4.5(d)).

Proof of Lemma 12.4.3: Note that we have chosen a "semicritical preliminary factorization" $\mathbb{D} = \mathbb{N}_{\mathrm{u}}^{\frown}\mathbb{M}_{\mathrm{u}}^{\frown}{}^{-1}$ to make the formulae in the proof (including those at the end of the lemma) simpler. By Lemma 12.3.8 (see Lemma 12.4.2), this causes no loss of generality.

The formula $\mathbb{V} = \overline{\mathbb{V}}\underline{\mathbb{V}}$ is from (A.9), and $\mathbb{X} = \underline{\mathbb{X}}\overline{\mathbb{X}}$ is its inverse.

(a) By Lemma 6.4.5(c), $\mathbb{N}_{\mathrm{u}}^{\frown}\mathbb{M}_{\mathrm{u}}^{\frown}{}^{-1}$ is a r.c.f. of (c). Obviously, $\mathbb{M}_{\mathrm{u}}^{\frown} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$. The rest follows as in Corollary 7.3.17.

By 2° of the proof of Lemma 12.4.2, the r.c.f. $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is as in (Factor1).

I The equivalence: Let $\widetilde{\mathbb{Q}}_1 \in \mathrm{TIC}(Y,U)$ and $\widetilde{\mathbb{Q}}_2 \in \mathrm{TIC}(U)$ be l.c. (i.e., let $\mathbb{Q} := \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ be a map with l.c. internal loop; cf. Definition 7.2.11). We shall show below the equivalence of (4BP1)–(4BP7)$\setminus$(4BP3) for $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ in the sense that given $n,m \in \{1,2,4,5,6,7\}$, there is $\mathbb{U}_n \in \mathcal{G}\mathrm{TIC}(U)$ s.t. $(\mathbb{U}_n\widetilde{\mathbb{Q}}_1, \mathbb{U}_n\widetilde{\mathbb{Q}}_n)$ satisfies (4BPn) iff there is $\mathbb{U}_m \in \mathcal{G}\mathrm{TIC}(U)$ s.t. $(\mathbb{U}_m\widetilde{\mathbb{Q}}_1, \mathbb{U}_m\widetilde{\mathbb{Q}}_n)$ satisfies (4BPm) (i.e., $\mathbb{Q}$ has a presentation satisfying (4BPn) iff $\mathbb{Q}$ has a presentation satisfying (4BPm)). Note that (4BP1) is independent on the presentation.

1° "(4BP1)⇔(4BP4)": This follows from Lemma 12.3.2.

2° "(4BP4)⇔(4BP5)": This follows directly from Theorem 11.3.6.

4° "(4BP6)⇔(4BP7)": Clearly (4BP7) is only a reformulation of (4BP6).

5° "(4BP7)⇔(4BP2)": If $(\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2)$ solves (4BP7), then $(\widetilde{\mathbb{U}}_2^{-1}\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{U}}_2^{-1}\widetilde{\mathbb{Q}}_2)$ solves (4BP2) (where $\widetilde{\mathbb{U}}_2$ is as in (4BP7)). Conversely, every solution of (4BP2) is a solution of (4BP7).

6° "(4BP5)⇒(d)&(4BP7)": We now assume (4BP5) (and hence (4BP1) and (4BP4) too) and will prove first the claims in part (d) and then (4BP7).

The uniqueness of the l.c.f. follows from Lemma 6.4.5(d). The first formula is equation $-\widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\frown}{}_{21} + \widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\frown}{}_{11} = I$ multiplied to the left by $\widetilde{\mathbb{Q}}_2^{-1}$.

Equations (12.53)–(12.56) follow from equations (11.89)–(11.92) of Theorem 11.3.6 (where we must use the spectral factor $\underline{\mathbb{X}} = \mathbb{M}^{-1}\mathbb{M}_{\mathrm{u}}^{\frown}$ of $\begin{bmatrix} \mathbb{N}_{\mathrm{u}}^{\frown}{}_{11} & \mathbb{N}_{\mathrm{u}}^{\frown}{}_{12} \\ 0 & I \end{bmatrix}$) and the last three equations are derived as follows:

$$\widetilde{\mathbb{U}}_2 := \widetilde{\mathbb{Q}}_2\mathbb{M}_{11} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{21} = \widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\frown}{}_{11} + \widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\frown}{}_{12}\mathbb{V}_{21} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\frown}{}_{21} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\frown}{}_{22}\mathbb{V}_{21} \tag{12.58}$$

$$= (\widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\frown}{}_{11} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\frown}{}_{21}) - (\widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\frown}{}_{22} - \widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\frown}{}_{12})\mathbb{V}_{21} \tag{12.59}$$

$= I - \mathbb{U}_o\mathbb{V}_{21}$, by the equations in (4BP5). Now $\widetilde{\mathbb{U}}_2 = I - \mathbb{U}_o\mathbb{V}_{21} \in \mathcal{G}\mathrm{TIC}(U)$, by (12.55). Similarly,

$$\widetilde{\mathbb{U}}_1 = -\widetilde{\mathbb{Q}}_2\mathbb{M}_{12} + \widetilde{\mathbb{Q}}_1\mathbb{N}_{22} = -\widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\frown}{}_{12}\mathbb{V}_{22} + \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\frown}{}_{22}\mathbb{V}_{22} = \mathbb{U}_o\mathbb{V}_{22}. \tag{12.60}$$

Therefore, $\widetilde{\mathbb{U}}_2^{-1}\widetilde{\mathbb{U}}_1 = (I - \mathbb{U}_o\mathbb{V}_{21})^{-1}\mathbb{U}_o\mathbb{V}_{22} = \widetilde{\mathbb{U}}_o$, whose norm is less than 1, hence (4BP5) implies (4BP7).

7° "(4BP2)⇒(4BP5)" (which completes the proof of the equivalence): Assume (4BP2). Then $\widetilde{\mathbb{G}} := \widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\frown}{}_{11} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\frown}{}_{21} = \widetilde{\mathbb{Q}}_2(\mathbb{M}_{11} - \mathbb{M}_{\mathrm{u}}^{\frown}{}_{12}\mathbb{V}_{21}) - \widetilde{\mathbb{Q}}_1(\mathbb{N}_{21} - \mathbb{N}_{\mathrm{u}}^{\frown}{}_{22}\mathbb{V}_{21}) = \widetilde{\mathbb{Q}}_2\mathbb{M}_{11} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{21} - \widetilde{\mathbb{Q}}_2\mathbb{M}_{12}\mathbb{V}_{22}^{-1}\mathbb{V}_{21} + \widetilde{\mathbb{Q}}_1\mathbb{N}_{22}\mathbb{V}_{22}^{-1}\mathbb{V}_{21} = I + \widetilde{\mathbb{U}}\mathbb{V}_{22}^{-1}\mathbb{V}_{21} \in \mathcal{G}\mathrm{TIC}(U)$, because $\|\mathbb{V}_{22}^{-1}\mathbb{V}_{21}\| < 1$, by (d) of the theorem.

Define $\widetilde{\mathbb{Q}}_2 := \widetilde{\mathbb{G}}^{-1}\widetilde{\mathbb{Q}}_2$, $\widetilde{\mathbb{Q}}_1 := \widetilde{\mathbb{G}}^{-1}\widetilde{\mathbb{Q}}_1$ to get a new l.c.f. of $\mathbb{Q}$, satisfying $\widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\frown}{}_{11} - \widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\frown}{}_{21} = I$, as required.

From the definitions of $\mathbb{N}$ and $\mathbb{M}$ (given in the lemma), we get $\widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}}^{\frown}{}_{22} - \widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}}^{\frown}{}_{12} = \widetilde{\mathbb{G}}^{-1}[\widetilde{\mathbb{Q}}_1\mathbb{N}_{22} - \widetilde{\mathbb{Q}}_2\mathbb{M}_{12}]\mathbb{V}_{22}^{-1} = (I + \widetilde{\mathbb{U}}\mathbb{V}_{22}^{-1}\mathbb{V}_{21})^{-1}[\widetilde{\mathbb{U}}]\mathbb{V}_{22}^{-1} = \widetilde{\mathbb{U}}(I + \mathbb{V}_{22}^{-1}\mathbb{V}_{21}\widetilde{\mathbb{U}})^{-1}\mathbb{V}_{22}^{-1} = \widetilde{\mathbb{U}}(\mathbb{V}_{22} + \mathbb{V}_{21}\widetilde{\mathbb{U}})^{-1} = \mathbb{U}_o$, if we set $\mathbb{U}_o := \mathbb{U}_1\mathbb{U}_2^{-1}$ and $\begin{bmatrix}\mathbb{U}_1\\ \mathbb{U}_2\end{bmatrix} := \underline{\mathbb{V}}\begin{bmatrix}\widetilde{\mathbb{U}}\\ I\end{bmatrix}$. Thus, (4BP3) is satisfied and hence the proof of the equivalence has been completed.

II — (b), (c), (d), (e):

(b) In I, we showed the equivalence in (b). Since $\mathbb{X}$ was arbitrary (and hence $\mathbb{N}$ and $\mathbb{M}$ too, by (12.49), which was established in Lemma 12.4.2), and (4BP1) is independent of $\mathbb{X}$, $\mathbb{N}$ and $\mathbb{M}$, hence so are (4BP2) and (4BP4)–(4BP7).

(c1) It follows directly from the equations in each (4BP$*$) that $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2$ are l.c., as claimed.

As one observes from the proof, conditions (4BP2) and (4BP4)–(4BP7) have same solutions $(\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2)$ modulo the multiplication to the left by an element of $\mathcal{G}\mathrm{TIC}(U)$. By Definition 7.2.11, this means that the maps $\mathbb{Q} := \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ corresponding to these solutions are equal (but there are usually more than one solution $\mathbb{Q}$); by 2° these maps are exactly the solutions of (4BP1) (i.e., the suboptimal DPF-stabilizing controllers). Thus, (c1) has been established.

(c2) This can be observed from part I.

(c3) 1° *Weaker assumptions:* In fact, it suffices to assume that $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \mathbb{X} \in \mathcal{A}$, that $\mathcal{A} \underset{a}{\subset} \mathrm{TIC}$ (see Definition 6.2.4), and that $\mathcal{A}$ is inverse-closed (i.e., $\mathcal{A} \cap \mathcal{G}\mathrm{TIC} = \mathcal{G}\mathcal{A}$).

The proof of Lemma 8.4.10 shows that if $\mathcal{A}$ is closed under spectral factorization (in the sense that $\widetilde{\mathbb{X}} \in \mathcal{G}\mathcal{A}$ whenever $\widetilde{\mathbb{D}} \in \mathcal{A}(U \times W, *)$, $\widetilde{J} = \widetilde{J}^* \in \mathcal{B}$, $\widetilde{S} \in \mathcal{G}\mathcal{B}$, $\widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}(U \times W)$ and $\widetilde{\mathbb{X}}^*\widetilde{S}\widetilde{\mathbb{X}} = \widetilde{\mathbb{D}}^*\widetilde{J}\widetilde{\mathbb{D}}$), then $\mathcal{A}$ is inverse-closed.

2° *Suitable $\mathcal{A}$'s:* On the other hand, Lemma 6.4.7(c) shows that $\mathrm{TIC}_{\exp}$ is closed under spectral factorization; obviously, so is $\widetilde{\mathcal{A}}$, so that we can take $\widetilde{\mathcal{A}} := \mathrm{TIC}_{\exp}$ or $\widetilde{\mathcal{A}} := \mathcal{A}$.

3° *The proof using* 1°*:* By the assumptions, we have $\mathbb{V}, \underline{\mathbb{X}}, \underline{\mathbb{V}}, \overline{\mathbb{X}}, \overline{\mathbb{V}} \in \mathcal{G}\mathcal{A}$. Therefore, $\mathbb{N}, \mathbb{M}, \mathbb{M}_{\mathrm{u}}^{\frown}, \mathbb{N}_{\mathrm{u}}^{\frown} \in \mathcal{A}$. Consequently, it is easy to verify from part I that (c3) holds, since different $\widetilde{\mathbb{Q}}_*$'s are obtained from each other and the above $\mathcal{A}$ maps by using only algebraic operations in TIC (here we again need the inverse-closedness of $\mathcal{A}$).

(d) This was established in 6°.

(e) Obviously, the proof below applies also with the changes listed in (e). □

Lemma 12.4.4 ($\mathbb{D}_+$ & (Factor2)) *The map* $\mathbb{D}_+ := \begin{bmatrix}\mathbb{N}_{21} & \mathbb{N}_{22}\\ I & 0\end{bmatrix}\begin{bmatrix}\mathbb{M}_{11} & \mathbb{M}_{12}\\ 0 & I\end{bmatrix}^{-1}$ *has a d.c. internal loop: we have the doubly coprime product*

$$\begin{bmatrix}\begin{bmatrix}\mathbb{M}_{11} & \mathbb{M}_{12}\\ 0 & I\end{bmatrix} & \begin{bmatrix}0 & -I\\ 0 & 0\end{bmatrix}\\ \begin{bmatrix}\mathbb{N}_{21} & \mathbb{N}_{22}\\ I & 0\end{bmatrix} & \begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}\end{bmatrix}^{-1} = \begin{bmatrix}\begin{bmatrix}0 & 0\\ 0 & I\end{bmatrix} & -\begin{bmatrix}0 & -I\\ 0 & 0\end{bmatrix}\\ -\begin{bmatrix}0 & \mathbb{N}_{22}\\ I & -\mathbb{M}_{12}\end{bmatrix} & \begin{bmatrix}I & -\mathbb{N}_{21}\\ 0 & \mathbb{M}_{11}\end{bmatrix}\end{bmatrix} \tag{12.61}$$

$\in \mathcal{G}\mathrm{TIC}((U\times W)\times(Y\times U),(U\times W)\times(U\times Y))$. *Moreover, Theorem 12.3.7(e1)&(e2) hold; in particular, we have (Factor2)⇔(Factor2Z).*

Proof: We start with the last equivalence, based on the proof of Theorem 3.8 of [Green].

Obviously, (12.61) is true, hence $\mathbb{D}_+$ has a d.c. internal loop (see Definition 7.2.11) and can be written as $\mathbb{D}_+ = \widetilde{\mathbb{X}}_+^{-1}\widetilde{\mathbb{Y}}_+$, where

$$\widetilde{\mathbb{X}}_+ := \begin{bmatrix}I & -\mathbb{N}_{21}\\ 0 & \mathbb{M}_{11}\end{bmatrix} \in \mathrm{TIC}(Y\times U),\quad \widetilde{\mathbb{Y}}_+ := \begin{bmatrix}0 & \mathbb{N}_{22}\\ I & -\mathbb{M}_{12}\end{bmatrix} \in \mathrm{TIC}(U\times W, Y\times U). \tag{12.62}$$

1° "(Factor2)⇒(Factor2Z)": Assume (Factor2). Then $\widetilde{\mathbb{M}}_+ = \mathbb{Z}^{-1}\widetilde{\mathbb{X}}_+$, $\widetilde{\mathbb{N}}_+ = \mathbb{Z}^{-1}\widetilde{\mathbb{Y}}_+$ for some $\mathbb{Z}\in\mathcal{G}\mathrm{TIC}(Y\times U)$, by Definition 7.2.11. Consequently,

$$\mathbb{Z}^{-1} = \begin{bmatrix}\widetilde{\mathbb{M}}_{+11} & \widetilde{\mathbb{N}}_{+11}\\ \widetilde{\mathbb{M}}_{+21} & \widetilde{\mathbb{N}}_{+21}\end{bmatrix} \text{ and } \mathbb{Z}^{-1}\mathbb{E} = \mathbb{Z}^{-1}\begin{bmatrix}\widetilde{\mathbb{Y}}_{+*2} & \widetilde{\mathbb{X}}_{+*2}\end{bmatrix} = \begin{bmatrix}\widetilde{\mathbb{N}}_{+*2} & \widetilde{\mathbb{M}}_{+*2}\end{bmatrix} = \mathbb{W}. \tag{12.63}$$

Therefore, (Factor2Z) holds.

2° "(Factor2Z)⇒(Factor2)": Assume (Factor2Z). Define $\widetilde{\mathbb{X}}_+, \widetilde{\mathbb{Y}}_+$ as above and set $\widetilde{\mathbb{M}}_+ = \mathbb{Z}^{-1}\widetilde{\mathbb{X}}_+$, $\widetilde{\mathbb{N}}_+ = \mathbb{Z}^{-1}\widetilde{\mathbb{Y}}_+$ to obtain $\widetilde{\mathbb{M}}_+^{-1}\widetilde{\mathbb{N}}_+ = \widetilde{\mathbb{X}}_+^{-1}\widetilde{\mathbb{Y}}_+$. Set $\mathbb{W} := \mathbb{Z}^{-1}\mathbb{E}$ to obtain that $\mathbb{W}J_1\mathbb{W}^* = J_1$, and that (12.63) holds. Then (Factor2) holds.

3° *Theorem 12.3.7(e1)&(e2):* Part II of the proof of Lemma 12.4.3 contains partial proofs of Theorem 12.3.7(e1)&(e2). The rest of (e1) and (e2) is obtained above. □

The factorization $\mathbb{D}_+ := \begin{bmatrix}\mathbb{N}_{21} & \mathbb{N}_{22}\\ I & 0\end{bmatrix}\begin{bmatrix}\mathbb{M}_{11} & \mathbb{M}_{12}\\ 0 & I\end{bmatrix}^{-1}$ is a r.c.f. when (Factor1$\widetilde{\mathcal{A}}$) holds:

Remark 12.4.5 ($\mathbb{D}_+$) *If (Factor1) holds, then* $\begin{bmatrix}\mathbb{N}_{21} & \mathbb{N}_{22}\\ I & 0\end{bmatrix} \in \mathrm{TIC}(U\times W, Y\times U)$ *and* $\begin{bmatrix}\mathbb{M}_{11} & \mathbb{M}_{12}\\ 0 & I\end{bmatrix} \in \mathrm{TIC}(U\times W)$ *are r.c. (even d.c.), by (12.61).* □

Lemma 12.4.6 *Lemmas 12.3.11 and 12.3.8 hold.*

Proof: 1.1° *Lemma 12.3.11(a)–(c):* Part II of the proof of Lemma 12.4.3 contains the proof of Lemma 12.3.11(a) and a partial proof of (b). The "(Factor2)" part of (b) is obtained as its (Factor1) part (note that the first columns of $\widetilde{\mathbb{M}}_+$ and $\widetilde{\mathbb{N}}_+$ are contained in $\mathbb{Z}^{-1}$, hence these depend on $\mathbb{D}$ only, where as their second columns are part of $\mathbb{W}$ and hence depend on $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}$ in the same way as $\mathbb{E}$ does).

Part (c) was established in Lemma 12.4.3.

Part (d): 1.2° By (Factor1X), the assumptions of Proposition 11.3.4 are satisfied and (FI3s) holds. By Lemma 11.3.11, the solution of (FI3s) for $\begin{bmatrix} \mathbb{N}_{u11} & \mathbb{N}_{u12} \\ 0 & I \end{bmatrix}$ (hence of (Factor1X)) can be chosen so that X is as in (d) (since $X \in \mathcal{GB}(U \times W)$, by Proposition 6.3.1(b1)).

1.3° Assume that $\mathbb{X}$ is as above and $\mathbb{N}_u, \mathbb{M}_u \in \mathrm{UR}$. Then $\mathbb{D} = \mathbb{N}_u \mathbb{M}_u{}^{-1} \in \mathrm{UR}$ and $\mathbb{N}, \mathbb{M}, \mathbb{E} \in \mathrm{UR}$, by (12.49) and Proposition 6.3.1(b1). Consequently, $X^{-1} = \begin{bmatrix} X_{11}^{-1} & * \\ 0 & X_{22}^{-1} \end{bmatrix}$, $M = \begin{bmatrix} * & * \\ 0 & X_{22}^{-1} \end{bmatrix}$; by Lemma A.1.1(b), $M_{11} \in \mathcal{GB}(U)$. If $D_{21} = 0$, then $N = DM = \begin{bmatrix} * & * \\ 0 & D_{22}X_{22}^{-1} \end{bmatrix}$, hence then $E = \begin{bmatrix} N_{22} & -N_{21} \\ -M_{12} & M_{11} \end{bmatrix} = \begin{bmatrix} N_{22} & 0 \\ -M_{12} & M_{11} \end{bmatrix}$.

1.4° *Part (e):* (Note that $\mathbb{D}, \mathbb{N}, \mathbb{M}, \mathbb{E} \in \mathrm{UR}$, by (d). Thus, any UR solution $\widetilde{\mathbb{M}}_+, \widetilde{\mathbb{N}}_+$ of (Factor2) corresponds to a UR solution $\mathbb{Z}$ of (Factor2Z) and vice versa, by Theorem 12.3.7(e2).)

By Theorem 12.3.7(e2), we have $(Z^{-1})_{22}M_{11} = \widetilde{M}_{+22} = W_{22}$ and $\mathbb{W} \in \mathrm{UR}$. By (Factor2) and Proposition 6.3.1(b1), $W_{22} \in \mathcal{GB}(U)$, hence $(Z^{-1})_{22} \in \mathcal{GB}(U)$, hence $Z_{11} \in \mathcal{GB}(Y)$, by Lemma A.1.1(c1). But $\widetilde{N}_{+21} = (Z^{-1})_{22}$, hence (e) holds.

2° *Lemma 12.3.8:* This is given in the proof of Lemma 12.4.3 for $\mathbb{N}_u, \mathbb{M}_u, \widetilde{\mathbb{N}_y}, \widetilde{\mathbb{M}_y}$. For (4BP1) and (Factor1[X]) independence on $\mathbb{N}, \mathbb{M}, \mathbb{X}, \widetilde{\mathbb{N}}_+, \widetilde{\mathbb{M}}_+, \mathbb{Z}$ is obvious; for (4BP2) it follows from Lemma 12.4.3(b).

Fix a pair $(\mathbb{N}, \mathbb{M})$ and a corresponding $\mathbb{E}$. Let $\mathbb{E}'$ correspond to some E given in Lemma 12.3.11(b). By Lemma 6.4.8(c), $EJ_1E^* = J_1$. By setting $R := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ we get $R^*\mathbb{E}'R = \begin{bmatrix} \mathbb{M}_{1*} \\ \mathbb{N}_{2*} \end{bmatrix} E$ and $R^*J_1R = -J_1$, hence

$$\mathbb{E}' J_1 \mathbb{E}'^* = -\begin{bmatrix} \mathbb{N}_{2*} \\ -\mathbb{M}_{1*} \end{bmatrix} J_1 \begin{bmatrix} \mathbb{N}_{2*} \\ -\mathbb{M}_{1*} \end{bmatrix}^*, \tag{12.64}$$

which is independent of E, i.e., depends on $\mathbb{D}$ only, hence so do the solutions $\mathbb{Z}$ of (Factor2Z[']). From the equivalences given in Theorem 12.3.7(e1), we obtain that also the solvability of (Factor2[']) depends on $\mathbb{D}$ only (but $\mathbb{D}_+$ depends on $\mathbb{N}, \mathbb{M}$, hence so do $\widetilde{\mathbb{N}}_+, \widetilde{\mathbb{M}}_+$). The claims concerning $\mathbb{Q}$ follow from Lemma 12.4.3(c). □

Now we can show that (4BP3) implies (4BP1)–(4BP7):

Lemma 12.4.7 ((Factor)⇒(4BP)) *Assume (Factor1) and (Factor2). Then (4BP1)–(4BP7) hold.*

Proof: We use the notation of Theorem 12.3.7(e1)–(e2) (see Lemma 12.4.4). Set $\begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} := \begin{bmatrix} 0 & I \end{bmatrix} \mathbb{Z}^{-1}$ to obtain

$$\begin{bmatrix} \widetilde{\mathbb{U}}_1 & \widetilde{\mathbb{U}}_2 \end{bmatrix} := \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \mathbb{E} = \begin{bmatrix} 0 & I \end{bmatrix} \mathbb{W} = \begin{bmatrix} \mathbb{W}_{21} & \mathbb{W}_{22} \end{bmatrix}. \tag{12.65}$$

From Lemma 12.3.11(a) we obtain that $\|\mathbb{W}_{22}^{-1}\mathbb{W}_{21}\| < 1$, hence (4BP7) is satisfied. Thus, (4BP1)–(4BP7) hold, by Lemma 12.4.3(b). □

We shall use the following to reduce (Factor2Z) to a (frequency-domain) H^∞ FICP:

Lemma 12.4.8 ($\exists\mathbb{P}$) *Assume (4BP2). Then there is* $\mathbb{P} \in \mathcal{G}\mathrm{TIC}(Y \times U)$ *s.t.* $\mathbb{P}\mathbb{E} = \begin{bmatrix} \mathbb{S} & 0 \\ \widetilde{\mathbb{U}} & I \end{bmatrix} \in \mathrm{TIC}(W \times U, Y \times U)$*, where* $\mathbb{S}\mathbb{S}^* \gg 0$*,* $\|\widetilde{\mathbb{U}}\| < 1$.

As the proof shows, $\widetilde{\mathbb{U}}$ is the one appearing in (4BP2).

Proof: Assume (4BP2). By Lemma 12.4.3(c1)&(b), $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ (from (4BP2)) stabilizes $\mathbb{D}$ (because it satisfies (4BP1)), and conditions (4BP1)–(4BP7) except possibly (4BP3) hold. In particular, $\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{M}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}_{\mathrm{y}\,21} \\ \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$, by Theorem 7.3.19(iv'), hence

$$\begin{bmatrix} \widetilde{\mathbb{M}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}_{\mathrm{y}\,21} \\ \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \in \mathcal{G}\mathrm{TIC}(Y \times U). \tag{12.66}$$

From the $(2,2)$- and $(2,1)$-blocks of equation $\widetilde{\mathbb{M}}_{\mathrm{y}}\mathbb{N}_{\mathrm{u}}^{\frown} = \widetilde{\mathbb{N}}_{\mathrm{y}}\mathbb{M}_{\mathrm{u}}^{\frown}$ we get

$$\widetilde{\mathbb{M}}_{\mathrm{y}\,22}\mathbb{N}^{\frown}_{\mathrm{u}\;22} - \widetilde{\mathbb{N}}_{\mathrm{y}\,21}\mathbb{M}^{\frown}_{\mathrm{u}\;12} = \widetilde{\mathbb{N}}_{\mathrm{y}\,22}, \;\; \widetilde{\mathbb{M}}_{\mathrm{y}\,22}\mathbb{N}^{\frown}_{\mathrm{u}\;21} - \widetilde{\mathbb{N}}_{\mathrm{y}\,21}\mathbb{M}^{\frown}_{\mathrm{u}\;11} = \widetilde{\mathbb{N}}_{\mathrm{y}\,22}\mathbb{M}^{\frown}_{\mathrm{u}\;21} = 0. \tag{12.67}$$

On the other hand, $\mathbb{E} = \begin{bmatrix} \mathbb{N}_{22} & -\mathbb{N}_{21} \\ -\mathbb{M}_{12} & \mathbb{M}_{11} \end{bmatrix} = \begin{bmatrix} \mathbb{N}^{\frown}_{\mathrm{u}\;22} & -\mathbb{N}^{\frown}_{\mathrm{u}\;21} \\ -\mathbb{M}^{\frown}_{\mathrm{u}\;12} & \mathbb{M}^{\frown}_{\mathrm{u}\;11} \end{bmatrix} \begin{bmatrix} \mathbb{V}_{22} & -\mathbb{V}_{21} \\ 0 & I \end{bmatrix}$ (here we have set $\mathbb{X} := \mathbb{M}^{-1}\mathbb{M}_{\mathrm{u}}$, $\mathbb{V} := \mathbb{X}^{-1}$). By combining this and (12.67) we get

$$\begin{bmatrix} \widetilde{\mathbb{M}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}_{\mathrm{y}\,21} \end{bmatrix} \mathbb{E} = \begin{bmatrix} \widetilde{\mathbb{N}}_{\mathrm{y}\,22} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{V}_{22} & -\mathbb{V}_{21} \\ 0 & I \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbb{N}}_{\mathrm{y}\,22}\mathbb{V}_{22} & -\widetilde{\mathbb{N}}_{\mathrm{y}\,22}\mathbb{V}_{21} \end{bmatrix}. \tag{12.68}$$

This combined with (4BP2) gives us $\begin{bmatrix} \widetilde{\mathbb{M}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}_{\mathrm{y}\,21} \\ \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \mathbb{E} = \begin{bmatrix} \widetilde{\mathbb{N}}_{\mathrm{y}\,22}\mathbb{V}_{22} & -\widetilde{\mathbb{N}}_{\mathrm{y}\,22}\mathbb{V}_{21} \\ \widetilde{\mathbb{U}} & I \end{bmatrix}$, where $\|\widetilde{\mathbb{U}}\| < 1$. Thus, $\mathbb{P}\mathbb{E} = \begin{bmatrix} \mathbb{S} & 0 \\ \widetilde{\mathbb{U}} & I \end{bmatrix}$, where

$$\mathbb{P} := \begin{bmatrix} I & \widetilde{\mathbb{N}}_{\mathrm{y}\,22}\mathbb{V}_{21} \\ 0 & I \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{M}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}_{\mathrm{y}\,21} \\ \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \in \mathrm{TIC}(Y \times U), \tag{12.69}$$

and $\mathbb{S} := \widetilde{\mathbb{N}}_{\mathrm{y}\,22}(\mathbb{V}_{22} + \mathbb{V}_{21}\widetilde{\mathbb{U}})$ is onto ($\mathbb{S}\mathbb{S}^* \gg 0$), because $\widetilde{\mathbb{N}}_{\mathrm{y}\,22}$ is onto, by Standing Hypothesis 12.3.1, and $\mathbb{V}_{22} + \mathbb{V}_{21}\widetilde{\mathbb{U}} \in \mathcal{G}\mathrm{TIC}(W)$ (because $\|\mathbb{V}_{22}^{-1}\mathbb{V}_{21}\widetilde{\mathbb{U}}\| < 1$, by Lemma 12.3.11(a)). By (12.66) and Lemma A.1.1(b1), $\mathbb{P} \in \mathcal{G}\mathrm{TIC}(Y \times U)$. □

Lemma 12.4.9 *Theorem 12.3.7(c) holds.*

Proof: 1° *(12.47):* Assume (Factor1) and (Factor2), so that (4BP2), (Factor1X) and (Factor2Z) hold, by Lemma 12.4.7 and Theorem 12.3.7(e1)&(e2) (see Lemma 12.4.4). We set $\mathbb{Z}^{-\mathrm{d}} := (\mathbb{Z}^{\mathrm{d}})^{-1}$ etc.

Set $\widetilde{\mathbb{Z}} := \mathbb{P}\mathbb{Z}$ to obtain $\widetilde{\mathbb{Z}}J_1\widetilde{\mathbb{Z}}^* = (\mathbb{P}\mathbb{E})J_1(\mathbb{P}\mathbb{E})^*$, where $\mathbb{P}$ is as in Lemma 12.4.8. Now equations $\mathbb{W} = \mathbb{Z}^{-1}\mathbb{E} = \widetilde{\mathbb{Z}}^{-1}\mathbb{P}\mathbb{E}$ and $(\mathbb{P}\mathbb{E})^{\mathrm{d}} = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix}$ imply that $\mathbb{W}^{\mathrm{d}} = \begin{bmatrix} * & * \\ (\widetilde{\mathbb{Z}}^{-\mathrm{d}})_{21} & (\widetilde{\mathbb{Z}}^{-\mathrm{d}})_{22} \end{bmatrix}$, hence $(\widetilde{\mathbb{Z}}^{-\mathrm{d}})_{22} = \mathbb{W}^{\mathrm{d}}_{22} \in \mathcal{G}\mathrm{TIC}(U)$, by (Factor2Z), hence $\widetilde{\mathbb{Z}}^{\mathrm{d}}_{11} \in \mathcal{G}\mathrm{TIC}(Y)$, by Lemma A.1.1(c1).

Now each pair $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in$ TIC corresponds to a unique pair $\overline{\mathbb{Q}}_1, \overline{\mathbb{Q}}_2 \in$ TIC defined by $\begin{bmatrix} \overline{\mathbb{Q}}_1^{\mathrm{d}} \\ \overline{\mathbb{Q}}_2^{\mathrm{d}} \end{bmatrix} := \mathbb{P}^{-\mathrm{d}} \begin{bmatrix} \widetilde{\mathbb{Q}}_1^{\mathrm{d}} \\ \widetilde{\mathbb{Q}}_2^{\mathrm{d}} \end{bmatrix}$. The pairs $(\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2)$ satisfying (4BP2) correspond to pairs $\overline{\mathbb{Q}}_1, \overline{\mathbb{Q}}_2$ satisfying

$$\begin{bmatrix} \widetilde{\mathbb{U}}^{\mathrm{d}} \\ I \end{bmatrix} = \mathbb{E}^{\mathrm{d}} \begin{bmatrix} \widetilde{\mathbb{Q}}_1^{\mathrm{d}} \\ \widetilde{\mathbb{Q}}_2^{\mathrm{d}} \end{bmatrix} = \mathbb{E}^{\mathrm{d}}\mathbb{P}^{\mathrm{d}} \begin{bmatrix} \overline{\mathbb{Q}}_1^{\mathrm{d}} \\ \overline{\mathbb{Q}}_2^{\mathrm{d}} \end{bmatrix}, \tag{12.70}$$

for some $\widetilde{\mathbb{U}}$ with $\|\widetilde{\mathbb{U}}\| < 1$ [≤ 1 for $\mathbb{Q}$'s s.t. $\|\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q})\| \leq \gamma$; see Lemma 12.4.3(e)].

From (12.70) and $\widetilde{\mathbb{D}} := \mathbb{E}^{\mathrm{d}}\mathbb{P}^{\mathrm{d}} = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$ we obtain that $\overline{\mathbb{Q}}_2 = I$, hence $\overline{\mathbb{Q}}_1^{\mathrm{d}}$ is as above iff it satisfies (the FICP) $\|\widetilde{\mathbb{U}}^{\mathrm{d}}\| := \|\widetilde{\mathbb{D}}_{11}\overline{\mathbb{Q}}_1^{\mathrm{d}} + \widetilde{\mathbb{D}}_{12}\| < 1$ [≤ 1].

Thus, we may apply Corollary 11.3.5 (with substitutions $\mathbb{D} \mapsto \widetilde{\mathbb{D}}$, $\gamma \mapsto 1$, $\mathbb{X} \mapsto \widetilde{\mathbb{Z}}^{\mathrm{d}}$; note that $\widetilde{\mathbb{D}}_{11}^*\widetilde{\mathbb{D}}_{11} = (\mathbb{S}^{\mathrm{d}})^*\mathbb{S}^{\mathrm{d}} = (\mathbb{S}\mathbb{S}^*)^{\mathrm{d}} \gg 0$, since $\mathbb{S}$ (and hence $(\mathbb{S}^{\mathrm{d}})^*$ is onto)) to obtain that all $\overline{\mathbb{Q}}_1^{\mathrm{d}}$'s of this form are given by $\overline{\mathbb{Q}}_1^{\mathrm{d}} = \mathbb{L}_1\mathbb{L}_2^{-1}$, where $\begin{bmatrix} \mathbb{L}_1 \\ \mathbb{L}_2 \end{bmatrix} := \widetilde{\mathbb{Z}}^{-1} \left[\begin{smallmatrix} \mathbb{L} \\ I \end{smallmatrix}\right]$ and $\|\mathbb{L}\| < 1$ [≤ 1] (hence $\mathbb{L}_2 \in \mathcal{G}\mathrm{TIC}(U)$, by Theorem 11.3.6). Writing it out, we have

$$\begin{bmatrix} \overline{\mathbb{Q}}_1^{\mathrm{d}} \\ I \end{bmatrix} = \begin{bmatrix} \mathbb{L}_1 \\ \mathbb{L}_2 \end{bmatrix} \mathbb{L}_2^{-1} = \widetilde{\mathbb{Z}}^{-\mathrm{d}} \begin{bmatrix} \mathbb{L} \\ I \end{bmatrix} \mathbb{L}_2^{-1}. \tag{12.71}$$

Thus, $\begin{bmatrix} \widetilde{\mathbb{Q}}_1^{\mathrm{d}} \\ \widetilde{\mathbb{Q}}_2^{\mathrm{d}} \end{bmatrix} = \mathbb{P}^{-\mathrm{d}}\widetilde{\mathbb{Z}}^{-\mathrm{d}} \left[\begin{smallmatrix} \mathbb{L} \\ I \end{smallmatrix}\right] \mathbb{L}_2^{-1} = \mathbb{Z}^{-\mathrm{d}} \left[\begin{smallmatrix} \mathbb{L} \\ I \end{smallmatrix}\right] \mathbb{L}_2^{-1}$.

By postmultiplying this by $\mathbb{L}_2$, we get another representative of $\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ (since $\mathbb{L}_2 \in \mathcal{G}\mathrm{TIC}(U)$), given by

$$\begin{bmatrix} \widetilde{\mathbb{Q}}_1' & \widetilde{\mathbb{Q}}_2' \end{bmatrix} = \mathbb{L}_2^{\mathrm{d}} \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} = \begin{bmatrix} \mathbb{L}^{\mathrm{d}} & I \end{bmatrix} \mathbb{Z}^{-1}, \tag{12.72}$$

But this "all solutions formula" is equal to (12.47) (see (12.50) for $\mathbb{Z}^{-1}$).

2° *Formula* $\mathbb{Q} = \mathcal{F}_\ell(\mathbb{T}, \mathbb{L})$*:* The formula $\mathbb{Q} = \mathcal{F}_\ell(\mathbb{T}, \mathbb{L})$ can be shown equal to (12.47) just by writing the two formulae out and simplifying slightly (note that $I - \mathbb{T}_{21}\mathbb{L} = I + \widetilde{\mathbb{N}}_{+11}\widetilde{\mathbb{N}}_{+21}^{-1}\mathbb{L}$ is invertible iff $\widetilde{\mathbb{Q}}_2 = \mathbb{L}\widetilde{\mathbb{N}}_{+11} + \widetilde{\mathbb{N}}_{+21} = (I + \mathbb{L}\widetilde{\mathbb{N}}_{+11}\widetilde{\mathbb{N}}_{+21}^{-1})\widetilde{\mathbb{N}}_{+21}$ is invertible, by Lemma A.1.1(f6), i.e., iff $\mathbb{Q}$ is well-posed (by Lemma 7.2.12(b))).

3° *Remark:* We needed $\mathbb{P}$ only to get a spectral factor (namely $\widetilde{\mathbb{Z}}^{\mathrm{d}}$) with invertible $(1,1)$-block and to establish the condition "$\mathbb{D}_1^* J \mathbb{D}_1 \gg 0$". Otherwise we could have applied Theorem 11.3.6 directly for $\mathbb{E}^{\mathrm{d}}$ instead of $\widetilde{\mathbb{D}}$.) □

If the coprimeness requirements of Standing Hypothesis 12.3.1 are satisfied "exponentially", then the existence of a solution the 4BP is equivalent to the existence of an exponentially stabilizing solution of the 4BP (at least when $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$, so that (4BP3) becomes necessary):

Proposition 12.4.10 (Exponentially stabilizing solutions of the (I/O) 4BP) *Assume that we can write "d.c.f. over* $\mathrm{TIC}_{\mathrm{exp}}$*", "r.c. over* $\mathrm{TIC}_{\mathrm{exp}}$*" and "l.c. over* $\mathrm{TIC}_{\mathrm{exp}}$*" to Standing Hypothesis 12.3.1 in place of "d.c.f.", "r.c." and "l.c.",* *respectively (this is the case if Hypothesis 12.5.1 holds and Σ is optimizable).*

If (4BP3) holds, then $\mathbb{N},\mathbb{M},\mathbb{X},\mathbb{Z},\mathbb{E}\in \mathrm{TIC}_{\mathrm{exp}}$, and all exponentially DPF-stabilizing suboptimal controllers (with internal loop) for $\mathbb{D}$ are parametrized by (12.47) with the additional requirement that $\mathbb{L}\in\mathrm{TIC}_{\mathrm{exp}}$.

Proof: One observes from the proof of Lemma 12.5.3 that the assumptions of the lemma are satisfied if Hypothesis 12.5.1 holds and Σ is optimizable.

1° *$\mathbb{Q}$ is suboptimal and exponentially DPF-stabilizing iff $\mathbb{Q}=\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ for some $\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2\in\mathrm{TIC}_{\mathrm{exp}}$ satisfying (4BP4):*

From Corollary 7.3.20(i') and Remark 7.3.24 we observe that $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ exponentially with internal loop iff $\mathbb{Q}=\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ and $\widetilde{\mathbb{Q}}_2\mathbb{M}_{\mathrm{u}\,11}-\widetilde{\mathbb{Q}}_1\mathbb{N}_{\mathrm{u}\,21}=I$ for some $\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2\in\mathrm{TIC}_{\mathrm{exp}}$ (cf. Lemma 12.3.2).

2° $\mathbb{X}^{\pm1},\mathbb{N},\mathbb{M},\mathbb{Z}^{\pm1},\mathbb{E}\in\mathrm{TIC}_{\mathrm{exp}}$*:* As in the proof of Lemma 12.4.3(c3), we observe that $\mathbb{X}^{\pm1},\mathbb{N},\mathbb{M}\in\mathrm{TIC}_{\mathrm{exp}}$, hence $\mathbb{E}\in\mathrm{TIC}_{\mathrm{exp}}$. By Lemma 6.4.7(c), it follows that $\mathbb{Z}\in\mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$. (by Lemma 12.3.11(b), this applies to all possible choices of $\mathbb{X},\mathbb{N},\mathbb{M},\mathbb{Z},\mathbb{E}$).

3° *Sufficiency:* Therefore, for each $\mathbb{L}\in\mathrm{TIC}_{\mathrm{exp}}$, we have $\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2\in\mathrm{TIC}_{\mathrm{exp}}$ in (12.47) (see (12.50) for $\mathbb{Z}^{-1}$). By Lemma 12.4.3(c3), $\mathbb{Q}:=\widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ corresponds to a solution of (4BP4) belonging to $\mathrm{TIC}_{\mathrm{exp}}$, hence $\mathbb{Q}$ is as in 1°.

4° *Necessity:* If $\mathbb{Q}=\underline{\mathbb{Q}}_2^{-1}\underline{\mathbb{Q}}_1$ is a suboptimal exponentially DPF-stabilizing controller (with an internal loop), then

$$\begin{bmatrix}\underline{\mathbb{Q}}_1 & \underline{\mathbb{Q}}_2\end{bmatrix}=\mathbb{U}\begin{bmatrix}\widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2\end{bmatrix}:=\begin{bmatrix}\mathbb{UL} & \mathbb{U}\end{bmatrix}\mathbb{Z}^{-1} \tag{12.73}$$

for some $\mathbb{U}\in\mathcal{G}\mathrm{TIC}$, by Lemma 6.4.5(d) and (12.47), hence then $\begin{bmatrix}\mathbb{UL} & \mathbb{U}\end{bmatrix}=\mathbb{Z}\begin{bmatrix}\underline{\mathbb{Q}}_1 & \underline{\mathbb{Q}}_2\end{bmatrix}\in\mathrm{TIC}_{\mathrm{exp}}$, hence $\mathbb{U}\in\mathrm{TIC}_{\mathrm{exp}}$ (recall from 2° that $\mathbb{Z}\in\mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$), hence $\mathbb{U}\in\mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$, by Lemma 2.2.7. Consequently, then $\mathbb{L}=\mathbb{U}^{-1}\mathbb{UL}\in\mathrm{TIC}_{\mathrm{exp}}$. □

Next we want to make $\mathbb{M}_{11}$ and $\mathbb{Z}_{11}$ invertible; therefore we need the following result:

Lemma 12.4.11 (X_{22} invertible) *Assume that $X\in\mathcal{GB}(U\times W)$, and that $\begin{bmatrix}T & I\end{bmatrix}X\left[\begin{smallmatrix}0\\ I\end{smallmatrix}\right]\in\mathcal{GB}(W)$ (resp. $X_{11}\in\mathcal{GB}$ and $\begin{bmatrix}T & I\end{bmatrix}X\left[\begin{smallmatrix}I\\ 0\end{smallmatrix}\right]=0$) for some $T\in\mathcal{B}(U,W)$ with $\|T\|<1$.*

Then there is $E\in\mathcal{GB}(U\times W)$ s.t. $E^*J_1E=J_1$, and $\widetilde{X}:=EX$ satisfies $\widetilde{X}_{22}\in\mathcal{GB}(W)$ (resp. $\widetilde{X}_{21}=0$ and $\widetilde{X}_{11},\widetilde{X}_{22}\in\mathcal{GB}$).

Thus, $\widetilde{X}^*J_1\widetilde{X}=X^*J_1X$.

Proof: We use below Lemma A.3.1(d)&(e2). Set $U:=(I-T^*T)^{-1/2}\gg 0$, $V:=(I-TT^*)^{-1/2}\gg0$. By Lemma A.1.1(f6)&(d1), $V^2T=TU^2$ and the inverse of

$$E:=\begin{bmatrix}U & 0\\ 0 & V\end{bmatrix}\begin{bmatrix}I & T^*\\ T & I\end{bmatrix}=\begin{bmatrix}U & UT^*\\ VT & V\end{bmatrix}\quad\text{is}\quad E^{-1}=\begin{bmatrix}U & -T^*V\\ -TU & V\end{bmatrix}. \tag{12.74}$$

By a simple computation one verifies that $E^*J_1E=J_1$.

If $\begin{bmatrix}T & I\end{bmatrix}X\left[\begin{smallmatrix}0\\ I\end{smallmatrix}\right]\in\mathcal{GB}(W)$, then $\widetilde{X}_{22}=V\begin{bmatrix}T & I\end{bmatrix}X\left[\begin{smallmatrix}0\\ I\end{smallmatrix}\right]\in\mathcal{GB}(W)$. If, instead, $\begin{bmatrix}T & I\end{bmatrix}X\left[\begin{smallmatrix}I\\ 0\end{smallmatrix}\right]=0$ and $X_{11}\in\mathcal{GB}$, then $\widetilde{X}_{21}=V\begin{bmatrix}T & I\end{bmatrix}X\left[\begin{smallmatrix}I\\ 0\end{smallmatrix}\right]=0$ and hence

$X_{11} = U\widetilde{X}_{11}$; therefore, $\widetilde{X}_{11} = U^{-1}X_{11} \in \mathcal{GB}$, and, consequently, $\widetilde{X}_{22} \in \mathcal{GB}$, by Lemma A.1.1(b2). $\square$

Lemma 12.4.12 (Well-posed $\mathbb{D}_+$) *If (Factor1) holds and $\mathbb{X}, \mathbb{M}_u \in$ ULR, then we can take $\mathbb{X}_{22}, \mathbb{M}_{11} \in \mathcal{G}TIC_\infty$ (and $X = \begin{bmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{bmatrix}$, $M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & X_{22}^{-1} \end{bmatrix}$ where $X_{11}, X_{22}, M_{11} \in \mathcal{GB}$); in particular, $\mathbb{D}_+$ becomes well-posed.*

Proof: (We use here Theorem 12.3.7(e1)&(e2) and Lemma 12.3.11(a)&(c); these have already been proved, in Lemmas 12.4.2 and 12.4.4.)

Let $\mathbb{X}$ satisfy (Factor1X). Now $X := \mathbb{X}(+\infty) \in \mathcal{GB}$ and $X_{11} \in \mathcal{GB}$, by Proposition 6.3.1(c). Clearly $\|X_{21}X_{11}^{-1}\| \leq \|\mathbb{X}_{21}\mathbb{X}_{11}^{-1}\| < 1$ (see Lemma 12.3.11(a)), hence with $T := -X_{21}X_{11}^{-1}$ we obtain from Lemma 12.4.11 an operator $E \in \mathcal{GB}$ s.t. $\widetilde{\mathbb{X}} := E\mathbb{X}$ satisfies $\widetilde{\mathbb{X}}^* J_1 \widetilde{\mathbb{X}} = \mathbb{X}^* J_1 \mathbb{X}$ and $\widetilde{X}_{11}, \widetilde{X}_{22} \in \mathcal{GB}$, $\widetilde{X}_{21} = 0$; hence $\widetilde{\mathbb{X}}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$, by Proposition 6.3.1(c).

By Lemma 12.3.11(c), also $\widetilde{\mathbb{X}}$ satisfies (Factor1X). The claims for $\mathbb{M} = \mathbb{M}_u \widetilde{\mathbb{X}}^{-1}$ follow from the above (the invertibility of M_{11} and hence that of $\mathbb{M}_{11}$ follows from Lemma A.1.1(b2)&(b1)). $\square$

Lemma 12.4.13 (Well-posed $\mathbb{Q}$) *Assume that (Factor1) and (Factor2Z) hold with $\mathbb{Z} \in \widetilde{\mathcal{A}}$, and let some well-posed $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ with $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}$ solve the 4BP. Then we can redefine $\mathbb{Z} \in \widetilde{\mathcal{A}}$ s.t. $(\mathbb{Z}^{-1})_{22} \in \mathcal{G}\mathrm{TIC}_\infty$ (i.e., $\widetilde{\mathbb{N}}_{+21} \in \mathcal{G}\mathrm{TIC}_\infty$).*

This also holds with ULR *in place of $\widetilde{\mathcal{A}}$.*

Proof: Let $\mathcal{A} = \widetilde{\mathcal{A}}$ or $\mathcal{A} =$ ULR (in fact, the lemma and this proof holds for $\mathcal{A}$ in place of $\widetilde{\mathcal{A}}$ whenever $\mathcal{B} \underset{a}{\subset} \mathcal{A} \underset{a}{\subset}$ ULR).

Set $\mathbb{L} := \begin{bmatrix} \widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2 \end{bmatrix} \mathbb{Z} \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathcal{A}$ as in Theorem 12.3.7(c).

Because $\mathcal{A} \subset$ ULR, we can set $T := \mathbb{L}(+\infty)$, $X := \mathbb{Z}^{-1}(+\infty)$, to obtain $\mathcal{GB} \ni \widetilde{\mathbb{Q}}_2(+\infty) = \begin{bmatrix} T & I \end{bmatrix} X \begin{bmatrix} 0 \\ I \end{bmatrix}$. By Proposition 6.3.1(c), $X \in \mathcal{GB}$. By Lemma 12.4.11 we get $(EX)_{22}$ invertible.

Thus, by setting $\mathbb{Z}' := \mathbb{Z}E^{-1} \in \mathcal{A}$ we get $(\mathbb{Z}'^{-1})_{22}(+\infty) = (EX)_{22} \in \mathcal{GB}$, and we see that $\mathbb{Z}'J_1\mathbb{Z}'^* = \mathbb{Z}J_1\mathbb{Z}^*$ ($= \mathbb{E}J_1\mathbb{E}^*$) and $\mathbb{W}' := \mathbb{Z}'^{-1}\mathbb{E} = E\mathbb{W}$ also satisfies $\mathbb{W}'J_1\mathbb{W}'^* = J_1$, by Lemma 6.4.8(c). $\square$

Lemma 12.4.14 *Theorem 12.3.7 holds.*

Proof: Parts (e1) and (e2) were shown in Lemma 12.4.4, and part (c) in Lemma 12.4.9. We prove (a), (b) and (d) below.

(a) 1° *"(4BP3)⇒(4BP2)⇒(4BP1)":* "(4BP3)⇒(4BP2)" is obtained from Lemma 12.4.7, and "(4BP2)⇒(4BP1)" from Lemma 12.4.3(b).

2° *"(4BP1)⇒(4BP2)" when* $(\begin{bmatrix} \mathbb{N}_{u11} & \mathbb{N}_{u12} \\ 0 & I \end{bmatrix}, J_\gamma) \in$ SpF*:* Assume (4BP1). Then $\mathbb{D}' := \begin{bmatrix} \mathbb{N}_{u11} & \mathbb{N}_{u12} \\ 0 & I \end{bmatrix}$ is minimax J_γ-coercive, by Lemma 12.4.1, hence J-coercive.

Thus, if $(\mathbb{D}', J_\gamma) \in$ SpF, then (Factor1X) (and hence (Factor1)) holds, by Lemma 11.4.3(b). Consequently, (4BP2) holds, by Lemma 12.4.3(b).

3° *"(4BP2)⇒(4BP3)" when* $(\mathbb{E}^{\mathrm{d}}, J_1) \in$ SpF*:* Assume that (4BP2) holds and $(\mathbb{E}^{\mathrm{d}}, J_1) \in$ SpF. As in p. 530 of [Green], we set $\begin{bmatrix}\overline{\mathbb{Q}}_1 & \overline{\mathbb{Q}}_2\end{bmatrix} := \begin{bmatrix}\widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2\end{bmatrix}\mathbb{P}^{-1}$ (here $\mathbb{P}$ is from Lemma 12.4.8 and $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2$ from (4BP2)) to obtain

$$\begin{bmatrix}\widetilde{\mathbb{U}} & I\end{bmatrix} = \begin{bmatrix}\widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2\end{bmatrix}\mathbb{E} = \begin{bmatrix}\overline{\mathbb{Q}}_1 & \overline{\mathbb{Q}}_2\end{bmatrix}\mathbb{P}\mathbb{E} = \begin{bmatrix}\overline{\mathbb{Q}}_1 & \overline{\mathbb{Q}}_2\end{bmatrix}\begin{bmatrix}\mathbb{S} & 0\\ \widetilde{\mathbb{U}} & I\end{bmatrix} = \begin{bmatrix}\overline{\mathbb{Q}}_1\mathbb{S} + \overline{\mathbb{Q}}_2\widetilde{\mathbb{U}}\\ \overline{\mathbb{Q}}_2\end{bmatrix}. \tag{12.75}$$

Thus, $\overline{\mathbb{Q}}_2 = I$ and $\|\overline{\mathbb{Q}}_1\mathbb{S} + \widetilde{\mathbb{U}}\| = \|\widetilde{\mathbb{U}}\| < 1$, i.e., $\|\mathbb{S}^{\mathrm{d}}\overline{\mathbb{Q}}_1^{\mathrm{d}} + \widetilde{\mathbb{U}}^{\mathrm{d}}\| < 1$, hence $\widetilde{\mathbb{D}} := \mathbb{E}^{\mathrm{d}}\mathbb{P}^{\mathrm{d}}$ is minimax J_1-coercive, by Lemma 11.3.10 (recall from Lemma 12.4.8 that $\widetilde{\mathbb{D}}_{11}^*\widetilde{\mathbb{D}}_{11} = (\mathbb{S}^{\mathrm{d}})^*(\mathbb{S}^{\mathrm{d}}) \gg 0$, as required by Hypothesis 11.3.1)), hence J_1-coercive, by Lemma 11.4.2, i.e., $\pi_+(\mathbb{E}^{\mathrm{d}}\mathbb{P}^{\mathrm{d}})^*J_1\mathbb{E}^{\mathrm{d}}\mathbb{P}^{\mathrm{d}}\pi_+$ is invertible. Therefore, also $\pi_+(\mathbb{E}^{\mathrm{d}})^*J_1\mathbb{E}^{\mathrm{d}}\pi_+$ is invertible, by Lemma 2.2.2(b)&(a1).

Since $(\mathbb{E}^{\mathrm{d}}, J_1) \in$ SpF, we have $(\mathbb{E}^{\mathrm{d}})^*J_1\mathbb{E}^{\mathrm{d}} = (\widetilde{\mathbb{Z}}^{\mathrm{d}})^*S\widetilde{\mathbb{Z}}^{\mathrm{d}}$ for some $\widetilde{\mathbb{Z}}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}(Y \times U)$ and $S \in \mathcal{G}\mathrm{TIC}(U)$. Consequently, $\widetilde{\mathbb{D}}^*J_1\widetilde{\mathbb{D}} = (\widetilde{\mathbb{Z}}^{\mathrm{d}}\mathbb{P}^{\mathrm{d}})^*S(\widetilde{\mathbb{Z}}^{\mathrm{d}}\mathbb{P}^{\mathrm{d}})$ and $\widetilde{\mathbb{Z}}^{\mathrm{d}}\mathbb{P}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}$. By Lemma 11.4.3(a), $\widetilde{\mathbb{D}}^*J_1\widetilde{\mathbb{D}} = \mathbb{R}^*J_1\mathbb{R}$ for some $\mathbb{R} \in \mathcal{G}\mathrm{TIC}(Y \times U)$ s.t. $\mathbb{R}_{11} \in \mathcal{G}\mathrm{TIC}(Y)$.

It follows that $(\mathbb{R}^{-1})_{22} \in \mathcal{G}\mathrm{TIC}(U)$, hence $(\mathbb{W}^{\mathrm{d}})_{22} \in \mathcal{G}\mathrm{TIC}(U)$, where $\mathbb{W}^{\mathrm{d}} := \widetilde{\mathbb{D}}\mathbb{R}^{-1}$ (we have $(\mathbb{W}^{\mathrm{d}})_{22} = \mathbb{R}_{22}^{-1}$, because $\widetilde{\mathbb{D}} = \left[\begin{smallmatrix}* & *\\ 0 & I\end{smallmatrix}\right]$, by Lemma 12.4.8). (Note that $\mathbb{W}^{\mathrm{d}} := \widetilde{\mathbb{D}}\mathbb{R}^{-1}$ is (J_1, J_1)-lossless, by Corollary 2.5.5(iii)&(i).)

Set $\mathbb{Z} := \mathbb{R}^{\mathrm{d}}\mathbb{P}^{-1}$, so that $\mathbb{W} = \mathbb{Z}^{-1}\mathbb{E}$ and hence (Factor2Z) (hence also (Factor2) holds.

(b) This is contained in Lemma 12.4.3(c1).

(d) By Definition 7.2.11 (and Lemma 7.2.12(b)), $\mathbb{Q} = \widetilde{\mathbb{Q}}_2^{-1}\widetilde{\mathbb{Q}}_1$ is well-posed iff $\widetilde{\mathbb{Q}}_2 \in \mathcal{G}\mathrm{TIC}_\infty$.

1° *If:* This follows from Lemma 12.4.13.

2° *Only if:* By Theorem 12.3.7(e2), we have $(\mathbb{Z}^{-1})_{22} = \widetilde{\mathbb{N}}_{+21}$. If this map is invertible and $\mathbb{Z} \in$ ULR, then $\mathbb{Z}^{-1} \in$ ULR, by Proposition 6.3.1(c), and we can take $\mathbb{L} = 0$ to obtain $\begin{bmatrix}\widetilde{\mathbb{Q}}_1 & \widetilde{\mathbb{Q}}_2\end{bmatrix} = \begin{bmatrix}(\mathbb{Z}^{-1})_{21} & (\mathbb{Z}^{-1})_{22}\end{bmatrix} \in$ ULR; in particular $\widetilde{\mathbb{Q}}_2 = (\mathbb{Z}^{-1})_{22} \in \mathcal{G}\mathrm{TIC}$. □

Lemma 12.4.15 *Lemma 12.3.10 holds.*

Proof: Assume that $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$. Then $(\left[\begin{smallmatrix}\mathbb{N}_{\mathrm{u}11} & \mathbb{N}_{\mathrm{u}12}\\ 0 & I\end{smallmatrix}\right], J_\gamma) \in$ SpF, hence we have "(4BP3)⇒(4BP2)⇔(4BP1)", by Theorem 12.3.7(a). If (4BP2) holds, then we have $\mathbb{X} \in \widetilde{\mathcal{A}}$, hence $\mathbb{E} \in \widetilde{\mathcal{A}}$, by Theorem 12.3.7(e2), hence $(\mathbb{E}^{\mathrm{d}}, J_1) \in$ SpF, hence (4BP3) holds, by Theorem 12.3.7(a). Thus, (4BP1)–(4BP3) are equivalent.

(a) Since $\mathbb{E} \in \widetilde{\mathcal{A}}$, we have $\mathbb{E}^{\mathrm{d}} \in \widetilde{\mathcal{A}}$, hence $\mathbb{Z}^{\mathrm{d}} \in \widetilde{\mathcal{A}}$, hence $\mathbb{Z} \in \widetilde{\mathcal{A}}$. It follows from Theorem 12.3.7(c)&(e2) that (a) holds.

(b) The first claim follows from Theorem 12.3.7(c), the second from the fact that $\mathbb{Z} \in \widetilde{\mathcal{A}} = \mathcal{G}\widetilde{\mathcal{A}}$ (and from (12.50)).

(c) By Lemma 12.4.12, we can take $\mathbb{M}_{11}, \mathbb{X}_{22} \in \mathcal{G}\mathrm{TIC}_\infty$ etc., hence $\mathbb{D}_+$ becomes well-posed. Consequently, $\widetilde{\mathbb{M}}_+ \in \mathcal{G}\mathrm{TIC}_\infty$, by Lemma 7.2.12(b). The d.c.f. (12.61) of $\mathbb{D}_+$ is over $\widetilde{\mathcal{A}}$, i.e., all its terms belong to $\widetilde{\mathcal{A}}$.

(d) This follows from Lemma 12.4.13 (because the converse is true by (12.47) with $\mathbb{L} = 0$).

(e) Choose $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2 \in \widetilde{\mathcal{A}}$ so that $\mathbb{Q}$ is DPF-stabilizing $\mathbb{D}$. By Theorem 7.3.19(iii), $\mathbb{Q}$ DF-stabilizes $\mathbb{D}_{21}$. Because $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2, \mathbb{N}_\mathrm{u}, \mathbb{M}_\mathrm{u} \in \widetilde{\mathcal{A}}$, it follows from Lemma 7.2.16 that the d.c.f. $\mathbb{D}_{21} = \mathbb{N}_{\mathrm{u}\,21}\mathbb{M}_{\mathrm{u}\,11}^{-1} = \widetilde{\mathbb{M}}_{\mathrm{y}\,22}^{-1}\widetilde{\mathbb{N}}_{\mathrm{y}\,21}$ is over $\widetilde{\mathcal{A}}$ if(f) $\widetilde{\mathbb{N}}_{\mathrm{y}\,21}, \widetilde{\mathbb{M}}_{\mathrm{y}\,22} \in \widetilde{\mathcal{A}}$. By (7.79), the "iff" in (e) follows analogously (because its converse is trivial).

(f) This is obvious (recall from Proposition 6.3.1(c) that $\mathbb{R} \in \mathcal{G}\mathrm{TIC}_\infty \Leftrightarrow \widehat{\mathbb{R}}(+\infty) \in \mathcal{GB}$ for any $\mathbb{R} \in \mathrm{TIC}_\infty \cap \mathrm{ULR}$, hence for any $\mathbb{R} \in \widetilde{\mathcal{A}}$).

(g) One observes this from the proof of Lemma 12.4.16 below. □

Lemma 12.4.16 *Theorem 12.3.6 holds.*

Proof: By Lemma 12.3.10(a)&(c), (Factor1) is now equivalent to (Factor1$\widetilde{\mathcal{A}}$).

The equivalence of (4BP1)–(4BP3) follows from Lemma 12.3.10; by Lemma 12.3.10(c)&(d), we can maintain the equivalence while strengthening the three assumptions to (4BP1$\widetilde{\mathcal{A}}$)–(4BP3$\widetilde{\mathcal{A}}$).

(We could equivalently drop the condition $\mathbb{M}_{11}(+\infty) \in \mathcal{GB}$ from (Factor1$\widetilde{\mathcal{A}}$), but we prefer having $\mathbb{D}_+$ well-posed in the formulation of (Factor2$\widetilde{\mathcal{A}}$).)

(a) This follows from Theorem 12.3.7(c) and Lemma 12.3.10(b).

(b) (In fact, it would suffice to assume that $\widetilde{\mathbb{N}}_{\mathrm{y}\,21}, \widetilde{\mathbb{M}}_{\mathrm{y}\,22} \in \widetilde{\mathcal{A}}$; cf. the proof of Lemma 12.3.10(e) above.)

"If" is trivial, "only if" follows from Lemma 7.2.16(a) (it provides a joint d.c.f. of $\mathbb{D}$ and $\mathbb{Q}$ in $\widetilde{\mathcal{A}}$, hence $\mathbb{Q} = \widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}} = \mathbb{Y}\mathbb{X}^{-1}$ for some $\widetilde{\mathbb{Y}}, \widetilde{\mathbb{X}}, \mathbb{X}, \mathbb{Y} \in \widetilde{\mathcal{A}}$; recall from Lemma 7.2.12(b) that $\mathbb{X}, \widetilde{\mathbb{X}} \in \mathcal{G}\mathrm{TIC}_\infty$).

(c) This follows from Lemma 12.3.9.

(d) By Lemma 12.3.11(e), the requirement $\widetilde{\mathbb{N}}_{+21} \in \mathcal{GB}(U)$ in (Factor2$\widetilde{\mathcal{A}}$) can always be satisfied, hence (Factor2$\widetilde{\mathcal{A}}$) is equivalent to (Factor2) when (Factor1$\widetilde{\mathcal{A}}$) (equivalently, (Factor1)) holds. Thus, (4BP3$\widetilde{\mathcal{A}}$) is equivalent to (4BP3), hence now (4BP1)–(4BP3) and (4BP1$\widetilde{\mathcal{A}}$)–(4BP3$\widetilde{\mathcal{A}}$) are all equivalent to each other. □

Notes

The methods of [Green] (or [CG97]) do not apply in the general case, as explained on p. 721. However, we have been able to use part of them by using suitable modifications.

Much of Lemma 12.4.3 was established for rational transfer functions in Section 3 of [Green]. Part of Lemma 12.4.11 is from Lemma 3.6 of [Green]. In the proof of Lemma 12.4.9, we have borrowed from [Green, p. 530] the idea to reduce the ASP (4BP2) to a H^∞ FICP. Green and [CG97] use the well-known matrix complementation properties of their respective transfer function classes. Since general H^∞ functions do not possess such properties, by Lemma 4.1.10, we have constructed a suitable reducing operator ($\mathbb{P}$) explicitly, in Lemma 12.4.8, by using (4BP2).

12.5 Proofs for Section 12.1 — 4BP $\mathcal{P}_X, \mathcal{P}_Y, \mathcal{P}_Z$

Conjecture 1. The proof of every result requires at least three auxiliary lemmas.

Addendum to Conjecture 1. This applies also to the proofs of auxiliary lemmas.

— K.M.

(Recall Standing Hypothesis 12.1.1.) In this section, we shall give proofs for the results of Section 12.1. Observe that, in most results of this section, the letters $\mathbb{X}$ and $\mathbb{M}$ correspond to "(1.)" (the $\mathcal{P}_X$-CARE) or to the corresponding IARE, not to (Factor1X) of Theorem 12.3.7; see each statement for details.

The classical assumption is that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \end{array}\right]$ is exponentially stabilizable and $\left[\begin{array}{c} \mathbb{A} \\ \hline \mathbb{C}_2 \end{array}\right]$ is exponentially detectable. By Lemma 13.3.17, in the discrete-time case this holds iff the system $\Sigma_{21} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_2 & \mathbb{D}_{21} \end{array}\right]$ is exponentially jointly stabilizable and detectable; the same is true in the continuous-time if, e.g., B_1 and C_2 are bounded (cf. Corollary 9.2.13(b)).

In that case, the corresponding pairs "$\left[\begin{array}{c|cc} \mathbb{K}_{\mathrm{u}\,1} & \mathbb{F}_{\mathrm{u}\,11} & \mathbb{F}_{\mathrm{u}\,12} \end{array}\right]$" and "$\left[\begin{array}{c} \mathbb{H}_{\mathrm{y}\,2} \\ \hline \mathbb{G}_{\mathrm{y}\,12} \\ \mathbb{G}_{\mathrm{y}\,22} \end{array}\right]$" (and interaction operator "$\mathbb{E}_{12}^{\mathrm{u\,y}}$") stabilize $\mathbb{A}$ and hence the whole Σ exponentially, by Lemma 13.3.8. However, sometimes the following, weaker assumption is enough for us:

Hypothesis 12.5.1 ($\mathbf{H}^\infty$ 4BP) *Assume that $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{WPLS}(U \times W, H, Z \times Y)$, and there are jointly stabilizing and detecting pairs*

$$\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{K}_{\mathrm{u}\,1} & \mathbb{F}_{\mathrm{u}\,11} & \mathbb{F}_{\mathrm{u}\,12} \\ 0 & 0 & 0 \end{array}\right] \quad \textit{and} \tag{12.76}$$

$$\left[\begin{array}{c} \mathbb{H}_{\mathrm{y}} \\ \hline \mathbb{G}_{\mathrm{y}} \end{array}\right] = \left[\begin{array}{cc} 0 & \mathbb{H}_{\mathrm{y}\,2} \\ \hline 0 & \mathbb{G}_{\mathrm{y}\,12} \\ 0 & \mathbb{G}_{\mathrm{y}\,22} \end{array}\right] \tag{12.77}$$

for Σ with some interaction operator $\begin{bmatrix} 0 & \mathbb{E}_{12}^{\mathrm{u\,y}} \\ 0 & 0 \end{bmatrix}$. Moreover, we make the nonsingularity assumptions

$$\mathbb{N}_{\mathrm{u}\,11}^{*}\mathbb{N}_{\mathrm{u}\,11} \gg 0, \quad \widetilde{\mathbb{N}}_{\mathrm{y}\,22}\widetilde{\mathbb{N}}_{\mathrm{y}\,22}^{*} \gg 0, \tag{12.78}$$

where $\mathbb{N}_{\mathrm{u}} := \mathbb{D}\mathbb{M}_{\mathrm{u}}$, $\widetilde{\mathbb{N}}_{\mathrm{y}} := \widetilde{\mathbb{M}}_{\mathrm{y}}\mathbb{D}$, $\mathbb{M}_{\mathrm{u}} := (I - \mathbb{F}_{\mathrm{u}\,11})^{-1}$, $\widetilde{\mathbb{M}}_{\mathrm{y}} := (I - \mathbb{G}_{\mathrm{y}\,22})^{-1}$.

Note also that joint stabilization is always doubly coprime, by Theorem 6.6.28, and that the interaction operator is necessarily of the above form (for pairs of the above form).

We strongly recommend for most readers to assume that the above pairs are exponentially jointly stabilizing and detecting, so that some of the proofs and results below become essentially simpler (and much of them can be ignored). This covers the case of an exponentially stabilizing controller, and the (rest of the) general case is not that important. Indeed, this assumption is necessary when one is interested in finding a suboptimal exponentially stabilizing controller (recall Definition 12.1.2):

Lemma 12.5.2 (Nonexp. 4BP & opt. ⇒ exp. 4BP) *Assume Hypothesis 12.5.1. The following are equivalent:*

(i) Σ is optimizable;

(ii) Σ is estimatable;

(iii) Σ is optimizable and estimatable

(iv) $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is exponentially stabilizing;

(v) $\left[\frac{\mathbb{H}_{\mathrm{y}}}{\mathbb{G}_{\mathrm{y}}}\right]$ is exponentially stabilizing;

(vi) $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ and $\left[\frac{\mathbb{H}_{\mathrm{y}}}{\mathbb{G}_{\mathrm{y}}}\right]$ are exponentially jointly stabilizing;

(vii) $\Sigma_{21} := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_2 & \mathbb{D}_{21} \end{array}\right]$ is exponentially jointly stabilizable and detectable;

(viii) Σ is exponentially DPF-stabilizable with internal loop;

(ix) Σ_{21} is exponentially DF-stabilizable with internal loop.

If (i) holds, then the systems $\Sigma_{\circlearrowleft}$, $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and $\Sigma_{\mathbb{T}}$ of Lemmas 12.5.15 and 12.5.16 and Proposition 12.5.19 are then exponentially stable (under the assumptions of those results).

If $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is strongly stabilizing for Σ, then the systems $\Sigma_{\circlearrowleft}$, $\Sigma_{\mathbb{E}^{\mathrm{d}}}{}^{\mathrm{d}}$ and $\Sigma_{\mathbb{T}}$ are strongly stable (under corresponding assumptions, as above).

Thus, when we can show that Hypothesis 12.5.1 is satisfied and Σ is optimizable, then, for solving the exponential H^∞ 4BP, it suffices to solve the I/O H^∞ 4BP of Section 12.3, i.e., the corresponding frequency-domain (or I/O map) problem, since then we obtain a solution of the exponential problem from Proposition 12.5.19.

An analogous claim obviously holds for the more general "nonexponential H^∞ 4BP" (see the discussion below Definition 12.1.2) instead of the exponential H^∞ 4BP, and also for the "strong H^∞ 4BP" where we require strong instead of exponential stability.

Proof of Lemma 12.5.2: 1° *The equivalence of (i)–(vi):* If Σ is estimatable, then $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ is exponentially stabilizing, by Theorem 6.7.15(c2), hence Σ is then optimizable. If Σ is optimizable, then $\left[\frac{\mathbb{H}_{\mathrm{y}}}{\mathbb{G}_{\mathrm{y}}}\right]$ is exponentially stabilizing, by the above and duality.

Thus, if (i) or (ii) holds, then so do (i)–(vi) (note that $\mathbb{A}_\flat = \mathbb{A}_L$ and $\mathbb{A}_\sharp = \mathbb{A}_{\widetilde{L}}$ in terms of (6.133), (6.170), (6.168) and (6.171), so that also $\mathbb{A}_L$ and $\mathbb{A}_{\widetilde{L}}$ are then exponentially stable). Moreover, then $\left[\begin{array}{c|c} \mathbb{K}_{u1} & \mathbb{F}_{u11} \\ 0 & 0 \end{array}\right]$ and $\left[\begin{array}{cc} 0 & \mathbb{H}_{\mathrm{y}2} \\ \hline 0 & \mathbb{G}_{\mathrm{y}22} \end{array}\right]$ are obviously jointly admissible, hence exponentially jointly stabilizing for $\Sigma_{21} = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \\ \hline \mathbb{C}_2 & \mathbb{D}_{21} \end{array}\right]$ (since $(\Sigma_{21})_L$ and $(\Sigma_{21})_{\widetilde{L}}$ have the same semigroups as Σ_L and $\Sigma_{\widetilde{L}}$), so that also (vii) holds.

Conversely, any of (iii)–(vii) obviously implies (i) or (ii) (hence (i)–(vii)).

Finally, (vi) implies (viii) and (ix), by Theorem 7.3.12(b1), "(viii)⇒(i)" follows from Theorem 7.3.12(a), and "(ix)⇒(i)" follows from Theorem 7.2.4(a) and Lemma 6.7.4.,

2° *$\Sigma_{\circlearrowleft}$, $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and $\Sigma_{\mathbb{T}}$ are exponentially stable:* By Lemma 12.5.15, $\Sigma_{\circlearrowleft}$ is exponentially stable, hence so is $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ (since $\mathbb{A}^{\mathrm{d}}_{\circlearrowleft}$ is). For $\Sigma_{\mathbb{T}}$ this is contained in Proposition 12.5.19.

3° *Strongly stable case:* See 2°. □

The theory of Section 12.3 can be applied under Hypothesis 12.5.1:

Lemma 12.5.3 (Hyp. 12.5.1 ⇒ Hyp. 12.3.1) *Hypothesis 12.5.1 implies Hypothesis 12.3.1 (with same $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{N}_{\mathrm{y}}}, \widetilde{\mathbb{M}_{\mathrm{y}}}$). Indeed, define Σ_{Total} by (12.87). Then*

$$\mathbb{X}_{\mathrm{u}} := I - \mathbb{G}_{\mathrm{y}} + \mathbb{D}\mathbb{Y}_{\mathrm{u}} = \begin{bmatrix} I & * \\ 0 & * \end{bmatrix}, \qquad \mathbb{Y}_{\mathrm{u}} := -\mathbb{M}_{\mathrm{u}}\mathbb{E}_{\mathrm{uy}} = \begin{bmatrix} 0 & -\mathbb{M}_{\mathrm{u}\,11}\mathbb{E}^{\mathrm{uy}}_{12} \\ 0 & 0 \end{bmatrix}, \tag{12.79}$$

$$\widetilde{\mathbb{X}_{\mathrm{y}}} := I - \mathbb{F}_{\mathrm{u}} + \widetilde{\mathbb{Y}_{\mathrm{y}}}\mathbb{D} = \begin{bmatrix} * & * \\ 0 & I \end{bmatrix} \quad \textit{and} \quad \widetilde{\mathbb{Y}_{\mathrm{y}}} := -\mathbb{E}_{\mathrm{uy}}\widetilde{\mathbb{M}_{\mathrm{y}}} = \begin{bmatrix} 0 & -\mathbb{E}^{\mathrm{uy}}_{12}\widetilde{\mathbb{M}_{\mathrm{y}}}_{22} \\ 0 & 0 \end{bmatrix} \tag{12.80}$$

complement $\mathbb{M}_{\mathrm{u}}, \mathbb{N}_{\mathrm{u}}, \widetilde{\mathbb{M}_{\mathrm{y}}}, \widetilde{\mathbb{N}_{\mathrm{y}}}$ to a d.c.f.; this d.c.f. satisfies Hypothesis 12.3.1.

We shall use this below without further mention.

Conversely, if $\mathbb{D} \in \mathrm{TIC}_\infty$ satisfies Hypothesis 12.3.1, then $\mathbb{D}$ has a realization satisfying Hypothesis 12.5.1, by Lemma 12.5.23.

Proof: Assume Hypothesis 12.5.1. Then $\mathbb{X}_{\mathrm{u}}, \mathbb{Y}_{\mathrm{u}}, \widetilde{\mathbb{X}_{\mathrm{y}}}, \widetilde{\mathbb{Y}_{\mathrm{y}}}$ in (12.79)–(12.80) are stable (being parts of (6.170) and (6.171)) and complement $\mathbb{M}_{\mathrm{u}}, \mathbb{N}_{\mathrm{u}}, \widetilde{\mathbb{M}_{\mathrm{y}}}, \widetilde{\mathbb{N}_{\mathrm{y}}}$ to a d.c.f.; this d.c.f. satisfies Hypothesis 12.3.1. Indeed, the $(1,1)$-block of the equation

$$\begin{bmatrix} \widetilde{\mathbb{X}} & -\widetilde{\mathbb{Y}} \\ -\widetilde{\mathbb{N}} & \widetilde{\mathbb{M}} \end{bmatrix} \begin{bmatrix} \mathbb{M} & \mathbb{Y} \\ \mathbb{N} & \mathbb{X} \end{bmatrix} = I \tag{12.81}$$

(from (6.172)) is $\widetilde{\mathbb{M}_{\mathrm{y}}}_{11}\mathbb{M}_{\mathrm{u}\,11} + (\mathbb{E}_\sharp)_{12}\mathbb{N}_{\mathrm{u}\,21} = I$, hence $\mathbb{N}_{\mathrm{u}\,21}$ and $\mathbb{M}_{\mathrm{u}\,11}$ are r.c. Analogously, $\widetilde{\mathbb{N}_{\mathrm{y}}}_{21}$ and $\widetilde{\mathbb{M}_{\mathrm{y}}}_{22}$ are l.c. □

Next we note that Hypothesis 12.5.1 is equivalent to the standard H^∞ 4BP assumptions (under the regularity assumption (A1)):

Lemma 12.5.4 ((A,B_1) & (A,C_2) ⇒ Hypothesis 12.5.1) *Assume (A1) of Theorem 12.1.5. Then Hypothesis 12.5.1 is satisfied with "exponentially jointly" in place of "jointly" iff (A,B_1) is optimizable, (A,C_2) is estimatable and (A2) of Theorem 12.1.5 holds.*

If Hypothesis 12.5.1 is satisfied, then necessarily $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{N}_{\mathrm{y}}}, \widetilde{\mathbb{M}_{\mathrm{y}}} \in \mathrm{MTIC}^{\mathrm{L}^1}_{\mathrm{exp}}$ (or in (γ') under the alternative "(IV)" in (A1)), and we can choose them so that $\widehat{\mathbb{M}_{\mathrm{u}}}(+\infty) = I$, $\widehat{\widetilde{\mathbb{M}_{\mathrm{y}}}}(+\infty) = I$.

Proof: 1° (A,B_1) & (A,C_2) $\Leftarrow$ *Hypothesis 12.5.1* This is contained in Lemma 12.5.2.

2° (A,B_1) & (A,C_2) $\Rightarrow$ *Hypothesis 12.5.1 except possibly (12.78):* By Corollary 9.2.13(b), there are $K \in \mathcal{B}(H,U)$ and $\mathtt{H} \in \mathcal{B}(Y,H)$ s.t. $A+B_1K$ and $A+\mathtt{H}C_2$ are exponentially stable. Extend Σ by K and $\mathtt{H}$ (with $F=0=G=E$) to satisfy Hypothesis 12.5.1 with "exponentially jointly" in place of "jointly" except possibly (12.78).

By (the proof of) Corollary 9.2.13(c), $\mathbb{N}_\mathrm{u}, \mathbb{M}_\mathrm{u}, \widetilde{\mathbb{N}_\mathrm{y}}, \widetilde{\mathbb{M}_\mathrm{y}} \in \mathrm{MTIC}^{\mathrm{L}^1}_{\exp}$ (and they are d.c. over $\mathrm{MTIC}^{\mathrm{L}^1}_{\exp}$) and $\widehat{\mathbb{M}_\mathrm{u}}(+\infty) = I$, $\widehat{\widetilde{\mathbb{M}_\mathrm{y}}}(+\infty) = I$. The case of the alternative "(IV)" in (A1) follows similarly (use (c3) and (b) instead of (c1) of Lemma 6.8.4).

3° *Conditions (12.78) $\Leftrightarrow$ (A2):* The operator K is exponentially stabilizing for $\Sigma_{11} := \left(\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11}\end{array}\right)$, with closed loop I/O map $\begin{bmatrix}\mathbb{N}_{\mathrm{u}\,11}\\ \mathbb{M}_{\mathrm{u}\,11}\end{bmatrix}$ (since $\mathbb{M}_\mathrm{u} := (I - \mathbb{F}_\mathrm{u})^{-1} =$
$\begin{bmatrix}(I-\mathbb{F}_{\mathrm{u}\,11})^{-1} & -(I-\mathbb{F}_{\mathrm{u}\,11})^{-1}\mathbb{F}_{\mathrm{u}\,12}\\ 0 & I\end{bmatrix}$).

By Lemma 8.4.11(a2), $\mathbb{N}_{\mathrm{u}\,11}^*\mathbb{N}_{\mathrm{u}\,11} \gg 0$ iff $\mathbb{N}_{\mathrm{u}\,11}$ is I-coercive; by Theorem 8.4.5(d), this is the case iff $\mathbb{D}_{11}$ is I-coercive over $\mathcal{U}_{\exp}$ (w.r.t. system Σ_{11}); by (e2)&(i)&(ii') of Proposition 10.3.2, this is the case iff $D_{11}^*D_{11} \gg 0$ and (12.11) holds; these are contained in the assumptions.

By dual arguments, we obtain that $\widetilde{\mathbb{N}_\mathrm{y}}_{22}\widetilde{\mathbb{N}_\mathrm{y}}_{22}^* \gg 0$ iff $D_{22}D_{22}^* \gg 0$ and (12.12) holds.

4° The final claims were observed in 2°. □

Lemma 12.5.5 ($\mathcal{P}_\mathrm{X}$ & $\mathcal{P}_\mathrm{Y}$ $\Rightarrow$ Hypothesis 12.5.1) *If conditions (A1), (A2), (1.) and (2.) of Theorem 12.1.5 are satisfied, then (A,B_1) is optimizable and (A,C_2) is estimatable.*

See Lemma 12.5.4 for more.

Proof: By Theorem 11.1.4(iii)&(i) (or Theorem 11.1.3(iii)&(i) in case (IV) of (A1)), the FICP for Σ_X has a solution; in particular, (A,B_1) is exponentially stabilizable. By duality, (A,C_2) is exponentially detectable. □

When solving $\mathcal{P}_\mathrm{X}$-part of the 4BP, we get the $\mathcal{P}_\mathrm{Y}$-part in the bargain due to duality:

Lemma 12.5.6 (Duality) *Hypotheses 12.1.1, 12.5.1, 12.3.1 and 12.0.1, and Definitions 12.1.2 and 12.3.3 are invariant under ("interchanged") duality.*

In particular, a map $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ is a suboptimal [exponentially] stabilizing DPF-controller for Σ (resp. for $\mathbb{D}$) iff $\mathbb{Q}^\mathrm{d} \in \mathrm{TIC}_\infty(U,Y)$ is a suboptimal [exponentially] stabilizing DPF-controller for Σ_d (resp. for $\mathbb{D}_\mathrm{d}$). □

(Part of this follows from Proposition 7.3.4(d), the rest is obvious (note also that $\mathbb{O}$ is well-posed iff $\mathbb{O}^\mathrm{d}$ is). Part of the lemma is contained in Lemma 12.3.4.)

Here (and elsewhere), $\mathbb{D}_\mathrm{d} := \left[\begin{smallmatrix}0&I\\I&0\end{smallmatrix}\right]\mathbb{D}^\mathrm{d}\left[\begin{smallmatrix}0&I\\I&0\end{smallmatrix}\right]$, $\mathbb{C}_\mathrm{d} := \mathbb{C}^\mathrm{d}\left[\begin{smallmatrix}0&I\\I&0\end{smallmatrix}\right]$, $\mathbb{B}_\mathrm{d} := \left[\begin{smallmatrix}0&I\\I&0\end{smallmatrix}\right]\mathbb{B}^\mathrm{d}$ and $\mathbb{A}_\mathrm{d} := \mathbb{A}^\mathrm{d}$, $\Sigma_\mathrm{d} := \left[\begin{array}{c|c}\mathbb{A}_\mathrm{d} & \mathbb{B}_\mathrm{d}\\ \hline \mathbb{C}_\mathrm{d} & \mathbb{D}_\mathrm{d}\end{array}\right]$ etc.; i.e., we take causal adjoints of each map or system, and interchange the rows and columns corresponding to U and W and to Z and Y.

If the 4BP has a solution, then the "I/O FICP" and its dual problem have well-posed solutions:

Lemma 12.5.7 ($\mathbf{H}^\infty$ 4BP $\Rightarrow$ $\mathbf{H}^\infty$ FICP) *Assume that there is a suboptimal stabilizing DPF-controller for Σ (or for $\mathbb{D}$). Then there are $\mathbb{U} \in \mathrm{TIC}(W,U)$ and $\widetilde{\mathbb{U}} \in \mathrm{TIC}(Y,Z)$ s.t. $\|\mathbb{D}_{11}\mathbb{U}+\mathbb{D}_{12}\|_{\mathrm{TIC}} < \gamma$ and $\|\mathbb{D}_{12}+\widetilde{\mathbb{U}}\mathbb{D}_{22}\|_{\mathrm{TIC}} < \gamma$.*

Assume that there is a suboptimal exponentially stabilizing DPF-controller for Σ. Then (A,B_1) is optimizable, hence $\mathcal{U}_{\exp}^{\Sigma_X}(x_0) \neq \emptyset$ for each $x_0 \in H$; $\gamma > \gamma_0$ for Σ_X (see Definition 11.1.2); and $\mathbb{U}$ can be chosen so that, in addition, $\left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right][\mathrm{L}^2(\mathbf{R}_+;W)] \subset \mathcal{U}_{\exp}^{\Sigma_X}(0)$. Analogous claims hold for Σ_Y.

Proof: 1° The first claim follows from Proposition 7.3.4(e) (the stability of $\mathbb{R}^{-1}$ and $\mathbb{H}^{-1}$ (hence of $\mathbb{U} := (\mathbb{R}^{-1})_{12}$ and $\widetilde{\mathbb{U}} := (\mathbb{H}^{-1})_{12}$) follows from Proposition 7.3.4(b)).

2° Let $\widetilde{\Sigma}$ be an exponentially stabilizing DPF-controller for Σ. By Proposition 7.3.4(e), $w \mapsto u$ is given by $\mathbb{U} := (\mathbb{R}^{-1})_{12} \in \mathrm{TIC}(W,U)$, and $\mathcal{F}_\ell(\mathbb{D},\mathbb{O}) = \mathbb{D}_{11}\mathbb{U}+\mathbb{D}_{12}$.

However, for each $w \in \mathrm{L}^2(\mathbf{R}_+;W)$ (and $x_0 = 0$, $u_L = 0$, $y_L = 0$), all signals in the combined closed-loop system (see Figure 12.1) are in L^2 (since $\widetilde{\Sigma}$ is exponentially stabilizing, so that Σ_I^o is exponentially stable); in particular,

$$\mathrm{L}^2(\mathbf{R}_+;H) \ni x = \mathbb{B}\tau \left[\begin{smallmatrix}u\\ w\end{smallmatrix}\right] = \mathbb{B}\tau \left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right] w \tag{12.82}$$

(because in each form of feedback (cf. Summary 6.7.1), the state, output and input are unique solutions of (6.122)–(6.124), by Proposition 6.6.2).

We conclude that $\left[\begin{smallmatrix}\mathbb{U}\\ I\end{smallmatrix}\right] w \in \mathcal{U}_{\exp}(0) = \mathcal{U}_{\exp}^{\Sigma_X}(0)$. Since w was arbitrary, we have $\gamma > \|\mathbb{D}_{11}\mathbb{U}+\mathbb{D}_{12}\| \geq \gamma_0$ (where $\gamma_0 := \gamma_{0,\Sigma_X}$ is defined with Σ_X in place of Σ). By Theorem 7.3.12(a), (A,B_1) is optimizable, hence so is (A,B); in particular, $\mathcal{U}_{\exp}^{\Sigma_X}(x_0) \neq \emptyset$ for all $x_0 \in H$. □

In finite-dimensional theory, one often looks for a map $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ that "stabilizes Σ", a system. However, it seems that in such theory one always assumes an optimizable and estimatable realization of Σ, and the definition of "$\mathbb{Q}$ stabilizes Σ" refers to this realization explicitly or implicitly. Nevertheless, we shall show below that such a concept has a meaningful definition for general $\Sigma \in$ WPLS, and that for "L^1-smooth systems" (and any discrete-time systems), this definition guarantees the existence of a realization of $\mathbb{Q}$ that stabilizes Σ exponentially (in the ordinary sense):

Indeed, instead of requiring the existence of a DPF-stabilizing system with internal loop, it suffices to require the existence of a DPF-stabilizing map with internal loop (cf. the difference between Figures 7.11 and 12.3); thus, also this formally weaker condition is equivalent to (1.)–(3.):

Remark 12.5.8 ("Suboptimal exponentially stabilizing $\mathbb{Q}$ (not $\widetilde{\Sigma}$) for Σ") *Instead of finding a suboptimal system that stabilizes Σ exponentially, we may require the existence of a suboptimal map (a (well-posed) map $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ or a map $\mathbb{Q}$ with internal loop) that stabilizes Σ exponentially.*

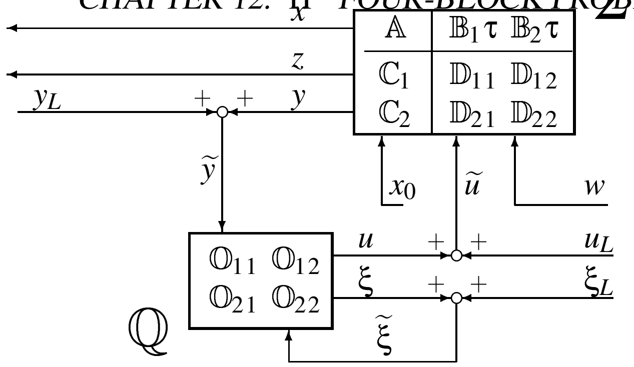

Figure 12.3: "DPF-controller $\mathbb{Q}$ with internal loop" for $\Sigma \in \mathrm{WPLS}(U \times W, H, Z \times Y)$

By this we mean the weakened form of Definition 7.3.1, where the second row and second column of Σ_I^o need not be exponentially stable (for an arbitrary realization of $\widetilde{\Sigma}$ of $\mathbb{Q}$; note that the realization affects only the second row and column of Σ_I^o). It follows that Σ_{21} is optimizable and estimatable.

Assume that (A1) of Theorem 12.1.5 holds.

Then any map $\mathbb{Q}$ that stabilizes Σ exponentially has a realization $\widetilde{\Sigma} \in \mathrm{WPLS}(U \times W \times \Xi, \widetilde{H}, Z \times Y \times \Xi)$ that stabilizes Σ exponentially (if $\mathbb{Q}$ is well-posed, then $\widetilde{\Sigma}$ can be chosen to be well-posed). In particular, the existence of a suboptimal exponentially stabilizing $\mathbb{Q}$ is equivalent to the existence of a suboptimal exponentially stabilizing controller $\widetilde{\Sigma}$ (because the converse is trivial; note that "(1.)–(3.)" is a third equivalent condition if also (A2) holds, and that then the controller can be taken well-posed whenever $D_{21} = 0$, by Theorem 12.1.8, even if $\mathbb{Q}$ were not well-posed).

Note that in discrete time (A1) becomes redundant, and that any of (I)–(IV) of Theorem 12.1.4(A1) implies (A1) of Theorem 12.1.5.

Proof: (We only sketch the proof. Note that analogous definitions could be made with other attributes than "exponentially", as well as for DF-stabilization (instead of DPF-stabilization studied in this chapter), but we have no use for such.)

1° *If $\mathbb{Q} \in \mathrm{TIC}_\infty(Y \times \Xi, U \times \Xi)$ DPF-stabilizes $\Sigma \in \mathrm{WPLS}(U \times W, H, Z \times Y)$ exponentially, then Σ_{21} is optimizable and estimatable:* Let $\widetilde{\Sigma}$ be a realization of $\mathbb{Q}$, let Σ_I^o be the corresponding closed-loop system (as in Definition 7.3.1), and let $\Sigma_{I,H}^o$ denote Σ_I^o with the second row and the second column removed.

By writing Σ_I^o out, one can verify that $\Sigma_{I,H}^o$ does not contain elements of $\widetilde{\Sigma}$ (other than its I/O map $\mathbb{Q}$). Indeed, for $\mathbb{D}_I^o$ this is trivial, for $\mathbb{C}_I^o$ we observe this from the fact that the left column of (7.74) does not contain such elements (since $(\mathbb{C}_I^o)_{31}$ consists of the elements of $(I - \mathbb{D}^o)^{-1}$ and $\mathbb{C}_2$ only), the formulae for $\mathbb{B}_I^o$ are analogous, and

$$\mathbb{A}_{I,H}^o := (\mathbb{A}_I^o)_{11} = \mathbb{A} + \mathbb{B}_1 \tau (\mathbb{C}_I^o)_{11}, \tag{12.83}$$

by (7.75); thus, also $\mathbb{A}_{I,H}^o$ is independent of $\widetilde{\Sigma}$. Therefore, the concept "$\mathbb{Q}$ stabilizes Σ exponentially" is independent of the realization $\widetilde{\Sigma}$ of $\mathbb{Q}$.

Assume that $\mathbb{Q}$ stabilizes Σ exponentially, i.e., that the four parts of $\Sigma_{I,H}^o$

satisfy 1.–4. of Definition 6.1.1 for some $\omega < 0$ (note that "$\mathbb{A}^o_{I,H}$" is not a semigroup in general). Then $(\mathbb{C}^o_I)_{11}$ is (exponentially) stable and $\mathbb{A}^o_{I,H}x_0 \in \mathrm{L}^2(\mathbf{R}_+;H)$ for all $x_0 \in H$. By (12.83), it follows that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}_1 \end{array}\right]$ is optimizable. By dual arguments, $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{C}_2 \end{array}\right]$ is estimatable.

2° *An optimizable and estimatable, hence exponentially stabilizing realization $\widetilde{\Sigma}$ of $\mathbb{Q}$ exists (under (A1)):* Assume that (A1) holds and that $\mathbb{Q}$ stabilizes Σ exponentially.

By 1°, Σ_{21} is optimizable and estimatable, hence the above claim holds, by Lemma 7.3.6(b1) (see also Theorem 7.3.11(c1)).

(This 2° applies both to (well-posed) $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ (giving (well-posed) $\widetilde{\Sigma} \in \mathrm{WPLS}(U,\widetilde{H},Y)$) and to $\mathbb{Q} \in \mathrm{TIC}_\infty(Y\times\Xi, U\times\Xi)$ (giving $\widetilde{\Sigma} \in \mathrm{WPLS}(U\times\Xi,\widetilde{H},Y\times\Xi)$).

3° Assume (A1). If $\widetilde{\Sigma}$ stabilizes Σ exponentially, then $\mathbb{Q}$ stabilizes Σ exponentially (if Σ^o_I of Definition 7.3.1 is exponentially stable, then so is the corresponding sub"system" "$\Sigma^o_{I,H}$"); the converse was given in 2°. Thus, we have the first equivalence.

Under (A2), we obtain the second equivalence from Theorem 12.1.5. If $D_{21} = 0$, then all suboptimal exponentially stabilizing controllers are equivalent to well-posed ones, by Theorem 12.1.8. □

We often wish to assume that $D_{21} = 0$. Usually, this is not a problem:

Lemma 12.5.9 (D_{21} is irrelevant) *The validity of Hypothesis 12.5.1 remains unchanged if one alters D_{21} (ceteris paribus). The same holds for Hypotheses 12.5.13 and 12.3.1, and for the assumptions of Theorem 12.1.11.*

Conditions (1.) and (4.) of Theorem 12.1.8 are independent of D_{21}; the same holds for (1.') and (4.') of Proposition 12.1.10. If Hypothesis 12.3.1 is satisfied, then (4BP1) is independent of D_{21}.

However, D_{21} does affect the well-posedness and the exact form of the suboptimal controllers; cf., e.g., Proposition 12.5.19(g).

Proof: 1° *Hypothesis 12.5.1:* Obviously, the same pairs will do (but $\mathbb{N}_\mathrm{u}$ and $\widetilde{\mathbb{N}}_\mathrm{y}$ are affected).

2° *Hypothesis 12.5.13:* Observe that Σ_X is independent of D_{21}.

3° *Hypothesis 12.3.1:* This follows from Lemma 7.3.23 and Proposition 7.3.14(i)&(iii) (or from equation (7.103).

4° *Theorem 12.1.11:* When $\mathbb{D}$ is replaced by $\mathbb{D}+R$, where $R = \left[\begin{smallmatrix} 0 & 0 \\ R_{21} & 0 \end{smallmatrix}\right] \in \mathcal{B}(U\times W, Z\times Y)$, the map $\mathbb{M}_\mathrm{u}$ of Hypothesis 12.5.1 is unaffected but $\mathbb{N}_\mathrm{u}$ is replaced by $\mathbb{N}_\mathrm{u} + R\mathbb{M}_\mathrm{u} \in \widetilde{\mathcal{A}}$. An analogous remark applies to $\widetilde{\mathbb{M}}_\mathrm{y}$ and $\widetilde{\mathbb{N}}_\mathrm{y}$.

4° *(1.), (4.), (1.') and (4.'):* This is obvious since corresponding operators are independent on D_{21} except for the r.c. condition in (1.'), which can be handled with Lemma 6.5.7(b).

5° *(4BP1):* Assume Hypothesis 12.3.1 (it is independent of D_{21}, by the last claim of Lemma 7.3.23). Let $\mathbb{D}'$ be equal to $\mathbb{D}$ with 0 in place of D_{21}.

We observe from Lemma 7.3.23 that there is a suboptimal stabilizing DPF-controller for $\mathbb{D}$ (necessarily with d.c. internal loop, by the hypothesis and

Theorem 7.3.19(i)&(i')) iff there is a suboptimal stabilizing DPF-controller for $\mathbb{D}'$. □

The factorizations of Theorem 12.3.7 correspond to system Σ_X and to system Σ_Z of (12.94), hence, by Theorem 9.9.10, to their Riccati equations, that is, to "(1.)" and "(4.)" of Theorem 12.1.8 (the system Σ_Y provides the dual (H^∞ filter problem) Riccati equation "(2.)" of Theorems 12.1.4 and 12.1.5):

Definition 12.5.10 (ARE systems) *We define the following systems and map:*

$$\Sigma_X := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C}_X & \mathbb{D}_X \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A} & \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & \mathbb{D}_{11} & \mathbb{D}_{12} \\ 0 & 0 & I \end{array}\right], \tag{12.84}$$

$$\Sigma_Y := \left[\begin{array}{c|c} \mathbb{A}_Y & \mathbb{B}_Y \\ \hline \mathbb{C}_Y & \mathbb{D}_Y \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A}^d & \mathbb{C}_2^d & \mathbb{C}_1^d \\ \hline \mathbb{B}_2^d & \mathbb{D}_{22}^d & \mathbb{D}_{12}^d \\ 0 & 0 & I \end{array}\right], \tag{12.85}$$

$$\mathbb{D}_d := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]\mathbb{D}^d\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \begin{bmatrix} \mathbb{D}_{22}^d & \mathbb{D}_{12}^d \\ \mathbb{D}_{21}^d & \mathbb{D}_{11}^d \end{bmatrix}. \tag{12.86}$$

See Lemma 12.5.16 for Σ_Z and $\Sigma_{\mathbb{E}^d}$.

Here we write out a few formulae:

Lemma 12.5.11 ($\Sigma_{\mathbf{Total}}$) *Assume Hypothesis 12.5.1. Then the pairs and $\mathbb{E}_{uy}$ of the hypothesis extend Σ to $\Sigma_{\mathrm{Total}} \in \mathrm{WPLS}(Z\times Y\times U\times W, H, Z\times Y\times U\times W)$, where*

$$\Sigma_{\mathrm{Total}} := \left[\begin{array}{c|cccc} \mathbb{A} & 0 & \mathbb{H}_{y\,2} & \mathbb{B}_1 & \mathbb{B}_2 \\ \hline \mathbb{C}_1 & 0 & \mathbb{G}_{y\,12} & \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{C}_2 & 0 & \mathbb{G}_{y\,22} & \mathbb{D}_{21} & \mathbb{D}_{22} \\ \mathbb{K}_{u\,1} & 0 & \mathbb{E}^{uy}_{12} & \mathbb{F}_{u\,11} & \mathbb{F}_{u\,12} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{12.87}$$

Moreover, $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ *stabilizes Σ_X to*

$$\Sigma_{X\flat} := \left[\begin{array}{c|cc} \mathbb{A}+\mathbb{B}_1\tau\mathbb{M}_{u\,11}\mathbb{K}_{u\,1} & \mathbb{B}_1\mathbb{M}_{u\,11} & \mathbb{B}_1\mathbb{M}_{u\,12}+\mathbb{B}_2 \\ \hline \mathbb{C}_1+\mathbb{N}_{u\,11}\mathbb{K}_{u\,1} & \mathbb{N}_{u\,11} & \mathbb{N}_{u\,11} \\ 0 & 0 & I \\ \mathbb{M}_{u\,11}\mathbb{K}_{u\,1} & \mathbb{M}_{u\,11}-I & \mathbb{M}_{u\,12} \\ 0 & 0 & 0 \end{array}\right] \tag{12.88}$$

here $\mathbb{M}_u := (I-\mathbb{F}_u)^{-1} = \left[\begin{smallmatrix} \mathbb{M}_{u\,11} & \mathbb{M}_{u\,12} \\ 0 & I \end{smallmatrix}\right]$.

Analogously, the pair $\left[\begin{array}{c|cc} \mathbb{H}_{y\,2}^{\ d} & \mathbb{G}_{y\,22}^{\ d} & \mathbb{G}_{y\,12}^{\ d} \\ 0 & 0 & 0 \end{array}\right]$ *stabilizes Σ_Y to*

$$\Sigma_{Y\flat} := \left[\begin{array}{c|cc} \mathbb{A}^d+\mathbb{C}_2^d\tau\widetilde{\mathbb{M}}_{y\,22}^{\ d}\mathbb{H}_{y\,2}^{\ d} & \mathbb{C}_2^d\widetilde{\mathbb{M}}_{y\,22}^{\ d} & \mathbb{C}_2^d\widetilde{\mathbb{M}}_{y\,21}^{\ d}+\mathbb{C}_1^d \\ \hline \mathbb{B}_2^d+\widetilde{\mathbb{N}}_{y\,22}^{\ d}\mathbb{H}_{y\,2}^{\ d} & \widetilde{\mathbb{N}}_{y\,22}^{\ d} & \widetilde{\mathbb{N}}_{y\,12}^{\ d} \\ 0 & 0 & I \\ \widetilde{\mathbb{M}}_{y\,22}^{\ d}\mathbb{H}_{y\,2}^{\ d} & \widetilde{\mathbb{M}}_{y\,22}^{\ d}-I & \widetilde{\mathbb{M}}_{y\,12}^{\ d} \\ 0 & 0 & 0 \end{array}\right] \tag{12.89}$$

(all elements above are from (6.171)); here $\widetilde{\mathbb{M}}_{\mathrm{yd}} := (I - \mathbb{G}_{\mathrm{y}}{}^{\mathrm{d}})^{-1} = \begin{bmatrix} (\widetilde{\mathbb{M}}_{\mathrm{yd}})_{11} & (\widetilde{\mathbb{M}}_{\mathrm{yd}})_{12} \\ 0 & I \end{bmatrix} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] \widetilde{\mathbb{M}_{\mathrm{y}}}^{\mathrm{d}} \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \begin{bmatrix} \widetilde{\mathbb{M}_{\mathrm{y}}}^{\mathrm{d}}_{22} & \widetilde{\mathbb{M}_{\mathrm{y}}}^{\mathrm{d}}_{12} \\ 0 & I \end{bmatrix}$.

Proof: The first claim is from Definition 6.6.21. One easily verifies the formulae. Since all elements of (12.88) are from (6.170) and all elements of (12.89) are from (6.171), it is obvious that these systems are stable. □

Now we are ready for the main part of the proof. We first show that (Factor1) is equivalent to the "$\mathcal{P}_{\mathrm{X}}$-CARE":

Lemma 12.5.12 ((Factor1) ⇔ $\mathcal{P}_{\mathrm{X}}$-CARE) *Assume that Hypothesis 12.5.1 holds and that* $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \mathrm{UR}$.

Then (Factor1) has a UR solution $\mathbb{N}, \mathbb{M}$ *iff the CARE for* Σ_{X} *and* J_γ *has a UR P-stabilizing solution* $(\mathcal{P}_{\mathrm{X}}, S_X, K_I)$ *s.t.* $\mathbb{D}(I-\mathbb{F})^{-1}$ *and* $(I-\mathbb{F})^{-1}$ *are [q.]r.c.,* $\mathcal{P}_{\mathrm{X}} \geq 0$, $S_{X11} \gg 0$ *and* $S_{X22} - S_{X21} S_{X11}^{-1} S_{X12} \ll 0$.

If $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ *is exponentially stabilizing, then (Factor1) has a UR solution* $\mathbb{N}, \mathbb{M}$ *iff the CARE for* Σ_{X} *and* J_γ *has a UR exponentially stabilizing solution* $(\mathcal{P}_{\mathrm{X}}, S_X, K_I)$ *s.t.* $\mathcal{P}_{\mathrm{X}} \geq 0$, $S_{X11} \gg 0$ *and* $S_{X22} - S_{X21} S_{X11}^{-1} S_{X12} \ll 0$.

In either case, the IARE for Σ_{X} *and* J_γ *has a UR P-stabilizing solution* $(\mathcal{P}_{\mathrm{X}}, J_1, \begin{bmatrix} \mathbb{K} & \big| & \mathbb{F} \end{bmatrix})$ *s.t.* $\mathbb{M} := (I-\mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$ *and* $\mathbb{X} := \mathbb{M}^{-1}\mathbb{M}_{\mathrm{u}}$ *satisfy (Factor1) and (Factor1X), and Hypothesis 12.5.13 is satisfied.*

Conversely, if Hypothesis 12.5.13 is satisfied, then the IARE for Σ_{X} *and* J_γ *has a unique UR P-stabilizing solution* $(\mathcal{P}_{\mathrm{X}}, J_1, \begin{bmatrix} \mathbb{K} & \big| & \mathbb{F} \end{bmatrix})$ *s.t.* $(I-\mathbb{F})^{-1} = \mathbb{M}$, $\mathbb{D}\mathbb{M} = \mathbb{N}$.

Proof: Now $\mathbb{D} = \mathbb{N}_{\mathrm{u}} \mathbb{M}_{\mathrm{u}}{}^{-1}$ is UR, by Proposition 6.3.1(b1). By Theorem 12.3.7(e1)&(e2), (Factor1) and (Factor1X) are equivalent, and $\mathbb{X}$ is UR iff $\mathbb{N}$ and $\mathbb{M}$ are UR

1° *(Factor1X) iff CARE for* $\Sigma_{\mathrm{X}\flat}$*:* By Proposition 11.3.4(f), (Factor1X) (or (FI3s)) is equivalent to (FI4s) for $\Sigma_{\mathrm{X}\flat}$. If we add the requirement that $\mathbb{X}$ (or $\mathbb{F}$) is UR, then, by Proposition 11.3.4(a), (FI4s) is equivalent to (FI5s), i.e., to the condition that the CARE for Σ_{X} and J_γ has a UR stable, P-SOS-stabilizing solution $(\mathcal{P}_{\mathrm{X}}, S_X, K_I)$ s.t. $\mathcal{P}_{\mathrm{X}} \geq 0$, $S_{X11} \gg 0$ and $S_{X22} - S_{X21} S_{X11}^{-1} S_{X12} \ll 0$.

2° *(Factor1X) iff CARE for* Σ_{X} *when* $\begin{bmatrix} \mathbb{K}_{\mathrm{u}} & \big| & \mathbb{F}_{\mathrm{u}} \end{bmatrix}$ *is exponentially stabilizing:* By Proposition 9.12.4 and Lemma 9.12.3(b), the exponentially stabilizing solutions of the CAREs for $(\Sigma_{\mathrm{X}\flat}, J_\gamma)$ and $(\Sigma_{\mathrm{X}}, J_\gamma)$ correspond to each other one-to-one (they have same $\mathbb{A}_{\circlearrowleft}$'s, by (b), hence one is exponentially stabilizing iff both are); by Theorem 9.8.12, they are equal. Since $\mathbb{F}_{\mathrm{u}}$ is UR, one is UR iff the other one is, by Proposition 9.12.4.

3° *(Factor1X) iff CARE for* Σ_{X}*:* Work as in 2°, let $\mathbb{F}_X$ and $\mathbb{F}_{X\flat}$ correspond to the solutions of the two CAREs, and set $\mathbb{X}_{X\flat} := I - \mathbb{F}_{X\flat}$ and $\mathbb{M}_X := (I - \mathbb{F}_X)^{-1}$.

By Lemma 9.12.3(b), condition (P) is preserved. The map $\mathbb{X}_{X\flat} := \mathbb{M}_X^{-1} \mathbb{M}_{\mathrm{u}}$ is in $\mathcal{G}$TIC iff $\mathbb{D}\mathbb{M}_X$ and $\mathbb{M}_X$ are r.c., by Lemma 6.4.5(c) (since $\mathbb{N}_{\mathrm{u}} = \mathbb{D}\mathbb{M}_{\mathrm{u}}$ and $\mathbb{M}_{\mathrm{u}}$ are r.c.). By Lemma 6.6.17(b)&(c), the solution for the CARE for $(\Sigma_{\mathrm{X}\flat}, J_\gamma)$ is stable and [SOS-]stabilizing iff $\begin{bmatrix} \mathbb{K}_{X\flat} & \big| & \mathbb{F}_{X\flat} \end{bmatrix}$ is r.c.-stabilizing; by Lemma 6.7.11(a2), this is the case iff $\begin{bmatrix} \mathbb{K}_X & \big| & \mathbb{F}_X \end{bmatrix}$ is q.r.c.-stabilizing for Σ (equivalently, r.c.-stabilizing, by Lemma 6.4.5(c)).

(Note that $(\mathcal{P}_{\mathrm{X}}, S_X, K_I)$ are the same in 2° and 3°.)

4° *X and M:* In 1°, we can choose X so that $X_{21} = 0$ and $X_{11}, X_{22} \in \mathcal{GB}$, by Proposition 11.3.4(e) (or by Lemma 11.3.13(i)&(iii') and Theorem 9.8.12), if we replace the CARE by the IARE and S_X by J_1. This is automatic if we are originally given $\mathbb{N}, \mathbb{M}$ satisfying Hypothesis 12.5.13 (the uniqueness claim follows from Theorem 9.8.12(b)(&(s1))).

The rest follows as above (note from 4° that $\mathbb{M}_X = \mathbb{M}_{\mathrm{u}} \mathbb{X}_{X^\flat}^{-1}$, hence now $\mathbb{M}_X$ becomes equal to $\mathbb{M}$, by (12.49)). □

We shall often assume the following:

Hypothesis 12.5.13 ((Factor1) with $X = \left[\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right]$) *Hypothesis 12.5.1 holds, $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in$ UR, and $\mathbb{N}, \mathbb{M}$ is an UR solution of (Factor1) s.t. $X_{21} = 0$, where $\mathbb{X} := \mathbb{M}^{-1}\mathbb{M}_{\mathrm{u}}$.*

(When this hypothesis holds, we denote by $(\mathcal{P}_{\mathrm{X}}, J_1, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ the solution of the $\mathcal{P}_{\mathrm{X}}$-IARE mentioned at the end of Lemma 12.5.12, and by $\Sigma_{\circlearrowleft}$ the corresponding (stable) closed-loop system.)

(The regularity assumptions might be somewhat (resp. completely) reduced, but this would require somewhat more complicated formulations of the results based on the above hypothesis (resp. and use of IAREs instead of CAREs as in (FI4s) instead of (FI5s) in Theorem 11.3.3).)

By (12.49), "$\mathbb{N}, \mathbb{M} \in$ UR" could be replaced by "$\mathbb{X} \in$ UR".

Lemma 12.5.14 *Make Hypothesis 12.5.13. Then $\mathbb{X}$ satisfies (Factor1X), $M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & X_{22}^{-1} \end{bmatrix}$, and $X_{11}, X_{22}, M_{11} \in \mathcal{GB}$. Moreover, then $\mathbb{D}, \mathbb{X}, \mathbb{E} \in$ UR.*

Proof: By Theorem 12.3.7(e2), $\mathbb{X}$ satisfies (Factor1X), hence $\mathbb{X}, \mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}_\infty$. By Lemma A.1.1(b), $\mathbb{M}_{\mathrm{u}\,11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$. By Proposition 6.3.1(b1), it follows that $X, X_{11}, M_{u11} \in \mathcal{GB}$, hence $X_{22}, (X^{-1})_{11}, (X^{-1})_{22} \in \mathcal{GB}(W)$ and $(X^{-1})_{21} = 0$, by Lemma A.1.1(b). By (12.49), $M = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right] X^{-1}$, hence also M and M_{11} are as above. By Proposition 6.3.1(b1), $\mathbb{D}, \mathbb{X}, \mathbb{E} \in$ UR. □

We need to define explicitly the closed-loop system corresponding to $\mathcal{P}_{\mathrm{X}}$, so that we can define the "$\mathcal{P}_{\mathrm{Z}}$-CARE" further below:

Lemma 12.5.15 ($\Sigma_{\circlearrowleft}$) *Make Hypothesis 12.5.13. Assume that $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right]$ is [[exponentially] strongly] stabilizing for Σ.*

Then there is a unique [[exponentially] strongly] P-stabilizing solution $(\mathcal{P}_{\mathrm{X}}, S_X, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ of the IARE for Σ_{X} and J_1, and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is determined uniquely by requiring that $\mathbb{F} = I - \mathbb{M}^{-1}$. The pair $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is [[exponentially] strongly] stabilizing for Σ too; we denote the corresponding closed-loop system by

$$\Sigma_{\circlearrowleft} := \left[\begin{array}{c|c} \mathbb{A}_{\circlearrowleft} & \mathbb{B}_{\circlearrowleft} \\ \hline \mathbb{C}_{\circlearrowleft} & \mathbb{D}_{\circlearrowleft} \\ \mathbb{K}_{\circlearrowleft} & \mathbb{F}_{\circlearrowleft} \end{array}\right], \tag{12.90}$$

so that $\mathbb{F}_{\circlearrowleft} = \mathbb{M} - I$, $\mathbb{D}_{\circlearrowleft} = \mathbb{N} = \mathbb{D}\mathbb{M}$.

It is important to note that $\Sigma_\circlearrowleft$ is a state-feedback perturbation of Σ, not of Σ_X, although K is determined by the IARE for Σ_X.

Proof: (Recall from Hypothesis 12.5.1 and Theorem 6.6.28 that $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ is r.c.-stabilizing for Σ even without the extra assumption of the lemma.)

By the assumption, $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ is [[exponentially] strongly] stabilizing for Σ, hence also for Σ_X. As in the proof of Lemma 12.5.12, we obtain the solution of the IARE mentioned above; being P-stabilizing, $\mathcal{P}_X$ is unique (see Theorem 9.8.12(b)&(s1); note that S_X and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is not unique in general before we fix $\mathbb{F}$). By Lemma 6.6.17(d)&(c), $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ is [[exponentially] strongly] stabilizing for Σ too. □

The third Riccati equation traditionally assosiated to the 4BP corresponds to the realization Σ_Z of $\begin{bmatrix} \mathbb{D}^d_{+12} & \mathbb{D}^d_{+22} \\ 0 & I \end{bmatrix} \in \mathrm{TIC}(Y \times U, W \times U)$ defined below:

Lemma 12.5.16 (Σ_Z and $\mathcal{P}_Z$-CARE) *Assume that Hypothesis 12.5.13 holds and that $\mathbb{M}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$. Then (see the hypothesis for $\Sigma_\circlearrowleft$)*

$$\left[\begin{array}{c|c} \widetilde{\mathbb{K}} & \widetilde{\mathbb{F}} \end{array}\right] := \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \mathbb{B}^d_{\circlearrowleft,1} & \mathbb{N}^d_{21} & I - \mathbb{M}^d_{11} \end{array}\right] \tag{12.91}$$

is a stable admissible state feedback pair for

$$\Sigma_{\mathbb{E}^d} := \left[\begin{array}{c|c} \mathbb{A}_{\mathbb{E}^d} & \mathbb{B}_{\mathbb{E}^d} \\ \hline \mathbb{C}_{\mathbb{E}^d} & \mathbb{D}_{\mathbb{E}^d} \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A}^d_\circlearrowleft & \mathbb{C}^d_{\circlearrowleft,2} & -\mathbb{K}^d_{\circlearrowleft,1} \\ \hline \mathbb{B}^d_{\circlearrowleft,2} & \mathbb{N}^d_{22} & -\mathbb{M}^d_{12} \\ -\mathbb{B}^d_{\circlearrowleft,1} & -\mathbb{N}^d_{21} & \mathbb{M}^d_{11} \end{array}\right] \tag{12.92}$$

$$:= \left[\begin{array}{c|cc} \mathbb{A}^d + \mathbb{K}^d \mathbb{M}^d \tau \mathbb{B}^d & \mathbb{C}^d_2 + (\mathbb{N}\mathbb{K})^d_2 & -(\mathbb{M}\mathbb{K})^d_1 \\ \hline (\mathbb{B}\mathbb{M})^d_2 & (\mathbb{D}\mathbb{M})^d_{22} & -\mathbb{M}^d_{12} \\ -(\mathbb{B}\mathbb{M})^d_1 & -(\mathbb{D}\mathbb{M})^d_{21} & \mathbb{M}^d_{11} \end{array}\right]. \tag{12.93}$$

The top three rows ("the $\left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \end{array}\right]$ part") of the corresponding closed-loop system are given by

$$\Sigma_Z := \left[\begin{array}{c|c} \mathbb{A}_Z & \mathbb{B}_Z \\ \hline \mathbb{C}_Z & \mathbb{D}_Z \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A}^d + \mathbb{K}^d_2 \tau \mathbb{X}^{-d}_{22} \mathbb{B}^d_2 & \mathbb{C}^d_2 + \mathbb{K}^d_2 \mathbb{X}^{-d}_{22} \mathbb{D}^d_{22} & -\mathbb{K}^d_1 + \mathbb{K}^d_2 \mathbb{X}^{-d}_{22} \mathbb{X}^d_{12} \\ \hline \mathbb{X}^{-d}_{22} \mathbb{B}^d_2 & \mathbb{X}^{-d}_{22} \mathbb{D}^d_{22} & \mathbb{X}^{-d}_{22} \mathbb{X}^d_{12} \\ 0 & 0 & I \end{array}\right] \tag{12.94}$$

$\in \mathrm{WPLS}(Y \times U, H, W \times U)$ (see (12.17) for corresponding generators). In particular,

$$\mathbb{D}_Z = \begin{bmatrix} \mathbb{X}^{-d}_{22} \mathbb{D}^d_{22} & \mathbb{X}^{-d}_{22} \mathbb{X}^d_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} \mathbb{D}^d_{+12} & \mathbb{D}^d_{+22} \\ 0 & I \end{bmatrix} \in \mathrm{TIC}(Y \times U, W \times U). \tag{12.95}$$

The system $\Sigma_{\mathbb{E}^d}$ is stable; it is exponentially stable iff Σ is optimizable (by Lemma 12.5.2). □

(This is obvious. Note from Lemma 12.5.14 and Proposition 6.3.1(c) that the assumption on $\mathbb{M}_{11}$ is redundant if $\mathbb{M} \in \mathrm{ULR}$.)

The CARE (resp. IARE) for Σ_Z and J_1 is called the *$\mathcal{P}_Z$-CARE* (resp. *$\mathcal{P}_Z$-IARE*).

Lemma 12.5.17 ($\mathcal{P}_Z$-IARE $\Leftrightarrow$ (Factor2)) *Assume that Hypothesis 12.5.13 holds and $\mathbb{M}_{11}\in\mathcal{G}\mathrm{TIC}_\infty(U)$. Consider the following conditions:*

(i) (Factor2Z) has a UR solution $\mathbb{Z}\in\mathcal{G}\mathrm{TIC}(Y\times U)$ s.t. $Z_{12}=0$ and $Z_{11},Z_{22}\in\mathcal{GB}$.

(ii) the CARE for Σ_Z and J_1 has a UR P-stabilizing solution $(\mathcal{P}_Z,S_Z,K_{Z,I})$ s.t. $\mathcal{P}_Z\geq 0$, $S_{Z11}\gg 0$, $S_{Z22}-S_{Z21}S_{Z11}^{-1}S_{Z12}\ll 0$, $(I-\mathbb{F}_{Z,I})(I-\widetilde{\mathbb{F}})\in\mathcal{G}\mathrm{TIC}(Y\times U)$ and $\mathbb{K}_{Z,I}+(I-\mathbb{F}_{Z,I})\widetilde{\mathbb{K}}$ is stable.

(iii) the CARE for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and J_1 has a UR stable, P-stabilizing solution $(\mathcal{P}_Z,S_E,K_E)$ s.t. $\mathcal{P}_Z\geq 0$, $S_{E11}\gg 0$ and $S_{E22}-S_{E21}S_{E11}^{-1}S_{E12}\ll 0$.

(ii') the CARE for Σ_Z and J_1 has a UR exponentially stabilizing solution $(\mathcal{P}_Z,S_Z,K_{Z,I})$ s.t. $\mathcal{P}_Z\geq 0$, $S_{Z11}\gg 0$ and $S_{Z22}-S_{Z21}S_{Z11}^{-1}S_{Z12}\ll 0$.

(iii') the CARE for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and J_1 has a UR exponentially stabilizing solution $(\mathcal{P}_Z,S_E,K_E)$ s.t. $\mathcal{P}_Z\geq 0$, $S_{E11}\gg 0$ and $S_{E22}-S_{E21}S_{E11}^{-1}S_{E12}\ll 0$.

Then (c1)–(d) below hold; if $D_{21}=0$, then also (a) and (b) hold:

(a) We have (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).

(b) If $\begin{bmatrix}\mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}}\end{bmatrix}$ is exponentially stabilizing, then (i)–(iii') are equivalent.

(c1) If (Factor2Z) has a solution, then (ii)–(iii) (and (ii') and (iii') if $\begin{bmatrix}\mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}}\end{bmatrix}$ is exponentially stabilizing) hold if we drop "UR" and the conditions on S_E and S_Z, and we replace "CARE" by "IARE".

(c2) If (Factor2Z) has a UR solution, then (ii) holds (and (ii') if $\begin{bmatrix}\mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}}\end{bmatrix}$ is exponentially stabilizing).

(d) If (ii) has a solution (or (ii') has and $\begin{bmatrix}\mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}}\end{bmatrix}$ is exponentially stabilizing), then $\mathbb{E}J_1\mathbb{E}^*=\mathbb{Z}J_1\mathbb{Z}^*$ for some $\mathbb{Z}\in\mathcal{G}\mathrm{TIC}(Y\times U)$ s.t. $\mathbb{E}^{\mathrm{d}}\mathbb{Z}^{-\mathrm{d}}$ is (J_1,J_1)-lossless and $Z_{11}\in\mathcal{GB}(Y)$.

If, in addition, $\dim U<\infty$, then $\mathbb{Z}$ satisfies (Factor2Z) (and (Factor2$\widetilde{\mathcal{A}}$) if $\mathbb{N}_{\mathrm{u}},\mathbb{M}_{\mathrm{u}}\in\widetilde{\mathcal{A}}$).

Recall from Theorem 12.3.7(a) that if $(\mathbb{E}^{\mathrm{d}},J_1)\in\mathrm{SpF}$, then one more equivalent condition is that the 4BP has a solution. We remind that most readers should ignore the nonexponential 4BP (and hence (ii) and (iii) and most of the proof below); it combines a less important problem with more complicated proofs and solutions.

Note that we could drop the regularity assumption in (i) and formulate (ii[']) and (iii[']) analogously to (FI4) (see Theorem 11.3.3).

In the proof of Theorem 12.1.8 and Proposition 12.1.10, we strengthen the equivalence of (Factor2) and (ii[']) (by establishing a converse to (c2) without assuming that $D_{21}=0$).

The condition in (ii) may seem complicated (it corresponds to $\mathbb{Z}^{\mathrm{d}}$ being a spectral factor of $(\mathbb{E}^{\mathrm{d}})^*J\mathbb{E}^{\mathrm{d}}$) compared to the simple r.c. condition of Lemma 12.5.12 (corresponding to $\mathbb{X}$ solving (Factor1)). The explanation is that the word "stable" in (iii) is, indeed, a r.c. condition (by Corollary 9.9.11), but since (12.91) need not be (q.)r.c.-stabilizing in general, the corresponding condition in (ii) is not a r.c. condition w.r.t. Σ_Z. This explains the corresponding difference in (1.') and (4.') of Proposition 12.1.10.

Proof of Lemma 12.5.17: (Much of this follows the lines of the proof of Lemma 12.5.12, we just need certain more complicated details. For each claim, we first give the proof where $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ is assumed to be exponentially stabilizing; most readers will skip the more complicated and less important part (with a mere stabilization assumption).)

Remarks and notation: Since $\mathbb{N}, \mathbb{M} \in \mathrm{UR}$, we have $\mathbb{D}, \mathbb{M}^{-1}, \mathbb{D}_+, \mathbb{D}_Z \in \mathrm{UR}$, As shown in the proof, the operators in (ii)–(iii') having same symbols are equal. The solutions of (ii)–(iii') (if any) are unique, by Theorem 9.8.12(b)&(s1).

We use the standard notation where $\mathbb{X} := I - \mathbb{F}$, $\mathbb{M} := \mathbb{X}^{-1}$, and $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ corresponds to K (with any subscripts). We also use the notation of Lemma 12.5.16.

The proof: We give the proofs in the order (c1), (c2), (d), (a), (b).

(c1) This follows from 2° and 4° (and 1° and 3° if $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ is exponentially stabilizing).

1° *(Factor2Z) $\Rightarrow$ weakened (ii') (when $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ is exponentially stabilizing):* Assume (Factor2Z). By Corollary 9.9.11 (and Theorem 9.9.10), the IARE for $\Sigma_{\mathbb{E}^d}$ and J_1 has an exponentially stabilizing solution $(\mathcal{P}_Z, J_1, \left[\begin{array}{c|c} \mathbb{K}_E & \mathbb{F}_E \end{array}\right])$, where $\mathbb{F}_E = I - \mathbb{Z}^d$ (note from Lemma 12.5.16 that $\Sigma_{\mathbb{E}^d}$ is exponentially stable).

From 3° we obtain an exponentially stabilizing solution $(\mathcal{P}_Z, J_1, \left[\begin{array}{c|c} \mathbb{K}_Z & \mathbb{F}_Z \end{array}\right])$ of the $\mathcal{P}_Z$-IARE. It only remains to be shown that $\mathcal{P}_Z \geq 0$.

By (9.224), we have $\mathbb{X}_Z = \mathbb{X}_E \widetilde{\mathbb{M}}$, hence $\mathbb{Z}^{-d} = \mathbb{M}_E = \widetilde{\mathbb{M}} \mathbb{M}_Z$ and

$$\mathbb{D}_{\circlearrowleft,Z} := \mathbb{D}_Z \mathbb{M}_Z = (\mathbb{E}^d \widetilde{\mathbb{M}}) \mathbb{M}_Z = \mathbb{E}^d \mathbb{Z}^{-d} =: \mathbb{W}^d \in \mathrm{TIC}(Y \times U, W \times U), \quad (12.96)$$

where $\mathbb{M}_Z := (I - \mathbb{F}_Z)^{-1}$, $\mathbb{M}_E = (I - \mathbb{F}_E)^{-1}$. By (12.96) and Lemma 12.3.11(a), $(\mathbb{D}_{\circlearrowleft,Z})_{22} = \mathbb{W}^d_{22} \in \mathcal{G}\mathrm{TIC}(U)$. But $(\mathbb{M}_Z)_{22} = (\mathbb{D}_{\circlearrowleft,Z})_{22}$ (since $\mathbb{D}_Z = \left[\begin{smallmatrix} * & * \\ 0 & I \end{smallmatrix}\right]$, by (12.95)), and $(J_1)_{22} = -I \ll 0$, hence it follows from Lemma 11.2.18 (see the remark in its proof) that $\mathcal{P}_Z \geq 0$ (and that there is a suboptimal H^∞-FI-pair for Σ_Z and $\gamma = 1$ and that Hypothesis 11.2.1 is satisfied for Σ_Z over $\mathcal{U}_{\mathrm{exp}}$).

2° *(Factor2Z) $\Rightarrow$ weakened (ii):* Assume (Factor2Z). (We start almost as in 1°, but the proof of "$\mathcal{P}_Z \geq 0$" must be written more explicitly.)

By Corollary 9.9.11 (and Theorem 9.9.10), the IARE for $\Sigma_{\mathbb{E}^d}$ and J_1 has a stable, P-stabilizing solution $(\mathcal{P}_Z, J_1, \left[\begin{array}{c|c} \mathbb{K}_E & \mathbb{F}_E \end{array}\right])$, where $\mathbb{F}_E = I - \mathbb{Z}^d$ (note from Lemma 12.5.16 that $\Sigma_{\mathbb{E}^d}$ is stable).

From 4° we obtain a P-stabilizing solution $(\mathcal{P}_Z, J_1, \left[\begin{array}{c|c} \mathbb{K}_Z & \mathbb{F}_Z \end{array}\right])$ of the $\mathcal{P}_Z$-IARE (by $\Sigma_{\circlearrowleft}$ we shall denote the corresponding (stable) closed-loop system) s.t. $(I - \mathbb{F}_{Z,I})(I - \widetilde{\mathbb{F}}) \in \mathcal{G}\mathrm{TIC}(Y \times U)$ and $\mathbb{K}_{Z,I} + (I - \mathbb{F}_{Z,I})\widetilde{\mathbb{K}}$ is stable. It only remains to be shown that $\mathcal{P}_Z \geq 0$.

By (9.224), we have $\mathbb{X}_Z = \mathbb{X}_E\widetilde{\mathbb{M}}$, hence $\mathbb{Z}^{-\mathrm{d}} = \mathbb{M}_E = \widetilde{\mathbb{M}}\mathbb{M}_Z$ and

$$\mathbb{D}_{\circlearrowleft,Z} := \mathbb{D}_Z\mathbb{M}_Z = (\mathbb{E}^{\mathrm{d}}\widetilde{\mathbb{M}})\mathbb{M}_Z = \mathbb{E}^{\mathrm{d}}\mathbb{Z}^{-\mathrm{d}} =: \mathbb{W}^{\mathrm{d}} \in \mathrm{TIC}(Y\times U, W\times U), \tag{12.97}$$

where $\mathbb{M}_Z := (I-\mathbb{F}_Z)^{-1}$, $\mathbb{M}_E = (I-\mathbb{F}_E)^{-1}$. By (12.96) and Lemma 12.3.11(a), $(\mathbb{D}_{\circlearrowleft,Z})_{22} = \mathbb{W}^{\mathrm{d}}_{22} \in \mathcal{G}\mathrm{TIC}(U)$. But $(\mathbb{M}_Z)_{22} = (\mathbb{D}_{\circlearrowleft,Z})_{22}$ (since $\mathbb{D}_Z = \left[\begin{smallmatrix} * & * \\ 0 & I\end{smallmatrix}\right]$, by (12.95)), and $(J_1)_{22} = -I \ll 0$,

In 1°, we obtained $\mathcal{P}_Z \geq 0$ at this point from Lemma 11.2.18. Since this time $\mathcal{P}_Z$ need not be exponentially stabilizing, we shall apply Lemma 11.2.18 for $\Sigma^{\curvearrowleft}$ (see below) instead of Σ_Z in this case.

The pair $\left[\begin{array}{c|c} \overline{\mathbb{K}} & \overline{\mathbb{F}}\end{array}\right] := \left[\begin{array}{c|cc} \mathbb{K}_{Z1} & \mathbb{F}_{Z11} & \mathbb{F}_{Z12} \\ 0 & 0 & 0\end{array}\right]$ is admissible for Σ_Z, because $\mathbb{F}_{Z11} \in \mathcal{G}\mathrm{TIC}_\infty(Y)$ (because $\mathbb{M}_{Z22} = (\mathbb{D}_{\circlearrowleft,Z})_{22} \in \mathcal{G}\mathrm{TIC}(U)$, as shown above); let $\Sigma^{\curvearrowleft}$ be the corresponding closed-loop system. Note that

$$\mathbb{D}^{\curvearrowleft} = \mathbb{D}_Z\overline{\mathbb{M}} = \begin{bmatrix} * & * \\ 0 & I\end{bmatrix}, \text{ where } \overline{\mathbb{M}} := (I-\overline{\mathbb{F}})^{-1} = \begin{bmatrix} * & * \\ 0 & I\end{bmatrix}. \tag{12.98}$$

By Lemma 9.12.3 $(\mathcal{P}, J_1, \left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right])$ is a solution of the IARE for $\Sigma^{\curvearrowleft}{}_0 := \left[\begin{array}{c|c} \mathbb{A}^{\curvearrowleft} & \mathbb{B}^{\curvearrowleft} \\ \hline \mathbb{C}^{\curvearrowleft} & \mathbb{D}^{\curvearrowleft}\end{array}\right]$ and J_1, where

$$\begin{aligned}\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right] &:= \left[\begin{array}{c|c} \mathbb{K} - \mathbb{X}\mathbb{K}^{\curvearrowleft} & I - \mathbb{X}(I - \left[\begin{smallmatrix} \mathbb{F}_{Z11} & \mathbb{F}_{Z12} \\ 0 & 0\end{smallmatrix}\right])^{-1}\end{array}\right] \\ &= \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline \mathbb{M}_{Z22}^{-1}\mathbb{K}_{\circlearrowleft 2} & \mathbb{M}_{Z22}^{-1}\mathbb{M}_{Z21} & I - \mathbb{M}_{Z22}^{-1}\end{array}\right]\end{aligned} \tag{12.99}$$

(the last equality follows by direct computation). But $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$ is stable (because $\Sigma_\circlearrowleft$ is stable and $\mathbb{M}_{Z22} \in \mathcal{G}\mathrm{TIC}(U)$), and so is $(I-\mathbb{F}_\natural)^{-1} = \left[\begin{smallmatrix} I & 0 \\ \mathbb{M}_{Z21} & \mathbb{M}_{Z22}\end{smallmatrix}\right]$, hence $\Sigma^{\curvearrowleft}$ is stable, by Corollary 6.6.9 (see Definition 6.6.10), hence $\left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural\end{array}\right]$ is r.c.-stabilizing, by Lemma 6.6.17(d).

Consequently, we can apply Lemma 11.2.18 to $\Sigma^{\curvearrowleft}$ to obtain that $\mathcal{P}_Z \geq 0$ (we have not established Hypothesis 11.2.1 for $\Sigma^{\curvearrowleft}$, but that hypothesis is redundant, as noted in the latter remark of the proof of Lemma 11.2.18; we did use the fact that $\mathbb{D}^{\curvearrowleft} = \left[\begin{smallmatrix} * & * \\ 0 & I\end{smallmatrix}\right]$, i.e., that Hypothesis 11.1.1 is satisfied).

3° *(c1)-form of (ii')⇔(iii'):* Given an exponentially stabilizing solution $(\mathcal{P}_Z, S_Z, \left[\begin{array}{c|c} \mathbb{K}_E & \mathbb{F}_E\end{array}\right])$ of the IARE for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and J_1, an exponentially stabilizing solution $(\mathcal{P}_Z, S_Z, \left[\begin{array}{c|c} \mathbb{K}_Z & \mathbb{F}_Z\end{array}\right])$ of the $\mathcal{P}_Z$-IARE (i.e., of the IARE for Σ_Z and J_1) is then determined by

$$\mathbb{X}_E = \mathbb{X}_Z\widetilde{\mathbb{X}} \quad\text{and}\quad \mathbb{K}_E = \mathbb{K}_Z + \mathbb{X}_E(\widetilde{\mathbb{M}}\widetilde{\mathbb{K}}) = \mathbb{K}_Z + \mathbb{X}_Z\widetilde{\mathbb{K}}, \tag{12.100}$$

by Lemma 9.12.3(b) (this has same closed-loop semigroup, hence also this is exponentially stabilizing). Thus, (ii') implies (iii') after the changes listed in (c). The converse follows analogously.

(For future use we note that both solutions are UR if one is, because $\widetilde{\mathbb{F}}$ is UR).

4° *(c1)-form of (ii)⇔(iii):* In the same way as in 3°, we observe from Lemma 9.12.3(b), that the P-admissible solutions $(\mathcal{P}_Z, S_Z, \left[\begin{array}{c|c} \mathbb{K}_Z & \mathbb{F}_Z\end{array}\right])$ and $(\mathcal{P}_Z, S_Z, \left[\begin{array}{c|c} \mathbb{K}_E & \mathbb{F}_E\end{array}\right])$ of the two IAREs correspond to each other one-to-one through (12.100).

By Lemma 6.6.17(c), $\left[\begin{array}{c|c} \mathbb{K}_E & \mathbb{F}_E\end{array}\right]$ is stable and stabilizing iff $\mathbb{K}_E$ is stable and $I - \mathbb{F}_E \in \mathcal{G}\mathrm{TIC}(Y \times U)$. By (12.100), this is the case iff $(I - \mathbb{F}_Z)(I - \widetilde{\mathbb{F}}) \in \mathcal{G}\mathrm{TIC}(Y \times U)$ and $\mathbb{K}_E = \mathbb{K}_Z + \mathbb{X}_Z\widetilde{\mathbb{K}}$ is stable. Thus, (ii) implies (iii).

Assume (iii), so that $\mathbb{M}_E$ and $\mathbb{K}_E$ are stable. Then also $\mathbb{M}_Z = \widetilde{\mathbb{X}}\mathbb{M}_E$ is stable. By the above, it only remains to show that $\mathcal{P}_Z$ is stabilizing for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and J_1. By Lemma 9.12.3(b), it sufficies to show that $\mathbb{M}_Z\mathbb{K}_Z$ is stable (since $\mathbb{M}_Z = \widetilde{\mathbb{X}}\mathbb{M}_E$ is stable and the rest of the closed-loop system is contained in that for Σ_Z). By (12.100),

$$\mathbb{M}_Z\mathbb{K}_Z = \mathbb{M}_Z\mathbb{K}_E - \mathbb{M}_Z\mathbb{X}_Z\widetilde{\mathbb{K}} = \mathbb{M}_Z\mathbb{K}_E - \widetilde{\mathbb{K}}, \tag{12.101}$$

which is stable (since also $\mathbb{K}_E$ and $\widetilde{\mathbb{K}}$ are stable). Thus, $\mathcal{P}_Z$ is stabilizing, hence (ii) holds.

(c2) Much of (c2) follows from (c1). The CARE and the condition on S_Z are obtained from 2° (and in 1° if $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right]$ is exponentially stabilizing) below.

(Note that (d) or (a)&(b) contains the converse implication if $D_{21} = 0$ or $\dim U < \infty$.)

1° *(ii') holds when $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right]$ is exponentially stabilizing:* It was noted at the end of (c1)1° that Hypothesis 11.2.1 is satisfied by Σ_Z (for $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{exp}}$) and that $1 > \gamma_{\mathrm{FI}}$ (for Σ_Z), hence $1 > \gamma_0$, by (11.12).

Since $\mathbb{Z} \in \mathrm{UR}$, we have $\mathbb{F}_E, \mathbb{M}_E \in \mathrm{UR}$, (because $\mathbb{F}_E = I - \mathbb{Z}^{\mathrm{d}}$, by (c1)1°) by Proposition 6.3.1(b1), hence $\mathbb{M}_Z = \widetilde{\mathbb{X}}\mathbb{M}_E \in \mathrm{UR}$. Consequently, the triple $(\mathcal{P}_Z, J_1, \left[\begin{array}{c|c} \mathbb{K}_Z & \mathbb{F}_Z \end{array}\right])$ of (c1)1°&(c1)3° corresponds to a (unique) exponentially stabilizing solution $(\mathcal{P}_Z, S_Z, K_{Z,I})$ of the $\mathcal{P}_Z$-CARE, by Corollary 9.9.8. By Proposition 11.2.19(d1), we have $S_{Z11} \gg 0$ and $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$.

2° *(ii) holds:* Almost as in 1° above, we obtain a unique UR r.c.-stabilizing solution $(\mathcal{P}, S_\natural, K_{\natural,I})$ of the CARE for $\Sigma^{\curvearrowleft}{}_0$ and J_1 (use (c1)2°&(c1)4° and Corollary 9.9.8), and $S_\natural = X_\natural^* J_1 X_\natural$, where $X_\natural = I - F_\natural$, and that $S_{\natural 11} \gg 0$ and $S_{\natural 22} - S_{\natural 21}S_{\natural 11}^{-1}S_{\natural 12} \ll 0$.

The solution $(\mathcal{P}, J_1, \left[\begin{array}{c|c} \mathbb{K}_\natural & \mathbb{F}_\natural \end{array}\right])$ of the CARE for $\Sigma^{\curvearrowleft}{}_0$ and J_1 corresponds analogously to a UR solution $(\mathcal{P}, S_Z, K_{Z,I})$ of the CARE for Σ_Z and J_1 (which also satisfies the (c1)-form of (ii)), where $S_Z = X^* J_1 X = \overline{X}^* S_\natural \overline{X}$, $\overline{X} := X_\natural^{-1} X = \left[\begin{smallmatrix} X_{11} & X_{12} \\ 0 & I \end{smallmatrix}\right]$. Consequently, also S_Z satisfies the required condition, by Lemma 11.3.13(i)&(ii'').

(d) We prove "(ii')⇒(i)" in 1° (and 2°) assuming that $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right]$ is exponentially stabilizing; the other case follows analogously (use (c1)4° instead of (c1)3°).

(N.B. By dropping the assumption $D_{21} = 0$ we lost the condition on S_E in (iii') (compared to (b)), hence do not longer know whether Z can be chosen as in (i) or whether W_{22} is invertible (the latter is true if $\dim U < \infty$).)

1° Assume (ii'). By Lemma 11.3.13(i)&(iii'), there is $\widetilde{Z} \in \mathcal{GB}(Y \times U)$ s.t. $\widetilde{Z} J_1 \widetilde{Z}^* = S_Z$. Apply Theorem 9.8.12(s1) to redefine the solution so that we obtain another UR nonnegative exponentially stabilizing solution $(\mathcal{P}_Z, J_1, \left[\begin{array}{c|c} \widetilde{\mathbb{K}}_Z & \widetilde{\mathbb{F}}_Z \end{array}\right])$ of the IARE for Σ_Z and J_1 s.t. $I - \widetilde{\mathbb{F}}_Z = \widetilde{Z}^*$.

Set $Z := \widetilde{X}^* \widetilde{Z}$, so that $Z S_Z Z^* = \widetilde{X}^* S_Z \widetilde{X} =: S_E$. By (c1)3°, we get an UR exponentially stabilizing solution $(\mathcal{P}_Z, J_1, \left[\begin{array}{c|c} \widetilde{\mathbb{K}}_E & \widetilde{\mathbb{F}}_E \end{array}\right])$ of the IARE for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$

and J_1 s.t. $\widetilde{X}_E = \widetilde{X}_Z\widetilde{X} = \widetilde{Z}^*\widetilde{X} = Z^*$. By Corollary 9.9.11, $\mathbb{Z}^{\mathrm{d}} := I - \widetilde{\mathbb{F}}_E \in \mathcal{G}\mathrm{TIC}(Y \times U)$. By Lemma 9.8.14, $\mathbb{W}^{\mathrm{d}} := \mathbb{E}^{\mathrm{d}}\mathbb{Z}^{-\mathrm{d}}$ is (J_1,J_1)-lossless. But $\widetilde{X} = \left[\begin{smallmatrix} I & 0 \\ * & * \end{smallmatrix}\right]$, by (12.91), hence $Z_{11} = \widetilde{Z}_{11} \in \mathcal{GB}(Y)$.

2° *Case* $\dim U < \infty$*:* Since $\mathbb{W}^{\mathrm{d}} := \mathbb{E}^{\mathrm{d}}\mathbb{Z}^{-\mathrm{d}}$ is (J_1,J_1)-lossless, we have $\mathbb{W}_{22} \in \mathcal{G}\mathrm{TIC}(U)$, by Proposition 2.5.4(1), hence (Factor2Z) is satisfied (hence so is (Factor2), hence also (4BP3), hence (4BP1) and (4BP2), by Theorem 12.3.7(a)&(e1))

(If $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}} \in \widetilde{\mathcal{A}}$, then $\mathbb{N}, \mathbb{M}, \widetilde{\mathbb{M}}_+, \widetilde{\mathbb{N}}_+ \in \widetilde{\mathcal{A}}$, by Lemma 12.3.10(a). Thus, then (Factor2$\widetilde{\mathcal{A}}$) is satisfied, since $\widetilde{\mathbb{M}}_{+22} = \mathbb{W}_{22} \in \mathcal{G}\mathrm{TIC}(U)$ and $\widetilde{N}_{+21} = (Z^{-1})_{22} \in \mathcal{GB}(U)$ (since $Z_{11} \in \mathcal{GB}(Y)$), by (12.50).)

(a) By (c2), (i) implies (ii). The converse follows from 2°, and "(ii)⇔(iii)" from 1°.

1° *(ii)⇔(iii):* Let S_Z be as in (ii) (and $\left[\begin{array}{c|c} \mathbb{K}_Z & \mathbb{F}_Z \end{array}\right]$ corresponds to a solution of the CARE, i.e., $F_Z = 0$), then $X_E = X_Z\widetilde{X} = \widetilde{X}$, where $(\mathcal{P}_Z, S_Z, \left[\begin{array}{c|c} \mathbb{K}_E & \mathbb{F}_E \end{array}\right])$ is as in (c1)4°.

By Corollary 9.9.8, the corresponding solution $(\mathcal{P}, S_E, K_E)$ of the CARE for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and J_1 has the signature operator $S_E := \widetilde{X}^* S_Z \widetilde{X}$. By Lemma 11.3.13(ii"), also S_E is as in (iii) (since $\widetilde{X} = \left[\begin{smallmatrix} I & 0 \\ 0 & M_{11} \end{smallmatrix}\right]$ due to the assumption that $D_{21} = 0$). Thus, (ii) implies (iii). The converse is analogous.

2° *(ii)⇒(i):* This follows from (d), since now $Z := \widetilde{X}^*\widetilde{Z}$, by (d)1°, where $\widetilde{X} = \begin{bmatrix} I & 0 \\ 0 & M_{11}^* \end{bmatrix}$ and $\widetilde{Z}$ is as in (i).

(b) The proof of (a) will do with slight changes. □

Lemma 12.5.18 ($\mathcal{P}_Z \Leftrightarrow \mathcal{P}_Y$ & $\rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2$) *Assume that Hypothesis 12.5.13 holds and $\mathbb{M}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$. Let $\mathcal{P}_X$ be the corresponding P-stabilizing solution of the $\mathcal{P}_X$-CARE (as in Lemma 12.5.12). Assume that the $\mathcal{P}_Z$-eIARE and the $\mathcal{P}_Y$-eIARE have internally P-stabilizing solutions $\mathcal{P}_Z$ and $\mathcal{P}_Y$, respectively.*

Then $\mathcal{P}_Z \geq 0$ iff $\mathcal{P}_Y \geq 0$ and $\rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2$. If $\mathcal{P}_Z \geq 0$, then (a)–(c) of Lemma 12.6.4 hold for the solutions of the eDAREs.

Proof: 1° *The assumptions of Lemma 12.6.4 are satisfied:* By Lemma 12.5.12, the IARE for Σ_X and J_γ has a UR P-stabilizing solution $(\mathcal{P}_X, J_1, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ s.t. $(I - \mathbb{F})^{-1} = \mathbb{M}$, $\mathbb{DM} = \mathbb{N}$. Since $\mathbb{M}_{22} = (\mathbb{X}^{-1})_{22}$ (by (12.49)), we have $\mathbb{M}_{22} \in \mathcal{G}\mathrm{TIC}_\infty(W)$ (because $\mathbb{X}_{11} \in \mathcal{G}\mathrm{TIC}$), hence $\mathbb{X}'_{11} \in \mathcal{G}\mathrm{TIC}_\infty(U)$, where $\mathbb{X}' := \mathbb{M}^{-1}$, by Lemma A.1.1(c1).

By discretization (see Proposition 9.8.7(a)), we observe that $(\mathcal{P}_X, J_1, \Delta^{\mathrm{S}}\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ a solution of the IARE for $\Delta^{\mathrm{S}}\Sigma_X$ and J_γ; the corresponding solution $(\mathcal{P}_X, S', K')$ of the DARE ("$\mathcal{P}_X$-DARE") satisfies the assumptions of Lemma 12.6.4, by Lemma 11.3.13(vi)&(i) (recall that $\mathbb{X}', \mathbb{X}'_{11} \in \mathcal{G}\mathrm{TIC}_\infty$ and that $\mathbb{X}'_{21}(\mathbb{X}'_{11})^{-1} = -\mathbb{M}_{22}^{-1}\mathbb{M}_{21} = \mathbb{X}_{21}(\mathbb{X}_{11})^{-1}$ (because $\mathbb{MX}' = I$ and $\mathbb{M}_{2*} = (\mathbb{X}^{-1})_{2*}$, by (12.49)), hence $\|\mathbb{X}'_{21}(\mathbb{X}'_{11})^{-1}\| < 1$, by Lemma 12.3.11(a)).

2° *The equivalence:* By discretization (see again Proposition 9.8.7(a)), we obtain that $\mathcal{P}_X$, $\mathcal{P}_Y$, $\mathcal{P}_X$ are P-stabilizing solutions of the $\mathcal{P}_X$-eDARE, $\mathcal{P}_Y$-eDARE and $\mathcal{P}_Z$-eDARE (the ones corresponding to $(\Delta^{\mathrm{S}}\Sigma_X, J_\gamma)$, $(\Delta^{\mathrm{S}}\Sigma_Y, J_\gamma)$ and $(\Delta^{\mathrm{S}}\Sigma_Z, J_1)$), respectively. (Cf. Lemma 12.6.1.)

If $\mathcal{P}_Z \geq 0$, then $\mathcal{P}_Y' := \gamma^2 \mathcal{P}_Z(I + \mathcal{P}_X\mathcal{P}_Z)^{-1}$ is an internally P-stabilizing solution of the $\mathcal{P}_Y$-eDARE and $\rho(\mathcal{P}_X\mathcal{P}_Y') < \gamma$, by Lemma 12.6.4(b)&(a) (recall 1°), hence $\mathcal{P}_Y' = \mathcal{P}_Y$, by Theorem 14.1.4(b). The converse follows analogously from Lemma 12.6.4.

3° *(a)–(c):* If $\mathcal{P}_Z \geq 0$, then (a)–(c) of Lemma 12.6.4 hold for the DAREs, by 1° (but those S_Z and S_Y may differ from those corresponding to the CAREs, etc.). □

We have already shown that all suboptimal I/O maps for $\mathbb{D}$ are given by (12.48). Now we construct a stable realization for $\mathbb{T}$, so that we can present the formula for all solutions in the standard form (cf. Theorem 12.1.8, which is based on this):

Proposition 12.5.19 ((Factor1&2) ⇒ 4BP & all solutions) *Make Hypothesis 12.5.1. Assume that (Factor1X) and (Factor2Z) hold (let $\mathbb{X}$ and $\mathbb{Z}$ be their solutions) and that $\mathbb{Z}_{11} \in \mathcal{G}\mathrm{TIC}_\infty(Y)$ (as in Theorem 12.3.7(d)). Define*

$$\begin{bmatrix} \mathbb{K}_{\mathbb{E}^{\mathrm{d}}} & \big| & \mathbb{F}_{\mathbb{E}^{\mathrm{d}}} \end{bmatrix} := \begin{bmatrix} -J_1^{-1}\pi_+(\mathbb{Z}^{-\mathrm{d}})^*\mathbb{D}_{\mathbb{E}^{\mathrm{d}}}^* J_1 \mathbb{C}_{\mathbb{E}^{\mathrm{d}}} & \big| & I - \mathbb{Z}^d \end{bmatrix}. \tag{12.102}$$

(as in (9.140); this is generated by $\begin{bmatrix} Z^*K_E & \big| & I - Z^* \end{bmatrix}$ if $(\mathcal{P}_Z, S_E, K_E)$ is a UR stable, P-stabilizing solution of the CARE for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ and J_1). Then $\begin{bmatrix} \mathbb{K}_{\mathbb{E}^{\mathrm{d}}} & \big| & \mathbb{F}_{\mathbb{E}^{\mathrm{d}}} \end{bmatrix}$ is an admissible state feedback pair for $\Sigma_{\mathbb{E}^{\mathrm{d}}}$; in particular, $\left[\begin{array}{c|c} \mathbb{A}_{\mathbb{E}^{\mathrm{d}}} & \mathbb{B}_{\mathbb{E}^{\mathrm{d}}} \\ \hline \mathbb{K}_{\mathbb{E}^{\mathrm{d}}} & \mathbb{F}_{\mathbb{E}^{\mathrm{d}}} \end{array}\right] \in \mathrm{WPLS}(Y \times U, H, Y \times U)$. Set $\mathbb{R} := I - \mathbb{F}_{\mathbb{E}^{\mathrm{d}}} = \mathbb{Z}^{\mathrm{d}}$. Since $\mathbb{R}_{11} = \mathbb{Z}_{11}^{\mathrm{d}} \in \mathcal{G}\mathrm{TIC}_\infty$, the output feedback operator $L := \left[\begin{smallmatrix} -I & 0 & I \\ 0 & 0 & 0 \end{smallmatrix}\right] \in \mathcal{B}(Y \times U \times Y, Y \times U)$ is admissible for

$$\Sigma_{\mathrm{alt}} := \left[\begin{array}{c|c} \mathbb{A}_{\mathrm{alt}} & \mathbb{B}_{\mathrm{alt}} \\ \hline \mathbb{C}_{\mathrm{alt}} & \mathbb{D}_{\mathrm{alt}} \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A}_{\mathbb{E}^{\mathrm{d}}} & (\mathbb{B}_{\mathbb{E}^{\mathrm{d}}})_1 & (\mathbb{B}_{\mathbb{E}^{\mathrm{d}}})_2 \\ \hline -(\mathbb{K}_{\mathbb{E}^{\mathrm{d}}})_1 & \mathbb{R}_{11} & \mathbb{R}_{12} \\ -(\mathbb{K}_{\mathbb{E}^{\mathrm{d}}})_2 & \mathbb{R}_{21} & \mathbb{R}_{22} \\ 0 & I & 0 \end{array}\right], \tag{12.103}$$

and the corresponding closed-loop system is given by

$$\Sigma_{\mathrm{alt},L} := \left[\begin{array}{c|c} \mathbb{A}_{\mathrm{alt},L} & \mathbb{B}_{\mathrm{alt},L} \\ \hline \mathbb{C}_{\mathrm{alt},L} & \mathbb{D}_{\mathrm{alt},L} \end{array}\right] = \left[\begin{array}{c|cc} \mathbb{A}_{\mathbb{T}}^{\mathrm{d}} & \mathbb{C}_{\mathbb{T}2}^{\mathrm{d}} & \mathbb{C}_{\mathbb{T}1}^{\mathrm{d}} \\ \hline 0 & I & 0 \\ \mathbb{B}_{\mathbb{T}1}^{\mathrm{d}} & \mathbb{T}_{21}^{\mathrm{d}} & \mathbb{T}_{11}^{\mathrm{d}} \\ \mathbb{B}_{\mathbb{T}2}^{\mathrm{d}} & \mathbb{T}_{22}^{\mathrm{d}} & \mathbb{T}_{12}^{\mathrm{d}} \end{array}\right] \tag{12.104}$$

$\in \mathrm{WPLS}(Y \times U, H, Y \times U \times Y)$. Then $\Sigma_{\mathbb{T}} := \left[\begin{array}{c|cc} \mathbb{A}_{\mathbb{T}} & \mathbb{B}_{\mathbb{T}1} & \mathbb{B}_{\mathbb{T}2} \\ \hline \mathbb{C}_{\mathbb{T}} & \mathbb{T}_{*1} & \mathbb{T}_{*2} \end{array}\right] \in \mathrm{WPLS}(U \times Y, H, U \times Y)$ is a realization of $\mathbb{T}$:=(12.48), and the following hold:

(a) If we delete the middle column and bottom row of $\Sigma_{\mathbb{T}}$, we obtain a realization $\Sigma_{\mathbb{T}_{12}}$ of $\mathbb{T}_{12}$. The system $\Sigma_{\mathbb{T}_{12}}$ is a well-posed suboptimal controller (the "central controller") for Σ.

(b) **(All well-posed $\Sigma_{\mathbb{Q}}$'s)** *All well-posed stabilizing suboptimal controllers $\Sigma_{\mathbb{Q}}$ for Σ are given by the connection of $\Sigma_{\mathbb{T}}$ and $\Sigma_{\mathbb{L}}$ in Figure 12.1 (cf. Remark 12.1.9), where the parameter $\mathbb{L}$ is as in (12.105) and $\Sigma_{\mathbb{L}}$ is any stable realization of $\mathbb{L}$.*

(c) **(All well-posed $\mathbb{Q}$'s)** *All well-posed stabilizing suboptimal controllers $\mathbb{Q} \in \mathrm{TIC}_\infty(Y,U)$ for $\mathbb{D}$ are given by*

$$\mathbb{Q} = \mathcal{F}_\ell(\mathbb{T},\mathbb{L}) \quad (\mathbb{L} \in \mathrm{TIC}(Y,U) \text{ is s.t. } \|\mathbb{L}\|_{\mathrm{TIC}} < 1 \text{ and } I - \mathbb{L}\mathbb{T}_{21} \in \mathcal{G}\mathrm{TIC}_\infty(U)). \tag{12.105}$$

(d) **(All (possibly ill-posed) solutions)** *By removing the condition $I - \mathbb{L}\mathbb{T}_{21} \in \mathcal{G}\mathrm{TIC}_\infty(U)$ we get all stabilizing suboptimal controllers (with internal loop) in any of (a)–(c).*

(e) **(All well-posed exponentially stabilizing solutions)** *If $\begin{bmatrix} \mathbb{K}_\mathrm{u} & \big| & \mathbb{F}_\mathrm{u} \end{bmatrix}$ is [exponentially] strongly stabilizing, then we can replace "stable" by "[exponentially] strongly stable" and "stabilizing" by "[exponentially] strongly stabilizing" everywhere in this proposition. [if we require in (12.105) that $\mathbb{L} \in \mathrm{TIC}_{\mathrm{exp}}(Y,U)$].*

(f) We can replace "$\|\mathbb{L}\|_{\mathrm{TIC}} < 1$" by "$\|\mathbb{L}\|_{\mathrm{TIC}} \le 1$" everywhere in this proposition if we replace "suboptimal" by "s.t. $\|\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})\| \le \gamma$".

(g) If we add some element $E \in \mathcal{B}(U,Y)$ to D_{21}, then the parametrizations in (d) are is unchanged except that we have to add to $\mathbb{Q}$ the output feedback through $-E$, as in Lemma 7.3.23 and Figure 7.12.

(h) If $\mathbb{Z}$ is SR (resp. UR, SVR, UVR, SLR, ULR), then Σ_{alt}, $\Sigma_{\mathrm{alt},L}$ and $\Sigma_{\mathbb{T}}$ are SR (resp. UR, SVR, UVR, SLR, ULR).

See Theorem 12.1.8 for a more classic "all controllers" result. As noted below Theorem 12.1.8, "all suboptimal stabilizing controllers" refers to all maps $\mathbb{Q} \in \mathrm{TIC}_\infty$ (or, in (d), all maps $\mathbb{Q}$ with internal loop modulo equivalence) $\mathbb{Q} : y \mapsto u$ s.t. $\|\mathcal{F}_\ell(\mathbb{D},\mathbb{Q})\|_{\mathrm{TIC}(W,Z)} < \gamma$ (with some realization), not to all systems having such an I/O map. That is, in our "all solutions" formula, we do not distinguish between two solutions having the same "I/O map".[1]

Indeed, any, e.g., l.c.-detectable realization of a $\mathbb{Q}$ satisfying (12.105) will do — on the other hand, any solution must be a realization of some $\mathbb{Q}$ satisfying (12.105). In the exponential case mentioned in (e), all possible $\Sigma_{\mathbb{Q}}$'s are exactly all optimizable and estimatable realizations of $\mathbb{Q}$'s satisfying (12.105), by (c) and Theorem 7.3.11(c1).

We recall from Definition 6.1.6 that for any $\mathbb{L} \in \mathrm{TIC}$ (resp. $\mathbb{L} \in \mathrm{TIC}_{\mathrm{exp}}$), there exists a strongly (resp. exponentially) stable realization $\Sigma_{\mathbb{L}}$ of $\mathbb{L}$.

We leave it to the reader to write out $\mathbb{A}_{\mathbb{T}}$, $\mathbb{B}_{\mathbb{T}}$ and $\mathbb{C}_{\mathbb{T}}$ in terms of Σ, $\mathbb{K}_\mathrm{u}$, $\mathbb{F}_\mathrm{u}$, $\mathbb{X}$ and $\mathbb{Z}$; see Theorem 12.1.8 for their generators (under the regularity assumption $\mathbb{N}_\mathrm{u}, \mathbb{M}_\mathrm{u} \in \widetilde{\mathcal{A}}$).

Proof of Proposition 12.5.19: (We note that Lemma 12.5.15 and the part of Lemma 12.5.16 concerning $\Sigma_{\mathbb{E}^\mathrm{d}}$ are valid also under the assumptions of this proposition, with the same proofs, hence $\Sigma_{\mathbb{E}^\mathrm{d}}$ is well defined.)

1° *The proof of initial claims:* By Theorem 9.9.10(g1)&(a2)&(c1), $\begin{bmatrix} \mathbb{K}_{\mathbb{E}^\mathrm{d}} & \big| & \mathbb{F}_{\mathbb{E}^\mathrm{d}} \end{bmatrix}$ is stabilizing and J-critical over $\mathcal{U}_{\mathrm{out}}$ for $\Sigma_{\mathbb{E}^\mathrm{d}}$. If the CARE for $\Sigma_{\mathbb{E}^\mathrm{d}}$ and J_1 has a UR stable, P-stabilizing solution $(\mathcal{P}_Z, S_E, K_E)$, then all J-critical pairs are generated by $\begin{bmatrix} QK_E & I-Q \end{bmatrix}$ ($Q \in \mathcal{GB}$), by Theorem

[1] Here we use quotes because $\mathbb{Q}$ need not be well-posed in general.

9.9.10(d1) and Theorem 9.8.12(b)&(s1), hence only $\begin{bmatrix} Z^*K_E & I - Z^* \end{bmatrix}$ can generate $\left[\begin{array}{c|c} * & I - \mathbb{Z}^{\mathrm{d}} \end{array}\right]$.

By Lemma A.1.1(c1), we have $(\mathbb{Z}^{-1})_{22} \in \mathcal{G}\mathrm{TIC}_\infty(U)$; by (12.50), $(\mathbb{Z}^{-1})_{22} = \widetilde{\mathbb{N}}_{+21}$. Set $\mathbb{R} := \mathbb{Z}^d = I - \mathbb{F}_{\mathbb{E}^{\mathrm{d}}}$, $\mathbb{G} := \mathbb{R}^{-1} = \mathbb{Z}^{-\mathrm{d}}$. If we denote "$\begin{bmatrix} \mathbb{T}^{\mathrm{d}}_{21} & \mathbb{T}^{\mathrm{d}}_{11} \\ \mathbb{T}^{\mathrm{d}}_{22} & \mathbb{T}^{\mathrm{d}}_{12} \end{bmatrix}$" in (12.104) by $\mathbb{H}$, then, by (6.125),

$$\mathbb{H}\begin{bmatrix} 0 & I_Y \\ I_U & 0 \end{bmatrix} = \begin{bmatrix} \mathbb{R}_{21} & \mathbb{R}_{22} \\ I & 0 \end{bmatrix} (I - \begin{bmatrix} I - \mathbb{R}_{11} & -\mathbb{R}_{12} \\ 0 & I \end{bmatrix})^{-1} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{12.106}$$

$$= \begin{bmatrix} \mathbb{R}_{21} & \mathbb{R}_{22} \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ \mathbb{R}_{11} & \mathbb{R}_{12} \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I \\ \mathbb{G}_{11} & \mathbb{G}_{12} \end{bmatrix} \begin{bmatrix} \mathbb{G}_{21} & \mathbb{G}_{22} \\ I & 0 \end{bmatrix}^{-1} \tag{12.107}$$

$$= \begin{bmatrix} 0 & I \\ \widetilde{\mathbb{M}}^{\mathrm{d}}_{+11} & \widetilde{\mathbb{M}}^{\mathrm{d}}_{+21} \end{bmatrix} \begin{bmatrix} \widetilde{\mathbb{N}}^{\mathrm{d}}_{+11} & \widetilde{\mathbb{N}}^{\mathrm{d}}_{+21} \\ I & 0 \end{bmatrix}^{-1} = \mathbb{T}^d \tag{12.108}$$

(write the formulae out or multiply (11.88) by $\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ to the left and right to obtain the third equality; the fourth follows from (12.50) and the last one from (12.48)), as claimed, i.e., the closed-loop I/O map $\mathbb{D}_{\mathrm{alt},L}$ contains $\mathbb{T}^{\mathrm{d}}$ (permuted), so that we can pick corresponding rows, interchange the columns and take the causal adjoint of the system to obtain a realization of $\mathbb{T}$, as stated in the proposition.

(a) This follows from (b) by taking $\mathbb{L} = 0$.

(c) This follows from Theorem 12.3.7(c) (which is applicable, by Lemma 12.5.3 and the fact that $\widetilde{\mathbb{N}}_{+21} = (\mathbb{Z}^{-1})_{22} \in \mathcal{G}\mathrm{TIC}_\infty$, by 1° above).

(d)&(f) Also these follow from Theorem 12.3.7(c) for (c); the proofs of (a) and (b) show that the same holds for them.

(g) This follows from Lemma 7.3.23 (note that in this case the well-posedness of a solution need not be equivalent to $I - \mathbb{L}\mathbb{T}_{21} \in \mathcal{G}\mathrm{TIC}_\infty$).

(h) Use Proposition 6.3.1(b2) (and Lemma 6.2.5). (Note also that if $\mathbb{Z} \in \widetilde{\mathcal{A}}$ (e.g., $\mathbb{E} \in \widetilde{\mathcal{A}}$ or $\mathbb{N}, \mathbb{M} \in \widetilde{\mathcal{A}}$), then $\mathbb{R} \in \widetilde{\mathcal{A}}$ and "$\mathbb{T} \in \widetilde{\mathcal{A}}^{-1}\widetilde{\mathcal{A}}$".)

(b)&(e) Assume that $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right]$ is [[exponentially] strongly] stabilizing for Σ. Let $\mathbb{L}$ be as in (c) or as in (d), and let $\Sigma_{\mathbb{L}}$ be an [[exponentially] strongly] stable realization of $\mathbb{L}$.

1° The system $\Sigma_{\mathbb{E}^{\mathrm{d}}}$ is [[exponentially] strongly*] stable, by Lemma 12.5.15, and Σ is [[exponentially] strongly] l.c.-detectable, by Theorem 6.6.28 [[shifted and Lemma 12.5.2(vi)]].

By (c), $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ (possibly with an internal loop; cf. (d)), hence $\Sigma_{\mathbb{Q}}$ I/O-DPF-stabilizes Σ. We shall show in 2°–3° $\Sigma_{\mathbb{Q}}$ DPF-stabilizes Σ [[exponentially] strongly]; this establishes (b) and (e) [[this includes also (c) modified as in (e), becase then $\mathbb{Q}$ DPF-stabilizes $\mathbb{D}$ exponentially]].

2° $\Sigma_{\mathbb{T}}$ *is [[exponentially] strongly] l.c.-detectable:* Since $\left[\begin{array}{c|cc} 0 & 0 & 0 \end{array}\right]$ is a [[exponentially] strongly*] r.c.-stabilizing state feedback pair for Σ_{alt}, the pair (6.188) is admissible for $\Sigma_{\mathrm{alt},L}$ (use substitutions $\mathbb{K}, \mathbb{F}, \mathbb{K}_L, \mathbb{F}_L, \mathbb{K}_\flat, \mathbb{F}_\flat \mapsto 0$, $\Sigma \mapsto \Sigma_{\mathrm{alt}}$, $\Sigma_L \mapsto \Sigma_{\mathrm{alt},L}$, $\Sigma_\flat \mapsto \begin{bmatrix} \Sigma_{\mathrm{alt}} \\ [0\ 0] \end{bmatrix}$ in the proof of Lemma 6.7.11(c)), and the corresponding closed-loop system is given by

$$\left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \\ \mathbb{K}_\flat & \mathbb{F}_\flat \end{array}\right] := \left[\begin{array}{c|cc} \mathbb{A}_{\mathrm{alt}} & \mathbb{B}_{\mathrm{alt}} & \\ \hline \mathbb{C}_{\mathrm{alt}} & \mathbb{D}_{\mathrm{alt}} & \\ -(\mathbb{K}_{\mathbb{E}^{\mathrm{d}}})_1 & \mathbb{R}_{11}-I & \mathbb{R}_{12} \\ 0 & 0 & 0 \end{array}\right]. \tag{12.109}$$

Therefore, (6.188) is admissible for $\Sigma^{\mathrm{d}}_{\mathbb{T}}\left[\begin{smallmatrix} I&0&0\\0&0&I\\0&I&0\end{smallmatrix}\right]$, and the corresponding closed-loop system is (12.109) with its second row (corresponding to $(\mathbb{C}_{\mathrm{alt}})_1$) removed; in particular, (6.188) is [strongly*] stabilizing. Consequently, corresponding maps "$\mathbb{N} := \mathbb{D}_\flat$" and "$\mathbb{M} := \mathbb{F}_\flat + I$" (see Definition 6.6.10) are given by

$$\mathbb{N}' = \begin{bmatrix} \mathbb{R}_{21} & \mathbb{R}_{22} \\ I & 0 \end{bmatrix}, \quad \mathbb{M}' = \begin{bmatrix} \mathbb{R}_{11} & \mathbb{R}_{12} \\ 0 & I \end{bmatrix}; \tag{12.110}$$

these are [[exponentially]] r.c. (because $\left[\begin{smallmatrix} 0&I\\0&0\end{smallmatrix}\right]\mathbb{N}' + \left[\begin{smallmatrix} 0&0\\0&I\end{smallmatrix}\right]\mathbb{M}' = I$). Thus, $\Sigma^{\mathrm{d}}_{\mathbb{T}}$ is [[exponentially] strongly*] r.c.-stabilizable (by Lemma 6.7.17, the permutation of columns does not matter), i.e., $\Sigma_{\mathbb{T}}$ is [[exponentially] strongly] l.c.-detectable.

3° $\Sigma_{\mathbb{Q}}$ *is [[exponentially] strongly] stabilizing:* Let Σ^o contain $\Sigma_{\mathbb{L}}$, $\Sigma_{\mathbb{T}}$ and Σ so that the static output feedback operator I corresponds to the connection $\mathcal{F}_\ell(\mathbb{D},\mathcal{F}_\ell(\mathbb{T},\mathbb{L}))$ (cf. (7.4) and Definition 7.3.1).

Since Σ (by 1°) and $\Sigma_{\mathbb{T}}$ (by 2°) and $\Sigma_{\mathbb{L}}$ (by assumption) are [[exponentially] strongly] l.c.-detectable, so is Σ^o, by Lemma 6.7.18 (applied twice). By Proposition 6.7.14(b)(2.), I is [[exponentially] strongly] stabilizing for Σ^o. Thus, all closed-loop maps in Figure 12.1 are [[exponentially] strongly] stable (because they are exactly the elements of Σ^o_I). As noted in 1°, this establishes (b) and (e).

(An alternative proof would apply Proposition 6.7.14(b)(1.)&(2.) for subsystems [[or the generators of Σ and $\Sigma_{\mathbb{T}}$ and optimizability and estimatability; cf. (12.20)]].) □

The $\mathcal{P}_{\mathrm{X}}$-DARE, $\mathcal{P}_{\mathrm{Y}}$-DARE and the coupling condition imply the existence of a suboptimal controller for Σ:

Lemma 12.5.20 (Sufficiency) *Assume that Hypothesis 12.5.1 is satisfied with $\mathbb{N}_{\mathrm{u}},\mathbb{M}_{\mathrm{u}},\widetilde{\mathbb{M}}_{\mathrm{y}},\widetilde{\mathbb{N}}_{\mathrm{y}} \in \widetilde{\mathcal{A}}$, that $\widetilde{\mathcal{A}}$ satisfy Hypothesis 8.4.8, and that conditions (1.)–(3.) of Lemma 12.1.12 are satisfied.*

Then Hypothesis 12.5.13 and conditions (1.) and (4.) of Theorem 12.1.8 are satisfied; in particular, then there are suboptimal exponentially stabilizing DPF-controllers.

Proof: (We use Hypothesis 8.4.8 only in 3°.)

1° *Hypothesis 12.5.13 and (Factor1$\widetilde{\mathcal{A}}$):* By (1.) and Lemma 12.5.12, Hypothesis 12.5.13 is satisfied and hence (Factor1) has a solution with $X_{11},X_{22},M_{11},M_{22} \in \mathcal{GB}$, by Lemma 12.5.14. By Lemma 12.3.10(a), $\mathbb{M},\mathbb{N},\mathbb{X} \in \widetilde{\mathcal{A}}$, hence (Factor1$\widetilde{\mathcal{A}}$) is satisfied.

2° *The $\mathcal{P}_Z$-CARE has an exponentially stabilizing solution $(\mathcal{P}_Z, S_Z, K_Z)$, where $\mathcal{P}_Z = \mathcal{P}_Y(I - \mathcal{P}_X\mathcal{P}_Y)^{-1} \geq 0$:* By 1° of the proof of Lemma 12.5.18, $\mathcal{P}_X$ is an exponentially stabilizing solution of the $\mathcal{P}_X$-DARE with S_X as in Lemma 12.6.4 (here we mean the DARE for $(\Delta^S\Sigma_X, J_\gamma)$). The situation with $\mathcal{P}_Y$ is analogous (by dual arguments).

By Lemma 12.6.4, the $\mathcal{P}_Z$-DARE (the DARE for $\Delta^S\Sigma_Z$ and J_1) has the exponentially stabilizing solution $\mathcal{P}_Z = \mathcal{P}_Y(I - \mathcal{P}_X\mathcal{P}_Y)^{-1} \geq 0$ with $S_{Z11} \gg 0$ and $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$ (recall "(3.)"). It follows from (the discretized, see Theorem 12.2.2) Lemma 12.5.17(ii')&(d) that $(\Delta^S\mathbb{E})J_1(\Delta^S\mathbb{E})^* = \mathbb{Z}_\Delta J_1 \mathbb{Z}_\Delta^*$ for some $\mathbb{Z}_\Delta \in \mathcal{G}\mathrm{tic}(Y_\Delta \times U_\Delta)$.

We deduce that $\pi_+\Delta^S(\mathbb{E}^d)^*J_1\Delta^S(\mathbb{E}^d)\pi_+ \in \mathcal{GB}$, hence $\pi_+(\mathbb{E}^d)^*J_1\mathbb{E}^d\pi_+ \in \mathcal{GB}(\mathrm{L}^2(\mathbf{R}_+;Y \times U))$, by Theorem 13.4.5(h2). Consequently, $(\mathbb{E}^d)^*J_1\mathbb{E}^d = \mathbb{R}^*S_E\mathbb{R}$ for some $\mathbb{R} \in \mathcal{G}\widetilde{\mathcal{A}}^d(Y \times U)$ and $S_E \in \mathcal{GB}(Y \times U)$.

By Corollary 9.9.11, $\mathbb{R}$ corresponds to an exponentially stabilizing solution $\mathcal{P}_Z'$ of the $\mathcal{P}_Z$-IARE, hence of the $\mathcal{P}_Z$-DARE too. By uniqueness (Theorem 9.8.12(a)), $\mathcal{P}_Z' = \mathcal{P}_Z$. By Proposition 9.8.10 and Remark 9.8.2, $\mathcal{P}_Z$ is an exponentially stabilizing solution of the $\mathcal{P}_Z$-CARE (since $\mathbb{R} \in \mathcal{G}\widetilde{\mathcal{A}}^d \subset \mathcal{G}\mathrm{ULR}$ and hence $R \in \mathcal{GB}(Y \times U)$, by Proposition 6.3.1(b1)).

3° *$S_{Z11} \gg 0$ and $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$ (so that $\mathcal{P}_Z$ solves Lemma 12.5.17(ii')):* Set $A = B = C = 0$ (but keep the D of Σ), so that the $\mathcal{P}_X$-DARE and $\mathcal{P}_Y$-DARE have unique exponentially stabilizing solutions, given by $\mathcal{P}_X = 0 = \mathcal{P}_Y = K_X = K_Y$, and S_X and S_Y are of the standard form, by the signature-conditions in (1.) and (2.). By Lemma 12.6.4(a)–(c), $\mathcal{P}_Z = 0$ is the (unique) exponentially stabilizing solution of the $\mathcal{P}_Z$-DARE and $S_{Z11} \gg 0$ and $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$. But S_X, S_Y and S_Z are the same as in our problem (because $B = C = 0$), hence this proves our claim.

(Alternatively, one could write a somewhat shorther (but still not short) proof by going through the same computations as in the proof of Lemma 12.6.4(c) (including those on pp. 321-326 of [IOW]; one can, e.g., take $A = B = C = 0$ but the D of Σ must be kept).)

4° *(1.) and (4.) of Theorem 12.1.8 are satisfied:* Condition (1.) is contained in the assumptions, and condition (4.) was established in 2°–3° above.

5° *The existence of a solution:* By 4°, the assumptions of Theorem 12.1.8 are satisfied, hence there are suboptimal exponentially stabilizing DPF-controllers for Σ. □

The above also holds under different assumptions:

Lemma 12.5.21 *Suppose that the assumptions (A1) and (A2) and conditions (1.)–(3.) of Theorem 12.1.5 are satisfied. Then the assumptions of Lemma 12.5.20 are satisfied for $\widetilde{\mathcal{A}} = \mathrm{MTIC}^{\mathrm{L}^1}_{\exp}$.*

Proof: Recall from Theorem 8.4.9 that $\widetilde{\mathcal{A}} = \mathrm{MTIC}^{\mathrm{L}^1}_{\exp}$ satisfies Standing Hypothesis 12.0.1 and Hypothesis 8.4.8.

By Lemmas 12.5.5 and 12.5.4, Hypothesis 12.5.1 is satisfied (even with "exponentially jointly" in place of "jointly"), $\mathbb{N}_u, \mathbb{M}_u, \widetilde{\mathbb{N}_y}, \widetilde{\mathbb{M}_y} \in \widetilde{\mathcal{A}}$ (or $\in(\gamma'$)

under "(IV)" of (A1)), and $\widehat{\mathbb{M}_\mathrm{u}}(+\infty)=I$, $\widehat{\widetilde{\mathbb{M}}_\mathrm{y}}(+\infty)=I$.

Note that conditions (1.)–(3.) of Theorem 12.1.5 are equal to those of Lemma 12.1.12 combined to conditions $S_X=D_X^*J_\gamma D_X$ and $S_Y=D_Y^*J_\gamma D_Y$. By Hypothesis 8.4.8 and Corollary 9.9.11, these additional conditions are now redundant. □

Lemma 12.5.22 (4BP: necessity) *Assume that there is a suboptimal exponentially stabilizing DPF-controller (possibly with internal loop), and that (A1) and (A2) of Theorem 12.1.5 hold. Then (1.)–(3.) of Theorem 12.1.5 hold.*

Proof: (The regularity condition (A1) on Σ is not superfluous, by, e.g., Example 11.3.7, but it may be weakened.)

Set $\widetilde{\mathcal{A}}:=\mathrm{MTIC}_{\mathrm{exp}}^{\mathrm{L}^1}$ (or $\widetilde{\mathcal{A}}=(\gamma')$ in case "(IV)" of (A1)), so that Standing Hypothesis 12.0.1 is satisfied, by Theorem 8.4.9(c).

1° *Hypothesis 12.5.1:* By Theorem 7.3.12(a), (A,B_1) is optimizable and (A,C_2) is estimatable. By Lemma 12.5.4, it follows that Hypothesis 12.5.1 is satisfied (even with "exponentially jointly" in place of "jointly").

2° By Lemma 12.5.3 and 1°, Hypothesis 12.3.1 is satisfied with $\mathbb{N}_\mathrm{u},\mathbb{M}_\mathrm{u},\widetilde{\mathbb{N}}_\mathrm{y},\widetilde{\mathbb{M}}_\mathrm{y}\in\widetilde{\mathcal{A}}$.

3° *(1.)–(3.) hold:* This follows from Theorem 12.1.11. □

Given just an I/O map, as in the frequency-domain problem of Section 12.3, we can choose a stabilizable realization as explained below (the advantage here is that we get the constructive formula (12.113)):

Lemma 12.5.23 *Assume Hypothesis 12.3.1. A strongly jointly stabilizable and detectable realization Σ for $\mathbb{D}$ can be chosen as follows.*

By the assumption, there are $\mathbb{Q}_1,\mathbb{Q}_2,\widetilde{\mathbb{Q}}_1,\widetilde{\mathbb{Q}}_2\in\mathrm{TIC}$ s.t.

$$\begin{bmatrix}\mathbb{M}_{\mathrm{u}\,11} & \mathbb{Q}_1\\ \mathbb{N}_{\mathrm{u}\,21} & \mathbb{Q}_2\end{bmatrix}^{-1}=\begin{bmatrix}\widetilde{\mathbb{Q}}_2 & -\widetilde{\mathbb{Q}}_1\\ -\widetilde{\mathbb{N}}_{\mathrm{y}\,21} & \mathbb{M}_{\mathrm{u}\,22}\end{bmatrix}\in\mathcal{G}\mathrm{TIC}. \tag{12.111}$$

It follows that $\begin{bmatrix}\mathbb{M}_\mathrm{u} & \mathbb{Q}_{\mathrm{DF1}}\\ \mathbb{N}_\mathrm{u} & \mathbb{Q}_{\mathrm{DF2}}\end{bmatrix},\begin{bmatrix}\widetilde{\mathbb{Q}}_{\mathrm{DF2}} & -\widetilde{\mathbb{Q}}_{\mathrm{DF1}}\\ -\widetilde{\mathbb{N}}_\mathrm{y} & \widetilde{\mathbb{M}}_\mathrm{y}\end{bmatrix}\in\mathcal{G}\mathrm{TIC}$, *where*

$$\mathbb{Q}_{\mathrm{DF2}}:=\begin{bmatrix}I & 0\\ 0 & \mathbb{Q}_2\end{bmatrix},\ \mathbb{Q}_{\mathrm{DF1}}:=\begin{bmatrix}0 & \mathbb{Q}_1\\ 0 & 0\end{bmatrix},\ \widetilde{\mathbb{Q}}_{\mathrm{DF2}}:=\begin{bmatrix}\widetilde{\mathbb{Q}}_2 & 0\\ 0 & I\end{bmatrix},\ \widetilde{\mathbb{Q}}_{\mathrm{DF1}}:=\begin{bmatrix}0 & \widetilde{\mathbb{Q}}_1\\ 0 & 0\end{bmatrix}. \tag{12.112}$$

Choose a strongly stable realization $\Sigma_\flat$ for $\begin{bmatrix}I-\mathbb{Q}_{\mathrm{DF2}} & \mathbb{N}_\mathrm{u}\\ \mathbb{Q}_{\mathrm{DF1}} & \mathbb{M}_\mathrm{u}-I\end{bmatrix}$. Then $\Sigma_{\mathrm{Total}}:=(\Sigma_\flat)_{\left[\begin{smallmatrix}0 & 0\\ 0 & -I\end{smallmatrix}\right]}$ satisfies Hypothesis 12.5.1 and its I/O map is given by (we denote the components of Σ_{Total} as in (12.87))

$$\begin{bmatrix}\mathbb{G}_\mathrm{y} & \mathbb{D}\\ \mathbb{E}_{\mathrm{uy}} & \mathbb{F}_\mathrm{u}\end{bmatrix}:=\begin{bmatrix}\begin{bmatrix}0 & \mathbb{D}_{11}\mathbb{Q}_1\\ 0 & I-\mathbb{Q}_2+\mathbb{D}_{21}\mathbb{Q}_1\end{bmatrix} & \mathbb{D}\\ \begin{bmatrix}0 & -\mathbb{M}_{\mathrm{u}\,11}^{-1}\mathbb{Q}_1\\ 0 & 0\end{bmatrix} & \begin{bmatrix}I-\mathbb{M}_{\mathrm{u}\,11}^{-1} & \mathbb{M}_{\mathrm{u}\,11}^{-1}\mathbb{M}_{\mathrm{u}\,12}\\ 0 & 0\end{bmatrix}\end{bmatrix}. \tag{12.113}$$

The pairs $\left[\begin{array}{c|c}\mathbb{K}_\mathrm{u} & \mathbb{F}_\mathrm{u}\end{array}\right]$ and $\begin{bmatrix}\mathbb{H}_\mathrm{y}\\ \mathbb{G}_\mathrm{y}\end{bmatrix}$ are actually strongly jointly stabilizing and detecting for Σ (with $\mathbb{E}_{\mathrm{uy}}$), these pairs are as in Hypothesis 12.5.1.

If there is a suboptimal stabilizing controller for $\mathbb{D}$, then there is a suboptimal strongly stabilizing controller for the above system, by Theorem 12.3.5. Also an exponential version of the above lemma and claim hold (use Remark 6.1.9 and the "exponential" assumptions of Proposition 12.4.10).

Proof: (Note that $\mathbb{Q} := \mathbb{Q}_1\mathbb{Q}_2^{-1} = \widetilde{\mathbb{Q}}_1^{-1}\widetilde{\mathbb{Q}}_2$ DPF-stabilizes $\mathbb{D}$ with d.c. internal loop, by Corollary 7.3.20(iii), whereas $\mathbb{Q}_{\mathrm{DF1}}\mathbb{Q}_{\mathrm{DF2}}^{-1} = \widetilde{\mathbb{Q}}_{\mathrm{DF1}}^{-1}\widetilde{\mathbb{Q}}_{\mathrm{DF2}}$ DF-stabilizes $\mathbb{D}$ with d.c. internal loop, by Theorem 7.2.14(iii); cf. also Lemma 7.3.10.)

We obtain (12.111) from Lemma 6.5.8. But from $\begin{bmatrix} \mathbb{M}_{\mathrm{u}} & \mathbb{Q}_{\mathrm{DF1}} \\ \mathbb{N}_{\mathrm{u}} & \mathbb{Q}_{\mathrm{DF2}} \end{bmatrix}$ we obtain $\begin{bmatrix} I & 0 \\ 0 & \mathbb{T} \end{bmatrix}$ by four permutations, where $\mathbb{T} := \begin{bmatrix} \mathbb{M}_{\mathrm{u}\,11} & \mathbb{Q}_1 \\ \mathbb{N}_{\mathrm{u}\,21} & \mathbb{Q}_2 \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$. Therefore, $\begin{bmatrix} \mathbb{M}_{\mathrm{u}} & \mathbb{Q}_{\mathrm{DF1}} \\ \mathbb{N}_{\mathrm{u}} & \mathbb{Q}_{\mathrm{DF2}} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$; analogously, $\begin{bmatrix} \widetilde{\mathbb{Q}}_{\mathrm{DF2}} & -\widetilde{\mathbb{Q}}_{\mathrm{DF1}} \\ -\widetilde{\mathbb{N}}_{\mathrm{y}} & \widetilde{\mathbb{M}}_{\mathrm{y}} \end{bmatrix} \in \mathcal{G}\mathrm{TIC}$.

The claims on strong joint stabilizability and detectability follow as in the proof of Theorem 6.6.28. $\square$

We use the rest of this section to study the connection between the CAREs and the factorizations of Section 12.3. We mainly just sketch the proofs (and some of the statements), since we do not use these results elsewhere.

Lemma 12.5.24 ($\mathcal{P}_\mathrm{X}, \mathcal{P}_\mathrm{Y}$ directly) *Assume Hypothesis 12.5.1. The IAREs corresponding to (1.') and (2.') have solutions $\mathcal{P}_\mathrm{X}$ and $\mathcal{P}_\mathrm{Y}$, respectively, iff (Factor1X) and (Factor2Y) hold, where the latter is given by*

(Factor2Y) There is $\mathbb{Y} \in \mathcal{G}\mathrm{TIC}(Y \times Z)$, s.t. $\mathbb{Y}^* J_1 \mathbb{Y} = \begin{bmatrix} \widetilde{\mathbb{N}}^{\mathrm{d}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}^{\mathrm{d}}_{\mathrm{y}\,12} \\ 0 & I \end{bmatrix}^* J_\gamma \begin{bmatrix} \widetilde{\mathbb{N}}^{\mathrm{d}}_{\mathrm{y}\,22} & \widetilde{\mathbb{N}}^{\mathrm{d}}_{\mathrm{y}\,12} \\ 0 & I \end{bmatrix}$ and $\mathbb{Y}_{11} \in \mathcal{G}\mathrm{TIC}(Y)$.

Assume (Factor1X) and (Factor2Y). Set $\mathbb{N} := \mathbb{N}_{\mathrm{u}}\mathbb{X}^{-1}$, $\widetilde{\mathbb{N}} := \mathbb{Y}_{\mathrm{d}}^{-1}\widetilde{\mathbb{N}_{\mathrm{y}}}$. Then

$$\mathcal{P}_\mathrm{X} = \mathbb{C}_{\flat\,1}^* (I - \mathbb{N}_{11}\pi_+\mathbb{N}_{11}^* + \mathbb{N}_{12}\pi_+\mathbb{N}_{12}^*)\,\mathbb{C}_{\flat\,1}, \tag{12.114}$$

$$\mathcal{P}_\mathrm{Y} = \mathbb{C}_{\mathrm{Y}\flat\,1}^* \left(I - \widetilde{\mathbb{N}}^{\mathrm{d}}_{22}\pi_+(\widetilde{\mathbb{N}}^{\mathrm{d}}_{22})^* + \widetilde{\mathbb{N}}^{\mathrm{d}}_{12}\pi_+(\widetilde{\mathbb{N}}^{\mathrm{d}}_{12})^*\right)\mathbb{C}_{\mathrm{Y}\flat\,1}, \tag{12.115}$$

where $\mathbb{C}_{\flat\,1} = \mathbb{C}_1 + \mathbb{N}_{\mathrm{u}\,11}\mathbb{K}_{\mathrm{u}\,1}$, $\mathbb{C}_{\mathrm{Y}\flat\,1} = \mathbb{B}_2^{\mathrm{d}} + \widetilde{\mathbb{N}_{\mathrm{y}}}^{\mathrm{d}}_{22}\mathbb{H}_{\mathrm{y}}{}^{\mathrm{d}}_{2}$.

See the remarks below Lemma 12.1.12 for (1.') and (2.'); by Lemma 12.5.12 (and Lemma 12.5.2 and duality), they are equivalent to (1.) and (2.) if Σ is exponentially stabilizable (but we need some regularity additions to get CAREs in place of IAREs).

Note that (Factor2Y) is not equivalent to (Factor2Z) but to the analogy of (Factor1X) for $\mathbb{D}_\mathrm{d}$.

Proof of Lemma 12.5.24: 1° *The equivalence:* By using Lemma 9.12.3, one can verify that the solutions of the IARE "(1.')" correspond 1-1 to the r.c.-stabilizing solutions of the IARE for $\Sigma_{\mathrm{X}\flat}$ and J_γ (see (12.88)); equivalently, to the solutions of (Factor1X) (of Theorem 12.3.7). By (9.141), we have

$$\mathcal{P}_\mathrm{X} = \begin{bmatrix} \mathbb{C}_{\flat\,1} \\ 0 \end{bmatrix}^* (J_\gamma - J_\gamma\mathbb{N}_X J_1^{-1}\pi_+\mathbb{N}_X^* J_\gamma) \begin{bmatrix} \mathbb{C}_{\flat\,1} \\ 0 \end{bmatrix}, \tag{12.116}$$

where $\mathbb{N}_X := \begin{bmatrix} (\mathbb{N}_u)_{11} & (\mathbb{N}_u)_{12} \\ 0 & I \end{bmatrix} \mathbb{X}^{-1}$.

The correspondence between (2.') and (Factor2Y) is analogous.

2° *(12.114) & (12.115):* We have $\mathbb{N}_X = \begin{bmatrix} \mathbb{N}_{11} & \mathbb{N}_{12} \\ * & * \end{bmatrix}$, where $\mathbb{N} := \mathbb{N}_u\mathbb{X}^{-1}$, hence

$$[\mathbb{N}_X J_1^{-1}\pi_+\mathbb{N}_X]_{11} = \mathbb{N}_{11}\pi_+\mathbb{N}_{11}^* - \mathbb{N}_{12}\pi_+\mathbb{N}_{12}^*. \tag{12.117}$$

By (12.116) and (12.89), $\mathcal{P}_X = \mathbb{C}_{\flat 1}^* \left[I - \mathbb{N}_X J_1^{-1}\pi_+\mathbb{N}_X^*\right]_{11} \mathbb{C}_{\flat 1}$. Combine this with (12.117) to obtain (12.114). Equation (12.115) is obtained analogously.

3° *A remark on* $\mathbb{P} := \mathbb{C}_{\flat 1}\mathbb{C}_{Y\flat}{}_1^*\boldsymbol{Я}$*:* Using Lemma 12.5.11 and the property "$\pi_+\mathbb{D}\pi_- = \mathbb{C}\mathbb{B}$" of Definition 6.1.1 applied to whole Σ_{Total}, we get

$$\begin{aligned}
\mathbb{P} &= (\mathbb{C}_1 + \mathbb{N}_{u\,11}\mathbb{K}_{u\,1})(\mathbb{B}_2 + \mathbb{H}_{y\,2}\widetilde{\mathbb{N}_y}_{22}) && (12.118)\\
&= \pi_+\mathbb{D}_{12}\pi_- + \mathbb{N}_{u\,11}\pi_+\mathbb{F}_{u\,12}\pi_- + \pi_+\mathbb{G}_{y\,12}\pi_-\widetilde{\mathbb{N}_y}_{22} + \mathbb{N}_{u\,11}\pi_+\mathbb{E}_{12}\pi_-\widetilde{\mathbb{N}_y}_{22} && (12.119)\\
&= \pi_+\mathbb{D}_{12}\pi_- + \mathbb{N}_{u\,11}\pi_+\mathbb{M}_u{}_{11}^{-1}\mathbb{M}_{u\,12}\pi_- + \pi_+\mathbb{N}_{u\,11}\pi_-\mathbb{M}_u{}_{11}^{-1}\mathbb{Q}_1\pi_-\widetilde{\mathbb{N}_y}_{22} && (12.120)
\end{aligned}$$

where the last identity uses equations (12.113) and $\mathbb{D}_{11} = \mathbb{N}_{u\,11}\mathbb{M}_u{}_{11}^{-1}$.

(The above components are in $\mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R}_-;W),\mathrm{L}^2_\omega(\mathbf{R}_+;Y))$, where $\omega \in \mathbf{R}$ is s.t. $\Sigma_{\text{Total}} \in \text{WPLS}_\omega$; their sum is stable (since so is the left-hand-side $\mathbb{P} := \mathbb{C}_{\flat 1}\mathbb{C}_{Y\flat}{}_1^*\boldsymbol{Я} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_-;W),\mathrm{L}^2(\mathbf{R}_+;Y)))$. Note this formula can be reduced to $\pi_+\mathbb{D}_{12}\pi_-$ if Σ is stable.) □

In, e.g., Theorem 12.1.4, we given necessary and sufficient conditions for the standard H^∞ 4BP in terms of the original system "(1.)–(3.)", whereas Theorem 12.1.8 uses both the original system and a perturbed system (condition "(4.')").

In Theorem 12.3.6, we have given analogous necessary and sufficient conditions ("(Factor1)" and "(Factor3") in terms of the original and a perturbed system for the corresponding frequency-domain problem (I/O map problem). The following remark contains the analogy of "(1.)–(3.)" for this problem; note that we again get sufficiency only for the exponential problem (see Remark 12.6.9 for the simpler discrete-time counterpart of this remark):

Remark 12.5.25 ($\rho(XY) < \gamma^2$: I/O formulation) *Assume Hypothesis 12.3.1, so that the d.c.f. (12.111) exists (we need its map $\mathbb{Q}_1 \in \mathrm{TIC}(Y,U)$ below for $\mathbb{P}$). Assume also that $\mathbb{N}_u,\mathbb{M}_u,\widetilde{\mathbb{N}_y},\widetilde{\mathbb{M}_y} \in \widetilde{\mathcal{A}}$. Then (i) implies (iii) (see below). Here*

$$\widetilde{\mathbb{O}} := \mathbb{P}^*\left(I - \mathbb{N}_{11}\pi_+\mathbb{N}_{11}^* + \mathbb{N}_{12}\pi_+\mathbb{N}_{12}^*\right)\mathbb{P}\left(I - \widetilde{\mathbb{N}}_{22}^*\pi_-\widetilde{\mathbb{N}}_{22} + \widetilde{\mathbb{N}}_{12}^*\pi_-\widetilde{\mathbb{N}}_{12}\right), \tag{12.121}$$

$\mathbb{P}$ is given by (12.120), $\mathbb{N} := \mathbb{N}\mathbb{X}^{-1}$ and $\widetilde{\mathbb{N}} := \mathbb{Y}_d^{-1}\widetilde{\mathbb{N}_y}$.

Assume, in addition, that we have exponential coprimeness in Hypothesis 12.3.1 (see Proposition 12.4.10), still with $\mathbb{N}_u,\mathbb{M}_u,\widetilde{\mathbb{N}_y},\widetilde{\mathbb{M}_y} \in \widetilde{\mathcal{A}}$, and that $\widetilde{\mathcal{A}}$ satisfied Hypothesis 8.4.8; assume that (12.111) is chosen accordingly. Then the following are equivalent:

(i) there is a suboptimal stabilizing controller for $\mathbb{D}$ (i.e., (4BP1) holds);

(ii) there is a suboptimal exponentially stabilizing controller for $\mathbb{D}$;

(iii) (Factor1X) and (Factor2Y) hold and $\rho(\widetilde{\mathbb{O}}) < \gamma$.

If (4BP1) holds and $D_{21} = 0$*, then each suboptimal stabilizing controller for* $\mathbb{D}$ *is equivalent to a well-posed one.*

We do not know whether (iii) implies (i) in the non-exponential case; the reasons for this are explained at the end of the proof of Lemma 12.1.12.

Proof: Note that now (4BP1)–(4BP3) are equivalent, by Lemma 12.3.10.

1° *"(i)⇒(iii)":* Assume (4BP1), so that (4BP3) and hence (Factor1X) holds. Then (4BP1) holds for $\mathbb{D}_{\mathrm{d}}$ too, by Proposition 7.3.4(d), hence so does (4BP3) for $\mathbb{D}_{\mathrm{d}}$, hence so does (Factor2Y) (since it equals "(Factor1X)" for $\mathbb{D}_{\mathrm{d}}$).

By Lemma 12.5.24, it follows that (1.') and (2.') have solutions (where Σ is chosen as in Lemma 12.5.23, so that Hypothesis 12.5.1 is satisfied). By Theorem 12.3.5(b), there is a suboptimal stabilizing DPF-controller for Σ, hence (3.) holds, by Lemma 12.1.12, i.e., $\rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma$. By (12.114), (12.115), (12.118) and Lemma A.3.3(s2), $\rho(\widetilde{\mathbb{O}}) = \rho(\mathcal{P}_X\mathcal{P}_Y)$.

2° *"(ii)⇒(i)":* This is trivial.

3° *"(iii)⇒(ii)":* Make now the additional assumptions in the remark. Since (12.111) was chosen to be in $\mathcal{G}\mathrm{TIC}_{\mathrm{exp}}$, we can use shifted Lemma 12.5.23 to satisfy Hypothesis 12.5.1 with $\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right]$ being exponentially stabilizing (cf. Lemma 12.5.2).

Assume (iii). By Lemma 12.5.24, it follows that (1.') and (2.') have solutions, hence so do (1.) and (2.). As in 1°, we observe that also (3.) holds, hence there is a suboptimal exponentially stabilizing controller for Σ by Theorem 12.1.11 (by Theorem 12.1.8, all such controllers are equivalent to well-posed controllers if $D_{21} = 0$). The I/O map of this controller is a suboptimal exponentially stabilizing controller for $\mathbb{D}$, hence (4BP1) holds.

Remark: In the discrete-time version of the remark, "still with $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{N}_{\mathrm{y}}}, \widetilde{\mathbb{M}_{\mathrm{y}}} \in \widetilde{\mathcal{A}}$, and that $\widetilde{\mathcal{A}}$ satisfied Hypothesis 8.4.8" becomes superfluous, since then Theorem 12.1.11 can be replaced by Theorem 12.2.1 and hence Hypothesis 8.4.8 is not required, so that $\widetilde{\mathcal{A}} = \mathrm{tic}_{\mathrm{exp}}$ becomes applicable (and we automatically have $\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{N}_{\mathrm{y}}}, \widetilde{\mathbb{M}_{\mathrm{y}}} \in \mathrm{tic}_{\mathrm{exp}}$ under this exponential coprimeness assumption). □

(See the notes on p. 706.)

12.6 Proofs for Section 12.2 — 4bp $\mathcal{P}_X, \mathcal{P}_Y, \mathcal{P}_Z$

Labor omnia vicit improbus
— Vergil (70–19 B.C.)

Recall Standing Hypothesis 12.1.1. Recall also that when referring to continuous time theory (as above), we assume that substitutions (13.63) are made.

The main result of this section is Lemma 12.6.4, which generalizes the "(1.)–(3.) iff (1.) and (4.)" proof of [IOW] to our generality, and which is needed also for the continuous-time proofs for Section 12.1. We start with a few results that define required symbols and show the correspondence between them and the symbols of [IOW]. At the end of the section, there are some results that clarify certain important properties of the (discrete-time) H^∞ 4bp.

The DARE (12.34) will be called the $\mathcal{P}_\mathrm{X}$*-DARE*, and the DARE (12.35) will be called the $\mathcal{P}_\mathrm{Y}$*-DARE*. In this section, we shall establish the connection between these two DAREs and a third one, $\mathcal{P}_\mathrm{Z}$*-DARE*, that will be defined below:

Lemma 12.6.1 ($\mathcal{P}_\mathrm{Z}$-DARE) *Assume that the $\mathcal{P}_\mathrm{X}$-DARE (12.34) has an internally P-stabilizing solution $(\mathcal{P}_\mathrm{X}, S_X, K_I)$ s.t. $S_{X11} \gg 0$ and $S_{X22} - S_{X12}^* S_{X11} S_{X12} \ll 0$.*

Then the corresponding IARE has another internally P-stabilizing solution $(\mathcal{P}_\mathrm{X}, J_1, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ s.t. $X_{21} = 0$, where $\mathbb{X} := I - \mathbb{F} \in \mathrm{tic}_\infty(U \times W)$.

Fix such a solution and set $\mathbb{M} := \mathbb{X}^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$. Then $X_{11}, X_{22} \in \mathcal{GB}$ and

$$K = XK_I = \begin{bmatrix} X_{11}K_{I1} + X_{12}K_{I2} \\ X_{22}K_{I2} \end{bmatrix}, \quad K_I = \begin{bmatrix} X_{11}^{-1}K_1 - X_{11}^{-1}X_{12}X_{22}^{-1}K_2 \\ X_{22}^{-1}K_2 \end{bmatrix}. \tag{12.122}$$

Moreover, Lemma 12.5.16 applies (even under these weaker assumptions; if Hypothesis 12.5.13 is satisfied, then its triple $(\mathcal{P}_\mathrm{X}, J_1, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ has the above properties), and the generating operators of Σ_Z defined by are given by

$$\left[\begin{array}{c|c} A_Z & B_Z \\ \hline C_Z & D_Z \end{array}\right] := \left[\begin{array}{c|cc} A^* + K_{I2}^* B_2^* & C_2^* + K_{I2}^* D_{22}^* & -K_{I1}^* X_{11}^* \\ \hline X_{22}^{-*} B_2^* & X_{22}^{-*} D_{22}^* & X_{22}^{-*} X_{12}^* \\ 0 & 0 & I \end{array}\right]. \tag{12.123}$$

*We define the $\mathcal{P}_\mathrm{Z}$-*DARE *as the DARE for Σ_Z and J_1. We note that (the symbols on the left ocrrespond to the notation of [IOW], pp. 307–; this will be explained later)*

$$Q_\times := C_Z^* J_\gamma C_Z = B_2 X_{22}^{-1} X_{22}^{-*} B_2^*, \tag{12.124}$$

$$\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] L_\times := D_Z^* J_\gamma C_Z = \begin{bmatrix} D_{22} \\ X_{12} \end{bmatrix} X_{22}^{-1} X_{22}^{-*} B_2^*, \tag{12.125}$$

$$\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] R_\times \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] := D_Z^* J_\gamma D_Z = \begin{bmatrix} D_{22} \\ X_{12} \end{bmatrix} X_{22}^{-1} X_{22}^{-*} \begin{bmatrix} D_{22}^* & X_{12}^* \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix}. \tag{12.126}$$

Note from Lemma 12.5.12 that here $\mathbb{X} = I - \mathbb{F} = \mathbb{M}^{-1}$ corresponds to $\mathcal{P}_\mathrm{X}$ (i.e., to (Factor1)), not to (Factor1X). Note also that (12.123) corresponds to $\left[\begin{array}{c|c} K & I - X \end{array}\right]$ (more exactly, to the state feedback pair $\left[\begin{smallmatrix} 0 \\ K_2 \end{smallmatrix} \middle| \begin{smallmatrix} 0 \\ -X_{21} \end{smallmatrix} \; \begin{smallmatrix} 0 \\ I - X_{22} \end{smallmatrix}\right]$ for (extended) Σ), where $X := \widehat{\mathbb{X}}(+\infty)$, not to K_I, although we have written it in terms of K_I and X, not in terms of K and X.

Proof: By Lemma 11.3.13(i)&(iii), there is $X \in \mathcal{GB}(U \times W)$ s.t. $X^* J_1 X = S_X$, $X_{21} = 0$. By Theorem 9.8.12(s1), $(\mathcal{P}_X, J_1, \begin{pmatrix} XK_I \mid I - X \end{pmatrix})$ is also a solution of the corresponding IARE with same stabilizability properties (note that $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$ is not unique but all possible choices are obtained parameterizes by $X \in \mathcal{B}(U \times W)$ s.t. $X_{21} = 0$ and $X^* J_1 X = S_X$).

Conversely, any internally P-stabilizing solution of the IARE with $X_{21} = 0$ is as above, by Theorem 9.8.12(b)&(s1), hence $X_{11}, X_{22} \in \mathcal{GB}$, by Lemma 11.3.13(b3). Therefore, (12.122) holds. The rest is straightforward. □

Next we make another remark on the correspondence of our notation and that of [IOW]:

Lemma 12.6.2 ($\mathcal{P}_X, \mathcal{P}_Y$) *For $\mathcal{P}_X$ and $\mathcal{P}_Y$ Riccati equations, we note that Σ_X satisfies*

$$Q_c := C_X^* J_\gamma C_X = C_1^* C_1, \tag{12.127}$$

$$\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] L_c^* := D_X^* J_\gamma C_X = \begin{bmatrix} D_{11}^* C_1 \\ D_{12}^* C_1 \end{bmatrix}, \tag{12.128}$$

$$\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] R_c \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] := D_X^* J_\gamma D_X = \begin{bmatrix} D_{11}^* D_{11} & D_{11}^* D_{12} \\ D_{12}^* D_{11} & D_{12}^* D_{12} - \gamma^2 I \end{bmatrix} \tag{12.129}$$

(here the first terms on each line refer to symbols of [IOW], p. 307–, to which we will refer later), and Σ_Y^{d} satisfies

$$Q_o := C_{Y^{\mathrm{d}}}^* J_\gamma C_{Y^{\mathrm{d}}} = B_2 B_2^*, \tag{12.130}$$

$$\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] L_o^* := D_{Y^{\mathrm{d}}}^* J_\gamma C_{Y^{\mathrm{d}}} = \begin{bmatrix} D_{22} B_2^* \\ D_{12} B_2^* \end{bmatrix}, \tag{12.131}$$

$$\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] R_o \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] := D_{Y^{\mathrm{d}}}^* J_\gamma D_{Y^{\mathrm{d}}} = \begin{bmatrix} D_{22} D_{22}^* & D_{22} D_{12}^* \\ D_{12} D_{22}^* & D_{12} D_{12}^* - \gamma^2 I \end{bmatrix}. \tag{12.132}$$

Let X be as in Lemma 12.6.1, so that $X_{21} = 0$. Then, $S_X K_I = X^* J_\gamma X K_I = X^* J_\gamma K$, hence $K = J_\gamma^{-1} X^{-*}(S_X K_I)$, hence

$$K = -J_\gamma^{-1} X^{-*}(-S_X K_I) = \begin{bmatrix} -X_{11}^{-*} & 0 \\ -\gamma^{-2} X_{22}^{-*} X_{12}^* X_{11}^{-*} & \gamma^{-2} X_{22}^{-*} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, \tag{12.133}$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{11}^* C_1 + B_1^* \mathcal{P}_X A \\ D_{12}^* C_1 + B_2^* \mathcal{P}_X A \end{bmatrix} \tag{12.134}$$

$$K_1 = -X_{11}^{-*}(D_{11}^* C_1 + B_1^* \mathcal{P}_X A), \quad K_2 = \gamma^{-2} X_{22}^{-*}(T_2 - X_{12}^* X_{11}^{-*} T_1) \tag{12.135}$$

□

(This is obvious. Note that we have $T_2^* = M_1$, $T_1^* = M_2$ in IOW-notation.)

We shall also need the following technical result:

Lemma 12.6.3 *Here (exceptionally) X, Y, Z refer to elements of $\mathcal{B}(H)$.*

(a) Let $\rho(XY) < 1$ and $X, Y \geq 0$. Then $I - XY \in \mathcal{GB}(H)$, $Z := Y(I - XY)^{-1} \geq 0$.

(b) Let $X, Z \geq 0$. Then $I + XZ \in \mathcal{GB}(H)$, $Y := Z(I + XZ)^{-1} \geq 0$, $\rho(XY) < 1$.

(c) If the assumptions of (a) or (b) hold, then the assumptions of both (a) and (b) hold, $\sigma(XY) \subset [0,1)$, $I+XZ = (I-XY)^{-1}$, $Y = Z(I+XZ)^{-1}$, *and* $Z = Y(I-XY)^{-1}$.

Proof: (a) Let $\rho(XY) < 1$. Then $1 \notin \sigma(XY)$, hence $I - XY \in \mathcal{GB}(H)$. Set $Z := Y(I-XY)^{-1}$, $W := I - Y^{1/2}XY^{1/2}$, so that $W = W^*$. Because $\rho(Y^{1/2}XY^{1/2}) = \rho(XY) < 1$, by Lemma A.3.3(s2), we have $\sigma(W) \subset 1 - (-1,1) = (0,2)$, in particular, $W \gg 0$. Therefore,

$$\langle Z(I-XY)x, (I-XY)x\rangle = \langle Yx, (I-XY)x\rangle = \langle Y^{1/2}x, WY^{1/2}x\rangle \geq 0 \quad \text{for all } x \in H. \tag{12.136}$$

Consequently, $Z \geq 0$. Moreover, $I + XZ = (I - XY + XY)(I-XY)^{-1} = (I - XY)^{-1}$, and $Z - ZXY = Y$ implies that $Y = Z(I+XZ)^{-1}$.

(b) By Lemma A.3.3(s2)&(s3), $\sigma(XZ) \cup \{0\} = \sigma(T) \cup \{0\}$, where $T := Z^{1/2}XZ^{1/2}$. But $T \geq 0$, hence $\sigma(XZ) \subset \mathbf{R}_+$, hence $\sigma(I+XZ) \subset [1,\infty)$; in particular, $I + XZ \in \mathcal{GB}(H)$. Moreover,

$$\langle Y(I+XZ)x, (I+XZ)x\rangle = \langle Zx, (I+XZ)x\rangle = \langle x, (Z+ZXZ)x\rangle \geq 0, \tag{12.137}$$

for all $x \in H$, i.e., for all $(I+XZ)x \in H$, hence $Y \geq 0$.

Furthermore, $\sigma(XY) = \sigma(I - (I+XZ)^{-1}) = 1 - 1/\sigma(I+XZ) \subset 1 - (0,1] = [0,1)$, hence $\rho(XY) < 1$. The final two equations are obtained from, e.g., $Y + YXZ = Z$.

(c) See the proofs of (b) and (a). □

Next we establish the connection between the $\mathcal{P}_\mathrm{Y}$-DARE and the $\mathcal{P}_\mathrm{Z}$-DARE, i.e., we show that "(1.) and (4.)" hold iff "(1.)–(3.)" hold (see Section 12.2):

Lemma 12.6.4 ($\mathcal{P}_\mathrm{Z} \Leftrightarrow \mathcal{P}_\mathrm{Y}$ & $\rho(\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Y}) < \gamma^2$) *Let the $\mathcal{P}_\mathrm{X}$-DARE have an internally P-stabilizing solution $(\mathcal{P}_\mathrm{X}, S_X, K_{X,I})$ s.t. $\mathcal{P}_\mathrm{X} \geq 0$, $S_{X11} \gg 0$ and $S_{X22} - S_{X21}S_{X11}^{-1}S_{X12} \ll 0$. Then the following are equivalent:*

(i) the $\mathcal{P}_\mathrm{Z}$-eDARE has a solution $(\mathcal{P}_\mathrm{Z}, S_Z, K_Z)$ s.t. $\mathcal{P}_\mathrm{Z} \geq 0$;

(ii) the $\mathcal{P}_\mathrm{Y}$-eDARE has a solution $(\mathcal{P}_\mathrm{Y}, S_Y, K_Y)$ s.t. $\mathcal{P}_\mathrm{Y} \geq 0$ and $\rho(\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Y}) < \gamma^2$.

Assume that (i) or (ii) (hence both) holds. Let $(\mathcal{P}_\mathrm{Z}, S_Z, K_Z)$ and $(\mathcal{P}_\mathrm{Y}, S_Y, K_Y)$ be the corresponding solutions (i.e., one is given and the other is the one constructed in the proof). Then the following hold:

(a) We have

$$\mathcal{P}_\mathrm{Z} = \mathcal{P}_\mathrm{Y}(\gamma^2 I - \mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Y})^{-1}, \qquad \mathcal{P}_\mathrm{Y} = \gamma^2\mathcal{P}_\mathrm{Z}(I + \mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Z})^{-1}, \tag{12.138}$$

$$I + \mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Z} = (I - \gamma^{-2}\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Y})^{-1}, \qquad A_{\circlearrowleft,\mathrm{Y}} = (I + \mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Z})A_{\circlearrowleft,\mathrm{Z}}(I + \mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Z})^{-1}, \tag{12.139}$$

where $A_{\circlearrowleft,\mathrm{Y}}$ and $A_{\circlearrowleft,\mathrm{Z}}$ are the corresponding closed-loop semigroup generators.

(b) $\mathcal{P}_\mathrm{Z}$ is [strongly/exponentially] internally P-stabilizing iff $\mathcal{P}_\mathrm{Y}$ is.

(c) If $S_{Y11} \gg 0$, $S_{Y22} - S_{Y21}S_{Y11}^{-1}S_{Y12} \ll 0$, then $S_{Z11} \gg 0$ and $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$.

Recall that an internally P-stabilizing solution is unique, and an internally exponentially stabilizing solution is exponentially stabilizing (see Theorem 14.1.4(b) and Lemma 13.3.8).

The $\mathcal{P}_{\mathrm{X}}$-DARE, $\mathcal{P}_{\mathrm{Y}}$-DARE and $\mathcal{P}_{\mathrm{Z}}$-DARE, i.e., the DAREs for $(\Sigma_{\mathrm{X}}, J_\gamma)$, $(\Sigma_{\mathrm{Y}}, J_\gamma)$ and $(\Sigma_{\mathrm{Z}}, J_1)$ are given in (12.34), (12.35) and (12.36), respectively. See (12.84), (12.85) and (12.94) for corresponding systems and (12.123) for the generators A_Z, B_Z, C_Z and D_Z.

Proof of Lemma 12.6.4: (As before, the eDAREs refer to corresponding DAREs without the requirement $S_* \in \mathcal{GB}$. Note also that we have made no coercivity assumptions on Σ (e.g., (12.32)–(12.33) (or "(A2)") need not hold a priori).)

We shall follow the (finite-dimensional, exponentially stabilizing) proof of [IOW] and extend it to our generality (in some parts of the proof we are forced to develope different methods due to infinite dimensions); the page numbers and "(10.*nnn*)"s below refer to IOW.

Let $(\mathcal{P}_{\mathrm{X}}, S_X, K_{X,I})$ be the internally P-stabilizing solution of the $\mathcal{P}_{\mathrm{X}}$-DARE. Fix some X of the form of Lemma 12.6.1, so that $X_{21} = 0$ (and $S_X = X^* J_1 X$). Set $K := X K_{X,I}$, so that $(\mathcal{P}_{\mathrm{X}}, J_1, \begin{pmatrix} K & \big| & I - X \end{pmatrix})$ is also internally P-stabilizing as in Lemma 12.6.1 (with the same $\mathbb{A}_{\circlearrowleft}$, $\mathbb{C}_{\circlearrowleft}$, $\mathbb{K}_{\circlearrowleft}$ as for $\begin{pmatrix} K_{X,I} & \big| & 0 \end{pmatrix}$).

We have the following correspondence of the [IOW]-notation (here on the left-hand-side) and ours

$$()^T = ()^*,\ \text{“} > 0\text{“} = '' \gg 0'',\ D = D\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right],\ B = B\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right],\ J = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] J_1 \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right], \tag{12.140}$$

$$V_c = \begin{bmatrix} V_{c11} & 0 \\ V_{c21} & V_{c22} \end{bmatrix} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] X \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right],\ W_c = -\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] K,\ F = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] K_{X,I} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] X^{-1} K, \tag{12.141}$$

$$R_\times = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] D_{Z^{\mathrm{d}}}^* J_1 D_{Z^{\mathrm{d}}} \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right],\ C_O = B_{Z^{\mathrm{d}}}^*,\ X = \mathcal{P}_{\mathrm{X}},\ Y = \mathcal{P}_{\mathrm{Y}},\ Z = \mathcal{P}_{\mathrm{Z}} \tag{12.142}$$

(note that $\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]^*$). Throughout the rest of this proof, symbols in quotation marks always refer to [IOW]-notation, in particular, the indices 1 and 2 corresponding to U and W are interchanged (as above). It follows that the $\mathcal{P}_{\mathrm{X}}$-DARE is equal to (note that $-SK_{X,I} = -X^* J_1 X K_{X,I} = -X^* J_1 K = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] V_c^* J W_c$) the equation system

$$W^* J W = W_{c2}^* W_{c2} - W_{c1}^* W_{c1} = A^* \mathcal{P}_{\mathrm{X}} A - \mathcal{P}_{\mathrm{X}} + C_1^* C_1 = \text{“}A^* X A - X + C_1^* C_1\text{“}, \tag{12.143}$$

$$V_c^* J V_c = \begin{bmatrix} V_{c21}^* V_{c21} - V_{c11}^* V_{c11} & V_{c21}^* V_{c22} \\ V_{c22}^* V_{c21} & V_{c22}^* V_{c22} \end{bmatrix} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] S_X \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \text{“} \begin{bmatrix} D_{11} D_{11}^* - I + B_1^* X B_1 & D_{11}^* D_{12} + B_1^* X B_2 \\ D_{12}^* D_{11} + B_2^* X B_1 & D_{12}^* D_{12} + B_2^* X B_2 \end{bmatrix} \text{“}, \tag{12.144}$$

$$W_c^* J V_c = \begin{bmatrix} W_{c2}^* V_{c21} - W_{c1}^* V_{c11} \\ W_{c2}^* V_{22} \end{bmatrix} = \text{“} \begin{bmatrix} D_{11}^* C_1 \\ D_{12}^* C_1 \end{bmatrix} + \begin{bmatrix} B_1^* X A \\ B_2^* X A \end{bmatrix} \text{“}. \tag{12.145}$$

Part I: case $\gamma = 1$:

1° *Symplectic pencils:* Note that the conditions (CD1), (CD2) and (CD4) on p. 308 hold, respectively, iff the $\mathcal{P}_{\mathrm{X}}$-DARE, $\mathcal{P}_{\mathrm{Y}}$-DARE and $\mathcal{P}_{\mathrm{Z}}$-DARE have exponentially stabilizing solutions with $S_{*11} \gg 0$ and $S_{*22} - S_{*21} S_{*11}^{-1} S_{*12} \ll 0$, by Lemma 11.1.7 (see also Lemma 12.6.2 and Lemma 12.6.1).

However, we start with only the assumption that the $\mathcal{P}_{\mathrm{X}}$-DARE has an internally P-stabilizing solution, and we shall obtain analogous results as in [IOW], by following their proof.

Let $S = X^*J_1X$ and $K_{X,I}$ correspond to $X_I = I$ (i.e., to the solution of the DARE), and let K correspond to X (i.e., $K = XK_{X,I}$). Then $V_c = RXR$ and $W_c = -RK$, where $R = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$ in the notation of [IOW].

Let (M_Y, N_Y) and (M_Z, N_Z) be the symplectic pencils (see Lemma 14.2.5) corresponding to (CD2) and (CD4), respectively, with third and fourth rows and columns of M_Z and N_Z interchanged, in particular,

$$M_Y = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & -A & 0 & 0 \\ 0 & -C_1 & 0 & 0 \\ 0 & -C_2 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_Z = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & -A - B_1 X_{22}^{-1} K_2 & 0 & 0 \\ 0 & -X_{12} X_{22}^{-1} K_2 + K_1 & 0 & 0 \\ 0 & -C_2 - D_{22} X_{22}^{-1} K_2 & 0 & 0 \end{bmatrix} \tag{12.146}$$

satisfy $M_Y \in \mathcal{B}(H \times H \times Z \times Y)$ and $M_Z \in \mathcal{B}(H \times H \times Y \times U)$.

By repeating the routine but loooong computations (do not try this at home) of pp. 315–317 of [IOW], we see that the extensions $M_Y', N_Y' \in \mathcal{B}(H \times H \times Z \times Y \times W \times U)$ of M_Y and N_Y, and $M_Z', N_Z' \in \mathcal{B}(H \times H \times U \times Y \times W \times Z)$ of M_Z and N_Z, and the operators U_a and W_a given there satisfy

$$U_a M_Z' W_a = M_Y', \quad U_a N_Z' W_a = N_Y', \tag{12.147}$$

and U_a and W_a are invertible (use Lemma A.1.1(b1)&(b2) and elementary row and column operations for this invertibility).

(Note: we must have $()^*$ in place of $()^T$. Note also the following misprints: the top row of the biggest matrix in (10.161) should be $\begin{bmatrix} -S_c & S_c V_{c21}^* V_{c22}^* \end{bmatrix}$, the W_2^T in (10.162) should be W_{c2}^T (i.e., W_{c2}^*), and "(10.165)" on line 5 of p. 317 should be "(10.164)".)

2° "$\rho(\mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}}) < 1$" *is necessary:* Let the $\mathcal{P}_{\mathrm{Z}}$-DARE have an internally P-stabilizing solution $\mathcal{P}_{\mathrm{Z}} \geq 0$. Then $I + \mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Z}} \in \mathcal{GB}(H)$, $\mathcal{P}_{\mathrm{Y}} := \mathcal{P}_{\mathrm{Z}}(I + \mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Z}})^{-1} \geq 0$ and $\rho(\mathcal{P}_{\mathrm{X}}\mathcal{P}_{\mathrm{Y}}) < 1$, by Lemma 12.6.3(a). Thus, we only have to show that $\mathcal{P}_{\mathrm{Y}}$ is the (unique) internally P-stabilizing solution of $\mathcal{P}_{\mathrm{Y}}$-eDARE.

Moreover, $N_{\mathrm{Z}} V_{\mathrm{Z}} = M_{\mathrm{Z}} V_{\mathrm{Z}} A_{\circlearrowleft,\mathrm{Z}}$ and hence $N_{\mathrm{Z}}' V_{\mathrm{Z}}' = M_{\mathrm{Z}}' V_{\mathrm{Z}}' A_{\circlearrowleft,\mathrm{Z}}$ (i.e., (10.166) and (10.169) hold), where

$$V_{\mathrm{Z}} = \begin{bmatrix} I \\ \mathcal{P}_{\mathrm{Z}} \\ K_{\mathrm{Z}} \end{bmatrix}, \; V_{\mathrm{Z}}' = \begin{bmatrix} V_{\mathrm{Z}} \\ 0 \\ -\Theta \mathcal{P}_{\mathrm{Z}} A_{\circlearrowleft,\mathrm{Z}} \end{bmatrix}, \tag{12.148}$$

Θ is as on p. 316, K_{Z} is the corresponding state feedback operator and $A_{\circlearrowleft,\mathrm{Z}}$ the corresponding closed-loop semigroup generator, by Lemma 14.2.5.

But (12.147) implies that $N_Y' W_a^{-1} V_{\mathrm{Z}}' = N_Y' W_a^{-1} V_{\mathrm{Z}}' A_{\circlearrowleft,\mathrm{Z}}$ (this is (10.170)), and

$$W_a = \begin{bmatrix} I & -\mathcal{P}_{\mathrm{X}} & 0 \\ 0 & I & 0 \\ * & * & * \end{bmatrix} \text{ implies that } W_a^{-1} = \begin{bmatrix} I & \mathcal{P}_{\mathrm{X}} & 0 \\ 0 & I & 0 \\ * & * & * \end{bmatrix}, \tag{12.149}$$

It follows that $W_a^{-1}V_Z' = \begin{bmatrix} I+\mathcal{P}_X\mathcal{P}_Z \\ \mathcal{P}_Z \\ * \end{bmatrix}$. Consequently,

$$N_Y'V_Y' = M_Y'V_Y'A_{\circlearrowleft,\mathrm{Y}}, \text{ where } V_Y' := W_a^{-1}V_Z'(I+\mathcal{P}_X\mathcal{P}_Z)^{-1} = \begin{bmatrix} I \\ \mathcal{P}_\mathrm{Y} \\ * \end{bmatrix} \tag{12.150}$$

and $A_{\circlearrowleft,\mathrm{Y}} = (I+\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Z})A_{\circlearrowleft,\mathrm{Z}}(I+\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Z})^{-1}$. The four top rows of $N_Y'V_Y' = M_Y'V_Y'A_{\circlearrowleft,\mathrm{Y}}$ are $N_YV_Y = M_YV_YA_{\circlearrowleft,\mathrm{Y}}$, where V_Y consists of the four top rows of V_Y'. By Lemma 14.2.5, $\mathcal{P}_\mathrm{Y}$ solves the $\mathcal{P}_\mathrm{Y}$-eDARE, and $A_{\circlearrowleft,\mathrm{Y}}$ is the corresponding closed-loop semigroup generator.

3° "$\rho(\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Y}) < 1$" *is sufficient:* (We go here 2° backwards.) Assume (ii). Then $\mathcal{P}_\mathrm{Z} := \mathcal{P}_\mathrm{Y}(I-\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Y})^{-1} \geq 0$, by Lemma 12.6.3(a), and $N_YV_Y = M_YV_YA_{\circlearrowleft,\mathrm{Y}}$ and hence $N_Y'V_Y' = M_Y'V_Y'A_{\circlearrowleft,\mathrm{Y}}$, where

$$V_Y = \begin{bmatrix} I \\ \mathcal{P}_\mathrm{Y} \\ K_\mathrm{Y} \end{bmatrix}, \; V_Y' = \begin{bmatrix} V_Y \\ X_aV_YA_{\circlearrowleft,\mathrm{Y}} \\ -X_bV_YA_{\circlearrowleft,\mathrm{Y}} \end{bmatrix}, \tag{12.151}$$

where X_a and X_b are the operators "X_1" and "X_2" of (10.158). By (12.149), we have $V_Z' := W_aV_Y'(I-\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Y})^{-1} = \begin{bmatrix} I \\ \mathcal{P}_\mathrm{Z} \\ * \end{bmatrix}$. But $N_Y'W_a^{-1}V_Z' = M_Y'W_a^{-1}V_Z'A_{\circlearrowleft,\mathrm{Z}}$, where $A_{\circlearrowleft,\mathrm{Z}} := (I-\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Y})A_{\circlearrowleft,\mathrm{Z}}(I-\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Y})^{-1}$, hence $N_Z'V_Z' = M_Z'V_Z'A_{\circlearrowleft,\mathrm{Z}}$, hence $N_ZV_Z = M_ZV_ZA_{\circlearrowleft,\mathrm{Z}}$, where $V_Z = \begin{bmatrix} I \\ \mathcal{P}_\mathrm{Z} \\ * \end{bmatrix}$ are the four top rows of V_Z'. Consequently, $\mathcal{P}_\mathrm{Z}$ is a solution of the $\mathcal{P}_\mathrm{Z}$-eDARE.

(a) Given (i) or (ii), we obtain the connecting formulae (12.138) and (12.139) from (2°, 3° and) Lemma 12.6.3.

(b) By (a) and Lemma A.4.2(h1), $A_{\circlearrowleft,\mathrm{Y}}$ is [strongly/exponentially] stable iff $A_{\circlearrowleft,\mathrm{Z}}$ is. Moreover, if $\mathcal{P}_\mathrm{Z}$ is internally P-stabilizing, so that $A_{\circlearrowleft,\mathrm{Z}}$ and $A_{\circlearrowleft,\mathrm{Y}}$ are bounded (by the above) and $\langle A_{\circlearrowleft,\mathrm{Z}}^n x_0, \mathcal{P}_\mathrm{Z}A_{\circlearrowleft,\mathrm{Z}}^n x_0\rangle = \|\mathcal{P}_\mathrm{Z}^{1/2}A_{\circlearrowleft,\mathrm{Z}}^n x_0\|_H^2 \to 0$, as $n \to +\infty$, for all $x_0 \in H$, then $\mathcal{P}_\mathrm{Z}A_{\circlearrowleft,\mathrm{Z}}^n x_0 = \mathcal{P}_\mathrm{Z}^{1/2}\mathcal{P}_\mathrm{Z}^{1/2}A_{\circlearrowleft,\mathrm{Z}}^n x_0 \to 0$, hence

$$\mathcal{P}_\mathrm{Y}A_{\circlearrowleft,\mathrm{Y}}^n x_0 = \mathcal{P}_\mathrm{Y}(I+\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Z})A_{\circlearrowleft,\mathrm{Z}}^n(I+\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Z})^{-1}x_0 = \mathcal{P}_\mathrm{Z}A_{\circlearrowleft,\mathrm{Z}}^n(I+\mathcal{P}_\mathrm{X}\mathcal{P}_\mathrm{Z})^{-1}x_0 \to 0, \tag{12.152}$$

for all $x_0 \in H$, hence also $\mathcal{P}_\mathrm{Y}$ is internally P-stabilizing (because $\{A_{\circlearrowleft,\mathrm{Y}}^n\}$ is bounded).

(c) We follow here closely pp. 321–326 of IOW. Now

$$\text{“}R_\times + C_OZC_O^*\text{''} = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right](D_{Z^\mathrm{d}}{}^*J_1D_{Z^\mathrm{d}} + B_{Z^\mathrm{d}}{}^*\mathcal{P}_\mathrm{Z}B_{Z^\mathrm{d}})\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]S_Z\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right], \tag{12.153}$$

where $C_O := \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]B_{Z^\mathrm{d}}^* = \left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]\begin{bmatrix} -X_{11}K_{I1} \\ C_2+D_{22}K_{I2} \end{bmatrix}$. We divide the proof in parts 1° and 2°.

1° $S_{Z11} \gg 0$*:* Make the definitions of p. 321. By p. 322, we have $\mathcal{V} \gg 0$ (use Lemma A.3.1(p2) for the final conclusion). Then

$$\text{''}\overline{D}_{22}\overline{D}_{22}^*\text{''} = D_{22}D_{22}^* + C_2\mathcal{P}_\mathrm{Y}C_2^* = S_{Y11} \gg 0. \tag{12.154}$$

Consequently, $S_{Z11} \gg 0$, because $S_{Z11} =\text{''}\, \overline{D}_{22}\mathcal{V}\overline{D}_{22}^*\text{''}$, by the computations on p. 322.

2° $S_{Z22} - S_{Z21}S_{Z11}^{-1}S_{Z12} \ll 0$, *equivalently (10.198) holds:* Now "$\overline{D}_{12}^*\overline{D}_{12} =$

$V_{c22}^T V_{c22}$", i.e.,

$$\text{“}\overline{D}_{12}^*\overline{D}_{12}'' = D_{11}^* D_{11} + B_1^* \mathcal{P}_X B_1 = (S_X)_{11} = X_{11}^* X_{11} \gg 0 \tag{12.155}$$

Because $Q :='' \overline{D}_{12} X_{11}^{-1}{}'' \in \mathcal{B}(U, Z\times H)$, so that $Q^*Q = I$, there is an unitary extension $U_c \in \mathcal{GB}(U\times V, Z\times H)$ of Q, by Lemma A.3.1(e3), where $V := \mathrm{Ran}(Q)^\perp$ is a closed subspace of $Z\times H$; thus, (10.200) holds. Similarly, $\overline{D}_{21}\overline{D}_{21}^* = S_{Y11} = Y_{11}^* Y_{11} \gg 0$, so that there is a unitary extension $U_o \in \mathcal{GB}(Y\times V', W\times H)$ of "$\overline{D}_{21}^T V_{o22}$"$= \overline{D}_{21}^* Y_{11}$, as in (10.201), where $V' = \mathrm{Ran}(\overline{D}_{21}^* Y_{11})^\perp \subset W\times H$. In (10.202) the partition is w.r.t. $\mathcal{B}(Y\times V', U\times V)$.

The rest is straightforward, hence (10.206)$\gg 0$ (i.e., $E \gg 0$ in (10.209), where $E \in \mathcal{GB}(Y\times V')$), hence (10.207)–(10.208) hold, by, e.g., Lemma A.3.1(d).

We arrive at the inequalities at the end of the proof. By Lemma A.3.3(s2), (10.208), and Lemma A.3.1(b1), the last inequality holds, hence (10.198)$=$ $-S_{Z22} + S_{Z21} S_{Z11}^{-1} S_{Z12} \gg 0$.

Part II: The general case ($\gamma > 0$): Apply Part I to $\mathrm{diag}(I, \gamma^{-1}I, I)\Sigma$, and then apply Lemma 12.6.5 (e.g., we have $\rho(\gamma^{-2}\mathcal{P}_X\mathcal{P}_Y) < 1 \Leftrightarrow \rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2$). □

For general $\gamma > 0$, we divide z by γ to reduce the 4bp for the case with $\gamma = 1$:

Lemma 12.6.5 ($\rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2 \neq 1$) *The 4bp for Σ and $\gamma > 0$ corresponds to the 4bp for $\mathrm{diag}(I, \gamma^{-1}I, I)\Sigma$ and 1 (i.e., we multiply $\left[\begin{array}{c|cc} \mathbb{C}_1 & \mathbb{D}_{11} & \mathbb{D}_{12}\end{array}\right]$ by γ^{-1}). Moreover, the solutions of the DAREs for the latter problem correspond to those for the original problem as follows (both solutions exist iff either exists; in the claims on $\mathcal{P}_Z$ we assume that the assumptions of Lemma 12.6.1 hold):*

$$(\mathcal{P}_{X\gamma}, S_{X\gamma}, K_{X\gamma}) = (\gamma^{-2}\mathcal{P}_X, \gamma^{-2}S_X, K_X), \tag{12.156}$$

$$\Sigma_{X\gamma\circlearrowleft} = \mathrm{diag}(I; \gamma^{-1}I, I, I, I)\cdot \Sigma_{X\circlearrowleft}; \tag{12.157}$$

$$(\mathcal{P}_{Y\gamma}, S_{Y\gamma}, K_{Y\gamma}) = (\mathcal{P}_Y, \begin{bmatrix} I & 0 \\ 0 & \gamma^{-1}I\end{bmatrix} S_Y \begin{bmatrix} I & 0 \\ 0 & \gamma^{-1}I\end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & \gamma I\end{bmatrix} K_Y), \tag{12.158}$$

$$\Sigma_{Y\gamma\circlearrowleft} = \mathrm{diag}(I; I, \gamma I, I, \gamma I)\cdot \Sigma_{Y\circlearrowleft}\cdot \mathrm{diag}(I; I, \gamma^{-1}I); \tag{12.159}$$

$$(\mathcal{P}_{Y\gamma}, S_{Z\gamma}, K_{Z\gamma}) = (\gamma^2\mathcal{P}_Z, \begin{bmatrix} \gamma I & 0 \\ 0 & I\end{bmatrix} S_Z \begin{bmatrix} \gamma I & 0 \\ 0 & I\end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & \gamma I\end{bmatrix} K_Z), \tag{12.160}$$

$$\Sigma_{Z\gamma\circlearrowleft} = \mathrm{diag}(I; \gamma I, \gamma I, I, \gamma I)\cdot \Sigma_{Z\circlearrowleft}\cdot \mathrm{diag}(I; I, \gamma^{-1}I); \tag{12.161}$$

in particular, $\rho(\mathcal{P}_X\mathcal{P}_Y) < \gamma^2 \Leftrightarrow \rho(\mathcal{P}_{X\gamma}\mathcal{P}_{Y\gamma}) < 1$, and the stabilizability of these solutions is invariant under this modification.

(This was used in the proof of Lemma 12.6.4.)

Proof: (In the lemma, $\Sigma_{X\gamma\circlearrowleft}$ is the closed-loop system corresponding to $\mathcal{P}_X$-DARE (hence to modified Σ_X and J_1); analogously for $\Sigma_{Y\gamma\circlearrowleft}$ and $\Sigma_{Z\gamma\circlearrowleft}$; in particular, the closed-loop semigroups are invariant.)

1° *$\mathcal{P}_X$-DARE and $\mathcal{P}_Y$-DARE:* (Note that here we have made no further assumptions than Standing Hypothesis 12.1.1.)

By writing the $\mathcal{P}_X$-DARE and the $\mathcal{P}_Y$-DARE out with substitutions $\Sigma \mapsto \mathrm{diag}(I, \gamma^{-1}I, I)\Sigma$ and $\gamma \mapsto 1$, one observes that the solutions of the modified and original $\mathcal{P}_X$-DARE and $\mathcal{P}_Y$-DARE correspond to each other through the above formulae.

2° *$\mathcal{P}_Z$-DARE:* The requirement that $X^*J_1X = S_X$ (which is implicit in Lemma 12.6.1) results in $\gamma^{-1}X$ in place of the original X in (12.123), which affects (the extended) Σ_Z as in (12.161), hence the claims on $\mathcal{P}_Z$-DARE can be verified in the same way as those on the $\mathcal{P}_X$-DARE.

3° The claims at the end of the lemma follow from the equations. □

We end this section by recording discrete-time counterparts of three important lemmas and a remark of Section 12.5.

First we note that Hypothesis 12.5.1 is weaker than standard H^∞ 4BP assumptions:

Lemma 12.6.6 ((A,B_1) & (A,C_2) ⇒ Hypothesis 12.5.1) *Assume that (A,B_1) is optimizable and (A,C_2) is estimatable.*

Then Hypothesis 12.5.1 is satisfied (even with "exponentially jointly" in place of "jointly") except possibly (12.78). Moreover, then condition (12.78) holds iff (12.32)–(12.33) are satisfied.

Proof: By Proposition 13.3.14 (and its proof), there are $K \in \mathcal{B}(H,U)$ and $\mathtt{H} \in \mathcal{B}(Y,H)$ s.t. $A + B_1K$ and $A + \mathtt{H}C_2$ are exponentially stable. Extend Σ by K and $\mathtt{H}$ (with $F = 0 = G = E$) to satisfy Hypothesis 12.5.1 with "exponentially jointly" in place of "jointly" (cf. the proof of Lemma 13.3.17(a)&(b)) except possibly (12.78).

But K is exponentially stabilizing for $\Sigma_{11} := \left(\begin{array}{c|c} A & B_1 \\ \hline C_1 & D_{11} \end{array}\right)$, with closed loop I/O map $\begin{bmatrix} \mathbb{N}_{u\,11} \\ \mathbb{M}_{u\,11} \end{bmatrix}$ (since $\mathbb{M}_u := (I - \mathbb{F}_u)^{-1} = \begin{bmatrix} (I-\mathbb{F}_{u\,11})^{-1} & -(I-\mathbb{F}_{u\,11})^{-1}\mathbb{F}_{u\,12} \\ 0 & I \end{bmatrix}$).

By Lemma 8.4.11(a2), $\mathbb{N}_{u\,11}^*\mathbb{N}_{u\,11} \gg 0$ iff $\mathbb{N}_{u\,11}$ is I-coercive; by Theorem 8.4.5(d), this is the case iff $\mathbb{D}_{11}$ is I-coercive over $\mathcal{U}_{\text{exp}}$ (w.r.t. system Σ_{11}); by Proposition 15.2.2(f1)&(i)&(ii), this is the case iff (12.32) holds.

By dual arguments, we obtain that $\widetilde{\mathbb{N}_y}_{22}\widetilde{\mathbb{N}_y}_{22}^* \gg 0$ iff (12.33) holds. □

Lemma 12.6.7 ($\mathcal{P}_X$ & $\mathcal{P}_Y$ ⇒ Hypothesis 12.5.1) *If conditions (1.) and (2.) of Theorem 12.2.1 are satisfied and (12.32)–(12.33) hold, then (A,B_1) is optimizable and (A,C_2) is estimatable.*

See Lemma 12.6.6 for more.

Proof: By Theorem 11.5.1(iii)&(i), (the ficp for Σ_X has a solution and) (A,B_1) is exponentially stabilizable. By dual arguments, (A,C_2) is exponentially detectable. □

Lemma 12.6.8 (Hypothesis 12.5.1 ⇒ (12.32)–(12.33)) *Hypothesis 12.5.1 is satisfied with $\left[\begin{array}{c|c} \mathbb{K}_u & \mathbb{F}_u \end{array}\right]$ and $\left[\begin{array}{c} \mathbb{H}_y \\ \hline \mathbb{G}_y \end{array}\right]$ being exponentially [jointly] stabilizing iff (A,B_1) is optimizable, (A,C_2) is estimatable and (12.32)–(12.33) are satisfied.*

Proof: "If": This follows from Lemmas 12.6.7 and 12.6.6.

"Only if" Assume that Hypothesis 12.5.1 holds and that $\left[\begin{array}{c|c} \mathbb{K}_\mathrm{u} & \mathbb{F}_\mathrm{u} \end{array}\right]$ is exponentially stabilizing. Then (A,B_1) is optimizable and (A,C_2) is estimatable, by Lemma 12.5.2. From the end of the proof of Lemma 12.6.6, we observe that (12.32) and (12.33) hold. □

In discrete-time, the necessary and sufficient conditions "(Factor1X) and (Factor2Z)" for the solvability of the frequency-domain H^∞ 4BP can be formulated in terms of original data without any regularity assumptions:

Remark 12.6.9 ($\rho(XY) < \gamma^2$: I/O formulation) *Assume that Hypothesis 12.3.1 is satisfied with exponential coprimeness (as in Proposition 12.4.10). Then (i)–(iii) of Remark 12.5.25 are equivalent.* □

(This was remarked at the end of the proof of Remark 12.5.25.)

However, also in discrete time, our result in the non-exponential case is onedirectional, as in Remark 12.5.25.

(See the notes on p. 711.)

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Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations

Volume 3/3 — Discrete-Time Theory & Appendices

Kalle Mikkola

$$
\begin{aligned}
x_{j+1} &= Ax_j + Bu_j\\
y_j &= Cx_j + Du_j
\end{aligned}
$$

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Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations

Volume 3/3 — Discrete-Time Theory & Appendices

Kalle Mikkola

Dissertation for the degree of Doctor of Science in Technology to be presented with due permission of the Department of Engineering Physics and Mathematics, for public examination and debate in Council Room at Helsinki University of Technology (Espoo, Finland) on the 18th of October, 2002, at 12 o'clock noon.

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

Kalle Mikkola: *Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations*; Helsinki University of Technology Institute of Mathematics Research Reports A452 (2002).

Abstract: *In this monograph, we solve rather general linear, infinite-dimensional, time-invariant control problems, including the* H^∞ *and LQR problems, in terms of algebraic Riccati equations and of spectral or coprime factorizations. We work in the class of (weakly regular) well-posed linear systems (WPLSs) in the sense of G. Weiss and D. Salamon.*
Moreover, we develop the required theories, also of independent interest, on WPLSs, time-invariant operators, transfer and boundary functions, factorizations and Riccati equations. Finally, we present the corresponding theories and results also for discrete-time systems.

AMS subject classifications: 42A45, 46E40, 46G12, 47A68, 49J27, 49N10, 49N35, 93-02, 93A10, 93B36, 93B52, 93C05, 93C55, 93D15

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Discrete-Time Maps and Systems (ti & wpls)

At any given moment, an arrow must be either where it is or where it is not. But obviously it cannot be where it is not. And if it is where it is, that is equivalent to saying that it is at rest.

— Zeno's (335–262 B.C.) paradox of the moving (still?) arrow

In this chapter, we present here briefly some facts on the discrete counterparts of WPLSs, which we call *discrete-time well-posed linear systems (wpls's)*. They are the systems governed by difference equations

$$\begin{cases} x_{j+1} = Ax_j + Bu_j, \\ \quad y_j = Cx_j + Du_j, \quad j \in \mathbf{Z}, \end{cases} \tag{13.1}$$

for $A, B, C, D \in \mathcal{B}$; see Definition 13.3.1 and Lemma 13.3.3 for definitions.

We show that almost all our continuous-time results have discrete-time analogies (see Theorem 13.3.13), and also many further results hold due to the boundedness of the generating operators (A, B, C, D). Roughly speaking, we write continuous-time results (and definitions) in lower case (e.g., $\mathrm{L}^2 \mapsto \ell^2$), as in (13.63).

In Section 13.1, we study bounded linear time-invariant maps $\ell^2_r(\mathbf{Z};U) \to \ell^2_r(\mathbf{Z};Y)$ ("$\mathrm{ti}_r(U,Y)$", where $\|u\|^2_{\ell^2_r(\mathbf{Z};U)} := \sum_k \|r^{-k}u_k\|^2_U$), and the corresponding transfer functions (this corresponds to Chapters 2 and 3). The Cayley transform is treated in Section 13.2.

In Section 13.3, we study wpls's (this corresponds to Chapter 6, also Chapters 4, 7 and 8 are treated in Theorem 13.3.13). The I/O maps of wpls's are exactly the causal maps in $\cup_{r>0}\mathrm{ti}_r$ (see (13.46)).

In Section 13.4, we show how to obtain wpls's from WPLSs, by discretization. This allows us to reduce several WPLS problems to wpls problems, which are often substantially simpler due to bounded input and output operators. (This differs from the Cayley transform of Lemma 13.2.1 and from the method of Lemma 13.1.4.)

Discrete-time Riccati equations (DAREs) and spectral factorization are treated in Chapter 14 (this corresponds to Chapters 9 and 5) and minimization problems in Chapter 15 (this corresponds to Chapter 10). Discrete-time H^∞ (and Nehari) problems are treated in Sections 11.5 and 12.2.

Also in this chapter, U, W, H, Y and Z denote Hilbert spaces of arbitrary dimensions and B denotes a Banach space.

13.1 Discrete-time I/O maps (tic)

The Priest's grey nimbus in a niche where he dressed discreetly. I will not sleep here tonight. Home also I cannot go. A voice, sweetened and sustained, called to him from the sea. Turning the curve he waved his hand. A sleek brown head, a seal's, far out on the water, round. Usurper.

— James Joyce (1882–1941), "Ulysses"

In this and the following section, we present results corresponding to Chapters 2 and 3; in particular, we extend the discrete-time Fourier multiplier and H$^\infty$ boundary function theorems to I/O maps over unseparable Hilbert spaces, in Lemmas 13.1.5 and 13.1.6. Our third main result is Lemma lticConvol(d), which treats time-invariant causal operators (tic$_{\text{loc}}$) that are "almost r-stable" (that map functions with finite support into ℓ^2_r). We also define ti and tic and treate their basic properties including adjoints, inverses, convolution forms and $\mathcal{Z}$-transforms. Further results are obtained through Theorem 13.3.13.

We start by presenting our notation. Let $S \subset \mathbf{Z}$, $p \in [1,\infty)$ and $r > 0$. Recall that $x : S \to B$ (equivalently, $x \in B^S$) means that x is a function from S to B, i.e., a B-valued sequence on S. We set $\|\{x_j\}_{j\in S}\|_{\ell^\infty_r(S;B)} := \sup_{j\in S}\|r^{-j}x_j\|_B$, $\ell^\infty := \ell^\infty_1$. We also define

$$\ell^p_r(S;B) := \{x : S \to B \,\big|\, \|x\|^p_{\ell^p_r} := \sum_{j\in S} \|r^{-j}x_j\|^p_B < \infty\} \tag{13.2}$$

and $\ell^p := \ell^p_1$. We have $\|x\|_{\ell^q_r} \le \|x\|_{\ell^p_r} \le \infty$ ($x : S \to B$, $\infty \ge q \ge p \ge 1$, $r > 0$) (proof: assume w.l.o.g. that $\|x\|_{\ell^p_r} = 1$, ...). For $S \subset \mathbf{N}$ we have (use Lemma B.3.13 for $p < q$)

$$\|x\|_{\ell^p_s(S;B)} \le M_{r/s,p,q}\|x\|_{\ell^q_r(S;B)} \quad (\infty > s > r > 0,\ p,q \in [1,\infty]). \tag{13.3}$$

By $\mathcal{BC}(U,Y)$ we denote compact linear operators $U \to Y$, and

$$\ell^1_{\mathcal{BC}}(S;\mathcal{B}(U,Y)) := \{T \in \ell^1(S;\mathcal{B}(U,Y)) \,\big|\, T_j \in \mathcal{BC}(U,Y) \text{ for all } j \ne 0\}, \tag{13.4}$$

$$\ell^1_\pm := \{T \in \ell^1 \,\big|\, T_j = 0 \text{ for all } \pm j < 0\}, \tag{13.5}$$

$$\ell^1_{\mathcal{BC},\pm} := \{T \in \ell^1_{\mathcal{BC}} \,\big|\, T_j = 0 \text{ for all } \pm j < 0\}. \tag{13.6}$$

Note that the vectors $e_k := \chi_{\{k\}}$ ($k \in S$) form the standard orthonormal base of the Hilbert space $\ell^2(S)$, and $\{x_\alpha e_k \,\big|\, \alpha \in \mathcal{A},\ k \in S\}$ is an orthonormal base for $\ell^2(S;H)$ whenever, $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ is an orthonormal base for H. Obviously,

$$\mathrm{c_c}(S;B) := \{(x_j)_{j\in S} \,\big|\, x_j = 0 \text{ for } j \text{ not in some finite subset of } S\} \tag{13.7}$$

is dense in $\ell^p_r(S;B)$. By Lemma B.4.15, $\ell^p_r(S;H)^* = \ell^q_{1/r}(S;H)$ when $1 \le p < \infty$ and $p^{-1} + q^{-1} = 1$.

The *convolution*

$$(a_j)_{j\in\mathbf{Z}} * (b_k)_{k\in\mathbf{Z}} := \Big(\sum_j a_j b_{n-j}\Big)_{n\in\mathbf{Z}} \tag{13.8}$$

is bilinear and bounded $\ell^1_r(\mathbf{Z};B)\times\ell^p_r(\mathbf{Z};B')\to\ell^p_r(\mathbf{Z};B'')$ whenever $B\times B'\to B''$ is bilinear and bounded (e.g., $B=\mathcal{B}(U,Y)$, $B'=\mathcal{B}(Y,Z)$ and $B''=\mathcal{B}(U,Z)$); see Lemma D.1.7. By the Fubini theorem, we have $(a*)^*=(\mathbf{Я}a^*)*$ in the sense that $\langle a*b,c\rangle=\langle b,(\mathbf{Я}a^*)*c\rangle$ when, e.g., $a\in\ell^1$, $b\in\ell^p$ and $c\in\ell^q$. We define the isometric isomorphism (multiplication operator) $r^{\cdot}\in\mathcal{B}(\ell^p_s,\ell^p_{rs})$ by

$$r^{\cdot}=((x_j)_{j\in S}\mapsto(r^jx_j)_{j\in S}) \tag{13.9}$$

Obviously, $(r^{\cdot}a)*b=r^{\cdot}(a*r^{-\cdot}b)$, hence (cf. Remark 13.3.9)

$$\mathcal{T}_r(a*):=r^{\cdot}(a*)r^{-\cdot}=(r^{\cdot}a)* \tag{13.10}$$

defines an isometric isomorphism $\mathcal{T}_r:\ell^1_s*\mapsto\ell^1_{rs}*$ (we identify ℓ^1_s* with ℓ^1_s as a Banach space). We identify $S\to B$ with $\{x:\mathbf{Z}\to B\,\big|\,x_j=0\text{ for all }j\in\mathbf{Z}\setminus S\}$.

The *left shift* $\tau=\tau^1$ is defined by $(\tau x)_i:=x_{i+1}$ for $x:\mathbf{Z}\to B$. We set $\mathbf{N}:=\{0,1,2,\dots\}$, $\mathbf{Z}_-:=\mathbf{Z}\setminus\mathbf{N}=\{-1,-2,-3,\dots\}$, so that $\pi^+:=\pi_{\mathbf{N}}$ maps $(\mathbf{Z}\to U)\to(\mathbf{N}\to U)$ and $\pi^++\pi^-=I$, where $\pi^-:=\pi_{\mathbf{Z}_-}$ (recall that $\pi_N u:=\chi_N u$ for all sequences u and sets N). We set $P_k u:=u_k$ ($k\in\mathbf{Z}$). The reflection $\mathbf{Я}$ is defined as in continuous time: $(\mathbf{Я}x)_i:=x_{-i}$, hence

$$\tau\mathbf{Я}=\mathbf{Я}\tau^{-1},\quad \mathbf{Я}\tau=\tau^{-1}\mathbf{Я},\quad \mathbf{Я}\pi^+\mathbf{Я}=\tau^{-1}\pi^-\tau,\quad \mathbf{Я}\pi^-\mathbf{Я}=\tau^{-1}\pi^+\tau. \tag{13.11}$$

However, the canonical discrete-time *reflection* is the one satisfying $\mathbf{Я}_{-1}\pi^+=\pi^-\mathbf{Я}_{-1}$, namely the one defined by $(\mathbf{Я}_{-1}x)_i:=x_{-1-i}$ (cf. Proposition 13.3.5). We have

$$\tau^{-1}\mathbf{Я}_{-1}=\mathbf{Я}=\mathbf{Я}_{-1}\tau,\quad \mathbf{Я}_{-1}\pi^+=\pi^-\mathbf{Я}_{-1},\quad \tau\mathbf{Я}_{-1}=\mathbf{Я}_{-1}\tau^{-1}. \tag{13.12}$$

Moreover, $\mathbf{Я}^*=\mathbf{Я}$ and $\mathbf{Я}_{-1}^*=\mathbf{Я}_{-1}$ on $\langle\cdot,\cdot\rangle_{\ell^2_r,\ell^2_{1/r}}$, $\mathbf{Я}^{-1}=\mathbf{Я}$, $\mathbf{Я}_{-1}^{-1}=\mathbf{Я}_{-1}$, and

$$\|\tau x\|_{\ell^p_r}=r\|x\|_{\ell^p_r},\quad \|\mathbf{Я}x\|_{\ell^p_r}=\|x\|_{\ell^p_{1/r}},\quad \|\mathbf{Я}_{-1}x\|_{\ell^p_r}=r\|x\|_{\ell^p_{1/r}}(x:\mathbf{Z}\to B,\ r>0). \tag{13.13}$$

We define the *Z-transform* $\widehat{u}:=\mathcal{Z}u$ of $u:\mathbf{Z}\to U$ by $\widehat{u}(z):=\sum_{j\in\mathbf{Z}}z^ju_j$ for those z for which the sum converges (one often uses z^{-1} instead of z to make the formulae more akin to their continuous-time counterparts at the cost of having to study functions holomorphic at infinity).

One easily verifies that the Z-transform maps $\ell^2_r(\mathbf{N};U)$ onto the Hardy space $\mathrm{H}^2_{r^{-1}}:=\mathrm{H}^2(r^{-1}\mathbf{D};U)$ through an isometric times $\sqrt{2\pi}$ isomorphism (i.e., $\|\widehat{u}\|_{\mathrm{H}^2_{1/r}}=\sqrt{2\pi}\|u\|_{\ell^2_r}$; use Lemma D.1.15 and scaling), and $\ell^1_r(\mathbf{N};U)$ into $\mathrm{H}^\infty_{1/r}:=\mathrm{H}^\infty(\mathbf{D}_{1/r};U)$ linearly and 1-1, with $\|\widehat{u}\|_{\mathrm{H}^\infty_{1/r}}\le\|u\|_{\ell^1_r}$ (note the exceptional meaning of H^∞_r (instead of $\mathrm{H}^\infty(\mathbf{C}^+_r;U)$) in this section; recall that $\mathbf{D}_r:=r\mathbf{D}=\{z\in\mathbf{C}\,\big|\,|z|<r\}$ and $\|\widehat{u}\|_{\mathrm{L}^2(r\partial\mathbf{D};U)}:=\int_0^{2\pi}\|\widehat{u}(re^{it})\|^2_U\,dt$, hence $\|1\|_2=\sqrt{2\pi}$). It follows that

$$\widehat{\mathbf{Я}u}(z)=\widehat{u}(1/z),\quad \widehat{\tau u}=z^{-1}\widehat{u}\qquad(u:\mathbf{Z}\to U). \tag{13.14}$$

We start by defining the discrete-time counterparts of TI and TIC (cf. Definitions 2.1.1 and 2.1.4):

Definition 13.1.1 (ti and tic) *Let $r > 0$. We define* $\mathrm{ti}_r(U,Y)$ *to be the (closed) subspace of operators* $\mathbb{E} \in \mathcal{B}(\ell^2_r(\mathbf{Z};U),\ell^2_r(\mathbf{Z};Y))$ *that are* time-invariant*, i.e.,* $\tau^1\mathbb{E} = \mathbb{E}\tau^1$.

We define $\mathrm{tic}_r(U,Y)$ *to be the (closed) subspace of operators* $\mathbb{D} \in \mathrm{ti}_r(U,Y)$ *that are* causal*, i.e.,* $\pi^-\mathbb{D}\pi^+ = 0$.

Finally, $\mathrm{tic}_{\mathrm{loc}}(U,Y)$ *is the set of linear maps* $\mathbb{D}^+ : U^{\mathbf{N}} \to Y^{\mathbf{N}}$ *that are time-invariant* ($\tau^{-1}\mathbb{D}^+ = \mathbb{D}^+\tau^{-1}$) *and causal* ($\pi_{\{0\}}\mathbb{D}^+\pi_{\{1,2,3,\dots\}} = 0$).

Maps in $\mathrm{ti} := \mathrm{ti}_1$ *are called* stable*; maps in* $\mathrm{ti}_{\exp} := \cup_{r<1}\mathrm{ti}_r$ *are called* exponentially stable*, and maps in* $\mathrm{ti}_\infty \setminus \mathrm{ti}$ *are called unstable, where* $\mathrm{ti}_\infty := \cup_{r>0}\mathrm{ti}_r$. *We set* $\mathrm{tic} := \mathrm{tic}_1$, $\mathrm{tic}_{\exp} := \mathrm{tic} \cap \mathrm{ti}_{\exp}$, $\mathrm{tic}_\infty := \mathrm{tic} \cap \mathrm{ti}_\infty$.

If $\mathbb{E} \in \mathrm{ti}_r(U,Y)$*, then its* (noncausal) adjoint $\mathbb{E}^*$ *is the* $\mathrm{ti}_{1/r}(Y,U)$ *map that satisfies*

$$\sum_{n\in\mathbf{Z}} \langle(\mathbb{E}u)(n), y(n)\rangle dt = \sum_{n\in\mathbf{Z}} \langle u(n), (\mathbb{E}^* y)(n)\rangle dt \quad (u \in \ell^2_r(\mathbf{Z};U),\ y \in \ell^2_{1/r}(\mathbf{Z};Y)), \tag{13.15}$$

and its causal adjoint *is* $\mathbb{E}^{\mathrm{d}} := \text{Я}\mathbb{E}^*\text{Я} = \text{Я}_{-1}\mathbb{E}^*\text{Я}_{-1} \in \mathrm{ti}_r(Y,U)$*, where* $(\text{Я}x)_i := x_{-i}$.

(In the literature, “exponentially stable” is often called “power stable”, but we prefer this analogy to continuous time.)

By Lemma 2.1.10 (see Theorem 13.3.13), we have $\mathrm{tic} = \mathrm{tic}_\infty \cap \mathrm{ti}$, $\mathrm{tic}_{\exp} = \cup_{r<1}\mathrm{tic}_r$, $\mathrm{tic}_\infty = \cup_{r>0}\mathrm{tic}_r$.

Let $\mathbb{E} \in \mathrm{ti}_r$, $r > 0$. One easily verifies that $\mathbb{E}^{\mathrm{d}} \in \mathrm{ti}_r$ is causal ($\in \mathrm{tic}_r$) iff $\mathbb{E}$ is. Obviously, $\tau^n\mathbb{E} = \mathbb{E}\tau^n$. for all $n \in \mathbf{Z}$ (and $\pi_{\{\dots,n-2,n-1,n\}}\mathbb{E}\pi_{\{n+1,n+2,n+3,\dots\}} = 0$ iff $\mathbb{E} \in \mathrm{tic}_\infty$).

If a map is causal and anti-causal, then it takes the form of a multiplication operator:

Lemma 13.1.2 (Static $\mathbb{D}$) *Let* $\mathbb{D}, \mathbb{D}^* \in \mathrm{tic}_\infty$*. Then* $\mathbb{D} \in \mathcal{B}$*. Moreover, the imbedding* $\mathcal{B} \mapsto \mathrm{TIC}$ *is isometric, preserves norms, and commutes with algebraic operations.* □

(The proof of Lemma 2.1.7 applies here too; see Remark 13.3.9 for a stability shift.)

If $\mathbb{D} \in \mathrm{tic}_\infty(U,Y)$, then, obviously, $\pi^+\mathbb{D}\pi^+ \in \mathrm{tic}_{\mathrm{loc}}(U,Y)$ and the map $\mathrm{tic}_\infty \mapsto \mathrm{tic}_{\mathrm{loc}}$ is injective, hence we can and will identify $\mathbb{D}$ and $\pi^+\mathbb{D}\pi^+$. Thus, $\mathrm{tic}_r \subset \mathrm{tic}_s \subset \mathrm{tic}_\infty \subset \mathrm{tic}_{\mathrm{loc}}$ when $0 < r < s < \infty$.

On the other hand, if $\mathbb{D}^+ \in \mathrm{tic}_{\mathrm{loc}}(U,Y)$, then, obviously, $\tau^{-n}\mathbb{D}^+ = \mathbb{D}^+\tau^{-n}$ and $\pi_{\{0,1,\dots,n\}}\mathbb{D}^+\pi_{\{n+1,n+2,n+3,\dots\}} = 0$ for all $n \in \mathbf{N}$. One easily verifies that $\mathrm{tic}_{\mathrm{loc}}$ maps correspond one-to-one to linear, causal ($\pi^+\mathbb{D}\pi^- = 0$), time-invariant ($\tau^n\mathbb{D} = \mathbb{D}\tau^n$ for all $n \in \mathbf{Z}$) maps between sequences $\mathbf{Z} \to U$ and $\mathbf{Z} \to Y$ whose supports are bounded from the left. Obviously, such a map belongs to tic_∞ iff it is bounded under some ℓ^2_r norm; we give another necessary and sufficient condition in (b) below:

Lemma 13.1.3 (tic maps are convolutions)

(a) The set $\mathrm{tic}_{\mathrm{loc}}(U,Y)$ *is exactly the set maps* $\mathbb{D}: U^{\mathbf{N}} \to Y^{\mathbf{N}}$ *that have a (necessarily unique) representation of the form*

$$\mathbb{D} = \sum_{j\in\mathbf{N}} T_j \tau^{-j}, \quad \text{i.e.,} \quad (\mathbb{D}u)_k = \sum_{j\in\mathbf{N}} T_j u_{k-j} \quad (u:\mathbf{N}\to U,\ k\in\mathbf{Z}), \tag{13.16}$$

equivalently, $\mathbb{D} = (T_j)_{j\in\mathbf{N}}*$, *where* $T_j \in \mathcal{B}(U,Y)$ *for all* $j \in \mathbf{N}$.

(b) Assume (13.16). Then $T_j = P_{\{i\}}\mathbb{D}P^*_{\{0\}}$ *for all* j, *and the following are equivalent:*

(i) $\mathbb{D} \in \mathrm{tic}_r(U,Y)$ *for some* $r > 0$;
(ii) $\|T_j\| \le Ms^j$ *for all* $j \in \mathbf{N}$ *and some* $s > 0$.

(c1) If (i) holds, then $\widehat{\mathbb{D}}(z) = \sum_{j\in\mathbf{N}} T_j z^j \in \mathrm{H}^\infty(\mathbf{D}_r;\mathcal{B}(U,Y))$ *and (ii) holds for* $s = r$ *and* $M = \|\mathbb{D}\|_{\mathrm{tic}_r}$.

(c2) Conversely, if (ii) holds, then (i) holds for any $r > s$ *(and* $\|\mathbb{D}\|_{\mathrm{tic}_r} \le M'_{r/s}M$*).*

(c3) If $\mathbb{D} \in \mathrm{tic}_r$, *then* $\mathbb{D} \in \ell^1_s(\mathbf{N};\mathcal{B}(U,Y))*$ *for all* $s > r$.

(d) Assume that $\mathbb{D} \in \mathrm{tic}_{\mathrm{loc}}$ *and* $r > 0$ *are s.t.* $\mathbb{D}u_0e_0 \in \ell^2_r$ *for all* $u_0 \in U$ *(equivalently,* $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D}_{1/r};\mathcal{B}(U,Y))$*). Let* $0 < s < r < t < \infty$. *Then* $\mathbb{D} \in \mathrm{tic}_t$, $\mathbb{D}[\ell^2_s(\mathbf{N};U) + \mathrm{c_c}] \subset \ell^2_r$, $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(r^{-1}\mathbf{D};\mathcal{B}(U,Y))$ *and*

$$\|\mathbb{D}\|_{\mathrm{tic}_t} \le M'_{t/r}\|\mathbb{D}\pi_{\{0\}}\| < \infty, \tag{13.17}$$

$$\|\mathbb{D}u\|_{\ell^2_r} \le \|\mathbb{D}\pi_{\{0\}}\|\|u\|_{\ell^1_r} \le M''_{r/s}\|\mathbb{D}\pi_{\{0\}}\|\|u\|_{\ell^2_s} \quad (u \in \ell^2_s(\mathbf{N};U)). \tag{13.18}$$

In particular, $\mathbb{D} \in \mathcal{B}(\ell^1_r(\mathbf{Z};U),\ell^2_r(\mathbf{Z};U))$, $\mathbb{D}^* \in \mathcal{B}(\ell^2_{1/r}(\mathbf{Z};U),\ell^\infty_{1/r}(\mathbf{Z};U))$, *and* $\mathbb{D}\pi_{[0,t)}u \to \mathbb{D}u$ *and* $\mathbb{D}^t u \to \mathbb{D}u$ *in* ℓ^2_r *for all* $u \in \ell^2_s(\mathbf{N};U) + \mathrm{c_c}$.

Thus, $\mathrm{tic}_{\mathrm{loc}}$ is the set of convolution operators having $\mathbf{N} \to \mathcal{B}(U,Y)$ kernels, and tic_r is its subset of maps that are bounded $\ell^2_r \to \ell^2_r$. If $\mathbb{D} \in \mathrm{tic}_{\mathrm{loc}}(U,Y)$ satisfies $\mathbb{D}[\mathrm{c_c}] \subset \ell^2_r$, then $\mathbb{D} \in \mathrm{tic}_{r'}$ for all $r' > r$, by (d).

As we will see from Definitions 13.3.1 and 13.3.4, (i) holds iff $\mathbb{D}$ has a wpls realization; hence all (linear, causal and time-invariant) maps satisfying (ii) have a wpls realization.

Proof of Lemma 13.1.3: (a) For all $u \in \mathbf{N} \to U$, we have

$$(\mathbb{D}u)_k = \sum_{j=0}^{k} (\mathbb{D}\pi_{k-j}u)_k = \sum_{j=0}^{k} (\pi_k\mathbb{D}\pi_{k-j}u)_k \tag{13.19}$$

$$= \sum_{j=0}^{k} (\tau^{-(k-j)}\pi_j\mathbb{D}\pi_0\tau^{k-j}u)_k = \sum_{j=0}^{k} P_j\mathbb{D}P_0^*u_{k-j}, \tag{13.20}$$

i.e., (13.16) holds for this u. The converse is obvious.

(b) This follows from (c1)&(c2).

(c1) Conversely, assume (i). The claim on $\widehat{\mathbb{D}}$ is obviously true. Let $\|u\|_U = 1$, so that $\|ue_0\|_{\ell_r^2} = 1$, where $e_0 := \chi_0$. Then, for any $j \in \mathbf{N}$,

$$\|\mathbb{D}\|_{\mathrm{tic}_r}^p \geq \|\mathbb{D}ue_0\|_{\ell_r^p}^p := \sum_k \|r^{-k}(\mathbb{D}ue_0)_k\|_Y^p \geq \|r^{-j}(\mathbb{D}ue_0)_j\|_Y^p = \|r^{-j}T_ju\|_Y^p, \tag{13.21}$$

hence $\|T_ju\|_Y \leq r^j\|\mathbb{D}\|_{\mathrm{tic}_r}$. Because u was an arbitrary unit vector, (ii) holds for $s = r$.

(c2) If (ii) holds and $r > s$, then (see Lemma D.1.7)

$$\|(T_j)_{j\in\mathbf{Z}} *\|_{\mathrm{tic}_r} \leq \|(T_j)_{j\in\mathbf{Z}}\|_{\ell_r^1} \leq M'_{r/s}M, \tag{13.22}$$

where $M'_{r/s} := \sum_{k\in\mathbf{N}}(r/s)^{-k} < \infty$. (Note that (i) does not have to hold $r = s$ (e.g., take $T_j = 1$ for all j).)

(c3) By (c1) (and (ii)) and (13.3), $\mathbb{D} \in \ell_r^\infty(\mathbf{N};\mathcal{B}(U,Y))* \subset \ell_s^1(\mathbf{N};\mathcal{B}(U,Y))*$ for any $s > r$.

(d) 1° $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}$: Obviously, $\mathbb{D}P_0^*[U] \subset \ell_r^2$ iff $(T_ju_0)_{j\in\mathbf{N}} \in \mathrm{L}^2$ for all $u_0 \in U$, i.e., iff $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{D}_{1/r};\mathcal{B}(U,Y))$. Thus, we have the equivalence.

2° Now $M := \|\mathbb{D}P_0^*\| = \|\mathbb{D}\pi_{\{0\}}\| < \infty$, by Lemma A.3.6 (with, e.g., $X_3 := \ell^2_{\mathrm{loc}} = (\mathbf{Z} \to B)$). Therefore, $\|T_j\| \leq Mr^j$ for all $j \in \mathbf{N}$, by (13.21). The first inequality follows from this.

3° Given $u \in \ell_r^1(\mathbf{N};U)$, we have

$$\|\mathbb{D}u_ke_k\|_{\ell_r^2} = \|\tau^{-k}\mathbb{D}u_ke_0\|_{\ell_r^2} \leq Mr^{-k}\|u_k\|_U, \tag{13.23}$$

by (13.13), hence $\|\mathbb{D}u\|_{\ell_r^2} \leq M\sum_k r^{-k}\|u_k\|_U = M\|u\|_{\ell_r^1} \leq M''\|u\|_{\ell_s^2}$, by (13.3). If $u \in \mathrm{c_c}$, then $\tau^{-n}u \in \ell_r^1(\mathbf{N};U)$ for some $n \in \mathbf{N}$, hence then $\mathbb{D}u \in \ell_r^2$.

4° Since $\mathbb{D} \in \mathcal{B}(\ell_r^1(\mathbf{Z};U),\ell_r^2(\mathbf{Z};U))$, by 3°, we have $\mathbb{D}^* \in \mathcal{B}(\ell^2_{1/r}(\mathbf{Z};U),\ell^\infty_{1/r}(\mathbf{Z};U))$, by Lemma B.4.15 (and Lemma A.3.24).

5° The last claim holds, because

$$\|\mathbb{D}u - \mathbb{D}^t u\|_{\ell_r^2} \leq \|\mathbb{D}u - \pi_{[0,t)}\mathbb{D}u\|_{\ell_r^2} + \|\pi_{[0,t)}(\mathbb{D}u - \mathbb{D}\pi_{[0,t)}u)\|_{\ell_r^2} \to 0, \tag{13.24}$$

as $t \to \infty$, for any $u \in \ell_s^2(\mathbf{N};U)$, since $\mathbb{D} \in \mathcal{B}(\ell_s^2(\mathbf{N};U),\ell_r^2)$. (For $u \in \mathrm{c_c}$, this is even easier.) □

We sometimes use the following lemma to derive discrete-time frequency-domain results from continuous-time ones:

Lemma 13.1.4 *Let $1 \leq p \leq \infty$. Define $\widehat{T}$ and $\widehat{T}^{-1}$ by*

$$(\widehat{T}\widehat{f})(\mathrm{e}^{-s}) := \widehat{f}(s) \qquad (s \in [0,+\infty) + i[-\pi,\pi)), \tag{13.25}$$

$$(\widehat{T}^{-1}\widehat{g})(s) := \widehat{g}(\mathrm{e}^{-s}) \qquad (s \in \overline{\mathbf{C}^+}). \tag{13.26}$$

Then $\widehat{T}$ maps $\mathrm{L}^p(i\mathbf{R};B)$ onto $\mathrm{L}^p(\partial\mathbf{D};B)$, $\mathrm{L}^p_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ onto $\mathrm{L}^p_{\mathrm{strong}}(\partial\mathbf{D};\mathcal{B}(U,Y))$, and $\mathrm{H}^p(\mathbf{C}^+;B)$ onto $\mathrm{H}^p(\mathbf{D};B)$ with norm ≤ 1, but none of these maps is one-to-one.

Moreover, $\widehat{T}\widehat{T}^{-1} = I$, and $\widehat{T}^{-1}$ maps $\mathrm{L}^\infty(\partial\mathbf{D};\mathcal{B}(U,Y))$ into $\mathrm{L}^\infty(i\mathbf{R};\mathcal{B}(U,Y))$ and $\mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D};\mathcal{B}(U,Y))$ into $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$ and $\mathrm{H}^\infty(\mathbf{C}^+;B)$ into $\mathrm{H}^\infty(\mathbf{D};B)$, *isometrically.*

Furthermore, $\widehat{T}(f\cdot g) = (\widehat{T}f)\cdot(\widehat{T}g)$ *and* $\widehat{T}^{-1}(f\cdot g) = (\widehat{T}^{-1}f)\cdot(\widehat{T}^{-1}g)$ *for all* f *and* g.

Finally, Lemmas 13.1.5 and 13.1.6 will show that $T \in \mathcal{B}(\mathrm{L}^2(J;U), \ell^2(N;U))$, $T \in \mathcal{B}(\mathrm{TI}(U,Y), \mathrm{ti}(U,Y))$ *and* $T \in \mathcal{B}(\mathrm{TIC}(U,Y), \mathrm{tic}(U,Y))$ *are onto, with norm* ≤ 1, *and* $T^{-1} \in \mathcal{B}(\ell^2(N;U), \mathrm{L}^2(J;U))$, $T^{-1} \in \mathcal{B}(\mathrm{ti}(U,Y), \mathrm{TI}(U,Y))$ *and* $T^{-1} \in \mathcal{B}(\mathrm{tic}(U,Y), \mathrm{TIC}(U,Y))$ *are isometries; where* Tu *is defined by* $\widehat{Tu} = \widehat{T}\widehat{u}$, *etc. and either* $J = \mathbf{R}_+$ & $N = \mathbf{N}$ *or* $J = \mathbf{R}$ & $N = \mathbf{Z}$. *Note that* $T(\mathbb{E}T^{-1}u) = (T\mathbb{E})u$ *etc.*

We also use two other methods to establish connections between discrete- and continuous-time maps; see Lemma 13.2.1, Theorem 13.2.3 and Section 13.4 for details.

Proof: Note that $\|\widehat{T}\widehat{f}\|_{\mathrm{L}^p(\partial\mathbf{D};B)} = \|\widehat{f}\|_{\mathrm{L}^p(i[-\pi,\pi);B)}$ and $\|\widehat{T}\widehat{f}\|_{\mathrm{L}^p(\partial\mathbf{D}_r;B)} = \|\widehat{f}\|_{\mathrm{L}^p(i[-\pi,\pi)-\log r;B)}$; the H^p claim follows from these, and the rest is obvious, because $s \mapsto \mathrm{e}^{-s}$ maps $[-\pi,\pi) \to \partial\mathbf{D}$ and $(0,+\infty) + i[-\pi,\pi) \to \mathbf{D}$, one-to-one and onto. □

The ti maps have $\mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D};*)$ transfer functions in the same way as the TI maps have $\mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};*)$ transfer functions:

Lemma 13.1.5 ($\widehat{\mathbf{ti}}_r = \mathbf{L}^\infty_{\mathbf{strong}}(r^{-1}\partial\mathbf{D},*)$) *Let* $\mathbb{E} \in \mathrm{ti}_r(U,Y)$. *Then there is a unique* transfer function $\widehat{\mathbb{E}}(z) \in \mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D}_{1/r};\mathcal{B}(U,Y))$ *s.t.* $\widehat{\mathbb{E}u} = \widehat{\mathbb{E}}\widehat{u}$ *for all* $u \in \ell^2_r(\mathbf{Z};U)$.

Moreover, the mapping $\mathbb{E} \mapsto \widehat{\mathbb{E}}$ *is an isometric isomorphism of* ti_r *onto* $\mathrm{L}^\infty_{\mathrm{strong}}$ *and it commutes with adjoints and compositions; in particular, it is an isometric* B^**-algebra isomorphism when* $U = Y$ *and* $r = 1$.

See Section 3.1 or Section F.1 for $\mathrm{L}^\infty_{\mathrm{strong}}$.

Proof: We take $r = 1$ to simplify the notation. Let $\widehat{\mathbb{F}} \in \mathcal{B}(\ell^2(\partial\mathbf{D};U), \ell^2(\partial\mathbf{D};Y))$ be the operator defined by $\widehat{\mathbb{F}}\widehat{u} := \widehat{\mathbb{E}u}$.

From Theorem F.1.7(b) we obtain easily the lemma except for the fact that each $\mathrm{ti}(U,Y)$ map has a transfer function; hence we study this claim only.

This claim is known in the case of separable U and $Y = U$ (e.g., Theorem 1 of [FS]), hence in the case of separable U and an arbitrary Y (because there is a separable subspace $Y_0 \subset Y$ s.t. $\mathbb{E} \in \mathrm{ti}(U,Y_0)$, i.e., $\mathbb{E}f \in \mathrm{L}^2(\mathbf{Z},Y_0)$ for all $f \in \mathrm{L}^2(\mathbf{Z};U)$, by Lemmas A.3.1(a3) and B.3.16). We could prove the unseparable result by the methods of Theorem 3.1.3(a1), but we have chosen to combine the latter with the separable case to obtain a shorter proof.

By the results mentioned above, for each closed, separable subspace $V \subset U$ there is a transfer function $\widehat{\mathbb{E}}_V \in \mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D};\mathcal{B}(U,Y))$, and for each $\mathbb{G} \in \mathrm{TI}(U,Y)$ a transfer function $\widehat{\mathbb{G}} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$.

Let $\mathbb{E} \in \mathrm{ti}(U,Y)$. Then $\mathbb{G}_{\mathbb{E}} : f \mapsto \mathcal{L}^{-1}\widehat{T}^{-1}\mathcal{Z}\mathbb{E}\mathcal{Z}^{-1}\widehat{T}\mathcal{L}f$ is obviously in linear and $\|\mathbb{G}_{\mathbb{E}}\|_{\mathcal{B}(\mathrm{L}^2)} \le 1$.

To prove that $\mathbb{G}_{\mathbb{E}} \in \mathrm{TI}(U,Y)$, take arbitrary $f \in \mathrm{L}^2(\mathbf{R};U)$ and $t \in \mathbf{R}$. Choose a closed separable subspace $V \subset U$ s.t. $f \in \mathrm{L}^2(\mathbf{R};V)$. Let $F_V :=$

$\widehat{T}^{-1}\widehat{\mathbb{E}}_V \in \mathrm{L}^\infty_{\mathrm{strong}}(i[-\pi,\pi);\mathcal{B}(V,Y))$ (here we used the separable result), so that $\mathcal{L}(\mathbb{G}_\mathbb{E} f) = \widehat{T}^{-1}(\widehat{\mathbb{E}}_V \widehat{T}\widehat{f}) = F_V \widehat{f}$.

But $F_V \mathrm{e}^{-t\cdot}\widehat{f} = \mathrm{e}^{-t\cdot}F_V\widehat{f}$, hence $\mathbb{G}_\mathbb{E}\tau(t)f = \tau(t)\mathbb{G}_\mathbb{E} f$, for all $t \in \mathbf{R}$. Therefore, $\mathbb{G}_\mathbb{E} \in \mathrm{TI}(U,Y)$, hence $\widehat{\mathbb{G}}_\mathbb{E} \in \mathrm{L}^\infty_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U,Y))$. Equation $\widehat{\mathbb{G}}_\mathbb{E}\widehat{f} = \widehat{T}^{-1}(\widehat{\mathbb{E}}_V\widehat{T}\widehat{f})$ implies that

$$(\widehat{T}\widehat{\mathbb{G}}_\mathbb{E})(\widehat{T}\widehat{f}) = \widehat{\mathbb{E}}_V\widehat{T}\widehat{f} = \mathcal{Z}(\mathbb{E}\mathcal{Z}^{-1}\widehat{T}\widehat{f}) \quad \text{for all } f,V. \tag{13.27}$$

Consequently, $\widehat{\mathbb{E}} := \widehat{T}\widehat{\mathbb{G}}_\mathbb{E} \in \mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D};\mathcal{B}(U,Y))$ is the transfer function of $\mathbb{E}$.

The "Moreover" claims are easy to prove, cf. the end of the proof of Theorem 3.1.3(a1). □

Lemma 13.1.6 ($\widehat{\mathrm{tic}}_r = \mathrm{H}^\infty_{1/r}$) *Let $\mathbb{D} \in \mathrm{tic}_r(U,Y)$. Then there is a unique $\widehat{\mathbb{D}}(z) \in \mathrm{H}^\infty(\mathbf{D}_{1/r};\mathcal{B}(U,Y))$ s.t. $\widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\widehat{u}$ for all $u \in \ell^2_r(\mathbf{N};U)$, namely the one defined in Lemma 13.1.3(c1). Moreover, the mapping $\mathbb{D} \mapsto \widehat{\mathbb{D}}$ is an isometric Banach algebra isomorphism of tic_r onto $\mathrm{H}^\infty_{1/r}$.*

In particular, $\mathrm{tic}_r \subset \mathrm{tic}_{r'}$ for $0 < r < r'$.

Furthermore, $\widehat{\mathbb{D}}$ has the (nontangential) boundary function $\widehat{\mathbb{D}} \in \mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D}_{1/r};\mathcal{B}(U,Y))$ in the sense of Theorem 3.3.1(c)&(e)&(f)

We call $D := \widehat{\mathbb{D}}(0) \in \mathcal{B}(U,Y)$ the *feedthrough operator* of $\mathbb{D}$.

Because of the last claim of the lemma, we may safely identify $\widehat{\mathbb{D}} \in \mathrm{H}^\infty$ and $\widehat{\mathbb{D}} \in \mathrm{L}^\infty_{\mathrm{strong}}$ (via an isometric Banach algebra isomorphism) and call both of them the transfer function of $\mathbb{D}$.

Analogously to the continuous case, we also identify $\mathbb{D} \in \mathrm{tic}_r$ and $\mathbb{D} \in \mathrm{tic}_{r'}$ if they (equivalently, their transfer functions) are extensions of a single tic (H^∞) map. Consequently, $\mathrm{tic}_r \subset \mathrm{tic}_{r'}$ for $r < r'$.

Proof of Lemma 13.1.6: Obviously, $\widehat{\mathbb{D}}\widehat{u} = \widehat{\mathbb{D}u}$ on $\mathbf{D}_{1/r}$ for all $\widehat{u} \in U$, hence for all $u \in \ell^2_r$, by time-invariance and continuity (alternatively, by [RR, Theorem 1.15B] and scaling). For $r = 1$, the isomorphism from Lemma D.1.15; the general case follows by scaling.

Thus, $\mathrm{tic}_r \subset \mathrm{tic}_{r'}$ follows from $\mathrm{H}^\infty_{1/r} \subset \mathrm{H}^\infty_{1/r'}$.

By (c) and (e) of Theorem 3.3.1, $\widehat{\mathbb{D}}$ has a boundary function ($\mathrm{L}^\infty_{\mathrm{strong}}$ equivalence class, to be exact); that function is obviously equal to the one given in Lemma 13.1.5. □

Invertibility in tic_∞ is equivalent to invertibility of the feedthrough operator:

Lemma 13.1.7 ($\mathbb{X}^{-1}$) *Let $\mathbb{X} \in \mathrm{tic}_\infty := \cup_{r>0}\mathrm{tic}_r$. Then $\mathbb{X} \in \mathcal{G}\mathrm{tic}_\infty \Leftrightarrow X := \widehat{\mathbb{X}}(0) \in \mathcal{GB}$.* □

This follows from the fact $\widehat{\mathbb{X}}$ is (boundedly) invertible on a neighborhood of 0 iff $X \in \mathcal{GB}$ (see Lemma A.3.3(A2)). Obviously, $\widehat{\mathbb{X}}^{-1}(0) = X^{-1}$, because $(\widehat{\mathbb{X}}\widehat{\mathbb{Z}})(0) = XZ$ for any $\mathbb{X} \in \mathrm{tic}_\infty(U,Y)$, $\mathbb{Z} \in \mathrm{tic}_\infty(Y,Z)$.

Lemma 13.1.8 ($\mathbb{D}^{\mathrm{d}}$) *Let $u \in \ell^2(\mathbf{Z};U)$ and $\mathbb{E} \in \mathrm{ti}(U,Y)$. Then $\widehat{\mathbf{Я}u}(z) = \widehat{u}(\bar{z})$ for $z \in \partial\mathbf{D}$. Moreover, $\widehat{\mathbb{E}^*}(z) = \widehat{\mathbb{E}}(z)^*$, $\widehat{\mathbf{Я}\mathbb{E}\mathbf{Я}}(z) = \widehat{\mathbb{E}}(\bar{z})$ and $\widehat{\mathbb{E}^{\mathrm{d}}}(z) = \widehat{\mathbb{E}}(\bar{z})^*$ for $z \in \partial\mathbf{D}$.*

If $\mathbb{D} \in \mathrm{tic}_r(U,Y)$, $r > 0$, then $\widehat{\mathbb{D}^{\mathrm{d}}}(z) = \widehat{\mathbb{D}}(\bar{z})^*$ for $z \in \mathbf{D}_r$, i.e., $\widehat{\mathbb{D}^{\mathrm{d}}}(s) = \sum_{n=0}^{\infty} D_n^* z^n$, where $\widehat{\mathbb{D}}(s) = \sum_{n=0}^{\infty} D_n z^n$ is the Taylor series of $\widehat{\mathbb{D}}$ (with $D_n \in \mathcal{B}(U,Y)$ for $n \in \mathbf{N}$).
□

(The proof is almost identical to that of Lemma 3.3.8 (with replacements (13.63)) and hence omitted.) See Definition 3.1.1 for $\widehat{\mathbb{E}}^*$ on $\partial\mathbf{D}$ (and Theorem 3.1.3(d), which is applicable on $\partial\mathbf{D}$ too, by Theorem 13.2.3). Note that $\widehat{\mathbb{E}}^*$ need *not* be the pointwise adjoint of an arbitrary representative of $\widehat{\mathbb{E}}$, which might be unbounded and nonmeasurable, by Example 3.1.4. Note also that $\bar{z} = 1/z$ on $\partial\mathbf{D}$.

Notes

Much of the convolution and Z-transform theory at the beginning of this section is probably well known, and so is Lemma 13.1.6 except the boundary function claim (in the unseparable case). Also Lemma 13.1.2 and Lemma 13.1.3(a) are well known (see, e.g., [Mal00] or [Sbook]). Further results on ti_∞ maps are given in Theorem 13.3.13; see the corresponding continuous-time chapters for further notes.

13.2 The Cayley transform ($\diamondsuit$, $\heartsuit$)

And thus in anguish Beren paid
for that great doom upon him laid,
the deathless love of Lúthien,
too fair for love of mortal Men;
and in his doom was Lúthien snared,
the deathless in his dying shared;
and Fate them forged a binding chain
of living love and mortal pain.

— J.R.R. Tolkien (1892–1973): "The Lay of Leithian"

In this section, we present standard and further results on the Cayley transform of functions and particularly on that of (stable) tic operators.

We will often use composition with the *Cayley function* to map $\mathrm{H}^\infty(\mathbf{D};B)$ one-to-one onto $\mathrm{H}^\infty(\mathbf{C}^+;B)$, $\mathcal{C}(\partial\mathbf{D};B)$ one-to-one onto $\mathcal{C}(i\mathbf{R}\cup\{\infty\};B)$, or $\mathrm{L}^\infty_{\text{strong}}(\partial\mathbf{D};B)$ one-to-one onto $\mathrm{L}^\infty_{\text{strong}}(i\mathbf{R};B)$:

Lemma 13.2.1 (Cayley function) *We define the* Cayley function *by*

$$\phi_{\text{Cayley}} : s \mapsto \frac{1-s}{1+s} \tag{13.28}$$

(a) We have $\phi_{\text{Cayley}}^{-1} = \phi_{\text{Cayley}}$ and $\phi_{\text{Cayley}}(s) = -\phi_{\text{Cayley}}(1/s)$ ($s \neq 0,-1$).

(b) ϕ_{Cayley} maps $\mathbf{C}^+ \to \mathbf{D}$, $i\mathbf{R}\cup\{\infty\} \to \partial\mathbf{D}$, and $\overline{\mathbf{C}^+}\cup\{\infty\} \to \overline{\mathbf{D}}$ one-to-one and onto (and continuously in both directions, i.e., it is a homeomorphism) (but $\mathbf{C}^+_\omega \not\to \mathbf{D}_r$ for any $\omega \neq 0$ or $r \neq 1$). Moreover, the positive direction on $i\mathbf{R}$ ($+i\infty$ to $-i\infty$) is mapped to the negative direction on $\partial\mathbf{D}$, and $f(\pm\infty) = -1 = f(\pm i\infty)$.

(c) If $it = \phi_{\text{Cayley}}(e^{i\theta})$ (i.e., $e^{i\theta} = \phi_{\text{Cayley}}(it)$), then $\frac{d\theta}{dt} = -2(1+t^2)^{-1}$.

(d) We have $\int_0^{2\pi}(f\circ\phi_{\text{Cayley}}^{-1})(e^{i\cdot})\,dm = \int_{\mathbf{R}} 2(1+t^2)^{-1}f(it)\,dt$ for measurable $f : i\mathbf{R} \to [0,+\infty]$ and for $f \in \mathrm{L}^1(i\mathbf{R};B)$, where B is a Banach space.

(e1) $f \mapsto f\circ\phi_{\text{Cayley}}$ maps $\mathrm{H}(\mathbf{D};B)$ one-to-one onto $\mathrm{H}(\mathbf{C}^+;B)$, $\mathrm{H}^\infty(\mathbf{D};B)$ one-to-one onto $\mathrm{H}^\infty(\mathbf{C}^+;B)$, $\mathcal{C}(\partial\mathbf{D};B)$ one-to-one onto $\mathcal{C}(i\mathbf{R}\cup\{\infty\};B)$, $\mathrm{L}^p(i\mathbf{R};B)$ one-to-one into $\mathrm{L}^p(\partial\mathbf{D};B)$, and $X(\partial\mathbf{D};B)$ one-to-one onto $X(i\mathbf{R};B)$, where $X = \mathrm{L}^\infty$, $X = \mathrm{L}^\infty_{\text{strong}}$, $X = \mathrm{L}^\infty_{\text{weak}}$ or $X = \mathrm{L}^p_{\text{loc}}$ ($1 \le p \le \infty$).

Moreover, this map preserves the $\|\cdot\|_\infty$ norm on the boundary, the supremum norm, and nontangential angles (except at $-1 \in \partial\mathbf{D}$).

(e2) Let $f \in \mathrm{H}(\mathbf{C}^+;B)$ and $g \in \mathrm{H}(\mathbf{D};B)$. Then $f \in \mathrm{H}^p(\mathbf{C}^+;B)$ iff $(z \mapsto (1+z)^{2/p}f(\frac{1-z}{1+z})) \in \mathrm{H}^p(\mathbf{D};B)$. Analogously, $g \in \mathrm{H}^p(\mathbf{D};B)$ iff $(s \mapsto (1+s)^{-2/p}g(\frac{1-s}{1+s})) \in \mathrm{H}^p(\mathbf{C}^+;B)$.

(f) If $f \in \mathrm{L}^1_{\text{loc}}(i\mathbf{R};B)$, then $ir \in \mathrm{Leb}(f)$ iff $\phi_{\text{Cayley}}^{-1}(ir) \in \mathrm{Leb}(f\circ\phi_{\text{Cayley}})$.

(There are several different Cayley functions in the literature, but they only differ by some additional constants in the above formulae. The advantage of our function is that it is the inverse of itself. See also Lemma 13.2.6.)

Proof: (a)–(c) These are obvious.

(d) (Note that if either side converges absolutely, then so does the other (replace f by $\|f\|_B$).) This follows from (c) and Lemma B.4.10.

(e1) 1° Because ϕ_{Cayley} and $\phi_{\mathrm{Cayley}}^{-1}$ are holomorphic, they preserve continuous and holomorphic functions, by Lemma D.1.2(b4). Trivially, the supremum norm is also preserved ($\partial\mathbf{D} \to i\mathbf{R} \cup \{\infty\}$ or $\mathbf{D} \to \mathbf{C}_+$), so $\mathrm{H}, \mathrm{H}^\infty, \mathcal{C}$ are already covered (note that $\partial\mathbf{D}$ and $i\mathbf{R} \cup \{\infty\}$ are compact).

2° By Lemma B.4.10 (applied to $\phi_{\mathrm{Cayley}}(-\cdot)$; cf. (c)), also the $\|\cdot\|_\infty$-norm is preserved and cases $X = \mathrm{L}^\infty$ and $X = \mathrm{L}^p_{\mathrm{loc}}$ are covered. If $B = \mathcal{B}(U,Y)$, then the cases $X = \mathrm{L}^\infty_{\mathrm{strong}}$, $X = \mathrm{L}^\infty_{\mathrm{weak}}$ follow from the case $X = \mathrm{L}^\infty$.

3° L^p*:* This follows from the theorem on p. 130 of [Hoffman] (whose proof applies also to vector-valued functions).

4° *Nontangential angles:* (See p. 967 for nontangential limits.) Because ϕ_{Cayley} is conformal $\mathbf{C} \setminus \{-1\} \mapsto \mathbf{C} \setminus \{-1\}$, the images of (small) nontangential cones are contained in nontangential cones, in both directions.

(e2) The (scalar case, with a different Cayley transform) proof the Theorem on p. 130 of [Hoffman] applies mutatis mutandis.

(f) This follows from Lemma B.5.5. (Here we have identified $\partial\mathbf{D}$ with $[-\pi,\pi)$ (via $e^{it} \mapsto t$); identification with $[0,2\pi)$ would affect the point $\phi_{\mathrm{Cayley}}^{-1}(i0) = 1$, unless we would use the periodic extension of $f \circ \phi_{\mathrm{Cayley}}^{-1} \circ e^{i\cdot}$ on $\mathbf{R}$.) □

The references of Theorem 5.1.6 will use the following fact (as the definition of $\widehat{\pi^+}$): The operator $\widehat{\pi^+}$ has the standard singular integral presentation

$$(\widehat{\pi^+}\widehat{f})(z) = \frac{\widehat{f}(z)}{2} + \frac{1}{2\pi i}\int_{\partial\mathbf{D}} \frac{\widehat{f}(s)}{s-z}\,ds, \tag{13.29}$$

for $\widehat{f} \in \mathrm{L}^2(\partial\mathbf{D};H)$; this follows by applying scalar case (from, e.g., [Garnett]) to Λf for each $\Lambda \in H^*$ (because $\Lambda\widehat{\pi^+} = \widehat{\pi^+}\Lambda$). One gets the corresponding presentation for $\widehat{\pi^-}$ analogously.

Next we shall construct an isomorphism $\heartsuit : \mathrm{TI} \to \mathrm{ti}$ that can be used to transform results from continuous time to discrete time and vice versa.

Definition 13.2.2 (TI ↔ ti) *We define the* (signal) Cayley transform $\diamondsuit$: $\mathrm{L}^2(\mathbf{R};U) \to \ell^2(\mathbf{Z};U)$ *by* $\widehat{\diamondsuit}\widehat{f} := \gamma \cdot (\widehat{f} \circ \phi_{\mathrm{Cayley}}^{-1})$*, i.e.,*

$$(\widehat{\diamondsuit}\widehat{f})(z) := \gamma(z)(\widehat{f} \circ \phi_{\mathrm{Cayley}}^{-1})(z) \quad (f \in \mathrm{L}^2(\mathbf{R};U)), \tag{13.30}$$

where $\gamma(z) = \sqrt{2}/(1+z)$*.*

The (map) Cayley transform $\heartsuit$ *is defined by* $\heartsuit\mathbb{E} := \diamondsuit_Y \mathbb{E} \diamondsuit_U^{-1} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};Y)$ *for* $\mathbb{E} : \mathrm{L}^2(\mathbf{R};U) \to \mathrm{L}^2(\mathbf{R};Y)$*.*

(See the proof of Theorem 13.2.3(a) for details.)

Next we show that the map $\diamondsuit$ is unitary ((a)), that $\widehat{\heartsuit} = \cdot \circ \phi_{\text{Cayley}}^{-1}$ ((b3)), and that $\widehat{\text{ti}} \circ \phi_{\text{Cayley}} = \widehat{\text{TI}}$ ((b2)):

Theorem 13.2.3 ($\heartsuit$: TI $\leftrightarrow$ ti)

(a) The map $\diamondsuit$ is an isometric isomorphism of $\mathrm{L}^2(\mathbf{R};U)$ *onto* $\ell^2(\mathbf{Z};U)$ *and of* $\mathrm{L}^2(\mathbf{R}_+;U)$ *onto* $\ell^2(\mathbf{N};U)$*).*

Indeed, for $u : \mathbf{Z} \to U$ *we have*

$$\|u\|_2^2 := \sum_n \|u_n\|_U^2 = (2\pi)^{-1} \int_0^{2\pi} \|\widehat{u}(\mathrm{e}^{i\theta})\|_U^2 \, d\theta \tag{13.31}$$

$$= (2\pi)^{-1} \int_{-\infty}^{+\infty} \|(\widehat{\diamondsuit}^{-1}\widehat{u})(it)\|_U^2 \, dt = \int_{-\infty}^{+\infty} \|(\diamondsuit^{-1}u)(t)\|_U^2 \, dt \tag{13.32}$$

Moreover, for $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ *the formula (13.30) holds on* $\mathbf{D}$ *too.*

(b1) The map $\heartsuit$ is an isometric isomorphism of $\mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$ *onto* $\mathcal{B}(\ell^2(\mathbf{Z};U),\ell^2(\mathbf{Z};Y))$*. Moreover, $\heartsuit$ commutes with adjoints and valid compositions of operators. Thus,*

$$\heartsuit(\mathbb{E}\mathbb{F}) = (\heartsuit\mathbb{E})(\heartsuit\mathbb{F}), \quad \heartsuit\mathbb{E}^{-1} = (\heartsuit\mathbb{E})^{-1}, \quad \heartsuit\mathbb{E}^* = (\heartsuit\mathbb{E})^*, \tag{13.33}$$

for $\mathbb{E} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))$ *and* $\mathbb{F} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};Y),\mathrm{L}^2(\mathbf{R};H))$*.*

(b2) The map $\heartsuit$ is an isometric isomorphism of TI *onto* ti *and of* TIC *onto* tic.

(b3) Let $\mathbb{E} \in \mathrm{TI}$, $\mathbb{F} := \heartsuit\mathbb{E}$*. Then* $\widehat{\mathbb{F}} = \widehat{\mathbb{E}} \circ \phi_{\text{Cayley}}^{-1}$ *in* $\mathrm{L}^\infty_{\text{strong}}$*, i.e., on* $\partial\mathbf{D}$ *(and in* H^∞*, i.e., on* $\mathbf{D}$*, if* $\mathbb{E} \in \mathrm{TIC}$*).*

(c1) $\diamondsuit\pi_\pm = \pi^\pm\diamondsuit$, $\heartsuit\pi_\pm = \pi^\pm\heartsuit$, $\diamondsuit\boldsymbol{Я} = \boldsymbol{Я}_{-1}\diamondsuit = \tau\boldsymbol{Я}\diamondsuit$, $\heartsuit\boldsymbol{Я} = \boldsymbol{Я}_{-1} = \tau\boldsymbol{Я}$.

(c2) $\widehat{\tau^k\diamondsuit} = \widehat{\diamondsuit}(\phi_{\text{Cayley}})^k\cdot$, $\widehat{\tau^k} = \widehat{\heartsuit}(\phi_{\text{Cayley}})^k\cdot$, $\widehat{\diamondsuit\tau(t)} = \mathrm{e}^{t\frac{1-z}{1+z}} \cdot \widehat{\diamondsuit}$, $\widehat{\heartsuit\tau(t)} = \mathrm{e}^{t\frac{1-z}{1+z}}\cdot$.

(c3) $\heartsuit L = L$ *for* $L \in \mathcal{B}(U,Y)$*.*

(c4) $\heartsuit\mathbb{E}^{\mathrm{d}} = (\heartsuit\mathbb{E})^{\mathrm{d}}$ *for* $\mathbb{E} \in \mathrm{TI}$*.*

(d) Let $\mathbb{E} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U))$*. Then* $\pi_\pm\mathbb{E}\pi_\pm$ *is invertible on* $\pi_\pm\mathrm{L}^2$ *iff* $\pi^\pm(\heartsuit\mathbb{E})\pi^\pm$ *is invertible on* $\pi^\pm\ell^2$*.*

(e) Let $\mathbb{E}, P \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U))$*. Then* $\heartsuit\mathbb{E} \ge 0$ *[$\gg 0$] on* $(\heartsuit P)\ell^2$ *iff* $\mathbb{E} \ge 0$ *[$\gg 0$] on* $P\mathrm{L}^2$*.*

Note that these results do not hold in the unstable case (see Lemma 13.2.1(b)). However, Section 13.4 treats another way to relate TI and ti, *discretization*, that handles also the unstable case.

Proof: (a) By Lemma 13.2.1(d), we have

$$\int_0^{2\pi} \|(\widehat{\diamondsuit}\widehat{u})(\mathrm{e}^{i\theta})\|^2 \, d\theta = \int_{\mathbf{R}} 2^{-1}|1+it|^2 \|\widehat{u}(it)\|^2 \frac{\mathrm{d}\theta}{\mathrm{d}t} \, dt = \int_{\mathbf{R}} \|\widehat{u}(it)\|^2 \, dt \tag{13.34}$$

for measurable $f : i\mathbf{R} \to U$.

This proves the first "=" sign in (13.31); the two following "=" signs are from Lemma D.1.15.

It follows that $\diamondsuit$ is an isometric isomorphism of L^2 onto ℓ^2. The fact that $\pi^+\diamondsuit = \diamondsuit\pi_+$ follows from the scalar case (given on, e.g., pp. 104–106 of [Hoffman]). (Indeed, our ϕ_{Cayley} uses an extra $z \mapsto -z$ on $\partial\mathbf{D}$ compared to that of [Hoffman], hence this transforms the ascending order (positive orientation) of $\partial\mathbf{D}$ to the ascending order on $-\infty\ldots+\infty$), because, obviously, $\Lambda\diamondsuit_U = \diamondsuit_{\mathbf{C}}\Lambda$ for $\Lambda \in U^*$. Note that if $u \in \mathrm{L}^2(\mathbf{R};U)$, then $u \in \mathrm{L}^2(\mathbf{R}_+;U) \Leftrightarrow \Lambda u \in \mathrm{L}^2(\mathbf{R}_+;\mathbf{C})$ for all $\Lambda \in U^*$, and that an analogous claim holds for ℓ^2.)

(b1) This follows from (a) except for $\heartsuit\mathbb{E}^* = (\heartsuit\mathbb{E})^*$, which is obtained as follows:

$$\langle\mathbb{E}^*u,v\rangle = \langle u,\mathbb{E}v\rangle = \langle\diamondsuit u,\heartsuit\mathbb{E}\diamondsuit v\rangle = \langle\diamondsuit^{-1}(\heartsuit\mathbb{E})^*\diamondsuit u,v\rangle. \tag{13.35}$$

(b2) This follows from (b1) and (b3).

(b3) Let $\mathbb{E} \in \mathrm{TI}(U,Y)$. Then

$$\widehat{\diamondsuit}\widehat{\mathbb{E}}\widehat{u} := \gamma((\widehat{\mathbb{E}}\widehat{u})\circ\phi_{\text{Cayley}}^{-1}) = (\widehat{\mathbb{E}}\circ\phi_{\text{Cayley}}^{-1})\gamma(\widehat{u}\circ\phi_{\text{Cayley}}^{-1}) = (\widehat{\mathbb{E}}\circ\phi_{\text{Cayley}}^{-1})\widehat{\diamondsuit}\widehat{u} \tag{13.36}$$

for all $u \in \mathrm{L}^2(\mathbf{R};U)$. Therefore, $\widehat{\diamondsuit}\widehat{\mathbb{E}}\widehat{\diamondsuit}^{-1} = \widehat{\mathbb{E}}\circ\phi_{\text{Cayley}}^{-1}$ (cf. also Lemma 13.1.6).

(c1) The first identity on $\pi_\pm$ was proved in the proof of (a1); the second identity follows from the first.. By Lemma 13.1.8,

$$\widehat{\text{Я}\diamondsuit u}(z) = \widehat{\diamondsuit u}(1/z) = \gamma(1/z)\widehat{u}(\frac{1-1/z}{1+1/z}) = z\gamma(z)\widehat{u}(-\frac{1-z}{1+z}) = z\widehat{\diamondsuit\text{Я}u}(z) \tag{13.37}$$

for $z \in \partial\mathbf{D}$. Because $\widehat{\tau} = z$ and $\text{Я}_{-1} = \tau\text{Я}$, we obtain the third identity; the fourth identity follows from the third.

(c2) Because $(\phi_{\text{Cayley}})^k\circ\phi_{\text{Cayley}} = z^k = \widehat{\tau^k}$, we have $\widehat{\diamondsuit}((\phi_{\text{Cayley}})^k\widehat{u}) = z^k\widehat{\diamondsuit}\widehat{u}$, i.e., the first (and hence the second) identity holds. Moreover, $\widehat{\tau(t)u} = e^{st}\widehat{u}$, hence $\widehat{\diamondsuit\tau(t)} = e^{t\phi_{\text{Cayley}}(z)}\widehat{\diamondsuit}$.

(c3) This is obvious.

(c4) This follows from (c1) and the formula $\heartsuit\mathbb{E}^* = (\heartsuit\mathbb{E})^*$ from (b1).

(d) By (b), $\mathbb{G}\pi_+\mathbb{E}\pi_+ = \pi_+ = \pi_+\mathbb{E}\pi_+\mathbb{G}$ for some $\mathbb{G} \in \mathcal{B}(\pi_\pm\mathrm{L}^2)$ (we may identify $\mathbb{G}$ with $\pi_+\mathbb{G}\pi_+ \in \mathcal{B}(\mathrm{L}^2)$) iff $(\heartsuit\mathbb{G})\pi^+(\heartsuit\mathbb{E})\pi^+ = \pi^+ = \pi^+(\heartsuit\mathbb{E})\pi^+(\heartsuit\mathbb{G})$.

(e) Now $\langle\heartsuit\mathbb{E}\heartsuit P\diamondsuit f,\heartsuit P\diamondsuit f\rangle \geq 0$ for all $\diamondsuit f \in \ell^2$ iff $\langle\mathbb{E}Pf,Pf\rangle \geq 0$ for all $f \in \mathrm{L}^2$. By replacing $\mathbb{E}$ by $\mathbb{E}-\varepsilon I$ we get the "$\gg 0$" claim. □

We remark that losslessness (Definition 2.5.1) is $\heartsuit$-invariant:

Corollary 13.2.4 *Let $J = J^* \in \mathcal{B}(Y)$ and $S = S^* \in \mathcal{B}(U)$. Let $\mathbb{E} \in \mathrm{TIC}$, $\mathbb{F} := \heartsuit\mathbb{E} \in$ tic.*

Then $\mathbb{F}$ is (J,S)-lossless iff $\mathbb{E}$ is (J,S)-lossless.

Proof: This follows from (13.33) and Theorem 13.2.3(e). □

It has been shown in Chapter 11 of [Sbook] that for every $\Sigma = \left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{wpls}_1$ with $\mathbb{A}$ contractive and $\mathbb{A}+I$ one-to-one, there is $\Sigma' = \left[\begin{array}{c|c}\mathbb{A} & \mathbb{B}\\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]' \in \mathrm{WPLS}_0$ s.t. $\mathbb{D}' = \heartsuit^{-1}\mathbb{D}$ (and conversely).

We now part extend Definition 13.2.2 for unstable $\mathbb{D}$ for later use (also much more is true):

Proposition 13.2.5 ($\heartsuit$: unstable $\mathbb{D}$) *If $\Omega \subset \mathbf{C}^+$ is open and $\widehat{\mathbb{D}} \in \mathrm{H}(\Omega; \mathcal{B}(U,Y))$, then we set $\mathbb{D}u := \mathcal{L}^{-1}\widehat{\mathbb{D}}\widehat{u}$, for all $u \in \mathrm{L}^2(\mathbf{R}_+;U)$ s.t. $\widehat{\mathbb{D}}\widehat{u} \in \mathrm{H}^2(\mathbf{C}^+;U)$. Moreover, we set $\heartsuit\mathbb{D} := \diamondsuit_Y \mathbb{D} \diamondsuit_U^{-1}$.*

Analogously, if $\Omega' \subset \mathbf{D}$ is open and $\widehat{\mathbb{F}} \in \mathrm{H}(\Omega'; \mathcal{B}(U,Y))$, then we set $\mathbb{F}u := \mathcal{Z}^{-1}\widehat{\mathbb{F}}\widehat{u}$ for all $u \in \ell^2(\mathbf{N};U)$ s.t. $\widehat{\mathbb{F}}\widehat{u} \in \mathrm{H}^2(\mathbf{D};U)$.

The following holds:

(b1) If $\mathbb{D}$ is as above, then $\widehat{\heartsuit\mathbb{D}} = \widehat{\mathbb{D}} \circ \phi_{\mathrm{Cayley}}^{-1}$.

(b2) $\heartsuit(\mathbb{D}\widetilde{\mathbb{D}}) = (\heartsuit\mathbb{D})(\heartsuit\widetilde{\mathbb{D}})$ for $\widetilde{\mathbb{D}} \in \mathrm{H}(\Omega; \mathcal{B}(H,U))$.

(b3) **($\widehat{\mathbb{D}}[\mathrm{H}^2] \subset \mathrm{H}^2 \Rightarrow \widehat{\mathbb{D}} \in \mathrm{H}^\infty$)** *If $\widehat{\mathbb{D}}$ is defined on whole $\mathrm{L}^2(\mathbf{R}_+;U)$, then $\widehat{\mathbb{D}} \in \mathrm{TIC}(U,Y)$ and $\heartsuit\mathbb{D}$ coincides with that of Definition 13.2.2.*

Proof: (Note that $\widehat{\mathbb{D}}\widehat{u} \in \mathrm{H}^2$ means that $\widehat{\mathbb{D}}\widehat{u}$ has a (unique, by Lemma D.1.2(e)) extension to $\mathbf{C}^+$, and that this extension is in H^2.)

(b1) If $u, \mathbb{D}u \in \mathrm{L}^2$, then $\widehat{(\heartsuit\mathbb{D})\diamondsuit u} = (\widehat{\mathbb{D}} \circ \phi_{\mathrm{Cayley}}^{-1})\widehat{\diamondsuit u}$, by (13.36). Conversely, if $(\widehat{\mathbb{D}} \circ \phi_{\mathrm{Cayley}}^{-1})\widehat{\diamondsuit u} \in \mathrm{H}^2$ for some $u \in \mathrm{L}^2(\mathbf{R}_+;U)$, then $\widehat{\mathbb{D}}\widehat{u} \in \mathrm{H}^2$, by (13.36) and Theorem 13.2.3(a).

(b2) This is obvious from the definition.

(b3) Fix and open $\Omega' \subset \Omega$ s.t. $\emptyset \neq \overline{\Omega'} \subset \Omega$, so that $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\Omega'; \mathcal{B}(U,Y))$. Now $\widehat{\mathbb{D}} \in \mathcal{B}(\mathrm{H}^2, \mathrm{H}^\infty(\Omega';Y))$, $\mathrm{H}^2 \underset{c}{\subset} \mathrm{H}^\infty(\Omega';Y)$ (by Lemma F.3.2(a)&(b)) and $\widehat{\mathbb{D}}[\mathrm{H}^2] \subset \mathrm{H}^2$, hence $\widehat{\mathbb{D}} \in \mathcal{B}(\mathrm{H}^2, \mathrm{H}^2)$, i.e., $\mathbb{D} \in \mathcal{B}(\mathrm{L}^2(\mathbf{R}_+;U), \mathrm{L}^2(\mathbf{R}_+;Y))$.

Since $\widehat{\mathbb{D}}\mathrm{e}^{\cdot t} = \mathrm{e}^{\cdot t}\widehat{\mathbb{D}}$ for all $t \in \mathbf{R}$, $\mathbb{D}$ commutes with translations, i.e., $\mathbb{D} \in \mathrm{TIC}(U,Y)$.

Obviously, the map $\heartsuit\mathbb{D}$ of this definition is equal to the restriction to $\ell^2(\mathbf{N};U)$ of the map $\heartsuit\mathbb{D}$ of Definition 13.2.2. □

Sometimes we wish to map ∞ to some other point of $\partial\mathbf{D}$ than to -1. Then we can combine the Cayley transform with a rotation:

Lemma 13.2.6 (Different Cayley) *Proposition 13.2.5 and Theorem 13.2.3 except possibly the claims on* **Я**, *τ and $()^{\mathrm{d}}$ (in (c1), (c2) and (c4)) hold even if we replace ϕ_{Cayley} by $\phi_{\mathrm{Cayley}} \circ \widehat{R}_\alpha$ for some $\alpha \in \partial\mathbf{D}$, where $(\widehat{R}\widehat{u})(z) := \widehat{u}(\alpha z)$ (equivalently, $Ru_k := \alpha^k u_k$ ($k \in \mathbf{Z}$)) for all z and u.*

Proof: Obviously, $\widehat{Ru}(z) = \sum_k z^k \alpha^k u_k = (\widehat{R}\widehat{u})(z)$ for all z and u, R is an isometric isomorphism on ℓ^2, $R\pi^+ = \pi^+ R$ etc. Part of the proposition and of the theorem follows directly from this and the rest (with the above exceptions) can easily be verified. □

Notes

Theorem 13.2.3(a) is essentially given in [RR], where one can find further information on this transform (alternatively, see Section 11.4 of [Sbook]).

13.3 Discrete-time systems ($wpls(U,H,Y)$)

Do you think when two representatives holding diametrically opposing views get together and shake hands, the contradictions between our systems will simply melt away? What kind of a daydream is that?

— Nikita Khrushchev (1894–1971)

In this section we present wpls's, the discrete counterparts of WPLSs. We will present the main definitions and results of the continuous-time part of this monograph converted to the discrete-time form; this section covers mainly the theory of Chapters 2–7. Thus, generators, $\mathcal{Z}$-transforms of maps, stability and feedback. Less straight-forward results (on stabilizability) are presented at the end of this section, and most of main results are contained in Theorem 13.3.13, which covers the discrete counterparts of almost all continuous-time results in this monographs (see the other chapters for details).

We start with the definition:

Definition 13.3.1 (wpls) *Let $r > 0$. An* r-stable discrete-time well-posed linear system (r-stable wpls) *on (U,H,Y) is a quadruple $\left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]$ of operators for which*

(1.) $A \in \mathcal{B}(H)$, and $\sup_{k\in\mathbf{N}} \|r^{-k}A^k\| < \infty$;

(2.) $\mathbb{B} \in \mathcal{B}(\ell^2_r(\mathbf{Z}_-;U),H)$ satisfies $\mathbb{B}\tau\pi^- = A\mathbb{B}$;

(3.) $\mathbb{C} \in \mathcal{B}(H,\ell^2_r(\mathbf{N};Y))$ satisfies $\mathbb{C}A = \pi^+\tau\mathbb{C}$;

(4.) $\mathbb{D} \in \mathrm{tic}_r(U,Y)$ and $\pi^+\mathbb{D}\pi^- = \mathbb{C}\mathbb{B}$;

we write $\left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{wpls}_r(U,H,Y)$ to express this, and we set $\mathrm{wpls} := \cup_{r>0}\mathrm{wpls}_r$.

If $\left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{wpls}$, and (3.) and (4.) hold for $r = 1$, then $\left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]$ is a stable-output system *($\left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{sos}$).*

If [$\left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right] \in \mathrm{wpls}_r$ and] $r^{-j}A^jx \to 0$ strongly (resp. weakly) as $j \to \infty$ for all $x \in H$, then [$\left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]$ and] A is strongly (resp. weakly) r-stable *and $\left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]$ is* strongly (resp. weakly) internally r-stable.

We call A the state map, *$\mathbb{B}$ the* reachability map, *$\mathbb{C}$ the* observability map *and $\mathbb{D}$ the* I/O map *of $\left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D}\end{array}\right]$; the map A (resp. $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$) is* r-stable *if (1.) (resp. (2.), (3.), (4.)) holds.*

The prefix "1-" is often omitted, e.g., systems in wpls_1 are called stable. *Systems in wpls_r for some $r < 1$ are called* exponentially stable*; similarly, if (1.) holds for some $r < 1$, we call A* exponentially stable.

Exponentially stable operators are often called power stable, but we wish to have our terminology compatible with the continuous-time notation; hence we sometimes also write $\mathbb{A}(t) := \mathbb{A}^t := A^t$.

We shall often use the basic identities $\mathbb{B}\tau^k\pi^- = A^k\mathbb{B}$, $\mathbb{C}A^k = \pi^+\tau^k\mathbb{C}$.

Lemma 13.3.2 ($\mathbf{wpls}_r \subset \mathbf{wpls}_{r'}$) *Let* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \text{wpls}_r$ *for some* $r > 0$. *Then* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \text{wpls}_{r'}$ *for all* $r' > r$. □

(The proof is analogous to that of Lemma 6.1.2 and omitted.) In fact, if A is r-stable, then $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ is r' stable for all $r' > r$, by Lemma 13.3.8.

One usually defines discrete systems by (13.40) below. This is not a problem, because wpls's correspond 1-1 to the solutions of (13.40):

Lemma 13.3.3 (Generators of a wpls)

(a) For each $\Sigma := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \text{wpls}$, *there is a unique quadruple of operators* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U, H \times Y)$, *called the* generators *of* $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$, *s.t. for* $x \in H$ *and* $u \in \mathrm{c_c}(\mathbf{N};U)$ *we have*

$$\begin{aligned} \mathbb{B}u &= \sum_{j=0}^{\infty} A^j B u_{-j-1} = \sum_{k=-\infty}^{-1} A^{-k-1} B u_k, \\ (\mathbb{C}x)_k &= CA^k x \quad (k \in \mathbf{N}), \\ (\mathbb{D}u)_k &= \sum_{j=0}^{\infty} CA^j B u_{k-j-1} + Du_k = \sum_{j=-\infty}^{k-1} (\tau^{-j-1}\mathbb{C}Bu_j)(k) + Du_k \quad (k \in \mathbf{Z}). \end{aligned} \tag{13.38}$$

Moreover, (13.38) hold for any $u \in \ell^2_{\text{loc}}([n,+\infty);U) + \ell^2_r(\mathbf{Z};U)$, $n \in \mathbf{Z}$ *and* r *is s.t.* $\Sigma \in \text{wpls}_r$. *We also have (here* $e_k := \chi_k$ $(k \in \mathbf{Z})$*)*

$$Bu = \mathbb{B}(ue_{-1})), \ \ Cx = (\mathbb{C}x)_0, \ \ Du = (\mathbb{D}(ue_0))_0 = \widehat{\mathbb{D}}(0) \text{ for } u \in U, \ x \in H. \tag{13.39}$$

Moreover, the unique solution (on $\mathbf{N}$*) of the difference equation pair*

$$\begin{cases} x_{j+1} = Ax_j + Bu_j, \\ \quad\ y_j = Cx_j + Du_j, \end{cases} \tag{13.40}$$

with initial value $x_0 \in H$ *and input* $u \in \mathrm{c_c}(\mathbf{N};U)$ *is given by*

$$\begin{bmatrix} x_j \\ y \end{bmatrix} = \begin{bmatrix} \mathbb{A}^j & \mathbb{B}\tau^j \\ \mathbb{C} & \mathbb{D} \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} \quad (j = 1,2,\ldots) \tag{13.41}$$

(formula (13.41) determines $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ *uniquely on* $H \times \mathrm{c_c}(\mathbf{N};U)$, *hence as a wpls).*

(b) Conversely, for each $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathcal{B}(H \times U, H \times Y)$, *the operators defined by (13.38) are the unique solution of (13.40) (and (13.41)), and they extend to a (unique) wpls. The resulting wpls is r-stable (and* $\mathbb{D} \in \ell^1_r$*) for any* $r > \rho(A)$ *(and for no* $r < \rho(A)$*). We call this wpls the* wpls generated by $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, *and we write* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$.

(c) Let $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \text{wpls}_r(U,H,Y)$. *Then (13.41) is the solution of (13.40) for any* $x_0 \in H$ *and* $u \in \mathbf{N} \to U$ *(the* initial value setting*). Similarly,*

$x_j = \mathbb{B}\tau^j u$, $y = \mathbb{D}u$ *is a solution of (13.40) for any* $u \in \ell^2_r(\mathbf{Z};U)$ *(the* time-invariant setting *satisfying* $x_j \to 0$ *as* $j \to -\infty$. *Moreover, we have*

$$\mathbb{B}^t u := \mathbb{B}\tau^t \pi^+ u = \sum_{k=0}^{t-1} A^k B u_{t-1-k} = \sum_{k=0}^{t-1} A^{t-1-k} B u_k \quad (u : Z \to U), \tag{13.42}$$

$$\mathbb{D} = D + C\mathbb{B}\tau \tag{13.43}$$

$$(\mathbb{D}^* u)_k = \sum_{j=0}^{\infty} B^*(A^*)^j A^* u_{n+j+1} + D^* u_k \tag{13.44}$$

$$(u \in \ell^2_{-r}(\mathbf{Z};U) \text{ or } u : (-\infty, n) \to U,\ n \in \mathbf{Z}). \tag{13.45}$$

Note that (b) shows that the whole wpls is exponentially stable iff A is (cf. Lemma 6.1.10); equivalently, iff $\rho(A) < 1$, i.e., iff $\sigma(A) \subset \mathbf{D}$ (see Lemma 13.3.7).

As above, we will denote the generators of operators and feedthrough operators of tic$_\infty$ operators by corresponding (ordinary) letters.

Proof: (a)&(b) Except for the claims proved below, the stable case of this follows from Section 4 of [S99], see [Mal00] for proofs, and the general case follows by scaling (see Remark 13.3.9); also the reader can easily verify the results.

The equation $\widehat{\mathbb{D}}(0) = D$ follows easily from (13.38).

Equations (13.41) define a wpls uniquely, because $\mathrm{c_c}$ is dense in ℓ^2.

The condition $r > \rho(A)$ implies that $\|r^{-k}A^k\| \le 1$ for big k, hence $\|r^{-k}A^k\|$ is then bounded (similarly, it is unbounded for $r < \rho(A)$). Replacing r by $r' \in (\rho(A), r)$ above, we see from (13.38) that $\mathbb{B}, \mathbb{C}, \mathbb{D}$ are r-stable and $\mathbb{D} \in \ell^1_r$.

(c) Obviously, (13.41) solves (13.40) in both cases. On the other hand, $\mathbb{B}\tau^j u = \mathbb{B}\pi^-\tau^j u$, and $\pi^-\tau^j u \to 0$ as $j \to -\infty$. The formulae for $\mathbb{B}^t$, $\mathbb{D}$ and $\mathbb{D}^*$ are straightforward. □

Any tic$_\infty$ map has a realization:

Definition 13.3.4 (Realization) *Let* $\mathbb{D} \in \mathrm{tic}_r(U,Y)$. *If* $\left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{wpls}(U,H,Y)$ *for some Hilbert space* H*, then we call* $\left[\begin{array}{c|c} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$ *(together with* H*) a* realization *of* $\mathbb{D}$.

We call the (strongly r-stable) system

$$\left[\begin{array}{c|c} \pi^+\tau & \pi^+\mathbb{D}\pi^- \\ \hline I & \mathbb{D} \end{array}\right] = \left(\begin{array}{c|c} \pi^+\tau^1 & \pi^+(\mathbb{D}\cdot e_{-1}) \\ \hline \pi_{\{0\}} & D \end{array}\right) \in \mathrm{wpls}_r(U, \ell^2_r(\mathbf{N};Y), Y) \tag{13.46}$$

the exactly observable realization *of* $\mathbb{D}$.

We now state the discrete version of dual systems. This requires (13.12), hence we have to use $\mathbf{Я}_{-1}$ instead of $\mathbf{Я}$ (recall that $(\mathbf{Я}_{-1}x)_k := x_{-1-k}$). Fortunately, $\mathbf{Я}_{-1}\mathbb{D}^*\mathbf{Я}_{-1} = \mathbf{Я}\tau^{-1}\mathbb{D}^*\tau\mathbf{Я} = \mathbf{Я}\mathbb{D}^*\mathbf{Я} =: \mathbb{D}^{\mathrm{d}}$; but for the duals of $\mathbb{B}$ and $\mathbb{C}$ the difference between $\mathbf{Я}$ and $\mathbf{Я}_{-1}$ is meaningful:

Proposition 13.3.5 (Dual system) *Let* $\left[\begin{smallmatrix} \mathbb{A}^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{wpls}_r$*,* $r > 0$*. Then its* (causal) dual system *(or (causal) adjoint system)*

$$\left[\begin{array}{c|c} A^{\cdot} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]^{\mathrm{d}} := \left[\begin{array}{c|c} (A^{\mathrm{d}})^{\cdot} & \mathbb{C}^{\mathrm{d}} \\ \hline \mathbb{B}^{\mathrm{d}} & \mathbb{D}^{\mathrm{d}} \end{array}\right] := \left[\begin{array}{c|c} (A^*)^{\cdot} & \mathbb{C}^* \boldsymbol{Я}_{-1} \\ \hline \boldsymbol{Я}_{-1}\mathbb{B}^* & \boldsymbol{Я}_{-1}\mathbb{D}^*\boldsymbol{Я}_{-1} \end{array}\right] \tag{13.47}$$

is also in wpls_r*. Moreover,* $\left(\left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]^{\mathrm{d}}\right)^{\mathrm{d}} = \left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]^{\mathrm{d}} = \left(\begin{smallmatrix} A^* & C^* \\ B^* & D^* \end{smallmatrix}\right)$*.*

Here the adjoints are taken with respect to the ℓ^2 inner product (i.e., without a weight function), e.g., for $\mathbb{C} \in \mathcal{B}(H, \ell^2_r(\mathbf{N};Y))$ we have that $\mathbb{C}^* \in \mathcal{B}(\ell^2_{1/r}(\mathbf{N};Y), H)$ and $\langle \mathbb{C}x, y\rangle = \langle x, \mathbb{C}^* y\rangle$ for $x \in H,\ y \in \ell^2_{1/r}(\mathbf{N};Y)$.

Note that $\ell^2_{1/r}$ is the dual of ℓ^2_r with respect to the (weightless) ℓ^2 inner product.

Proof of Proposition 13.3.5: Using (13.12) and (13.11) one can verify that (1.)–(4.) of Definition 13.3.1 hold (e.g., $\mathbb{C}^* \boldsymbol{Я}_{-1} \in \mathcal{B}(\ell^2_r(\mathbf{Z}_-;Y), H)$, and equation $A^*\mathbb{C}^* \boldsymbol{Я}_{-1} = (\pi^+\tau\mathbb{C})^* \boldsymbol{Я}_{-1} = \cdots = \mathbb{C}^* \boldsymbol{Я}_{-1}\tau\pi^-$ is easily verified).

The claim on generators follows easily from (13.38). □

Next we write out the symbols ("Z-transforms") of the components of a wpls:

Lemma 13.3.6 *Let* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}_r(U,H,Y)$*,* $r > 0$*. Then, for* $|z| < 1/\rho(A)$ *and* $u_0 \in U$*, we have*

$$\widehat{\mathbb{A}}(z) = (I - zA)^{-1} = \sum_{k=0}^{\infty} A^k z^k, \tag{13.48}$$

$$\widehat{\mathbb{B}}(z) = z(I - zA)^{-1}B = (z^{-1} - A)^{-1}B = \sum_{k=0}^{\infty} A^k z^{k+1} B, \tag{13.49}$$

$$\widehat{\mathbb{C}}(z) = C(I - zA)^{-1} = \sum_{k=0}^{\infty} CA^k z^k, \tag{13.50}$$

$$\widehat{\mathbb{D}}(z) = D + Cz(I - zA)^{-1}B = D + C(z^{-1} - A)^{-1}B \tag{13.51}$$

$$= D + \sum_{k=0}^{\infty} CA^k B z^{k+1} = D + \widehat{\mathbb{C}}(z)Bz = D + C\widehat{\mathbb{B}}(z) \tag{13.52}$$

$$\widehat{\mathbb{B}}(z)u_0 = \mathbb{B}(z^{-\cdot}u_0), \quad \mathbb{D}z^{-\cdot}u_0 = z^{-\cdot}\widehat{\mathbb{D}}(z)u_0. \tag{13.53}$$

in the sense that $\widehat{\mathbb{A}x_0} = \widehat{\mathbb{A}}x_0$*,* $\widehat{\mathbb{B}\tau u} = \widehat{\mathbb{B}}\widehat{u}$*,* $\widehat{\mathbb{C}x_0} = \widehat{\mathbb{C}}x_0$ *and* $\widehat{\mathbb{D}u} = \widehat{\mathbb{D}}\widehat{u}$ *on* $\mathbf{D}_{1/r}$ *for all* $x_0 \in H$ *and* $u \in \ell^2_r(\mathbf{N};U)$*.*

Thus, $\mathbb{C}$ is stable iff $\widehat{\mathbb{C}} \in \mathrm{H}^2_{\mathrm{strong}}$, i.e., iff $C(I - zA)^{-1}x_0 \in \mathrm{H}^2(\mathbf{D};Y)$ for all $x_0 \in H$. Analogously, $\mathbb{B}$ is stable iff $B^*(I - zA^*)^{-1}x_0 \in \mathrm{H}^2(\mathbf{D};U)$ for all $x_0 \in H$.

Proof: The equations for $\widehat{\mathbb{A}}, \widehat{\mathbb{B}}, \widehat{\mathbb{C}}, \widehat{\mathbb{D}}$ are straightforward.

Set then $u := z^{-\cdot}u_0$, so that $u \in \ell^2_r(\mathbf{Z};U) + \ell^2_{\mathrm{loc}}(\mathbf{N};U)$. Obviously, $\widehat{\mathbb{B}}(z)u_0 = \mathbb{B}(z^{-\cdot}u_0)$, hence

$$\widehat{\mathbb{D}}(z)u_0 = Du_0 + C\mathbb{B}u = (\mathbb{D}u)_0. \tag{13.54}$$

Since $\tau^k u = z^{-k}u$, we have $(\mathbb{D}u)_k = (\mathbb{D}\tau^k u)(0) = z^{-k}\widehat{\mathbb{D}}(z)u_0$ ($k \in \mathbf{Z}$). □

Lemma 13.3.7 (Exp. stable) *The following are equivalent for $A \in \mathcal{B}(H)$:*

(i) A is exponentially stable, i.e., $\sup \|r^{-k}A^k\| < \infty$ *for some* $r < 1$.

(ii) $A^{\cdot}x_0 \in \mathrm{L}^2(\mathbf{R}_+;H)$ *for all* $x_0 \in H$*;*

(ii') $(s \mapsto (I-sA)^{-1}x_0) \in \mathrm{H}^2(\mathbf{D};\mathcal{B}(H))$ *for all* $x_0 \in H$*;*

(iii) $\|\sum_0^\infty A^k\phi_k\|_H \le M\|\phi\|_2$ *for all* $\phi \in \mathrm{c_c}(\mathbf{N};H)$*;*

(iv) $\rho(A) < 1$*, where* $\rho(A) := \lim_{k\to\infty}\|A^k\|^{1/k} = \inf_{k\to\infty}\|A^k\|^{1/k} = \max|\sigma(A)| \le \|A\|$*;*

(v) $\sigma(A) \subset \mathbf{D}$.

The value $\rho(A)$ is called the *spectral radius* of A (see Lemma A.3.3).

Proof: By Lemma A.3.3(r1)&(s1), we have (iv)$\Leftrightarrow$(v). Equivalence "(i)$\Leftrightarrow$(iv)" is almost trivial. We obtain "(ii)$\Leftrightarrow$(ii')" from (13.48) and "(ii)$\Leftrightarrow$(i)" from [W89d] (which shows that the weak form of (ii) is sufficient). Implication "(i)$\Leftrightarrow$(iii)" follows as in the proof of Lemma A.4.5. □

Lemma 13.3.8 (Stability) *Let* $\Sigma = \left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{wpls}(U,H,Y)$ *and* $0 < r < r' < \infty$. *Then*

(a1) Σ is exponentially stable iff A is exponentially stable.

(a2) If A is r-stable (or $\rho(A) \le r$), then $\Sigma \in \mathrm{wpls}_{r'}$*,* $\mathbb{B}\tau \in \mathrm{tic}_{r'}(U,H)$*, and* $\mathbb{D} \in \ell^1_{r'}(\mathbf{N};\mathcal{B}(U,Y))*$.

(b1) If $\mathbb{B}$ is r-stable, then $\mathbb{B}\tau$ and $\mathbb{D}$ is r'-stable.

(b2) If $\mathbb{B}\tau$ is r-stable, then $\mathbb{B}$ and $\mathbb{D}$ are r-stable.

(b3) If $\mathbb{C}$ is r-stable, then $\mathbb{D} \in \mathrm{tic}_{r'} \cap \mathcal{B}(\ell^1_r,\ell^2_r)$, $\mathbb{D}^* \in \mathcal{B}(\ell^2_{1/r}(\mathbf{Z};U),\ell^\infty_{1/r}(\mathbf{Z};U))$, $\widehat{\mathbb{D}} \in \mathrm{H}^2_{\mathrm{strong}}(r^{-1}\mathbf{D};\mathcal{B}(U,Y))$ *and Lemma 13.1.3(d) applies.*

Thus, Σ is r-stable for all $r > \rho(A)$.

Proof: (a1) If Σ is exponentially stable, then so is A, by definition. Assume that A is exponentially stable, i.e., that $\|A^k\| \le Mr^k$ for all $k \in \mathbf{N}$ for some $M < \infty$, $r < 1$. By using Lemma 13.3.3(b), one easily verifies that $\|\mathbb{A}x_0\|_2 \le M(1-r^2)^{-1}\|x_0\|^2$ and $\left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right] \in \mathrm{wpls}$ is exponentially stable.

(a2) By Remark 13.3.9, we can w.l.o.g. assume that $r' = 1$, hence we obtain this from (a1) for Σ (see Lemma 13.3.7(iv)&(i) for $\rho(A)$). By applying (a1) with $C = I$ and $D = 0$, we get $\mathbb{B}\tau =$"$\mathbb{D}$"$\in \mathrm{tic}(U,H)$.

Finally, choose $s \in (r,1)$ to obtain $\mathbb{D} \in \mathrm{tic}_{s'}$ (by (a1)). Then $\mathbb{D} \in \ell^1_t(\mathbf{N};\mathcal{B}(U,Y))$ for any $t > s$, particularly for $t = 1$, by Lemma 13.1.3(c3).

(b2) This follows from (13.43) for $\mathbb{D}$; take $C = I, D = 0$ to get $\mathbb{B}\tau = \mathbb{D}$.

(b3) Assume that $\mathbb{C}$ is r-stable. By (13.38), $\mathbb{D}u_0e_0 = Du_0e_0 + \tau^{-1}\mathbb{C}Bu_0e_0 \in \ell^2_r$ for each $u_0 \in U$, hence we get the claims from Lemma 13.1.3(d).

(b1) If $\mathbb{B}$ is r-stable, then so is $\mathbb{B}^{\mathrm{d}}$ and hence then $\mathbb{D}^{\mathrm{d}}$ and $\mathbb{D}$ are r'-stable, by (b3); application with $C = I$ and $D = 0$ shows that also $\mathbb{B}\tau$ is r'-stable. □

Now we are able to present the discrete counterpart of Remark 6.1.9 (see (13.9) for $r^{\cdot} : (x_j) \mapsto (r^jx_j)$):

Remark 13.3.9 (Stability shift) *Let* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}_s(U,H,Y)$. *Then the* stability shift *(or scaling operator)* $\mathcal{T}_r : \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} rA & rB \\ C & D \end{smallmatrix}\right)$ *satisfies*

$$\mathcal{T}_r \left[\begin{array}{c|c} A^{\cdot} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] = \left(\begin{array}{c|c} rA & rB \\ \hline C & D \end{array}\right) = \left[\begin{array}{c|c} (rA)^{\cdot} & \mathbb{B}r^{-\cdot} \\ \hline r^{\cdot}\mathbb{C} & r^{\cdot}\mathbb{D}r^{-\cdot} \end{array}\right] \in \mathrm{wpls}_{rs}(U,H,Y). \quad (13.55)$$

Thus, $\mathcal{T}_r : \mathrm{wpls}_s \mapsto \mathrm{wpls}_{rs}$ *is a bijection.*

Moreover, $\mathcal{T}_r : \mathbb{E} \mapsto r^{\cdot}\mathbb{E}r^{-\cdot}$ *is an isometric isomorphism* $\mathrm{ti}_r(U,Y) \to \mathrm{ti}_{rs}(U,Y)$ *as well as* $\ell^1_s(\mathbf{Z};\mathcal{B}(U,Y))* \to \ell^1_{rs}(\mathbf{Z};\mathcal{B}(U,Y))*$. □

(We leave the simple proof to the reader (cf. (13.10)).)

We let $\mathcal{T}_r$ also denote its components (note that this is in accordance with $\mathcal{T}_r\mathbb{E} := r^{\cdot}\mathbb{E}r^{-\cdot}$ for $\mathbb{E} = \mathbb{D} \in \mathrm{tic}$).

In Sections 6.6–6.7 and Chapter 7, we reduced all kinds of feedbacks to static output feedback for WPLSs. Next we shall do the same for wpls's. As in Section 6.6, we replace the input u by $u_L + Ly$, where u_L is an external input and $L \in \mathcal{B}(Y,U)$ is a static feedback operator (see Figure 6.2) to obtain equations

$$\begin{cases} x_{j+1} = Ax_j + B(Ly_j + (u_L)_j), \\ \quad y_j = Cx_j + D(Ly_j + (u_L)_j), \quad j \in \mathbf{Z}. \end{cases} \quad (13.56)$$

These are algebraically the same as (6.123)–(6.124), in particular, they have a unique solution (i.e., they are well-posed) iff $I - DL$ is invertible. If that is the case, we call the feedback admissible:

Definition 13.3.10 (Admissible static output feedback) *Let* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}(U,H,Y)$. *An operator* $L \in \mathcal{B}(Y,U)$ *is called an* admissible (static) output feedback operator *for* $\left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ *if* $I - L\mathbb{D} \in \mathcal{G}\mathrm{tic}_\infty(U)$.

We call L r-stabilizing *if* $\Sigma_L \in \mathrm{wpls}_r$ *etc., as in Definition 6.6.4.*

By Lemmas A.1.1(f6) and 13.1.7, each of the conditions "$I - LD \in \mathcal{GB}(U)$", "$I - DL \in \mathcal{GB}(Y)$", and $I - \mathbb{D}L \in \mathcal{G}\mathrm{tic}_\infty(Y)$ is equivalent to $I - L\mathbb{D} \in \mathcal{G}\mathrm{tic}_\infty(U)$.

The corresponding closed-loop system is given below:

Lemma 13.3.11 *Let* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}_s(U,H,Y)$ *and* $I - LD \in \mathcal{GB}(U)$. *Then*

$$\left(\begin{array}{c|c} A_L & B_L \\ \hline C_L & D_L \end{array}\right) := \left(\begin{array}{c|c} A + BL(I-DL)^{-1}C & B(I-LD)^{-1} \\ \hline (I-DL)^{-1}C & (I-DL)^{-1}D \end{array}\right) \quad (13.57)$$

$$= \left[\begin{array}{c|c} A_L & \mathbb{B}_L \\ \hline \mathbb{C}_L & \mathbb{D}_L \end{array}\right] := \left[\begin{array}{c|c} A + BL(I-DL)^{-1}C & \mathbb{B}(I-L\mathbb{D})^{-1} \\ \hline (I-\mathbb{D}L)^{-1}\mathbb{C} & (I-\mathbb{D}L)^{-1}\mathbb{D} \end{array}\right] \in \mathrm{wpls}(U,H,Y). \quad (13.58)$$

Moreover, $A_L^j - A^j = \mathbb{B}\tau^j L\mathbb{C}_L = \mathbb{B}\tau^j L(I - \mathbb{D}L)^{-1}\mathbb{C} = \mathbb{B}_L L\tau^j\mathbb{C}$ *for* $j \geq 0$.

Proof: By solving (13.56), we obtain (13.57). By Lemma 13.3.3(b), the operators (13.57) generate a wpls, whose state map is necessarily A_L, and whose reachability, observability, and I/O maps $\mathbb{B}'_L, \mathbb{C}'_L, \mathbb{D}'_L$ can be found by solving

$$\begin{bmatrix} x_j \\ y \end{bmatrix} = \begin{bmatrix} (A_L)^j & \mathbb{B}'_L\tau^j \\ \mathbb{C}'_L & \mathbb{D}'_L \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} \quad (j = 1, 2, \ldots). \quad (13.59)$$

But from (13.59) and "$u = u_L + Ly$" we obtain

$$\begin{bmatrix} x_j \\ y \end{bmatrix} = \begin{bmatrix} A^j + \mathbb{B}\tau^j L\mathbb{C}_L & \mathbb{B}_L\tau^j \\ \mathbb{C}_L & \mathbb{D}_L \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} \quad (j = 1, 2, \ldots). \tag{13.60}$$

Thus, $\mathbb{B}_L' = \mathbb{B}_L$, $\mathbb{C}_L' = \mathbb{C}_L$ and $\mathbb{D}_L' = \mathbb{D}$, and $A_L^j - A^j = \mathbb{B}\tau^j L\mathbb{C}_L$; the last equation follows Lemma A.1.1(f6). □

Thus, the formula for state feedback, defined as in Definition 6.6.10, takes the following form:

Lemma 13.3.12 *A state feedback pair* $\begin{pmatrix} K \mid F \end{pmatrix}$ *is admissible for* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in$ wpls *iff* $I - F \in \mathcal{GB}$. *If this is the case, then the resulting closed-loop system is given by (here* $M := (I - F)^{-1}$*)*

$$\Sigma_\flat := \left[\begin{array}{c|c} A^{\cdot} + \mathbb{B}\tau(\cdot)\mathbb{M}\mathbb{K} & \mathbb{B}\mathbb{M} \\ \hline \mathbb{C} + \mathbb{D}\mathbb{M}\mathbb{K} & \mathbb{D}\mathbb{M} \\ \mathbb{M}\mathbb{K} & \mathbb{M} - I \end{array}\right] = \left(\begin{array}{c|c} A + BMK & BM \\ \hline C + DMK & DM \\ MK & M - I \end{array}\right). \tag{13.61}$$

□

The pair $\begin{pmatrix} MK \mid 0 \end{pmatrix}$, where $M := (I - F)^{-1}$, is equivalent to $\begin{pmatrix} K \mid F \end{pmatrix}$ in the sense that it is admissible or stabilizing iff $\begin{pmatrix} K \mid F \end{pmatrix}$ is, and the resulting closed-loop system is

$$\left[\begin{array}{c|c} A + BMK & B \\ \hline C + DMK & D \\ MK & 0 \end{array}\right]. \tag{13.62}$$

We identify $K \in \mathcal{B}(H, U)$ as a *state feedback operator* to the (admissible) state feedback pair $\begin{bmatrix} K \mid 0 \end{bmatrix}$. Thus, $K \in \mathcal{B}(H, U)$ is *exponentially stabilizing* iff $A + BK$ is exponentially stable, etc. See also Lemma 13.3.16.

Also other definitions of Section 6.6 can be converted to the discrete time case analogously; the results can be converted in a similar way:

Theorem 13.3.13 (WPLS results hold for wpls's) *If we make the following replacements (here CT refers to continuous and DT to discrete time):*

$$\begin{aligned}
&\text{WPLS} \mapsto \text{wpls},\ \ \text{SOS} \mapsto \text{sos},\ \ \text{TI} \mapsto \text{ti},\ \ \text{WR},\text{SR},\text{ULR},\text{TIC} \mapsto \text{tic}\\
&\text{MTIC}^*_* \mapsto \heartsuit\text{MTIC}^*_*,\ \ \mathrm{L}^2 \mapsto \ell^2,\ \ C_c^\infty \mapsto \mathrm{c_c},\ \ \pi_+ \mapsto \pi^+,\ \ \pi_- \mapsto \pi^-, \text{Я} \mapsto \text{Я}_{-1};\\
&\mathbf{R} \mapsto \mathbf{Z},\ \ \mathbf{R}_+ \mapsto \mathbf{N},\ \ \mathbf{R}_- \mapsto \mathbf{Z}_-,\ \ i\mathbf{R} \mapsto \partial\mathbf{D} \setminus \{-1\},\ \ i\mathbf{R} \cup \{\infty\} \mapsto \partial\mathbf{D},\ \ \mathbf{C}^+ \mapsto \mathbf{D},\ \ \overline{\mathbf{C}^+} \cup \{\infty\} \mapsto \overline{\mathbf{D}};\\
&[t_1,t_2] \mapsto [t_1,t_2-1],\ \ \int_{t_1}^{t_2} \mapsto \sum_{t_1}^{t_2-1};\\
&\mathbb{A} \mapsto A^{\cdot},\ \ \mathbb{A}(t) \mapsto A^t,\ \ \tau(t) \mapsto \tau^t;\ \ C_{\mathrm{w}}, C_{\mathrm{s}}, C_{\mathrm{L,w}}, C_{\mathrm{L,s}}, C_{\mathrm{c}} \mapsto C\ \ \textit{etc.};\\
&\textit{any regularity assumption/statement on a map or system} \ \mapsto\ \textit{a true assumption/statement}\\
&\textit{(the same applies to the boundedness of input and output operators)};\\
&\mathrm{Dom}(A), H_B, H^*_{C,K}, H_{\pm 1}, H^*_{\pm 1} \mapsto H;\\
&\textit{“[e]IARE”,”[e]CARE”,”[e]}B^*_{\mathrm{w}}\textit{-CARE”} \ \mapsto\ \textit{“[e]DARE”},\\
&\textit{[e]IARE,[e]CARE (the equations)} \ \mapsto\ \textit{[e]DARE (the corresponding DT equation)};\\
&S = D^*JD \mapsto S = D^*JD + B^*\mathcal{P}B \quad \textit{(similarly for anything based on equation } S = D^*JD\textit{)};\\
&(s-A)^{-1} \mapsto (I-sA)^{-1},\ \ (s-A)^{-1}B \mapsto s(I-sA)^{-1};\\
&(\mathcal{P}, S, [\ \mathbb{K} \mid \mathbb{F}\ ]) \mapsto (\mathcal{P}, S, K) \quad \textit{(for solutions of the eIARE)},\\
&\text{Stability indices } \omega \text{ and Laplace/Z-transform arguments } s\text{:}\\
&“\omega \ge 0'' \mapsto “\omega \ge 1'',\ \ “\omega > 0 \mapsto “\omega > 1'',\\
&“\omega = 0'' \mapsto “\omega = 1'',\ \ “\omega \ne 0'' \mapsto “\omega \in (0,\infty) \setminus \{1\}'';\\
&“\mathrm{Re}\, s > \omega'' \mapsto'' s \in \mathbf{D}''_{1/\omega},\quad 0\textit{-stabilizing} \mapsto 1\textit{-stabilizing};\ \ s = +\infty \mapsto s = 0\\
&\mathrm{e}^{\pm\omega t} \mapsto \omega^{\pm t},\ \ \mathrm{e}^{\pm\omega\cdot} \mapsto \omega^{\pm\cdot},\ \ \omega + \alpha \mapsto \alpha\omega,\ \ s - \alpha \mapsto \alpha s,\ \ ir \mapsto \mathrm{e}^{ir}
\end{aligned} \tag{13.63}$$

(naturally, the above changes apply also any other stability index (resp. transform argument, element of $i\mathbf{R}$*, time value) in place of* ω *(resp. s, ir, t), any other system in place of* Σ *etc.), then the following definitions are still applicable and the following results (among others) still hold:*

Lemma A.4.2(h1), Proposition E.1.8; Sections 2.1 (note that Lemma 2.1.15 now says that $(\mathbb{D}(s^{\cdot}u_0))(k) = s^k\widehat{\mathbb{D}}(s)u_0$ *for all* $\mathbb{D} \in \mathrm{tic}_r(U,Y)$*,* $k \in \mathbf{Z}$*,* $s \in r\mathbf{D}$*,* $u_0 \in U$*), 2.2, 2.4 and 2.5.*

Chapter 4 except possibly Lemmas 4.1.3 and 4.1.5.

Sections 6.4 and 6.5 except Lemma 6.5.10(c) and possibly the claims on p.r.c. (probably also they are true); Section 6.6 and 6.7 except Proposition 6.6.18 (in fact, even 6.6.18 it is true except for its parts that are meaningless in the discrete-time case) and Example 6.6.23.

See Theorems 14.1.3, 15.1.1, 11.5.2 and 12.2.2 for Chapters 8–12; (mainly) Section 14.3 for Chapter 5 and Lemma 13.3.19 for Lemma 6.3.20.

Of course, also the (non-italic) text between subsections is almost completely applicable too (although the regularity problems disappear in this discrete-time case).

Moreover, most MTIC results can also be rewritten for discrete time for classes ℓ^1_+*, $\mathrm{tic}_{\mathrm{exp}}$ etc. (see Lemma 14.3.5) in place of MTIC classes (but the “$S \ne D^*JD$” requirement of Hypothesis 8.4.8 is not satisfied by these classes). There are some

CARE results that implicitly or explicitly have D^*JD in place of S. As explained above, the CAREs must be replaced by DAREs, hence this term must always be replaced by $S := D^*JD + B^*\mathcal{P}B$ while writing the results in their discrete-time forms (thus, most "D^*JD" terms and their simplified forms must be replaced by "$D^*JD + B^*\mathcal{P}B$", whereas any $\lim_{s\to+\infty} B_{\mathrm{w}}^*\mathcal{P}(s-A)^{-1}B$ terms may be removed; this makes results such as Theorem 10.2.9 and Proposition 9.9.12(c)(3.) much less useful in their discrete-time forms (since they are based on "$S = D^*JD$")). Most of the time the reader need not be concerned about this since this has been explicitly written into the results following the theorems listed above.

As noted around Example 14.2.9, there is no discrete equivalent for the B_{w}^*-CARE theory of Section 9.2 (in particular, we almost always have $S \neq D^*JD$); the same holds largely for the $\mathrm{Dom}(A_{\mathrm{crit}})$-CARE theory of Section 9.7 (since now $\mathrm{Dom}(A_{\mathrm{crit}}) = H = \mathrm{Dom}(A)$; note that most of the theory holds with S in place of D^*JD).

Proof of Theorem 13.3.13: All proofs hold in discrete time too, mutatis mutandis, usually the discrete time versions become simpler. Thus, the references from discrete time to continuous time are always non-essential. However, some results are proved in discrete time only, and the results are then transferred to continuous time by discretization. Therefore, if one wishes to verify the proofs linearly, one should verify the entire monograph in its discrete time form before verifying the continuous time forms (alternatively, one could read both settings simultaneously but go somewhat further in discrete time in such places).

There is a short cut: by using Theorem 13.2.3, one can convert the results corresponding to TIC maps only. Some other results are implied by Remark 6.5.11 (mainly the ones concerning TIC_∞ maps only). By discretization, one can convert uniqueness results from discrete time and existence results from continuous time.

For the rest, one must make the corresponding changes in proofs too. Some proofs contain references which either can be replaced by the discrete results of this monograph or whose proofs must be verified in the same way; we mention that we have verified for the discrete case [S97b, Lemma 21], all of [S98a] and [S98c] (including the parts of [S98b] that are contained in [S98c] (and more), in particular, Subsections 1–3.4, 3.9(i), 4.1–4.7, and Chapter 5 apply) (with replacements (13.63), both with Remark 6.1.15 and without it.

All this is quite straightforward (the explicit results above contain all the nonstraightforward parts). We sketch below the hardest proofs:

The proof of Lemma 2.2.7 does not need the reference to Lemma D.1.8 in this (discrete) case.

The Corona Theorem for $\mathcal{A} = \mathrm{ti}$ follows directly (use Theorem 13.2.3(b1) for (iv)); case $\mathcal{A} = \ell^1$ follows as shown in the proof of case $\mathcal{A} = \mathrm{MTI_d}$ (take $\mathcal{A} = \heartsuit^{-1}\widetilde{\mathcal{A}}$, prove the theorem, then make the replacements).

Proposition 4.1.7: "(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(iv')$\Leftrightarrow$(iv")" follows from Theorem 13.2.3, the rest by transforming the original proof.

Lemma 4.1.8: If $\mathbb{D} \in \mathrm{tic}(U,Y)$ and $\mathbb{D}^*\mathbb{D} \not\geq \varepsilon I$ for any $\varepsilon > 0$, then we can construct $\widehat{u} \in \mathrm{H}^2_1 \setminus \mathrm{H}^2$ as in the proof, but for $\mathbb{F} := T^{-1}\mathbb{D}$, with $|r_1| < \pi/2$ (see

Lemma 13.1.4).

Then $v := Tu$ satisfies $\widehat{v} \in \mathrm{H}(\mathbf{D};U)$, hence $v \in \mathrm{H}^2(r\mathbf{D};U)$ for all $r < 1$, but $\|\widehat{v}\|_2 = \|\widehat{u}\|_{\mathrm{L}^2([-\pi,\pi);U)} = \infty$. However, $\mathbb{D}v = T(\mathbb{F}u) \in \mathrm{L}^2$. Thus, then $\mathbb{D}$ is not quasi-left-invertible. This shows (a); the rest can be shown as in the original proof.

Theorem 6.7.10(d): 1° *(ii)⇒(i):* Assume that $s(I-sA)^{-1}B \in \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(U,H))$ (i.e., $\mathbb{B}\tau \in \mathrm{tic}$) and that Σ is optimizable, hence exponentially stabilizable, by Proposition 13.3.14. Thus, there is $K \in \mathcal{B}(H,U)$ s.t. $\mathbb{A}_\circlearrowleft := \mathbb{A} + \mathbb{B}\tau\mathbb{K}_\circlearrowleft$ is exponentially stable, hence

$$(I-sA)^{-1} = (I-sA_\circlearrowleft)^{-1} - s(I-sA)^{-1}BK(I-sA_\circlearrowleft)^{-1} \in \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(H)). \tag{13.64}$$

Thus, A is exponentially stable, by Lemma 13.3.7(ii'). 2° *(viii)⇒(v):* This follows from Proposition 13.3.14. (The rest of the proof of Theorem 6.7.10 does not require clarification.)

The DT version of part of Chapter 9 is verified in Theorem 14.1.3.

We recommend reading "$[t_1,t_2]$" as "$[t_1,t_2)$" (i.e., $[t_1,t_2-1]$), so that it holds in both discrete-time and continuous-time cases. □

Since B and C are always bounded in discrete-time, several aspects of system theory become as simple as for finite-dimensional systems:

Proposition 13.3.14 (Opt. ⇔ exp. stab.) *A wpls is optimizable iff it is exponentially stabilizable. Thus, a wpls is estimatable iff it is exponentially detectable.*

Proof: Assume that $\Sigma \in \mathrm{wpls}(U,H,Y)$ is optimizable. By Exercise 6.34(i) of [CZ] (with $C = I$), there is $K \in \mathcal{B}(H,U)$ s.t. $A+BK$ is stable (use Lemma 13.3.7(ii) and the fact that $(A+BK)^{\cdot}x_0 \in \ell^2$ for all $x_0 \in H$). The converse is obvious, and the dual claim follows, by duality. □

We have $u, y \in \ell^2 \Rightarrow x \in \ell^2$ for estimatable systems:

Theorem 13.3.15 ($u, y \in \ell^2 \Rightarrow x \in \ell^2$) *Let* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}(U,H,Y)$ *be estimatable. Then there is $M < \infty$ s.t. if $u \in \ell^2(\mathbf{R}_+;U)$ and $x_0 \in H$ are s.t. $y := \mathbb{C}x_0 + \mathbb{D}u \in \ell^2$, then $x := \mathbb{A}x_0 + \mathbb{B}\tau u \in \ell^2$ and $\|x\|_2 \le M(\|x_0\|_H + \|u\|_2 + \|y\|_2)$.*

Proof: Because Σ is exponentially detectable, we have $\mathbb{A}x_0 + \mathbb{B}\tau u = \mathbb{A}_\sharp x_0 + \mathbb{B}_\sharp \tau u - \mathbb{H}_\sharp \tau y$, where $\Sigma_\sharp$ is the closed-loop system (6.168) corresponding to an exponentially stabilizing output injection pair $\left[\begin{smallmatrix} \mathbb{H} \\ \mathbb{G} \end{smallmatrix}\right]$, hence $M := \|\mathbb{A}_\sharp\|_{\mathcal{B}(H,\ell^2)} + \|\mathbb{B}_\sharp\tau\|_{\mathrm{tic}} + \|\mathbb{H}_\sharp\tau\|_{\mathrm{tic}} < \infty$, by Lemma 13.3.8. □

It is easy to identify a stabilizing state feedback operator to an exponentially stable system:

Lemma 13.3.16 (*K*) *Let $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}(U,H,Y)$ be exponentially stable and $K \in \mathcal{B}(H,U)$. Then the following are equivalent:*

(i) K is I/O-stabilizing;

(i') K is output-stabilizing;

(i") K is input-stabilizing;

(ii) K is exponentially r.c.-stabilizing;

(iii) $\sigma(A+BK) \subset \mathbf{D}$, i.e., $\rho(A+BK) < 1$;

(iv) $I - Kz(I-zA)^{-1}B \in \mathcal{GB}(U)$ for $z \in \overline{\mathbf{D}}$.

Proof: Obviously, (i')$\Leftarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(i). By Lemma 6.7.9, (i') implies (i).

Since $\mathbb{A}$ is exponentially stable, so are $\mathbb{D}$ and $\mathbb{F}$. Therefore, (i) holds iff $(I-\mathbb{F})^{-1} \in$ tic; or equivalently, iff $I-\mathbb{F} \in \mathcal{G}$tic. But $I-\mathbb{F} \in \mathcal{G}$tic implies that the closed-loop system is exponentially stable, i.e., that (iii) holds, by Corollary 6.6.9. On the other hand, condition $(I-\mathbb{F})^{-1} \in$ tic is equivalent to (iv), because the boundedness of $\widehat{\mathbb{X}}^{-1}$ follows from the compactness of $\overline{\mathbf{D}}$ (see also Lemma D.1.2(b2)).

If (iii) holds, then $I-\mathbb{F} \in \mathcal{G}\mathrm{tic}_{\mathrm{exp}}$, hence then (ii) holds. If (i") holds, then (iii) holds, by Lemma 6.6.8(c). □

Lemma 13.3.17 (Jointly stabilizing *K* & *H*) *Let $\Sigma \in \mathrm{wpls}(U,H,Y)$. Then the following holds:*

(a) Any admissible state feedback and output injection pairs for Σ are jointly admissible.

(b) Any output-stabilizing and exponentially detecting pairs for Σ are exponentially jointly r.c.- and l.c.-stabilizing.

In particular, the following are equivalent:

(i) Σ is exponentially jointly r.c.-stabilizable and l.c.-detectable;
(ii) Σ is optimizable and estimatable;
(iii) Σ is output-stabilizable and estimatable;
(iv) Σ is optimizable and input-detectable.

(c) Let Σ be estimatable. Then any I/O-stabilizing pair for Σ is r.c.-I/O-stabilizing.

Recall that in classical articles (those with $\dim H < \infty$, i.e., with rational transfer functions) the word "stabilizing" means usually "exponentially stabilizing", hence for them one usually makes the prefix "r.c.-" (etc.) redundant by assuming the system to be "detectable" (then any exponentially stabilizing state feedback pair is exponentially r.c.-stabilizing, by, e.g., Lemma 6.6.26 and (shifted) Theorem 6.6.28).

Proof: (a) Assume that $\begin{pmatrix} K \mid F \end{pmatrix}$ and $\left(\frac{H}{G}\right)$ are admissible. Then $\Sigma' := \left(\begin{array}{c|cc} A & H & B \\ \hline C & G & D \\ K & 0 & F \end{array}\right) \in$ wpls, by Lemma 13.3.3, hence $\begin{pmatrix} K \mid F \end{pmatrix}$ and $\left(\frac{H}{G}\right)$ are jointly admissible.

(b) By Proposition 13.3.14, we have (i)$\Rightarrow$(ii)$\Rightarrow$(iii); by duality, (iii)$\Leftrightarrow$(iv), so that only (iii)$\Rightarrow$(i) remains to be proved.

Let $\begin{bmatrix} K \mid F \end{bmatrix}$ and $\left[\frac{H}{G}\right]$ be as in (iii). By Lemma 6.7.9, $\begin{bmatrix} K \mid F \end{bmatrix}$ is exponentially stabilizing. Thus, the (closed-loop) state maps $A_\flat := A+B(I-F)^{-1}K$ and $A+H(I-G)^{-1}C$ are exponentially stable, hence so are the closed-loop systems of Σ' corresponding to $L = \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right]$ and $L = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, by Lemma 13.3.7. Therefore, (i) holds (the coprimeness follows from the exponentially stable discrete form (cf. Remark 13.3.9 and Theorem 13.3.13) of Theorem 6.6.28.

(c) Let $\begin{pmatrix} K \mid F \end{pmatrix}$ and $\left(\frac{H}{G}\right)$ be corresponding pairs; by (a), they are jointly admissible. Thus, if we define $\widetilde{\mathbb{X}}, \widetilde{\mathbb{Y}}, \mathbb{X}, \mathbb{Y}$ by (6.172), then $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{Y}}$ are exponentially stable and $\widetilde{\mathbb{X}}\mathbb{M} - \widetilde{\mathbb{Y}}\mathbb{N} = I$ (because $\Sigma_\sharp$ is exponentially stable). □

Lemma 13.3.18 ($u, x \in \ell^2 \Rightarrow y \in \ell^2$) *Let* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{wpls}(U,H,Y)$. *If* $u, x \in \pi^+\ell^2$, *then* $y \in \pi^+\ell^2$ *and* $\|y\|_2 \le M(\|u\|_2 + \|x\|_2)$, *where* $x_0 \in H$ *is arbitrary,* $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] := \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B}\tau \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right]$, *and* $M := \max\{\|C\|, \|D\|\}$. □

(This follows from equation $y = Cx + Du$, (equation (13.40)).)

The discrete-time version of Lemma 6.3.20 is obvious, but we shall record it for future use:

Lemma 13.3.19 *Let* $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{wpls}(U,H,Y)$. *Assume that* $r > 0$, $u \in \ell^2_r(\mathbf{N};U)$, *and* $x := \mathbb{B}\tau u \in \ell^2_r(\mathbf{N};H)$. *Then* $(z^{-1} - A)\widehat{x}(z) = B\widehat{u}(z) \in H$ *for a.e.* $z \in r^{-1}\partial\mathbf{D}$.

Assume, in addition, that $y := \mathbb{D}u \in \ell^2_r$. *Then* $\widehat{y} = C\widehat{x} + D\widehat{u} \in Y$ *a.e. on* $r^{-1}\partial\mathbf{D}$. *In particular, for* $r = 1$ *and* $J \in \mathcal{B}(Y)$ *we have*

$$\langle \mathbb{D}u, J\mathbb{D}u \rangle_{\ell^2(\mathbf{Z};Y)} = (2\pi)^{-1} \left\langle \begin{bmatrix} \widehat{x} \\ \widehat{u} \end{bmatrix}, \kappa \begin{bmatrix} \widehat{x} \\ \widehat{u} \end{bmatrix} \right\rangle_{\mathrm{L}^2(\partial\mathbf{D};Y)}, \tag{13.65}$$

where $\kappa := \begin{bmatrix} C & D \end{bmatrix}^* J \begin{bmatrix} C & D \end{bmatrix}$. □

(The proof is a much simpler version of the proof of Lemma 6.3.20, and hence omitted.)

Notes

In the form (13.40), the wpls's in have been studied for several decades; two of the cornerstones being [KFA] and [Fuhrmann81]; whose Sections III.1–III.5 contain a further study on their realizations theory.

Olof Staffans [S99] has formulated stable wpls's essentially as in Definition 13.3.1. Jarmo Malinen [Mal00] has defined wpls's with different domain and range spaces and presented a theory on them; his results include part of Lemma 13.3.3 and Lemma 13.3.12.

The monographs [SF], [RR], [Nikolsky] and [FF] contain rather general operator theory and harmonic analysis, but many of their results are applicable for wpls's. The article [S01] and Chapter 11 of [Sbook] contain applications of [SF] to both continuous-time system (especially for ones with contractive semigroups) and discrete-time systems. In Chapter 11 of [Sbook] Staffans shows how to use the Cayley transform to convert a complete WPLS to a wpls or vice versa, whereas we have only treated the Cayley transform of the I/O map (Theorem 13.2.3); that chapter was written two years after this one.

Observe that Theorem 13.3.13 (and the theorems mentioned right below it) contains the discrete-time variants of most continuous-time results of this monograph. Also for most other continuous-time results the discrete-time variants are true and rather easily verified (usually the same proofs apply, mutatis mutandis). Much of our theory is well known in the finite-dimensional case (see, e.g., [LR] or [IOW]).

13.4 Time discretization (Δ^{S} : WPLS $\to$ wpls)

Our problem is within ourselves. We have found the means to blow the world physically apart. Spiritually, we have yet to find the means to put the world's pieces back together again.

— Thomas E. Dewey (1902–1971) US lawyer, politician

In this section, we shall present *discretization*, a method to convert a WPLS to a wpls (Theorem 13.4.4). In other chapters of this book, we often use discretization to deduce properties of WPLSs from those of wpls's, because the latter ones have bounded generators and can hence be more easily explored. Note that discretization differs from the methods of Theorem 13.2.3 (the Cayley transform) and of Lemma 13.1.4.

The principle is well-known, and it has been used (implicitly) to deduce that any semigroup control system (as defined in [Sal89]) is a WPLS (i.e., that a locally L^2-bounded system is actually bounded w.r.t. L^2_ω for some $\omega \in \mathbf{R}$).

Theorem 13.4.4 describes the preservation of properties of systems and Theorem 13.4.5 of those of I/O maps.

As mentioned above, U, W, H, Y and Z denote Hilbert spaces of arbitrary dimensions.

Definition 13.4.1 *For* $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)$ *we define its* discretization $\Delta^{\ell^2} u : \mathbf{Z} \to \mathrm{L}^2([0,1);U)$ *by* $(\Delta^{\ell^2} u)_n := \pi_{[0,1)}\tau(n)u$ $(n \in \mathbf{Z})$.

Note that this discretization is completely different from the Cayley transform $\heartsuit$: TIC $\leftrightarrow$ tic of Theorem 13.2.3.

The map Δ^{ℓ^2} is obviously a linear map of $\mathrm{L}^2_{\mathrm{loc}}$ one-to-one and onto "$\ell^2_{\mathrm{loc}}(\mathbf{Z};\mathrm{L}^2([0,1);U))$", the space of all sequences $\mathbf{Z} \to \mathrm{L}^2([0,1);U)$. We identify Δ^{ℓ^2} with its restrictions (to, e.g., $\mathrm{L}^2_\omega(\mathbf{R};U) \to \ell^2_{\mathrm{e}^\omega}(\mathbf{Z};U)$ or to $\mathrm{L}^2_\omega(\mathbf{R}_+;U) \to \ell^2_{\mathrm{e}^\omega}(\mathbf{N};U)$ for some $\omega \in \mathbf{R}$).

It will be shown in Theorem 13.4.5 that for $\omega \in \mathbf{R}$, $r := \mathrm{e}^\omega$ and $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)$ we have $u \in \mathrm{L}^2_\omega \Leftrightarrow \Delta^{\ell^2} u \in \ell^2_r$, and that Δ^{ℓ^2} is an isomorphism of L^2_ω onto ℓ^2_r, i.e., $\Delta^{\ell^2} \in \mathcal{GB}(\mathrm{L}^2_\omega, \ell^2_r)$. Before going into further technical details we define the discretization of systems:

Definition 13.4.2 *For* $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$ *we define its* discretization

$$\Delta^{\mathrm{S}}\Sigma := \left[\begin{array}{c|c} \Delta^{\mathrm{S}}\mathbb{A} & \Delta^{\mathrm{S}}\mathbb{B} \\ \hline \Delta^{\mathrm{S}}\mathbb{C} & \Delta^{\mathrm{S}}\mathbb{D} \end{array}\right] := \left[\begin{array}{c|c} \mathbb{A}(1)^{\cdot} & \mathbb{B}(\Delta^{\ell^2})^{-1} \\ \hline \Delta^{\ell^2}\mathbb{C} & \Delta^{\ell^2}\mathbb{D}(\Delta^{\ell^2})^{-1} \end{array}\right] \in \mathrm{wpls}(U_\Delta, H, Y_\Delta), \tag{13.66}$$

where $U_\Delta := \mathrm{L}^2([0,1);U)$, $Y_\Delta := \mathrm{L}^2([0,1);Y)$.

If $\Sigma \in \mathrm{WPLS}$, then $\Delta^{\mathrm{S}}\Sigma \in \mathrm{wpls}$; the converse is not true without additional assumptions:

Proposition 13.4.3 *Let $\omega \in \mathbf{R}$, $r := \mathrm{e}^{\omega}$. If $\Sigma \in \mathrm{WPLS}_{\omega}(U,H,Y)$, then $\Delta^{\mathrm{S}}\Sigma \in \mathrm{wpls}_r(U_\Delta,H,Y_\Delta)$.*

Conversely, assume that $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{wpls}_r(U_\Delta,H,Y_\Delta)$. Then $\Sigma := \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] := \Delta^{\mathrm{S}^{-1}}\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{WPLS}(U,H,Y)$ iff $A = \mathbb{A}(1)$ for some C_0-semigroup $\mathbb{A}$, and

$$\mathbb{A}^t\mathbb{B} = \mathbb{B}\tau^t, \qquad \mathbb{C}\mathbb{A}^t = \pi_+\tau^t\mathbb{C}, \qquad \tau^t\mathbb{D} = \mathbb{D}\tau^t \qquad (t \in (0,1)). \tag{13.67}$$

If this is the case, then $\Sigma \in \mathrm{WPLS}_{\omega}$.

However, Δ^{S} does not map WPLS onto wpls; in fact, none of the four Tauberian conditions above is redundant:

We have $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{wpls}$ but $\Delta^{\mathrm{S}}\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \notin \mathrm{WPLS}$ when, e.g., 1) A is s.t. it does not have a square root "$\mathbb{A}(1/2)$" or 2) D is s.t. it is not "causal" on $\pi_{[0,1)}\mathrm{L}^2$ (see also Theorem 13.4.5(f)), or 3) $H = \mathrm{L}^2(\mathbf{R}_+)$, $\mathbb{A} = \pi_+\tau$, $Cx_0 := x_0(1-\cdot) \in \mathcal{B}(H,\mathrm{L}^2([0,1)))$ (use the dual of "3)" for the $\mathbb{B}$-condition; we can take $B = 0 = D = C$ in 1), $A = 0 = B = C$ in 2) or $B = 0 = D$ in 3) to guarantee that only one condition is violated).

Proof of Proposition 13.4.3: The other claims are obvious, so we only sketch the proof of the converse claim.

By Theorem 13.4.5, we have $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2_{\omega},H)$, $\mathbb{C} \in \mathcal{B}(H,\mathrm{L}^2_{\omega})$, $\pi_+\mathbb{D}\pi_- = \mathbb{C}\mathbb{B}$, $\pi_-\mathbb{D}\pi_+ = 0$, $\tau^t\mathbb{D} = \mathbb{D}\tau^t$ and $\mathbb{C}\mathbb{A}^t x_0 = \pi_+\tau^t\mathbb{C}x_0$ for $t \in \mathbf{Z}$. Combine this with the assumptions to the get the axioms of Definition 6.1.1 satisfied (by density and continuity, the axioms hold for ω iff they hold for some $\omega' \in \mathbf{R}$). □

Above we used the fact that $\Sigma \in \mathrm{WPLS}_{\omega}(U,H,Y) \Leftrightarrow \Delta^{\mathrm{S}}\Sigma \in \mathrm{wpls}_{\mathrm{e}^{\omega}}(U_\Delta,H,Y_\Delta)$ (when Σ is known to be a WPLS):

Theorem 13.4.4 *Let $\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{WPLS}(U,H,Y)$, $\omega \in \mathbf{R}$, $r := \mathrm{e}^{\omega}$. Then the following holds:*

(a1) The equations $\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathbb{A}(t)x_0 + \mathbb{B}\tau(t)u \\ \mathbb{C}x_0 + \mathbb{D}u \end{bmatrix}$ become equivalent to $\begin{bmatrix} x_j \\ y \end{bmatrix} = \begin{bmatrix} A^j & \mathbb{B}\tau^j \\ \mathbb{C} & \mathbb{D} \end{bmatrix}\begin{bmatrix} x_0 \\ u \end{bmatrix}$ (i.e., to $\begin{cases} x_{j+1} = Ax_j + Bu_j \\ y_j = Cx_j + Du_j \end{cases}$), and $\begin{bmatrix} x(t) \\ y \end{bmatrix} = \begin{bmatrix} \mathbb{B}\tau(t)u \\ \mathbb{D}u \end{bmatrix}$ become equivalent to $\begin{bmatrix} x_n \\ (y_\cdot) \end{bmatrix} = \begin{bmatrix} \Delta^{\mathrm{S}}\mathbb{B}\tau^n(u_\cdot) \\ \Delta^{\mathrm{S}}\mathbb{D}(u_\cdot) \end{bmatrix}$, where the discrete and continuous time input, state and output correspond to each other through

$$u_n = \pi_{[0,1)}\tau(n)u, \;\; x_n = x(n), \;\; y_n = \pi_{[0,1)}\tau(n)y, \tag{13.68}$$

(for all n), i.e., $(u_\cdot) = \Delta^{\ell^2}u$, $x_\cdot := x(\cdot)$, $(y_\cdot) = \Delta^{\ell^2}y$.

(In both settings, we must have $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)$; in the initial value setting we assume that $\pi_-u = 0$ and $x_0 \in H$, in the time-invariant setting we must have $\pi_-u \in \mathrm{L}^2_{\alpha}$ (equivalently, $\pi_-\Delta^{\mathrm{S}}u \in \ell^2_{\mathrm{e}^r}$), where α is s.t. $\mathbb{B}$ and $\mathbb{D}$ are α-stable. Also (a2) and (a3) use the same assumptions.)

(a2) In both settings described in (a1), we have

$$u \in \mathrm{L}^2_{\omega'} \Leftrightarrow \Delta^{\mathrm{S}} u \in \ell^2_{r'}, \qquad y \in \mathrm{L}^2_{\omega'} \Leftrightarrow \Delta^{\mathrm{S}} y \in \ell^2_{r'}, \qquad x, u \in \mathrm{L}^2_{\omega'} \Leftrightarrow x, \Delta^{\mathrm{S}} u \in \ell^2_{r'} \tag{13.69}$$

for any $\omega' \in \mathbf{R}$, $r' := \mathrm{e}^{\omega'}$. *(Here "*$x \in \ell^2_{r'}$*" means that the restriction* $(x(n))_{n\in\mathbf{Z}}$ *of* x *to* $\mathbf{Z}$ *(that is, "the discretized state") belongs to* $\ell^2_{r'}$*.)*

(a3) Let $\omega' \in \mathbf{R}$, $r' := \mathrm{e}^{\omega'}$. *Then there is* $M = M_{\omega'} \in (0,\infty)$ *s.t. in both settings described in (a1), we have*

$$\|x\|_{\mathrm{L}^2_{\omega'}(J;H)} + \|u\|_{\mathrm{L}^2_{\omega'}(J;U)} \le M(\|x\|_{\ell^2_{r'}(N;H)} + \|\Delta^{\mathrm{S}} u\|_{\ell^2_{r'}(N;U)}) \tag{13.70}$$

$$\|x\|_{\ell^2_{r'}(N+1;H)} + \|\Delta^{\mathrm{S}} u\|_{\ell^2_{r'}(N;U)} \le M(\|x\|_{\mathrm{L}^2_{\omega'}(J;H)} + \|u\|_{\mathrm{L}^2_{\omega'}(J;U)}). \tag{13.71}$$

Here we must take $J = \mathbf{R}_+$, $N = \mathbf{N}$ *in the initial value setting and* $J = \mathbf{R}$, $N = \mathbf{Z}$ *in the time-invariant setting (there is no bound for* $\|x_0\|_H$ *in the initial value setting, hence the "*$N+1$*").*

(b1) The generators of $\Delta^{\mathrm{S}}\Sigma$ *are given by*

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := \left(\begin{array}{c|c} \mathbb{A}(1) & \mathbb{B}\tau(1)\pi_{[0,1)} \\ \hline \pi_{[0,1)}\mathbb{C} & \pi_{[0,1)}\mathbb{D}\pi_{[0,1)} \end{array}\right) \in \mathcal{B}(H \times U_\Delta, H \times Y_\Delta). \tag{13.72}$$

(b2) We have $D \in \mathcal{GB} \Leftrightarrow \mathbb{D} \in \mathcal{G}\mathrm{TIC}_\infty$.

(c) The discretization Δ^{S} *commutes with valid compositions and inversions of operators.*

(See also Theorem 13.4.5. Note that $\Delta^{\ell^2\,-1}\mathbb{B}_{\mathrm{d}}$ *is not valid for* $\mathbb{B}_{\mathrm{d}} \in \mathcal{B}(\ell^2_r, H)$*, hence neither is* $\Delta^{\mathrm{S}\,-1}(\mathbb{B}_{\mathrm{d}}\tau)$*; on the other hand,* $(\Delta^{\mathrm{S}}\mathbb{B})\tau = (u \mapsto \mathbb{B}\tau\Delta^{\ell^2\,-1}u) = \Delta^{\mathrm{S}}(\mathbb{B}\tau) \in \mathrm{tic}_\infty$. *Note also that the discretization of* $\left[\begin{smallmatrix} A & B \\ I & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{A} & \mathbb{B}\tau \end{smallmatrix}\right]$ *is not* $\left(\begin{smallmatrix} A_{\mathrm{d}} & B_{\mathrm{d}} \\ I & 0 \end{smallmatrix}\right)$ *in general, where* $\left(\begin{array}{c|c} A_{\mathrm{d}} & B_{\mathrm{d}} \end{array}\right) = \Delta^{\mathrm{S}}\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$*).)*

(d1) We have $\Sigma \in \mathrm{WPLS}_\omega(U,H,Y) \Leftrightarrow \Delta^{\mathrm{S}}\Sigma \in \mathrm{wpls}_r(U_\Delta, H, Y_\Delta)$.

(d2) Σ *and* $\Delta^{\mathrm{S}}\Sigma$ *have the same stability properties (see Definitions 6.1.3 and 13.3.1).*

To be exact, a component of Σ *is [exponentially/strongly/weakly]* ω*-stable iff the corresponding component of* $\Delta^{\mathrm{S}}\Sigma$ *is [exponentially/strongly/weakly]* r*-stable*

(e1) An output feedback operator $L \in \mathcal{B}(Y,U)$ *is admissible [stabilizing] for* Σ *iff* L *is admissible [stabilizing] for* $\Delta^{\mathrm{S}}\Sigma$. *If* L *is admissible, then* $(\Delta^{\mathrm{S}}\Sigma)_L = \Delta^{\mathrm{S}}\Sigma_L$.

An analogous result holds for other forms of feedback for Σ *(but not for those for* $\Delta^{\mathrm{S}}\Sigma$*, because the discretization* Δ^{S} : WPLS $\to$ wpls *is not onto):*

Any dynamic feedback (resp. state feedback, output injection) for Σ *is admissible [stabilizing] for* Σ *iff its discretization is admissible [stabilizing] for* $\Delta^{\mathrm{S}}\Sigma$.

The prefices "I/O-", "SOS-", "weakly", "strongly", "exponentially", "q.r.c.-" and "q.l.c.-" apply (whereas "r.c.-", "l.c.-", "d.c.-" and "jointly" *possibly do not).*

(e2) $\Delta^{\mathrm{S}}\Sigma$ *has all the stabilizability properties of* Σ.

(e3) $\Delta^{\mathrm{S}}\Sigma$ *is optimizable (resp. estimatable) iff* Σ *is optimizable (resp. estimatable).*

(f1) **(J-critical control over $\mathcal{U}$)** *Let* $x_0 \in H$ *and* $J = J^* \in \mathcal{B}(Y)$. *Then* $\mathcal{U}_{\mathrm{out}}^{\Delta^{\mathrm{S}}\Sigma}(x_0) = \Delta^{\ell^2}\mathcal{U}_{\mathrm{out}}^{\Sigma}(x_0)$, $\mathcal{U}_{\mathrm{exp}}^{\Delta^{\mathrm{S}}\Sigma}(x_0) = \Delta^{\ell^2}\mathcal{U}_{\mathrm{exp}}^{\Sigma}(x_0)$, $\mathcal{U}_{\mathrm{sta}}^{\Delta^{\mathrm{S}}\Sigma}(x_0) \supset \Delta^{\ell^2}\mathcal{U}_{\mathrm{sta}}^{\Sigma}(x_0)$, $\mathcal{U}_{\mathrm{str}}^{\Delta^{\mathrm{S}}\Sigma}(x_0) \supset \Delta^{\ell^2}\mathcal{U}_{\mathrm{str}}^{\Sigma}(x_0)$, *where the superindex corresponds to the underlying system.*

The same equalities and inclusions also hold for the subsets of the corresponding J-critical controls. Thus if such controls exist for Σ and for each $x_0 \in H$, then corresponding J-critical cost operators are equal for Σ and $\Delta^{\mathrm{S}}\Sigma$.

In particular, if $\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *is J-critical for Σ and J over* $\mathcal{U}_{\mathrm{out}}$ *(resp.* $\mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}, \mathcal{U}_{\mathrm{exp}}$*), then* $\Delta^{\mathrm{S}}\left[\begin{array}{c|c}\mathbb{K} & \mathbb{F}\end{array}\right]$ *is J-critical for Σ and J over* $\mathcal{U}_{\mathrm{out}}$ *(resp.* $\mathcal{U}_{\mathrm{sta}}, \mathcal{U}_{\mathrm{str}}, \mathcal{U}_{\mathrm{exp}}$*).*

(f2) **(J-critical control over $\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}$)** *Let* $\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}$ *be as in Definition 8.3.2 and let* $J = J^* \in \mathcal{B}(Y)$. *Then*

$$\Delta^{\ell^2}\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}(x_0) = \mathcal{U}^{\mathrm{e}^{\vartheta},\Delta^{\mathrm{S}}\Sigma}_{\left[\mathbb{Q}\ \mathbb{R}(\Delta^{\ell^2})^{-1}\right]}(x_0) \quad (x_0 \in H); \tag{13.73}$$

the same holds for corresponding subsets of J-critical controls. Moreover, this maps $\mathcal{U}_{\mathrm{out}}^{\Sigma} \mapsto \mathcal{U}_{\mathrm{out}}^{\Delta^{\mathrm{S}}\Sigma}$ *and* $\mathcal{U}_{\mathrm{exp}}^{\Sigma} \mapsto \mathcal{U}_{\mathrm{exp}}^{\Delta^{\mathrm{S}}\Sigma}$.

(g) *The map $\mathbb{D}$ is [positively] J-coercive over* $\mathcal{U}_{\mathrm{exp}}^{\Sigma}$ *(resp.* $\mathcal{U}_{\mathrm{out}}^{\Sigma}$, $\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}$*) iff* $\Delta^{\mathrm{S}}\mathbb{D}$ *is [positively] J-coercive over* $\mathcal{U}_{\mathrm{exp}}^{\Delta^{\mathrm{S}}\Sigma}$ *(resp.* $\mathcal{U}_{\mathrm{out}}^{\Delta^{\mathrm{S}}\Sigma}$, $\mathcal{U}^{\mathrm{e}^{\vartheta},\Delta^{\mathrm{S}}\Sigma}_{\left[\mathbb{Q}\ \mathbb{R}(\Delta^{\ell^2})^{-1}\right]}$*) (cf. (f1)–(f2)).*

If $\mathbb{D}$ is [positively] J-coercive over $\mathcal{U}_{\mathrm{str}}^{\Sigma}$ *(resp.* $\mathcal{U}_{\mathrm{sta}}^{\Sigma}$*), then* $\Delta^{\mathrm{S}}\mathbb{D}$ *is [positively] J-coercive over* $\mathcal{U}_{\mathrm{str}}^{\Delta^{\mathrm{S}}\Sigma}$ *(resp.* $\mathcal{U}_{\mathrm{sta}}^{\Delta^{\mathrm{S}}\Sigma}$*).*

Because Δ^{S} maps WPLS into wpls (but not onto), we can use the theorem to obtain continuous-time analogies of uniqueness results (including equality of formulae and stability of operators) only, not of existence results (including words “jointly”, “r.c.”, ...). E.g., the interaction operator and coprime multipliers provided by Lemma 13.3.17 need not be images of any continuous-time operators (their preimages need not be time-invariant). Cf. the proof of (e1).

By (13.83), we could have made Δ^{ℓ^2} (and hence $\Delta^{\mathrm{S}} \in \mathcal{B}(\mathrm{TIC}_\omega, \mathrm{tic}_r)$ too) isometric $\mathrm{L}^2_\omega \to \ell^2_r$ by using $\mathrm{L}^2_\omega([0,1);U)$-valued sequences instead of $\mathrm{L}^2([0,1);U)$-valued ones. However, we have chosen the latter ones in order to make the discretization independent of ω (cf. Lemma 13.3.2).

Proof of Theorem 13.4.4: (a1) These claims are quite obvious.

(a2) The claims on u and y follow from Theorem 13.4.5(a1); the last one follows from (a3).

(a3) (As obvious from the proof, in fact any $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R}_+;U)$ will do. On the other hand, to see that we cannot get a bound for $\|x_0\|$ in the initial value setting, let $H=\ell^2(\mathbf{N})$ and $\mathbb{A}^t e_n = \mathrm{e}^{-nt} e_n$ ($n\in\mathbf{N}$), so that $\|\mathbb{A}e_n+\mathbb{B}0\|_{\mathrm{L}^2}^2 = 1/2n$ even though $\|e_n\|_{\ell^2}=1$.)

By isomorphism claim of Theorem 13.4.5(a1), we have $\|u\|_{\mathrm{L}^2_{\omega'}(J;U)} \le M''\|\Delta^{\mathrm{S}}u\|_{\ell^2_{r'}(N;U)}$ and $\|\Delta^{\mathrm{S}}u\|_{\ell^2_{r'}(N;U)} \le M''\|u\|_{\mathrm{L}^2_{\omega'}(J;U)}$ for some $M''<\infty$ and all $u\in\mathrm{L}^2_{\omega'}$. Therefore, we only need upper bounds for the norms of x.

Set $M' := \max_{0\le t\le 1}\{\|\mathbb{A}(t)\|_{\mathcal{B}(H)}, \|\mathbb{B}^t\|_{\mathcal{B}(\mathrm{L}^2,H)}\}$. Let x_0, u and x be as in (either setting of) (a1) (in the initial value setting we extend them by zero on $\mathbf{R}_-$ and $\mathbf{Z}_-$).

1° Assume that $x,\Delta^{\mathrm{S}}u\in\ell^2_{r'}$, so that $u\in\mathrm{L}^2_{\omega'}$. Then, by (6.9),

$$\|x(n+t)\| \le \|\mathbb{A}^t x(n)\| + \|\mathbb{B}^t\tau^n u\| \le M'\big(\|x(n)\| + \|\pi_{[n,n+1)}u\|_2\big), \tag{13.74}$$

hence $\|\pi_{[n,n+1)}x\|_2 \le M'\big(\|x(n)\|+\|\pi_{[n,n+1)}u\|\big)$, for any $n\in N$. Consequently,

$$\|\Delta^{\mathrm{S}}x\|_{\ell^2_{r'}} \le M'\big(\|x\|_{\ell^2_{r'}} + \|\Delta^{\mathrm{S}}u\|_{\ell^2_{r'}}\big). \tag{13.75}$$

Thus, $M := \sqrt{2}M'M''+M''$ will do for the first inequality (note that $2(a^2+b^2)\ge(a+b)^2$; the addition $+M''$ is for u).

2° Let $x,u\in\mathrm{L}^2_{\omega'}$, so that $\Delta^{\mathrm{S}}u\in\ell^2_{r'}$. Then

$$\|x(n+1)\| \le \min_{0\le t\le 1}\|\mathbb{A}^t x(n+1-t)\| + \max_{0\le t\le 1}\|\mathbb{B}^t\tau^{n+1-t}u\|\big) \tag{13.76}$$

$$\le M'\big(\|\pi_{[n,n+1)}x\|_2 + \|\pi_{[n,n+1)}u\|_2\big), \tag{13.77}$$

(because $\min_{0\le t\le 1}\|x(n+1-t)\| \le \|\pi_{[n,n+1)}x\|_2$). Therefore, $\|x\|_{\ell^2_{r'}(N+1;H)} \le M'\|\Delta^{\mathrm{S}}x\|_{\ell^2_{r'}(N;H)} + \|\Delta^{\mathrm{S}}x\|_{\ell^2_{r'}(N;U)}$. Thus, M will do for the second inequality too.

(b1) This follows from (a) (alternatively, from (13.39)).

(b2) This follows from Lemma 13.1.7 (which says that $D\in\mathcal{GB}\Leftrightarrow\Delta^{\mathrm{S}}\mathbb{D}\in\mathcal{G}\mathrm{tic}_\infty$) and (c) (which says that $\mathbb{D}\in\mathcal{G}\mathrm{TIC}_\infty\Leftrightarrow\Delta^{\mathrm{S}}\mathbb{D}\in\mathcal{G}\mathrm{tic}_\infty$).

(c) This is obvious from the definition (e.g., $(\Delta^{\mathrm{S}}\mathbb{C})(\Delta^{\mathrm{S}}\mathbb{B}) = \Delta^{\mathrm{S}}(\mathbb{C}\mathbb{B})$); note that to $\mathbb{A}(t)$ this applies for $t\in\mathbf{Z}$ only.

(d1) This follows from (d2) (recall that we have assumed that $\Sigma\in$ WPLS).

(d2) 1° $\mathbb{C}$ *and* $\mathbb{D}$*:* Obviously, $\Delta^{\mathrm{S}}\mathbb{C}[H]\subset\ell^2_r\Leftrightarrow\mathbb{C}[H]\subset\mathrm{L}^2_\omega$, and $\Delta^{\mathrm{S}}\mathbb{D}[\ell^2_r]\subset\ell^2_r\Leftrightarrow\mathbb{D}[\mathrm{L}^2_\omega]\subset\mathrm{L}^2_\omega$, so that the stability of $\mathbb{C}$ (resp. $\mathbb{D}$) equals that of $\Delta^{\mathrm{S}}\mathbb{C}$ (resp. $\Delta^{\mathrm{S}}\mathbb{D}$) (see Lemma 6.1.12).

2° $\Delta^{\mathrm{S}}\mathbb{A}$ *and* $\Delta^{\mathrm{S}}\mathbb{B}$ *are at least as stable as* $\mathbb{A}$ *and* $\mathbb{B}$*:* If $\mathrm{e}^{-\omega t}\mathbb{A}^t x_0$ (resp. $\mathrm{e}^{-\omega t}\mathbb{B}\tau^t u$) is bounded or converges to zero strongly or weakly, as $t\to+\infty$, then so does $r^{-n}\mathbb{A}^n x_0$ (resp. $r^{-n}\mathbb{B}\tau^n u$), as $\mathbf{N}\ni n\to+\infty$. Thus, we only have to show that $\Delta^{\mathrm{S}}\mathbb{A}$ (resp. $\Delta^{\mathrm{S}}\mathbb{B}$) cannot be more stable than $\mathbb{A}$ (resp. $\mathbb{B}$).

3° $\Delta^{\mathrm{S}}\mathbb{A}$ *is as stable as* $\mathbb{A}$*:* If $\|r^{-n}\mathbb{A}^n\|\le M$ for all n, then

$$\|\mathrm{e}^{-\omega(n+h)}\mathbb{A}^{n+h}\| = \|r^{-n}\mathbb{A}^n\mathrm{e}^{-\omega h}\mathbb{A}^h\| \le M\max_{h\in[0,1]}\|\mathrm{e}^{-\omega h}\mathbb{A}^h\|. \tag{13.78}$$

Thus, the r-stability of A implies the ω-stability of $\mathbb{A}$. From (13.78) one also observes that if $r^{-n}A^n x_0\to 0$ strongly, then $\mathrm{e}^{-\omega t}\mathbb{A}(t)x_0\to 0$ strongly.

Assume then that A is weakly r-stable, i.e., that $\|r^{-n}A^n\|\le M$ for all n and $\langle z_0, r^{-n}A^n x_0\rangle\to 0$ for $x_0,z_0\in H$.

Let $x_0, z_0 \in H$ be arbitrary. Set $K := \{\mathrm{e}^{-\omega h}\mathbb{A}(h)x_0 \,|\, h \in [0,1]\}$, and $T_n x := \langle z_0, r^{-n}A^n x\rangle$ (thus, $T_n \in H^*$). It follows that $T_n x \to 0$ uniformly on K, by Lemma A.3.4(H2), hence $\langle z_0, \mathrm{e}^{-\omega t}\mathbb{A}(t)x_0\rangle \to 0$, as $t \to +\infty$. Thus, $\mathbb{A}$ is weakly ω-stable.

4° $\Delta^{\mathrm{S}}\mathbb{B}$ *is as stable as* $\mathbb{B}$*:* If $\mathbb{B}(\Delta^{\ell^2})^{-1} \in \mathcal{B}(\ell^2_r; H)$, then $\mathbb{B} \in \mathcal{B}(\mathrm{L}^2_\omega; H)$, since $\Delta^{\ell^2} \in \mathcal{GB}(\mathrm{L}^2_\omega, \ell^2_r)$. Thus, $\Delta^{\mathrm{S}}\mathbb{B}$ is r-stable iff $\mathbb{B}$ is ω-stable.

Assume that $\mathbb{B}(\Delta^{\ell^2})^{-1}$ is strongly r-stable. Let $u \in \mathrm{L}^2_\omega(\mathbf{R}; U)$. Since $K := \{\mathrm{e}^{-\omega h}\tau^h u \,|\, h \in [0,1]\} \subset \mathrm{L}^2_\omega$ is compact and $\mathrm{e}^{-\omega n}\mathbb{B}\tau^n v \to 0$, as $\mathbf{N} \ni n \to \infty$, for all $v \in K$ (in fact, for all $v \in \mathrm{L}^2_\omega$), this convergence is uniform on K, hence $\|\mathrm{e}^{-\omega(n+h)}\mathbb{B}\tau^{(}n+h)u\| < \varepsilon$ for all $n > N_\varepsilon$ and $h \in [0,1]$, so that $\mathbb{B}$ is strongly ω-stable.

The map $\mathbb{B}$ is weakly ω-stable whenever $\mathbb{B}(\Delta^{\ell^2})^{-1}$ is weakly r-stable, as one observes by adding an arbitrary $\Lambda \in H^*$ before $\mathbb{B}$ in the above proof.

(e1) This follows from (c), (d2) and Theorem 13.4.5(g) (it is enough to verify this for static output feedback, because other forms of feedback and injection can be reduced to static output feedback, as in Summary 6.7.1.

(Note that prefices "r.c.", "l.c.", "d.c." and "jointly" would require existence of certain kinds of TIC or tic operators. Because $\Delta^{\mathrm{S}}\mathbb{X} \in \mathrm{tic}_\infty \not\Rightarrow \mathbb{X} \in \mathrm{TIC}_\infty$ (cf. Theorem 13.4.5), existence results for $\Delta^{\mathrm{S}}\Sigma$ do not necessarily tell anything about Σ. If $\left[\ \mathbb{K} \,\middle|\, \mathbb{F}\ \right]$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ admissible for Σ, then so are $\Delta^{\mathrm{S}}\left[\ \mathbb{K} \,\middle|\, \mathbb{F}\ \right]$ and $\Delta^{\mathrm{S}}\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$, they are even jointly admissible, by Lemma 13.3.17(a), but we do not know whether $\left[\ \mathbb{K} \,\middle|\, \mathbb{F}\ \right]$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ are then jointly admissible. Even if they were jointly admissible with some $\mathbb{E} \in \mathrm{TIC}_\infty$, and $\Delta^{\mathrm{S}}\left[\ \mathbb{K} \,\middle|\, \mathbb{F}\ \right]$ and $\Delta^{\mathrm{S}}\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ were jointly stabilizing, we do not know whether $\left[\ \mathbb{K} \,\middle|\, \mathbb{F}\ \right]$ and $\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ would then be jointly stabilizing (unless $\Delta^{\mathrm{S}}\left[\ \mathbb{K} \,\middle|\, \mathbb{F}\ \right]$ and $\Delta^{\mathrm{S}}\left[\frac{\mathbb{H}}{\mathbb{G}}\right]$ are known to be jointly stabilizing with some $\Delta^{\mathrm{S}}\mathbb{E}'$, where $\mathbb{E}' \in \mathrm{TIC}_\infty$).)

(e2) This follows easily from (e1) and (d2). (Note that the converse holds (at least) for static feedback.)

(e3) For optimizability this follows from (f). For estimatability, one could very carefully verify this directly, but the easiest way is to note that the final state estimation problems (FSEPs) for Σ and $\Delta^{\mathrm{S}}\Sigma$ are obviously equivalent. (By Theorem 5.3 of [WR00], the FSEP for Σ has a solution iff Σ is estimatable; it is even easier to verify this in discrete time.)

(f1)&(f2) The equalities and inclusions follow from (a2); also the rest of (f) follows easily from (a)–(d1) and Theorem 13.4.5(a1)&(a2). (Note that $\Delta^{\ell^2}x$ may be bounded even if x is unbounded, hence we can state mere inclusions for $\mathcal{U}_{\mathrm{sta}}$ and $\mathcal{U}_{\mathrm{str}}$ in both (f1) and (f2).)

(g) This follows from the facts that the norms $\|\cdot\|_{\mathcal{U}^{\vartheta}_{[\mathbb{Q}\ \mathbb{R}]}}$ and $\|\Delta^{\ell^2}\cdot\|_{\mathcal{U}^{\mathrm{e}^{\vartheta},\Delta^{\mathrm{S}}\Sigma}_{[\mathbb{Q}\ \mathbb{R}(\Delta^{\ell^2})^{-1}]}}$ are obviously equivalent (use Theorem 13.4.5(a1)) (they are equal if $\vartheta = 0$), and that, similarly, $\|\cdot\|'_{\mathcal{U}_{\exp}} = \|\Delta^{\ell^2}\cdot\|'_{\mathcal{U}_{\exp}}$, $\|\cdot\|'_{\mathcal{U}_{\mathrm{out}}} = \|\Delta^{\ell^2}\cdot\|'_{\mathcal{U}_{\mathrm{out}}}$, $\|\cdot\|'_{\mathcal{U}_{\mathrm{sta}}} \geq \|\Delta^{\ell^2}\cdot\|'_{\mathcal{U}_{\mathrm{sta}}}$ and $\|\cdot\|'_{\mathcal{U}_{\mathrm{str}}} \geq \|\Delta^{\ell^2}\cdot\|'_{\mathcal{U}_{\mathrm{str}}}$ (see Lemma 8.4.2 for the norms) □

We end this section by listing the basic properties of the discretization of I/O maps:

Theorem 13.4.5 ($\mathbf{L^2_\omega(R;\mathit{U}) \cong \ell^2_{e^\omega}(Z;L^2([0,1);\mathit{U}))}$) *Let $\omega \in \mathbf{R}$, and set $r := \mathrm{e}^\omega$. Then the following holds:*

(a1) Let $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)$. Then $u \in \mathrm{L}^2_\omega \Leftrightarrow \Delta^{\ell^2} u \in \ell^2_r$, and $u \in \mathrm{L}^2_{\mathrm{c}} \Leftrightarrow \Delta^{\ell^2} u \in \mathrm{c_c}$.

Moreover, Δ^{ℓ^2} is an isomorphism of L^2_ω onto ℓ^2_r (i.e., $\Delta^{\ell^2} \in \mathcal{GB}(\mathrm{L}^2_\omega, \ell^2_r)$). For $\omega = 0$ this isomorphism is an isometry.

(a2) We have $\langle \Delta^{\ell^2} f, \Delta^{\ell^2} g\rangle = \langle f, g\rangle$ for $f \in \mathrm{L}^2_\omega(\mathbf{R};U)$, $g \in \mathrm{L}^2_{-\omega}(\mathbf{R};U)$.

(b1) The map $\Delta^{\mathrm{S}}_{U\to Y} : \mathbb{E} \mapsto \Delta^{\ell^2}_Y \mathbb{E}(\Delta^{\ell^2}_U)^{-1}$ is an isomorphism of $\mathcal{B}(\mathrm{L}^2_\omega, \mathrm{L}^2_\omega)$ onto $\mathcal{B}(\ell^2_r, \ell^2_r)$. For $\omega = 0$ this isomorphism is an isometry.

(b2) Moreover, Δ^{S} commutes with (anticausal) adjoints and valid compositions of operators. Thus,

$$\Delta^{\mathrm{S}}\mathbb{E}^* = (\Delta^{\mathrm{S}}\mathbb{E})^*, \;\; \Delta^{\mathrm{S}}(\mathbb{E}\mathbb{F}) = (\Delta^{\mathrm{S}}\mathbb{E})(\Delta^{\mathrm{S}}\mathbb{F}), \;\;\; \Delta^{\mathrm{S}}\mathbb{E}^{-1} = (\Delta^{\mathrm{S}}\mathbb{E})^{-1} \tag{13.79}$$

for $\mathbb{E} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U), \mathrm{L}^2_\omega(\mathbf{R};Y))$ and $\mathbb{F} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};Y), \mathrm{L}^2_\omega(\mathbf{R};H))$. (But $\Delta^{\mathrm{S}}\mathbb{E}^{\mathrm{d}} \neq (\Delta^{\mathrm{S}}\mathbb{E})^{\mathrm{d}}$ in general.)

(b3) The map Δ^{S} is an isomorphism of TI *into* ti *and of* TIC *into* tic.

(c) $\Delta^{\ell^2}\pi_\pm = \pi^\pm\Delta^{\ell^2}$, $\;\Delta^{\mathrm{S}}\pi_\pm = \pi^\pm$, $\;\Delta^{\ell^2}\tau(n) = \tau^n\Delta^{\ell^2}$, $\;\Delta^{\mathrm{S}}\tau(n) = \tau^n$.

(d) Let $\mathbb{E} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U))$. Then $\pi_\pm\mathbb{E}\pi_\pm$ is invertible on $\pi_\pm\mathrm{L}^2_\omega$ iff $\pi^\pm(\Delta^{\mathrm{S}}\mathbb{E})\pi^\pm$ is invertible on $\pi^\pm\ell^2_r$.

(e) Let $\mathbb{E}, P \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U))$. Then $\Delta^{\mathrm{S}}\mathbb{E} \geq 0$ [$\gg 0$] on $(\Delta^{\mathrm{S}}P)\ell^2$ iff $\mathbb{E} \geq 0$ [$\gg 0$] on $P\mathrm{L}^2$.

(f) Δ^{S} is an isomorphism of $\mathrm{TI}_\omega(U,Y)$ into $\mathrm{ti}_r(\mathrm{L}^2([0,1);U), \mathrm{L}^2([0,1);Y))$. Moreover, if $\mathbb{E} \in \mathrm{TI}_{\omega'}(U,Y)$ for some $\omega' \in \mathbf{R}$, then

$$\mathbb{E} \in \mathrm{TI}_\omega(U,Y) \Leftrightarrow \Delta^{\mathrm{S}}\mathbb{E} \in \mathrm{ti}_r, \;\; \mathbb{E} \in \mathcal{G}\mathrm{TI}_\omega \Leftrightarrow \Delta^{\mathrm{S}}\mathbb{E} \in \mathcal{G}\mathrm{ti}_r, \tag{13.80}$$

$$\mathbb{E} \in \mathrm{TIC}_\omega \Leftrightarrow \Delta^{\mathrm{S}}\mathbb{E} \in \mathrm{tic}_r, \;\; \mathbb{E} \in \mathcal{G}\mathrm{TIC}_\omega \Leftrightarrow \Delta^{\mathrm{S}}\mathbb{E} \in \mathcal{G}\mathrm{tic}_r, \tag{13.81}$$

$$\mathbb{E} \in \mathcal{B}(U,Y) \Leftrightarrow \Delta^{\mathrm{S}}\mathbb{E} \in \mathcal{B}(U_\Delta, Y_\Delta). \tag{13.82}$$

However, the time-invariance (resp. staticity) of $\Delta^{\mathrm{S}}\widetilde{\mathbb{E}}$ does not imply that of $\widetilde{\mathbb{E}}$ for general $\widetilde{\mathbb{E}} \in \mathcal{B}(\mathrm{L}^2, \mathrm{L}^2)$.

(g) Operators $\mathbb{N} \in \mathrm{TIC}(U,Y)$ and $\mathbb{M} \in \mathrm{TIC}(U)$ are q.r.c. iff $\Delta^{\mathrm{S}}\mathbb{N}$ and $\Delta^{\mathrm{S}}\mathbb{M}$ are q.r.c.

(h1) Let $\mathbb{D} \in \mathrm{TI}(U,Y)$, $J = J^* \in \mathcal{B}(Y)$. Then $\mathbb{D}$ is minimax J-coercive iff $\Delta^{\mathrm{S}}\mathbb{D}$ minimax J-coercive.

(h2) Let $\mathbb{D} \in \mathrm{TIC}(U,Y)$, $J = J^* \in \mathcal{B}(Y)$. Then $\mathbb{D}$ is [positively] J-coercive over $\mathcal{U}_{\mathrm{out}}$ iff $\Delta^{\mathrm{S}}\mathbb{D}$ [positively] J-coercive over $\mathcal{U}_{\mathrm{out}}$.

*(m) **($\mathbf{MTIC_d \to \ell^1}$)** Let $T > 0$. Let $\mathbb{E} := \sum_j A_j\delta_{jT}* \in \mathcal{B}(\mathrm{L}^2(\mathbf{R};U), \mathrm{L}^2(\mathbf{R};Y))$ and $\mathbb{F} := \sum_j A_j e_j* \in \mathcal{B}(\ell^2, \ell^2)$, where $A_j \in \mathcal{B}(U,Y)$, $\sum_j \|A_j\| < \infty$, and*

$e_j = \chi_{\{j\}} \in \ell^1(\mathbf{Z})$ (i.e., $(e_j * g)_k = g(k-j)$ for all $g \in \ell^2(\mathbf{Z};U_\Delta)$). Redefine $\Delta_U^{\ell^2} u := (\tau(nT)\pi_{[nT,nT+1)}u)$.

Then $\Delta^{\mathrm{S}}\mathbb{E} = \mathbb{F}$, and $\widehat{\mathbb{E}}(-it/T) = \widehat{\mathbb{F}}(\mathrm{e}^{it})$ for $t \in \mathbf{R}$; in particular, $\widehat{\mathbb{E}}(ir + i2\pi/T) = \widehat{\mathbb{F}}(\mathrm{e}^{-iTr})$ for all $r \in \mathbf{R}$. If $A_j = 0$ for $j < 0$, then we also have $\widehat{\mathbb{E}}(s) = \widehat{\mathbb{E}}(s + i2\pi/T) = \widehat{\mathbb{F}}(\mathrm{e}^{-Ts})$ for all $s \in \overline{\mathbf{C}^+} \cup \{\infty\}$.

The results hold also when we use $[0,T)$ for arbitrary $T > 0$ instead of $T = 1$ (this is illustrated in (m)); except for (e), (a2) and the adjoint formula of (b2), we can even let U and Y be arbitrary Banach spaces.

However, $\boldsymbol{\text{Я}}_{-1}\Delta^{\mathrm{S}} \neq \Delta^{\mathrm{S}}\boldsymbol{\text{Я}} \neq \boldsymbol{\text{Я}}\Delta^{\mathrm{S}}$ and $\Delta^{\mathrm{S}}\mathbb{E}^{\mathrm{d}} \neq (\Delta^{\mathrm{S}}\mathbb{E})^{\mathrm{d}}$ in general.

Because of the (algebraic and topologic) isomorphism, things such as exponential stability and coprimeness are preserved under the discretization (in both ways).

The formula $s \mapsto \mathrm{e}^{-sT}$ in (m) maps any strip of $\overline{\mathbf{C}^+} \cup \{\infty\}$ of height $2\pi/T$ one-to-one and onto $\overline{\mathbf{D}}$. However, for general $\mathbb{D} \in \mathrm{TIC}$, the connection between $\widehat{\mathbb{D}}$ and $\widehat{\Delta^{\mathrm{S}}\mathbb{D}}$ seems to be rather complicated.

Note also the differences to Theorem 13.2.3: the transform Δ^{ℓ^2} (or Δ^{S}) treats also the unstable case and commutes with time-shifts, but it does not map TI onto ti, it does not commute with time reflection, and U and Y are different from U_Δ and Y_Δ.

Thus, the Cayley transform is usually better for transferring stable I/O results, whereas the discretization can be used to transform uniqueness results (including equality results) from discrete time to continuous time (and existence results in the other direction), including the unstable results and those concerning more than just the I/O maps of systems.

Also systems (not merely I/O maps) can be mapped to each other by using the Cayley transform (see, e.g., p. 212–213 and 331–332 of [CZ]). However, the transform of $\left(\begin{array}{c|c} A_{\mathrm{d}} & B_{\mathrm{d}} \\ \hline C_{\mathrm{d}} & D_{\mathrm{d}} \end{array}\right)$ requires that $I + A_{\mathrm{d}} \in \mathcal{GB}$, and we only know the preservation of I/O-stability and exponential stability, not, e.g., internal [P-]stability. Moreover, this does not apply to continuous-time systems with unbounded generators.

Proof of Theorem 13.4.5: (a1) Clearly $u \mapsto (\Delta_U^{\ell^2} u)_k \in \mathrm{L}^2([0,1);U)$ is linear, continuous and onto. Obviously, $u \in \mathrm{L}^2_{\mathrm{c}} \Leftrightarrow \Delta^{\ell^2} u \in \mathrm{c}_{\mathrm{c}}$. For $u \in \mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)$ we have

$$\|\Delta^{\ell^2} u\|^2_{\ell^2_r} = \sum_{n\in\mathbf{Z}} \|r^n \pi_{[0,1)}\tau(n)u\|_2^2 = \sum_{n\in\mathbf{Z}} \int_n^{n+1} \|\mathrm{e}^{-\omega n} u(t)\|^2\,dt, \tag{13.83}$$

$\|u\|^2_{\mathrm{L}^2_\omega} = \sum_{n\in\mathbf{Z}} \int_n^{n+1} \|\mathrm{e}^{-\omega t} u(t)\|^2\,dt$, and the quotient $\mathrm{e}^{-\omega t}/\mathrm{e}^{-\omega n} = \mathrm{e}^{-\omega(t-n)}$ is between 1 and $\mathrm{e}^{-\omega} = r^{-1}$. Therefore, (a1) holds. (Note that there are no norm equivalence constants that would suit for every $\omega \in \mathbf{R}$.)

(a2) Now $\langle \Delta^{\ell^2} f, \Delta^{\ell^2} g\rangle_{\ell^2_r,\ell^2_{-r}} := \sum_n \int_n^{n+1} \langle f,g\rangle_U\,dt = \langle f,g\rangle_{\mathrm{L}^2_\omega,\mathrm{L}^2_{-\omega}}$ (cf. (13.83)).

(b1) This follows from (a1) and (a2): Δ^{S} has the inverse $\mathcal{B}(\ell^2(\mathbf{Z};Y_\Delta),\ell^2(\mathbf{Z};U_\Delta)) \ni \mathbb{E} \mapsto (\Delta_U^{\ell^2})^{-1}\mathbb{E}\Delta_Y^{\ell^2} \in \mathcal{B}(\mathrm{L}^2,\mathrm{L}^2)$ and it is isometric by the equation $\|\Delta^{\mathrm{S}}\mathbb{F}\Delta_U^{\ell^2} f\| = \|\Delta_U^{\ell^2}\mathbb{F}f\| = \|\mathbb{F}f\|$, valid for $f \in \mathrm{L}^2(\mathbf{R};U)$, $\mathbb{F} \in \mathcal{B}(\mathrm{L}^2,\mathrm{L}^2)$.

(b2) This is obvious from the definition except for $\Delta^{\mathrm{S}}\mathbb{E}^* = (\Delta^{\mathrm{S}}\mathbb{E})^*$, which holds because the equation $\langle \mathbb{E}f, g\rangle = \langle f, \mathbb{E}^* g\rangle$ is equivalent to $\langle (\Delta^{\mathrm{S}}\mathbb{E})\Delta^{\ell^2} f, \Delta^{\ell^2} g\rangle = \langle \Delta^{\ell^2} f, (\Delta^{\mathrm{S}}\mathbb{E}^*)\Delta^{\ell^2} g\rangle$, by (a2).

(b3) This follows from (b1) and (g).

(c) The $\Delta_U^{\ell^2}$ formulae are obvious, the Δ^{S} formulae follow. It is obvious that Δ^{ℓ^2} does not commute with time reflection.

(d) By (b), $\mathbb{G}\pi_+\mathbb{E}\pi_+ = \pi_+ = \pi_+\mathbb{E}\pi_+\mathbb{G}$ for some $\mathbb{G} \in \mathcal{B}(\pi_\pm \mathrm{L}^2_\omega)$ (we may identify $\mathbb{G}$ with $\pi_+\mathbb{G}\pi_+ \in \mathcal{B}(\mathrm{L}^2_\omega)$) iff $(\Delta^{\mathrm{S}}\mathbb{G})\pi^+(\Delta^{\mathrm{S}}\mathbb{E})\pi^+ = \pi^+ = \pi^+(\Delta^{\mathrm{S}}\mathbb{E})\pi^+(\Delta^{\mathrm{S}}\mathbb{G})$.

(e) Now $\langle \Delta^{\mathrm{S}}\mathbb{E}\Delta^{\mathrm{S}} P\Delta^{\ell^2} f, \Delta^{\mathrm{S}} P\Delta^{\ell^2} f\rangle \geq 0$ for all $\Delta^{\ell^2} f \in \ell^2$ iff $\langle \mathbb{E}Pf, Pf\rangle \geq 0$ for all $f \in \mathrm{L}^2$. By replacing $\mathbb{E}$ by $\mathbb{E} - \varepsilon I$ we get the "$\gg 0$" claim.

(f) By (c), Δ^{S} preserves time-invariance, hence (see (b) too) Δ^{S} is an isomorphism (into).

Because of (b), the "$\Rightarrow$" parts of (13.80) and the first "$\Leftarrow$" are trivial. The second "$\Leftarrow$" follows from the first, because $\mathbb{E}^{-1} \in \mathcal{GB}(\mathrm{L}^2)$ inherits the time-invariance of $\mathbb{E}$ (since $\tau \in \mathcal{GB}$).

Because $\Delta^{\mathrm{S}}\pi_-\mathbb{E}\pi_+ = \pi^-(\Delta^{\mathrm{S}}\mathbb{E})\pi^+$ (by (b)), the next two equivalences follow.

By the above results, $\mathbb{E} \in \mathrm{TIC} \cap \mathrm{TIC}^* \Leftrightarrow \mathbb{E} \in \mathrm{tic} \cap \mathrm{tic}^*$. By Lemmas 2.1.7 and 13.1.2, $\mathrm{TIC} \cap \mathrm{TIC}^* = \mathcal{B}$ and $\mathrm{tic} \cap \mathrm{tic}^* = \mathcal{B}$.

The counter-example is obtained by choosing a static $\Delta^{\mathrm{S}}\mathbb{E}$ so that it is not "time-invariant on $\pi_{[0,1)}\mathrm{L}^2$", e.g., take $E = \pi_{[0,1)}\tau^{1/2}\pi_{[0,1)} \in \mathcal{B}(U_\Delta)$, $(\mathbb{E}u)(t) := Eu(t)$ $(t \in \mathbf{Z})$.

(g) This follows from (a1).

(h1) This follows from (a1), (e), and Definition 11.4.1.

(h2) This follows from (a1), (e) and Lemma 8.4.11(a1)&(a2).

(m) We have

$$\begin{aligned}
[\Delta^{\mathrm{S}}(A_j\delta_{jT}*)]\Delta_U^{\ell^2} f &= \Delta_U^{\ell^2} A_j\delta_{jT} * f = \Delta_U^{\ell^2} A_j\tau(-jT)f \\
&= (\tau((k-j)T)\pi_{[(k-j)T,(k-j+1)T)}A_j f)_{k\in\mathbf{Z}} \\
&= A_j((\Delta_U^{\ell^2} f)_{k-j})_{k\in\mathbf{Z}} = A_j e_j * \Delta_U^{\ell^2} f,
\end{aligned}$$

for each $f \in \mathrm{L}^2(\mathbf{R};U)$, i.e., for each $\Delta_U^{\ell^2} f \in \ell^2(\mathbf{Z};U_\Delta)$. Therefore $\Delta^{\mathrm{S}}\mathbb{E} = \mathbb{F}$.

We have $(\mathcal{L}\delta_{jT})(s) = \mathrm{e}^{-jTs} = (\mathrm{e}^{-Ts})^j = (\mathcal{Z}e_j)(\mathrm{e}^{-Ts})$ for each $j \in \mathbf{Z}$ and $s \in i\mathbf{R}$ (or for each $s \in \overline{\mathbf{C}^+} \cup \{\infty\}$ if $A_j = 0$ for $j < 0$), hence $(\mathcal{L}\mathbb{E})(s) = (\mathcal{Z}\mathbb{F})(\mathrm{e}^{-Ts})$ for such s; in particular $(\mathcal{L}\mathbb{E})(-it/T) = (\mathcal{Z}\mathbb{F})(\mathrm{e}^{-it})$ for all $t \in \mathbf{R}$ (set $s := -it/T$). □

Just to simplify the notation, we have used $T = 1$ above, although the discretization could be written for a general $[0,T)$:

Remark 13.4.6 (Discretization over $[0,T)$) *As obvious from the proofs, all results of this section could be formulated for discretization over $[0,T)$ ($T>0$) instead of $[0,1)$ (e.g., $(\Delta^{\ell^2}u)_n := \pi_{[0,T)}\tau(nT)u$ ($n\in\mathbf{Z}$), hence Δ^{ℓ^2} maps $\mathrm{L}^2_{\mathrm{loc}}(\mathbf{R};U)\to \ell^2_{\mathrm{loc}}(\mathbf{Z};\mathrm{L}^2([0,T);U))$).* □

(An alternative proof would be to compress time; the operator $\mathcal{T}_T \in (\mathrm{L}^2_\omega, \mathrm{L}^2_{\omega/T})$ defined by $(\mathcal{T}_T u)(t) := u(Tt)$ is an isomorphism with inverse $\mathcal{T}_{1/T}$; see [Sbook] for details.)

Notes

Time discretization has more or less implicitly been used in [Sal89] and [W94a]. Some basic facts are given in Section 2.4 of [Sbook], but most of this section seems to be new.

Chapter 14

Riccati Equations (DARE)

The inherent vice of capitalism is the unequal sharing of blessings;
the inherent virtue of socialism is the equal sharing of misery.
— Winston Churchill (1874–1965)

In this chapter, we shall define and explore *Discrete-time Algebraic Riccati Equations (DAREs)*. Thus, this chapter is the discrete-time counterpart of Chapter 9 (cf. Theorem 14.1.3).

In Section 14.1, we study the basic properties of DAREs. In Section 14.2, we list certain auxiliary lemmas and further results. Section 14.3 contains some results on discrete-time spectral factorization.

See Chapter 15 for more on positive DAREs (those with positive cost (for zero initial state) or $S \geq 0$). That chapter also treats minimization (LQR) problems. The H^∞ full-information control problem is treated in Sections 11.5 and 11.6, and the H^∞ four-block problem in Sections 12.2 and 12.6.

Part of the continuous-time results are proved using this chapter, hence it is important to remember that, logically, one should verify all results of this monograph in discrete time (cf. Theorem 13.3.13) before verifying them in continuous time.

We shall work under the discrete-time counterpart of Hypothesis 9.0.1:

Standing Hypothesis 14.0.1 *Throughout this chapter and Chapter 15, we assume that* $\Sigma = \left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}(U,H,Y)$ *and* $J = J^* \in \mathcal{B}(Y)$. *The letters U, H and Y denote Hilbert spaces of arbitrary dimensions.*

We also assume that $\begin{bmatrix} \mathbb{Q} & \mathbb{R} \end{bmatrix}$, Z^{u} *and* Z^{s} *are as in (discrete-time) Definition 8.3.2,* $\left[\begin{smallmatrix} A^{\cdot} & \mathbb{B} \\ \mathbb{Q} & \mathbb{R} \end{smallmatrix}\right] \in \mathrm{wpls}(U,H,\widetilde{Y})$ *for some Hilbert space* $\widetilde{Y}$, *and that* $\pi_+\tau^t z \in Z^{\mathrm{s}} \Leftrightarrow z \in Z^{\mathrm{s}}$ $(z \in Z^{\mathrm{u}},\ t \in \mathbf{N})$.

The reader may again ignore the latter paragraph of the hypothesis and read $\mathcal{U}_*^*$ as any of $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$, $\mathcal{U}_{\mathrm{str}}$, $\mathcal{U}_{\mathrm{exp}}$; see the comments around Hypothesis 9.0.1 for details.

14.1 Discrete-time Riccati equations (DARE)

Duggan's Equality: To every Ph.D. there is an equal and opposite Ph.D.

Traditionally optimal control problems are solved by solving corresponding DAREs. An optimal exponentially stabilizing state feedback operator exists iff the DARE has an exponentially stabilizing solution, in which case the operator and the optimal cost operator form the solution of the DARE. In this section, we shall extend this to infinite-dimensional systems and also generalize this to other forms of stabilization than exponential stabilization.

We shall introduce the DAREs and their basic properties, including the uniqueness of a $\mathcal{U}_*^*$-stabilizing solution. In Theorem 14.1.5, we show that $\mathcal{U}_*^*$-stabilizing solutions of the DARE correspond one-to-one to the optimal state feedback operators as in the classical setting. In Theorem 14.1.6, we show that whenever there is a unique optimal control, this control can be given in a state feedback form.

Under the standard coercivity assumption, an optimal control is necessarily unique, hence of state feedback form, hence the unique solution of the DARE, by the above. Thus, in Corollary 14.1.7, we can state that the three above conditions are equivalent and that they hold iff the system is stabilizable. In Theorem 14.2.7, we include the "converse" (for $\mathcal{U}_{\text{exp}}$): the DARE has an exponentially stabilizing solution iff the standard coercivity assumption holds and the system is exponentially stabilizable (for $\dim U < \infty$ a third equivalent condition is that there is a unique optimal control).

We start by defining the DARE. Our definition is equivalent to the definition of the IARE in the sense explained in Remark 14.1.2.

Definition 14.1.1 (DARE) *We call $\mathcal{P}$ (or $(\mathcal{P},S,K)$) a* solution of the Discrete-time Algebraic Riccati Equation (DARE) *(induced by Σ and J; we denote this by $\mathcal{P} \in \mathrm{DARE}(\Sigma,J)$) iff*

$$\begin{cases} \mathcal{P} = A^*\mathcal{P}A + C^*JC - K^*SK, \\ S = D^*JD + B^*\mathcal{P}B, \\ SK = -(D^*JC + B^*\mathcal{P}A), \end{cases} \tag{14.1}$$

$\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $K \in \mathcal{B}(H,U)$, and $S \in \mathcal{GB}(U)$.

A solution of the DARE is called stabilizing *(resp.* r.c.-stabilizing, stable, ...*) if $\begin{pmatrix} K \mid 0 \end{pmatrix}$ is a stabilizing (resp. r.c.-stabilizing, stable, ...) state feedback operator for Σ (see Definition 6.6.10 for further prefices and suffices).*

We use prefices "$\mathcal{U}_*^*$-", "P-" and "PB-" as in Definition 9.8.1.

The solution $\mathcal{P}$ (or $(\mathcal{P},S,K)$) of the extended DARE (eDARE) *is defined analogously except that we do not require $S \in \mathcal{GB}(U)$. For solutions of the eDARE we denote $\mathcal{P} \in \mathrm{eDARE}(\Sigma,J)$ (or $(\mathcal{P},S,K) \in \overline{\mathrm{eDARE}}(\Sigma,J)$).*

We call K the state feedback operator *and S the* signature (or sensitivity) operator *corresponding to the solution. We define $\begin{bmatrix} \mathbb{K} \mid \mathbb{F} \end{bmatrix}$, $\mathbb{X}$, $\mathbb{M}$, $\mathbb{N}$ and $\Sigma_{\circlearrowleft}$ as in Definitions 9.1.3 and 9.1.4.*

Note that necessarily $S = S^*$, and that $\begin{pmatrix} K \mid 0 \end{pmatrix}$ is an admissible state-feedback pair for Σ (by Lemma 13.3.12).

A solution $\mathcal{P}$ of the eDARE determines S uniquely and K modulo the addition of an operator $\Delta K \in \mathcal{B}(H, \mathrm{Ker}(S))$ (hence uniquely if S is one-to-one). By Remark 9.8.8, the operators S and K can be eliminated from the eDARE too.

We have required that "$F = 0$" in the [e]DARE; this simplification does not reduce generality:

Remark 14.1.2 (DARE vs. IARE) *If we replace "$\begin{pmatrix} K \mid 0 \end{pmatrix}$" by "$\begin{pmatrix} K \mid F \end{pmatrix}$" in the definition of the [e]DARE, where we require that $I - F \in \mathcal{GB}(U)$, then the solutions of the [e]DARE become exactly the admissible solutions of the [e]IARE. Moreover, such solutions are exactly the triples $(\mathcal{P}, X^*SX, \begin{pmatrix} X^{-1}K \mid I - X \end{pmatrix})$, where $(\mathcal{P}, S, K)$ is a (original) solution of the [e]DARE and $X \in \mathcal{GB}(U)$.* □

The left columns of the corresponding closed-loop systems are equal, by (13.61 and (13.62), hence both solutions correspond to same closed-loop state and outputs (in the absence of a closed-loop input). See also Theorem 9.8.12(s1).

The results of Chapter 9 hold in discrete time too:

Theorem 14.1.3 (Chapters 8–9 apply) *The results of Sections 8.3, 8.4, 9.1, 9.8–9.10 (except the "we do not know" part of Lemma 9.9.7(d)), Lemma 9.11.1, Section 9.12, and Definition 9.14.1–Theorem 9.14.3 hold modulo the changes given in (13.63).*

(Note that (13.63) makes Hypothesis 8.4.8 useless, since it removes any "$S = D^*JD$" claims. This is because in discrete time one almost never has $S = D^*JD$, not even for $\heartsuit\mathrm{MTIC}^{\mathrm{L}^1}$ (because $\widehat{\mathbb{D}}(0) \neq \widehat{\mathbb{D}}(-1)$ in general); see also the comments around Example 14.2.9.)

Moreover, in Theorem 9.9.10 we have $u_{\mathrm{crit}}(x_0)(t) = Kx(t)$ in (f1) and $K = K' + K_\natural$ in (g2) if we take $M' = I = M$ (i.e., $F' = 0 = F$).

Also almost everything of Section 9.13 can easily be adapted for [e]DAREs, and most of the rest can be obtained directly by discretization. Many of the results of Chapter 9 not mentioned above have counterparts in this chapter. Also the B_{w}^*-CARE results of Section 9.2 can be written in their discrete-time forms, but since we lose "$S = D^*JD$", many of these results become rather useless.

Recall from Theorem 9.9.1 that there is a J-critical state feedback pair for Σ iff the eIARE has a $\mathcal{U}_*^*$-stabilizing solution.

If this is the case, then $\mathcal{J}(x_0, u_{\mathrm{crit}}(x_0)) = \langle x_0, \mathcal{P}x_0\rangle$, and u_{crit} is given by the feedback K; the cost with closed-loop input $u_\circlearrowleft \in \ell^2([0,t);U)$ $(t < \infty)$ is given by $\langle x_0, \mathcal{P}x_0\rangle + \langle u_\circlearrowleft, Su_\circlearrowleft\rangle$, by Theorem 9.9.1(h); hence the name for the signature operator S.

Lemma 14.3.5 provides us some additional classes satisfying (the converted) Hypothesis 8.4.7.

Proof of Theorem 14.1.3: One can first verify Sections 8.3 and 8.4, then Lemma 14.2.1, then 9.10 (and 9.11.1 from 9.11), then 9.8 (including Theorem 14.1.4) and 9.9, then the rest of Chapter 9. One can avoid references "backwards" by verifying each claim in its discrete and continuous time forms before proceeding to the next one.

Note that $K = -S^{-1}\pi_{\{0\}}\mathbb{N}^* JCA^{\cdot} \in \mathcal{B}(H,U)$ in Theorem 9.9.10(g1). $\square$

Note that $\mathcal{U}_{\text{out}}(x_0) := \{u \in \ell^2(\mathbf{N};U) \,|\, \mathbb{C}x_0 + \mathbb{D}u \in \ell^2(\mathbf{N};Y)\}$ etc., when we write (8.29)–(8.32) out (applying (13.63)).

A strongly internally stabilizing solution of the DARE is unique:

Theorem 14.1.4 ($\mathcal{P}$ is unique) *A solution $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$ of the eIARE is unique to the following extent:*

(a) If the eDARE (14.1) has a strongly internally stabilizing solution, then that solution is unique among internally stabilizing solutions.

(b) There is at most one internally P-stabilizing solution of the eDARE.

(c) If the eDARE has an internally r-stabilizing solution for some $r < 1$, then any other solution is (internally) at most $1/r$-stabilizing.

(d) The eDARE has at most one P-q.r.c.-SOS-stabilizing solution.

(e) The eDARE has at most one $\mathcal{U}_*^*$-stabilizing solution.

Naturally, $\mathcal{P}$ determines $S := D^*JD + B^*\mathcal{P}B$ uniquely and K modulo the addition of an operator $\Delta K \in \mathcal{B}(H, \mathrm{Ker}(S))$.

Note from Theorem 9.9.1(a1)&(e), that each a $\mathcal{U}_*^*$-stabilizing solution $(\mathcal{P}, S, K)$ determines a J-critical state feedback operator for Σ, and vice versa. The corresponding S is unique; the operator K is unique iff S is one-to-one.

Thus, if S is not one-to-one. then the feedback operator K solving the eDARE with $\mathcal{P}$ and S is not unique, but it may be that just one K is $\mathcal{U}_*^*$-stabilizing, i.e., that the $\mathcal{U}_*^*$-stabilizing solution triple $(\mathcal{P}, S, K)$ is unique, and that the J-critical control in state feedback form ($\mathbb{K}_{\circlearrowleft} x_0$) is unique (for each x_0) (see Example 9.13.6 for an example).

Proof of Theorem 14.1.4: The claim on S and K is obvious, so we only need to prove the uniqueness of $\mathcal{P}$.

(a) Let $\mathcal{P}_1, \mathcal{P}_2$ be a stabilizing solutions; let $\mathcal{P}_1$ be strongly stabilizing. For $k = 1,2$, let S_k and K_k correspond to $\mathcal{P}_k$. By the definition of K and S (see eDARE), we have

$$R := C^*JDK_1 - K_2^*D^*JC + A^*\mathcal{P}_2BK_1 - K_2^*B^*\mathcal{P}_1A + K_2^*(B^*\mathcal{P}_2B - B^*\mathcal{P}_1B)K_1 \tag{14.2}$$

$$= -K_2^*S_2K_1 + K_2^*S_1K_1 + K_2^*(S_2 - S_1)K_1 = 0. \tag{14.3}$$

Therefore, by the eDARE, we have

$$\mathcal{P}_1 - \mathcal{P}_2 = A^*(\mathcal{P}_1 - \mathcal{P}_2)A + K_2^*S_2K_2 - K_1^*S_1K_1 \tag{14.4}$$

$$= A^*(\mathcal{P}_1 - \mathcal{P}_2)A + (C^*JD + A^*\mathcal{P}_1B)K_1 - K_2^*(D^*JC + B^*\mathcal{P}_2A) \tag{14.5}$$

$$= (A^* + K_2^*B^*)(\mathcal{P}_1 - \mathcal{P}_2)(A + BK_1) + R \tag{14.6}$$

$$= A_{\circlearrowleft 2}^*(\mathcal{P}_1 - \mathcal{P}_2)A_{\circlearrowleft 1}. \tag{14.7}$$

Multiply equation $\mathcal{P}_1 - \mathcal{P}_2 = A_{\circlearrowleft 2}^*(\mathcal{P}_1 - \mathcal{P}_2)A_{\circlearrowleft 1}$ by $(A_{\circlearrowleft 2}^*)^k$ to the left and $A_{\circlearrowleft 1}^k$ to the right, and use the resulting chain of equations for $k = 0, \ldots, n-1$ to obtain

$$\mathcal{P}_1 - \mathcal{P}_2 = (A_{\circlearrowleft 2}^*)^n(\mathcal{P}_1 - \mathcal{P}_2)A_{\circlearrowleft 1}^n \quad (n \in \mathbf{N}). \tag{14.8}$$

Because $(A^*_{\circlearrowleft 2})^n(\mathcal{P}_1 - \mathcal{P}_2)$ is bounded and $A^n_{\circlearrowleft 1}x_0 \to 0$ as $n \to +\infty$, for any $x_0 \in H$, we have $\mathcal{P}_1 x_0 - \mathcal{P}_2 x_0 = 0$ for all $x_0 \in H$, hence $\mathcal{P}_1 = \mathcal{P}_2$.

(b) Now $\mathcal{P}_k A^n_{\circlearrowleft k}x_0 \to 0$ as $n \to +\infty$, for all $x_0 \in H$, $k = 1,2$, by (P3), and $\|A^n_{\circlearrowleft k}\|$ is bounded, so the result follows as above.

(c) If $M, r > 1$ and $s > 1/r$ are s.t. $\|A^n_{\circlearrowleft 1}x_0\| \le Mr^{-n}$, $\|A^n_{\circlearrowleft 2}x_0\| \le Ms^{-n}$, then

$$|\langle x_0, (\mathcal{P}_1 - \mathcal{P}_2)x_0\rangle| \le (\|\mathcal{P}_1\| + \|\mathcal{P}_2\|)M^2(rs)^{-n} \quad \text{for all } \ n \in \mathbf{N}, \tag{14.9}$$

hence then $|\langle x_0, (\mathcal{P}_1 - \mathcal{P}_2)x_0\rangle| = 0$ (in the proof of (a)).

(e) (resp. (d)) By Proposition 9.10.2(c) (resp. (e2)), the $\mathcal{U}^*_*$-stabilizing (resp. P-SOS-r.c.-stabilizing) solutions of the eIARE correspond one-to-one to the J-critical (resp. SOS-r.c.-stabilizing and J-critical over $\mathcal{U}_{\text{out}}$) state feedback operators (see Lemma 14.2.1).

The J-critical cost $\langle x_0, \mathcal{P}x_0\rangle$ over $\mathcal{U}^*_*$ (resp. over $\mathcal{U}_{\text{out}}$) is independent of the J-critical control $\mathbb{K}_{\circlearrowleft}x_0$, by Lemma 8.3.8, hence $\mathcal{P}$ is unique. □

From Theorem 9.9.1(a1) one can observe that any J-critical state feedback operator K corresponds to a $\mathcal{U}^*_*$-stabilizing solution $\mathcal{P}$ of the eIARE, and vice versa. Due to bounded input and output operators (B and C), in discrete-time we can go further:

Theorem 14.1.5 (J-critical $K \Leftrightarrow$ DARE) *There is a J-critical state feedback operator K iff the eDARE has a $\mathcal{U}^*_*$-stabilizing solution $(\mathcal{P}, S, K)$.*

Moreover, such Ks are exactly those associated with the solutions, with S being the corresponding signature operator and $\mathcal{P}$ being the J-critical cost operator satisfying (9.139)). □

(This follows from the above and Lemma 14.2.1.) Also most of Theorem 14.1.6 holds, as one can observe from Theorem 9.9.1. Recall from Theorem 14.1.4 that $\mathcal{P}$ is unique but K need not be. Also recall from Remark 14.1.2 how each K corresponds to a set of pairs $\begin{pmatrix} K \mid F \end{pmatrix}$ with same $\mathcal{P}$.

In most standard settings, any J-critical control is unique. Such a control can always be given by some state feedback operator:

Theorem 14.1.6 (Unique J-critical control is of the feedback form) *There is a unique J-critical control for each $x_0 \in H$ iff the eDARE has a $\mathcal{U}^*_*$-stabilizing solution $(\mathcal{P}, S, K)$ with S one-to-one.*

Assume that this is the case. Then the following hold:

(a) The $\mathcal{U}^*_*$-stabilizing solution $(\mathcal{P}, S, K)$ is unique.

(b1) The J-critical control is determined by $u_{\text{crit}}(x_0) = \mathbb{K}_{\circlearrowleft}x_0$, i.e., by $u_{\text{crit}}(x_0)_n = K(A+BK)^n x_0$ ($n \in \mathbf{N}$), where $\Sigma_{\circlearrowleft}$ is the closed-loop system corresponding to $\begin{pmatrix} K \mid 0 \end{pmatrix}$.

(b2) Conversely, $Kx_0 = u_{\text{crit}}(x_0)_0$ and $\mathcal{P} = \mathbb{C}^*_{\circlearrowleft} J \mathbb{C}_{\circlearrowleft}$ (and $S = D^*JD + B^*\mathcal{P}B$).

(c) If Σ is exponentially stable, then K is exponentially r.c.-stabilizing.

(d) Theorem 9.9.1(f1)–(k) apply.

See Theorem 9.8.5 for $\mathcal{U}_*^*$-stabilizing solutions. Note in particular, that a solution is $\mathcal{U}_{\mathrm{exp}}$-stabilizing iff $\rho(A+BK) < 1$, by Lemma 13.3.7 (and Theorem 9.8.5).

Theorem 9.9.6 contains an analogy of the above theorem for WPLSs (i.e., for continuous time) with bounded B. For very irregular WPLSs, the "only if" part may fail (at least we have to give up eCAREs; presumably even eIAREs).

In general, all J-critical state feedback operators correspond bijectively to $\mathcal{U}_*^*$-stabilizing solutions of the eDARE as in the above theorem, by Proposition 9.9.1.

Proof of Theorem 14.1.6: The equivalence follows from Theorem 9.9.1(a1)&(e2), because a unique (for each x_0) J-critical control u_{crit} is necessarily of the feedback form: if K is defined by (b2), then $\mathbb{A}_{\circlearrowleft} = \mathbb{A}_{\mathrm{crit}}$, $\mathbb{K}_{\circlearrowleft} = \mathbb{K}_{\mathrm{crit}}$, and $\mathbb{C}_{\circlearrowleft} = \mathbb{C}_{\mathrm{crit}}$, by (13.61).

(a) By Theorem 9.9.1(f1), $\mathcal{P}$ is unique, hence so are S and K (because S is one-to-one).

(b2)&(d) These follow from Theorem 9.9.1.

(b1) This follows from formula $u_{\mathrm{crit}}(x_0) = \mathbb{K}_{\mathrm{crit}} x_0 = \mathbb{K}_{\circlearrowleft} x_0$.

(c) This follows from Theorem 8.3.9(a5) (see also (13.61). □

Thus, if Σ is J-coercive and the finite cost condition is satisfied, then the optimal control corresponds to a (unique) $\mathcal{U}_*^*$-stabilizing solution of the DARE:

Corollary 14.1.7 (J-coercive ⇒ DARE) *Assume that Σ is J-coercive. Then the following are equivalent:*

(i) there is a [unique] J-critical control over $\mathcal{U}_*^*(x_0)$ for each $x_0 \in H$;

(ii) $\mathcal{U}_*^*(x_0) \neq \emptyset$ for each $x_0 \in H$;

(iii) the DARE has a $\mathcal{U}_*^*$-stabilizing solution. □

(This follows from Theorem 14.1.6 and Theorem 8.4.3, since necessarily $S \in \mathcal{GB}(U)$, by Lemma 9.10.3.) See also Theorem 14.2.7.

(See p. 829 for notes.)

14.2 DARE — further results

Everyone wants results, but no one is willing to do what it takes to get them.

— Dirty Harry

In this section, we present auxiliary lemmas and further results on DAREs. These include the relations between all (not necessarily stabilizing) solutions of the eDARE, symplectic "matrix" pencils, DAREs connected to the internal and/or output stability of a system, equivalence between J-coercivity and the existence of a unique optimal control, and H^2 "spectral factors".

In previous section, we used the fact that the solutions of the [e]DARE are exactly the admissible solutions of the [e]IARE:

Lemma 14.2.1 (eDARE ⇔ eIARE) *Let $S \in \mathcal{B}(U)$ and $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$. Let $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$ be an admissible state feedback pair for Σ, and let $\Sigma_\circlearrowleft := \left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right] \in$ $\mathrm{wpls}(U,H,Y\times U)$ be the corresponding closed-loop system. Set $\mathbb{M} := (I-\mathbb{F})^{-1}$, $\mathbb{N} := \mathbb{D}\mathbb{M} = \mathbb{D}_\circlearrowleft$. If equations*

$$\mathbb{K}^{t\,*}S\mathbb{K}^t = \mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}^t - \mathcal{P} + \mathbb{C}^{t\,*}J\mathbb{C}^t \tag{14.10}$$

$$\mathbb{X}^{t\,*}S\mathbb{X}^t = \mathbb{D}^{t\,*}J\mathbb{D}^t + \mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t, \tag{14.11}$$

$$\mathbb{X}^{t\,*}S\mathbb{K}^t = -\left(\mathbb{D}^{t\,*}J\mathbb{C}^t + \mathbb{B}^{t\,*}\mathcal{P}\mathbb{A}^t\right). \tag{14.12}$$

hold for $t=1$, then they hold for each $t \in \mathbf{N}$.

Thus, by (13.62), $(\mathcal{P}, X^*SX, X^{-1}K)$ (here $X := I-F$) is a [stabilizing] solution the eDARE iff $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is an admissible [stabilizing] solution of the eIARE. All prefices and suffices apply.

If $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ is an admissible solution of the eIARE with closed-loop system $\Sigma_\circlearrowleft$, then the closed-loop system corresponding to $\left(\begin{array}{c|c} X^{-1}K & 0 \end{array}\right)$ is $\left[\begin{array}{c|c} A_\circlearrowleft & B_\circlearrowleft X \\ \hline C_\circlearrowleft & D_\circlearrowleft X \\ K_\circlearrowleft & 0 \end{array}\right]$, by (13.61) and (13.62).

Proof: We assume that (14.10)–(14.12) are satisfied for $t=1$ (i.e., that the eDARE is satisfied).

1° (14.10): Let $x \in H$. Apply (14.10) with $t=1$ to $A^k x$ ($k \in \mathbf{N}$) to obtain

$$\langle A^{k+1}x, \mathcal{P}A^{k+1}x\rangle - \langle A^k x, \mathcal{P}A^k x\rangle + \langle CA^k x, JCA^k x\rangle = \langle KA^k x, SKA^k x\rangle. \tag{14.13}$$

But, by (13.38), we have $(\mathbb{C}x)_k = CA^k x$ and $(\mathbb{K}x)_k = KA^k x$, hence we can add (14.13) for $k = 0,1,2,\ldots,t-1$ to get (14.10).

2° (14.11): Let $0 \neq u \in \mathrm{c_c}(\mathbf{N};U)$. Set $x := \mathbb{B}\tau u$ to obtain that $x_{j+1} = Ax_j + Bu_j$ and $(\mathbb{D}u)_j = Cx_j + Du_j$ ($j \in \mathbf{N}$), by Lemma 13.3.3(c). Similarly, $\mathbb{X}u = u - Kx$. Thus, by (14.12) with $t=1$, we have (here the inner products are taken on U or H, and the subscript j refers to "time", i.e., to the argument of x

and u):

$$\langle \mathbb{X}u, S\mathbb{X}u\rangle_j = \langle u, Su\rangle_j - \langle u, SKx\rangle_j - \langle SKx, u\rangle_j + \langle Kx, SKx\rangle_j \tag{14.14}$$

$$= \langle u, Su\rangle_j + \langle u, (D^*JC + B^*\mathcal{P}A)x\rangle_j + \langle (D^*JC + B^*\mathcal{P}A)x, u\rangle_j \tag{14.15}$$

$$+ \langle x, A^*\mathcal{P}Ax\rangle_j - \langle x, \mathcal{P}x\rangle_j + \langle x, C^*JCx\rangle = g_j + h_j \tag{14.16}$$

on $\mathbf{Z}$, where

$$g_j := \langle u, D^*JDu\rangle_j + \langle u, D^*JCx\rangle_j + \langle D^*JCx, u\rangle_j + \langle x, C^*JCx\rangle_j = \langle \mathbb{D}u, J\mathbb{D}u\rangle_j, \tag{14.17}$$

$$h_j := \langle u, (S - D^*JD)u\rangle_j + \langle u, B^*\mathcal{P}Ax\rangle_j + \langle B^*\mathcal{P}Ax, u\rangle_j + \langle x, A^*\mathcal{P}Ax\rangle_j - \langle x, \mathcal{P}x\rangle_j \tag{14.18}$$

$$= \langle Ax + Bu, \mathcal{P}(Ax, Bu)\rangle_j - \langle x_j, \mathcal{P}x_j\rangle = \langle x_{j+1}, \mathcal{P}x_{j+1}\rangle - \langle x_j, \mathcal{P}x_j\rangle. \tag{14.19}$$

Therefore,

$$\sum_0^j \langle \mathbb{X}u, S\mathbb{X}u\rangle_j - \sum_0^j \langle \mathbb{D}u, J\mathbb{D}u\rangle_j = \sum_0^j \langle \mathbb{X}u, S\mathbb{X}u\rangle_j - \sum_0^j g_j = \sum_0^j h_j = \langle x_{j+1}, \mathcal{P}x_{j+1}\rangle. \tag{14.20}$$

But this is (14.11) for $t = j+1$ applied to u (because $x := \mathbb{B}\tau^j u$). Because u and j were arbitrary, equation (14.11) holds, by density and continuity.

3° (14.12): For $t \in \mathbf{N}$, we set $\mathbb{T}_t := \mathbb{X}^{t\,*} S\pi_{[0,t)}\mathbb{K} + \pi_{[0,t)}\big(\mathbb{D}^* J\pi_{[0,t)}\mathbb{C} + \tau^t\mathbb{B}^*\mathcal{P}A^t\big) \in \mathcal{B}(H, \ell^2([0,t);U))$, as in Lemma 9.11.6. We must prove that $\mathbb{T} \equiv 0$, hence it is enough to show that

$$f_t := f_t(u, x_0) := \langle u, \mathbb{T}_t x_0\rangle_{\ell^2} = 0 \tag{14.21}$$

for arbitrary $t \in \mathbf{N}$, $u \in \ell^2((0,t);U)$ and $x_0 \in H$. Let u, x_0 be as above, and set

$$x_t := A^t x_0, \quad z_t := \mathbb{B}\tau^t u \tag{14.22}$$

to obtain $x_{t+1} = Ax_t$, $z_{t+1} = Az_t + Bu_t$, $(\mathbb{X}u)_t = u_t - Kz_t$, $(\mathbb{D}u)_t = Du_t + Cz_t$, $(\mathbb{C}x_0)_t = Cx_t$, $(\mathbb{K}x_0)_t = Kx_t$ (see Lemma 13.3.3). Thus, for $t \in \mathbf{N}$, we have (recall that $\mathbb{N}v = \mathbb{D}u$)

$$f_{t+1} = \sum_0^t \langle (\mathbb{X}u)_n, S(\mathbb{K}x_0)_n\rangle_U + \sum_0^t \langle (\mathbb{D}u)_n, J(\mathbb{C}x_0)_n\rangle_Y\,dt + \langle z_{t+1}, \mathcal{P}x_{t+1}\rangle_H. \tag{14.23}$$

Therefore, (we set here $f_{-1} = 0$ so that this holds for $t = -1$ too)

$$f_{t+1} - f_t = \langle u_t - Kz_t, SKx_t\rangle + \langle Du_t + Cz_t, JCx_n\rangle + \langle Az_t + Bu_t, \mathcal{P}Ax_t\rangle - \langle z_t, \mathcal{P}x_t\rangle = 0, \tag{14.24}$$

by (14.10)–(14.12) with $t = 1$. Consequently, $f \equiv 0$ and $\mathbb{T} \equiv 0$.

4° *eIARE⇔eDARE:* Obviously, the eIARE with $t = 1$ is exactly the eDARE. □

The "K" of a $\mathcal{U}^*_*$-stabilizing solution of the eDARE is the optimal state feedback operator. Conversely, given a unique optimal control, we obtain the corresponding solution of the eDARE as follows:

Lemma 14.2.2 *Let there be a unique J-critical control $u_{\mathrm{crit}}(x_0)$ for each x_0, and let Σ_{crit} be as in Theorem 8.3.9.*

Set $Kx_0 := u_{\mathrm{crit}}(x_0)_0$ ($x_0 \in H$). Then Σ_{crit} is the left column of the closed-loop system corresponding to the state feedback pair $\left(\begin{array}{c|c} K & 0 \end{array} \right)$ for Σ. Thus, $\mathcal{P} := \mathbb{C}_{\mathrm{crit}}^* J \mathbb{C}_{\mathrm{crit}}$, $S := D^*JD + B^*\mathcal{P}B$ and K constitute the unique $\mathcal{U}_*^*$-stabilizing solution of the eDARE.

Proof: By (13.39), the generators of $\mathbb{K}_{\mathrm{crit}}$, $\mathbb{C}_{\mathrm{crit}}$ and $\mathbb{A}_{\mathrm{crit}}$ are $K, C+DK$ and $A+BK$, respectively; hence the first claim follows from (13.62) (with $M = I$). By Proposition 9.10.2(c), $(\mathcal{P}, S, K)$ is $\mathcal{U}_*^*$-stabilizing. □

From Lemma 9.12.3 (which also contains further results) we obtain the connection between the DAREs for the open- and closed-loop systems (note that no stabilization is required here):

Lemma 14.2.3 *Let $\left(\begin{array}{c|c} K' & F' \end{array} \right)$ be admissible for Σ, i.e., $(M')^{-1} := I - F' \in \mathcal{GB}(U)$, and let $\Sigma_\flat$ be the corresponding closed-loop system*

Then $(\mathcal{P}, S, K) \in \overline{\mathrm{eDARE}}(\Sigma, J) \Leftrightarrow (\mathcal{P}, M'^* S M', M'^{-1}K - K') \in \overline{\mathrm{eDARE}}(\left(\begin{array}{c|c} A_\flat & B_\flat \\ \hline C_\flat & D_\flat \end{array} \right), J)$. □

In particular, $\mathrm{eDARE}(\Sigma, J) = \mathrm{eDARE}(\Sigma_\flat^1, J)$ and $\mathrm{DARE}(\Sigma, J) = \mathrm{DARE}(\Sigma_\flat^1, J)$ (see Definition 14.1.1 for the notation), where (see Lemma 13.3.12)

$$\Sigma_\flat^1 := \left(\begin{array}{c|c} A_\flat & B_\flat \\ \hline C_\flat & D_\flat \end{array} \right) := \left(\begin{array}{c|c} A+BK_\flat & BM' \\ \hline C+DK_\flat & DM' \end{array} \right), \quad K_\flat = M'K'. \tag{14.25}$$

By setting $F' = 0$ we see, that if we perturbate Σ by a feedback K' and the resulting system can be optimized by feedback $K_\natural$, then the original system can be optimized by feedback $K = K' + K_\natural$.

From Lemma 9.12.5 we obtain that difference between two solutions of the eDARE solves the eDARE for the corresponding $\mathbb{X}$ system (still no stabilization assumed):

Lemma 14.2.4 *Let $(\mathcal{P}_1, S_1, K_1) \in \overline{\mathrm{eDARE}}(\Sigma, J)$.*

Then $(\mathcal{P}_2, S_2, K_2) \in \overline{\mathrm{eDARE}}(\Sigma, J) \Leftrightarrow (\mathcal{P}_2 - \mathcal{P}_1, S_2, K_2) \in \overline{\mathrm{eDARE}}(\left(\begin{array}{c|c} A & B \\ \hline -K_1 & I \end{array} \right), S_1)$. □

Thus, for $(\mathcal{P}, S, K) \in \overline{\mathrm{eDARE}}(\Sigma, J)$ we have $\mathrm{eDARE}(\Sigma, J) = \mathcal{P} + \mathrm{eDARE}(\left(\begin{array}{c|c} A & B \\ \hline -K & I \end{array} \right), S)$. Note that $\left(\begin{array}{c|c} A & B \\ \hline -K & I \end{array} \right)$ is a realization of "$\mathbb{X}$".

Matrix pencils can sometimes be used to simplify certain computations. The following lemma relates eDAREs to so called *extended symplectic matrix pencils*:

Lemma 14.2.5 (Symplectic Pencil) *Let $\mathcal{P} = \mathcal{P}^* \in \mathcal{B}(H)$, $K \in \mathcal{B}(H, U)$, and $A_\circlearrowleft \in \mathcal{B}(H)$. Set $S := D^*JD + B^*\mathcal{P}B$, $V := \begin{bmatrix} I_H \\ \mathcal{P} \\ K \end{bmatrix}$. Then the following are equivalent:*

(i) The eDARE for Σ *and* J *has the solution* $(\mathcal{P}, S, K)$*, and* $A_\circlearrowleft = A + BK$.

(ii) We have $M_{\Sigma,J} V A_\circlearrowleft = N_{\Sigma,J} V$*, where*

$$M_{\Sigma,J} := \begin{bmatrix} I_H & 0 & 0 \\ 0 & -A^* & 0 \\ 0 & -B^* & 0 \end{bmatrix}, \; N_{\Sigma,J} := \begin{bmatrix} A & 0 & B \\ C^*JC & -I_H & C^*JD \\ D^*JC & 0 & D^*JD \end{bmatrix} \in \mathcal{B}(H \times H \times U). \tag{14.26}$$

(iii) Some $Z \in \mathcal{GB}(H)$ *and* $V' = \left[\begin{smallmatrix} Z \\ * \\ * \end{smallmatrix}\right] \in \mathcal{B}(H, H \times H \times U)$ *satisfy* $M_{\Sigma,J} V' A'_\circlearrowleft = N_{\Sigma,J} V'$*, and* $V = V' Z^{-1}$ *and* $A_\circlearrowleft = Z A'_\circlearrowleft Z^{-1}$.

Thus, we have three equivalent problems; the second one will be our tool in in Section 12.6 for the proof of the H$^\infty$ Four-Block Problem.

Proof: 1° "(i)⇔(ii)": Write $M_{\Sigma,J} V A_\circlearrowleft = N_{\Sigma,J} V$ out to obtain the equation $A_\circlearrowleft = A + BK$ and a system which becomes the eDARE after this substitution.

2° "(ii)⇔(iii)": This is obvious. □

Next we present the discrete-time counterpart of Lemma 9.12.2, i.e., the connection between the internal and/or output stability of a system and the solvability of certain Riccati equations (the solutions need not be stabilizing a priori):

Lemma 14.2.6 ($\mathbb{A}/\mathbb{C}$ is stable ⇔ DARE)

(a) Assume that $J \gg 0$*. Then* $\mathbb{C}$ *is stable iff there is* $\mathcal{P} \in \mathcal{B}(H)$ *s.t.* $\mathcal{P} \geq 0$ *and*

$$\mathcal{P} \geq A^* \mathcal{P} A + C^* J C. \tag{14.27}$$

(b) Assume that $\mathbb{C}$ *is stable. Then* $\mathcal{P} = \mathbb{C}^* J \mathbb{C}$ *is a solution of* $\mathcal{P} = A^* \mathcal{P} A + C^* J C$*, and* $\widetilde{\mathcal{P}} \geq \mathcal{P}$ *for any* $\widetilde{\mathcal{P}} \geq 0$ *that solves (14.27).*

In particular, if $J \geq 0$*, then* $\mathcal{P} = \mathbb{C}^* J \mathbb{C}$ *is the smallest nonnegative solution of (14.27).*

(c) Assume, that $\mathbb{A}$ *is strongly stable and* $\mathbb{C}$ *stable. Then* $\mathcal{P} = \mathbb{C}^* J \mathbb{C}$ *is the unique solution (in* $\mathcal{B}(H)$*) of* $\mathcal{P} = A^* \mathcal{P} A + C^* J C$.

(d) The map $\mathbb{A}$ *is exponentially stable iff* $\mathcal{P} \gg A^* \mathcal{P} A$ *for some nonnegative* $\mathcal{P} \in \mathcal{B}(H)$ *(and any such* $\mathcal{P}$ *necessarily satisfies* $\mathcal{P} \gg 0$*).* □

(The proof is analogous to that of Lemma 9.12.2 (use also Lemma 15.5.1) and hence omitted.) Naturally, we can again use duality to get corresponding results for (A, B).

We can now show that for $\mathcal{U}_{\exp}$, J-coercivity is equivalent to the existence of a unique optimal control (if, e.g., $\dim U < \infty$):

Theorem 14.2.7 ($\mathcal{U}_{\exp}$: Unique optimum⇔DARE⇔ J-coercive) *Conditions (i)–(iii) are equivalent.*

(i) There is a unique J*-critical control over* $\mathcal{U}_{\exp}(x_0)$ *for each* $x_0 \in H$*, and* $S := D^* J D + B^* \mathcal{P} B \in \mathcal{GB}(U)$.

(ii) The DARE has an exponentially stabilizing solution.

(iii) $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\rm exp}$, and Σ is exponentially stabilizable.

If $\dim U < \infty$, or $D^*JD \gg 0$ and $J \ge 0$, then the condition $S \in \mathcal{GB}(U)$ is redundant in (i).

If S is as in (i) (or (ii)), then $S \gg 0$ iff $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\rm exp}$ (iff the J-critical control is minimizing).

Naturally, $\mathcal{P} := \mathbb{C}_{\rm crit}^* J \mathbb{C}_{\rm crit}$ in (i) ($= \mathbb{C}_{\circlearrowleft}^* J \mathbb{C}_{\circlearrowleft}$ in (ii)). See, e.g., Theorem 14.1.6 for further details.

Proof: 1° Equivalence "(i)⇔(ii)" follows from Theorem 14.1.6, and implication "(iii)⇒(i)" from Theorems 8.4.3 and 14.1.6 and Lemma 9.10.3.

2° *(ii)⇒(iii):* Assume (ii). Trivially, then Σ is exponentially stabilizable, so that we only have to prove that $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\rm exp}$.

By Theorem 9.9.1(k), we have $\mathbb{M}\ell^2(\mathbf{N};U) = \mathcal{U}_{\rm exp}(0)$. Choose $\varepsilon_S > 0$ s.t. $\|Su_0\| \ge \varepsilon_S \|u_0\|$ for all $u_0 \in U$. By Lemma 6.1.10, we have

$$M := \max\{\|\mathbb{M}\|_{\rm tic}, \|\mathbb{B}_{\circlearrowleft}\tau\|_{\mathcal{B}(\ell^2,H)}\} < \infty. \tag{14.28}$$

Assume that $0 \ne u \in \mathcal{U}_{\rm exp}(0)$, and set $u_{\circlearrowleft} := \mathbb{M}^{-1}u \in \ell^2(\mathbf{N};U)$. Set $v_{\circlearrowleft} := Su_{\circlearrowleft}/\|Su_{\circlearrowleft}\|_2 \in \ell^2(\mathbf{N};U)$, $v := \mathbb{M}v_{\circlearrowleft} \in \mathcal{U}_{\rm exp}(0)$, so that $\|v_{\circlearrowleft}\|_2 \le 1$ and $\langle v_{\circlearrowleft}, Su_{\circlearrowleft}\rangle \ge \varepsilon_S \|u_{\circlearrowleft}\|_2$.

Then $\|v\|'_{\mathcal{U}_{\rm exp}} := \max\{\|v\|_2, \|\mathbb{B}\tau v\|_2\} \le M$ (since $\mathbb{B}\tau v = \mathbb{B}_{\circlearrowleft}\tau v_{\circlearrowleft}$) and $\|u\|'_{\mathcal{U}_{\rm exp}} \le M\|u_{\circlearrowleft}\|_2$ (see Lemma 8.4.2). Since

$$\langle \mathbb{D}v, J\mathbb{D}u\rangle = \langle \mathbb{D}_{\circlearrowleft}v_{\circlearrowleft}, J\mathbb{D}_{\circlearrowleft}u_{\circlearrowleft}\rangle = \langle v_{\circlearrowleft}, Su_{\circlearrowleft}\rangle \ge \varepsilon_S\|u_{\circlearrowleft}\|_2 \ge \varepsilon_S M^{-2}\|u\|'_{\mathcal{U}_{\rm exp}}\|v\|'_{\mathcal{U}_{\rm exp}}, \tag{14.29}$$

and u was arbitrary, we have shown that $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\rm exp}$, by Lemma 8.4.2.

3° *Redundancy:* If $\mathcal{J} \ge 0$ (e.g., $J \ge 0$), then necessarily $\mathcal{P} \ge 0$, hence then $D^*JD \gg 0$ implies that $S \gg 0$.

If there is a unique J-critical control over $\mathcal{U}_{\rm exp}(x_0)$ for each $x_0 \in H$, then S is one-to-one, by Theorem 14.1.6, so that $\dim U < \infty$ makes condition $S \in \mathcal{GB}(U)$ redundant.

4° $S \gg 0$*:* This follows from Theorem 9.9.1(k) and Lemma 10.2.2. □

Now we state the discrete-time counterpart of Lemma 9.12.8. This corresponds to the fact that, in the indefinite case, the "spectral factors" may become unstable ($\widehat{\mathbb{X}} \in \mathrm{H}^2 \setminus \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(U))$ for $\dim U < \infty$).

Lemma 14.2.8 ($\mathbb{B}, \mathbb{D}$ stable ⇒ $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$ & $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}^2$) *Assume that $\mathbb{B}$ and $\mathbb{D}$ are stable, $\vartheta = 1$, and $(\mathcal{P}, S, \begin{bmatrix} \mathbb{K} & \mathbb{F} \end{bmatrix})$ is a $\mathcal{U}_*^*$-stabilizing solution of the eIARE. Set $\mathbb{M}^{-1} := \mathbb{X} := I - \mathbb{F}$, $\mathbb{N} := \mathbb{D}\mathbb{M}$, $\widehat{\mathbb{X}}^{\rm d} := \widehat{\mathbb{X}}(\bar{\cdot})^*$. Then*

(a1) $\mathbb{N}, \mathbb{M}, \mathbb{X} \in \mathrm{tic}_r(U,*)$ for all $r > 1$.

(a2) $\widehat{\mathbb{N}}, \widehat{\mathbb{M}}, \widehat{\mathbb{X}}^{\rm d} \in \mathrm{H}^2_{\rm strong}(\mathbf{D};\mathcal{B}(U,*))$; in particular, $\widehat{\mathbb{X}} \in \mathcal{G}\mathrm{H}(\mathbf{D};\mathcal{B}(U))$.

(b1) $\mathbb{N}, \mathbb{M}, \mathbb{X}^* \in \mathcal{B}(\ell^1(\mathbf{Z};U), \ell^2(\mathbf{Z};*))$, $\mathbb{N}^*, \mathbb{M}^*, \mathbb{X} \in \mathcal{B}(\ell^2(\mathbf{Z};*), \ell^\infty(\mathbf{Z};U))$, and $\mathbb{X}^*\pi^\pm\mathbb{M}^*, \mathbb{M}\pi_{[-T,t)}, \mathbb{X}^*\pi_{[-T,t)} \in \mathcal{B}(\ell^2(\mathbf{Z};U))$ for all $T, t \in \mathbf{N}$.

(b2) $\mathbb{M}\pi^+\mathbb{X}, \pi_{[-T,t)}\mathbb{X} \in \mathcal{B}(\ell^1(\mathbf{Z};U), \ell^2(\mathbf{Z};U)) \cap \mathcal{B}(\ell^2_r(\mathbf{Z};U)) \cap \mathcal{B}(\ell^2(\mathbf{N};U))$ *for each* $r > 1$, *and* $\mathbb{M}\pi^+\mathbb{X}$ *and* $\pi_{[-T,t)}\mathbb{X}$ *have a continuous extensions to* $\mathcal{B}(\ell^2(\mathbf{Z};U))$.

(c1) $\langle \mathbb{N}u, J\mathbb{N}v\rangle = \langle u, Sv\rangle$ *for all* $u, v \in \ell^1(\mathbf{Z};U)$.

(c2) $\mathbb{X}^*\pi_{[-T,t)}S\mathbb{X}u \to \mathbb{D}^*J\mathbb{D}u$ *in* $\ell^2(\mathbf{Z};U)$, *as* $t, T \to +\infty$, *if* $\mathbb{B}$ *is strongly stable and* $u \in \ell^2(\mathbf{Z};U)$.

(d) **($\dim U < \infty \Rightarrow \widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$)** *If* $\dim U < \infty$, *then* $\widehat{\mathbb{X}}, \widehat{\mathbb{M}} \in \mathrm{H}^2(\mathbf{D};\mathcal{B}(U)) \cap \mathrm{L}^2(\partial\mathbf{D};\mathcal{B}(U))$, *and* $\widehat{\mathbb{X}} \in \mathcal{GB}(U)$ *and* $\widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ *a.e. on* $\partial\mathbf{D}$.

(e) **($(\pi^+\mathbb{D}^*J\mathbb{D}\pi^+)^{-1} = \mathbb{M}\pi^+S^{-1}\mathbb{M}^*$)** *If* $\mathbb{T} := \pi^+\mathbb{D}^*J\mathbb{D}\pi^+$ *is invertible on* $\ell^2(\mathbf{N};U)$ *and* $\mathbb{B}$ *is strongly stable, then* $S \in \mathcal{GB}(U)$ *and* $\mathbb{T}^{-1} = \mathbb{M}\pi^+S^{-1}\mathbb{M}^* \in \mathcal{GB}(\ell^2(\mathbf{N};U))$.

(f) If Σ *is exponentially stable, then* $\mathbb{X} \in \mathcal{G}\mathrm{tic}_r(U)$ *for some* $r < 1$, *and* $\mathbb{N}^*J\mathbb{N} = S$ *and* $\mathbb{D}^*J\mathbb{D} = \mathbb{X}^*S\mathbb{X}$, *i.e.,*

$$\widehat{\mathbb{D}}^*J\widehat{\mathbb{D}} = \widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} \quad on \;\; \partial\mathbf{D}. \tag{14.30}$$

Recall that $\vartheta = 1$ for $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$, $\mathcal{U}_{\mathrm{str}}$ and $\mathcal{U}_{\mathrm{exp}}$, and that $\ell^2(\mathbf{N};U) \subset \ell^2_r$ and $\ell^2_{r'}(\mathbf{N};U) \subset \ell^2$ for $r' \le 1 \le r$. See also Proposition 9.12.7(b).

Proof: (a1)&(a2)&(b1) These follow from Lemma 13.3.8(b3), since $\mathbb{K}_{\circlearrowleft}$ (due to $\vartheta = 1$), $\mathbb{C}_{\circlearrowleft}$ and $\mathbb{B}^{\mathrm{d}}$ are stable, except that we have to show that $\mathbb{X}^*\pi^{\pm}\mathbb{M}^*$ are stable:

$$\mathbb{M}\pi^+\mathbb{X} = \mathbb{M}\pi^+\mathbb{X}(\pi^+ + \pi^-) = I\pi^+ + \mathbb{M}\mathbb{K}\mathbb{B} = \pi^+ + \mathbb{K}_{\circlearrowleft}\mathbb{B}, \tag{14.31}$$

which is stable, hence the map $\mathbb{X}^*\pi^+\mathbb{M}^* \in \mathcal{B}(\ell^2, \ell^2_r)$ (note that $\pi_+\ell^\infty \underset{c}{\subset} \ell^2_r$) has its range in ℓ^2, hence $\mathbb{X}^*\pi^+\mathbb{M}^* \in \mathcal{B}(\ell^2)$, by Lemma A.3.6. We observe that $\mathbb{X}^*\pi^-\mathbb{M}^* = I - \mathbb{X}^*\pi^+\mathbb{M}^*$ is also stable.

(b2) This follows from (b1) (note that $\ell^2(\mathbf{N};U) \subset \ell^2_r$ and that $\mathbb{X}$ itself is not necessarily defined on $\ell^2(\mathbf{Z}_-;U)$).

(c1) By Lemma 9.10.1(c2), we have $\langle \mathbb{N}u, J\mathbb{N}v\rangle = \langle u, Sv\rangle$ for all $u, v \in \mathrm{c_c}$, hence for all $u, v \in \ell^1$, by density.

(c2) This is Proposition 9.12.7(b).

(d) By Theorem 3.3.1(e)&(a4), $\widehat{\mathbb{X}}, \widehat{\mathbb{M}} \in \mathrm{H}^2(\mathbf{D};\mathcal{B}(U)) \cap \mathrm{L}^2(\partial\mathbf{D};\mathcal{B}(U))$. By continuity, $\widehat{\mathbb{X}}\widehat{\mathbb{M}} = I = \widehat{\mathbb{M}}\widehat{\mathbb{X}}$ and $\widehat{\mathbb{N}} = \widehat{\mathbb{D}}\widehat{\mathbb{M}}$ a.e. on $\partial\mathbf{D}$ (since these hold on $\partial\mathbf{D}$); in particular, $\widehat{\mathbb{X}} \in \mathcal{GB}(U)$ a.e. on $\partial\mathbf{D}$.

Since $\langle u_0e_0, Sv_0e_0\rangle_{\ell^2} = \langle \mathbb{N}u_0e_0, J\mathbb{N}v_0e_0\rangle_{\ell^2}$, i.e., $\langle u_0, Sv_0\rangle_{\mathrm{L}^2(\partial\mathbf{D};U)} = \langle \widehat{\mathbb{N}}u_0, J\widehat{\mathbb{N}}v_0\rangle_{\mathrm{L}^2(\partial\mathbf{D};U)}$ for all $u_0, v_0 \in U$, we must have $\widehat{\mathbb{N}}^*J\widehat{\mathbb{N}} = S$ a.e. on $\partial\mathbf{D}$, i.e., $\widehat{\mathbb{X}}^*S\widehat{\mathbb{X}} = \widehat{\mathbb{D}}^*J\widehat{\mathbb{D}}$ a.e. on $\partial\mathbf{D}$.

(e) By Lemma 9.9.7(c5), $S \in \mathcal{GB}(U)$. Set $\mathbb{E} := \mathbb{D}^*J\mathbb{D}$. Let $u \in \ell^1(\mathbf{Z};U)$. Since $\mathbb{M}, \mathbb{X}^* \in \mathcal{B}(\ell^1, \ell^2)$, by Lemma 14.2.8, we obtain from (c2) that

$$\mathbb{E}\mathbb{M}u = \lim_{T,t\to+\infty} \mathbb{X}^*\pi_{[0,t)}S\mathbb{X}\mathbb{M}u = \lim_{T,t\to+\infty} \mathbb{X}^*\pi_{[0,t)}Su = \mathbb{X}^*Su, \tag{14.32}$$

hence $\mathbb{E}\mathbb{M} = \mathbb{X}^*S \in \mathcal{B}(\ell^1, \ell^2)$. Consequently, $\mathbb{M}^*\mathbb{E} = S\mathbb{X} \in \mathcal{B}(\ell^2, \ell^\infty)$, by Lemma B.4.15.

Let $r > 1$. Since $S^{-1}\mathbb{M}^*\mathbb{E} = \mathbb{X}$, we have $\pi_+ S^{-1}\mathbb{M}^*\mathbb{T} = \pi_+\mathbb{X}\pi_+ \in \mathcal{B}(\ell^2(\mathbf{N};U),\ell^\infty(\mathbf{N};U)) \cap \mathcal{B}(\ell^2_r(\mathbf{N};U))$. But $\pi_+\mathbb{M}\pi_+\mathbb{X}\pi_+ = \pi_+$ on ℓ^2_r, hence $\pi_+\mathbb{M}\pi_+ S^{-1}\mathbb{M}^*\mathbb{T} = \pi_+$ on ℓ^2_r. The invertibility of $\mathbb{T}^{-1}$ on $\ell^2(\mathbf{N};U) \subset \ell^2_r(\mathbf{N};U)$ implies that $\mathbb{M}\pi_+ S^{-1}\mathbb{M}^* u = \mathbb{T}^{-1}u$ for all $u \in \ell^2(\mathbf{N};U)$; in particular, $\mathbb{M}\pi_+ S^{-1}\mathbb{M}^* \in \mathcal{GB}(\ell^2(\mathbf{N};U))$.

Remarks: We have $\mathbb{M} = \mathbb{E}^{-1}\mathbb{X}^* S \in \mathcal{B}(\ell^1,\ell^2)$; in particular, $\mathbb{M}\pi_+ = \mathbb{T}^{-1}\mathbb{X}^* S \in \mathcal{B}(\ell^1,\ell^2)$.

If $\mathcal{U}_*^* \equiv \mathrm{L}^2$ (e.g., $\Sigma \in$ sos and $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$) then $\mathbb{T} \in \mathcal{GB}(L^2(\mathbf{R}_+;U))$ implies that the eIARE a solution, by Proposition 8.3.10 and Theorem 14.1.6.

(f) The exponential stability of A (i.e., of Σ) implies that of $\mathbb{D}$ and $\mathbb{X}$. By Theorem 9.9.1(d)&(g2), $\Sigma_\circlearrowleft$ is exponentially stable (hence so are $\mathbb{N}$ and $\mathbb{M} = \mathbb{X}^{-1}$) and $\mathbb{N}^* J\mathbb{N} = S$, hence $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$. This implies (14.30) (by, e.g., (the discrete-time version of) Theorem 3.1.3(a1)). □

In the finite-dimensional CARE theory, one always has "$S = D^*JD$", but for DAREs this is virtually never true. In Section 9.2, the equality "$S = D^*JD$" was extended to all WPLSs with a bounded B and beyond; for $\mathcal{U}_{\mathrm{exp}}$ an alternative condition was that $\mathbb{A}B, C_{\mathrm{w}}\mathbb{A}, C_{\mathrm{w}}\mathbb{A}B \in \mathrm{L}^1_{\mathrm{loc}}$ (see Theorem 9.2.18; see Remark 9.9.14(b) for further sufficient conditions). None of these holds for discrete-time systems:

Example 14.2.9 [$\boldsymbol{S \neq D^*JD}$] Let $A = 0$, $B = I$, $C = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, $J = I$ (so that $\widehat{\mathbb{D}}(z) = \left[\begin{smallmatrix} z \\ I \end{smallmatrix}\right]$ or $\mathbb{D} = \left[\begin{smallmatrix} \tau^{-1} \\ I \end{smallmatrix}\right] \in \mathrm{tic}_{\mathrm{exp}} \subset \ell^1 *$). Then the [e]DARE becomes

$$S = 1 + \mathcal{P},\ SK = 0,\ -\mathcal{P} + 1 = K^*SK, \tag{14.33}$$

with the unique solution $\mathcal{P} = I$, $S = 2$, $K = 0$, hence $\mathbb{X} = I = X$. Therefore, $D^*JD = I \neq 2I = S\ (= X^*SX)$. ◁

In particular, we have no equivalent for the B_{w}^*-CARE theory of Section 9.2. Moreover, we have no decent equivalent for the $\mathrm{MTIC}^{\mathrm{L}^1}$ (or $\mathrm{MTI}_{T\mathbf{Z}}$) theory; indeed, there does not seem to exist any useful discrete-time classes that would satisfy Hypothesis 8.4.8 (after (13.63)), as noted around Lemma 14.3.5. Note that even if we use $\heartsuit\widetilde{\mathcal{A}}$ for $\widetilde{\mathcal{A}} := \mathrm{MTIC}^{\mathrm{L}^1}$ or something similar, we only know that $\widehat{\mathbb{D}}(-1)^*J\widehat{\mathbb{D}}(-1) = S$ (since $\phi_{\mathrm{Cayley}}(+\infty) = -1$), which does not imply that $D^*JD = \widehat{\mathbb{D}}(0)^*J\widehat{\mathbb{D}}(0) = S$.

Notes for Sections 14.1–14.2

Historical remarks and references for DAREs for finite-dimensional systems can be found from, e.g., [LR]. We have recently become aware of the fact that also the eDARE for finite-dimensional systems has been studied extensively, by V. Popov, V. Ionescu, C. Oară and M. Weiss [Popov] [IW] [IOW] and others, under the name DTARS. See [IOW] also for symplectic pencils for finite-dimensional systems and [HI] for a Popov function approach to time-varying discrete-time linear systems.

Notes for infinite-dimensional positive DARE results can be found on p. 840. In the indefinite case, Jarmo Malinen [Mal97] (alternatively, see [Mal00]) has shown the equivalence of the existence of a spectral factorization and the existence of a P-I/O-stabilizing solution of the DARE (see Corollary 9.9.11 and Proposition 9.8.11 for a stronger result) for stable reachable systems; at the same time we published the corresponding continuous-time result in [Mik97b] (without reachability assumptions, for general regular WPLSs).

In Section 4.6 of [Mal00], Malinen shows that the partial ordering of the different P-stable solutions of the DARE by "$\le$" matches with the partial ordering of the range spaces of the corresponding "pseudospectral factors" by "$\subset$". He assumes that the system is stable and has a uniformly positive Popov operator (i.e., $\mathbb{D}^* J \mathbb{D} \gg 0$), the input operator B is a Hilbert–Schmidt operator, and U and Y are separable. Malinen also establishes a connection to invariant subspaces of A^* (Chapter 5, under essentially the same assumptions).

Lemma 14.2.4 is from [Mal00], and the proof of Theorem 14.1.4(a)–(c) follows the classical approach (see, e.g., Proposition 13.5.1 of [LR]).

14.3 Spectral and coprime factorizations

Molecule, *n.:*
The ultimate, indivisible unit of matter. It is distinguished from the corpuscle, also the ultimate, indivisible unit of matter, by a closer resemblance to the atom, also the ultimate, indivisible unit of matter ... The ion differs from the molecule, the corpuscle and the atom in that it is an ion ...
— Ambrose Bierce (1842–1914), "The Devil's Dictionary"

Spectral and coprime factorization have been treated in Sections 6.4 and 8.4 and in Chapter 5 (recall Theorem 13.3.13). In this section, we shall supplement those results by some discrete-time-specific additional results.

A *spectral factorization* of $\mathbb{E} \in \mathrm{ti}(U)$ is an equation of form $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$, where $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{tic}(U)$. Through the $\mathcal{Z}$-transform, this means writing some $\mathbb{E} \in \mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D}; \mathcal{B}(U))$ (i.e., some strongly bounded, strongly measurable operator-valued function on the unit circle) as $\widehat{\mathbb{Y}}^*\widehat{\mathbb{X}}$, where $\widehat{\mathbb{X}}, \widehat{\mathbb{Y}} \in \mathcal{G}\mathrm{H}^\infty(\mathbf{D}; \mathcal{B}(U))$ (i.e., $\widehat{\mathbb{Y}}$ and $\widehat{\mathbb{X}}$ are (the nontangential limits at the circle of) operator-valued bounded, boundedly invertible holomorphic functions on the unit disc). (If U is unseparable, then the adjoint $\widehat{\mathbb{Y}}^*$ cannot be taken pointwise for an arbitrary representative of the boundary function; see Definition 3.1.1 for details.)

Given a stable system $\left[\begin{smallmatrix} A & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ (or and I/O map $\mathbb{D}$) and a cost operator J corresponding to some optimal control problem (such as LQR or H^∞), a spectral factorization of the Popov operator $\mathbb{D}^* J \mathbb{D}$ leads to formulae for the optimal controller and cost, as shown in Corollary 8.3.11. Under certain assumptions, also unstable problems can be reduced to stable ones, hence spectral factorizations have become an important tool in solving control problems.

For finite-dimensional exponentially stable systems, the existence of a spectral factorization is equivalent to the invertibility of the corresponding Popov Toeplitz operator (aka. J-coercivity). In Theorems 14.3.2 and 14.3.4 we extend this fact to infinite-dimensional systems and weaken the exponential stability assumption to the requirement that the convolution kernel in ℓ^1 (i.e., absolutely summable). In Corollary 14.3.3, we extend the corresponding unstable result.

We start this section by listing the basic properties of spectral factorization. By using Theorem 13.2.3 (and reformulating the proof of (c)), we obtain the following from Lemma 5.2.1:

Lemma 14.3.1 (SpF) *Let* $\mathbb{E} \in \mathrm{ti}(U)$.

(a) Then $\mathbb{E} \gg 0$ *iff* $\mathbb{E}$ *has the spectral factorization* $\mathbb{E} = \mathbb{X}^*\mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{tic}(U)$.

If this is the case, then all spectral factorizations of form $\mathbb{E} = \mathbb{Z}^*\mathbb{Z}$ *are given by* $\mathbb{E} = (L\mathbb{X})^*(L\mathbb{X})$, *where* $L \in \mathcal{G}\mathcal{B}(U)$ *is unitary.*

Assume now that $\mathbb{E} \in \mathrm{ti}(U)$ *has a spectral factorization* $\mathbb{E} = \mathbb{Y}^*\mathbb{X}$ *for some* $\mathbb{X}, \mathbb{Y} \in \mathcal{G}\mathrm{tic}(U)$. *Then we have the following:*

(b) The Toeplitz operator $\pi^+\mathbb{E}\pi^+$ is invertible on $\pi^+\ell^2$, and $\pi^+\mathbb{X}^{-1}\pi^+\mathbb{Y}^{-*}\pi^+$ is its inverse.

(c) If $\mathbb{E} \in \mathrm{ti}_r \cap \mathrm{ti}_{1/r}$ for some $r < 1$, then $\mathbb{Y}, \mathbb{X} \in \mathcal{G}\mathrm{tic}_{1-\varepsilon}(U,Y) \cap \mathrm{tic}_r(U,Y)$ for some $\varepsilon > 0$; in particular, then $\mathbb{X}^{\pm 1}$ and $\mathbb{Y}^{\pm 1}$ are exponentially stable.

(d) If $\mathbb{E} = \mathbb{E}^*$, then $\mathbb{Y} = \mathbb{X}^* S$ for some $S = S^* \in \mathcal{GB}(U)$; thus, then $\mathbb{E} = \mathbb{X}^* S\mathbb{X}$.

If, in addition, $\mathbb{E} \in \mathrm{ti}_\omega(U)$ for some $\omega \neq 0$, then $\mathbb{X} \in \mathcal{G}\mathrm{tic}_{-\varepsilon}(U)$ for some $\varepsilon > 0$.

(e) The map $\mathbb{E}^{\mathrm{d}} := \text{Я}\mathbb{E}\text{Я} \in \mathrm{TI}(U)$ has the co-spectral factorization *$\mathbb{E}^{\mathrm{d}} = \mathbb{X}^{\mathrm{d}}(\mathbb{Y}^{\mathrm{d}})^*$ ($\mathbb{X}^{\mathrm{d}}, \mathbb{Y}^{\mathrm{d}} \in \mathcal{G}\mathrm{tic}(U)$).*

(f) All spectral factorizations of $\mathbb{E}$ are given by $\mathbb{E} = (L^{-*}\mathbb{Y})^*(L\mathbb{X})$, where $L \in \mathcal{GB}(U)$. □

For exponentially stable discrete-time I/O maps, we have the equivalence (without any $\widetilde{\mathcal{A}}$ assumptions) between the invertibility of the Popov Toeplitz operator and the existence of a spectral factorization:

Theorem 14.3.2 (tic$_{-\varepsilon}$: Popov⇔SpF) *Let $\mathbb{D} \in \mathrm{tic}_{\exp}(U,Y)$. Then $\pi_+\mathbb{D}^* J\mathbb{D}\pi_+$ is invertible iff $\mathbb{D}^* J\mathbb{D} = \mathbb{X}^* S\mathbb{X}$ for some $\mathbb{X} \in \mathcal{G}\mathrm{tic}_{\exp}(U)$ and $S \in \mathcal{GB}(U)$.*

This does not hold with tic in place of $\mathrm{tic}_{\exp}$, by Example 8.4.13 (the map $\mathbb{X}$ may become slightly unstable; see Lemma 14.2.8 for details)).

Proof: (We give a system-theoretic proof to obtain a constructive formula for $\mathbb{X}$ and S in terms of an arbitrary realization (or of a solution of the DARE). Combine Lemma 13.3.8(a2), Theorem 14.3.4 and Lemma 14.3.1(d) to obtain an alternative proof.)

1° *The equivalence:* The necessity ("if") follows from Lemma 14.3.1(b), so assume that $T := \pi_+\mathbb{D}^* J\mathbb{D}\pi_+$ is invertible in $\mathcal{B}(\pi_+\mathrm{L}^2)$. Then there is a unique critical control, by Proposition 8.3.10.

Let $\Sigma := \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ be an exponentially stable realization of A (see Definition 13.3.4). By Theorem 14.1.6, the eDARE for Σ and J has a unique PB-output-stabilizing solution $(\mathcal{P}, S, K)$, S is one-to-one, and Σ_{crit} is the left column of $\Sigma_{\circlearrowleft}$ corresponding to $\left(\begin{array}{c|c} K & 0 \end{array}\right)$. In particular K is exponentially stabilizing. Consequently, $\mathbb{X}$ and $\mathbb{X}^{-1}$ are exponentially stable, hence K is exponentially r.c.-stabilizing, by Lemma 13.3.16, and $\mathbb{X}^* S\mathbb{X}$ is a spectral factorization, by Theorem 9.9.10(a1).

2° *Remark:* If we use the shift realization from Definition 13.3.4, then the formula for $\widehat{\mathbb{X}}(z) = I - K(z^{-1} - A)^{-1}B$ becomes

$$\widehat{\mathbb{X}}(z)u_0 = I + K\sum_{k=0}^{\infty} z^{k+1}\pi_+\tau^{k+1}\mathbb{D}u_0 e_0 \tag{14.34}$$

(by (13.49)), where $K = ((\pi^+\mathbb{D}J\mathbb{D}\pi^+)^{-1}\pi^+\mathbb{D}^* J\mathbb{C})_0$, by (8.45) (and (13.39)).

(Of course, $K = -S^*(D^* JC + B^*\mathcal{P}A)$, where $S = D^* JD + B^*\mathcal{P}B$ and $\mathcal{P}$ is given by (8.46).) □

This leads to the following corollary in the unstable case:

Corollary 14.3.3 (*J*-coercive⇔inner r.c.f.) *Let $\mathbb{D}$ have an exponential [q.]r.c.f. Then $\mathbb{D}$ is J-coercive iff $\mathbb{D}$ has an exponential $(J,*)$-inner [q.]r.c.f.*

Proof: Necessity follows from Corollary 8.4.14(a). Conversely, if $\mathbb{D} = \mathbb{N}\mathbb{M}^{-1}$ is an exponential [q.]r.c.f. and $\mathbb{D}$ is J-coercive, then $\mathbb{N}$ is J-coercive, by Lemma 8.4.11(b1), hence $\pi_+\mathbb{N}^*J\mathbb{N}\pi_+$ is invertible, hence $\mathbb{N}^*J\mathbb{N}$ has an exponential spectral factorization $\mathbb{X}^*S\mathbb{X}$, by Theorem 14.3.2, hence $\mathbb{D}$ has the (exponentially stable) $(J,*)$-inner r.c.f. $(\mathbb{N}\mathbb{X}^{-1})(\mathbb{M}\mathbb{X}^{-1})^{-1}$, by the proof of Lemma 6.4.8(b). □

In discrete time, ℓ^1 takes the role of MTIC as the family of I/O maps that admit spectral factorization (though without the "$D^*JD = X^*SX$" property):

Theorem 14.3.4 (ℓ^1 admits SpF) *Let either $\mathcal{A} = \ell^1*$ and $\widetilde{\mathcal{A}} = \ell^1_+*$, or $\mathcal{A} = \ell^1_{\mathcal{B}C}*$ and $\widetilde{\mathcal{A}} = \ell^1_{\mathcal{B}C,+}*$.*

Let $\mathbb{E} \in \mathcal{A}(U)$. The Toeplitz operator $\pi^+\mathbb{E}\pi^+ \in \mathcal{B}(\pi^+\ell^2)$ is invertible iff $\mathbb{E}$ has a spectral factorization

$$\mathbb{E} = \mathbb{Y}^*\mathbb{X}, \quad \textit{where } \mathbb{X}, \mathbb{Y} \in \mathcal{G}\widetilde{\mathcal{A}}. \tag{14.35}$$

If, in addition, $r < 1$ and $\mathbb{E} \in \mathcal{A}_r \cap \mathcal{A}_{1/r}$, then $\mathbb{Y}, \mathbb{X} \in \mathcal{G}\widetilde{\mathcal{A}}_{1-\varepsilon}(U,Y) \cap \widetilde{\mathcal{A}}_r(U,Y)$ for some $\varepsilon > 0$.

In particular, if $\mathbb{E} = \mathbb{E}^* \in \ell^1$ is "exponentially ℓ^1", i.e., $\mathbb{E} \in \ell^1_r$ for some $r < 1$ too, then its (possible) spectral factors are "exponentially ℓ^1". The "in addition" claim also holds for $\mathcal{A} =$ ti and $\widetilde{\mathcal{A}} =$ tic (since $\text{ti}_{\text{exp}}(\mathbf{N};*) \subset \ell^1(\mathbf{N};*)$; use also Lemma 14.3.1(d)).

Proof: The factorization equivalence follows from Theorem 5.1.3. The proof of the additional claim is analogous to that of Proposition 5.2.2. □

Since Hypothesis 8.4.7 (converted as in (13.63)) often appears in our continuous-time results, we shall list three discrete-time classes satisfying this hypothesis (we also note that also the Cayley transforms of continuous-time classes satisfying the original hypothesis will do):

Lemma 14.3.5 (Hypothesis 8.4.7) *Let $\mathcal{A} = \ell^1_+$, $\mathcal{A} = \ell^1_{\mathcal{B}C,+}$ or $\mathcal{A} = \text{tic}_{\text{exp}}$. Then (unconverted) Hypothesis 8.4.7 holds for $\heartsuit^{-1}\mathcal{A}(U)$ as well as for the class $\heartsuit^{-1}\mathcal{A}_{\text{exp}}(U)$, where $\mathcal{A}_{\text{exp}}(U)$ is the class*

$$\mathcal{A}_{\text{exp}}(U) := \cap_{r<0}\mathcal{A}_r = \cap_{r>1}\{r^{-\cdot}\mathbb{D}r^{\cdot} \,\big|\, \mathbb{D} \in \mathcal{A}(U)\} \tag{14.36}$$

of exponentially stable $\mathcal{A}(U)$ maps.

Finally, both $\mathcal{A}(U)$ and $\mathcal{A}_{\text{exp}}(U)$ satisfy Hypothesis 8.4.7 (converted as in (13.63)); and so does $\heartsuit[\widetilde{\mathcal{A}}(U)]$ whenever $\widetilde{\mathcal{A}}(U)$ satisfies the unconverted Hypothesis 8.4.7.

(Note $\heartsuit^{-1}[\mathcal{A}_{\text{exp}}(U,Y)] \subset \text{TIC}_{\text{exp}}(U,Y)$.) The (first) claim on the unconverted hypothesis is not as important, it is needed only when one wants to use Theorem 13.2.3 to obtain discrete-time results from (stable) continuous-time results.

By taking $\mathbb{D} = \tau^{-1} \in \mathrm{tic}_{\exp}(U) \subset \ell^1(\mathbf{N};\mathcal{B}(U))*$, we obtain that $\mathbb{D}^* I \mathbb{D} = I = I^* II$ (i.e., $\mathbb{X} = I = X$, $S = I$) although $D = 0$; thus we observe that non of the above classes satisfies inverse Cayley transformed Hypothesis 8.4.8 (however, since in (13.63) we ignore any simplifications of form $S = D^* JD$, this is not a problem). See also the remarks around Example 14.2.9.

Recall from Remark 13.3.9 that $\mathcal{T}_r \ell^1 = r^{\cdot} \ell^1 r^{-\cdot} = \ell^1_r$.

We note (but do not need) the fact that the feedthrough operator of $\heartsuit^{-1}\mathbb{D} \in \heartsuit^{-1}[\ell^1] \subset \mathrm{UHPR} \subset \mathrm{TIC}$ is $\mathcal{L}(\heartsuit^{-1}\mathbb{D})(+\infty) = \widehat{\mathbb{D}}(-1) = \sum_{k=0}^{\infty}(-1)^k A_k \in \mathcal{B}$ if $\widehat{\mathbb{D}}(z) = \sum_{k=0}^{\infty} A_k z^k \in \ell^1$.

Proof of Lemma 14.3.5: Condition (2.) of Hypothesis 8.4.7 follows from Theorem 5.1.3 for $\mathcal{A}$, and from Theorem 14.3.2 for $\mathrm{tic}_{\exp}$. Conversion (13.63) makes condition (1.) trivially true. This proves the latter ("converted") claim (use Theorem 13.2.3 for $\heartsuit[\widetilde{\mathcal{A}}(U)]$).

To prove the former claim it remains to establish uniform half-plane-regularity of $\heartsuit^{-1}\mathbb{D}$ for any $\mathbb{D} \in \mathcal{A}(U,Y)$, by Theorem 13.2.3. But $\widehat{\mathbb{D}} \subset \mathcal{C}(\overline{\mathbf{D}};\mathcal{B}(U,Y))$, hence $\heartsuit\widehat{\mathbb{D}}$ is uniformly half-plane-regular, by Lemma 2.6.2, for any $\mathbb{D} \in \mathcal{A}(U,Y)$. $\square$

(See the notes on pp. 829, 141 and 148.)

Chapter 15

Quadratic Minimization

Large increases in cost with questionable increases in performance can be tolerated only in race horses and women.

— Lord Kalvin

This chapter is mostly the discrete-time counterpart of Chapter 10. In Section 15.1, we first note that the discrete-time form of Chapter 10 is true, and then we present some further discrete-time specific results on minimization and H^2 control problems; this corresponds to Sections 10.1, 10.2 and 10.4. Most results become as simple as for WPLSs with bounded input and output operators (the main exception is that $S \neq D^*JD$ in general).

Section 15.2 contains the discrete-time variant of Section 10.3, i.e., the relations between different classical coercivity assumptions.

In Section 15.3, we present discrete-time variants of Section 10.6, namely necessary and sufficient conditions for the Popov operator $\mathbb{D}^*J\mathbb{D}$ to be uniformly positive ($\geq \varepsilon I$ for some $\varepsilon > 0$). Section 15.4 is the discrete-time counterpart of Section 10.5, containing extended generalized forms of The Strict Bounded (Real) Lemma and The Strictly Positive (Real) Lemma. In Section 15.5, we show that any strongly stabilizing solution of a positive DARE (or of the corresponding Riccati inequality) is the maximal one.

Throughout this chapter, we assume that Standing Hypotheses 14.0.1 and 10.6.6 hold (the latter is satisfied by, e.g., $\widetilde{\mathcal{A}}_+ = \text{tic}$).

15.1 J-critical control and minimization

Everything takes longer, costs more, and is less useful.
— Erwin Tomash

We first state that the discrete-time versions of Chapter 10 hold, and then we give more elegant discrete-time-specific results. For example, there is a unique minimizing exponentially stabilizing control iff the eDARE has an exponentially stabilizing solution with positive signature operator, by Corollary 15.1.3. Under suitable coercivity assumptions, one need not check whether the solution of the eDARE is stabilizing; see, e.g., Corollary 15.1.6; the special case for an LQR is given in Corollary 15.1.7. The minimizing state feedback operator always solves also the H^2 state feedback control problem, as shown in Theorem 15.1.8. For finite-dimensional systems, all this this is essentially known, and a special case of Corollary 15.1.7 was treated in [CZ].

We start by verifying that the results of Chapter 10 hold in their discrete-time forms too:

Theorem 15.1.1 (Chapter 10 applies) *The results of Chapter 10 hold modulo the changes given in (13.63).*

Standing Hypothesis 10.6.6 is satisfied by, e.g., $\widetilde{\mathcal{A}}_+ = \mathrm{tic}$. *Hypothesis 10.6.1(1.)–(6.) are satisfied by all* $\mathbb{D} \in \mathrm{tic}$ *(when we interpret (13.63) so that the requirement "*$(S=)X^*X = D^*JD$*" is removed).*

Recall that CAREs and B_{w}^*-CAREs must be replaced by DAREs; e.g., the LQR-CARE and the LQR-B_{w}^*-CARE become the "LQR-DARE" (15.5). Naturally, this also applies to simplified forms of CAREs (i.e., to CAREs under simplifying assumptions); e.g., (10.3) becomes (15.6). Analogously, equations such as $S = D^*JD$ or $S = R + D^*QD$ must be replaced by $S = D^*JD + B^*\mathcal{P}B$ or by $S = R + D^*QD + B^*\mathcal{P}B$, respectively.

To make things clear, we have rewritten most main results of Chapter 10 into this chapter; due to bounded input and output operators, most of these results are stronger and more elegant than their continuous-time counterparts. The main differences are that a unique minimizing (or J-critical) control is always of state feedback form (as in continuous time in the case of bounded B), and that we need no regularity considerations (nor corresponding assumptions).

This section contains most main results of Sections 10.1 and 10.2, starting from the most general ones. See Section 15.2 for the discrete-time form of Section 10.3, Theorem 15.1.8 for the H^2 problem (Section 10.4), Section 15.3 for Section 10.6, and Section 15.4 for Section 10.5.

Proof of Theorem 15.1.1: The same proofs apply mutatis mutandis. For Hypothesis 10.6.1(1.)–(6.), recall that (13.63) removes any regularity requirements; see the proof of Lemma 10.6.2(c)(8.) for "(6.)" (and recall that (13.63) replaces the B_{w}^*-CARE by the DARE). □

By combining Theorem 14.1.6 and Theorem 9.9.1(a2), we obtain:

Theorem 15.1.2 (Unique minimum) *The following are equivalent:*

(i) there is a unique minimizing control for each $x_0 \in H$*;*

(ii) $\mathcal{J}(0,\cdot) \geq 0$*, and the eDARE has a [unique]* $\mathcal{U}_*^*$*-stabilizing solution with* $S > 0$*;*

(iii) there is a unique minimizing state feedback operator for Σ*.*

If $(\mathcal{P}, S, K)$ *is as in (ii), then the minimizing state feedback is given by* $u_{\min}(t) = Kx(t)$*, the minimal cost is* $\mathcal{J}(x_0, u_{\min}) = \langle x_0, \mathcal{P}x_0 \rangle$*, and the cost for closed-loop input* $u_{\circlearrowleft} \in \mathrm{c_c}(\mathbf{N}; U)$ *is given by*

$$\langle x_0, \mathcal{P}x_0 \rangle_H + \langle u_{\circlearrowleft}, S u_{\circlearrowleft} \rangle_{\ell^2(\mathbf{N};U)}. \tag{15.1}$$

□

(The condition $\mathcal{J}(0,\cdot) \geq 0$ is redundant at least for $\mathcal{U}_*^* = \mathcal{U}_{\exp}$, by Theorem 9.9.1(k). See Theorem 9.9.1 for more on $(\mathcal{P}, S, K)$.)

A solution with $S \geq 0$ would still be minimizing (and $\mathcal{P}$, S and K might all be unique) but the minimizing control would no longer be unique; see Theorem 9.9.1(f2) for details.

Combine Theorem 8.4.2 and Theorem 9.9.1(k) to the above theorem to obtain

Corollary 15.1.3 *There is a unique minimizing control over* $\mathcal{U}_{\exp}$ *for each* $x_0 \in H$ *iff the eDARE has an exponentially stabilizing solution with* $S > 0$*.* □

If $\mathbb{D}$ is positively J-coercive, then the existence of any allowable control implies the existence of a unique minimizing control:

Corollary 15.1.4 (Coercive minimization) *Assume that there is* $\varepsilon > 0$ *s.t.*

$$\mathcal{J}(0,u) \geq \varepsilon \|u\|_{\mathcal{U}_*^*}^2 \qquad (u \in \mathcal{U}_*^*(0)). \tag{15.2}$$

Assume that Z^{s} *is reflexive (e.g., that* $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$ *or* $\mathcal{U}_*^* = \mathcal{U}_{\exp}$*). Then the following are equivalent:*

(i) there is a unique minimizing control over $\mathcal{U}_*^*(x_0)$ *for each* $x_0 \in H$*;*

(ii) $\mathcal{U}_*^*(x_0) \neq 0$ *for each* $x_0 \in H$*;*

(iii) the DARE has a $\mathcal{U}_*^*$*-stabilizing solution* $(\mathcal{P}, S, K)$*.*

If (iii) holds, then $S \gg 0$*,* K *is the unique* J*-critical state feedback operator, and the minimal cost is given by* $\mathcal{J}(x_0, u_{\min}(x_0)) = \langle x_0, \mathcal{P}x_0 \rangle$ $(x_0 \in H)$*.* □

See Lemma 8.4.2 for most common $\|\cdot\|_{\mathcal{U}_*^*}$'s. See Proposition 10.3.1 for equivalent conditions for (15.2) in case $\mathcal{U}_*^* = \mathcal{U}_{\mathrm{out}}$.

Proof: By Lemma 8.2.3(c2), D is positively J-coercive, hence Theorems 8.2.5 and 8.1.10 apply. By Corollary 8.1.8, the unique J-critical control is strictly minimizing (on $\mathcal{U}(x)$). □

A special case of this is the standard LQR problem; even in somewhat more general LQR setting, the existence of a solution becomes equivalent to positive J-coercivity:

Corollary 15.1.5 (Standard minimization) *Assume that $J \geq 0$ and $D^*JD \gg 0$. Then the following are equivalent:*

(i) There is a minimizing $u \in \mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$.

(ii) The DARE has a $\mathcal{U}_{\text{exp}}$-stabilizing solution $(\mathcal{P}, S, K)$.

(iii) Σ is exponentially stabilizable, and $\mathbb{D}$ is J-coercive (i.e., any (hence all) of (i)–(iii) of Proposition 15.2.2 holds).

If (ii) holds, then $\mathcal{P} \geq 0$, $S \gg 0$, K is the unique J-critical state feedback operator, and the minimal cost is given by $\mathcal{J}(x_0, u_{\min}(x_0)) = \langle x_0, \mathcal{P}x_0 \rangle$ $(x_0 \in H)$.

If we remove the assumptions $J \geq 0$ and $D^*JD \gg 0$, then we must assume that u is unique in (i), and $S := D^*JD + B^*\mathcal{P}B \gg 0$ in (i) and (ii) above; then also (i') and (ii') of Proposition 15.2.2 become merely sufficient in (iii). □

(This follows from Proposition 15.2.2(e)&(f1) and Theorem 14.2.7.)

See Proposition 15.2.2 and the comments following it for additional equivalent conditions. However, even for $\mathcal{U}_*^* = \mathcal{U}_{\text{exp}}$, there may be a minimizing control even without J-coercivity, see Example 9.13.3 (which can easily be modified to a discrete-time example).

For the standard LQR cost function $\mathcal{J}(x_0, u) := \|y_1\|_2^2 + \|u\|_2^2$ (i.e., $\mathbb{C} = \left[\begin{smallmatrix} \mathbb{C}_1 \\ 0 \end{smallmatrix}\right]$, $\mathbb{D} = \left[\begin{smallmatrix} \mathbb{D}_1 \\ I \end{smallmatrix}\right]$, $J = I$) and other similar ones, we have the following:

Corollary 15.1.6 (Coercive minimization: $\mathcal{P}_+$ and $\mathcal{P}_-$) *Assume that $J \gg 0$ and $D^*JD \gg 0$, and that there is $\varepsilon > 0$ s.t.*

$$\mathcal{J}(x_0, u) := \langle \mathbb{C}x_0 + \mathbb{D}u, J(\mathbb{C}x_0 + \mathbb{D}u) \rangle_{\ell^2(\mathbf{N};Y)} \geq \varepsilon \|u\|_2^2 \quad (x_0 \in H,\ u : \mathbf{N} \to U). \tag{15.3}$$

Then the following are equivalent:

(i) the eDARE has a nonnegative solution;

(ii) **(FCC)** *$\mathcal{U}_{\text{out}}(x_0) \neq \emptyset$ (i.e., $\inf_u \mathcal{J}(x_0, u) < \infty$) for all $x_0 \in H$,*

(iii) **($\mathcal{U}_{\text{out}}$-min)** *the eDARE has a smallest nonnegative solution $(\mathcal{P}_-, S_-, K_-)$; $S_- \gg 0$; and K_- is strictly minimizing over all $u : \mathbf{N} \to \infty$ (hence over $\mathcal{U}_{\text{out}}$).*

Assume (iii). Then

(a) **($\mathcal{U}_{\text{exp}}$-min)** *The following are equivalent:*

(i') there is a minimizing control over $\mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$;

(ii') the [e]DARE has an exponentially stabilizing solution.

If (i')–(ii') hold, then eDARE has (a) greatest solution $(\mathcal{P}_+, S_+, K_+)$, $S_+ \geq S_- \gg 0$, and K_+ is strictly minimizing over $\mathcal{U}_{\text{exp}}$.

(b) If $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is strongly stable (resp. Σ is exponentially q.r.c.-stabilizable or exponentially detectable; e.g., $C^*C \gg 0$), then $\mathcal{P}_-$ is the unique nonnegative solution of the eDARE, and it is strongly stabilizing (resp. exponentially q.r.c.-stabilizing).

Thus, then this solution is strictly minimizing over $\{u : \mathbf{N} \to U\}$, $\mathcal{U}_{\text{out}}$, $\mathcal{U}_{\text{sta}}$ and $\mathcal{U}_{\text{str}}$ (resp. and $\mathcal{U}_{\text{exp}}$).

Thus, if $C^*C \gg 0$, then, by (b), any nonnegative solution is the unique nonnegative solution, minimizing over all $u : \mathbf{N} \to U$ and exponentially stabilizing.

By Theorem 14.1.4(e)&(a), $\mathcal{P}_-$ is the unique $\mathcal{U}_{\text{out}}$-stabilizing solution of the eDARE (if (iii) holds), and $\mathcal{P}_+$ is the unique stabilizing solution of the eDARE (if (ii') holds).

See Theorem 9.2.10 (or Section 10.1) for a continuous-time counterpart.

Proof: 1° *(iii)⇒(i)⇒(ii):* Obviously, (iii) implies (i). Since any nonnegative solution of the eDARE is SOS-stabilizing, by Proposition 10.7.3(d), we have "(i)⇒(ii)".

2° *(ii)⇒(iii):* Assume (ii). Since $\mathbb{D}$ is J-coercive over $\mathcal{U}_{\text{out}}$, there is a unique minimizing control over $\mathcal{U}_{\text{out}}(x_0)$ for each $x_0 \in H$, by Theorem 8.4.3, and this control corresponds to the smallest nonnegative solution $(\mathcal{P}_-, S_-, K_-)$ of the eDARE, as shown in (the proof of) Theorem 9.9.1(a2). Since $S_- \geq D^*JD \gg 0$, condition (iii) holds.

(a) If $(\mathcal{P}_+, S_+, K_+)$ is an exponentially stabilizing solution of the eDARE (as in (ii')), then K_+ is minimizing over $\mathcal{U}_{\text{exp}}$, by Theorem 9.8.5 and Theorem 9.9.1(a2), $S_+ \geq S_- \gg 0$ and $\mathcal{P}_+$ is the greatest solution of the eDARI, by Theorem 15.5.2.

To complete to proof of (a), we assume (i') and prove (ii'). The minimizing control for $x_0 = 0$ is obviously unique, namely $u = 0$. Therefore, the minimizing control over $\mathcal{U}_{\text{exp}}(x_0)$ is unique for any $x_0 \in H$, by Lemma 8.3.8. Consequently, there is a $\mathcal{U}_{\text{exp}}$-stabilizing solution of the eDARE, by Theorem 14.1.6.

(b) 1° Assume that $\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \end{array}\right]$ is strongly stable. Let $\mathcal{P} \geq 0$ be a solution of the eDARE with closed-loop system $\Sigma_{\circlearrowleft}$. Then $\mathcal{P}$ is SOS-stabilizing, by Proposition 10.7.3(d), so that $\mathbb{A}_{\circlearrowleft} = \mathbb{A} + \mathbb{B}\mathbb{K}_{\circlearrowleft}$ and $\mathbb{B}_{\circlearrowleft} = \mathbb{B}\mathbb{M}$ are strongly stable, by Theorem 6.7.15(d); thus, $\mathcal{P}$ is strongly stabilizing. Since the strongly stabilizing solution is unique, by Theorem 14.1.4(a), $\mathcal{P}$ must be equal to $\mathcal{P}_-$ (and $\mathcal{P}_+$).

2° Assume that Σ is exponentially q.r.c.-stabilizable or exponentially detectable. Since any nonnegative solution is SOS-stabilizing, it is exponentially q.r.c.-stabilizing, by (b1) or (c1) of Theorem 6.7.15, hence unique, hence equal to $\mathcal{P}_-$ (and $\mathcal{P}_+$).

3° Since $\mathcal{P}_-$ is $\mathcal{U}_{\text{out}}$-stabilizing, it satisfies (PB) over $\mathcal{U}_{\text{out}}$, hence over the smaller classes $\mathcal{U}_{\text{sta}}$, $\mathcal{U}_{\text{str}}$ and $\mathcal{U}_{\text{exp}}$ too. Since $\mathcal{P}_-$ is strongly (resp. exponentially) stabilizing, we have $\mathbb{K}_{\circlearrowleft}x_0 \in \mathcal{U}_{\text{str}}(x_0) \subset \mathcal{U}_{\text{sta}}(x_0) \subset \mathcal{U}_{\text{out}}(x_0)$ (resp. and $\mathbb{K}_{\circlearrowleft}x_0 \in \mathcal{U}_{\text{exp}}(x_0)$) for all $x_0 \in H$ for the corresponding $\mathbb{K}_{\circlearrowleft}$, so that $\mathcal{P}_-$ is also $\mathcal{U}_{\text{sta}}$-stabilizing and $\mathcal{U}_{\text{str}}$-stabilizing (resp. and $\mathcal{U}_{\text{exp}}$-stabilizing). □

By substitutions $J := \left[\begin{smallmatrix} Q & 0 \\ 0 & R \end{smallmatrix}\right]$, $\mathbb{C} \mapsto \left[\begin{smallmatrix} \mathbb{C} \\ 0 \end{smallmatrix}\right]$, $\mathbb{D} \mapsto \left[\begin{smallmatrix} \mathbb{D} \\ I \end{smallmatrix}\right]$, we obtain the cost function

$$\mathcal{J}(x_0, u) := \sum_{k=0}^{\infty} (\langle y_k, Qy_k \rangle_Y + \langle u_k, Ru_k \rangle_U) \tag{15.4}$$

and hence the following corollary (of Corollary 15.1.6):

Corollary 15.1.7 (LQR: $\sum_{j=0}^{\infty}(\|y_j\|_Y^2 + \|u_j\|_U^2)$) *Let $R, Q \gg 0$. Then the following are equivalent:*

(i) there is a $\langle y, Qy\rangle_{\ell^2} + \langle u, Ru\rangle_{\ell^2}$-minimizing control over all $u : \mathbf{N} \to U$ for each $x_0 \in H$;

(ii) for each $x_0 \in H$ there is $u \in \ell^2(\mathbf{N};U)$ s.t. $y \in \ell^2$;

(iii) the DARE

$$\begin{cases} \mathcal{P} = A^*\mathcal{P}A + C^*QC - K^*SK, \\ S = R + D^*QD + B^*\mathcal{P}B, \\ K = -S^{-1}(D^*QC + B^*\mathcal{P}A), \end{cases} \tag{15.5}$$

has a nonnegative solution $\mathcal{P}$.

If (iii) holds, then the smallest nonnegative solution is minimizing over all $u : \mathbf{N} \to U$.

There is a minimizing control over $\mathcal{U}_{\mathrm{exp}}$ iff the DARE has an exponentially stabilizing solution $\mathcal{P}_+$; such a solution is strictly minimizing over $\mathcal{U}_{\mathrm{exp}}$ and the greatest nonnegative solution of the DARE.

If Σ is exponentially detectable (e.g., $C^*C \gg 0$), then the DARE has at most one nonnegative solution, and such a solution is necessarily strictly minimizing over $\mathcal{U}_{\mathrm{exp}}$. □

(This follows from Corollary 15.1.6. Above, as elsewhere, $x_{n+1} := Ax_n + Bu_n$, $y_n := Cx_n + Du_n$, and $\mathcal{U}_{\mathrm{exp}}(x_0) = \{u \in \ell^2(\mathbf{N};U) \,\big|\, x \in \ell^2\}$. Note that the cost is finite for $u \in \mathcal{U}_{\mathrm{out}}(x_0)$ only.)

Thus, in the LQR problem, we only have to find a maximal or minimal solution, depending whether we wish to require the state to be stable or not. Then we shall check whether this condition satisfies (PB) for $\mathcal{U}_{\mathrm{out}}$ or whether it is exponentially stabilizing; see Theorem 10.1.4(b1)&(b2).

If $D = 0$, then (15.5) reduces to

$$\mathcal{P} = A^*\mathcal{P}A + C^*QC - A^*\mathcal{P}B(R + B^*\mathcal{P}B)^{-1}B^*\mathcal{P}A. \tag{15.6}$$

and the minimizing state feedback reduces to $u = -(R + B^*\mathcal{P}B)^{-1}B^*\mathcal{P}Ax$,

The minimizing state feedback operator always solves also the H^2 state feedback and full-information control problems:

Theorem 15.1.8 ($\mathbf{H^2}$ problem) *Assume that there is a minimizing state feedback operator K over $\mathcal{U}_*^*$. Let $B_2 \in \mathcal{B}(W,H)$,*

Then K solves the H^2 problem (strictly if K is strictly minimizing), i.e., it leads to the minimization of the cost $\|\mathbb{D}u + \mathbb{C}B_2 w_0\|_{\ell^2(\mathbf{N};Y)}$ $(= (2\pi)^{-1/2}\|\widehat{\mathbb{D}}\widehat{u} + C(I - zA)^{-1}B_2\|_{\mathrm{H}^2(\mathbf{D};Y)})$ (where $u_k := Kx_k$ for all $k \in \mathbf{N}$), for each $w_0 \in W$; see Figure 10.2. □

(This is Theorem 10.4.2 in its discrete-time form.) Note from Theorem 15.1.1 that also the rest of Section 10.4 holds in its discrete-time form.

Notes

Finite-dimensional minimization problems have been studied extensively, see e.g., [LR] for rather up-to-date results and historical remarks. See also the notes to the sections of Chapter 10.

A classical article on the stable infinite-dimensional discrete-time minimization problems is [Helton76b], where William Helton uses a spectral factorization approach with the factors allowed to be noninvertible, so that also some singular cost functions (with $D^*JC = 0$) are covered, at the cost of less direct results. Under certain assumptions on reachability and cost, Helton shows that a certain positive "eDARE" has a solution iff the Popov operator $\mathbb{D}^*J\mathbb{D}$ can be written as $\mathbb{X}^*\mathbb{X}$ for some $\mathbb{X} \in$ tic.

A standard LQR result can be found from Exercise 6.34 of [CZ] (close to the LQR case of Corollary 15.1.4).

The finite-dimensional discrete-time H^2-problem is explained and treated on pp. 271–274 of [IOW].

15.2 Standard assumptions in discrete time

The explanation requiring the fewest assumptions is the most likely to be correct.

— William of Occam, (c. 1285 – c. 1349)

This is the discrete-time equivalent of Section 10.3, with essentially the same results.

Positive J-coercivity is the standard coercivity assumption in minimization problems (e.g., LQR and H^2), and it is also posed in H^∞ problems on a part of the system. In this section, we present several equivalent (or sufficient or necessary) conditions for positive J-coercivity over $\mathcal{U}_{\mathrm{out}}$ or $\mathcal{U}_{\mathrm{exp}}$. The former means that the output is coercive w.r.t. to the input ($\|y\|_{\ell^2} \ge \varepsilon\|u\|_{\ell^2}$ for some $\varepsilon > 0$ whenever $u, y \in \ell^2$ (and $x_0 = 0$); see Proposition 10.3.1), and the latter means that the output is coercive w.r.t. to both the input and the state ($\|y\|_{\ell^2} \ge \varepsilon\left(\|u\|_{\ell^2} + \|x\|_{\ell^2}\right)$ for some $\varepsilon > 0$ whenever $u, x, y \in \ell^2$ (and $x_0 = 0$); see Proposition 15.2.2).

We start with the simpler one, namely $\mathcal{U}_{\mathrm{out}}$:

Proposition 15.2.1 ($\boldsymbol{\mathcal{U}_{\mathrm{out}}}$: $\boldsymbol{y \in \ell^2 \Rightarrow u \in \ell^2}$) *Proposition 10.3.1 holds also in its discrete-time form. However, Proposition 10.3.2 becomes Proposition 15.2.2, since now S takes the role of D^*JD.*

Proof: (We only need to prove Proposition 10.3.1, since Proposition 15.2.2 is proved below.)

(a)&(c)&(d) The original proofs apply (mutatis mutandis).

(b) Assume that $\dim U \times H \times Y < \infty$ and that $\Sigma = \left[\begin{smallmatrix} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}(U, H, Y)$. Since now $\widehat{\mathbb{D}}$ is rational, so is $\widehat{\mathbb{F}} := \heartsuit^{-1}\widehat{\mathbb{D}}$ (by Proposition 13.2.5(b3); use Lemma 13.2.6 to guarantee that $\|\widehat{\mathbb{F}}(\infty)\| < \infty$ if -1 is among the poles of $\widehat{\mathbb{D}}$). Consequently, $\widehat{\mathbb{F}}$ has a finite-dimensional realization (see, e.g., Section 6.4 of [LR] or Theorem 1.13.2 of [IOW]), and we can apply Proposition 10.3.1(b) to this realization to obtain the same claims (in their discrete-time forms) for Σ. $\square$

The case for $\mathcal{U}_{\mathrm{exp}}$ is more tricky, though yet essentially simpler than that in the continuous time:

Proposition 15.2.2 ($\boldsymbol{\mathcal{U}_{\mathrm{exp}}}$: $\boldsymbol{y \in \ell^2 \Rightarrow u, x \in \ell^2}$) *Let $\Sigma := \left[\begin{smallmatrix} A & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right] \in \mathrm{wpls}(U, H, Y)$. Let $J = J^* \in \mathcal{B}(Y)$, and set*

$$\kappa(x_0, u_0) := \langle (Cx_0 + Du_0), J(Cx_0 + Du_0) \rangle, \qquad \mathcal{J}(0, u) := \langle \mathbb{D}u, J\mathbb{D}u \rangle. \tag{15.7}$$

We have the following implications between the conditions (i)–(iii') given below:

(a) Each of conditions (i)–(iii') is invariant under admissible state feedback (in the sense that if $\Sigma_\flat$ is the corresponding closed-loop system, then $\left[\begin{smallmatrix} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \end{smallmatrix}\right]$ satisfies (i) (resp. (i'), ...) iff $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$ satisfies (i) (resp. (i'), ...)).

(b) If Σ is estimatable, then $\mathcal{U}_{\text{out}} = \mathcal{U}_{\text{exp}}$; hence then (i) becomes equivalent to (ai) of Proposition 10.3.1.

(c) (i')⇒(i)⇔(i")⇐(ii)⇐(iii)⇒(iii'); and (ii')⇒(ii) (without further assumptions).

(d) **(dim$<\infty$)** *Assume that $\dim U \times H \times Y < \infty$. Then (iii)⇔(iii')⇐(vii).*

Assume, in addition, that Σ is exponentially stabilizable. Then (i), (i"), (ii), (iii), (iii'), (v) and (vi) are equivalent to each other (and to (ai) and (bii)–(biv) of Proposition 10.3.1 if Σ is exponentially detectable). Moreover, in (ii), (ii'), (iii) and (iv), we may replace "$z \in \partial\mathbf{D}$" by "$z \in E$", where $E \subset \partial\mathbf{D}$ is dense.

(e) Assume Σ is exponentially stabilizable, $D^*JD \gg 0$ and $J \geq 0$.

Then (i)–(iii), (v) and (vi) are equivalent, and the word "unique" is redundant in (v).

(f1) If Σ is exponentially stabilizable, then (i')⇔(ii'), and (i)⇔(i")⇔(ii)⇔(iii)⇔(vi)⇒(iii')&(v).

(f2) If Σ is exponentially stabilizable and $\dim U < \infty$, then (i)⇔(v) (see also (f1)).

(i) $\mathcal{J}(0,u) \geq \varepsilon\left(\|u\|_2^2 + \|\mathbb{B}\tau u\|_2^2\right)$ for some $\varepsilon > 0$ and all $u \in \mathcal{U}_{\text{exp}}(0)$;
i.e., $\mathbb{D}$ is positively J-coercive over $\mathcal{U}_{\text{exp}}$.

(i') $\mathcal{J}(0,u) \geq \varepsilon\|\mathbb{B}\tau u\|_2^2$ for some $\varepsilon > 0$ and all $u \in \mathcal{U}_{\text{exp}}(0)$, and $D^*JD \gg 0$.

(i") $\mathcal{J}(0,u) \geq \varepsilon\left(\|u\|_2^2 + \|\mathbb{B}\tau u\|_2^2 + \|\mathbb{D}u\|_2^2\right)$ for some $\varepsilon > 0$ and all $u \in \mathcal{U}_{\text{exp}}(0)$.

(ii) There is $\varepsilon > 0$ s.t.

$$(z-A)x_0 = Bu_0 \implies \kappa(x_0,u_0) \geq \varepsilon(\|x_0\|_H^2 + \|u_0\|_U^2) \quad (x_0 \in H,\ u_0 \in U,\ z \in \partial\mathbf{D}). \tag{15.8}$$

(ii') $D^*JD \gg 0$ and there is $\varepsilon > 0$ s.t.

$$(z-A)x_0 = Bu_0 \implies \kappa(x_0,u_0) \geq \varepsilon\|x_0\|_H^2 \quad (x_0 \in H,\ u_0 \in U,\ z \in \partial\mathbf{D}). \tag{15.9}$$

(iii) There is $\varepsilon > 0$ s.t. $T_z^* \left[\begin{smallmatrix} I & 0 \\ 0 & J \end{smallmatrix}\right] T_z \geq \varepsilon I$ $(z \in \partial\mathbf{D})$ on $H \times U$, where $T_z := \left[\begin{smallmatrix} A-z & B \\ C & D \end{smallmatrix}\right]$.

(iii') $T_z^* \left[\begin{smallmatrix} I & 0 \\ 0 & J \end{smallmatrix}\right] T_z > 0$ $(z \in \partial\mathbf{D})$. Equivalently,

$$z \in \partial\mathbf{D} \ \&\ \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \neq \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in H \times U \ \&\ (z-A)x_0 = Bu_0 \implies \kappa(x_0,u_0) > 0. \tag{15.10}$$

(iv) There is $\varepsilon > 0$ s.t. $\langle u_0, \widehat{\mathbb{D}}(z)^* J \widehat{\mathbb{D}}(z) u_0\rangle \geq \varepsilon\left(\|u_0\|_U^2 + \|(z^{-1}-A)^{-1}Bu_0\|_H^2\right)$ for a.e. $z \in \partial\mathbf{D}$.

(v) There is a unique minimizing $u \in \mathcal{U}_{\text{exp}}(x_0)$ for each $x_0 \in H$.

(vi) The DARE has an exponentially stabilizing solution with $S \gg 0$.

(vii) (C,A) has no unobservable nodes on $\partial\mathbf{D}$, $J = I$, $D^*D > 0$ and $D^*C = 0$.

By Example 15.2.3, $D^*JD \gg 0$ (hence neither (i') and (ii')) is not implied by any of (i)–(vi) except (trivially) by (i') and (ii'). (Intuitively, this follows from the fact that $(z^{-1}-A)^{-1}B = z$ is bounded away from zero on $\partial\mathbf{D}$ in the example; the same cannot happen for $(\cdot - A)^{-1}B$ on $i\mathbf{R}$ (for WPLSs).)

Example 15.2.3 Let $\Sigma := \left(\frac{A \mid B}{C \mid D}\right) := \left(\frac{0 \mid 1}{1 \mid 0}\right)$. Then $\Sigma \in \mathrm{wpls}(\mathbf{C},\mathbf{C},\mathbf{C})$ is exponentially stable and $\mathbb{D} = \tau^{-1}$ (note that $y_n = x_n = u_{n-1}$ for all n).

Moreover, Σ and $J := 1$ do not satisfy (i') nor (ii') (since $D^*JD = 0$) but do satisfy the other assumptions of Proposition 15.2.2. $\triangleleft$

Thus, the "correct" assumption in (i') and (ii') would be "$S \gg 0$", not "$D^*JD \gg 0$", but the former cannot be formulated without $\mathcal{P}$, hence we prefer having sufficient conditions instead of necessary and sufficient conditions.

This important difference makes several DARE results either "weaker" or more complicated than their CARE counterparts. To make things simpler, we shall often assume that $D^*JD \gg 0$ & $J \geq 0$ even when this condition is not necessary.

If $J \gg 0$ (equivalently, $J = I$), we get the following equivalent forms of above conditions:
(i) $\|\mathbb{D}u\|_2 \geq \varepsilon(\|\mathbb{B}\tau u\|_2 + \|u\|_2)$;
(ii) $(z-A)x_0 = Bu_0 \Rightarrow \|Cx_0 + Du_0\|_Y \geq \varepsilon(\|x_0\|_H + \|u_0\|_U)$;
(ii') $D^*D \gg 0$, and $(z-A)x_0 = Bu_0 \Rightarrow \|Cx_0 + Du_0\|_Y \geq \varepsilon\|x_0\|_H$;
(iii') $T_z := \begin{bmatrix} A-z & B \\ C & D \end{bmatrix}$ has a full column rank (i.e., $T_z^* T_z \geq \varepsilon I$) on $H \times U$ for all $z \in \partial\mathbf{D}$;
(iv) $\|\widehat{\mathbb{D}}(z)u_0\| \geq \varepsilon(\|u_0\| + \|(z^{-1}-A)^{-1}Bu_0\|$ for a.e. $z \in \partial\mathbf{D}$ and all $u_0 \in U$
(for some $\varepsilon > 0$).

Proof of Proposition 15.2.2: The original proofs apply mutatis mutandis (this requires slightly more than (13.63)).

In fact, the proofs become much easier in most cases, since we do not have to worry about regularity; use Lemma 13.3.19 in place of Lemma 6.3.20.

(Note that we would have to replace z by z^{-1} outside (iv) and (iv') to have all z's correspond to each other.)

We have combined the modified versions of (g1) and (g2) with (c).

The first condition in (vii) means that $\mathrm{Ker}(\begin{bmatrix} z-A \\ C \end{bmatrix}) = \{0\}$ for all $z \in \partial\mathbf{D}$.

In 2.1° of the proof of (f), use the Cayley transform of the function $\widehat{f}$ provided by Lemma D.1.24.

The only "new" claim is the redundancy of uniqueness in (e). However, when $J \geq 0$ and $D^*JD \gg 0$, then

$$\langle y, Jy\rangle_{\mathrm{L}^2} = \langle Cx + Du, J(Cx + Du)\rangle_{\mathrm{L}^2} \geq \langle u, D^*JDu\rangle \geq \varepsilon\|u\|_2^2 \quad (u \in \ell^2(\mathbf{N};U)) \tag{15.11}$$

for some $\varepsilon > 0$, where $x := \mathbb{B}\tau u$. Consequently, then $\mathcal{J}(0,\cdot)$ has a unique minimum at $u = 0$. It follows Lemma 8.3.8 that the word "unique" is redundant in (v). $\square$

(See the notes on p. 583.)

15.3 Positive DAREs

I have yet to see any problem, however complicated, which, when you looked at it in the right way, did not become still more complicated.
— Poul Anderson

Now we present discrete-time equivalents of the main results of Section 10.6, namely necessary and sufficient conditions, in terms of Riccati equations and inequalities, for the Popov operator $\mathbb{D}^* J \mathbb{D}$ to be (stable and) uniformly positive ($\geq \varepsilon I$ for some $\varepsilon > 0$). The reader may find the equivalent condition "$\widehat{\mathbb{D}}^* J \widehat{\mathbb{D}} \geq \varepsilon I$ on $\partial\mathbf{D}$" (in $\mathrm{L}^\infty_{\mathrm{strong}}(\partial\mathbf{D}; \mathcal{B}(U))$) more familiar. The proofs are analogous to their continuous-time counterparts and hence omitted. See Section 10.6 for additional remarks too.

We first note that uniform positivity is equivalent to the existence of a positive spectral factorization, as well as to the existence of an I/O-stabilizing solution of the DARE:

Lemma 15.3.1 ($\mathbb{D}^* J \mathbb{D} \gg 0 \Leftrightarrow$ SpF $\Leftrightarrow$ DARE) *Let* $\Sigma = \left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right] \in \mathrm{wpls}(U, H, Y)$ *be strongly stable and* $J = J^* \in \mathcal{B}(Y)$. *Then the following are equivalent:*

(i) $\mathbb{D}^* J \mathbb{D} \gg 0$.

(ii) $\mathbb{D}^* J \mathbb{D} = \mathbb{X}^* \mathbb{X}$ *for some* $\mathbb{X} \in \mathcal{G}\mathrm{tic}(U)$.

(iii) The DARE has an I/O-stabilizing solution $\mathcal{P} = \mathcal{P}^*$, *with* $S \gg 0$.

(iv) The DARE has a solution $\mathcal{P} = \mathcal{P}^*$ *s.t.* $S \gg 0$, *and* $\widehat{\mathbb{M}}(z) = I + K(z^{-1} - (A + BK))^{-1} B$ *is in* $\mathrm{H}^\infty(\mathbf{D}; \mathcal{B}(U))$.

Moreover, if a solution $\mathcal{P}$ *of (iii) or (iv) exists, then a stable and strongly stabilizing solution* $\mathcal{P}$ *of the DARE exists with* $S \gg 0$.

If Σ *is exponentially stable, then we have one more equivalent condition:*

(v) The DARE has a solution $\mathcal{P} = \mathcal{P}^*$ *with* $S \gg 0$ *and* $\sigma(A + BK) \subset \mathbf{D}$.

Condition (vi) is sufficient for $\mathbb{D}^* J \mathbb{D} \geq 0$*:*

(vi) The DARE has a solution $\mathcal{P} = \mathcal{P}^*$ *with* $S \geq 0$. □

In many applications, we have $C^* J C \leq 0$ and $\Sigma \in \mathrm{SOS}$, and this allows us to drop any stability assumptions and avoid checking any stabilization conditions, as long as (non-uniform) nonnegativity ($\mathbb{D}^* J \mathbb{D} \geq 0$) is sufficient for us. Indeed, then the nonnegativity of the Popov operator is "practically equivalent" (cf. Proposition 15.4.2) to the existence of a nonpositive solution to the Riccati inequality:

Proposition 15.3.2 ($\mathbb{D}^* J \mathbb{D} \geq 0 \Leftrightarrow$ DARI) *Assume that* $C^* J C \leq 0$. *Then we have (ii)⇒(iii)⇒(i)⇐(iv) for the following conditions:*

(i) $\mathbb{D}^{t\,*} J \mathbb{D}^t \geq 0$ *for all* $t \geq 0$.

(ii) There is $\mathcal{P} \leq 0$ *s.t.* $S := D^* J D + B^* \mathcal{P} B \gg 0$ *and*

$$\begin{bmatrix} A^* \mathcal{P} A - \mathcal{P} + C^* J C & A^* \mathcal{P} B + C^* J D)^* \\ B^* \mathcal{P} A + D^* J C & S \end{bmatrix} \geq 0. \tag{15.12}$$

(iii) There is $\mathcal{P} \leq 0$ *s.t.*

$$S := D^*JD + B^*\mathcal{P}B \gg 0, \quad \textit{and} \tag{15.13}$$

$$(B^*\mathcal{P}A + D^*JC)^*S^{-1}(B^*\mathcal{P}A + D^*JC) \leq A^*\mathcal{P}A - \mathcal{P} + C^*JC. \tag{15.14}$$

(iv) $\mathbb{D} \in \mathrm{tic}$ *and* $\mathbb{D}^*J\mathbb{D} \geq 0$.

Moreover, the following hold:

(a) If $\mathbb{D} \in \mathrm{tic}$*, then we have (i)⇔(iv).*

(b) If $\Sigma \in \mathrm{sos}$ *and* $\mathbb{D}^*J\mathbb{D} \gg 0$*, then (i)–(iv) hold (in fact, we can have equality in (10.81)).*

(c) If $\mathbb{D} \in \mathrm{tic}$ *and* $\mathbb{B}$ *is strongly stable, then we can replace "*$\mathcal{P} \leq 0$*" by* $\mathcal{P} = \mathcal{P}^*$ *everywhere in this proposition.* □

(Use Lemma 15.5.1 in the proof of "(iii)⇒(i)".)

Similarly, uniform positivity ($\mathbb{D}^*J\mathbb{D} \geq \varepsilon I$) with exponential stability is equivalent to the existence of a nonpositive solution to the uniform Riccati inequality (still assuming that $C^*JC \leq 0$). This time neither the system nor the solution need be stabilizing (a priori):

Theorem 15.3.3 ($\mathbb{D}^*J\mathbb{D} \gg 0$ ⇔ DARI) *Assume that* $C^*JC \leq 0$*. Then the following are equivalent:*

(i) Σ *is exponentially stable and* $\mathbb{D}^*J\mathbb{D} \gg 0$.

(ii) There is $\mathcal{P} \leq 0$ *s.t.*

$$\begin{bmatrix} A^*\mathcal{P}A - \mathcal{P} + C^*JC & A^*\mathcal{P}B + C^*JD \\ B^*\mathcal{P}A + D^*JC & D^*JD + B^*\mathcal{P}B \end{bmatrix} \gg 0. \tag{15.15}$$

(iii) There is $\mathcal{P} \leq 0$ *s.t.* $S := D^*JD + B^*\mathcal{P}B \gg 0$ *and*

$$(B^*\mathcal{P}A + D^*JC)^*S^{-1}(B^*\mathcal{P}A + D^*JC) \ll A^*\mathcal{P}A - \mathcal{P} + C^*JC. \tag{15.16}$$

Moreover, any solution of (ii) or (iii) satisfies $\mathcal{P} < 0$ *(and there is an exponentially stabilizing solution if (i) holds).* □

(See the notes on p. 595.)

15.4 Real lemmas

Progress means replacing a theory that is wrong with one more subtly wrong.

In this section, we present the Bounded Real Lemma (in two forms) and the Strict Positive Real Lemma; see Section 10.5 for an introduction and corresponding continuous-time results. The proofs are analogous to their continuous-time counterparts and hence omitted.

We start with necessary and sufficient conditions for the norm of the I/O map to be less than a given constant:

Theorem 15.4.1 (Generalized Strict Bounded Real Lemma) *Assume that $\gamma > 0$. Then the following are equivalent:*

(i) Σ is exponentially stable and $\|\mathbb{D}\| < \gamma$.

(ii) There is $\mathcal{P} \le 0$ s.t.

$$\begin{bmatrix} A^*\mathcal{P}A - \mathcal{P} - C^*C & A^*\mathcal{P}B - C^*D \\ B^*\mathcal{P}A - D^*C & \gamma^2 I - D^*D + B^*\mathcal{P}B \end{bmatrix} \gg 0. \tag{15.17}$$

(iii) There is $\mathcal{P} \le 0$ s.t. $S := \gamma^2 I - D^*D + B^*\mathcal{P}B \gg 0$ and

$$(B^*\mathcal{P}A - D^*C)^* S^{-1}(B^*\mathcal{P}A - D^*C) \ll A^*\mathcal{P}A - \mathcal{P} - C^*C. \tag{15.18}$$

Moreover, any solution of (ii) or (iii) satisfies $\mathcal{P} < 0$ (and there is an exponentially stabilizing solution if (i) holds). □

As before, $\|\mathbb{D}\| := \|\mathbb{D}\|_{\mathrm{ti}} := \|\mathbb{D}\|_{\mathrm{tic}} := \|\mathbb{D}\|_{\mathcal{B}(\ell^2(\mathbf{Z};U),\ell^2(\mathbf{Z};Y))}$.

For $\Sigma \in$ sos, we have an "almost equivalence" that can be used to find an estimate of the norm of $\mathbb{D}$:

Proposition 15.4.2 (Nonexp. $\|\mathbb{D}\|_{\mathrm{tic}} < \gamma$) *Assume $\gamma > 0$.*

If (ii) or (iii) holds, then $\mathbb{D} \in$ tic and $\|\mathbb{D}\| \le \gamma$.

Conversely, if $\Sigma \in$ sos and $\|\mathbb{D}\| < \gamma$, then (ii) and (iii) hold (also with "$=$" in place of "$\ge$").

Here we have referred to the following conditions:

(ii) There is $\mathcal{P} \le 0$ s.t. $\gamma^2 I - D^*D + B^*\mathcal{P}B \gg 0$ and

$$\begin{bmatrix} A^*\mathcal{P}A - \mathcal{P} - C^*C & A^*\mathcal{P}B - C^*D \\ B^*\mathcal{P}A - D^*C & \gamma^2 I - D^*D + B^*\mathcal{P}B \end{bmatrix} \ge 0. \tag{15.19}$$

(iii) There is $\mathcal{P} \le 0$ s.t. $S := \gamma^2 I - D^*D + B^*\mathcal{P}B \gg 0$ and

$$(B^*\mathcal{P}A - D^*C)^* S^{-1}(B^*\mathcal{P}A - D^*C) \le A^*\mathcal{P}A - \mathcal{P} - C^*C. \tag{15.20}$$

Moreover, (ii) and (iii) are equivalent. If $\mathbb{B}$ is strongly stable, then we can replace "$\mathcal{P} \le 0$" by $\mathcal{P} = \mathcal{P}^*$ everywhere in this proposition. □

Theorem 15.4.3 (Generalized Strictly Positive (Real) Lemma) *The following are equivalent:*

(i) Σ *is exponentially stable and* $\mathbb{D}$ *is strictly positive (i.e.,* $\mathbb{D}+\mathbb{D}^* \gg 0$*);*

(ii) There is $\mathcal{P} \le 0$ *s.t.*

$$\begin{bmatrix} A^*\mathcal{P}A-\mathcal{P} & A^*\mathcal{P}B+C^* \\ B^*\mathcal{P}A+C & D+D^*+B^*\mathcal{P}B \end{bmatrix} \gg 0. \tag{15.21}$$

(iii) There is $\mathcal{P} \le 0$ *s.t.* $S := D+D^*+B^*\mathcal{P}B \gg 0$ *and*

$$(B^*\mathcal{P}A+C)^*S^{-1}(B^*\mathcal{P}A+C) \ll A^*\mathcal{P}A-\mathcal{P}. \tag{15.22}$$

Moreover, any solution of (ii) or (iii) satisfies $\mathcal{P} < 0$ *(and there is an exponentially stabilizing solution if (i) holds).* □

Notes for Sections 15.3 and 15.4

The restriction of Theorem 15.4.1 to finite-dimensional systems is essentially contained in Section 7.7 of [IOW] (multiply the inequalities by -1 and replace $\mathcal{P}$ by $-\mathcal{P}$); that result is the most general discrete-time "real lemma" that we have found in the literature. In particular, also the others are contained in our results. See also the notes on p. 595.

15.5 Riccati inequalities and the maximal solution

Like a great poet, Nature knows how to produce the greatest effects with the most limited means.
— Heinrich Heine

In this section, we extend to infinite-dimensional systems the standard Riccati inequality result (cf., e.g., [LR], Theorem 13.1.1 for discrete time and Theorem 9.1.1 for continuous time): if a Riccati inequality of a strongly stabilizable system has any solution with a uniformly positive signature operator, then it has a greatest solution. Moreover, the solutions of standard minimization problems correspond to this solution (assuming that we require strong or exponential stabilization).

Traditionally, this greatest solution has been called modestly "maximal solution", but we use the more specific term "greatest solution". See Theorem 9.8.13 and Corollary 9.2.11 for continuous-time analogies.

Lemma 15.5.1 (DARI) *We call* $(\mathcal{P},S,K)$ *a solution of the* extended Discrete-time Algebraic Riccati inequality (eDARI) *if* $\mathcal{P}=\mathcal{P}^*\in\mathcal{B}(H)$*,* $S\in\mathcal{B}(U)$*,* $K\in\mathcal{B}(H,U)$ *and*

$$\begin{cases} \mathcal{P}\le A^*\mathcal{P}A+C^*JC-K^*SK, \\ S=D^*JD+B^*\mathcal{P}B, \\ SK=-(D^*JC+B^*\mathcal{P}A). \end{cases} \tag{15.23}$$

Such a solution necessarily satisfies

$$\mathbb{K}^{t\,*}S\mathbb{K}^t\le\mathbb{A}^{t\,*}\mathcal{P}\mathbb{A}^t-\mathcal{P}+\mathbb{C}^{t\,*}J\mathbb{C}^t \quad \text{and} \tag{15.24}$$

$$\mathbb{X}^{t\,*}S\mathbb{X}^t\le\mathbb{D}^{t\,*}J\mathbb{D}^t+\mathbb{B}^{t\,*}\mathcal{P}\mathbb{B}^t. \tag{15.25}$$

for all $t\in\mathbf{N}$*. Inequality (15.24) holds even if we drop the assumptions that* $\mathcal{P}=\mathcal{P}^*$*,* $S=D^*JD+B^*\mathcal{P}B$ *and* $SK=-(D^*JC+B^*\mathcal{P}A)$*.* □

(The proof of Lemma 14.2.1 applies with certain "$=$" symbols replaced by "$\le$" or "$\ge$" symbols. Note that the third equation of the IARE is lost due to its unsymmetry.)

Theorem 15.5.2 (Greatest solution $\mathcal{P}_+$ of DARE and DARI) *Assume that* $\Sigma\in\mathrm{wpls}(U,H,Y)$ *is strongly (*$\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$*-)stabilizable, and that the eDARI has a solution* $(\mathcal{P},S,K)$ *s.t.* $S\gg 0$*.*

Then the DARE has a solution $(\mathcal{P}_+,S_+,K_+)$ *s.t.* $S_+\gg 0$ *and* $\mathcal{P}_+\ge\mathcal{P}$ *for all solutions* $\mathcal{P}=\mathcal{P}^*\in\mathcal{B}(H)$ *of the eDARI having* $S\ge 0$*.*

Moreover, if $\mathcal{P}$ *is a strongly* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$*-stabilizing solution of the eDARE, then* $\mathcal{P}=\mathcal{P}_+$*.*

Corollary 15.5.3 (Greatest solution $\mathcal{P}_+$ of the DARE) *If the DARE has a strongly stabilizing (or strongly* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$*-stabilizing) solution s.t.* $S\gg 0$*, then this solution is the greatest solution of the eDARE having* $S\ge 0$*.*

In particular, if $D^*JD\gg 0$ *and the DARE has a strongly* $\left[\begin{smallmatrix}\mathbb{A} & \mathbb{B}\\ \mathbb{C} & \mathbb{D}\end{smallmatrix}\right]$*-stabilizing solution* $\mathcal{P}\ge 0$*, then* $\mathcal{P}\ge\mathcal{P}'$ *for any nonnegative solution* $\mathcal{P}'$ *of the eDARE (or the eDARI).* □

(The corollary follows from Theorem 15.5.2.)

The example $A = 1$, $B = 0 = C$, $D = 1 = J$, $\mathcal{P} \in \mathbf{R}$ shows that "strongly" is not redundant in the above corollary; modify (or discretize) Example 9.13.12(b) to observe that "strongly" cannot be replaced by "weakly" (in the infinite-dimensional case).

N.B. The number of all self-adjoint solutions of a Riccati equation can be infinite (the simplest example is the two-dimensional CARE $\mathcal{P}^2 = I$ with self-adjoint solutions $\mathcal{P} = \left[\begin{smallmatrix} \sin\theta & i\cos\theta \\ i\cos\theta & \sin\theta \end{smallmatrix}\right]$ $(\theta \in [0, 2\pi))$).

Proof of Theorem 15.5.2:

0° *Notes on the theorem:* We have required above a solution of the eDARI to be self-adjoint. We have also assumed the existence of a solution of the eDARI that has $S \gg 0$. Still, the operator $\mathcal{P}_+$ need not be positive (take, e.g., $J = \left[\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right], C = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right], D = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right], A = -I$, so that $S = I$ and $\mathcal{P}_+ = -\int_0^\infty e^{-2t} I\, dt \ll 0$ is the unique solution of the DARE, by Lemma 9.12.2(c)).

Note that we have allowed instead of strong stabilizability the weaker condition of strong $\left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$-stabilizability: there must be a state feedback operator K_0 for Σ s.t. $\left(\begin{array}{c|c} A+BK_0 & B \\ \hline C+DK_0 & D \end{array}\right)$ is strongly stable; the third row (generated by $\left(\begin{array}{c|c} K_0 & 0 \end{array}\right)$) of the corresponding closed-loop system Σ_0 need not be stable (the same applies to the "moreover" claim).

1° *About the proof:* This is Theorem 13.1.1 of [LR] except that 1. it assumes that $\dim U, \dim H, \dim Y < \infty$, 2. it requires Σ to be exponentially stabilizable, 3. it assumes that $D^*JD \in \mathcal{GB}(U)$, 4. its statements incorrectly do not (the proof does) require $S \gg 0$ for solutions $\mathcal{P}$ that are to be shown to satisfy $\mathcal{P} \leq \mathcal{P}_+$ (note that we do assume $S \geq 0$), 5. it also states that $\mathcal{P}_+$ is "almost stabilizing" (our 10° is formally weaker, but if $\dim H < \infty$, then this implies that $\mathcal{P}_+$ is "almost stabilizing", i.e., that $\rho(A_+) \leq 1$, as noted in 8°).

We shall follow the proof from [LR], mutatis mutandis.

It is easy to bypass 3. (see below); to bypass 1., we have to use additional tricks for the stability of A_n $(n \in \mathbf{N})$ (see 4° below); by developing these tricks further we can bypass 2. too.

The symbols of [LR] map to those of ours as follows: $X_k \mapsto \mathcal{P}_k$, $\widetilde{X} \mapsto \mathcal{P}$, $L_k \mapsto -K_k$, $R \mapsto D^*JD$, $Q \mapsto C^*JC$, $C \mapsto D^*JC$, $E \mapsto -SK$.

2° $\mathcal{P}_0 \geq \mathcal{P}$*:* For each $n \in \mathbf{N}$, we shall denote by Σ_n the strongly stable closed-loop system of Σ corresponding to the state feedback operator K_n (specified below).

Let K_0 be strongly stabilizing for Σ (or at least s.t. $\left(\begin{array}{c|c} A+BK_0 & B \\ \hline C+DK_0 & D \end{array}\right)$ becomes stable). By Lemma 9.12.2(c), the operator $\mathcal{P}_0 := \mathbb{C}_0^* J \mathbb{C}_0$ is the unique solution of the DARE $\mathcal{P}_0 = A_0^* \mathcal{P}_0 A_0 - C_0^* J C_0$. It is straightforward to verify that

$$(\mathcal{P}_0 - \mathcal{P}) - A_0^*(\mathcal{P}_0 - \mathcal{P})A_0 = \mathcal{R}(\mathcal{P}) + (K_0 - K)^* S (K_0 - K) \geq 0 \tag{15.26}$$

for any solution $\mathcal{P}$ of the eDARI s.t. $S \geq 0$, where $\mathcal{R}(\mathcal{P}) := A^*\mathcal{P}A - \mathcal{P} + C^*JC - K^*SK \geq 0$. Multiply (15.26) by $(A_0^*)^k$ to the left and by A_0^k to the right, and add the results for $k = 0, 1, \ldots, n$ to obtain that $\mathcal{P}_0 - \mathcal{P} \geq (A_0^*)^{n+1}(\mathcal{P}_0 - \mathcal{P})A_0^{n+1} \to 0$, as $n \to +\infty$, so that $\mathcal{P}_0 \geq \mathcal{P}$.

3° *Induction —* $\mathcal{P}_n$*:* Let $n \in \mathbf{N} + 1$. Assume (inductively) that $\mathcal{P}_0 \geq \mathcal{P}_1 \geq$

$\cdots \geq \mathcal{P}_{n-1}$ s.t. $\mathcal{P}_k \geq \mathcal{P}$ $(k \leq n-1)$ for all solutions $\mathcal{P}$ of the eDARI having $S \geq 0$ (in particular, $S_k := D^*JD + B^*\mathcal{P}_k B \geq S \gg 0$, where S is as in the assumptions) and $\left[\begin{smallmatrix} \mathbb{A}_k & \mathbb{B}_k \\ \mathbb{C}_k & \mathbb{D}_k \end{smallmatrix}\right] := \left(\begin{smallmatrix} A+BK_k & B \\ C+DK_k & D \end{smallmatrix}\right)$ is strongly stable; these are the top two rows of the closed-loop system Σ_K corresponding to the state feedback operator

$$K_k := -S_{k-1}^{-1}(D^*JC + B^*\mathcal{P}_{k-1}A) \quad (k \geq 1) \tag{15.27}$$

for $k \leq n-1$. Define K_n and Σ_n as above.

4° *Induction —* $\left(\begin{smallmatrix} A_n & B \\ C_n & D \end{smallmatrix}\right)$ *is strongly stable:* We obtain for any solution $\mathcal{P}$ of the DARI, after a straightforward computation, that

$$\begin{aligned} &(\mathcal{P}_{n-1} - \mathcal{P}) - A_n^*(\mathcal{P}_{n-1} - \mathcal{P})A_n \\ &\quad = (K_n - K_{n-1})^* S_{n-1}(K_n - K_{n-1}) + \mathcal{R}(\mathcal{P}) + (K_n - K)^* S(K_n - K). \end{aligned} \tag{15.28}$$

Multiply this by $(A_n^*)^k$ to the left and by A_n^k to the right, and add the results for $k = 0, 1, \ldots, m$ to obtain that

$$(\mathcal{P}_{n-1} - \mathcal{P}) \geq (A_n^*)^{m+1}(\mathcal{P}_{n-1} - \mathcal{P})A_n^{m+1} + \sum_{k=0}^{m} (A_n^*)^k (K_n - K_{n-1})^* S_{n-1}(K_n - K_{n-1})A_n^k \tag{15.29}$$

for all $m \in \mathbf{N}$ (recall $\mathcal{P}_{n-1} - \mathcal{P} \geq 0$). Since $S_{n-1} \gg 0$, it follows that $\|TA_n^{\cdot}\|_{\mathcal{B}(H,\mathrm{L}^2)} < \infty$, where $T := K_n - K_{n-1}$.

Now add $(A_n^*)^k \cdot (15.28) \cdot A_n^k |z|^{2k+2}$ for $k = 0, 1, \ldots, m$ to obtain that

$$\sum_{k=0}^{m} (A_n^*)^k T^* S_{n-1} T A_n^k |z|^{2k+2} \tag{15.30}$$

$$\leq \sum_{k=0}^{m} (A_n^*)^k (\mathcal{P}_{n-1} - \mathcal{P})A_n^k |z|^{2k+2} - \sum_{k=0}^{m} (A_n^*)^{k+1} (\mathcal{P}_{n-1} - \mathcal{P})A_n^{k+1} |z|^{2k+2} \tag{15.31}$$

$$= (\mathcal{P}_{n-1} - \mathcal{P})|z|^2 - \sum_{k=1}^{m} (A_n^*)^k (\mathcal{P}_{n-1} - \mathcal{P})A_n^k |z|^{2k}(1 - |z|^2) \tag{15.32}$$

$$- (A_n^*)^{m+1}(\mathcal{P}_{n-1} - \mathcal{P})A_n^{m+1}|z|^{2m+2} \tag{15.33}$$

$$\leq (\mathcal{P}_{n-1} - \mathcal{P})|z|^2 \leq (\mathcal{P}_{n-1} - \mathcal{P}) \qquad (|z| \leq 1). \tag{15.34}$$

Let Σ_n be the closed-loop system for Σ_{n-1} corresponding to the state feedback operator $T := K_n - K_{n-1}$, i.e.,

$$\Sigma_n' := \left[\begin{array}{c|c} \mathbb{A}_n & \mathbb{B}_n \\ \hline \mathbb{C}_n & \mathbb{D}_n \\ \mathbb{T}_n & \mathbb{E}_n \end{array}\right] := \left(\begin{array}{c|c} A_n & B \\ \hline C_n & D \\ T & 0 \end{array}\right) \tag{15.35}$$

(indeed, from the generators we see that the two top rows of Σ_n' are equal to those of Σ_n; observe also that $A_n = A + BK_n = A_{n-1} + BT$, $C_n = C + DT$). Below (15.29) we noted that $\mathbb{T}_n = TA_n^{\cdot}$ is stable. By Lemma 6.6.8, it follows that $\mathbb{A}_n = \mathbb{A}_{n-1} + \mathbb{B}_{n-1}\tau\mathbb{T}_n$ is strongly stable, From $B^*\cdot(15.30)\cdot B \le B^*(\mathcal{P}_{n-1}-\mathcal{P})B$ we obtain (since $S_{n-1}\gg 0$) that

$$\|\sum_{k=0}^{m} TA_n^k z^{k+1}B\|_{\mathcal{B}(U)} \le M_n < \infty \quad (m\in\mathbf{N}). \tag{15.36}$$

By (13.52), this implies that $\widehat{\mathbb{E}}_n \in \mathrm{H}^\infty(\mathbf{D};\mathcal{B}(U))$, i.e., that $\mathbb{E}_n \in \mathrm{tic}(U)$. Consequently, $\mathbb{B}_n = \mathbb{B}_{n-1}(\mathbb{E}_n+I)$ and $\mathbb{D}_n = \mathbb{D}_{n-1}(\mathbb{E}_n+I)$ are stable. Therefore, also $\mathbb{C}_n = \mathbb{C}_{n-1}+\mathbb{D}_{n-1}\mathbb{T}_n$ is stable. Now we have shown that two top rows of Σ_n are strongly stable.

5° $\mathcal{P}_n \ge \mathcal{P}$*:* By Lemma 9.12.2(c), the operator $\mathcal{P}_n := \mathbb{C}_n^* J\mathbb{C}_n$ is the unique solution of the DARE $\mathcal{P}_n = A_n^*\mathcal{P}_nA_n + C_n^*JC_n$. If $\mathcal{P}$ is a solution of the eDARI having $S\ge 0$, then (as in 2°)

$$(\mathcal{P}_n-\mathcal{P}) - A_n^*(\mathcal{P}_n-\mathcal{P})A_n = \mathcal{R}(\mathcal{P}) + (K_n-K)S(K_n-K) \ge 0, \tag{15.37}$$

hence $\mathcal{P}_n - \mathcal{P} \ge (A_n^*)^{m+1}(\mathcal{P}_n-\mathcal{P})A_n^{m+1} \to 0$ (as in 2°), as $m\to+\infty$, so that $\mathcal{P}_n \ge \mathcal{P}$.

6° $\mathcal{P}_{n-1} \ge \mathcal{P}_n$*:* One can analogously to p. 310 of [LR] compute that

$$(\mathcal{P}_{n-1}-\mathcal{P}_n) - A_n^*(\mathcal{P}_{n-1}-\mathcal{P}_n)A_n = (K_n-K_{n-1})^*S_{n-1}(K_n-K_{n-1}) \ge 0. \tag{15.38}$$

As above, we obtain that $\mathcal{P}_{n-1}-\mathcal{P}_n \ge (A_n^*)^{m+1}(\mathcal{P}_{n-1}-\mathcal{P}_n)A_n^{m+1} \to 0$, hence $\mathcal{P}_{n-1} \ge \mathcal{P}_n$.

7° $\mathcal{P}_+$*:* Let $\mathcal{P}$ be a solution of the eDARI. Then $\mathcal{P}_n \ge \mathcal{P}_{n+1} \ge \mathcal{P}$ $(n\in\mathbf{N})$, so that there is a unique $\mathcal{P}_+ \in \mathcal{B}(H)$ s.t. $\mathcal{P}_n x_0 \to \mathcal{P}_+ x_0$ for all $x_0\in H$, by Lemma A.3.1(b5). Since $\mathcal{P}_n = \mathcal{P}_n^* \ge \mathcal{P}$ for all n, we have $\mathcal{P}_+ = \mathcal{P}_+^* \ge \mathcal{P}$.

8° $S_n \to S_+,\ S_n^{-1}\to S_+^{-1}, K_n\to K_+, A_n\to A_+$*, strongly:* Set $S_+ := D^*JD + B^*\mathcal{P}_+B \ge S \gg 0$ (where $S\gg 0$ corresponds to the solution in the assumptions), $K_+ := S_+^{-1}(D^*JC + B^*\mathcal{P}_+A)$. For all $n\in\mathbf{N}$, we have $S_n \ge S_+ \ge \varepsilon I$ for some $\varepsilon>0$, hence $0 \ll S_n^{-1} \le \varepsilon^{-1}I$, by Lemma A.3.1(b1), hence $\|S_n^{-1}\| \le \varepsilon^{-1/2} < \infty$. Therefore, $S_n^{-1} \to S_+^{-1}$ strongly, as $n\to+\infty$, by Lemma A.3.1(b5). Therefore, $K_n \to K_+$ strongly (see Lemma A.3.1(j3)), hence $A_n\to A_+$, $C_n \to C_+$ strongly, as $n\to+\infty$, where Σ_+ is the closed-loop system corresponding to K_+ (i.e., $A_+ := A+BK_+$, $C_+ := C+DK_+$).

(Note that if $\dim H<\infty$, then "strong equals uniform", hence then $\rho(A_+)\le 1$ (since $\rho(A_n)\le 1$ for all $n\in\mathbf{N}$), but even then we may have $A_+ = I$ as in Example 13.2.1 of [LR], where $C=0$, $A=B=R=I$, and hence $\mathcal{P}_n = (2^{n+1}-1)^{-1}I$ $(n\in\mathbf{N})$.)

9° $\mathcal{P}_+$ *solves the DARE:* By 8° and 5°, we have $\mathcal{P}_+ = A_+^*\mathcal{P}_+A_+ + C_+^*JC_+$; combine this to the definitions of K_+ and S_+ to observe that $\mathcal{P}_+$ is a solution of the DARE, by Lemma 9.10.1(b4)(iv).

10° $\mathcal{P}_+ = \mathcal{P}$*:* Let $\mathcal{P}$ be a strongly stabilizing (at least for the top two rows) solution of the eDARE. Then we can choose $\mathcal{P}_0 := \mathbb{C}_\circlearrowleft^* J\mathbb{C}_\circlearrowleft = \mathcal{P}$ in 2°. It follows that $\mathcal{P} = \mathcal{P}_0 \ge \mathcal{P}_+ \ge \mathcal{P}$, hence $\mathcal{P} = \mathcal{P}_+$.

11° *Remark — exponentially stable A_n:* One observes from 4° (and Lemma 13.3.7(ii)) that if A_0 is exponentially stable, then so is A_n for all $n\in\mathbf{N}$. □

Notes

The classical result (e.g., Theorem 13.1.1 of [LR]) assumes that $\mathcal{P}$ is exponentially stabilizing, hence our result seems to be new (a generalization) also for finite-dimensional systems.

In continuous-time, the classical result (e.g., Theorem 9.1.1 of [LR]) was extended to WPLSs with bounded B and C by Ruth Curtain and Leiba Rodman in [CR], which also contains comparison theorems for the solutions of two different Riccati inequalities (for two different WPLSs).

Laurence Dumortier has enhanced some of the results of [CR] (for WPLSs having bounded B and C and finite-dimensional U and Y). In [Dumortier], she has shown that the maximal solution $\mathcal{P}_+$ of the standard LQR CARE stabilizes all strictly intact poles of A as well as all observable ones, but it leaves the critical (meaning $\sigma(A) \cap i\mathbf{R}$) nonobservable poles unstable. Thus, at least in this special case, $\mathcal{P}_+$ is exponentially stabilizing iff $\mathrm{Ker}(\mathbb{C}) \cap \sigma(A) \cap i\mathbf{R} = \emptyset$.

Unbounded input and output operators make the classical proof virtually impossible. This is surely the reason why there does not seem to be any earlier results for WPLSs having an unbounded input or output operator. We have obtained such results, Corollary 9.2.11 and Theorem 9.8.13, by reducing them to the results of this section, using discretization. This way we obtain analogous results for the "IARI" in the general case (replace the first equation of the IARE by an inequality). However, this way we have obtained results for the CARE (or for the IARE or for the "IARI") only, not for the corresponding inequality ("CARI"), since the "CARI" (9.217) does not lead to the IARI, as explained in Proposition 9.11.9. Therefore, a "CARI" result would require a continuous-time proof.

It seems that such a proof would be possible for bounded B, as well as for the parabolic systems of Hypothesis 9.5.1, since in both of these cases we would have $\mathrm{Dom}(A_n) = \mathrm{Dom}(A)$ for each n in the proof (generalize the proof of Theorem 9.1.1 of [LR]; note that each K_n would become bounded). These might be worth of further investigations.

Conclusions

Everything that can be invented has been invented.
— Charles Duell, Director of U.S. Patent Office, 1899

In the preceding chapters, we have presented a generalization of much of the finite-dimensional time-invariant linear systems and optimization theory to WPLSs and presented some results that seem to be new even for finite-dimensional systems.

Nevertheless, by viewing the literature on finite-dimensional systems (see, e.g., [IOW], [LR], [GL], ...), one observes that there are still numerous problems waiting for solutions, such as the measurement feedback (dynamic feedback) variants of the LQR and H^2 problems and nondeterministic problems like the Kalman filter problem and the Linear Quadratic Gaussian control problem. Moreover, there are several more "infinite-dimensional" problems that only have been solved for some subsets of WPLSs. It seems that several of these can be rather easily solved in the WPLS framework (the Riccati equation formulations requiring some regularity assumptions, e.g., those of Lemma 6.8.5) by using the tools and methods of this book.

Moreover, as indicated in the notes to several sections of this book, we have met several old and new subjects worth of a separate study. Among the most important tasks is that of finding further sufficient conditions for a system to have any unique (or at least J-coercive) optimal control in regular state feedback form (see Remark 9.9.14 for most known ones). One might also wish to find further conditions that lead to simplify the continuous-time Riccati equations (see Theorem 9.9.6 and Sections 9.2 and 9.5 for known ones).

Even the systems theory for WPLSs can still be further developed, although a rather mature and extensive presentation can be found in [Sbook]. One important piece of the development is the combination of the western (Weiss and others) and eastern (Arov and others) WPLS theories; part of this has already been done (see Chapter 11 of [Sbook] or [S01]). One might also wish to apply the methods of this monograph to time-variant systems (see the notes on p. 8.5).

All the above corresponds to the abstract mathematical theory, which naturally needs a more concrete supplementary, such as numerical methods and applications to concrete systems arising in other sciences, far beyond the scope of this book.

Appendix A

Algebraic and Functional Analytic Results

Algebraic symbols are used when you do not know what you are talking about.

— Philippe Schnoebelen

In this appendix, we present a number of algebraic and functional analytic results that are needed in the main parts of this monograph.

In Section A.1, we mainly present algebraic results, such as "$(I-AB)^{-1} = I + A(I-BA)^{-1}B$" or "$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$", that are valid for matrices and more general linear operators or elements of certain rings.

In Section A.2, we very briefly introduce metric spaces and other topological spaces. In Section A.3, we list standard and extended concepts and facts about Hilbert and Banach spaces and Banach algebras. In Section A.4, we present strongly continuous (C_0) semigroups.

In the main part of the monograph, all vector spaces (e.g., Banach spaces) are assumed to be complex. However, in the appendices we generally assume that the scalar field is **K**, which the reader may read as either **C** or **R**. In Appendix D and Sections A.4 and F.3, we assume that $\mathbf{K} = \mathbf{C}$, as explicitly stated there; in the other sections in appendices we always state explicitly any such exceptions.

Thus, the concepts "vector space", "Banach space" and "Hilbert space" mean spaces of the corresponding type over the scalar field **K** (in particular, if some spaces in a theorem are assumed to be Banach spaces, all of them must be Banach spaces over the same **K** (= **R** or **C**)).

A.1 Algebraic auxiliary results ($\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$)

If $A \in \mathcal{B}(X_1 \times X_2, Y_1 \times Y_2)$, where X_1, X_2, Y_1, Y_2 are Banach spaces, then A can be written as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where $A_{ij} \in \mathcal{B}(X_j, Y_i)$, as shown in Lemma A.1.1(a4) below. There are many simple, well-known algebraic rules how to handle this kind of *operator matrices*; e.g., if $A_{12} = 0$ and the diagonal blocks (operators) A_{11} and A_{22} are invertible, then so is A (see (b1)).

In Lemma A.1.1 we present a number of such results in a more general setting (we only need the ring operations and units); see the remarks following the lemma for generalizations.

Recall that "&" means "and", and that Banach spaces are topological vector spaces. Moreover, $X \times Y := \{(x,y) \mid x \in X,\ y \in Y\}$. If X and Y are normed spaces, we use the norm $\|(x,y)\|_{X\times Y} := (\|x\|_X^2 + \|y\|_Y^2)^{1/2}$; if X and Y are inner product spaces, we use the inner product $\langle (x,y),(x',y')\rangle_{X\times Y} := \langle x,x'\rangle_X + \langle y,y'\rangle_Y$; use induction for $\prod_{k=1}^n X_k := X_1 \times X_2 \times \cdots \times X_n$. Finally, $X^n := \prod_{k=1}^n X$.

Lemma A.1.1 (Operator matrix lemma) *We assume that (1.), (2.), (3.), (4.) or (5.) holds, where*

(1.) $\mathcal{X}$ is the collection of all vector spaces, and $\mathcal{A}(X,Y) = \mathrm{Hom}(X,Y)$ is the set of vector homomorphisms (i.e., linear mappings) $X \to Y$.

(2.) $\mathcal{X}$ is the collection of all topological vector spaces, and $\mathcal{A}(X,Y) = \mathcal{B}(X,Y)$ is the set of continuous linear mappings $X \to Y$.

(3.) $\mathcal{X}$ is the collection of all Banach spaces, and $\mathcal{A}$ is any of the symbols Hom, $\mathcal{B}$, $\mathcal{C}(\Omega;\mathcal{B}(\cdot,\cdot))$, $\mathrm{H}(\Omega;\mathcal{B}(\cdot,\cdot))$, $\mathrm{H}^\infty(\Omega;\mathcal{B}(\cdot,\cdot))$, $[\mathcal{B}+]\mathrm{H}^p_\infty$, $[\mathcal{B}+]\mathrm{H}^p_{\mathrm{strong},\infty}$, TI_ω, TIC_ω, $\mathrm{L}^\infty(Q,\mathcal{B}(\cdot,\cdot))$, *and* $\mathrm{L}^\infty_{\mathrm{strong}}(Q,\mathcal{B}(\cdot,\cdot))$, *where $\Omega \subset \mathbf{C}$ is open, Q and μ are as on p. 907, and $\omega \in \mathbf{R} \cup \{\infty\}$.*

(For $*\mathrm{TI}*$ and $\mathrm{H}*$ we require the elements of $\mathcal{X}$ to be complex Banach spaces; this applies (4.) and (5.) too.)

(4.) $\mathcal{X}$ is the collection of all complex Banach spaces, and $\mathcal{A}$ is any of the symbols defined in Definitions 2.6.1 and 2.6.3, except that if $\mathcal{A}$ is a symbol with a specified atomgroup $\mathbf{S}$, we require that $\mathbf{S} = \mathbf{S} - \mathbf{S} \subset \mathbf{R}$; cf. Definition 2.6.3 and Theorem 2.6.4.

(5.) $\mathcal{X}$ is the collection of all Banach spaces, and $\mathcal{A}$ is any of the symbols SR, SLR, SHPR, SVR, UR, ULR, UHPR *and* UVR, *(where these symbols are as in Definition 6.2.3) or $\mathcal{A}$ is the intersection* TIC_ω *and any of the above symbols for some $\omega \in \mathbf{R} \cup \{\infty\}$.*

We use the following notation: By $\mathcal{A}$ we mean any of the sets $\mathcal{A}(X,Y)$ $(X,Y \in \mathcal{X})$; the group operation and identity operator in (any) $\mathcal{A}$ are denoted by $+$ and $I = I_{\mathcal{A}}$, respectively. If $A \in \mathcal{A}(X,Y)$ and $B \in \mathcal{A}(Y,X)$ are s.t. $AB = I_{\mathcal{A}(Y)}$, *then we write $A_{\text{right}}^{-1} = B$ and $B_{\text{left}}^{-1} = A$; if, in addition, $BA = I_{\mathcal{A}(X)}$, then we write $A^{-1} = B$. By "$\exists A^{-1}$" we mean that A has an inverse (as above). Similarly, "$\exists A_{\text{right}}^{-1}$" (resp. "$\exists A_{\text{left}}^{-1}$") means that A has a right (resp. left) inverse.*

We assume that $X_k, Y_k, Z_k \in \mathcal{X}$ for all $k \in \mathbf{N}$.

With the above assumptions and notation, the following holds:

(a1) If $A \in \mathcal{A}(X_1, Y_1)$ has a left inverse B and a right inverse C, then $B = BAC = C$ is the unique inverse of A.

(a2) If $\exists A^{-1}, B^{-1}$, then $\exists (AB)^{-1} = B^{-1}A^{-1}$, when $A \in \mathcal{A}(X_1, Y_1)$ and $B \in \mathcal{A}(Y_1, Z_1)$.

(a3) If $\dim X_1 = \dim Y_1 < \infty$, then any $A \in \mathcal{A}(X_1, Y_1)$ is left (resp. right) invertible iff it is invertible.

(a4) Let $n, m, N \in \{1, 2, 3, \ldots\}$, and let $X := X_1 \times \cdots \times X_n$, $Y := Y_1 \times \cdots \times Y_m$ $Z := Z_1 \times \cdots \times Z_N$. Let $P_i : Y \to Y_i$ be the canonical projection and $L_j : X_j \to X$ the canonical imbedding. Then $P_i \in \mathcal{A}(Y, Y_i)$ and $L_j \in \mathcal{A}(X_j, X)$.

Let, in addition, $A \in \mathcal{A}(X, Y)$. Then $A_{ij} := P_i A L_j \in \mathcal{A}(X_j, Y_i)$ for all i, j, and the representation

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{bmatrix}, \tag{A.1}$$

satisfies the standard matrix multiplication rules $(A(x_1, \ldots, x_n))_i = \sum_{j=1}^n A_{ij} x_j \in Y_i$ for $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$, $i = 1, \ldots, n$, and $(AB)_{ik} = \sum_k A_{ij} B_{jk}$ for $B \in \mathcal{A}(Y, Z)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$, $k = 1, \ldots, N$. Moreover, $(A + A')_{ij} := A_{ij} + A'_{ij}$ for $A' \in \mathcal{A}(X, Y)$, and $(-A)_{ij} := -A_{ij}$.

Conversely, if(f) $A_{ij} \in \mathcal{A}(X_j, Y_i)$ for all i, j, then we can define $A \in \mathcal{A}(X, Y)$ by setting $A := \sum_{i,j} P_i^* A_{ij} L_j^*$; equivalently, $(A(x_1, \ldots, x_n))_i := \sum_{j=1}^n A_{ij} x_j \in Y_i$. We denote this by (A.1).

If X_j and Y_i are Hilbert spaces for all i, j and (A.1) holds, then $(A^*)_{ij} = A_{ji}^*$ for all i, j.

In parts (b1)–(h1) we assume that $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathcal{A}(X_1 \times X_2, Y_1 \times Y_2)$ and $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathcal{A}(Y_1 \times Y_2, X_1 \times X_2)$, and that also the other terms (operators) belong to the $\mathcal{A}$'s that are compatible with the formulae.

(b1) $\exists A_{11}^{-1}, A_{22}^{-1} \implies \exists \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$ & $\exists \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}$.

*If $\exists (A_{11})_{\text{right}}^{-1}, (A_{22})_{\text{right}}^{-1}$ (resp. $\exists (A_{11})_{\text{left}}^{-1}, (A_{22})_{\text{left}}^{-1}$), then the above inverse matrices exist as right (resp. left) inverses, and the above formulae hold (*mutatis mutandis*).*

(b2) Conversely, if $\exists \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} =: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, *then* $\exists A_{11}^{-1}$, A_{22}^{-1} *iff any of the conditions (1)–(7) holds, where:*

(1) $\exists (A_{11})^{-1}_{\text{right}}$*; (2)* $\exists (A_{22})^{-1}_{\text{left}}$*; (3)* $B_{21} = 0$*; (4)* $A_{12} = 0$*; (5)* $\dim X_1 = \dim Y_1 < \infty$*; (6)* $\dim X_2 = \dim Y_2 < \infty$*; (7)* X_1, X_2, Y_1, Y_2 *are Banach spaces,* $\mathcal{A} = \mathcal{B}$*, and* $A_{11} \in \mathcal{GB} + \mathcal{BC}$ *or* $A_{22} \in \mathcal{GB} + \mathcal{BC}$.

(c1) Let $\exists \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} =: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$. *Then* $\exists B_{11}^{-1} \Leftrightarrow \exists A_{22}^{-1} \Leftrightarrow \exists A_{22}^{-1} = B_{22} - B_{21} B_{11}^{-1} B_{12} \Leftrightarrow \exists B_{11}^{-1} = A_{11} - A_{12} A_{22}^{-1} A_{21}$.

If $\dim X_1 < \infty$ *or* $\dim X_2 < \infty$*, then, in addition,* $\exists (A_{22})^{-1}_{\text{left}} \Leftrightarrow \exists (A_{22})^{-1}_{\text{right}} \Leftrightarrow \exists A_{22}^{-1} \Leftrightarrow \exists (B_{11})^{-1}_{\text{left}} \Leftrightarrow \exists (B_{11})^{-1}_{\text{right}} \Leftrightarrow \exists B_{11}^{-1}$.

If $\exists B_{11}^{-1}$ *and* $A^{-1} = B$*, then the formulae of (d1) hold,* $B_{21} B_{11}^{-1} = -A_{22}^{-1} A_{21}$*, and*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} I & B_{12} \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix}^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ B_{21} & B_{22} \end{bmatrix}. \tag{A.2}$$

(c2) If $\exists \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}_{\text{right}} =: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, *then* $\exists (B_{11})^{-1}_{\text{right}} \Longrightarrow \exists (A_{22})^{-1}_{\text{right}} = B_{22} - B_{21} (B_{11})^{-1}_{\text{right}} B_{12}$
and $\exists (A_{22})^{-1}_{\text{left}} \Longrightarrow \exists (B_{11})^{-1}_{\text{left}} = A_{11} - A_{12} (A_{22})^{-1}_{\text{left}} A_{21}$.
If $\exists \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1}_{\text{left}} =: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, *then* $\exists (B_{11})^{-1}_{\text{left}} \Longrightarrow \exists (A_{22})^{-1}_{\text{left}} = B_{22} - B_{21} (B_{11})^{-1}_{\text{left}} B_{12}$
and $\exists (A_{22})^{-1}_{\text{right}} \Longrightarrow \exists (B_{11})^{-1}_{\text{right}} = A_{11} - A_{12} (A_{22})^{-1}_{\text{right}} A_{21}$.

(c3) $\exists \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} \Leftrightarrow \exists \begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix}^{-1}$.

(c4) Let $\exists \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} =: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ *and* $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathrm{TIC}$.
If $\exists A_{11}^{-1} \in \mathrm{TIC}_\infty$*, then* $\exists (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} = B_{22} \in \mathrm{TIC}$*; if* $\exists A_{22}^{-1} \in \mathrm{TIC}_\infty$*, then* $\exists (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} = B_{11} \in \mathrm{TIC}$.

(d1) **(Schur decomposition)** *Let* $\exists A_{11}^{-1}$*. Then*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1} A_{12} \\ 0 & A_{22} - A_{21} A_{11}^{-1} A_{12} \end{bmatrix} \tag{A.3}$$

$$= \begin{bmatrix} I & 0 \\ A_{21} A_{11}^{-1} & A_{22} - A_{21} A_{11}^{-1} A_{12} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & I \end{bmatrix} \tag{A.4}$$

$$= \begin{bmatrix} I & 0 \\ A_{21} A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} - A_{21} A_{11}^{-1} A_{12} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1} A_{12} \\ 0 & I \end{bmatrix} \tag{A.5}$$

$$= \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} - A_{21} A_{11}^{-1} A_{12} \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1} A_{12} \\ 0 & I \end{bmatrix}. \tag{A.6}$$

Therefore, $\exists A^{-1} \Leftrightarrow \exists (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1}$ *(see also Lemma 11.3.13). If, in* *addition,* $\exists A^{-1}$*, then*

$$A^{-1} = \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}^{-1} \begin{bmatrix} A_{11} & 0 \\ A_{21} & I \end{bmatrix}^{-1} \tag{A.7}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}^{-1} \tag{A.8}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ (A^{-1})_{21} & (A^{-1})_{22} \end{bmatrix} \tag{A.9}$$

$$= \begin{bmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \tag{A.10}$$

$$= \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix}, \tag{A.11}$$

In particular, $\exists \left[\begin{smallmatrix} I & A_{12} \\ A_{21} & I \end{smallmatrix}\right]^{-1} \Leftrightarrow \exists (I - A_{12}A_{21})^{-1} \Leftrightarrow \exists (I - A_{21}A_{12})^{-1}$*, and the possible inverse is necessarily*

$$\begin{bmatrix} I & A_{12} \\ A_{21} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I - A_{12}A_{21})^{-1} & -A_{12}(I - A_{21}A_{12})^{-1} \\ -(I - A_{21}A_{12})^{-1}A_{21} & (I - A_{21}A_{12})^{-1} \end{bmatrix}. \tag{A.12}$$

(d2) $\exists (A_{11})_{\text{left}}^{-1}$ & $\exists (A_{22} - A_{21}(A_{11})_{\text{left}}^{-1}A_{12})_{\text{left}}^{-1} \implies \exists A_{\text{left}}^{-1}$*, (and (A.4), (A.8 and (A.9) hold for these left inverses);*

$\exists (A_{11})_{\text{right}}^{-1}$ & $\exists (A_{22} - A_{21}(A_{11})_{\text{right}}^{-1}A_{12})_{\text{right}}^{-1} \implies \exists \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]_{\text{right}}^{-1}$ *(and (A.3) and (A.7) hold for these right inverses);*

$\exists (A_{11})_{\text{left}}^{-1}$ & $\exists \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]_{\text{right}}^{-1} \implies \exists (A_{22} - A_{21}(A_{11})_{\text{left}}^{-1}A_{12})_{\text{right}}^{-1}$;

$\exists (A_{11})_{\text{right}}^{-1}$ & $\exists \left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]_{\text{left}}^{-1} \implies \exists (A_{22} - A_{21}(A_{11})_{\text{right}}^{-1}A_{12})_{\text{left}}^{-1}$.

(e1) **(Coprime)** *Given* A_1, A_2, B_1*, there is* B_2 *s.t.* $\left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} B_1 & B_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$ *iff* $A_1B_1 = I$ & $A_2B_1 = 0$ & $A_2B_2' = I$ *for some* B_2'*. If the latter holds, then* $B_2 := (I - B_1A_1)B_2'$ *is as above (but not necessarily unique).*

(e2) Let $\left[\begin{smallmatrix} B_1 & B_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right] = I$*. Then* $\left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} B_1 & B_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \Leftrightarrow A_1B_1 = I$ & $A_2B_2 = I$.

(e3) Let $\left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} B_1 & B_2 \end{smallmatrix}\right]$*. Then, for a given* A_1'*, there is* B_2' *s.t.* $\left[\begin{smallmatrix} A_1' \\ A_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} B_1 & B_2' \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$ *iff* $A_1'B_1 = I$*. If* $A_1'B_1 = I$*, then* $\left[\begin{smallmatrix} A_1' \\ A_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} B_1 & B_2' \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \Leftrightarrow B_2' = (I - B_1A_1')A_2$ & $\left[\begin{smallmatrix} B_1 & B_2' \end{smallmatrix}\right] \left[\begin{smallmatrix} A_1' \\ A_2 \end{smallmatrix}\right] = I$.

(e4) Let $\left[\begin{smallmatrix} A_1 \\ A_2 \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} B_1 & B_2 \end{smallmatrix}\right]$*. Then, for a given* A_2'*, there are* A_1', B_2' *s.t.* $\left[\begin{smallmatrix} A_1' \\ A_2' \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} B_1 & B_2' \end{smallmatrix}\right]$ *iff* $A_2'B_1 = 0$ *and* $\exists (A_2'B_2)^{-1}$*. If such a solution* A_1', B_2' *exists, then all*

solutions are given by

$$\left(\begin{bmatrix} I & T \\ 0 & I \end{bmatrix}\begin{bmatrix} A_1 \\ A_2' \end{bmatrix}\right)^{-1} = \begin{bmatrix} B_1 & B_2(A_2'B_2)^{-1} \end{bmatrix}\begin{bmatrix} I & -T \\ 0 & I \end{bmatrix}, \quad T \in \mathcal{A}. \qquad \text{(A.13)}$$

(e5) If $\left[\begin{smallmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{smallmatrix}\right]\left[\begin{smallmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$ *and* $\exists (A_{22})_{\text{left}}^{-1}, (B_{11})_{\text{right}}^{-1}$ *(resp.* $\exists (A_{11})_{\text{left}}^{-1}, (B_{22})_{\text{right}}^{-1}$*), then* $\exists A_{22}^{-1}, B_{11}^{-1}$ *(resp.* $\exists A_{11}^{-1}, B_{22}^{-1}$*) and* $BA = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] = AB$.

As above, in parts (f1)–(h1) we require that the operators belong to the $\mathcal{A}$*'s that are compatible with the formulae; that is,* $x \in \mathcal{A}(X,Y)$*,* $y, q \in \mathcal{A}(Y,X)$*,* $z \in \mathcal{A}(X)$ *and* $w \in \mathcal{A}(Y)$*, where* $X, Y \in \mathcal{X}$.

(f1) $\exists y^{-1} = (xy)^{-1}x$ *if* $\exists x^{-1}, (xy)^{-1}$ *or if* $\exists (xy)^{-1}, (yx)^{-1}$.

(f2) $\exists x^{-1} \Leftrightarrow \exists (x^n)^{-1}$ *for* $n \in \{1, 2, 3, \ldots\}$.

(f3) $(x+y)^{-1} - x^{-1} = -(x+y)^{-1}yx^{-1}$ *if* $\exists (x+y)^{-1}, x^{-1}$.

(f4) $z(I-z)^{-1} = (I-z)^{-1}z$ *&* $I + z(I-z)^{-1} = (I-z)^{-1}$ *if* $\exists (I-z)^{-1}$.
$I + z(I-z)_{\text{right}}^{-1} = (I-z)_{\text{right}}^{-1}$ *if* $\exists (I-z)_{\text{right}}^{-1}$.

(f5) $z(I+z)^{-1} = (I+z)^{-1}z$ *&* $z(I+z)^{-1} = I - (I+z)^{-1}$ *if* $\exists (I+z)^{-1}$.

(f6) $\exists (I-xy)^{-1} \Leftrightarrow \exists (I-yx)^{-1} = I + y(I-xy)^{-1}x$ *&* $y(I-xy)^{-1} = (I-yx)^{-1}y$.
$\exists (I-xy)_{\text{left}}^{-1} \Leftrightarrow \exists (I-yx)_{\text{left}}^{-1} = I + y(I-xy)_{\text{left}}^{-1}x$ *(analogously for* $()_{\text{right}}^{-1}$*).*

(f7) $\{y \,|\, \exists (I-xy)^{-1}\} = \{(I+qx)^{-1}q \,|\, \exists (I+qx)^{-1}\}$ *and* $\{q \,|\, \exists (I+qx)^{-1}\} = \{y(I-xy)^{-1} \,|\, \exists (I-xy)^{-1}\}$*, for a fixed x.*

(g1) $\exists (I \pm yw^{-1}x)^{-1} = I \mp y(w \pm xy)^{-1}x$ *if* $\exists w^{-1}, (w \pm xy)^{-1}$.

(g2) $\exists (z + yw^{-1}x)^{-1} = z^{-1} - z^{-1}y(w + xz^{-1}y)^{-1}xz^{-1}$ *if* $\exists w^{-1}, z^{-1}, (w + xz^{-1}y)^{-1}$.

(h1) Let $x, y, z \in \mathcal{A}(X)$ *be invertible,* $X \in \mathcal{X}$*. Then* $xyz = y^{-1} \Leftrightarrow zyx = y^{-1}$.

The claims in the lemma are "the best possible ones", i.e., there is nothing superfluous in the conditions and nothing (that one would expect) missing in the conclusions. For any candidate "better" claims there are counter-examples, even in the case where both X and Y are the Hilbert space $\ell^2(\mathbf{N})$ (and when $\mathcal{A}$ is any the symbols listed in (1.)–(5.)).

Recall that if $A \in \mathcal{B}(X,Y)$, where X and Y are Hilbert spaces (and $\mathcal{A} = \mathcal{B}$), then $\exists A_{\text{left}}^{-1} \Leftrightarrow A^*A \gg 0$ (and $\exists A_{\text{right}}^{-1} \Leftrightarrow AA^* \gg 0$), by Lemma A.3.1(c1) (if $\mathcal{A} = \text{Hom}$, then $\exists A_{\text{left}}^{-1} \Leftrightarrow \text{Ker}(A) = 0$).

Part (c4) is an example about how to apply the claims of the lemma in two different $\mathcal{A}$'s.

One often needs to apply the lemma with induction, e.g., an upper triangular matrix is invertible if its diagonal blocks (operators) are, by (b1) (e.g., consider $\left[\begin{smallmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{smallmatrix}\right]$ first partitioned as $\left[\begin{array}{c|cc} A_{11} & A_{12} & A_{13} \\ \hline 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{array}\right]$).

Exchanging two rows (resp. columns) of an operator matrix corresponds to exchanging the corresponding columns (resp. rows) of its inverse matrix.

In case of linear operators between vector spaces, multiplication of the kth row of an operator matrix by a scalar α corresponds to the multiplication of the kth column of its (left, right) inverse by α^{-1}.

(To be completely rigorous, we mention that we have tacitly used the convention that the zero mapping is the only homomorphism between vector spaces with different scalar fields. However, we do not even think that anybody would like to use the lemma for such homomorphisms.)

Finally, we note that we often define operators by using the converse part of (a4).

Proof of Lemma A.1.1: It is obvious that the assumptions (algebraic laws) of Remark A.1.3 are satisfied in case (1.) as well as when $\mathcal{A}(X,Y)$ is the set of functions $S \to \mathcal{B}(X,Y)$, for a fixed set S. Thus, the same laws hold for cases (2.)–(5.) too, except that one has to verify that $\mathcal{A}$ is closed under addition and multiplication. For $\mathrm{L}^\infty_{\text{strong}}$, this verification is given in Lemma F.1.3(b); for H^p_∞ and $\mathrm{H}^p_{\text{strong},\infty}$ this follows from Lemma F.3.5. For SR_ω, UR_ω and ULR_ω this is contained in Lemma 6.2.5 for complex Hilbert spaces, and the general case is analogous. Classes of Definitions 2.6.1 and 2.6.3 will also do, by Lemma 2.6.2 and Theorem 2.6.4. For all other classes this is straightforward.

Although we have assumed U and Y to be Hilbert spaces in the definition of $\mathrm{TI}(U,Y)$ and its subspaces, there is no need for this in the definition itself. Thus we have been able to state (3.)–(5.) for arbitrary complex Banach spaces (assuming the definitions to be extended correspondingly).

(a1)&(a2) These are obvious.

(a3) Case "Hom" is obvious, case "$\mathcal{B}$" is [Rud73, Theorem 2.12(b)], case "TI" is Lemma 2.2.1(b) (because any finite-dimensional Banach spaces are Hilbert spaces) and implies the other cases (because in them $\mathcal{A}$ is a subclass of TI; see also Theorem 2.1.2 for H^∞).

(a4) In cases (1.)–(5.) the assumptions of Lemma A.1.1 are clearly satisfied. The claims "$P_i \in \mathcal{A}(Y,Y_i)$" and "$L_j \in \mathcal{A}(X_j,X)$" are clearly true as well.

The first matrix multiplication claim follows from $\sum_j A_{ij}x_j = \sum_j P_iAL_jx_j = P_iA\sum_j L_jx_j = P_iAx$, and the second from

$$P_iABx = \sum_j A_{ij} \sum_k B_{jk}y_k = \sum_k (\sum_j A_{ij}B_{jk})y_k. \tag{A.14}$$

The final claims are even easier.

(b1) This is obvious.

(b2) (3)&(4) $B_{11}A_{11} = I = A_{11}B_{11}$, $A_{22}B_{22} = I = B_{22}A_{22}$, hence $\exists A_{11}^{-1}, A_{22}^{-1}$. (1)&(2) $A_{22}B_{21} = 0 = B_{21}A_{11}$, so, if $\exists (A_{11})^{-1}_{\text{right}}$ tai $\exists (A_{22})^{-1}_{\text{left}}$, then $B_{21} = 0$, so the claim follows from (3). (7) $B_{11}A_{11} = I_{X_1}$ and $A_{22}B_{22} = I_{Y_2}$, so if A_{11} or A_{22} belongs to $\mathcal{GB} + \mathcal{BC}$ (and X_1, X_2, Y_1, Y_2 are Banach spaces), then it is invertible, by Lemma A.3.4(B3), hence then the claim follows from (1) or (2). (5)&(6) Work as in (7).

(Similarly, $\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}$.)

"A counter-example" ($B_{12} = 0$ is not enough): $\exists \begin{bmatrix} R & P \\ 0 & L \end{bmatrix}^{-1} =: \begin{bmatrix} L & 0 \\ P & R \end{bmatrix}$ (one can interpret these as the left and right shift on $\ell^2(\mathbf{Z}) = \ell^2(\mathbf{Z}_-) \times \ell^2(\mathbf{N})$, respectively); here R [L] is the right [left] shift on $\ell^2(\mathbf{N})$ and $P := I - RL$ is the projection to the first element; note that $RL = I - P$, $LR = I = P + RL$ and $LP = 0 = PR$.

(c1) & (c2) Assume that $\exists B_{11}^{-1}$ (alternatively, as a right inverse only) $AB = I \implies 0 = A_{21}B_{11} + A_{22}B_{21} \implies A_{21} = -A_{22}B_{21}B_{11}^{-1}$, $I = A_{21}B_{12} + A_{22}B_{22} = -A_{22}B_{21}B_{11}^{-1}B_{12} + A_{22}B_{22} = A_{22}(B_{22} - B_{21}B_{11}^{-1}B_{12})$ from this we get $A_{22}^{-1}A_{21} = -B_{21}B_{11}^{-1}$ if $\exists A_{22}^{-1}, B_{11}^{-1}$), in particular, $\exists (A_{22})_{\text{right}}^{-1}$.

Similarly, $I = BA$ implies that $0 = B_{11}A_{12} + B_{12}A_{22}$, hence $A_{12} = -B_{11}^{-1}B_{12}A_{22}$, and $I = B_{21}A_{12} + B_{22}A_{22} = B_{22}A_{22} - B_{21}B_{11}^{-1}B_{12}A_{22} = (B_{22} - B_{21}B_{11}^{-1}B_{12})A_{22}$ (this holds for merely left-invertible B_{11} too), hence $\exists A_{22}^{-1} = B_{22} - B_{21}B_{11}^{-1}B_{12}$.

The converse claims (assuming the invertibility of A_{22} are analogous.

If $\dim X_2 < \infty$, then $\exists (A_{22})_{\text{left}}^{-1} \Leftrightarrow \exists A_{22}^{-1} \Leftrightarrow \exists (A_{22})_{\text{right}}^{-1}$; if $\dim X_1 < \infty$, then (c2) implies that $\exists (A_{22})_{\text{left}}^{-1} \Leftrightarrow \exists (B_{11})_{\text{left}}^{-1} \Leftrightarrow \exists B_{11}^{-1} \Leftrightarrow \exists A_{22}^{-1}$ etc.

(c3) This follows from equation $\begin{bmatrix} A_{11} & -A_{12} \\ -A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

(c4) This follows from (c1) by setting $\mathcal{A} = \mathrm{TIC}_\infty$ (we present this application for an easy reference).

(d1) $\begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}$, hence these three operator matrices must be invertible (because two of them are, by the assumptions), therefore so is $A_{22} - A_{21}A_{11}^{-1}A_{12}$, by (b2)(1). Thus,

$$\begin{aligned}
\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} &= \begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}^{-1} \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \\
&= \begin{bmatrix} I & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ 0 & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \\
&= \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{bmatrix}.
\end{aligned}$$

The case for $\begin{bmatrix} I & A_{12} \\ A_{21} & I \end{bmatrix}$ follows from this and (f6).

(We may have $\nexists A_{22}^{-1}$, e.g., $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$.)

(d2) If $\exists (A_{11})_{\text{left}}^{-1}$, then (A.4) applies; if $\exists (A_{11})_{\text{right}}^{-1}$, then (A.3) applies; the other claims follow from these (and (b1)).

(e1) It is obvious that the three conditions are necessary. Conversely, by taking $B_2' = (I - B_1A_1')A_2$ we get $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \end{bmatrix} = I$.

(e2) It is obvious that the three conditions are necessary. For the converse, assume that $B_1A_1 + B_2A_2 = I$ and $A_1B_1 = I$. By (d1), the invertibility of $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} I & A_1B_2 \\ A_2B_1 & I \end{bmatrix}$ follows from that of $I - A_2B_1I^{-1}A_1B_2 = I - A_2(I - B_2A_2)B_2 = I - A_2B_2 + A_2B_2A_2B_2 = I$.

(e3) The condition $FB_1 = I$ is obviously necessary. Conversely, for $E = (I - B_1F)B_2$, one soon verifies that $\begin{bmatrix} F \\ A_2 \end{bmatrix} \begin{bmatrix} B_1 & E \end{bmatrix} = I$ and $\begin{bmatrix} B_1 & E \end{bmatrix} \begin{bmatrix} F \\ A_2 \end{bmatrix} = I$, hence

$\begin{bmatrix} B_1 & E \end{bmatrix} = \begin{bmatrix} F \\ A_2 \end{bmatrix}^{-1}$ is unique.

(e4) The condition $GB_1 = 0$ is obviously necessary. Conversely, assume $GB_1 = 0$. Then $\begin{bmatrix} Z \\ G \end{bmatrix}$ is invertible iff $\begin{bmatrix} Z \\ G \end{bmatrix} \begin{bmatrix} B_1 & B_2 \end{bmatrix} = \begin{bmatrix} ZB_1 & ZB_2 \\ GB_1 & GB_2 \end{bmatrix} = \begin{bmatrix} ZB_1 & ZB_2 \\ 0 & GB_2 \end{bmatrix}$ is, i.e., iff ZB_1 and GB_2 are; therefore the condition $\exists (GB_2)^{-1}$ is necessary. It is also sufficient, because we can choose $Z = A_1$, $W = B_2$.

Whenever $\begin{bmatrix} Z \\ G \end{bmatrix}^{-1} = \begin{bmatrix} B_1 & W \end{bmatrix}$, we have $ZB_1 = I$ & $GB_2 = I$, hence then $\begin{bmatrix} Z \\ G \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \begin{bmatrix} I & ZB_2 \\ 0 & I \end{bmatrix}^{-1}$; therefore all solutions are of the form (A.13); conversely, it is obvious that each $T \in \mathcal{A}(Y_2, Y_1)$ determines a solution.

(e5) We have $\exists A_{22}^{-1}, B_{11}^{-1}$, by (c2). Now $A_{22}^{-1}A_{21} = B_{21}B_{11}^{-1}$, so we obtain the (right-)invertibility of B from (d2), because $B_{22} - B_{21}B_{11}^{-1}B_{12} = B_{22} - A_{22}^{-1}A_{21}B_{12} = A_{22}^{-1}$ is invertible. The other case is obtained from this by permuting the rows and columns of A and B.

(f1) Clearly $(xy)^{-1}x$ is a left inverse of y, so we only have to show that $y(xy)^{-1}x$ is invertible (hence equal to I).

1° $xy(xy)^{-1}xx^{-1} = I \Longrightarrow y(xy)^{-1}x = I$. 2° If $(yx)y(xy)^{-1}x = yx$ is invertible, then so is $y(xy)^{-1}x$.

(f2) Set $y := x$, $x := x^{n-1}$ in (f1), and use induction.

(f3) $\Leftrightarrow I - (x+y)x^{-1} = -yx^{-1} \Leftrightarrow -yx^{-1} = -yx^{-1}$.

(f4) $I + z(I-z)^{-1} = (I - z + z)(I-z)^{-1} = (I-z)^{-1} = (I-z)^{-1}(I - z + z) = I + (I-z)^{-1}z$.

(f5) Work as in (f4).

(f6) $[I + y(I-xy)^{-1}x][I - yx] = I - yx + y(I-xy)^{-1}x - y(I-xy)^{-1}xyx = I + y[-x + (I-xy)^{-1}(I-xy)x] = I$ & $[I - yx][I + y(I-xy)^{-1}x] = I - yx + y(I-xy)^{-1}x - yxy(I-xy)^{-1} = I + y[-x + (I-xy)(I-xy)^{-1}x] = I$. $(I-yx)^{-1}y = [I + y(I-xy)^{-1}x]y \overset{(f4)}{=} y + yxy(I-xy)^{-1} = y[I + xy(I-xy)^{-1}] = y[(e - xy + xy)(I-xy)^{-1}] = y(I-xy)^{-1}$.

N.B: $y(I-y)^{-1}_{\text{left}} \neq (I-y)^{-1}_{\text{left}}y$ when $y = I - R$, and R and $L = R^{-1}_{\text{left}}$ are the right and left translation on $\ell^2(\mathbf{N})$, respectively (because $(I-R)L \neq L(I-R)$).

(f7) 1° $y := (I+qx)^{-1}q \Longrightarrow I - xy = I - xq(I+xq)^{-1}$ $[(f6)]$ $= (I + xq - xq)(I+xq)^{-1} = (I+xq)^{-1}$, hence $\exists (I-xy)^{-1} = (I+xq)$.
2° $q := y(I-xy)^{-1} \Longrightarrow I + qx = I + yx(I-yx)^{-1} = (I-yx)^{-1}$ resp.
3° Let q be s.t. $\exists (I+qx)^{-1}$. $y := y_q := (I+qx)^{-1}q$. Then $q_{y_q} := y(I-xy)^{-1} = y(I+xq) = (I-qx)^{-1}q(I+xq) = (I-qx)^{-1}(I+qx)q = q$, hence each such q is determined by the y_q it determines, i.e., $q = q_{y_q}$, hence $q \mapsto y_q$ is an injection. Similarly, $\exists q := q_y := y(I-xy)^{-1} \Longrightarrow y_{q_y} = (I+qx)^{-1}q = (I-yx)y(I-xy)^{-1} = y$, hence each such y is determined by the q_y it determines; therefore this correspondence is bijective.

(g1) Set $z = I$ in (g2) (and "$y = \pm y$").

(g2) Set $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} := \begin{bmatrix} z & y \\ \mp x & w \end{bmatrix} =: \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}^{-1}$ in (d1). The formula $B_{11} = A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}$ is obtained from (d1). Part (c1) implies that $\exists B_{11}^{-1} = A_{11} - A_{12}A_{22}^{-1}A_{21}$.

(h1) Assume that $xyz = y^{-1}$. Then $zyxyz = zyy^{-1} = z$, hence $zyxy = I$, hence $zyx = y^{-1}$. The converse is obtained analogously. □

Remark A.1.2 *Lemma A.1.1 is also valid in case*

(5.) $\mathcal{X}$ is the collection of groups, and $\mathcal{A}(X,Y)$ is the set of group homomorphisms $X \to Y$,

provided we replace the assumptions of the form "$\dim X = \dim Y < \infty$ *by "Inv*(X,Y)*", and "If X_1 or X_2 is finite-dimensional" in (c1) by "If Inv(X_1,X_1) or Inv(X_2,X_2)", where Inv(X,Y) is the assumption on X and Y that*

any $A \in \mathcal{A}(X,Y)$ is left (resp. right) invertible iff it is invertible. □

Except for the projection–imbedding claims, the proof of Lemma A.1.1 is based only on the properties listed below, hence its conclusions are true under a wider set of circumstances:

Remark A.1.3 *Let $\mathcal{X}$ be a set. Let $\mathcal{A}(X,Y)$ be a group with a zero $0 = 0_{\mathcal{A}(X,Y)}$, for all $X,Y \in \mathcal{X}$, and let for each $X,Y,Z \in \mathcal{X}$ there be an operation $\mathcal{A}(X,Y) \times \mathcal{A}(Y,Z) \to \mathcal{A}(X,Z)$ be defined in such a way that this operation is associative and distributive:*

$$(AB)C = A(BC) \text{ and } (A+A')(B+B') = AB + AB' + A'B + A'B' \tag{A.15}$$

for all $A, A' \in \mathcal{A}(X,Y)$, $B, B' \in \mathcal{A}(Y,Z)$, $C \in \mathcal{A}(X,Z)$; $\mathcal{A}(X)$ is a ring with a unit $I = I_{\mathcal{A}(X)}$ (we allow for $I = 0$, that is, for $\mathcal{A}(X) = \{0\}$); and the rules

$$A0 = 0 = 0B, \quad A = AI, \quad IB = B \qquad (A \in \mathcal{A}(X,Y),\ B \in \mathcal{A}(Y,Z)) \tag{A.16}$$

are obeyed.

As explained below, the structure $(\mathcal{A}, \mathcal{X})$ determines naturally another structure that also satisfies the above assumptions: If $n,m,N \in \{1,2,3,\ldots\}$, $A_{ij}, C_{ij} \in \mathcal{A}(X_j,Y_i)$ $B_{jk} \in \mathcal{A}(Y_i,Z_k)$ ($i = 1,\ldots,m$, $j = 1,\ldots,n$, $k = 1,\ldots,N$), we denote by A the $\mathcal{A}$ matrix

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{bmatrix}; \tag{A.17}$$

similarly for B and C. For such representations, we use the standard matrix operation rules, that is, $(AB)_{ij} := \sum_k A_{ik}B_{kj}$, $(A+C)_{ij} := A_{ij} + C_{ij}$, $(-A)_{ij} := -A_{ij}$.

For these $\mathcal{A}$ and $\mathcal{X}$, all the conclusions of Lemma A.1.1 except (a4) hold if we make the Inv(X,Y) replacements of Remark A.1.2 and replace all expressions of the form $\mathcal{A}(X_1 \times X_2, Y_1 \times Y_2)$, $X_1, X_2, Y_1, Y_2 \in \mathcal{X}$, by the set of the corresponding $\mathcal{A}$-matrices. □

In particular, Lemma A.1.1 applies when $\mathcal{A}(X)$ is a ring with a unit and $\mathcal{X} = \{X\}$ (here X need not stand for anything).

Notes

Many of the formulae of this section and some additional ones are often used in control theory; some of them and further formulae are presented in many matrix calculus textbooks.

A.2 Topological spaces

Noli turbare circulos meos!

— Archimedes (287–212 BC)

In this section, we very briefly introduce metric spaces and other topological spaces and present some well-known lemmas. For most of this monograph, basic knowledge on Hilbert and Banach spaces (Section A.3) is sufficient. Therefore, the reader may skip this section unless (s)he wishes to go through also the proofs of certain auxiliary results.

The details presented here are sufficient for most of our purposes, but a reader wishing to know more may consult any book on topological spaces; also many books on functional analysis (e.g., [Rud86] or [Rud73]) contain the basic theory of topological spaces.

Since any metric space is a topological space, it is adviseable to visualize the topological spaces as metric ones (or as $\mathbf{R}^2$) to get an intuitive picture on the general case (all concepts defined here are direct generalizations of those defined for metric spaces). Most spaces that we meet are metric (see, e.g., [Rud76] for the theory of metric spaces).

The most important nonmetrizable topologies (see Exercises 2.1 and 3.15 of [Rud73]) are the weak and weak* topologies of infinite-dimensional Banach spaces.

A *topology* on a set Q is a collection $\mathcal{T}$ of subsets of Q s.t. $\emptyset, Q \in \mathcal{T}$, and $\mathcal{T}$ is closed under finite interjections and arbitrary unions. We call the pair $(Q,\mathcal{T})$ (or just Q when there is no ambiguity about $\mathcal{T}$ or when we do not need to specify it) a *topological space*.

The elements of $\mathcal{T}$ are called *open* and their complements are called *closed*. If $E \subset Q$, then $E^o := \cup\{V \in \mathcal{T} \,\big|\, V \subset E\}$ is the *interior* and $\bar{E} := \cap\{F \,\big|\, F^c \in \mathcal{T}$ and $E \subset F\}$ is the *closure* of E. We call $\partial E := \bar{E} \cap \overline{E^c}$ the *boundary* of E. A set $K \subset Q$ is *compact* if $\mathcal{V} \subset \mathcal{T}$ and $K \subset \cup\mathcal{V}$ imply that $K \subset \cup\mathcal{V}'$ for some finite $\mathcal{V}' \subset \mathcal{V}$.

Let also $(Q_2,\mathcal{T}_2)$ be a topological space. Then $f : Q \to Q_2$ is *continuous* (i.e., $f \in \mathcal{C}(Q;Q_2)$) if $f^{-1}[V] \in \mathcal{T}$ for any open $V \in \mathcal{T}_2$. If f is a continuous bijection and also its inverse is continuous, then f is called a *homeomorphism*. The sets Q and Q_2 are called *homeomorphic* if there is a homeomorphism $Q \to Q_2$. The set $Q \times Q_2$ is usually equipped by its *product topology*, which is the smallest topology containing $\mathcal{T} \times \mathcal{T}_2$.

We equip any subset E of Q with the topology $\{V \cap E \,\big|\, V \in \mathcal{T}\}$ *inherited from* Q (unless something else is indicated).

A sequence $\{q_n\} \subset Q$ *converges to* $q \in Q$, i.e., $\lim_{n\to+\infty} q_n = q$, iff, for each open set $V \ni q$, there is $N_V \in \mathbf{N}$ s.t. $q_n \in V$ for all $n \geq N_V$. If this is the case, we also say that $\{q_n\}$ *converges in* Q.

If $\mathcal{T}$ and $\mathcal{T}'$ are topologies on Q and $\mathcal{T} \subset \mathcal{T}'$, then $\mathcal{T}$ is *weaker* than $\mathcal{T}'$ and $\mathcal{T}'$ is *stronger* than $\mathcal{T}$. It obviously follows that if T is a nonempty collection of topologies on Q and $\mathcal{T}_0$ is the weakest element of T, then $\mathcal{T}_0 = \cap_{\mathcal{T}\in T}\mathcal{T}$.

A *neighborhood* of $q_0 \in Q$ means an open set containing q_0. A point $q_0 \in Q$ is called a *limit point* of $E \subset Q$ if $q_0 \in \overline{E \setminus \{q_0\}}$, equivalently, if every neighborhood of q_0 contains an element of $E \setminus \{q_0\}$.

A set $E \subset Q$ is *disconnected* if there are nonempty sets $A, B \subset Q$ s.t. $E = A \cup B$ and $A \cap \bar{B} = \emptyset = \bar{A} \cap B$. Otherwise E is *connected*. An *interval* is a nonempty connected subset of $\mathbf{R}$.

Let Q be a set. A function $d : Q \times Q \to [0, +\infty)$ is called a *metric* if 1. $d(x,y) = 0 \Leftrightarrow x = y$, 2. $d(x,y) = d(y,x)$, and 3. $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in Q$. A set equipped with a metric is called a *metric space* (thus, we call Q or (Q,d) a metric space if the above conditions are satisfied). A topological space (or a topology) is called *metrizable* if it is induced by some metric.

If d is a metric on a set Q, then we usually equip Q with the topology induced by d, which consists of arbitrary unions of open balls $D(q,r) := \{q' \in Q \,\big|\, d(q',q) < r\}$ ($q \in Q$, $r > 0$). It follows that d becomes continuous $Q \to \mathbf{R}$.

We recall from [Rud76] that if (Q,d) and (Q',d') are metric spaces, $q_0 \in Q$, $q_0' \in Q'$ and $f : Q \to Q'$, then $\lim_{q \to q_0} f(q) = q_0'$ iff $\lim_{n \to \infty} f(q_n) = q_0'$ whenever $q_n \to q_0$ (this is not true for all topological spaces Q, not even for all TVSs). A map $f : Q \to Q'$ satisfying $d(x,y) = d'(f(x), f(y))$ for all $x, y \in Q$ is called an *isometry* (or *isometric*). If $f : Q \to Q'$ is s.t. for all $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ s.t. $d(x,y) < \delta_\varepsilon \Longrightarrow d'(f(x), f(y)) < \varepsilon$ for all $x, y \in Q$, then f is *uniformly continuous*.

Let (Q,d) be a metric space. A sequence $\{q_n\} \subset Q$ is a *Cauchy-sequence* in Q if for each $\varepsilon > 0$, there is $N_\varepsilon \in \mathbf{N}$ s.t. $d(q_n, q_m) < \varepsilon$ for all $n, m \geq N_\varepsilon$. It easy to show that any converging sequence is a Cauchy-sequence. If any Cauchy-sequence in Q converges in Q, then the metric space Q is called *complete*.

A compact subset of a metric space is closed and bounded; the converse holds for subsets of $\mathbf{R}^n$ (or of $\mathbf{C}^n$). If $K \subset Q$ is compact and $f \in C(K, Q')$, then $f[K]$ is compact in Q'.

If Q is a metric space and $K, E \subset Q$, then we set $d(q,K) := \inf_{q' \in K} d(q,q')$, $d(E,K) := \inf_{q \in E,\, q' \in K} d(q,q')$; if K is compact and nonempty and $q \notin K$, then $d(q,K) = \min_{q' \in K} d(q,q') > 0$. If $a \in Q$, then, obviously, $d(a,E) = 0$ iff $a \in \bar{E}$.

Lemma A.2.1 *Let $\emptyset \neq K \subset V \subset Q$, where Q is a metric space, V is open and K is compact.*

(a) If $f : K \to \mathbf{R}$ is continuous, then $\min_{q \in Q} f(q)$ and $\max_{q \in Q} f(q)$ exist.

(b) If $f : K \to Q'$ is continuous, where Q' is a metric space, then f is uniformly continuous, i.e., for all $\varepsilon > 0$ there is $\delta > 0$ s.t. $x, y \in K \& d(x,y) < \delta \Longrightarrow d(f(x), f(y)) < \varepsilon$.

(c) If $V \neq Q$, then $d(K, V^c) = d(a, V^c) > 0$ for some $a \in K$.

(d) If $V \neq Q = \mathbf{R}^n$, then there are $a \in K$ and $b \in V^c$ s.t. $d(a,b) = d(K, V^c) := \inf_{x \in K, y \in V^c} d(x,y)$.

Proof: (a)&(b) These are Theorems 4.16 and 4.19 of [Rud76], respectively.

(c) Assume that $V \neq Q$. Then $d(\cdot, V^c)$ attains a minimum on K, by (a), hence $d(K, V^c) = d(a, V^c)$ for some $a \in K$. Since $a \notin V^c = \overline{V^c}$, we have $d(a, V^c) > 0$.

(d) Assume that $V \neq Q = \mathbf{R}^n$. Choose $x \in K$. Choose $R > 0$ s.t. $K \subset \bar{D_R}$ and $\bar{D_R} \cap V^c \neq \emptyset$, where $D_R := \{q \in Q \,\big|\, |q| < R\}$. The set $F := \bar{D_{3R}} \cap V^c$ is closed and bounded, hence compact. By (a), there is

$a \in K$ s.t. $d(a,F) = d(K,F)$. By (a), there is $b \in F'$ s.t. $d(a,b) = d(a,F)$. But $d(K,V^c) = \min\{d(K,F), d(K,V^c \setminus F)\} = d(K,F)$ (since $d(K,F) < 2R$ and $d(K,V^c \setminus F) > 2R$) and $d(K,F) = d(a,b)$. □

The following well-known fact is sometimes handy:

Lemma A.2.2 *Let $V \subset \mathbf{R}^n$ be open. Then V is the union of an at most countable number of disjoint open, connected sets.*

Thus, open $V \subset \mathbf{R}$ is the union of an at most countable number of disjoint open intervals.

Proof: Let $\{q_k\}_{k \in \mathbf{N}} \subset \mathbf{R}^n$ be dense (e.g., enumerate $\mathbf{Q}^n$). For each $k \in \mathbf{N}$, take $V_k := \emptyset$ if $q_k \notin V$ or if $q_k \in V_j$ for some $j < k$; otherwise let V_k be the connected component of V that contains q_k. Then $V = \cup_{k \in \mathbf{N}} V_k$, and the sets V_k are open (because, obviously, $\partial V_k \subset \partial V$). □

The following lemma often allows one to work on an open set with a compact closure instead of a general open set:

Lemma A.2.3 (Compact exhaustion of Ω) *Let $\Omega \subset \mathbf{R}^n$ be open. Set $K_k := \{q \in \Omega \mid |q| \leq k \ \& \ d(q, \Omega^c) \geq 1/k\}$ ($k \in \mathbf{N}+1$). Then each K_k is a compact subset of Ω, $K_1 \subset K_2 \subset \cdots$, $\Omega = \cup_k K_k^o$, and each compact $K \subset \Omega$ is contained in some K_k^o.*

Note that each K_k is compact.

Proof: This quite obvious. If $K \subset \Omega$ is compact, then some finite subset of sets K_k^o contains K, hence some K_k contains K. □

Notes

All of this is well known, see any book on topology (e.g., [Bredon], [Kelley] or even [Rud86]) for more.

A.3 Hilbert and Banach spaces

> *Mathematicians often resort to something called Hilbert space, which is described as being n-dimensional. Like modern sex, any number can play.*
>
> — James Blish, "The Quincunx of Time"

In this section, we present certain standard definitions and useful facts on Hilbert and Banach spaces; see any text on functional analysis (e.g., [Rud86] or [Rud73]) for their basic properties (and for details for most facts presented below).

Recall that in this appendix, the scalar field of any vector space is assumed to be $\mathbf{K}$ (that is $\mathbf{R}$ or $\mathbf{C}$). If $\mathbf{K}=\mathbf{R}$, then, naturally, conjugation $\alpha \mapsto \bar{\alpha}$ becomes the identity operator on $\mathbf{K}$, conjugate-linear is the same as linear, and sesquilinear is the same as bilinear.

A set A is closed under a function if the function maps the elements of A into A.

Let $\mathcal{T}$ be a topology for a vector space X. If $\{x\}^c \in \mathcal{T}$ for each $x \in X$, and sum and scalar multiplication on X are continuous, then X (or $(X,\mathcal{T})$) is called a *topological vector space (TVS)*. Most important examples of TVSs are Banach spaces, and we need other TVSs only in some external references. See, e.g., [Rud73] for more on TVSs.

A *normed space* is a vector space X equipped with a function (*norm*) $\|\cdot\|$: $X \to [0,+\infty)$ satisfying $\|\alpha x\| = |\alpha|\|x\|$, $\|x+y\| \le \|x\|+\|y\|$ and $\|x\|=0 \Rightarrow x=0$ for all $x \in X,\ \alpha \in \mathbf{K}$. We often write $\|\cdot\|_X := \|\cdot\|$ to distinguish between the norms of different normed spaces.

An *inner product space* is a vector space H equipped with a function (*inner product*) $\langle\cdot,\cdot\rangle : H \times H \to \mathbf{K}$ satisfying $\langle y,x\rangle = \overline{\langle x,y\rangle}$, $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$, $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$, $\langle x,x\rangle \ge 0$ and $\langle x,x\rangle = 0 \Rightarrow x=0$ for all $x,y,z \in H,\ \alpha \in \mathbf{K}$. We often write $\langle\cdot,\cdot\rangle_H := \langle\cdot,\cdot\rangle$ to distinguish between the inner products of different normed spaces.

We equip any inner product space with norm $\|x\| = \langle x,x\rangle^{1/2}$ (it follows that any inner product space is a normed space). We equip any normed space X by the metric $d(x,y) := \|x-y\|_B$ (it follows that any normed space is a metric space).

A complete normed space is called a *Banach space*. A complete inner product space is called a *Hilbert space* (in particular, any Hilbert space is a Banach space).

The space $\mathbf{K}^n$ ($n \in 1+\mathbf{N}$) is equipped with the canonical inner product $\langle x,y\rangle_{\mathbf{K}^n} := \sum_{k=1}^n x_k \bar{y}_k$.

Let X and Y be normed spaces. We set $\|x\|_X := +\infty$ for $x \notin X$. We equip $X \times Y$ with the product topology, i.e., with the norm $\|(x,y)\|_{X\times Y}^2 := \|x\|_X^2 + \|y\|_Y^2$ (or some equivalent norm, such as $\max\{\|x\|_X, \|y\|_Y\}$ or $(\|x\|_X^2+\|y\|_Y^2)^{1/2}$).

By $\mathcal{B}(X,Y)$ we denote the normed space of continuous (i.e., bounded) linear operators $L : X \to Y$ with norm $\|L\|_{\mathcal{B}(X,Y)} := \sup_{\|x\|_X \le 1} \|Lx\|_Y$, and vector operations $(\alpha L + \beta L')x := \alpha(Lx) + \beta(L'x)$ ($L, L' \in \mathcal{B}(X,Y)$, $\alpha,\beta \in \mathbf{K}$, $x \in X$). We usually write $Lx := L(x)$ when L is linear. One easily verifies that $\mathcal{B}(X,Y)$ is a Banach space iff Y is a Banach space.

The space $X^* := \mathcal{B}(X,\mathbf{K})$ is called the *dual space of Xdual space (X^*)* (see also Remark A.3.22), and we set $X^{**} := (X^*)^*$. We identify $x \in X$ and the element

$x^{**} : \Lambda \mapsto \Lambda x$ of X^{**}. If all elements of X^{**} are of this form, then X is called *reflexive* (and then X is isometrically isomorphic to X^{**}, hence a Banach space). Any Hilbert space is reflexive.

Let X and Y be normed spaces. To each $T \in \mathcal{B}(X,Y)$ corresponds a unique $T^{\mathrm{B}} \in \mathcal{B}(Y^*,X^*)$ s.t. $y^*(Tx) = (T^{\mathrm{B}}y^*)x$ for all $x \in X$ and $y^* \in Y^*$; moreover, $\|T^{\mathrm{B}}\| = \|T\|$. We call T^{B} the *Banach adjoint* of T.

Let X and Y be Hilbert spaces. To each $T \in \mathcal{B}(X,Y)$ corresponds a unique $T^{\mathrm{H}} \in \mathcal{B}(Y,X)$ s.t. $\langle Tx,y\rangle_Y = \langle x,T^{\mathrm{H}}y\rangle_X$ for all $x \in X$ and $y \in Y$; moreover, $\|T^{\mathrm{H}}\| = \|T\|$. We call T^{H} the *Hilbert adjoint* of T. In a Banach space context, T^* denotes T^{B}, whereas in a Hilbert space context, T^* denotes T^{H} (unless we use pivot spaces, see Definition A.3.23 and Lemma A.3.24).

Let H be a Hilbert space and let B be a Banach space. A set $E \subset H$ is *orthonormal* if $\langle x,y\rangle = 0$ whenever $x,y \in E$, $x \neq y$ and $\langle x,x\rangle = 1$ for all $x \in E$. If $\operatorname{span} E$ is dense in H, then E is an *orthonormal basis* of H. An operator $P \in \mathcal{B}(B)$ is a *projection* if $P^2 = P$ (here $P^2 := PP$). A projection $P \in \mathcal{B}(H)$ is an *orthogonal projection* if $P = P^*$ (equivalently, $\operatorname{Ran}(P) = \operatorname{Ker}(P)^{\perp}$, by Theorem 12.14 of [Rud73]).

By $\mathcal{BC}(X,Y)$ we denote the set of linear mappings $T : X \to Y$ that are *compact*, that is, such that $\overline{\{Tx \,|\, x \in X,\ \|x\| < 1\}}$ is compact. It follows that $\mathcal{BC}(X,Y)$ is a subspace of $\mathcal{B}(X,Y)$.

The *weak topology* of X is the weakest topology on X on which each $\Lambda \in X^*$ is continuous. The *weak* topology* of X^* is the weakest topology on X on which each of the maps $x : \Lambda \mapsto \Lambda x$ $(x \in X)$ is continuous. We do not use these two topologies except when we explicitly say so. (If X is finite-dimensional, then $X = X^*$ and the weak, weak* and original (normed) topologies of X and X^* coincide with the standard Euclidean topology of X. In general, weak and weak* topologies need not be even metrizable.) See, e.g., [Rud73] for further details.

We set $\mathcal{GB}(X,Y) := \{L \in \mathcal{B}(X,Y) \,|\, LT = I_Y \,\&\, TL = I_X \text{ for some } T \in \mathcal{B}(Y,X)\}$ (and we write $T := L^{-1}$ for T,L as above) hence $\mathcal{GB}(X)$ becomes the subgroup of invertible operators ($\mathcal{G}$ for "group"). It follows that $L \in \mathcal{GB}(X,Y)$ iff $L \in \mathcal{B}(X,Y)$ and L is onto and one-to-one (i.e., the inverse is necessarily bounded), by (part (c3)(ii) of) Lemma A.3.4(F1).

For $L \in \mathcal{B}(X,Y)$ we set $\sigma(L) := \{\zeta \in \mathbf{K} \,|\, \zeta - L \notin \mathcal{GB}(X,Y)\}$, $\rho(L) := \sup|\sigma(L)|$ (see Lemma A.3.3; recall that $\zeta - L := \zeta I - L$).

An element of $\mathcal{GB}(X,Y)$ is called a *(Banach) isomorphism* of X *onto* Y. Whenever the range of LX is closed in Y (hence a Banach space itself if Y is complete) and L is as isomorphism of X onto LX, then L is an isomorphism of X *into* Y. (Note the standard abuse of language: we do not require an isomorphism to be an *isometry*, that is, to satisfy $\|Lx\| = \|x\|$ for all $x \in X$; a more rigorous term would be "a topological vector space isomorphism".)

Thus, if X and Y are Hilbert spaces, then a map $L \in \mathcal{B}(X,Y)$ is an isometric isomorphism of X onto Y iff L is *unitary*, i.e., $L^*L = I = LL^*$ (note that we often use same I (resp. 0) for identity (resp. zero) mappings in different groups.

Two norms, say $\|\cdot\|$ and $\|\cdot\|'$, defined on a normed space X are called *equivalent* if they define same topology. This is the case iff there are $\varepsilon, M \in (0,\infty)$ s.t. $\varepsilon\|x\| \leq \|x\|' \leq M\|x\|$ for all $x \in X$.

Let H be a Hilbert space. By $\langle x,y\rangle_H$ (often just $\langle x,y\rangle$), we denote the (sesquilinear, i.e., $\langle \alpha x, \beta y\rangle = \alpha\bar{\beta}\langle x,y\rangle$) inner product $H \times H \to \mathbf{K}$. It is well known that $H^* = \{x \mapsto \langle x,y\rangle_H \,|\, y \in H\}$, hence H is reflexive. The mapping $\Lambda_x : y \mapsto \langle y,x\rangle_H$ is conjugate-linear (i.e., $\Lambda_{\alpha x} = \bar{\alpha}\Lambda_x$, $\Lambda_{x+y} = \Lambda_x + \Lambda_y$), isometric and onto, hence $\Lambda \in \mathcal{GC}(H,H^*)$ (an element of $\mathcal{GC}$ is called a *homeomorphism*).

Let $x,y \in H$, $A \subset H$ and $B \subset H$. If $\langle x,y\rangle = 0$, then we write $x \perp y$; if $a \perp b$ for all $a \in A$ and $b \in B$, then we write $A \perp B$. Obviously, $A^\perp := \{x \in H \,|\, x \perp A\}$ is a closed subspace of H.

Let $T,S \in \mathcal{B}(H)$. We call T *nonnegative* [*positive*] and write $T \geq 0$ [$T > 0$] if $\langle Tx,x\rangle \geq 0$ [> 0] for all $x \in H \setminus \{0\}$. By $T \geq S$ we mean that $T = T^*$, $S = S^*$, and $T - S \geq 0$. By $T \gg 0$ we mean that $T \geq \varepsilon I$ for some $\varepsilon > 0$; in this case we say that T is *uniformly positive* (note that $(x_k) \mapsto (x_k/k)$ is is positive and one-to-one but not uniformly positive on $\ell^2(\mathbf{N})$). If $T^*T \gg 0$, then we say that T is *coercive* (or bounded from below). These relations (among others) are studied within the following lemma.

Lemma A.3.1 (Hilbert spaces) *Let H, U and Y be complex Hilbert spaces (much of this holds for real ones too).*

(a1) Each orthonormal base of H has the same cardinality; we denote this cardinality by $\dim H$.[1]

(a2) H is isomorphic to $\ell^2(\mathcal{E})$ iff $\mathcal{E}$ is a set of cardinality $\dim H$.

(a3) If $T \in \mathcal{B}(U,H)$, then $\dim \overline{\mathrm{Ran}(T)} \leq \dim U$.

(a4) The following are equivalent:

(i) $\dim U \leq \dim H$;
(ii) $T^*T \gg 0$ for some $T \in \mathcal{B}(U,H)$;
(iii) $T^*T = I$ for some $T \in \mathcal{B}(U,H)$;
(iv) some $S \in \mathcal{B}(H,U)$ is onto.

(a5) The following are equivalent:

(i) $\dim U = \dim H$;
(ii) $\mathcal{GB}(U,H) \neq \emptyset$;
(iii) $T^*T = I_U$ and $TT^* = I_H$ for some $T \in \mathcal{B}(U,H)$.

(a6) Let $\dim U \geq \dim H < \infty$. *Then $T \in \mathcal{B}(U,H)$ is invertible iff $T^*T \gg 0$.*

(b1)$[\mathbf{T} \gg \mathbf{0}]$ *Let $T \in \mathcal{B}(H)$ and $E \in \mathcal{GB}(H)$. Then $T \gg 0 \overset{\text{def}}{\Leftrightarrow} T \geq \varepsilon I$ for some $\varepsilon > 0 \Leftrightarrow T = T^*$ & $\sigma(T) \subset (0,\infty) \Leftrightarrow T = P^2$ for some $P \gg 0 \Leftrightarrow T = X^2$ for some $X \in \mathcal{GB}(H) \Leftrightarrow E^*TE \gg 0 \Leftrightarrow T \in \mathcal{GB}$ & $T \geq 0 \Leftrightarrow T \in \mathcal{GB}$ & $T^{-1} \geq 0$.*

If $T \geq \varepsilon I$, then $\varepsilon^{-1} \geq T^{-1} \geq \|T\|^{-1} I \gg 0$.

(b2)$[\mathbf{T} > \mathbf{0}]$ *We have $T > 0$ iff $T \geq 0$ and* $\mathrm{Ker}(T) = \{0\}$.

[1] We also use the standard notation $\dim B < \infty$ [$\dim B = \infty$] to mean that a vector space B is [in]finite-dimensional; if B is a Hilbert space, then, obviously, $\dim B < \infty$ iff $\dim B$ is finite.

(b3)$[\mathbf{T} \geq \mathbf{0}]$ *Let* $T \in \mathcal{B}(H)$. *Then* $T \geq 0$ *iff* $T = T^*$ *and* $\sigma(T) \subset [0,\infty)$.

(b4) Let $T \geq 0$. *Then there is a unique* $T^{1/2} \geq 0$ *s.t.* $(T^{1/2})^2 = T$. *Moreover,* $T^{1/2} \gg 0$ *(resp.* > 0*) iff* $T \gg 0$ *(resp.* > 0*). Furthermore,* $ST = TS \Leftrightarrow ST^{1/2} = T^{1/2}S$ *for all* $S \in \mathcal{GB}$.

(b5) Let $T_n \geq T_{n+1} \geq A$ *for all* $n \in \mathbf{N}$ *for some* $A = A^* \in \mathcal{B}(H)$. *Then there is* $T \geq A$ *s.t.* $T_n x \to Tx$ *for all* $x \in H$.

(b6)$[\mathbf{T} = \mathbf{T}^*]$ *Let* $T \in \mathcal{B}(H)$. *Then* $T = T^*$ *iff* $T^*T = TT^*$ *and* $\sigma(T) \subset \mathbf{R}$.

(b7) Assume that $S, T \in \mathcal{GB}(H)$ *and* $ST = TS$. *If* $T \geq S \gg 0$, *then* $S^{-1} \geq T^{-1} \gg 0$; *if* $T > S \gg 0$, *then* $S^{-1} > T^{-1} \gg 0$; *if* $T \gg S \gg 0$, *then* $S^{-1} \gg T^{-1} \gg 0$.

(b8) If $T \in \mathcal{B}(H)$, $\varepsilon > 0$, *and* $|\langle Tx, x\rangle| \geq \varepsilon \|x\|^2$ *for all* $x \in H$, *then* $T \in \mathcal{GB}(H)$.

(b9) If $I \geq T \gg 0$, *then* $\|I - T\| < 1$.

(c1)$[\mathbf{R}^*\mathbf{R} \gg \mathbf{0}]$ *Let* $R \in \mathcal{B}(U,H)$. *The following are equivalent:*

- *(i)* $\|Rx\| \geq \varepsilon\|x\|$ *for all* x *for some* $\varepsilon > 0$, *i.e.,* R *is* coercive *("uniformly bounded from below");*
- *(ii)* $R^*R \geq \varepsilon^2 I$ *for some* $\varepsilon > 0$;
- *(iii)* $\mathrm{Ran}(R^*) = U$, *i.e.,* R^* *is* onto;
- *(iv)* $\mathrm{Ran}(R)$ *is closed and* $\mathrm{Ker}(R) = \{0\}$;
- *(v) there is* $L \in \mathcal{B}(H,U)$ *s.t.* $LR = I$;
- *(vi) there is some closed subspace* $H_1 \subset H$ *and some* $S \in \mathcal{B}(H_1,H)$ *s.t.* $\begin{bmatrix} R & S \end{bmatrix} \in \mathcal{GB}(U \times H_1, H)$;
- *(vii)* $X^*X = R^*R$ *for some* $X \gg 0$;
- *(viii)* $P \gg 0 \Longrightarrow R^*PR \gg 0$;
- *(viii')* $\|G - I_H\| < 1 \Longrightarrow R^*GR \in \mathcal{GB}(U)$;
- *(ix)* $\|RX\|_{\mathcal{B}(*,H)} \geq \varepsilon \|X\|_{\mathcal{B}(*,*)}$ *for some* $\varepsilon > 0$ *whenever* X *is linear (in particular,* X *is bounded iff* RX *is bounded);*
- *(x)* $R \in \mathcal{GB}(U, \mathrm{Ran}(R))$;
- *(xi) There is* $\varepsilon > 0$ *s.t. for all* $x \in H \setminus \{0\}$ *there is* $y \in H \setminus \{0\}$ *s.t.* $\langle y, Rx\rangle \geq \varepsilon \|x\|\|y\|$.

Moreover, if (i) holds, then the following hold:

- *(1)* $\|(R^*R)^{-1}\| \leq \varepsilon^{-2}$ *and* $\|(R^*R)^{-1}R^*\| \leq \varepsilon^{-1}$
- *(1') if* $R \in \mathcal{GB}$, *then* $\|R^{-1}\| \leq \varepsilon^{-1}$.
- *(2) For any* $r > 0$ *we can choose* H_1 *and* S *in (vi) so that* $\|\begin{bmatrix} R & S \end{bmatrix}\| \leq \max\{\|R\|, r\}$ *and* $\|\begin{bmatrix} R & S \end{bmatrix}^{-1}\| \leq \max\{\varepsilon^{-1}, r^{-1}\}$. *If also* $\dim U < \infty$ *and* $H = U \times U'$, *then we can take* $S \in \mathcal{B}(U', H)$ *without affecting the above norms.*
- *(3) If* $R^*R = I$, *then we can have* $\begin{bmatrix} R & S \end{bmatrix}$ *unitary (see (e3)).*

However $R^*R \gg 0$ & $G \in \mathcal{GB} \not\Rightarrow R^*GR \in \mathcal{GB}$ *in general (cf. (viii'); even* $\sigma(G) \subset [0, +\infty)$ *is not sufficient).*

(c2) Let $M \in \mathcal{B}(\mathbf{K}^m)$, $N \in \mathcal{B}(\mathbf{K}^m, \mathbf{K}^n)$. *There is* $L \in \mathcal{B}(\mathbf{K}^n, \mathbf{K}^m)$ *such that* $M + LN$ *is invertible iff* $M^*M + N^*N > 0$.

(c3) Let $R \in \mathcal{B}(U,H)$. *Then the following are equivalent:*

(i) $R \in \mathcal{GB}(U,H)$ *(i.e., R is invertible)*
(ii) R is injective and onto;
(iii) R is coercive and has a dense range;
(iv) $R^*R > 0$ *and* $RR^* \gg 0$*;*
(v) $R^*R \gg 0$ *and* $RR^* \gg 0$*;*
(vi) $R^* \in \mathcal{GB}(H,U)$.

If $\dim U = \dim H < \infty$, *then one more equivalent condition is that R is one-to-one (equivalently,* $\det R \neq 0$ *or R is onto).*

(c4) Let $R = R^* \in \mathcal{B}(H)$. *Then* $R \in \mathcal{GB}(H) \Leftrightarrow R^*R \gg 0$.

(c6) Let $R \in \mathcal{B}(H)$ *and* $R^*R = RR^*$. *Then* $\|R\| = \sup_{\|x\|=1} |\langle x, Rx\rangle| = \rho(R)$.

(c7) Let $R \in \mathcal{B}(H,U)$. *Then* $\mathrm{Ker}(R^*) = \mathrm{Ran}(R)^\perp$.

(c8) Let $R \in \mathcal{B}(H,U)$. *Then* $\mathrm{Ker}(R^*R) = \mathrm{Ker}(R)$ *and* $\overline{\mathrm{Ran}(R)} = \overline{\mathrm{Ran}(RR^*)}$.

(c9) Let $R \in \mathcal{B}(U,H)$. *Then the following are equivalent:*

(i) R is one-to-one (i.e., $\mathrm{Ker}(R) = \{0\}$*);*
(ii) R^* *has a dense range;*
(iii) $R^*R > 0$

If $\dim U < \infty$, *then (i)–(iii) hold iff* $R^*R \gg 0$ *(cf. (c1)).*

(c10) Let $A \subset H$. *Then* $A^\perp$ *is a closed subspace of H and* $(A^\perp)^\perp = \overline{\mathrm{span}(A)}$.

(c11) $\sup_{|\alpha|=1} \|\alpha x + y\| \geq \|x\|^2 + \|y\|^2$ *(*$x, y \in H$*).*

In (d)–(f) we assume that $A \in \mathcal{B}(U,Y)$, $B \in \mathcal{B}(U,H)$. *In (d)–(e2) we assume that* $\gamma > 0$ *(the* TI_ω *claims concern cases when U and Y are* L^2_ω *spaces for some* $\omega \in \mathbf{R} \cup \{\infty\}$*).*

(d) We have $\gamma^2 - B^*B \geq 0 \Leftrightarrow \|B\| \leq \gamma$. *Analogously,* $\gamma^2 - B^*B \gg 0 \Leftrightarrow \|B\| < \gamma$.

(e1) $\|Ax\|^2 - \gamma^2\|Bx\|^2 \leq 0$ *for all* $x \in H \Leftrightarrow \gamma^2 B^*B \geq A^*A \Leftrightarrow A = LB$ *for some* $\|L\| \leq \gamma$
(if $A, B \in \mathrm{TI}_\omega$, *then we can take* $L \in \mathrm{TI}_\omega$*).*

(e2) The following are equivalent:

(i) $\gamma^2 B^*B - A^*A \gg 0$*;*
(i') $\|Ax\|^2 - \gamma^2\|Bx\|^2 \leq -\varepsilon\|x\|^2$ *for all* $x \in H$*;*
(i") $\gamma^2 B^*B \gg A^*A \geq 0$*;*
(ii) $A = LB$ *for some* $\|L\| < \gamma$ *and* $B^*B \gg 0$*;*
(iii) $B^*B \gg 0$ *and* $\|AB^{-1}_{\mathrm{left}}\| < \gamma]$, *where* $B^{-1}_{\mathrm{left}} := (B^*B)^{-1}B^*$.

If $B \in \mathcal{GB}$, *then (i) (hence (i)–(iii)) is also equivalent to* $\|AB^{-1}\| < \gamma$. *If (i) holds and* $\dim U \geq \dim H < \infty$, *then* $B \in \mathcal{GB}$.

In (ii) we can take $L = AB^{-1}_{\mathrm{left}}$*; this way we get* $L \in \mathrm{TI}_\omega$, *if* $B, A \in \mathrm{TI}_\omega$.

(e3) *Let* $B^*B = I$. *Then* $\overline{B} := \left[\begin{smallmatrix} B & 0 \\ 0 & I \end{smallmatrix}\right] \in \mathcal{GB}(U \times H_1, H)$ *is an* unitary extension *of* B, *where* $H_1 := \operatorname{Ran}(B)^\perp \subset H$.

(f) *Suppose that* $\dim U < \infty$. *Then* $A = LB$ *for some* $L \in \mathcal{B}(H,Y) \Leftrightarrow \operatorname{Ker}(B) \subset \operatorname{Ker}(A) \Leftrightarrow \operatorname{Ran}(A^*) \subset \operatorname{Ran}(B^*) \Leftrightarrow A^* = B^*L^*$ *for some* $L^* \in \mathcal{B}(F,H) \Leftrightarrow$ *(e1) holds for some* $\gamma > 0$.

(g1) *Let* $A, B \in \mathcal{B}(H)$. *If* $\langle x, Ax\rangle = \langle x, Bx\rangle$ *for all* $x \in H$, *then* $A = B$.

(g2) *Let* $A_t \in \mathcal{B}(H)$ $(t \in \mathbf{R}_+)$. *If* $\lim_{t\to+\infty}\langle x, A_t x\rangle = 0$ *for all* $x \in H$, *then* $\lim_{t\to+\infty}\langle y, A_t x\rangle = 0$ *for all* $x, y \in H$.

(g3) *Let* X *be a vector space. Let* $A, B : X \to U$, *and* $C, D : X \to Y$ *be linear. If* $\langle Ax, Bx\rangle = \langle Cx, Dx\rangle$ *for all* $x \in X$, *then* $\langle Ax, Bz\rangle = \langle Cx, Dz\rangle$ *for all* $x, z \in X$.

In (h1)–(k2) we assume that $x, y, x_n, y_n \in H$ *for all* $n \in \mathbf{Z}_+$, $T_n, T \in \mathcal{B}(H,Y)$ *for all* $n \in \mathbf{Z}_+$, *and* $S_n, S \in \mathcal{B}(Y,U)$ *for all* $n \in \mathbf{Z}_+$. *We write* $x_n \rightharpoonup x$ *if* $\langle x_n, z\rangle \to \langle x, z\rangle$ *for all* $z \in H$ *(i.e., if* $x_n \to x$ *weakly) (here always* $n, m, k \to +\infty$*).*

We say that T_n *converges to* T uniformly *(resp.* strongly, weakly*), if* $\|T_n - T\| \to 0$ *(resp.* $T_n x \to Tx$ *for all* $x \in H$, $T_n x \rightharpoonup Tx$ *for all* $x \in H$*).*

(h1) *If* $T_n \to T$ *weakly, then* $\{T_n\}$ *is uniformly bounded.*

(h2) *If* $T_n x$ *converges for all* $x \in H$, *then the limiting operator* $\widehat{T} : x \mapsto \lim_n T_n x$ *satisfies* $\widehat{T} \in \mathcal{B}(H,Y)$ *and* $\|\widehat{T}\| \le \liminf_n \|T_n\| < \infty$.

(h3) *If* $\langle y, T_n x\rangle_Y$ *converges for all* $x \in H$, $y \in Y$, *then there is* $\widehat{T} \in \mathcal{B}(H,Y)$ *s.t.* $T_n \to T$ *weakly,* $\|\widehat{T}\| \le \liminf_n \|T_n\| \le \sup_n \|T_n\| < \infty$.

(i1) *If* $x_n \rightharpoonup x$, *then* $\{x_n\}$ *is uniformly bounded and* $\|x\| \le \liminf_n \|x_n\| < \infty$.

(i2) *If* $x_n \to x$ *and* $y_m \rightharpoonup y$, *then* $\langle x_n, y_m\rangle \to \langle x, y\rangle$.

(i3) *If* $\{\langle x_n, y\rangle\}$ *converges for all* $y \in H$, *then* $\{x_n\}$ *converges weakly.*

(i4) *If(f)* $x_n \rightharpoonup x$ *and* $\|x_n\| \to \|x\|$, *then* $x_n \to x$.

(j1) $S_n \to S$ *weakly iff* $S_n^* \to S^*$ *weakly.*

(j2) *If* $S_n \to S$ *weakly and* $T_n \to T$ *strongly, then* $S_n T_n \to ST$ *weakly (but even for* $U = H$ *we may have* $T_n S_n \not\to TS$*).*

(j3) *If* $S_n \to S$ *strongly,* $T_m \to T$ *strongly, and* $x_k \to x$ *strongly, then* $S_n T_m \to ST$ *strongly and* $S_n T_m x_k \to STx$ *strongly.*

(j4) *Let* $T_n \in \mathcal{GB}(H,Y)$ *for all* n, $T_n \to T$ *strongly and* $T_n^{-1} \to P$ *strongly for some* $P \in \mathcal{B}(Y,H)$. *Then* $T \in \mathcal{GB}(H,Y)$ *and* $T^{-1} = P$.

(j5) *Conversely, let* $T_n \in \mathcal{GB}(H,Y)$ *for all* n, *let* $T_n \to T$ *strongly, and let* $\{T_n^{-1}\}$ *be uniformly bounded. Then* $T \in \mathcal{GB}$ *iff* $T_n^{-1} \to P$ *strongly for some* $P \in \mathcal{B}(Y,H)$.

(k1) *Let* $\dim Y < \infty$ *and* $T \in \mathcal{B}(U,Y)$. *Then* $T_n \to T$ *strongly iff* $T_n \to T$ *uniformly.*

(k2) *Let* $\dim U < \infty$ *and* $T \in \mathcal{B}(U,Y)$. *Then* $T_n \to T$ *strongly iff* $T_n \to T$ *weakly.*

In (p1)–(q) we assume that $\left[\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right] \in \mathcal{B}(H_1 \times H_2)$, *where* H_1 *and* H_2 *are Hilbert spaces.*

(p1) Let $AX = B$. *Then* $\left[\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right] \geq 0$ *iff* $A \geq 0$ & $D - X^*AX \geq 0$. *Moreover,* $\left[\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right] \gg 0$ *iff* $A \gg 0$ & $D - X^*AX \gg 0$.

(p2) Let $\exists A^{-1}$. *Then* $\left[\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right] \geq 0$ *iff* $A \geq 0$ & $D - B^*A^{-1}B \geq 0$.

(p3) Let $\dim H_2 < \infty$. *Then* $\left[\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right] \geq 0$ *iff for some* $X \in \mathcal{B}(H_2, H_1)$ *we have* $AX = B$, $D - X^*AX \geq 0$ *and* $A \geq 0$.

(p4) We have $\left[\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right] \gg 0$ *iff* $A \gg 0$ & $D - B^*A^{-1}B \gg 0$.

(q) Let $P \in \mathcal{B}(U, H_1)$, $Q \in \mathcal{B}(U, H_2)$ *and* $D \leq 0$. *If* $T := \left[\begin{smallmatrix} P^* & Q^* \end{smallmatrix}\right] \left[\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right] \left[\begin{smallmatrix} P \\ Q \end{smallmatrix}\right] > 0$ $[\gg 0]$, *then* $P^*P > 0$ $[\gg 0]$.

If $\varepsilon > 0$ *and* $T \geq \varepsilon I$, *then* $P^*P \geq \delta I$, *where* δ *only depends on* ε, $\|Q\|$, $\|A\|$ *and* $\|B\|$.

If $\varepsilon > 0$, $D \leq -\varepsilon I$ *and* $T \geq 0$, *then* $P^*P \geq \delta Q^*Q$ *and* $Q = LP$ *for some* $L \in \mathcal{B}(H_1, H_2)$ *s.t.* $\|L\|_{\mathcal{B}} \leq \delta^{-1/2}$, *where* δ *only depends on* ε, $\|A\|$ *and* $\|B\|$.

(s) Let $T \in \mathcal{B}(U, Y)$, *let* $U_1 \subset U$ *be a finite-dimensional subspace, and let* $\dim T[U_1] = \dim U_1 < \infty$. *Then* T *is of the form* $T = \left[\begin{smallmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{smallmatrix}\right] \in \mathcal{B}(U_1 \times U_1^\perp, Y_1 \times Y_1^\perp)$, *where* $Y_1 := T[U_1]$, *and* $T_{11} \in \mathcal{GB}(U_1, Y_1)$. *Moreover,* T *is (resp. right-, left-)invertible iff* T_{22} *is (resp. right-, left-)invertible.*

(P) Let H_1 *be a closed subspace of* H. *Let* P *be the orthogonal projection of* H *onto* $H_1^\perp$. *Let* $x \in H$. *Then* Px *is the unique element of minimum norm on* $x + H_1$.

As a curiosity, we remark that (c1) demonstrates that finding a left inverse (v), a complement (vi) or a spectral factor (vii) is easy, if we do not have to worry about causality (in contrast to the Corona and spectral factorization theorems).

Results concerning convergence of sequences (such as Monotone Convergence Theorem or (h1)–(k2) above) are applicable for limits of functions between any metric (or “first countable”) spaces (recall from Theorem 4.2 of [Rud76] that $f(t) \to q$ as $t \to T$ iff $f(t_n) \to q$ for each sequence $\{t_n\}$ converging to T (with $t_n \neq T$ for all n)). We shall use this fact without further mention.

Proof of Lemma A.3.1: (We often apply Banach adjoint results for Hilbert adjoints, see Remark A.3.20 for the justification.)

(a1)&(a2) These follow from [Rud86, 4.19].

(a3) If $\dim U < \infty$, these claims are easy to prove, so assume $\dim U = \infty$ and let $\{u_a\}_{a \in A}$ be an orthonormal base of U. Then the cardinality of $Q := \{\sum_{k=1}^n q_k u_{a_k} \,\big|\, q_k \in \mathbf{Q} + i\mathbf{Q},\ a_k \in A \text{ for all } k\}$ is $A =: \dim U$ and Q is dense in U (obviously the cardinality of any dense set is at least that of A).

Obviously, TQ is dense in $\overline{\operatorname{Ran}(T)}$, hence $\dim \overline{\operatorname{Ran}(T)}$ is at the cardinality of Q, i.e., $\dim U$.

(a4) “(iv)⇒(i)” follows from (a3). “(iv)⇔(ii)” is [Rud73, Theorem 4.15]. “(ii)⇒(iii)” Follows by taking $T' := T(T^*T)^{-1/2}$ (cf. (b)). “(iii)⇒(ii)” is trivial. “(i)⇒(iv)” Let $\{u_a\}_{a \in A}$ and $\{h_a\}_{a \in B}$ be bases of U and H, respectively, and $A \subset B$. Set $S \sum_{a \in B} \alpha_a h_a := \sum_{a \in A} \alpha_a u_a$.

(a5) “(i)⇒(iii)&(ii)” Construct S as in “(i)⇒(iv)” above, with $A = B$. The converses follow from (a2).

(a6) Let $T^*T \gg 0$. From (a4) we get that $\dim U = \dim H$. Because T is injective, it is invertible. The converse is trivial.

(b1)&(b3)&(b6) These follow from [Rud73, Chapter 12] and straightforward computations.

(b2) Let $T \geq 0$. Assume that $\langle x, Tx \rangle = 0$. Then $\|T^{1/2}x\|^2 = \langle T^{1/2}x, T^{1/2}x \rangle = 0$ (see (b4)), hence then $Tx = T^{1/2}T^{1/2}x = 0$. Consequently, $x \in \mathrm{Ker}(T)$; the converse is trivial.

(b4) This follows from Theorem 6.2.10 of [Aupetit] (alternatively, from Chapters 11–12 of [Rud73]) and (c3)&(c9).

(b5) (The convergence need not be uniform; e.g., $\pi_{[n,n+1)} \geq \pi_{[n+1,\infty)} \to 0$ strongly but not uniformly on $\ell^2(\mathbf{N})$.)

W.l.o.g. we assume that $A = 0$ (use $T_n \mapsto T_n - A$). Set $S(x) := \lim_n \langle x, T_n x \rangle \geq 0$ ($x \in H$). Now $S(x+y) - S(x) - S(y) = \lim_n 2\,\mathrm{Re}\langle y, T_n x \rangle$ exists for all $x, y \in H$. Apply this to iy to see that $R(y,x) := \lim_n \langle y, T_n x \rangle$ exists for all $x, y \in H$. Obviously, R is sesquilinear and $\|R(y,x)\| \leq \|T_1\|((\|x\| + \|y\|) + \|x\| + \|y\|)$, hence $R(y,x) = \langle y, Tx \rangle$ for some $T \in \mathcal{B}(H)$, by Theorem 12.8 of [Rud73].

Now $\|(T_n - T)^{1/2}x\|^2 = \langle x, (T_n - T)x \rangle \to 0$, hence $\|(T_n - T)x\| \to 0$, for all $x \in H$, since $0 \leq T_n - T \in \mathcal{B}(H)$. Thus, $T_n x \to Tx$ strongly.

(b7) By (b1), $S^{-1}, T^{-1} \gg 0$. Apply twice (b4) to obtain $S^{1/2}T^{1/2} = T^{1/2}S^{1/2}$. It follows that $T^{-1/2}S^{1/2} = S^{1/2}T^{-1/2}$. Let $x \in H$ be arbitrary, and set $y := S^{-1/2}x$. Then

$$\begin{aligned} \langle x, T^{-1}x \rangle &= \langle S^{1/2}y, T^{-1}S^{1/2}y \rangle = \langle T^{-1/2}y, ST^{-1/2}y \rangle \\ &\leq \langle T^{-1/2}y, TT^{-1/2}y \rangle = \langle y, y \rangle = \langle x, S^{-1}x \rangle. \end{aligned} \tag{A.18}$$

Thus, $T^{-1} \leq S^{-1}$. Analogously, if $T > S$, then $T^{-1} < S^{-1}$; if $T \gg S$, then $T^{-1} \ll S^{-1}$.

(b8) Now $\|Tx\| \geq \varepsilon\|x\|$ and $\|T^*x\| \geq \varepsilon\|x\|$ for all $x \in H$, hence this follows from (c3)(i)&(iv). (The converse of (b8) is obviously not true.)

(b9) If $I \geq T \gg \varepsilon I$, then $\langle (I-T)x, x \rangle \leq (1-\varepsilon)\|x\|^2$ for all $x \in H$, hence $\|I - T\| \leq 1 - \varepsilon$, by (c6).

(c1) The equivalence of (i)–(viii) is obtained as follows:

"(i)⇔(ii)": $\|Rx\|^2 = \langle x, R^*Rx \rangle$. "(i)⇔(iii)": See [Rud73, 4.15]. "(i)⇒(iv)": Clearly $\mathrm{Ker}(R) = \{0\}$. If $\{Rx_n\}$ is a Cauchy-sequence, then so is $\{x_n\}$. "(iv)⇒(x)": See [Rud73, 2.12b]. "(x)⇒(i)": See [Rud73, 2.12c]. "(i)⇒(v)": Take $L := (R^*R)^{-1}R^*$ (note: $\|L\| \leq \|R\|/\varepsilon$). "(v)⇒(i)": $\|Rx\| \geq \|x\|/\|L\|$ for all $x \in U$. "(vii)⇔(ii)": See [Rud73, 12.33] ($X = X^* \gg 0$ is unique). "(viii)⇒(ii)": Take $P = I$. "(ii)⇒(viii)": $\langle x, R^*PRx \rangle \geq \varepsilon_P\|Rx\| \geq \varepsilon_P\varepsilon_R\|x\|$ for all x. "(i)⇒(xi)": Take $y = Rx$. "(xi)⇒(i)": Obviously.

"(vi)⇒(v)": If $\begin{bmatrix} L \\ M \end{bmatrix} \begin{bmatrix} R & S \end{bmatrix} = I_{H \times H_1}$, then $LR = I_H$.

"(iii)⇒(vi) & (2)": Let $H_2 := \mathrm{Ran}(R)$, $H_1 := H_2^{\perp}$, and write $R =: \begin{bmatrix} T \\ 0 \end{bmatrix} \in \mathcal{B}(U, H_2 \times H_1)$ (i.e., $T := P_{H_2}R$), so that $T \in \mathcal{GB}(U, H_2)$, because it is 1-1 and onto.

Choose some $r > 0$. Then $S = \begin{bmatrix} 0 \\ rI \end{bmatrix}$ complements R, and the inverse is $\begin{bmatrix} T^{-1} & 0 \\ 0 & r^{-1}I \end{bmatrix}$. The inequalities in (2) follow as in (C1).

If $\dim U < \infty$ and $H = U \times U'$, then $\dim H_2 = \dim U$ implies that $\dim H_1 = \dim U'$, so we may choose some isometric isomorphism $I' \in \mathcal{GB}(U', H_1)$ and replace S by SI' to obtain the rest of (2).

"(ix)$\Leftrightarrow$(i)": Let H' be a Hilbert space, let $X : H' \to H$ be linear, and let (i) hold. Then $\|RXy\| \geq \varepsilon\|Xy\|$ for all $y \in H'$, hence then $\|RX\| \geq \varepsilon\|X\|$. For the converse, let $x \in H$ be given, take some $H' \neq \{0\}$, $y_0 \in H' \setminus \{0\}$, $\Lambda \in H'^*$ with $\Lambda y_0 = \|y_0\|$, $\|\Lambda\| = 1$ [Rud73, 3.3Cor], and define $Xy := x\Lambda y$. Then $\|X\| = \|x\|$, $\|RX\| = \|Rx\|\|\Lambda\| = \|Rx\|$, hence then $\|Rx\| \geq \varepsilon\|X\| = \varepsilon\|x\|$, i.e., (i) holds.

"(viii')$\Rightarrow$(ii)": Set $G := I_H$. "(x)$\Rightarrow$(viii')": Set $G' := P^*GP = I - P^*FP$, where $F := I - G$ and $P \in \mathcal{GB}(H, H_R)$ is the orthogonal projection onto $H_R := \mathrm{Ran}(R)$. Since $\|P^*FP\| \leq \|F\| < 1$, we have $G'|_(H_R) \in \mathcal{GB}(H_R)$, hence $R^*GR = R^*G'R \in \mathcal{GB}(U)$ (since $R \in \mathcal{GB}(H, H_R)$).

(Note: condition $\sigma(G) \subset \mathbf{R}^+$ would not be sufficient in (viii'): if $G' = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$, $G = \left[\begin{smallmatrix} G' & 0 \\ 0 & I \end{smallmatrix}\right] \in \mathcal{B}(H)$ and $H := \ell^2 =: U$, then $\sigma(G) = \{1\}$ but $R^*GR(1,0,0,\ldots) = 0$, where $R : (a,b,c,\ldots) \mapsto (a,-a,b,c,\ldots)$, so that $R^*R \geq I \gg 0$.)

(1)&(1') From (ii) and (b1) we obtain $(R^*R)^{-1} \leq \varepsilon^{-2}$. Now $\langle R(R^*R)^{-1}x, R(R^*R)^{-1}x\rangle = \langle (R^*R)^{-1}x, x\rangle \leq \varepsilon^{-2}\|x\|^2$, hence $\|R(R^*R)^{-1}\| \leq \varepsilon^{-1}$. If $R \in \mathcal{GB}$, then $(R^*R)^{-1}R^* = R^{-1}$.

(2) This was shown in "(iii)$\Rightarrow$(vi) & (2)".

(3) Let $r = 1$ in "(iii)$\Rightarrow$(vi) & (2)".

The final remark was justified before "(1)&(1')" above.

(c2) By (c1), $M^*M + N^*N > 0$ is necessary (of course, $\mathrm{Ker}(\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]) = \{0\}$). Assume that $M^*M + N^*N > 0$. Set $U_2 := \mathrm{Ker}(M)$, $U_1 := U_2^\perp$. Let $x_1, \ldots, x_k$ be a base of U_2, let $x_{k+1}, \ldots, x_m$ be a base of U_1, and let $y_1, \ldots, y_k$ be a base of $(MU_1)^\perp$. Let $S\sum_{j=1}^m \alpha_j x_j := \sum_{j=1}^k \alpha_j y_j$. Then $M + S$ maps $\{x_1, \ldots, x_m\}$ to a base of $\mathbf{K}^m$, hence it is invertible. Now $\mathrm{Ker}(N) \subset U_1 = U_2^\perp = \mathrm{Ker}(S)$, hence, by (f), $S = LN$ for some $L \in \mathcal{B}(\mathbf{K}^n, \mathbf{K}^m)$.

(c3) 1° *(i)–(vi):* This equivalence follows easily from (c1).

2° *(i)$\Leftrightarrow$(vii):* If $R \in \mathcal{GB}(H)$, we can take $y = Rx$. Conversely, $\langle y, Rx\rangle \geq \varepsilon\|x\|\|y\|$ implies that $\|Rx\| \geq \varepsilon\|x\|$, hence $R^*R \gg 0$, by (c1), and analogously $RR^* \gg 0$, so that $R \in \mathcal{GB}(H)$, by "(v)$\Leftrightarrow$(i)".

3° *The last claim:* This is given in almost any matrix calculus textbook.

(c4) This follows from (c3).

(c6)&(c7) These are Theorems 12.25, 11.28(b) and 12.10 of [Rud73] (slightly modified).

(c8) If $x \in \mathrm{Ker}(R^*R)$, then $\|Rx\|^2 = \langle x, R^*Rx\rangle = 0$, hence then $x \in \mathrm{Ker}(R)$. Thus, $\mathrm{Ker}(R^*R) = \mathrm{Ker}(R)$. Consequently, $\mathrm{Ran}(R^*R)^\perp = \mathrm{Ran}(R^*)^\perp$, by (c7), hence $\overline{\mathrm{Ran}(R^*)} = (\mathrm{Ran}(R^*)^\perp)^\perp = \overline{\mathrm{Ran}(RR^*)}$. Because $R^{**} = R$, also the latter claim holds.

(c9) By (c8), we have (i)$\Leftrightarrow$(ii). But (i) holds iff $\|Rx\|^2 = \langle Rx, Rx\rangle > 0$ for all $x \in U$, i.e., iff (iii) holds.

(c10) This is an easy exercise.

(c11) This is obvious (and this is not true for, e.g., $H = \mathrm{L}^\infty(\mathbf{R})$).

(d) The first claim follows from $\gamma^2 - B^*B \geq 0 \Leftrightarrow \gamma^2\|x\|^2 - \|Bx\|^2 \geq 0$ for all $x \in H$. The second follows by replacing γ by some $\gamma - \varepsilon$.

(e1) (This is from [RR, Lemma 1.14].) Assume $\|Ax\|^2 \leq \gamma^2\|Bx\|^2$. Then we can define $L_0 \in \mathcal{B}(H_1, Y)$ (where $H_1 := \overline{B[U]} \subset H$) by $L_0(Bx) := Ax$ to get $\|L_0\| \leq \gamma$. Let P be the orthonormal projection $H \to H_1$ and define $L := L_0 P \in \mathcal{B}(H, Y)$ to get $A = LB$ and $\|L\| = \|L_0\| \leq \gamma$. The other direction is straightforward.

If $B, A \in \mathrm{TI}_\omega$, then $\tau_r H_1 = H_1$ for all $r \in \mathbf{R}$ ($\tau_r Bx = B\tau_r x$) and $L_0 \in \mathrm{TI}_\omega$, hence then $L \in \mathrm{TI}_\infty$ too (if $P : H \to H_1$ is orthonormal, then $L = L_0 P \in \mathrm{TI}_\omega$) (N.B. Even if $A, B \in \mathrm{TIC}_\omega$, we may have to take $L \notin \mathrm{TIC}_\infty$, e.g., $A = I \wedge B = \tau_{-1} \implies L = \tau_1$. Note also that we may have that $B = 0 = C = L$, i.e., that $\nexists B_{\mathrm{left}}^{-1}$).

(e2) "(i)$\Leftrightarrow$(i')$\Leftrightarrow$(i")" and "(iii)$\Rightarrow$(ii)" are obvious. "(ii)$\Rightarrow$(i)" follows from $A^*A = B^*L^*LB \ll B^*B\gamma^2$, which holds by (c1)(viii).

"(i')$\Rightarrow$(iii)": Assume (i') and set $S := (B^*B)^{-1}B^*$. Then $\|ASy\|^2 - \gamma^2\|y\|^2 \leq -\varepsilon\|y\|^2$ for all $y \in \mathrm{Ran}(B)$, i.e., $\|ASy\| \leq (\gamma - \varepsilon')\|y\|$ for all $y \in \mathrm{Ran}(B)$. Because $Sy = 0$ for all $y \in \mathrm{Ran}(B)^\perp = \mathrm{Ker}(B^*)$ and $\mathrm{Ran}(B)$ is closed, this holds for all $y \in H$,

If $B \in \mathcal{GB}$, then $B_{\mathrm{left}}^{-1} = L = B^{-1}$. If (i) holds and $\dim U \geq \dim H < \infty$, then $B \in \mathcal{GB}$, by (a6).

(e3) See the proof of (c1)(3).

(f) The proof goes as that of (e1), except that L_0 is necessarily continuous, because it is linear and $\dim H_1 < \infty$ (note that now $\mathrm{Ker}(B) = \mathrm{Ran}(B^*)^\perp$).

(g2) Now $2\,\mathrm{Re}\langle y, A_t x\rangle = \langle (x+y), A_t(x+y)\rangle - \langle x, A_t x\rangle - \langle y, A_t y\rangle \to 0$. If $\mathbf{K} = \mathbf{C}$, we apply this to y and iy to obtain (g2).

(g1)&(g3) The proofs are analogous to that of (g2).

(h1) This (i.e., that $\|T_n\| \leq M$ for all n for some $M < \infty$) follows from Theorems 2.6 and 3.18 of [Rud73].

(h2) By Theorem 2.8 of [Rud73], $\widehat{T} \in \mathcal{B}(H, U)$. The bound is obtained from $\|\widehat{T}x\| \leq \liminf_n \|T_n\|\|x\| = (\liminf_n \|T_n\|)\|x\|$ for all $x \in H$. By (h1), $\liminf_{n\to+\infty} \|T_n\| < \infty$.

(h3) By (i3), $\widehat{T}x := \text{w-lim}_n T_n x \in Y$ exists for all $x \in H$. Obviously, $\widehat{T} : H \to Y$ is linear. By Theorem 3.18 of [Rud73], $\{T_n x\}$ is bounded for each $x \in H$, hence $M := \sup_n \|T_n\| < \infty$, by Theorem 2.6 of [Rud73]. The inequalities follow from this, hence $\widehat{T} \in \mathcal{B}(H, Y)$ and $T_n \rightharpoonup T$.

(i0) *If $T_n \to T$ strongly and $x_m \to x$, then $T_n x_m \to Tx$*: we have $T_n x_m - Tx = T_n(x_m - x) + (T_n - T)x \to 0$, because $\{T_n\}$ is uniformly bounded, by (h1).

(i1)&(i3) Set $T_n := \langle \cdot, x_n\rangle$ and apply (h1)&(h2) (recall that any $T \in \mathcal{B}(H, \mathbf{K})$ is of form $\langle \cdot, x\rangle_H$ for some $x \in H$).

(i2) As above, $\langle x_n, y_n\rangle - \langle x, y\rangle = \langle x_n - x, y_n\rangle + \langle x, y_n - y\rangle \to 0$, because $\{y_n\}$ is uniformly bounded (alternatively, apply (i0) for $T_n := \langle \cdot, y_n\rangle$, $T := \langle \cdot, y\rangle \in \mathcal{B}(H, \mathbf{K})$).

(i4) This holds because $\langle x - x_n, x - x_n\rangle = \|x\|^2 + \|x_n\|^2 - 2\,\mathrm{Re}\langle x, x_n\rangle \to 0$.

(j1) This is obvious.

(j2) Now $\langle z, S_n T_n x\rangle = \langle S_n^* z, T_n x\rangle \to \langle S^* z, Tx\rangle = \langle z, STx\rangle$ for all x, z, by (i2) and (j1). (But $T_n S_n = I \to I \neq 0 = TS$ for $T_n := \tau(n) \to 0$, $S_n := \tau(-n) \to 0$ on $\ell^2(\mathbf{N}) =: H =: Y =: U$.)

(j3) The first claim follows from (i0), the second from the first and (i0).

(j4) By (j3), $x = T_n^{-1}T_n x \to PTx$ for all $x \in H$, and $y = T_n T_n^{-1} y \to TPy$ for all $y \in Y$, hence $\exists T^{-1} = P$.

(If $T_n \to T$ strongly and $T_n^{-1} \to P$ weakly, then T and P need not be invertible (and if either is, then both are (by (j2), we have $PT = I$, as above, hence $P, T \in \mathcal{GB}$) and $T_n^{-1} \to P$ strongly, by (h1) and (j5)). An example of this is given by $T_n x := (x_{n+1}, x_0, x_1, \cdots, x_n, x_{n+2}, x_{n+3}, \cdots) \in \mathcal{B}(\ell^2(\mathbf{N}))$.)

(j5) "Only if" is given in (j4), so we only prove "if": Let $T \in \mathcal{GB}$. Then $T_n^{-1}y - T^{-1}y = T_n^{-1}(T - T_n)T^{-1}y \to 0$ for all $y \in Y$.

(In fact, if $T, T_n \in \mathcal{GB}$ and $T_n \to T$ strongly, then $T_n^{-1} \to T^{-1}$ iff $\{T_n^{-1}\}$ is uniformly bounded (this is not the case in general (take $T_n = I - 2^{-1/n^3}\tau(n)$ to obtain $\|T_n^{-1}(j^{-2/3})_{j\in\mathbf{Z}}\| \to +\infty$)), by (j5) and (h1).)

(k1) Cf. the proof of (k2).

(k2) Set $m := \dim U$, so that $T = \begin{bmatrix} T^1 & T^2 & \cdots & T^m \end{bmatrix}$. Let $x \in U$. Then $T_n x \to Tx$ (strongly) $\Leftrightarrow T_n^k x \to T^k x$ for all $k = 1, \ldots, m \Leftrightarrow T_n x \rightharpoonup Tx$.

(p1) $\left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]^* \left[\begin{smallmatrix} A & B \\ B^* & D \end{smallmatrix}\right] \left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & 0 \\ 0 & D - X^*AX \end{smallmatrix}\right]$, which is ≥ 0 [$\gg 0$] iff $A, D - X^*AX \geq 0$ [$\gg 0$].

(p2) Set $X := A^{-1}B$ in (p1) to obtain $D - X^*AX = D - B^*(A^{-1})^*AA^{-1}B = D - B^*A^{-1}B$.

(p3) "$\Longleftarrow$": from (p1). "$\Longrightarrow$": Obviously $\mathrm{Ker}(A) \subset \mathrm{Ker}(B^*)$, hence $AX = B$ for some linear (hence bounded, because $\dim H_2 < \infty$) X. The rest follows from (p1).

(p4) See the proofs of (p2) and (p1).

(q) (W.l.o.g. we assume that $U \neq \{0\}$.) 1° *Case* $T > 0$*:* Set $M := \max\{\|Q\|, \|A\|, \|B\|, 1\}$. If $x \in B$ is s.t. $Px = 0$, then $\langle x, Tx\rangle = \langle Qx, DQx\rangle \leq 0$; thus, $T > 0$ implies that $\mathrm{Ker}(P) = \{0\}$, i.e., that $P^*P > 0$.

2° *Case* $T \geq \varepsilon I$*:* Assume that $T \geq \varepsilon I$, $\varepsilon > 0$. Choose $\delta > 0$ s.t. $M\delta^2 + 2\delta M^2 \leq \varepsilon$. If $x \in B$ is s.t. $\|Px\| < \delta$, then $\langle x, Tx\rangle < \delta^2 M + 2\delta M^2 + 0 \leq \varepsilon$ (since $T = P^*AP + P^*BQ + Q^*B^*P + Q^*D^*Q$), a contradiction; hence $P^*P \geq \delta^2 I$.

3° *Case* $D \leq -\varepsilon I$*:* Assume that $D \leq -\varepsilon I$. Then, for any $x \in U$ s.t. $\|x\|_U = 1$, we must have

$$0 \leq |\langle x, Tx\rangle| \leq Mp^2 + 2Mpq - \varepsilon q^2, \tag{A.19}$$

where $p := \|Px\|$, $q := \|Qx\|$, $M := \max\{\|A\|, \|B\|\}$. Thus, if $p = 0$, then $q = 0$, and if $p > 0$, then $0 \leq M + 2Mr - \varepsilon r^2$, where $r := q/p \geq 0$, so that $r \leq \delta^{-1/2}$ for some $\delta := \delta_{M,\varepsilon} > 0$.

Therefore, $p \geq q/r \geq \delta^{1/2}q$, or $\langle Px, Px\rangle \geq \delta\langle Qx, Qx\rangle$, for any $x \in U$. The claim on L follows from (e1).

(s) $\dim T[U_1] = \dim U_1$ means that T is coercive on U_1, i.e., that $T_{11}^* T_{11} \gg 0$. If P_2 is the orthogonal projection of Y onto $Y_1^\perp$, then $P_2 T = 0$ on U_1, hence T is of the form claimed in the lemma. The last claim follows from Lemma A.1.1(b).

(P) The minimum of $\|x - z\|$ ($z \in H_1$) is obtained at $z = (x - Px)$, by Theorem 4.11 of [Rud86]. $\square$

We now show that each $T = T^* \in \mathcal{B}(H)$ can be written as $T = T_+ - T_-$, where $T_\pm \geq 0$:

Lemma A.3.2 (f(T)) *Let $T \in \mathcal{B}(H)$ and $T^*T = TT^*$, where H is a complex Hilbert space. Then there is a (canonical) algebra homomorphism $f \mapsto f(T)$. from the set of all bounded Borel functions on $\sigma(T)$ to $\mathcal{B}(H)$. Moreover, this algebra homomorphism satisfies (here f and g are bounded Borel functions):*

(a) $1(T) = I$, $I(T) = T$, $f(T)g(T) = fg(T)$ *and* $\bar{f}(T) = f(T)^*$, *hence* $f(T)^*f(T) = f(T)f(T)^* = |f|^2(T)$.

(b) $\|f(T)\| \leq \sup|f|$ *(if* $f \in C(\sigma(T))$*, then* $\|f(T)\| = \sup|f|$*).*

(c1) If $f \in C(\sigma(T))$*, then* $\sigma(f(T)) = f(\sigma(T))$.

(c2) We have $\mathrm{Ker}(T) = \chi_{\{0\}}(T)$.

(d) If $S \in \mathcal{B}(H)$ *and* $ST = TS$*, then* $Sf(T) = f(T)S$.

(e1) If $fg = 0$*, then there are closed subspaces* $H_\pm \subset H$ *s.t.* $\mathrm{Ker}(f(T)), \mathrm{Ran}(f(T)) \subset H_+$, $\mathrm{Ker}(g(T)), \mathrm{Ran}(g(T)) \subset H_-$, $H = H_+ + H_-$ *and* $H_+ \cap H_- = \emptyset$.

(e2) If $T = T^*$*, then* $f(T) = f(T)^*$*, and we can have* $H_+ = H_-^\perp$ *in (e1).*

(f1) If $T = T^*$*, then there are orthogonal projections* $P_+, P_0, P_- \in \mathcal{B}(H)$ *s.t.* $I = P_+ + P_0 + P_-$, $T_\pm := TP_\pm = P_\pm T = P_\pm TP_\pm$ *satisfy* $\pm T_\pm > 0$ *on* $H_\pm := \mathrm{Ran}(P_\pm)$, $T = T_+ + T_-$*. Consequently,* $H_0 := \mathrm{Ran}(P_0) = \mathrm{Ker}(T)$*, and*

$$T = \begin{bmatrix} \widetilde{P}_+ & \widetilde{P}_0 & \widetilde{P}_- \end{bmatrix} \begin{bmatrix} \widetilde{T}_+ & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \widetilde{T}_- \end{bmatrix} \begin{bmatrix} \widetilde{P}_+ \\ \widetilde{P}_0 \\ \widetilde{P}_- \end{bmatrix}, \tag{A.20}$$

where $\widetilde{P}_\pm \in \mathcal{B}(H, H_\pm)$, $\widetilde{P}_0 \in \mathcal{B}(H, H_0)$ *and* $\widetilde{T}_\pm \in \mathcal{B}(H_\pm)$ *have the same values on their domains as the corresponding operators without tildes (note that* $J := \begin{bmatrix} P_+ & P_0 & P_- \end{bmatrix} = J^{-*} \in \mathcal{GB}(H)$*, i.e., J is unitary).*

(f2) If $T = T^* \in \mathcal{GB}(H)$*, then* $P_0 = 0$ *and* $\widetilde{T}_+ \gg 0 \gg \widetilde{T}_-$ *in (f1); thus, then* $T = \begin{bmatrix} \widetilde{P}_+ & \widetilde{P}_- \end{bmatrix} \begin{bmatrix} \widetilde{T}_+ & 0 \\ 0 & \widetilde{T}_- \end{bmatrix} \begin{bmatrix} \widetilde{P}_+ \\ \widetilde{P}_- \end{bmatrix}$

Note that if $H = \mathrm{L}^2(\mathbf{R}; U)$ and $T \in \mathrm{TI}(U)$, then $f(T) \in \mathrm{TI}(U)$, by (d).

Proof: We obtain the initial claims, (a) and (b) from Theorem 12.24 of [Rud73]. Claim (c1) follows from Theorems 13.27(c) and 12.22(b) of [Rud73], (c2) from Theorem 12.29(a) of [Rud73], and (d) from 12.24 of [Rud73].

(e1) Set $E := \{z \in \sigma(T) \mid f(z) \neq 0\}$, $P_+ := \chi_E(T)$, $P_- := I - P_+ = \chi_{E^c}(T)$, $H_\pm := \mathrm{Ran}(P_\pm)$. Then $P_\pm = P_\pm^2$ (i.e., $P_\pm$ are projections), $P_+P_- = 0 = P_-P_+$, $P_+ + P_- = I$, $f = P_+fP_+$ and $g = P_-gP_-$, by (a). Therefore, $\mathrm{Ran}(P_\pm) = \mathrm{Ker}(P_\mp)$, $H_+ + H_- = H$, $H_+ \cap H_- = \emptyset$ and $H_\pm$ are closed (see Section 5.15 of [Rud73]).

(e2) By Lemma A.3.1(b6), we have $\sigma(T) \subset \mathbf{R}$, hence $f = \bar{f}$, so that $f(T)^* = f(T)$. In (e1) we have $P_\pm = P_\pm^*$ (i.e., $P_\pm$ is orthogonal), hence $H_+ = H_-^\perp$, by Theorem 12.14(c) of [Rud73].

(f1) (The statement means that $\widetilde{T}_+ x := T_+ x := Tx$ for all $x \in H_+$ etc.) Set $P_\pm := \chi_{\pm(0,+\infty)}(T)$, $P_0 := \chi_{\{0\}}(T)$, so that $T_\pm = (s \mapsto s\chi_{\pm(0,+\infty)}(s))(T)$ etc. Now we obtain the claims easily from (c1), (c2), (e1) and (e2) (e.g., by (c1) we have $\pm T_\pm \geq 0$ on H and $0 \neq Tx = P_\pm T_\pm P_\pm x$ for $x \in H_\pm$, hence $\pm T_\pm > 0$ on $H_\pm$).

(f2) If $T \in \mathcal{GB}$, then $J^{-1}TJ \in \mathcal{GB}$, hence then $\widetilde{T}_\pm \in \mathcal{GB}(H_\pm)$, hence $\pm\widetilde{T}_\pm \gg 0$, by Lemma A.3.1(b1). □

By a *Banach algebra* we mean a complex Banach space A equipped with a multiplication $A \times A \to A$ (not necessarily commutative) and possessing a *unit* $I = I_A$ that satisfy $xI = Ix = x$, $x(yz) = (xy)z$, $(x+y)z = xz+yz$, $x(y+z) = xy+xz$, $\alpha(xy) = (\alpha x)y = x(\alpha y)$, $\|xy\| \leq \|x\|\|y\|$ and $\|I\| = 1$ for all $x,y,z \in A$, $\alpha \in \mathbf{C}$. (In the literature, Banach algebras are not always required to possess a unit.)

Some examples of Banach algebras are $\mathcal{B}(B)$, $\mathrm{TI}_\omega(B)$, $\mathrm{TIC}_\omega(B)$ or $\mathrm{H}^\infty(\Omega;B)$ for any complex Banach space B, real number ω and open set $\Omega \subset \mathbf{C}$.

Let A be a Banach algebra. We define $\mathcal{G}A := \{x \in A \,|\, xy = I = yx$ for some $y \in A\}$. For $x \in A$ we define the *spectrum* $\sigma(x) := \{\zeta \in \mathbf{C} \,|\, \zeta - x \notin \mathcal{G}A\}$ and the *spectral radius* $\rho(x) := \sup|\sigma(x)|$ of x; by $\zeta - x$ we mean $\zeta I - x$, where I is the identity operator on A. Note that these definitions coincide with those made earlier for $\mathcal{B}(B)$. We also set $x^0 := I$, $\mathrm{e}^x := \sum_{n=0}^\infty x^n/n!$ for $x \in A$.

Lemma A.3.3 (Banach algebras) *Let A be a Banach algebra and $x,y,h \in A$. Then we have the following:*

(A0) If $x \in \mathcal{G}A$ and $\|x\| < 1$, then $\exists (I-x)^{-1} = \sum_{k=0}^\infty x^k$.

(A1) Let $x \in \mathcal{G}A$, and set $M := \|x^{-1}\|$. If $\|h\| < 1/2M$, then $x+h \in \mathcal{G}A$ and

$$\|(x+h)^{-1} - x^{-1}\| < 2M^3\|h\|^2 + M^2\|h\| < M. \tag{A.21}$$

(A2) The set $\mathcal{G}A$ of invertible elements is open in A, and $x \mapsto x^{-1}$ is a continuous bijection $\mathcal{G}A \to \mathcal{G}A$.

(A3) If $\{x_n\} \subset \mathcal{G}A$ and $x_n \to x \notin A \setminus \mathcal{G}A$, as $n \to \infty$, then $\|x_n^{-1}\| \to \infty$.

(A4) Let $x \in A \setminus \mathcal{G}A$. If $M > 0$, then there is $\delta > 0$ s.t. $y \in \mathcal{G}A$ & $\|y - x\| < \delta \Rightarrow \|y^{-1}\| > M$.

(r1) $\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n} = \max|\sigma(x)| \leq \|x\|$.

(s1) $\sigma(x) \subset \mathbf{C}$ *is compact and nonempty.*

(s2) $\sigma(xy) \cap \{0\} = \sigma(yx) \cup \{0\}$ *and* $\rho(xy) = \rho(yx)$.

(s3) If p is a nonconstant polynomial, then $\sigma(p(x)) = \{p(\zeta) \,|\, \zeta \in \sigma(x)\}$.

(s4) If $x \in \mathcal{G}A$, then $\sigma(x^{-1}) = \{\zeta^{-1} \,|\, \zeta \in \sigma(x)\}$.

Proof: (A0)–(A4) See Theorems 10.7, 10.11, 10.12 and 10.17 of [Rud73]. (A4) follows from (A3).

(r1)&(s1)&(s4) See Theorems 10.13 and 10.28 of [Rud73].

(s2)&(s3) See Theorems 3.1.2 and 3.2.4 of [Aupetit]. □

We remind the reader that a *B^*-algebra* (also called as a C^*-algebra), to which we sometimes refer, is a Banach algebra A with an *involution* $(\cdot)^* : A \to A$ satisfying $(x+y)^* = x^* + y^*$, $(\alpha x)^* = \widehat{\alpha} x^*$, $(xy)^* = y^* x^*$, $x^{**} = x$, and $\|xx^*\| = \|x\|^2$

for all $x, y \in A$, $\alpha \in \mathbf{C}$. Thus, by a *B^*-algebra isomorphism $T : A \mapsto A'$* we mean that T is an algebra isomorphism and a Banach space isomorphism, and that $(Tx)^* = Tx^*$ for all $x \in A$.

The Banach algebras $\mathcal{B}(H)$, $\mathrm{TI}(H)$, $\mathcal{C}_\mathrm{b}(\Omega;\mathcal{B}(H))$ and $\mathrm{L}^\infty(Q;\mathcal{B}(H))$ are B^*-algebras whenever H is a complex Hilbert space, Ω is a topological space and Q is as on p. 907. See [Rud73] for more on B^*-algebras.

Let $x \in B$ and $\Lambda \in B^*$, where B is a Banach space. Then we write $\langle \Lambda, x\rangle_{\langle B^*,B\rangle} := \Lambda x =: \langle x, \Lambda\rangle_{\langle B,B^*\rangle}$ for the bilinear pairing between B and B^* (except that in Hilbert space contexts we use conjugate-linear scalar multiplication and hence we must set $\langle x, \Lambda\rangle_{\langle B,B^*\rangle} := \Lambda x =: \overline{\langle \Lambda, x\rangle}_{\langle B^*,B\rangle}$, so that pairing is still linear in its first argument but no more linear (but conjugate-linear) in its second argument (unless $\mathbf{K} = \mathbf{R}$), so that the pairing coincides with the inner product for Hilbert spaces; see Remark A.3.22 for details).

Lemma A.3.4 (Banach spaces) *Let B_1, B_2, B_3 and B_4 be Banach spaces and denote by $\mathcal{B}(B_i,B_j)$ the continuous and by $\mathcal{BC}(B_i,B_j)$ the compact linear mappings $B_i \to B_j$. Let $\mathcal{GB}(B_i,B_j)$ be the set of invertible elements of $\mathcal{B}(B_i,B_j)$ ($i, j = 1,2,3,4$).*

(B1) $\mathcal{BC}(B_1,B_2)$ is a Banach space.

Let $S \in \mathcal{B}(B_1,B_2)$, $K \in \mathcal{BC}(B_2,B_3)$, and $T \in \mathcal{B}(B_3,B_4)$. Then $SK, KT \in \mathcal{BC}$ (in particular, $\mathcal{BC}(B_1)$ is an ideal of $\mathcal{B}$). Moreover, $K \in \mathcal{BC} \Leftrightarrow K^* \in \mathcal{BC}$.

If $\dim B_1 = \infty$, then $\mathcal{BC}(B_1,B_2) \cap \mathcal{GB}(B_1,B_2) = \emptyset$. If $K[B_2]$ is closed, then $\dim K[B_2] < \infty$. If $\dim B_1 < \infty$ or $\dim B_2 < \infty$, then $\mathcal{BC}(B_1,B_2) = \mathcal{B}(B_1,B_2)$.

If $\{S_n\} \subset \mathcal{B}(B_1,B_2)$, $\dim S_n[B_1] < \infty$ for all n, and $S_n \to S$ as $n \to \infty$, then $S \in \mathcal{BC}(B_1,B_2)$. Conversely, if B_2 is a Hilbert space, then $\mathcal{BC}(B_1,B_2)$ is the closure of finite-dimensional operators.

(B2) Let U and Y be Hilbert spaces and $K \in \mathcal{BC}(U,Y)$. Then there are sequences $\{u_n\}_{n=1}^\infty \subset U$ and $\{y_n\}_{n=1}^\infty \subset Y$ s.t. when P_n [P'_n] is the orthogonal projection of U [Y] onto $\mathrm{span}(u_1,\ldots,u_n)$ [$\mathrm{span}(y_1,\ldots,y_n)$], we have $P'_n K P_n \to K$, as $n \to \infty$.

If $U = Y$, we can choose the sequence $\{u_n\} \subset U$ so that $P_n K P_n \to K$, as $n \to \infty$.

(B3) Let $C \in \mathcal{BC}(B_1,B_2)$ and $L \in \mathcal{B}(B_1,B_2)$. Then $L(I-C) = I$ iff $(I-C)L = I$. Moreover, $I - L \in \mathcal{BC}$, if $L(I-C) = I$.

(B4) The presentation $G+K$ of an operator $G+K \in \mathcal{GB}+\mathcal{BC}(B_1,B_2) := \{G+K \mid G \in \mathcal{GB}(B_1,B_2),\ K \in \mathcal{BC}(B_1,B_2)\}$ is unique unless $\dim B_1 = \dim B_2 < \infty$.

Moreover, $G+K$ is left-invertible iff it is right-invertible. If $G+K$ is invertible, then $(G+K)^{-1} = G^{-1} + K'$ for some $K' \in \mathcal{BC}$.

The class $\mathcal{GB}+\mathcal{BC}$ is closed w.r.t. composition and adjungation; in particular, $\mathcal{GB}+\mathcal{BC}(B_1)$ is a group. Obviously, $\mathcal{GB} \subset \mathcal{GB}+\mathcal{BC}$.

Each $A \in \mathcal{GB}+\mathcal{BC}(B_1,B_2)$ is a Fredholm operator, i.e., $\mathrm{Ran}(A)$ is closed, $\dim \mathrm{Ker}(A) < \infty$, and $\dim B_1/\mathrm{Ran}(A) < \infty$.

(B5) Let B_1 be complex. The set $\alpha I + K \in \mathbf{C}I + \mathcal{BC}(B_1)$ is a closed subalgebra of the Banach algebra $\mathcal{BC}(B_1)$, and this subalgebra is also closed w.r.t. inverses and adjoints. The presentation of an element of $\mathbf{C}I + \mathcal{BC}(B_1)$ is unique unless $\dim B_1 < \infty$.

Furthermore, $\alpha I + K$ is left-invertible iff it is right-invertible, and if $\alpha I + K$ is invertible, then $(\alpha I + K)^{-1} = \alpha^{-1} I + K'$ for some $K' \in \mathcal{BC}$.

If $\dim B_1 = \infty$ and $K \in \mathcal{BC}(B_1)$, then $\|\lambda I + K\| \geq |\lambda|$ for all $\lambda \in \mathbf{C}$.

(C1) Let $\left[\begin{smallmatrix} S & T \\ U & V \end{smallmatrix}\right] \in \mathcal{B}(B_1 \times B_2, B_3 \times B_4)$. Then $\left\| \left[\begin{smallmatrix} S & 0 \\ 0 & V \end{smallmatrix}\right] \right\| = \max(\|S\|, \|V\|)$, $\left\| \begin{bmatrix} S & T \end{bmatrix} \right\| \leq (\|S\|^2 + \|T\|^2)^{1/2}$, and $\left\| \left[\begin{smallmatrix} S \\ U \end{smallmatrix}\right] \right\| \leq (\|S\|^2 + \|U\|^2)^{1/2}$; more generally $\|(L_{ij})_{i=1,\ldots,n;\, j=1,\ldots,m}\| \leq \|(\|L_{ij}\|)_{i=1,\ldots,n;\, j=1,\ldots,m}\|_{\mathbf{K}^{n\times m}}$.

(D1) Let $R \in \mathcal{B}(B_1, B_2)$ and $\|Rx\| \geq \varepsilon \|x\|$ for all $x \in X$. Then R is one-to-one and its range is closed, in particular, $R \in \mathcal{GB}$ iff its range is dense. Moreover, if $R \in \mathcal{GB}$, then $\|R^{-1}\| \leq 1/\varepsilon$.

(E1) **(Closed Graph Theorem)** *Let $T : B_1 \to B_2$ be linear. Then $T \in \mathcal{B}(B_1, B_2)$ iff $x_n \to 0 \,\&\, Tx_n \to y \Rightarrow y = 0$ (for all $\{x_n\} \subset B_1,\ y \in B_2$).*

(F1) Claims (j3)–(k2) and (c3)(i)–(iii)&(vi) of Lemma A.3.1 hold for Banach spaces too.

(G1) Let $A : \mathrm{Dom}(A) \to B_2$ be linear, where $\mathrm{Dom}(A)$ is a subspace of B_1. Equip $\mathrm{Dom}(A)$ with norm $\|x\|_A^2 := \|x\|_{B_1}^2 + \|Ax\|_{B_2}^2$. Then $\mathrm{Dom}(A)$ is a normed space, and $A \in \mathcal{B}(\mathrm{Dom}(A), B_2)$. If B_1 and B_2 are inner product spaces, then so is $\mathrm{Dom}(A)$, with inner product $\langle x, y\rangle_A := \langle x, y\rangle_{B_1} + \langle Ax, Ay\rangle_{B_2}$. Moreover, $\mathrm{Dom}(A)$ is complete iff A is closed.

We recall from [Rud73] that A in (G1) is called closed *iff $x_n \to x$ and $Ax_n \to y$ imply that $x \in \mathrm{Dom}(A)$ and $Ax = y$ (this holds if $\{(x, Ax) \,|\, x \in \mathrm{Dom}(A)\}$ is a closed subset of $B_1 \times B_2$).*

If $B_1 = B_2$ and B_1 is complex, then we also define $\sigma(A)^c := \{\lambda \in \mathbf{C} \,|\, \exists (\lambda - A)^{-1} \in \mathcal{B}(B_1)\}$ (that is, $\lambda \notin \sigma(A)$ iff there is $T \in \mathcal{B}(B_1)$ s.t. $(\lambda I - A)T = I_{B_1}$ and $T(\lambda I - A) = I_{\mathrm{Dom}(A)}$).

(G2) Let $A \in \mathcal{B}(B_1, B_3)$. Let $B_2 \subset B_3$, continuously. Set $\mathrm{Dom}(A) := \{x \in B_1 \,|\, Ax \in B_2\}$. Then $A_{|\mathrm{Dom}(A)} : \mathrm{Dom}(A) \to B_2$ is closed and (G1) applies.

(G3) If A is as in (G1), $B_1 = B_2$ is complex and $\sigma(A)^c \neq \emptyset$, then A is closed and $x \mapsto \|(\alpha - A)x\|_{B_1}$ is an equivalent norm on $\mathrm{Dom}(A)$.

(H1) Let $F : \mathbf{R}_+ \to \mathcal{B}(B_1, B_2)$, $M < \infty$ be s.t. $\|F(t)\| \leq M$ for all $t \in \mathbf{R}_+$, and let the span of $X \subset B_1$ be dense. If $F(t)x \to 0$, as $t \to +\infty$, for all $x \in X$, then $F(t)x \to 0$ for all $x \in B_1$.

(H2) Let $F : \mathbf{R}_+ \to \mathcal{B}(B_1, B_2)$, $M < \infty$ be s.t. $\|F(t)\| \leq M$ for all $t \in \mathbf{R}_+$. If $K \subset B_1$ is compact and $F(t)x \to 0$, as $t \to +\infty$, for all $x \in K$, then $F(t)x \to 0$ uniformly for $x \in K$.

(I1) Let $B_1 \subset B_2$ be a subspace, $\Lambda \in B_2^*$, $x \in B_1$ and $\Lambda x \neq 0$. Then $B_1 = B_1 \cap \mathrm{Ker}(\Lambda) + \mathbf{K}x$.

(J1) **(Bilinear)** *Let $S : B_1 \times B_2 \to B_3$ be bilinear. Then S is continuous iff there is $M < \infty$ s.t.*

$$\|S(x,y)\|_{B_3} \le M\|x\|_{B_1}\|y\|_{B_2} \quad \text{for all } x \in B_1,\ y \in B_2. \tag{A.22}$$

Assume that this is the case. Then there is a unique $T \in \mathcal{B}(B_2, \mathcal{B}(B_1, B_3))$ s.t. $S(x,y) = (Ty)x$ for all $x \in B_1$, $y \in B_2$. Moreover, $\|T\|_{\mathcal{B}(B_2,\mathcal{B}(B_1,B_3))}$ is the minimal number M satisfying (A.22).

(K1) If $x \in B$, then there is $\Lambda \in B^*$ s.t. $\|\Lambda\| \le 1$ and $\Lambda x = \|x\|_B$.

(L1) If $\sum_{n\in\mathbf{N}} \|x_n\|_{B_1} < \infty$, then $\sum_{n\in\mathbf{N}} x_n$ converges absolutely, and $\|\sum_{n\in\mathbf{N}} x_n\|_{B_1} \le \sum_{n\in\mathbf{N}} \|x_n\|_{B_1}$.

(M1) Let $T \in \mathcal{B}(B_1, B_2)$. If B_1 is reflexive, then $T = T^{**}$.

(N1) Let $T \in \mathcal{B}(B_1, B_2)$. Then Ran$(T)$ *is dense in B_2 iff T^* is one-to-one.*

(N2) If B_1 is reflexive, then $T \in \mathcal{B}(B_1, B_2)$ is one-to-one iff Ran(T^*) *is dense in B_1^*.*

(N3) Let $T \in \mathcal{B}(B_1, B_2)$. Then $T[B_1] = B_2$ iff $\|T^*y^*\|_{B_1^*} \ge \varepsilon\|y^*\|_{B_2^*}$ for some $\varepsilon > 0$ and all $y^* \in B_2^*$.

(N4) **($\|\mathbf{Tx}\| \ge \varepsilon\|\mathbf{x}\|$)** *Let $T \in \mathcal{B}(B_1, B_2)$. Then the following are equivalent:*

- *(i) $\|Tx\| \ge \varepsilon\|x\|$ for all x for some $\varepsilon > 0$, i.e., T is* coercive *("uniformly bounded from below");*
- *(iii)* Ran$(T^*) = B_1^*$, *i.e., T^* is* onto*;*
- *(iv)* Ran(T) *is closed and* Ker$(T) = \{0\}$*;*
- *(ix) $\|TX\|_{\mathcal{B}(*,H)} \ge \varepsilon\|X\|_{\mathcal{B}(*,*)}$ for some $\varepsilon > 0$ whenever X is linear (in particular, X is bounded iff TX is bounded);*
- *(x) $T \in \mathcal{GB}(U, \mathrm{Ran}(T))$;*
- *(xi) There is $\varepsilon > 0$ s.t. for all $x \in B_1 \setminus \{0\}$ there is $y^* \in B_2^* \setminus \{0\}$ s.t. $\langle y^*, Tx\rangle_{\langle B_2^*, B_2\rangle} \ge \varepsilon\|x\|\|y^*\|$.*

We can replace B_2^* by any of its norming subsets in (xi).

(N5) **($\mathbf{T} \in \mathcal{GB}(\mathbf{U},\mathbf{H})$)** *Let $T \in \mathcal{B}(U,H)$. Then the following are equivalent:*

- *(i) $T \in \mathcal{GB}(U,H)$ (i.e., T is invertible)*
- *(ii) T is one-to-one and onto;*
- *(iii) T is coercive and has a dense range;*
- *(vi) $T^* \in \mathcal{GB}$.*

Moreover, each of (i)–(xi) implies that $(T^*)^{-1} = (T^{-1})^*$.

(O1) **(Uniform Boundedness Principle)** *Let $\mathcal{A} \subset \mathcal{B}(B, B_2)$. If $\{\|Ax\|_{B_2}\}_{A\in\mathcal{A}}$ is bounded for each $x \in B$, then there is $M < \infty$ s.t. $\|A\|_{\mathcal{B}(B,B_2)} \le M$ for all $A \in \mathcal{A}$.*

(P1) **(Completion)** *If X is a normed space, then there is a Banach space $\bar{X}$ (the* completion *of X) s.t. X is a dense subspace of $\bar{X}$ (with the same topology). If X is an inner product space, then $\bar{X}$ is a Hilbert space.*

(Q1) **($\dim X < \infty$)** *If X is a normed space and $n := \dim X < \infty$, then X is isomorphic to $\mathbf{K}^n$ (with equivalent norms).*

(R1) The Banach space B_1 is reflexive iff B_1^* is reflexive.

(R2) If B_1^* is reflexive, then B_1 is reflexive.

(R3) The Banach space B_1 is reflexive and separable iff B_1^* is reflexive and separable.

(S1) **(Subspaces)** *Let X be a closed subspace of a normed space Y. If Y is complete (resp. reflexive, separable), then so is X.*

By (J1), the space of bilinear continuous mappings $B_1 \times B_2 \to B_3$ with norm $\inf\{M \,|\, M$ satisfies (A.22)$\}$ is a Banach space isometrically isomorphic to $\mathcal{B}(B_2, \mathcal{B}(B_1, B_3))$ (and to $\mathcal{B}(B_1, \mathcal{B}(B_2, B_3))$).

Proof: (B1) All these claims are given in Chapter 4 of [Rud73] (Theorems 4.18 and 4.19 in particular) or easily deducible from them. Although the results of [Rud73] concern the Banach space adjoint $(\cdot)^{\mathrm{B}}$ only, (e.g., $K \in \mathcal{BC}(B_1, B_2) \Leftrightarrow K^{\mathrm{B}} \in \mathcal{BC}(B_2^*, B_1^*)$), they are applicable for the Hilbert space adjoint $(\cdot)^{\mathrm{H}}$ too (when the Banach spaces concerned happen to be Hilbert spaces), because H is isomorphic to H^* as a real Hilbert space [Rud73, 12.5] (note that each complex Hilbert space is also a real Hilbert space), hence $K^{\mathrm{H}} \in \mathcal{BC}(B_2, B_1) \Leftrightarrow K^{\mathrm{B}} \in \mathcal{BC}(B_2^*, B_1^*)$, $K^{\mathrm{H}} \in \mathcal{GB}(B_2, B_1) \Leftrightarrow K^{\mathrm{B}} \in \mathcal{GB}(B_2^*, B_1^*)$ etc. The last claim can be proved as the first one in (B2).

(B2) 1° Let $F := \{u \in U \,|\, \|u\| \le 1\}$. For each $n \in \mathbf{N}$, choose $m_n \in \mathbf{N}$, $y_{n,1}, y_{n,2}, \ldots, y_{n,m_n} \in K[F]$ s.t. the balls $\{y \in Y \,|\, \|y - y_{n,k}\| < 1/n\}$ cover $K[F]$. If P_n'' is the orthogonal projection of Y onto $\mathrm{span}(y_{n,1}, y_{n,2}, \ldots, y_{n,m_n})$, then clearly $\|y - P_n''y\| < 1/n$ for each $y \in K[F]$, i.e., $\|Ku - P_n''Ku\| < 1/n$ whenever $\|u\| \le 1$. Let $y_1', y_2', \ldots$ be the sequence $y_{1,1}, y_{1,2}, \ldots, y_{1,m_1}, y_{2,1}, \ldots$ so that $\|(I - P_n')K\| \to 0$, as $n \to \infty$.

2° Choose $\{u_k\} \subset U$ analogously for $K^* \in \mathcal{BC}(Y, U)$. Then $\|K(I - P_n)\| = \|(I - P_n)K^*\| \to 0$, hence $\|K - P_n'KP_n\| = \|(I - P_n')K + P_n'[K(I - P_n)]\| \to 0$, as $n \to \infty$, as required.

3° The sequence $u_1, y_1, u_2, y_2, \ldots$ satisfies the requirements of the last claim, because the projections P_n'' thus obtained satisfy $\mathrm{Ran}(I - P_{2n}'') = \mathrm{Ker}(P_{2n}'') \subset \mathrm{Ker}(P_n) \cap \mathrm{Ker}(P_n')$

(B3) See [Rud73, Exercise 4.9].

(B4) 1° If (there is) $G \in \mathcal{GB}(B_1, B_2)$, then $G[B_1] = B_2$, and B_1 and B_2 have the same dimension ($\in \mathbf{N} \cup \{\infty\}$).

If $K \in \mathcal{BC}(B_1, B_2)$, then [Rud73, Theorem 4.18(a)] implies that $\dim K[B_1] < \infty$; hence then $\mathcal{GB} \cap \mathcal{BC} = \{0\}$ unless $\dim B_1 = \dim B_2 < \infty$. If $G + K = G' + K' \in \mathcal{GB} + \mathcal{BC}$, then $\mathcal{GB} \ni G - G' = K' - K \in \mathcal{BC}$, hence the uniqueness.

2° Set $C := -G^{-1}K \in \mathcal{BC}$. By (B3) and the invertibility of G, we have $I = S(G + K) = SG(I - C) \Leftrightarrow (I - C)SG = I \Leftrightarrow G(I - C)S = I$ for any $S \in \mathcal{B}$, hence $G + K$ is right-invertible iff it is left-invertible.

Let now $\exists (G + K)^{-1} = S \in \mathcal{B}$. Set $(I - C') := (I - C)^{-1}$ to obtain $C' \in \mathcal{BC}$, by (B3). Then $S = (I - C)^{-1}G^{-1} = G^{-1} - C'G^{-1} \in \mathcal{BGC}$.

3° Obviously, $G^* + K^* \in \mathcal{GB} + \mathcal{BC}$ and $(G+K)(G'+K') = GG' + (GK' + KG' + KK') \in \mathcal{GB} + \mathcal{BC}$ when $G+K \in \mathcal{GB} + \mathcal{BC}(B_1, B_2)$ and $G'+K' \in \mathcal{GB} + \mathcal{BC}(B_2, B_3)$.

4° Fredholm: If $A = I + K$, $K \in \mathcal{BC}$, then the claim follows from Theorems 4.23 and 4.25 of[Rud73]. If $G + K' \in \mathcal{GB} + \mathcal{BC}(B_1, B_2)$, then $G + K'$ is a Fredholm operator, because so is $A := I + K'G^{-1}$ (by what we have noted above) and we have $G + K' = AG$.

(B5) Most claims follows easily from (B4). For the norm inequality, let $K \in \mathcal{BC}(B_1)$ and $\dim B_1 = \infty$. If we had $\|Kx\| \geq \varepsilon\|x\|$ for all $x \in B_1$ for some $\varepsilon > 0$, then K^* would be onto [Rud73, 4.15] and compact, which is impossible [Rud73, 4.18(b)].

If $\|Kx\| < \varepsilon\|x\|$, then $\|\lambda x + Kx\| \geq (|\lambda| - \varepsilon)\|x\|$; because this kind of an x exists for an arbitrary $\varepsilon > 0$, we have $\|\lambda I + K\| \geq |\lambda|$.

It follows that if $\{\lambda_n I + K_n\}$ is a Cauchy sequence, then so is $\{\lambda_n\}$, hence then $\lambda_n \to \lambda$ for some $\lambda \in \mathbf{C}$, which in turn implies that also $\{K_n\}$ is a Cauchy sequence, thus converging to some $K \in \mathcal{BC}$. (If, instead, $\dim B_1 < \infty$, then $I + \mathcal{BC} = \mathcal{BC} = \mathcal{B}$.)

(C1) This is easy: if $x \in B_1 \times \ldots \times B_m$ and we replace each term by its norm in the vector Lx, we get a product of the form $\mathbf{K}^{n\times m} \cdot \mathbf{K}^m$, from which the upper bound is obtained (the converse does not hold, e.g., $\| \begin{bmatrix} A & B \end{bmatrix} \| = 1 < \sqrt{2} = \| \begin{bmatrix} 1 & 1 \end{bmatrix} \|$, when $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$).

We have used here the 2-norm $\|(b,b')\|^2 := \|b\|^2 + \|b'\|^2 = |(\|b\|, \|b'\|)|_{\mathbf{K}^2}$ (which is always compatible with the product topology and the canonical one in case of Hilbert spaces); of course the results are different for other norms. Also the space $\mathbf{K}^{n\times m}$ is equipped with its canonical ($\mathcal{B}(\mathbf{K}^m, \mathbf{K}^n)$) norm (i.e., $\|A\|_{\mathbf{K}^{n\times m}} := \sup_{\|x\|\leq 1} \|Ax\|$; if $\mathbf{K} = \mathbf{C}$, then one can show that $\|A\|_{\mathbf{K}^{n\times m}} = \max \sigma(A^*A)^{1/2}$).

(D1) The first claim is straightforward, the equivalence follows from the open mapping theorem [Rud73, 2.12(b)], and the inequality from the formula $\|x\| = \|RR^{-1}x\| \geq \varepsilon\|R^{-1}x\|$ for all $x \in H$.

(E1) "Only if" is obvious; "if" follows from Theorem 2.15 of [Rud73].

(F1) The same proofs apply (see the references in the proof of (h1)).

(G1) The claims are quite obvious (note that $\{x_n\}$ is a Cauchy-sequence in $\mathrm{Dom}(A)$ iff $\{x_n\}$ and $\{Ax_n\}$ converge to some $x \in B_1$ and $y \in B_2$, respectively).

(G2) Let $\{x_n\} \subset \mathrm{Dom}(A)$, $x_n \to x$ in B_1, and $Ax_n \to y$ in B_2. Then $Ax_n \to y$ in B_3 too, by also $Ax_n \to Ax$ in B_3, hence $Ax = y$ and $x \in \mathrm{Dom}(A)$.

(G3) One easily verifies the first claim. If $\alpha \in \sigma(A)^c$ and $\{x_n\}$ is a Cauchy-sequence in $\mathrm{Dom}(A)$, then $(\alpha - A)x_n \to y$ for some $y \in B_1$, hence $x_n \to (\alpha - A)^{-1}y =: x$, because $(\alpha - A)^{-1} \in \mathcal{B}(B_1)$. Consequently, then $x \in (\alpha - A)^{-1}[B_1] = \mathrm{Dom}(A)$ and $(\alpha - A)x = y$, so that $x_n \to x$ in $\mathrm{Dom}(A)$. Thus, $\mathrm{Dom}(A)$ is complete.

(H1) Let $x \in B_1$. Replace X by its span. Choose $x' \in X$ s.t. $\|x - x'\| < \varepsilon/2M$. If $\|F(t)x'\| < \varepsilon/2$, then $\|F(t)x\| < \varepsilon$.

(H2) Let $\varepsilon > 0$ be given. Set $\delta := \varepsilon/2M$. Then there are $n \in \mathbf{N}$ and $x_1, \ldots, x_n \in K$ s.t. $K \subset \cup_{k=1}^n D(x_k, \delta)$. Choose $T_1, \ldots, T_n < \infty$ s.t. $\|F(t)x_k\| < \varepsilon/2$ for all $t > T_k$, for all $k = 1, \ldots, n$.

If $x \in K$ and $t > \max\{T_1, \dots, T_n\}$, choose k s.t. $\|x - x_k\| < \delta$ to observe that $\|F(t)x\| \le \|F(t)x_k\| + \|F(t)(x - x_k)\| < \varepsilon/2 + M\delta = \varepsilon$.

(I1) Obviously, $B_1 \cap \mathrm{Ker}(\Lambda) + \mathbf{K}x \subset B_1$. For the converse, assume that $z \in B_1$. Choose $\alpha \in \mathbf{K}$ s.t. $\Lambda z = \alpha\Lambda x$. Then $\alpha x - z \in \mathrm{Ker}(\Lambda) \cap B_1$.

(J1) 1° Assume that S is continuous. Then there are $\varepsilon, \varepsilon' > 0$ s.t. $S[D^1_\varepsilon \times D^2_{\varepsilon'}] \subset D^3_1$, where $D^k_r := \{x \in B^k \,|\, \|x\| < r\}$. By bilinearity, we can take $M := (\min\{\varepsilon, \varepsilon'\})^{-2}$.

2° Assume that $M < \infty$ satisfies (A.22). (If $M = 0$, then $S = 0$, hence we assume that $M > 0$.) Then, for any $\varepsilon > 0$, we have $S[D^1_\delta \times D^2_\delta] \subset D^3_\varepsilon$, where $\delta := (M/\varepsilon)^{-1/2}$. Thus, S is continuous at zero. By bilinearity, we have

$$\begin{aligned} S(x+x', y+y') - S(x,y) &= S(x+x', y+y') - S(x+x', y) + S(x+x', y) - S(x,y) \\ &= S(x+x', y') + S(x', y) \end{aligned} \tag{A.23}$$

for all $x, x' \in B_1$, $y, y' \in B_2$. Given $(x,y) \in B_1 \times B_2$ and $\varepsilon > 0$, it obviously follows that $\|S(x+x', y+y') - S(x,y)\|_{B_3} < \varepsilon$ for $\|x'\|, \|y'\| < \delta := \varepsilon/2M(R+1)$, where $R := \max\{\|x\|, \|y\|\}$. Therefore, S is continuous.

3° Assume that $M < \infty$ satisfies (A.22). For each $y \in Y$, the map $T_y : x \mapsto S(x,y)$ is in $\mathcal{B}(B_1, B_3)$, and $T : y \mapsto T_y$ is linear and $\|T_y\|_{\mathcal{B}(B_1,B_3)} \le M\|y\|_{B_2}$, hence T is as in (J1) and $\|T\| \le M$.

We have $\|S(x,y)\|_{B_3} \le \|Ty\|\|x\| \le \|T\|\|y\|\|x\|$, thus $\|T\|$ satisfies (A.22) in place of M; because $\|T\| \le M$ for any other such number M, $\|T\|$ is the infimal one.

(K1) See Corollary 3.3 of [Rud73].

(L1) This is an easy exercise.

(M1) For all $x \in B_1 = B_1^{**}$ and $y \in B_2^*$, we have

$$\langle y^*, Tx\rangle_{\langle B_2^*, B_2\rangle} = \langle T^*y^*, x\rangle_{\langle B_1^*, B_1\rangle} = \langle T^*y^*, x\rangle_{\langle B_1^*, B_1^{**}\rangle} = \langle y^*, T^{**}x\rangle_{\langle B_2^*, B_2^{**}\rangle}. \tag{A.24}$$

Thus, $T^{**}x \in B_2^{**}$ is the element $y^* \mapsto \langle y^*, Tx\rangle_{B_2^*, B_2}$ of B_2^{**}, which is identified to $Tx \in B_2$. I.e., the range of $T^{**} \in \mathcal{B}(B_1^{**}, B_2^{**}) = \mathcal{B}(B_1, B_2^{**})$ is contained in the closed subspace B_2 of B_2^{**}.

(N1) This is Corollary 4.12(b) of [Rud73].

(N2) By (N1), $\mathrm{Ran}(T^*)$ is dense iff T^{**} is one-to-one. But $T^{**} = T$, by (M1).

(N3) This is Theorem 4.15 of [Rud73].

(N4) One easily verifies the implication "(i)⇒(iv)". We obtain "(ix)⇔(i)⇐(xi)" and "(iv)⇒(x)⇒(i)" as in Lemma A.3.1(c3), "(i)⇔(iii)" from Theorems 4.14 and 4.12(c) of [Rud73], and "(i)⇒(xi) from (K1).

(N7) "(iii)⇔(i)⇒(ii)" follows from (N4)(x), "(ii)⇒(vi)" from (N3) and Theorems 4.12(c)&4.14 (since then T^* is coercive and onto) "(vi)⇒(iii)" from (N3)&(N4)(iii).

Finally, if $y \in B_2$ and $y^* \in B_2^*$, then $\langle (T^{-1})^*T^*y^*, y\rangle = \langle y^*, TT^{-1}y\rangle = \langle y^*, y\rangle$, hence $(T^{-1})^*T^* = I$, hence $(T^{-1})^* = (T^*)^{-1}$.

(O1) See, e.g., [Rud86], Theorem 5.8.

(P1) This is well-known (any metric space has a completion — a complete metric space in which the space is complete, by Exercise 24 of [Rud76]; one easily verifies that vector operations, norm and inner product can be extended to the completion).

(Q1) By Theorem 1.21 of [Rud73], every vector isomorphism (i.e., linear bijection) from $\mathbf{K}^n$ to a TVS is a homeomorphism, hence a TVS isomorphism.

(R1)&(S1) See Exercice 4.1(d)&(f) of [Rud73].

(R2) See [Adams, Theorem 1.14].

(R3) This follows from (R1) and (R2). □

Sometimes one needs concepts such as "self-adjoint" also for spaces other than inner product spaces. Such extended concepts are treated below (see Remark A.3.22 for $(\cdot)^{\mathrm{d}}$):

Lemma A.3.5 *Let $J, S \in \mathcal{B}(B, B^{\mathrm{d}})$, where B is a Banach space. Then (we use below sesquilinear pairing)*

(a) If $\langle x, Jx\rangle = \langle x, Sx\rangle$ for all $x \in B$, then $J = S$.

(b) We have $\overline{\langle y, Jx\rangle_{\langle B^{\mathrm{dd}}, B^{\mathrm{d}}\rangle}} = \langle Jx, y\rangle_{\langle B^{\mathrm{d}}, B^{\mathrm{dd}}\rangle} = \langle x, J^{\mathrm{d}}y\rangle_{\langle B, B^{\mathrm{d}}\rangle}$ for all $x, y \in B$.

(c1) The following are equivalent:

(i) $J = J^{\mathrm{d}}|_B$;

(ii) $\langle y, Jx\rangle_{\langle B, B^{\mathrm{d}}\rangle} = \overline{\langle x, Jy\rangle_{\langle B, B^{\mathrm{d}}\rangle}}$ $(=: \langle Jy, x\rangle_{\langle B^{\mathrm{d}}, B^{\mathrm{dd}}\rangle})$ for all $x, y \in B$;

(iii) $\langle x, Jx\rangle = \overline{\langle x, Jx\rangle}$ for all $x \in B$.

(Thus, "$J \geq 0$" (i.e., $\langle x, Jx\rangle \geq 0$ for all $x \in B$) implies that $J = J^{\mathrm{d}}|_B$.)

(c2) Assume that $J = J^{\mathrm{d}}|_B$. Then T is invertible iff T is onto (iff T is coercive and B is reflexive).

(d) Then the following are equivalent:

(i) "$J \gg 0$" (i.e., $\langle x, Jx\rangle_{\langle B, B^{\mathrm{d}}\rangle} \geq \varepsilon\|x\|^2$ for some $\varepsilon > 0$ and all $x \in B$);

(ii) "$J \geq 0$" and J is coercive.

Moreover, in either case $J = J^{\mathrm{d}}|_B$ (hence $J \in \mathcal{GB}(B, B^*)$ iff B is reflexive).

Proof: (We apply $()^{\mathrm{B}}$ results freely for $()^{\mathrm{d}}$; see Remark A.3.20 for justification. Note that J, S are linear to B^{d}, hence not to B^{B} (unless $\mathbf{K} = \mathbf{R}$ or $B = \{0\}$); cf. Remark A.3.22.)

(a) The proof of Theorem 12.7 of [Rud73] shows that $\langle x, (J-S)y\rangle = 0$ for all $x, y \in B$, hence $(J-S)y = 0 \in B^{\mathrm{d}}$ for all $y \in B$, hence $J - S = 0$.

(b) This is obvious (see Remark A.3.22).

(c1) We obtain "(i)⇔(iii)" from (a) and (b). Since $\langle y, J^{\mathrm{d}}x\rangle := \langle Jy, x\rangle$, we have "(i)⇔(ii)"

(c2) The map J is onto (i.e., $J[B] = B^{\mathrm{d}}$) iff $J^{\mathrm{d}} \in \mathcal{B}(B^{\mathrm{dd}}, B^{d})$ is coercive, by Lemma A.3.4(N3). In either case, $J = J^{\mathrm{d}}|_B$ is coercive, hence $J, J^{\mathrm{d}} \in \mathcal{GB}$, by (N5)(iii)&(i)&(iv). Thus, $J[B] = B^{\mathrm{d}} = J^{\mathrm{d}}[B^{\mathrm{dd}}]$, hence $B = B^{\mathrm{dd}}$, i.e., B is reflexive.

Finally, if(f) B is reflexive, then $J^{\mathrm{d}} = J^{\mathrm{d}}|_B = J$, hence then coercivity implies "onto", by (N3).

(d) (Note that (i) or (ii) implies that J is an isomorphism onto its range, by Lemma A.3.4(N4).)

Trivially, (i) implies (ii) and (ii) implies that (c1)(iii) holds. Assume then (ii), so that $\|Jx\| \geq \delta\|x\|$ for all $x \in X$ and some $\delta > 0$. Set $\varepsilon := \delta^2/4\|J\|$.

Let $x \in B$ be arbitrary. There is $y \in B$ s.t. $\|y\| = 1$ and $\langle y, Jx\rangle \geq \delta\|x\|/2$. By (c1)(ii) we have

$$0 \leq \langle (rx+y), J(rx+y)\rangle = \langle y, Jy\rangle + 2r\operatorname{Re}\langle x, Jy\rangle + r^2\langle x, Jx\rangle \quad (r \in \mathbf{R}). \tag{A.25}$$

Thus, (A.25) is a real polynomial with at most one root, hence $(2\operatorname{Re}\langle x, Jy\rangle)^2 - 4\langle x, Jx\rangle\langle y, Jy\rangle \leq 0$, hence

$$\langle x, Jx\rangle \geq \operatorname{Re}\langle x, Jy\rangle/\langle y, Jy\rangle \geq (\delta/2)^2\|x\|^2/\|J\| = \varepsilon\|x\|^2. \tag{A.26}$$

□

We end this section with some auxiliary lemmas. We often use the following lemma to show the continuity of a continuous mapping under a stronger norm:

Lemma A.3.6 ($TX_1 \subset X_2 \Rightarrow T \in \mathcal{B}(X_1, X_2)$) *Assume that X_1 and X_2 are Banach spaces, X_3 is a TVS, and $X_2 \underset{c}{\subset} X_3$. If $T \in \mathcal{B}(X_1, X_3)$ and $TX_1 \subset X_2$, then $T \in \mathcal{B}(X_1, X_2)$.*

Here $X_2 \underset{c}{\subset} X_3$ means that $X_2 \subset X_3$ continuously, i.e., there is $M < \infty$ s.t. $\|x\|_{X_3} \leq \|x\|_{X_2}$ for all $x \in X_2$. A continuous inclusion is called an *embedding* (or an imbedding). E.g., $\mathrm{L}^\infty([0,1]) \underset{c}{\subset} \mathrm{L}^1([0,1])$. (Sometimes one means by an embedding any continuous injective linear mapping; the name *inclusion* requires injectivity and some sense of unique (canonical) identification.)

It follows that $X_1 \underset{c}{\subset} X_2 \underset{c}{\subset} X_3 \implies X_1 \underset{c}{\subset} X_3$ and that $X_1 \underset{c}{\subset} X_2$ & $T \in \mathcal{B}(X_2, X_3) \implies T \in \mathcal{B}(X_1, X_3)$.

Proof of Lemma A.3.6: Let $x_n \to x$ in X_1 and $Tx_n \to 0$ in X_2. Then $Tx_n \to 0$ in X_3, hence $Tx = 0$, hence $T \in \mathcal{B}(X_1, X_2)$, by Lemma A.3.4(E1). □

By setting $T := I$ and applying Lemma A.3.6, we obtain the following:

Corollary A.3.7 (Inclusions are continuous) *Assume that X_1 and X_2 are Banach spaces and X_3 is a TVS. If $X_1 \underset{c}{\subset} X_3$, $X_2 \underset{c}{\subset} X_3$ and $X_1 \subset X_2$, then $X_1 \underset{c}{\subset} X_2$.*

In particular, if $X_1 = X_2$ as sets and $X_1, X_2 \underset{c}{\subset} X_3$, then X_1 and X_2 have equivalent norms (i.e., the same topology). □

Sometimes we wish to apply the above corollary with $X_3 = \mathrm{L}^p_{\mathrm{loc}}(\Omega; B)$, where $\Omega \subset \mathbf{R}^n$ is open or $\Omega \subset \mathbf{Z}^n$ (in the latter case $\mathrm{L}^p_{\mathrm{loc}}(\Omega; B)$ becomes the set of all functions $\Omega \to B$ with the topology of pointwise convergence). For that purpose, we note that $\mathrm{L}^p_{\mathrm{loc}}$ is a TVS (even a Fréchet space) with the topology induced by the seminorms $\|\cdot\|_{\mathrm{L}^p(K;B)}$ ($K \subset \Omega$, K is compact); see Appendix B.3 (or Chapter 1 of [Rud73]) for details.

Definition A.3.8 (Norming set) *A set $C \subset B^*$ is a* norming set (for B) *(or a determining set) if $\|x\|_B = \sup_{\Lambda \in C} |\Lambda x|$ for each $x \in B$. A subspace $X \subset B^*$ is a* norming subspace (for B) *if it contains a norming set.*

Note that $\|\Lambda\| \leq 1$ for each $\Lambda \in C$ above; obviously, we can redefine C s.t. $\|\Lambda\| = 1$ for each $\Lambda \in C$ unless $B = \{0\}$.

We may take X to be B^* or a dense subspace of B^*; if $B = B_2^*$, then we may take $X = B_2$.

Lemma A.3.9 *Let B be separable. Then B and B^* possess countable norming sets.* □

(This is Theorem 2.8.5 of [HP].)

Lemma A.3.10 (Extension by density) *Let X be a normed space and Y a Banach space. Let X_0 be a dense subspace of X (with the same norm) and let $T \in \mathcal{B}(X_0, Y)$. Then there is a unique $\bar{T} \in \mathcal{B}(X, Y)$ s.t. $\bar{T}x = Tx$ for all $x \in X_0$. Moreover, $\|\bar{T}\| = \|T\|$.* □

The lemma is approximately Proposition 2.3.1 of [Rauch]; the proof is straightforward (if $\{x_n\} \subset X_0$ is a Cauchy sequence, then so is $\{Tx_n\}$, etc.), and we omit it. Note that one can identify X_0^* and X^* (as well as $\mathcal{B}(X_0, Y)$ and $\mathcal{B}(X, Y)$) by the above lemma.

Lemma A.3.11 (Norm-preserving extension) *Let $T \in \mathcal{B}(X, B_2)$, where X is a subspace of B. If $B_2 = \mathbf{K}^n$ or B is a Hilbert space, then T has an extension $S \in \mathcal{B}(B, B_2)$ s.t. $\|S\| = \|T\|$.*

(Note, in contrast, that $I \in \mathcal{B}(\mathrm{H}^1, \mathrm{H}^1)$ has no extension in $\mathcal{B}(\mathrm{L}^1, \mathrm{H}^1)$, by p. 154 of [Hoffman]. However, whenever $B_2 = \mathbf{K}^n$, we can extend all n components of T to get a continuous extension of T (possibly with a greater norm).)

Proof: If $B_2 = \mathbf{K}$, any Hahn–Banach extension of T will do. If B is a Hilbert space, then we can extend T to $S \in \mathcal{B}(\bar{X}, B_2)$ by Lemma A.3.10, and set $S = 0$ on $X^\perp$. □

Not all continuous linear mappings can be extended (e.g., $\mathrm{H}^1(\mathbf{D}$ is not complemented in $\mathrm{L}^1(\partial\mathbf{D})$, by p. 130 of [Rud73]):

Lemma A.3.12 *Let X be a subspace of B. Then the following are equivalent:*

(i) For each Banach space B_2 and operator $T \in \mathcal{B}(X, B_2)$ there is an extension $S \in \mathcal{B}(B, B_2)$ of T;

(ii) $\bar{X}$ is complemented in B;

(iii) there is a continuous projection $B \to \bar{X}$.

We say that a closed subspace $M \subset B$ is *complemented in* B if there is a closed subspace $N \subset B$ s.t. $B = M + N$ and $M \cap N = \{0\}$.

Proof: By Lemma A.3.10, we assume that X is closed, w.l.o.g.

(ii)$\Leftrightarrow$(iii) This is Theorem 5.16 of [Rud73].

(i)$\Leftrightarrow$(iii) If (iii) holds and T is given, set $S = TP$, where $P \in \mathcal{B}(B,X)$ is a projection, to establish (i). Conversely, if (i) holds, there is a continuous extension of $I \in \mathcal{B}(X,X)$ to an operator $P \in \mathcal{B}(B,X)$, which must satisfy $PP = IP = P$. $\square$

Lemma A.3.13 ($\mathcal{B}(Y,Z) \underset{c}{\subset} \mathcal{B}(X,Z)$) *Let X, Y and Z be normed spaces. If $X \underset{c}{\subset} Y$ densely, then $\mathcal{B}(Y,Z) \underset{c}{\subset} \mathcal{B}(X,Z)$.*

Trivially, if $Y \underset{c}{\subset} Z$, then $\mathcal{B}(X,Y) \underset{c}{\subset} \mathcal{B}(X,Z)$.

Proof: Choose $M < \infty$ s.t. $\|x\|_Y \le M\|x\|_X$ $(x \in X)$. Let $T \in \mathcal{B}(Y,Z)$. Then $T_{|X} \in \mathcal{B}(X,Z)$ determines T uniquely by density, hence $\mathcal{B}(Y,Z) \subset \mathcal{B}(X,Z)$. Moreover,

$$\|Tx\|_Z \le \|T\|_{\mathcal{B}(Y,Z)}\|x\|_Y \le M\|T\|_{\mathcal{B}(Y,Z)}\|x\|_X \quad (x \in X), \tag{A.27}$$

hence $\|T\|_{\mathcal{B}(X,Z)} \le M\|T\|_{\mathcal{B}(Y,Z)}$, i.e., the inclusion is continuous. $\square$

Lemma A.3.14 (Bounds for a projection) *Let $Y \neq X$ be a closed subspace of the Banach space X and $\varepsilon > 0$. Then there is $x \in X$ s.t. $\|x\| = 1$ and $d(x,Y) := \inf_{y\in Y}\|x-y\| > (1+\varepsilon)^{-1}$. Consequently, $\|P_1\| < 1+\varepsilon$ and $\|P_0\| < 2+\varepsilon$, where $P_0 : (y+\alpha x) \to y$ and $P_1 : (y+\alpha x) \to x$, for $y \in Y$ and $\alpha \in \mathbf{K}$.*

Moreover, there is $\Lambda \in X^*$ s.t. $\|\Lambda\| = 1$, $\Lambda = 0$ on Y, and $\|\Lambda x\| > (1+\varepsilon)^{-1}$.

Proof: The existence of x follows from [Rud73, Lemma 4.22].

Choose $\delta < \varepsilon$ s.t. $d(x,Y) > (1+\delta)^{-1}$. For $\alpha \neq 0$ we have

$$\|y+\alpha x\| = |\alpha|\|y/\alpha + x\| > (1+\delta)^{-1}|\alpha| \tag{A.28}$$

and $\|P_1(y+\alpha x)\| = \|\alpha x\| = |\alpha| < (1+\delta)\|y+\alpha x\|$, hence $\|P_1\| \le 1+\delta < 1+\varepsilon$. Consequently, $\|P_0\| = \|I - P_1\| < 1+1+\varepsilon$.

Set $\Lambda'\alpha x := \alpha$ for $\alpha \in \mathbf{K}$, so that $\Lambda' = 1$. Set $\Lambda'' := \|P_0\|^{-1}\Lambda' P_0 \in X_0^*$, so that $\|\Lambda''\| = 1$ and $\Lambda'' x = \|P_0\|^{-1} > (1+\varepsilon)^{-1}$. Now Λ can be taken to be a Hahn–Banach extension of Λ''. $\square$

We now show how norms can be induced and coinduced:

Lemma A.3.15 (Induction) *Let X be a vector space, Y a normed space and $T \in \mathrm{Hom}(X,Y)$ s.t. $\mathrm{Ker}(T) = \{0\}$. Set $\|x\|_X := \|Tx\|_Y$ for each $x \in X$.*

Then X becomes a normed space, $T \in \mathcal{B}(X,Y)$, and T is an isometry. If Y is an inner product space, then so is X.

Assume that $X' := X$ is a Banach space under some norm $\|\cdot\|'_X$, and that $T \in \mathcal{B}(X',Y)$ and $\|Tx\|_Y \ge \varepsilon\|x\|'_X$ for all $x \in X$ and some $\varepsilon > 0$. If Y is complete (resp. reflexive), then so is X, and then X is isometrically isomorphic to a closed subspace of Y.

Proof: 1° Obviously, $\|\cdot\|_X$ is a norm on X and T is an isometry (hence $T \in \mathcal{B}(X,Y)$). If $\langle\cdot,\cdot\rangle$ is an inner product on Y, then $\langle T\cdot, T\cdot\rangle$ is obviously an inner product on X and $\langle Tx, Tx\rangle = \|x\|_X^2$ for all $x \in X$.

Remark: If $X \subset Y$ and we set $T = I$, then $\|\cdot\|_X = \|\cdot\|_{Y|_X}$; in particular, X need not be complete even if Y were complete.

2° *When Y is complete:* If $\{x_n\}$ is Cauchy in X, then $\{Tx_n\}$ is Cauchy in Y, hence $\{x_n\}$ is Cauchy in X', hence then $x_n \to x$ in X' for some $x \in X$. By continuity, $Tx_n \to Tx$ in Y, hence $x_n \to x$ in X.

It follows that T is an isometric isomorphism of X to a closed subspace of Y.

3° *When Y is reflexive:* Being reflexive, Y is complete, hence so is X. Therefore, X is reflexive, by Lemma A.3.4(S1) and 2°. □

Lemma A.3.16 (Coinduction) *Let X and Y be normed spaces and $T \in \mathcal{B}(X,Y)$. Set $Z := T[X]$. Then $\|z\|_Z := \inf\{\|x\|_X \mid x \in X,\ z = Tx\}$ is a norm on Z. With this norm, we have $\|T\|_{\mathcal{B}(X,Z)} \le 1$ and $\|\cdot\|_Y \le \|T\|_{\mathcal{B}(X,Y)}\|\cdot\|_Z$. If X is complete or a Hilbert space, then so is Z.*

Proof: Because the space $N := \mathrm{Ker}(T) \subset X$ is closed, the quotient space X/N with norm $\|x+N\|_{X/N} := \inf\|x+N\|_X$ is a normed space, and it is complete if X is complete, by Theorem 1.41 of [Rud73].

Set $S(x+N) := Tx$ for $x \in X$. Then $S : X/N \to Z$ is obviously a linear bijection, and $\|S(x+N)\|_Z = \|x+N\|_{X/N}$ for all $x \in X$. It follows that $\|\cdot\|_Z$ is a norm and it makes S an isometric isomorphism. In particular, if X is complete, then Z is complete.

For each $x \in X$, we have

$$\|Tx\|_Y = \inf_{Tx'=Tx}\|Tx'\|_Y \le \|T\|_{\mathcal{B}(X,Y)} \inf_{Tx'=Tx}\|x'\|_X = \|T\|_{\mathcal{B}(X,Y)}\|Tx\|_Z, \quad \text{(A.29)}$$

hence $\|\cdot\|_Y \le \|T\|_{\mathcal{B}(X,Y)}\|\cdot\|_Z$. Trivially, $\|Tx\|_Z \le \|x\|_X$ for $x \in X$.

Assume now that X is a Hilbert space. Define $P \in \mathcal{B}(N^\perp, X/N)$ by $Px := x+N$. Then $\|Px\|_{X/N} = \|x\|_H$ for all $x \in N^\perp$. It follows that X/N is a Hilbert space (because so is $N^\perp$ and P is an isometric isomorphism of $N^\perp$ onto X/N), hence so is Z (with $\langle z,z'\rangle_Z := \langle x,x'\rangle_X$, where $\{x\} = T^{-1}(z)\cap N^\perp$ and $\{x'\} = T^{-1}(z')\cap N^\perp$). □

One sometimes wishes to define spaces such as $\mathrm{L}^1(\mathbf{R}) + \mathrm{L}^\infty(\mathbf{R})$; a necessary condition is, of course, that both original spaces lie in a single vector space. To guarantee that the resulting space is a normed space, we require a bit more.

Normed spaces X and Y are called *sum-compatible*, or (X,Y) is called a *sum-compatible pair*, if $X \underset{c}{\subset} Z$, $Y \underset{c}{\subset} Z$ for some TVS Z.

Lemma A.3.17 ($X+Y$) *Let X and Y be sum-compatible normed spaces. Then $X+Y$ is a normed space with norm $\|z\|_{X+Y} := \inf_{z=x+y}(\|x\|_X + \|y\|_Y)$.*

This operation is commutative and associative: $X+Y = Y+X$ and $(X+Y)+X_2 = X+Y(+X_2)$, isometrically, when X_2 is sum-compatible with X and with Y. Moreover, $X \underset{c}{\subset} X+Y$ and $Y \underset{c}{\subset} X+Y$.

If X and Y are Hilbert spaces, then so is $X+Y$ (with an equivalent norm). If X and Y are complete, then so is $X+Y$.

The following are equivalent:

(i) X and Y are sum-compatible;

(ii) X and Y are vector subspaces of some third vector space, and $\|\cdot\|_{X+Y}$ is a norm on $X+Y$.

Indeed, (ii) implies that we can take $Z = X+Y$ in (i), and the above lemma provides the converse (i)⇒(ii). Thus, $X+Y$ is well defined iff X and Y are compatible.

Proof: This is contained in Lemma 2.3.1 of [BL], except the claims on commutativity, associativity and embeddings, which are obvious, and the claim on Hilbert spaces, which we prove below:

One easily verifies that $T(z,z') := z+z'$ defines an element of $T \in \mathcal{B}(X\times Y, X+Y)$. The norm of $X+Y = [T(X\times Y)]$ is clearly that coinduced by T (see Lemma A.3.16), hence $X+Y$ is a Hilbert space, by Lemma A.3.16. □

Lemma A.3.18 ($X\cap Y$) *Let X and Y be normed spaces that are subspaces of some vector space Z. Then $X\cap Y$ is a normed space with norm $\|x\|_{X\cap Y} :=$* $\max(\|x\|_X, \|x\|_Y)$.

This operation is commutative and associative: $X\cap Y = Y\cap X$ and $(X\cap Y)\cap X_2 = X\cap(Y\cap X_2)$, isometrically, when $X_2 \subset Z$. Moreover, $X\cap Y \underset{c}{\subset} X$ and $X\cap Y \underset{c}{\subset} Y$.

If X and Y are inner product spaces, then so is $X\cap Y$ (with inner product $\langle x,y\rangle_X + \langle x,y\rangle_Y$ and the corresponding norm $\|x\| := (\|x\|_X^2 + \|x\|_Y^2)^{1/2}$, equivalent to $\max\{\|x\|_X, \|x\|_Y\}$*).*

If $x = y$ whenever $x_n \to x$ in X and $x_n \to y$ in Y (this holds whenever Z is a TVS and $X \underset{c}{\subset} Z$ and $Y \underset{c}{\subset} Z$), and X and Y are complete, then $X\cap Y$ is complete.

Thus, also the spaces $X := \mathrm{L}^p(Q;B)$ and $Y := \mathrm{L}^q(Q;B)$ of Definition B.3.1 will do when $p,q \in [1,\infty]$ (if limits in L^p and L^q must be equal a.e., hence equal as elements of $\mathrm{L}^p \cap \mathrm{L}^q$, by Theorem B.3.2.

Proof: All these claims are quite obvious (cf. Lemma 2.3.1 of [BL]). (The inclusions to Z make sum, scalar multiplication and $X\cap Y$ well defined.) □

Lemma A.3.19 *Let X_1 and X_2 be sum-compatible normed spaces, and let Y be a normed space.*

(a) If $X_1 \underset{c}{\subset} X_2$, then $X_1 \cap X_2 = X_1$ and $X_1 + X_2 = X_2$ as TVSs (i.e., with equivalent norms).

(b) If $X_0 \subset X_1 \cap X_2$ is dense in X_1 and in X_2, then X_0 is dense in $X_1 + X_2$.

(c1) If $X_k \underset{c}{\subset} Y$ $(k = 1,2)$, then $X_1 + X_2 \underset{c}{\subset} Y$.

(c2) If $Y \underset{c}{\subset} X_k$ $(k = 1,2)$, then $Y \underset{c}{\subset} X_1 \cap X_2$.

Note that the assumptions of (a) and (c1) imply the sum-compatibility of X_1 and X_2.

Proof: (a) Now $\|\cdot\|_{X_2} \le M\|\cdot\|_{X_1}$ on X_1 for some $M \in [1,\infty)$, hence $\|x\|_{X_1} \le \|x\|_{X_1 \cap X_2} \le M\|x\|_{X_1}$. Obviously, $\|z\|_{X_1+X_2} \le \|z\|_{X_2}$. But if $z = x+y$, then $\|z\|_{X_2} \le M\|x\|_{X_1} + \|y\|_{X_2}$. Therefore, $\|z\|_{X_2} \le M\|z\|_{X_1+X_2}$.

(b) Let $x+y \in X_1+X_2$. Then $\|(x+y)-(x'+y')\|_{X_1+X_2} \le \|x-x'\|_{X_1} + \|y-y'\|_{X_2}$ for any $x', y' \in X_0$.

(c1)&(c2) These are quite obvious too. □

We deduce that sum-compatibility is *not* an equivalence relation, because it is not transitive: Set $Y := \mathrm{span}(\{x_n\}_{n\in\mathbf{N}}) = \mathrm{c_c}(\mathbf{N})$, where $x_n := ne_0 + e_n$ $(n \in \mathbf{N})$, with the norm $\|\sum_k \alpha_k x_k\|_Y := \sum_k |\alpha_k|^2$ (thus Y becomes isometrically isomorphic a subspace of $\ell^2(\mathbf{N})$). Set $X := \ell^2(\mathbf{N})$. Then $V := \mathrm{span}(\{e_7\})$ is sum-compatible with X and Y, but X is not sum-compatible with Y: The function $\|\cdot\|_{X+Y}$ is not a norm on $X+Y$, because $\|e_0\|_{X+Y} = n^{-1}\|x_n - e_n\|_{X+Y} \le 2/n$ for all $n \in \mathbf{N}+1$, i.e., $\|e_0\| = 0$, although $e_0 \ne 0$. (One can also replace Y by its completion; then all three spaces become Hilbert spaces.)

Remark A.3.20 $((\mathbf{K}=\mathbf{C}) \mapsto (\mathbf{K}=\mathbf{R}))$ *Complex normed spaces can be considered as real normed spaces having same topological properties, as shown in Lemma A.3.21. It follows that several results on real normed spaces can be applied to complex normed spaces; in particular, we obtain results for duals and adjoints (both $()^{\mathrm{d}}$ and $()^{\mathrm{B}}$ (and $()^{\mathrm{H}}$), see Remark A.3.22) on complex normed spaces from results for duals and adjoints given on real normed spaces.*

We apply this fact without further mention (this has been done in Lemmas A.3.1 and A.3.5). □

Here $()^{\mathrm{B}}$ (resp. $()^{\mathrm{d}}$) refers to standard (resp. "conjugate-linear") duals and adjoints in Banach spaces; $()^{\mathrm{H}}$ to the adjoints and duals in Hilbert spaces (thus $()^{\mathrm{H}}$ coincides with $()^{\mathrm{d}}$ on Hilbert spaces when we identify a Hilbert space and its dual).

Analogously, a complex vector space is also a real vector space. Next we record the invariance of certain properties under this modification:

Lemma A.3.21 $(\mathbf{X_R})$ *Let X be a complex vector space. Let $X_{\mathbf{R}}$ be X with scalars restricted to $\mathbf{R}$. Then $X_{\mathbf{R}}$ is a vector space.*

Assume, in addition, that X and Y are TVSs. Then $X_{\mathbf{R}}$ is a TVS, $X_{\mathbf{R}}^* = \mathrm{Re}\,X^* = \mathrm{Re}\,X^{\mathrm{d}}$ and $\mathcal{B}(X,Y) \subset \mathcal{B}(X_{\mathbf{R}}, Y_{\mathbf{R}})$. If X is normed, then the same norm is a norm on X too and $\|\mathrm{Re}\,\Lambda\| = \|\Lambda\|$ for all $\Lambda \in X^*$, in particular, $X_{\mathbf{R}}^* = \mathrm{Re}\,X^* = \mathrm{Re}\,X^{\mathrm{d}}$ isometrically. If X and Y are normed, then $\mathcal{B}(X,Y) \subset \mathcal{B}(X_{\mathbf{R}}, Y_{\mathbf{R}})$ linearly and isometrically; thus, if $T \in \mathcal{B}(X,Y)$, then $T_{\mathbf{R}} := T \in \mathcal{B}(X_{\mathbf{R}}, Y_{\mathbf{R}})$ and $\|T_{\mathbf{R}}\|_{\mathcal{B}(X_{\mathbf{R}};Y_{\mathbf{R}})} = \|T\|_{\mathcal{B}(X;Y)}$; moreover, then $(T^*)_{\mathbf{R}} = (T^{\mathrm{d}})_{\mathbf{R}}$ (i.e., $T^* = T^{\mathrm{d}}$ as elements of $\mathcal{B}(Y_{\mathbf{R}}^*, X_{\mathbf{R}}^*)$).

Moreover, X is complete (resp. metrizable, separable, normable, reflexive) iff $X_{\mathbf{R}}$ is. If $\langle\cdot,\cdot\rangle$ is an inner product in X, then $\mathrm{Re}\langle\cdot,\cdot\rangle$ is an inner product in

$X_{\mathbf{R}}$. *Finally,* $(X_{\mathbf{R}})_{\mathbf{R}} = X_{\mathbf{R}}$ *and* $\mathbf{C}_{\mathbf{R}} = \mathbf{R}^2$ *as TVSs, but* $\mathcal{B}(\mathbf{C}_{\mathbf{R}}) = \mathbf{R}^{2\times 2}$ *whereas* $\mathcal{B}(\mathbf{C}) = \mathbf{C}$.

Analogously, if Y is a real vector space (resp. [separable] TVS, normed space, Banach space, Hilbert space), then so is $Y_{\mathbf{C}} := Y + iY$ (with the topology and addition of Y^2 and natural scalar multiplication ($i(y+iy') := -y'+iy$)), but we do not need this.

We note that if X is normed, then $(X_{\mathbf{R}})^* = (X^*)_{\mathbf{R}} = (X^{\mathrm{d}})_{\mathbf{R}}$ as Banach spaces (if we identify Λ and $\operatorname{Re}\Lambda$); analogously, $X_{\mathbf{R}}^{**} = (X^*)_{\mathbf{R}}^* = (X^{**})_{\mathbf{R}}$, which establishes the reflexivity claim.

Proof of Lemma A.3.21: It is not hard to verify the above claims, some of which are given in Section 3.1 of [Rud73].

By $X_{\mathbf{R}}^* = \operatorname{Re} X^* = \operatorname{Re} X^{\mathrm{d}}$ we mean that $X_{\mathbf{R}}^* = \{\operatorname{Re}\Lambda \,|\, \Lambda \in X^* =: X^{\mathrm{B}}\} = \{\operatorname{Re}\Lambda \,|\, \Lambda \in X^{\mathrm{d}}\}$; cf. Remark A.3.22 ($X$ need not be normed; the identity map between vector spaces X^{B} and X^{d} is nevertheless a conjugate-linear bijection, hence the identity map between vector spaces $\operatorname{Re} X^{\mathrm{B}}$ and $\operatorname{Re} X^{\mathrm{d}}$ is a linear bijection, i.e., a vector space isomorphism (an isometric Banach space isomorphism if X is normed)). $\square$

As noted above, the Hilbert space adjoint $T^{\mathrm{H}} \in \mathcal{B}(Y,H)$ ("T^*") of a bounded linear operator $T \in \mathcal{B}(H,Y)$ is identical to the Banach space adjoint $T^{\mathrm{B}} \in \mathcal{B}(Y^*,H^*)$ ("T^*") when we identify the Hilbert spaces with their duals; however, this identification does not preserve scalar multiplication (in the nontrivial case with $\mathbf{K} = \mathbf{C}$): $(\alpha T)^{\mathrm{H}} = \bar{\alpha} T^{\mathrm{H}}$, whereas $(\alpha T)^{\mathrm{B}} = \alpha T^{\mathrm{B}}$.

But this follows from the fact that multiplication by α in Y corresponds to multiplication by $\bar{\alpha}$ in Y^*; by defining the multiplication in Y^* by the latter, we can make these two concepts identical:

Remark A.3.22 (Conjugate-linear dual X^{d} (or X^*)) *Let X and Y be normed spaces. Usually, the dual $X^* =: X^{\mathrm{B}}$ is equipped with scalar multiplication $(\alpha\Lambda)x := \alpha(\Lambda x)$ ($\alpha \in \mathbf{K},\ \Lambda \in X^*,\ x \in X$). However, X^* becomes a Banach space (denoted by X^{d}) also with the "conjugate-linear" scalar multiplication $(\alpha\Lambda)x := \bar{\alpha}(\Lambda x)$, i.e.,*

$$(\bar{\alpha}\Lambda)x =: \langle x, \bar{\alpha}\Lambda\rangle_{\langle X,X^{\mathrm{d}}\rangle} := \langle \alpha x, \Lambda\rangle_{\langle X,X^{\mathrm{d}}\rangle} = \alpha\langle x,\Lambda\rangle_{\langle X,X^{\mathrm{d}}\rangle} := \alpha(\Lambda x) =: \alpha\overline{\langle \Lambda, x\rangle_{\langle X^{\mathrm{d}},X\rangle}} \tag{A.30}$$

for all $x \in X,\ \Lambda \in X^*,\ \alpha \in \mathbf{K}$ (note that we have set $\langle \Lambda, x\rangle_{\langle X^{\mathrm{d}},X\rangle} := \overline{\Lambda x} = \overline{\langle x,\Lambda\rangle_{\langle X,X^{\mathrm{d}}\rangle}}$).

Then the identity mapping $I := I_{X^{\mathrm{d}}} : X^{\mathrm{d}} \to X^{\mathrm{B}}$ is a conjugate-linear (i.e., $I(\Lambda+\Lambda') = I\Lambda + I\Lambda'$, but $I\alpha\Lambda = \bar{\alpha}I\Lambda$) isometry of X^{d} onto X^{B}. In particular, all set-theoretic, topological and metric properties of X^{B} and X^{d} are identical. Moreover, if $T \in \mathcal{B}(X,Y)$ and we set

$$\langle x, T^{\mathrm{d}}y^{\mathrm{d}}\rangle_{\langle X,X^{\mathrm{d}}\rangle} := \langle Tx, y^{\mathrm{d}}\rangle_{\langle Y,Y^{\mathrm{d}}\rangle} \quad (x \in X,\ y^{\mathrm{d}} \in Y^{\mathrm{d}}), \tag{A.31}$$

then $T^{\mathrm{d}} \in \mathcal{B}(Y^{\mathrm{d}},X^{\mathrm{d}})$, $\|T^*\|_{\mathcal{B}(Y^{\mathrm{B}},X^{\mathrm{B}})} = \|T^{\mathrm{d}}\|_{\mathcal{B}(Y^{\mathrm{d}},X^{\mathrm{d}})} = \|T\|_{\mathcal{B}(Y,X)}$ and $T^{\mathrm{d}} = I_{X^{\mathrm{d}}}^* T^* I_{Y^{\mathrm{d}}}$. Thus, $\mathcal{B}(Y^{\mathrm{B}},X^{\mathrm{B}})$ and $\mathcal{B}(Y^{\mathrm{d}},X^{\mathrm{d}})$ are isometrically isomorphic.

The conjugate-linear adjoint satisfies algebraic laws $(ST)^{\mathrm{d}} = T^{\mathrm{d}}S^{\mathrm{d}}$ *and* $(\alpha S + \beta T)^{\mathrm{d}} = \bar{\alpha}S^{\mathrm{d}} + \bar{\beta}T^{\mathrm{d}}$.

The map $x \mapsto (\Lambda \mapsto \overline{\Lambda x})$ *is a (canonical) linear isometry* $X \to X^{\mathrm{dd}} := (X^{\mathrm{d}})^{\mathrm{d}}$ *(cf. the canonical linear isometry* $X \ni x \mapsto (\Lambda \mapsto \Lambda x) \in X^{\mathrm{BB}} := (X^{\mathrm{B}})^{\mathrm{B}}$*). If* X *is reflexive, then* $X^{\mathrm{dd}} := (X^{\mathrm{d}})^{\mathrm{d}}$ *is isometrically isomorphic to* X *(and we identify the two; analogously, we identify* X^{BB} *and* X *through the latter isometry when we use "linear duals").*

If X *is a Hilbert space, then* X^{d} *becomes isometrically isomorphic to* X *(and we identify the two). In particular, if* X *and* Y *are Hilbert spaces and* $T \in \mathcal{B}(X,Y)$*, then* T^{d} *becomes the Hilbert space adjoint of* T. □

Thus, $T^{\mathrm{d}} = T^{\mathrm{B}}$ except that their (otherwise equal) domain and range spaces have differently defined scalar multiplication.

The name "conjugate-linear dual" is somewhat misleading: we stress that also X^{d} is an ordinary Banach space over $\mathbf{K}$; in particular, the scalar multiplication (and addition) satisfies standard vector space axioms. The choice $(\alpha\Lambda)x := \alpha\Lambda x$ is certainly not the only one to provide a vector space structure for X^*, rather it is an unnatural one for any complex Hilbert space.

Note that the pairing $X \times X^{\mathrm{d}} \to \mathbf{K}$ becomes sesquilinear, i.e., it is linear in its first and conjugate-linear in its second argument; the same applies to $X^{\mathrm{d}} \times X \to \mathbf{K}$ (whereas $X \times X^* \to \mathbf{K}$ and $X^* \times X \to \mathbf{K}$ are bilinear).

In Hilbert space context we use Hilbert adjoints instead of Banach adjoints (i.e., $T^* := T^{\mathrm{d}}$ when T maps between Hilbert (or inner product) spaces; otherwise $T^* := T^{\mathrm{B}}$), following the standard convention. Usually, $X^* := X^{\mathrm{B}}$ (for normed spaces X), but in pivot space context we use sesquilinear pairings, i.e., $X^* := X^{\mathrm{d}}$; see Definition A.3.23 for details (this should be clear from the context). Of course, there is no difference when $\mathbf{K} = \mathbf{R}$.

Another common convention in infinite-dimensional control theory is to use pivot spaces (see Definition A.3.23), to have the pairing between spaces and their duals coincide to the inner product (or its extension) in a *pivot space*, and hence dependent only on the elements to be paired, not on the spaces.

To illustrate this, let $f \in \mathrm{L}^2_\omega(\mathbf{R})$, $g \in \mathrm{L}^2_{-\omega}(\mathbf{R}) = \mathrm{L}^2_\omega(\mathbf{R})^*$. Then we use $\mathrm{L}^2(\mathbf{R})$ as the *pivot space* by setting

$$\langle f, g\rangle_{\langle \mathrm{L}^2_\omega, \mathrm{L}^2_{-\omega}\rangle} := \int_{\mathbf{R}} \langle f(t), g(t)\rangle_{\mathbf{K}}\, dt := \int_{\mathbf{R}} \overline{g(t)} f(t)\, dt; \tag{A.32}$$

in particular, $\langle f, \alpha g\rangle = \bar{\alpha}\langle f, g\rangle$ (thus, αg corresponds to $\bar{\alpha}\Lambda$, where Λ is the corresponding element in the linear dual of L^2_ω). If $f \in \mathrm{L}^2_\omega \cap \mathrm{L}^2$ and $g \in \mathrm{L}^2_{-\omega} \cap \mathrm{L}^2$, then it follows that $\langle f, g\rangle_{\langle \mathrm{L}^2_\omega, \mathrm{L}^2_{-\omega}\rangle} = \langle f, g\rangle_{\mathrm{L}^2}$ (here the right-hand-side refers to inner product; or, equivalently, to the pairing $\mathrm{L}^2 \times (\mathrm{L}^2)^{\mathrm{d}} \to \mathbf{K}$).

Moreover, if by Y^{B} we denote the linear dual (i.e., the dual space in the ordinary sense) of $Y := \mathrm{L}^2_\omega$, then Y^{B} can be identified with $\mathrm{L}^2_{-\omega}$, by $Y \times Y^{\mathrm{B}} \ni (f,g) \mapsto \int_{\mathbf{R}} fg\, dm$. Thus, then the canonical identification $Y^{\mathrm{B}} \to Y^* := Y^{\mathrm{d}}$ becomes $Y^{\mathrm{B}} \ni g \mapsto \bar{g} \in Y^*$, whereas the identification $J : Y^* \to Y$ (corresponding to the inner product in Y instead of L^2) is given by $Jg := e^{2\omega\cdot} g$. Both these identifications are bijective isometries, but the former is conjugate-linear whereas the latter is linear (hence an isometric isomorphism), and neither is the isometric isomorphism (w.r.t. to the pivot space) illustrated above and defined below.

Thus, we take adjoints and duals w.r.t. a pivot space instead of initial and range spaces, hence only the pivot space remains the dual of itself. Formally this goes as follows:

Definition A.3.23 (Pivot space) *Let H be a Hilbert space. When we say that we use H as a* pivot space*, we mean the following:*

We use "conjugate-linear duals" (see Remark A.3.22) and Banach adjoints instead of Hilbert adjoints for any appropriate spaces.

If X is a normed space s.t. $X \cap H$ is dense in X, and if $y \in H$ is s.t. the operator $\langle \cdot, y \rangle_H : X \cap H \ni x \mapsto \langle x, y \rangle_H \in \mathbf{K}$ is continuous in the $\| \cdot \|_X$ norm, i.e.,

$$|\langle x, y \rangle_H| \leq M_y \|x\|_X \quad (x \in X), \tag{A.33}$$

then we identify y with the unique continuous extension of $\langle \cdot, y \rangle_H$ in X^* (thus, then $y \in H \cap X^*$).

Thus, $H \cap X^*$ becomes identified with a dense subspace of X^*, and $\langle x, y \rangle_{\langle X, X^* \rangle} = \langle x, y \rangle_H$ for all $x \in X \cap H,\ y \in H \cap X^*$.

Some aspects of pivot spaces are treated in [Keu] with further details, but [Keu] only mentions the cases $X \subset H$ and $H \subset X$, hence it would require one to treat L^2_ω by parts (on $\mathbf{R}_\pm$).

Lemma A.3.24 *Definition A.3.23 is well posed.*

Let X and H be as in Definition A.3.23. Then $H^* = H$ (this is an isometric isomorphism, in particular, $\langle x, y \rangle_{\langle H, H^* \rangle} = \langle x, y \rangle_H$ for all $x, y \in H$), and the identification $H \cap X^* \to X^*$ is linear and one-to-one.

Assume that Y is a normed space and $H \underset{c}{\subset} Y$ densely. Then $Y^* \underset{c}{\subset} H$ densely and $H \underset{c}{\subset} Y \underset{c}{\subset} Y^{**}$.

Assume that $Z \underset{c}{\subset} H$ is a normed space and dense in H. Then $H \underset{c}{\subset} Z^*$. If Z is reflexive, then H is dense in Z^* and Z becomes identified to Z^{**} in the pivot space sense (with the canonical identification), thus, then $Z = Z^{**} \underset{c}{\subset} H \underset{c}{\subset} Z^* = Z^{***}$.

Assume that also $V \underset{c}{\subset} H$ is a dense (in H) normed space, and $T \in \mathcal{B}(H) \cap \mathcal{B}(V, Z)$. Then $T^* \in \mathcal{B}(Z^*, V^*) \cap \mathcal{B}(H)$. If $T = T^* \in \mathcal{B}(H) \cap \mathcal{B}(V, Z)$, then $T = T^* \in \mathcal{B}(V, Z) \cap \mathcal{B}(Z^*, V^*)$.

If, in addition, $Z \subset V$ densely, then $H \underset{c}{\subset} V^* \underset{c}{\subset} Z^*$.

By Lemma A.3.13, we have for any normed space N that if $T \in \mathcal{B}(N, Z)$, then $T^* \in \mathcal{B}(Z^*, N^*) \underset{c}{\subset} \mathcal{B}(H, N^*)$; if $T \in \mathcal{B}(Y, N)$, then $T^* \in \mathcal{B}(N^*, Y^*) \underset{c}{\subset} \mathcal{B}(N^*, H)$.

Note that for H, nothing has changed, but the duals of its proper (Hilbert) subspaces are no longer identified with their duals, although the two spaces are still isometrically isomorphic, by Remark A.3.22.

Following the standard convention, we shall use L^2 (resp. H) as the pivot space when taking adjoints in L^2_ω (resp. in $\mathrm{Dom}(A)$ or $\mathrm{Dom}(A^*)$, where A generates a C_0-semigroup in H). This is illustrated in (6.2) and Definition 6.1.17.

Finally we note that given a fixed pivot space H and its dense subspaces $B_1, B_2, V_1, V_2 \underset{c}{\subset} H$, and elements $w \in B_1 \cap B_2$, $w^* \in B_1^* \cap B_2^*$, we have $\langle w, w^*\rangle_{B_1,B_1^*} = \langle w, w^*\rangle_{B_2,B_2^*}$, i.e., the pairing is independent of the spaces chosen (as long as the pairing is defined). Therefore, the adjoints of an operator $F \in \mathcal{B}(B_1,V_1) \cap \mathcal{B}(B_2,V_2)$ (w.r.t. the pivot space H) coincide on $V_1^* \cap V_2^*$ (i.e., $\langle w, F^*v^*\rangle_{\langle B_1,B_1^*\rangle} := \langle Fw, v^*\rangle_{V_k,V_k^*} = \langle w, F^*v^*\rangle_{\langle B_2,B_2^*\rangle}$ for $k = 1,2$ and $v^* \in V_1^* \cap V_2^*$).

Proof of Lemma A.3.24: $1°$ By density, the identification of "$H \cap X^{*}$" to X^* is well defined; in particular, this identification is injective and linear (recall that $X^* = X^{\mathrm{d}}$).

$2°$ *Case $H \underset{c}{\subset} Y$ densely:* By density, the canonical embedding $Y^* \subset H^* = H$ one-to-one. Obviously, it is linear and continuous. Because the identity (inclusion) operator $I \in \mathcal{B}(H,Y)$ is one-to-one (recall that inclusions are required to be one-to-one), the range of the canonical embedding $I^* \in \mathcal{B}(Y^*,H^*) = \mathcal{B}(Y^*,H)$ is dense in H, by Lemma A.3.4(N2) (replace Y by its completion; here $I^*\Lambda = \Lambda_{|H}$ for $\Lambda \in Y^*$; note that $\langle Ih, \Lambda\rangle_{\langle Y,Y^*\rangle} = \langle h, I^*\Lambda\rangle_{\langle H,H^*\rangle} = \langle h, I^*\Lambda\rangle_H$ for $h \in H,\ \Lambda \in Y^*$).

Consequently, $H \underset{c}{\subset} Y^{**}$, by Lemma A.3.13. By definition, $\langle Ih, y^*\rangle_{\langle Y,Y^*\rangle} = \langle h, I^*y^*\rangle_H = \langle Jh, y^*\rangle_{\langle Y,Y^*\rangle}$ for all $h \in H,\ y^* \in Y^*$, where J is the (identity) inclusion $H \to Y^{**}$ (of Definition A.3.23). Therefore, $Ih = Jh$ ($h \in H$), hence the completions Y and Y^{**} of H become equal, and the identification JI^{-1} extends to the linear isometry $T : Y \to Y^{**}$ that satisfies $\langle Iy, y^*\rangle_{\langle Y^{**},Y^*\rangle} = \langle y, y^*\rangle_{\langle Y,Y^*\rangle}$ for all $y \in Y,\ y^* \in Y^*$. Thus, the identification $H \to Y$ is the same whether we consider Y as itself or as a subspace of Y^{**}.

$3°$ *Case $Z \underset{c}{\subset} H$ densely:* By Lemma A.3.13, we have $H \underset{c}{\subset} Z^*$. Assume that Z is reflexive. Then H is dense in Z^*, by Lemma A.3.4(N2), hence $Z^{**} \underset{c}{\subset} H$, by $1°$. But the elements of Z^{**} are those $h \in H$, for which $|\langle z^*, h\rangle_H| \le M\|z^*\|_{Z^*}$ for all $z^* \in H$ (i.e., those for which $\langle \cdot, h\rangle_H : H \to \mathbf{K}$ extends continuously to $Z^* \to \mathbf{K}$); reflexivity implies that this holds only for $h \in Z$, hence $Z = Z^{**}$ (and $\langle z^*, z\rangle_{\langle Z^*,Z^{**}\rangle} = \langle z^*, z\rangle_{\langle Z^*,Z\rangle}$ for all $z^* \in H$, hence for all $z^* \in Z^*$, by density). By $2°$, we have $Z^* = Z^{***}$.

$4°$ *Case $T \in \mathcal{B}(H) \cap \mathcal{B}(V,Z)$:* (By Lemma A.3.6, this is the case iff $T \in \mathcal{B}(H)$ is s.t. $T[V] \subset Z$.) If $z^* \in H$ and $x_0 \in V$, then

$$\langle x_0, T^*z^*\rangle_{\langle V,V^*\rangle} = \langle Tx_0, z^*\rangle_{\langle Z,Z^*\rangle} = \langle Tx_0, z^*\rangle_H = \langle x_0, T^{\mathrm{H}}z^*\rangle_H, \tag{A.34}$$

where $T^{\mathrm{H}} \in \mathcal{B}(H)$ (resp. $T^* \in \mathcal{B}(Z^*,V^*)$) denotes the adjoint of $T \in \mathcal{B}(H)$ (resp. $T \in \mathcal{B}(V,Z)$), hence then $T^*z^* = T^{\mathrm{H}}z^* \in H$. Thus, $T^*{}_{|H} = T^{\mathrm{H}}$, so that we can write $T^* \in \mathcal{B}(Z^*,V^*) \cap \mathcal{B}(H)$. The $T = T^*$ claim is obvious.

$5°$ *$H \underset{c}{\subset} V^* \underset{c}{\subset} Z^*$:* By Lemma A.3.13, we have $\mathcal{B}(V,\mathbf{K}) \underset{c}{\subset} \mathcal{B}(Z,\mathbf{K})$. We have shown above that $H \underset{c}{\subset} V^*$. □

Notes

As indicated in the proofs, most results of this section are well known at least to some extent. Sum-compatibility, $X \cap Y$ and $X + Y$ are defined in [BL], which also contains parts of the corresponding lemmas.

For further results on Hilbert and Banach spaces or Banach algebras, see any textbook on functional analysis; standard references include [Yosida], [Rud73], [Rud86] and [HP].

A.4 C_0-Semigroups

Could Hamlet have been written by committee, or the Mona Lisa painted by a club? Creative ideas don't spring from groups. They spring from individuals.

— Alfred Whitney Griswold (1906–1963)

In this section, we present basic facts on strongly continous (C_0) semigroups. Throughout the section, the Hilbert and Banach spaces are assumed to be complex.

The solution of $x' = Ax$, $x(0) = x_0$ is $\mathrm{e}^{At}x_0$ for $x_0 \in B$, $A \in \mathcal{B}(B)$ and a Banach space B. For more general control systems, the operator A need not be bounded, hence the (C_0-)semigroup "$t \mapsto \mathrm{e}^{At}$" of a control system is usually more complicated:

Definition A.4.1 (Semigroups) *A C_0-*semigroup *on a complex Banach space B is a function $\mathbb{A} : [0,\infty) \to \mathcal{B}(B)$ having the following properties:*

$$\mathbb{A}(t+s) = \mathbb{A}(t)\mathbb{A}(s) \quad \textit{for } t,s \geq 0; \tag{A.35}$$

$$\mathbb{A}(0) = I; \tag{A.36}$$

$$\|\mathbb{A}(t)x_0 - x_0\| \to 0 \quad \textit{as } t \to 0+ \quad \forall x_0 \in B. \tag{A.37}$$

The (infinitesimal) generator *A of $\mathbb{A}$ is defined by*

$$Ax := \lim_{t\to 0+} \frac{1}{t}\left(\mathbb{A}(t) - I\right)x; \tag{A.38}$$

$\mathrm{Dom}(A) \subset B$ *is the set of $x \in B$ for which the limit Ax exists.*

If $s \in \mathbf{C}$ is s.t. $s - A := sI - A$ maps $\mathrm{Dom}(A)$ *one-to-one onto B, and the* resolvent $(s-A)^{-1}$ *is bounded ($(s-A)^{-1} \in \mathcal{B}(B)$), then $s \in \sigma(A)^c$; this is the definition of the* spectrum $\sigma(A)$ *of A.*

We also define the growth rate $\omega_A := \inf_{t>0}\left(t^{-1}\log\|\mathbb{A}(t)\|\right)$ *of $\mathbb{A}$.*

Let $\omega \in \mathbf{R}$. We call $\mathbb{A}$ ω-stable *if $t \mapsto \|\mathrm{e}^{-\omega t}\mathbb{A}(t)\|$ is bounded on $\mathbf{R}_+$. We call $\mathbb{A}$* strongly (resp. weakly) ω-stable *if $\mathrm{e}^{-\omega t}\mathbb{A}(t) \to 0$ strongly (resp. weakly) as $t \to +\infty$. We call $\mathbb{A}$* exponentially stable *if $\mathbb{A}$ is ω-stable for some $\omega < 0$. By* [strongly/weakly] stable *we mean* [strongly/weakly] 0-stable.

The "C_0" condition (A.37) ("strongly continuous at 0") is equivalent to strong continuity (see (a1) below). In this monograph, all semigroups will be C_0-semigroups, hence we usually call them just semigroups. Sometimes we write $\mathbb{A}^t$ for $\mathbb{A}(t)$.

Note that A is the strong right derivative of $\mathbb{A}$ at 0, and recall that "$\mathbb{A}(t) \to 0$ strongly (resp. weakly)" means that $\mathbb{A}(t)x \to 0$ for all $x \in B$ ($\Lambda\mathbb{A}(t)x \to 0$ for all $x \in B$ and $\Lambda \in B^*$).

We need a number of facts on C_0-semigroups:

Lemma A.4.2 *Assume that B and B_2 are complex Banach spaces, $\mathbb{A}$ is a C_0-semigroup on B, A be the generator of $\mathbb{A}$ and $t \geq 0$. Then the following hold:*

(a1) $\mathbb{A}(\cdot)x \in C([0,\infty);B)$ *for all $x \in X$.*

(a2) $A \in \mathcal{B}(B)$ *iff* $\mathbb{A}$ *is uniformly continuous (i.e., iff* $\mathbb{A} \in \mathcal{C}([0,\infty);\mathcal{B}(B))$*). If* $A \in \mathcal{B}(B)$*, then* $\mathbb{A}(t) = \mathrm{e}^{At}$ $(t \geq 0)$*.*

(a3) $A \in \mathcal{B}(B)$ *iff* $\mathbb{A}$ *is (uniformly) Bochner-measurable (see Definition B.2.1).*

(a4) $\|\mathbb{A}(\cdot)\|$ *and* $\mathbb{A}f$ *are measurable for any measurable* $f : \mathbf{R}_+ \to B$*.*

(b) $\cap_{n\in\mathbf{N}} \mathrm{Dom}(A^n)$ *is dense in* B*.*

(c1) $\exists(\mathbb{A}x)'(t) = A\mathbb{A}(t)x = \mathbb{A}(t)Ax$ *for all* $x \in \mathrm{Dom}(A)$*.*

(c2) $\int_0^t \mathbb{A}(r)x\,dr \in \mathrm{Dom}(A)$ *and* $A\int_0^t \mathbb{A}(r)x\,dr = \mathbb{A}(t)x - x$ *for all* $x \in H$*. In particular,* $\int_0^t \mathbb{A}(r)\cdot dr \in \mathcal{B}(H,\mathrm{Dom}(A))$*.*

(c3) $\mathbb{A}(t)x = \lim_{s\to+\infty} \mathrm{e}^{tsA(s-A)^{-1}}x$ *for all* $x \in H$*.*

(c4) $\mathbb{A}^t[\mathrm{Dom}(A^n)] \subset \mathrm{Dom}(A^n)$ *for all* $n \in \mathbf{N}$*.*

(c5) $\mathbb{A}x \in \mathcal{C}^k(\mathbf{R}_+;\mathrm{Dom}(A^{n-k}))$ *and* $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathbb{A}x = A^k\mathbb{A}x = \mathbb{A}A^kx$ $(n \in \mathbf{N},\ k = 0,1,\ldots,n,\ x \in \mathrm{Dom}(A^n))$*.*

(d) The set $\sigma(A)$ *is closed, and* $\operatorname{Re} s > \omega_A \Rightarrow s \in \sigma(A)^c$*.*

(e1) $\omega_A = \lim_{t\to+\infty}\left(t^{-1}\log\|\mathbb{A}(t)\|\right) < \infty$*.*

(e2) If $\omega > \omega_A$*, then there is* $M_\omega > 0$ *s.t.* $\|\mathbb{A}(t)\| \leq M_\omega \mathrm{e}^{\omega t}$ *for all* $t \geq 0$*.*

(f) If B *is a Hilbert space, then* $\mathbb{A}^*$ *is a* C_0*-semigroup on* B *and* A^* *is its generator.*

(g1) If $T \in \mathcal{B}(B)$*, then* $A + T$ *is a* C_0*-semigroup on* B*.*

(g2) The generator of $\mathrm{e}^{\alpha\cdot}\mathbb{A}(\cdot)$ *is* $\alpha I + A$ *with* $\mathrm{Dom}(\alpha + A) = \mathrm{Dom}(A)$*, and* $\omega_{\alpha I+A} = \alpha + \omega_A$*.*

(h1) If $T \in \mathcal{GB}(B,B_2)$*, then* $\widetilde{\mathbb{A}} := T\mathbb{A}T^{-1}$ *is a* C_0*-semigroup on* B_2*, and its generator* $\widetilde{A}$ *is given by* $\widetilde{A} = TAT^{-1}$*,* $\mathrm{Dom}(\widetilde{A}) = T\,\mathrm{Dom}(A)$*.*

Moreover, $M^{-1}\|\mathbb{A}\| \leq \|\widetilde{\mathbb{A}}\| \leq M\|\mathbb{A}\|$*, where* $M := \|T\|\|T^{-1}\|$*, hence* $\omega_{\widetilde{A}} = \omega_A$*. For* $\omega \in \mathbf{R}$*, the semigroup* $\widetilde{\mathbb{A}}$ *is [strongly (resp. weakly)]* ω*-stable iff* $\mathbb{A}$ *is.*

(h2) Let $R \in \mathcal{B}(B_2,B)$ *and* $L \in \mathcal{B}(B,B_2)$*. Then* $\mathbb{A}_{LR} := L\mathbb{A}R$ *is a* C_0*-semigroup on* B_2 *iff* $LR = I$ *and* $P\mathbb{A}^tP\mathbb{A}^sP = P\mathbb{A}^{t+s}P$ *for all* $t,s > 0$*, where* $P := RL \in \mathcal{B}(B)$*.*

Assume this. Then $\omega_{A_{LR}} \leq \omega_A$ *and* $(s - A_{LR})^{-1} = L(s-A)^{-1}R$ *for* $s \in \mathbf{C}^+_{\omega_A}$*. Moreover, if* $y \in B_2$ *and* $Ry \in \mathrm{Dom}(A)$*, then* $y \in \mathrm{Dom}(A_{LR})$ *and* $A_{LR}y = LARy$*.*

(i) If also $\widetilde{\mathbb{A}}$ *is a* C_0*-semigroup on* B *and* $A \subset \widetilde{A}$*, then* $A = \widetilde{A}$*.*

By (c3), $\mathbb{A}$ is uniquely determined by A (the converse is trivial).

By (a3), we must not write $\left(\int_0^t \mathbb{A}(r)\,dt\right)x$ in (c2); see Appendix B for (Bochner) measurability and integration; alternatively, the strong integration theory of Section F.2 should be used.

By (d), we have $\omega_A \geq \sup\operatorname{Re}\sigma(A)$. For differentiable semigroups we have $\omega_A = \sup\operatorname{Re}\sigma(A)$, but this is not the case in general (see Example 5.14 of [CZ] for a counter-example).

Proof of Lemma A.4.2: The canonical references on semigroups are [Pazy] and [HP], but [Rud73], [CZ], [Prüss93] and [Sbook] also contain many of the above results. We only treat here the least known ones.

For (a3)&(a4) we note that on pp. 304–306 of [HP] it is stated that a uniformly measurable iff it is uniformly continuous (and that if $\mathbb{A}$ is weakly continuous on $(0,\infty)$, then it is strongly continuous on $(0,\infty)$ and hence strongly measurable on $\mathbf{R}_+$), and that for a C_0 semigroup $\mathbb{A}$, the function $\|\mathbb{A}(\cdot)\|$ is lower semicontinuous, hence measurable (the measurability of $\mathbb{A}f$ follows from Lemma F.1.3(a)).

We give below a sketch of the proofs for (h) and (i).

(h1) Now $t^{-1}(\mathbb{A}(t)x-x)\to 0 \Leftrightarrow t^{-1}(T\mathbb{A}(t)T^{-1}Tx-Tx)\to 0$, the claims on $\widetilde{A}$ follow from this. The claims on M are obvious; use them and (e1) for ω_A. The stability claims are again obvious.

(h2) If $L\mathbb{A}R$ is a semigroup and $t,s\ge 0$, then $L\mathbb{A}^tRL\mathbb{A}^sR = L\mathbb{A}^{t+s}R$, hence then $P\mathbb{A}^tP\mathbb{A}^sP = P\mathbb{A}^{t+s}P$. Conversely, assume that $P\mathbb{A}^tP\mathbb{A}^sP = P\mathbb{A}^{t+s}P$, hence $L\mathbb{A}^tRL\mathbb{A}^sR = L\mathbb{A}^{t+s}R$, for all $t,s>0$. Obviously, $\mathbb{A}^t_{LR}x = L\mathbb{A}^tRx \to LRx = x$, as $t\to 0+$, hence $\mathbb{A}_{LR}$ has the C_0-property. (Note also that $P^2 = R(LR)L = P$, i.e., P is a projection.)

Obviously, $\omega_{A_{LR}}\le\omega_A$. For $s\in\mathbf{C}^+_{\omega_A}$ and $x\in B$, we have $(s-A_{LR})^{-1}x = L\int_0^\infty e^{-st}\mathbb{A}^tRx\,dt = L(s-A)^{-1}Rx$, by Lemma A.4.4(f).

If $y\in B_2$ and $Ry\in\mathrm{Dom}(A)$, then $Lt^{-1}[\mathbb{A}^tRy-Ry]\to LARy$, hence then $y\in Dom(A)$ and $A_{LR}y = LARy$.

(i) By $A\subset\widetilde{A}$ we mean that $\mathrm{Dom}(A)\subset\mathrm{Dom}(\widetilde{A})$ and $A=\widetilde{A}$ on $\mathrm{Dom}(A)$. Let $\omega>\omega_A,\omega_{\widetilde{A}}$. Then $\omega I-\widetilde{A}$ maps $\mathrm{Dom}(A)$ one-to-one onto B and $\mathrm{Dom}(\widetilde{A})$ one-to-one onto B, hence $\mathrm{Dom}(\widetilde{A}) = \mathrm{Dom}(A)$, hence $A=\widetilde{A}$, hence $\mathbb{A}=\widetilde{\mathbb{A}}$, by (c3). □

The celebrated Hille–Yosida Theorem gives necessary and sufficient conditions for an operator to generate a C_0 semigroup:

Theorem A.4.3 (Hille–Yosida) *A linear operator $A:\mathrm{Dom}(A)\to B$ is the generator of a C_0-semigroup $\mathbb{A}$ s.t. $\|\mathbb{A}(t)\|_B\le Me^{\omega t}$ $(t\ge 0)$ for some $M<\infty$ iff*

(i) A is closed and $\mathrm{Dom}(A)$ is dense in B;

(ii) $(\omega,\infty)\subset\sigma(A)^c$ and

$$\|(s-A)^{-n}\|\le M(s-\omega)^{-n}\quad \textit{for all}\quad s>\omega,\ n=1,2,3,\ldots \tag{A.39}$$

□

(See, e.g., Theorem I.5.3 of [Pazy] for the proof.)

The resolvents of the generators of semigroups has been studied extensively, here we list some important facts:

Lemma A.4.4 *Let $\mathbb{A}$ be a C_0-semigroup on a complex Banach space B, and let A be its generator. Define $H_1 := \mathrm{Dom}(A)$ and H_{-1} as in Lemma 6.1.16. Then, for all $x\in H$, the following holds:*

(a) $(s-A)^{-1}-(r-A)^{-1}=(r-s)(s-A)^{-1}(r-A)^{-1}\in \mathrm{H}(\mathbf{C}^+_{\omega_A};\mathcal{B}(H_{-1},H_1))$ *for a fixed* $r\in\sigma(A)^c$ *(this is called the* resolvent equation*).*

(b) $(s-A)^{-1}\in \mathrm{H}(\mathbf{C}^+_{\omega_A};\mathcal{B}(H,H_1))$.

(c1) $\|(s-A)^{-1}\|_{\mathcal{B}(H)}\le M/(\mathrm{Re}\, s-\omega_A)$ *(*$s\in\mathbf{C}^+_{\omega_A}$*) for some* $M<\infty$.

(c2) $\|s(s-A)^{-1}\|_{\mathcal{B}(H)}\le M_r$ *and* $\|(s-A)^{-1}\|_{\mathcal{B}(H,H_1)}\le M_r$ *(*$s\ge r$*), when* $r>\omega_A$.

(c3) $(s-A)^{-1}\in \mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(H))\cap \mathrm{H}(\mathbf{C}^+_r;\mathcal{B}(H,H_1))\cap \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_r;\mathcal{B}(H))$ *for* $r>\omega_A$.

(d1) $\mathrm{Dom}(A)\ni s(s-A)^{-1}x\longrightarrow x$ *in* H*, as* $s\to+\infty$.

(d2) $H\ni A(s-A)^{-1}x\longrightarrow 0$ *in* H*, as* $s\to+\infty$.

(d3) $\mathrm{Dom}(A)\ni(s-A)^{-1}x\longrightarrow 0$ *in* $\mathrm{Dom}(A)$*, as* $s\to+\infty$.

(d4) $s(s-A)^{-1}(s-A)^{-1}x\to 0$ *in* $\mathrm{Dom}(A)$*, as* $s\to+\infty$.

(d5) $s(s-A)^{-1}s(s-A)^{-1}x\to x$ *in* H*, as* $s\to+\infty$.

(e1) The limits in (d1)–(d5) exist also as $\mathrm{Re}\, s\to+\infty$ *and* $s\in\{re^{i\theta}\,|\,r\ge 0,\ |\theta|<\pi/2-\varepsilon\}$ *for any* $\varepsilon>0$.

(e2) The limits in (d1)–(d5) exist also as $s\in\Sigma_{\theta,\omega}$*,* $|s|\to\infty$ *for some* $\theta>\pi/2$ *if the semigroup generated by* A *is analytic and* $\omega>\omega_A$.

(f) $(s-A)^{-1}x=\int_0^\infty e^{-st}\mathbb{A}(t)x\,dt$ *(*$s\in\mathbf{C}^+_{\omega_A}$*).*

(g) $A\mathbb{A}^t=\mathbb{A}'(t)=\mathbb{A}^tA$ *and* $A(s-A)^{-1}=s(s-A)^{-1}-I=(s-A)^{-1}A$ *on* $\mathrm{Dom}(A)$*, and* $(s-A)^{-1}\mathbb{A}^t=\mathbb{A}^t(s-A)^{-1}$ *and* $(s-A)^{-1}(r-A)^{-1}=(r-A)^{-1}(s-A)^{-1}$ *on* H*, for all* $s,r\in\sigma(A)^c$*,* $t\ge 0$.

We recall that $s\to+\infty$ refers to the limit at $+\infty$ along $\mathbf{R}$. Note that the results given for H can applied on H_1 and H_{-1} too if we restrict of extend A accordingly, because the three A's are isomorphic, as noted at Lemma 6.1.16. E.g., it follows from (a) that we also have $H\ni s(s-A)^{-1}x\longrightarrow x$, as $s\to+\infty$ for all $x\in H_{-1}$.

See Appendix D for holomorphic functions ($\mathrm{H}(\mathbf{C}^+_r)$).

Proof of Lemma A.4.4: (a) The resolvent equation (in $\mathcal{B}(H,H_1)$) is readily computed. The "$\in \mathrm{H}(\mathbf{C}^+_{\omega_A};\mathcal{B}(H_{-1},H_1))$" claim follows from (e).

(b) Equation (d1) implies that $\exists\ \frac{d}{ds}(s-A)^{-1}=-(s-A)^{-2}$, even in $\mathcal{B}(H,H_1)$ (use Lemma D.1.1).

(c1) This holds by the Hille–Yosida Theorem [Pazy, Theorem 1.5.3],

(c2) Because $s/(s-\omega_A)=1+\omega_A/(s-\omega_A)\le r/(r-\omega_A)$ ($s>r>\omega_A\ge 0$), and $s/(s-\omega_A)<1$ ($s>\omega_A<0$), we can set $M_r:=M\max\{1,r/(r-\omega_A)\}$ to obtain $\|s(s-A)^{-1}\|\le M_r$, by (c1). Because $A(s-A)^{-1}=s(s-A)^{-1}-I$, we can replace M_r by M_r+1 to obtain the second inequality.

(Obviously, we can allow s to belong to any sector $S_{r,T}:=\{s=r+z\,|\,\mathrm{Re}\, z>0,\ z/\mathrm{Re}\, z\le T\}$, or rectangular $R_{r,T}:=\{s\in\mathbf{C}^+_r\,|\,\mathrm{Im}\, s\le T\}$, where $r>\omega_A$, $T<\infty$.)

(c3) This follows from (a), (b) and Lemma A.4.5(i)&(v) (cf. Remark 6.1.9).

(d1)&(d2) Choose some $r>\omega_A$. Define $r_{x,s}:=\|x-s(s-A)^{-1}x\|_H=\|A(s-A)^{-1}x\|_H$. For $x\in H_1$ we have $r_{x,s}=\|(s-A)^{-1}Ax\|<M_r\|Ax\|/s\to 0$, by (c2), hence $r_{x,s}\to 0$ for all $x\in H$, by the uniform boundedness of $s(s-A)^{-1}$ and the density of $\mathrm{Dom}(A)$ in H. Thus (d1) and (d2) are true.

(d3) By (d2), $\|(s-A)^{-1}x\|_{\mathrm{Dom}(A)} := \|(s-A)^{-1}x\|_H + \|A(s-A)^{-1}x\|_H \to 0+0=0$.

(d4) This follows from (d3) and (c2).

(d5) Apply (d1) and Lemma A.3.1(j3) (see Lemma A.3.4(F1)).

(e1) This can be seen from the above proofs with slight modifications, because $s/\operatorname{Re} s = (\cos\theta)^{-1}$ is bounded.

(e2) The above proofs of (b1)–(b5) hold also when $s \in \Sigma_{\theta,\omega}$, $|s| \to \infty$ if the semigroup generated by A is analytic and θ is as in Lemma 9.4.2(a).

(f) See p. 20 of [Pazy].

(g) We obtain $(s-A)^{-1}\mathbb{A}^t = \mathbb{A}^t(s-A)^{-1}$ from (a) and Lemma A.4.2(c3). Use (a) and Lemma A.4.2(c1) for the others. □

We will also need the following "extended Datko's Theorem":

Lemma A.4.5 (Datko) *The following are equivalent for a C_0-semigroup $\mathbb{A}$ on H:*

(i) $\mathbb{A}$ is exponentially stable;

(ii) $\mathbb{A}(\cdot)x_0 \in \mathrm{L}^2(\mathbf{R}_+;H)$ for all $x_0 \in H$ (or equivalently, for all x_0 in a dense subset of H);

(iii) $\|\int_0^\infty \mathbb{A}(s)\phi(s)\,ds\|_H \le M\|\phi\|_2$ for all $\phi \in \mathcal{C}_c^\infty((0,\infty);H)$;

(iv) $(s-A)^{-1} \in \mathrm{H}^\infty(\mathbf{C}^+;\mathcal{B}(H))$;

(v) $(s-A)^{-1} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(H))$;

(vi) $\mathbb{A}^*$ satisfies some (hence all) of (i)–(vi).

Note also that $\mathbb{A}$ is exponentially stable iff $\mathbb{A}^*$ is. As the proof shows (see Theorem B.4.12), it is enough that (iii) holds for $\phi \in \mathcal{C}_c^\infty((0,\infty);H_0)$, for some norming subspace $H_0 \subset H$. Also (ii) can be weakened: by Theorem 1.1 of [W88], it suffices that for some $p \in [1,\infty)$ we have $\int_0^\infty |\langle x_1, \mathbb{A}(t)x_0\rangle|^p\,dt < \infty$ for all $x_0, x_1 \in H$. See also [Sbook, Theorem 3.11.8] for a generalization.

Proof of Lemma A.4.5: 1° Obviously, $\mathbb{A}^*$ is exponentially stable iff $\mathbb{A}$ is; in particular, we only need to establish the equivalence between (i)–(v).

2° The equivalence (i)⇔(iv) is shown in [Prüss84] (and in Theorem 3.11.6 of [Sbook]); the equivalence (ii)⇔(v) follows from the Plancherel Theorem.

3° If (i) holds, then $\|\mathbb{A}\| \in \mathrm{L}^2$ (cf. Lemma A.4.2(a4)), hence then (ii) and (iii) hold. The implication (ii)⇒(i) is [Pazy, Theorem 4.4.1, p. 116], so we assume (iii) and prove (i) to complete the equivalence.

Let $x_0 \in H$ and set $f := \mathbb{A}(\cdot)^* x_0 : \mathbf{R}_+ \to H$. Then

$$|\int_0^\infty \langle f(s), \phi(s)\rangle_H\,ds| = |\int_0^\infty \langle x_0, \mathbb{A}(s)\phi(s)\rangle_H\,ds| = |\langle x_0, \int_0^\infty \mathbb{A}(s)\phi(s)\,ds\rangle_H| \le M\|x_0\|_H\|\phi\|_2 \tag{A.40}$$

for all $\phi \in \mathcal{C}_c^\infty((0,\infty);H)$, hence $\|f\|_2 \le M\|x_0\|_H$, by Theorem B.4.12. Because $x_0 \in H$ was arbitrary, (ii) and hence (i) holds for $\mathbb{A}^*$; therefore, also $\mathbb{A}$ is exponentially stable.

4° *The "or equivalently" claim in (ii):* Assume that $\mathbb{A}x_0 \in \mathrm{L}^2$ for all $x_0 \in X$, where $X \subset H$ is dense. Then this map has a unique extension $\widetilde{\mathbb{A}} \in \mathcal{B}(H, \mathrm{L}^2(\mathbf{R};H))$, by Lemma A.3.10. Choose $\omega > \omega_A$, so that

$\mathbb{A} \in \mathcal{B}(H, \mathrm{L}^2_\omega(\mathbf{R}_+; H))$. Then $\widetilde{\mathbb{A}} = \mathbb{A}$ as elements of $\mathbb{A} \in \mathcal{B}(H, \mathrm{L}^2_\omega(\mathbf{R}_+; H))$, by density, hence (ii) holds ("for all $x_0 \in H$"). $\square$

We finish this section by presenting some standard conventions in control theory:

Lemma A.4.6 ($W \subset H \subset X^*$, $X \subset H \subset W^*$) *Let A generate a C_0-semigroup on a Hilbert space H. Fix $\alpha \in \sigma(A)^c$.*

Set $W := (\alpha - A)^{-1}H = \mathrm{Dom}(A)$ (with norm $\|(\alpha - A) \cdot \|_H$) and $X := (\bar{\alpha} - A^*)^{-1}H = \mathrm{Dom}(A^*)$ (with norm $\|(\bar{\alpha} - A^*) \cdot \|_H$). Then W^* can be identified with the completion of H w.r.t. $\|(\alpha - A)^{-1} \cdot \|_H$, and X^* can be identified with the completion of H w.r.t. $\|(\bar{\alpha} - A^*)^{-1} \cdot \|_H$; in particular, for any $w \in W$, $x \in W^*$ we have $\langle w, x\rangle_{W,W^*} = \langle(\alpha - A)w, (\bar{\alpha} - A^*)^{-1}x\rangle_H$ ($= \langle w, x\rangle_H$ when $x \in H$).

Moreover, (extended) $\alpha - A$ is an isometric isomorphism $W \to H$ and $H \to X^*$, and A (and its extension to H and restriction to $\mathrm{Dom}(A^2)$) generates isomorphic C_0-semigroups on W, H and X^*.

Furthermore, $\mathrm{Dom}(A) = \{x \in H \mid Ax \in H\}$, and $\beta - A \in \mathcal{GB}(W, H)$ for any $\beta \in \sigma(A^c)$. In particular all above spaces and their topologies are independent on $\alpha \in \sigma(A)^c$.

We shall set $H_1 := \mathrm{Dom}(A) := W$, $H^*_{-1} := W^*$ in Chapter 6.

Proof: This is well known (see Lemma A.4.6 or p. 532 of [Weiss-C] (or [S97b, Section 7] or [Sbook])), so only sketch part of the proof. By Lemma A.3.4(G3)&(G1), the norm on $W = \mathrm{Dom}(A)$ is equivalent to the graph norm on $\mathrm{Dom}(A)$ (in particular, W and its topology are independent on $\alpha \in \sigma(A)^c$), and W is a Banach space. In particular, $(\beta - A)^{-1} \in \mathcal{GB}(H, W)$ for any $\beta \in \sigma(A^c)$.

Let $\beta \in \sigma(A^c)$. Because $\beta - A \in \mathcal{GB}(W, H)$, we have $\bar{\beta} - A^* \in \mathcal{GB}(H, W^*)$ (see Lemma A.3.24; thus $A^* \in \mathcal{B}(X, H) \cap \mathcal{B}(H, W^*)$). It follows that the norm of W^* becomes equivalent to $\|(\bar{\beta} - A^*)^{-1} \cdot \|_H$, hence W^* (as a TVS) is the completion of H w.r.t. this norm. The rest of the proof follows the same lines.

$\square$

Notes

Most facts in this section are well known. The canonical references on semigroups are [Pazy] and [HP], but the list of suitable references for C_0-semigroup theory would be endless, including [Rud73], [CZ], [Prüss93] and [Sbook]. The notes for Chapter 3 of [Sbook] and those for Chapter 5 of [CZ] contain historical remarks on C_0-semigroups.

Appendix B

Integration and Differentiation in Banach Spaces

Bring me my bow of burning gold!
Bring me my arrows of desire!
Bring me my spear! O clouds unfold!
Bring me my chariot of fire!
— William Blake (1757–1827)

In this appendix, we arm us with magic weapons to fight evil integral equations in final frontiers of unexplored Banach spaces. We treat (Bochner) integration, differentiation, function spaces $\mathcal{C}$ and L^p $(p \in [1,\infty])$ and similar concepts for functions with values in Banach spaces. We extend standard and extended results on scalar-valued functions for vector-valued ones.

Lebesgue measurability, integration and L^p spaces in the scalar-valued case are treated in Section B.1. In the rest of this appendix we treat the extensions of these concepts to the vector-valued case.

In Section B.2, we treat Bochner measurable functions $f : Q \to B$, where B is a Banach space and Q is a positive measure space. In Section B.3, we define and study the L^p and $\mathcal{C}$ spaces of such functions. In Section B.4, a generalization of the Lebesgue integral (*the Bochner integral*) is defined and studied for such functions. The reader might wish to have just a look at the beginnings of the sections mentioned above and skip the rest of this appendix until suggested to look up a specific fact by a proof in the main part of this monograph.

Differentiation of integrals and Lebesgue points are treated in Section B.5, vector-valued distributions ($\mathbb{D}'(\Omega;B) := \mathcal{B}(\mathcal{D}(\Omega);B)$) are treated in Section B.6, and Sobolev spaces ($\mathrm{W}^{k,p}(\Omega;B)$ and $\mathrm{W}_0^{k,p}(\Omega;B)$) in Section B.7.

Throughout this appendix, B, B_2 and B_3 denote Banach spaces with scalar field $\mathbf{K}$ ($\mathbf{K} = \mathbf{C}$ or $\mathbf{K} = \mathbf{R}$), U, H, and Y denote Hilbert spaces, μ is a complete positive measure on a set Q, and $\mathfrak{M}$ is the corresponding σ-algebra.

We use the (standard) terminology of [Rud86]: a *positive measure* on a set Q is a function $\mu : \mathfrak{M} \to [0,+\infty]$ (or the pair $(\mathfrak{M},\mu)$ or the triple $(Q,\mathfrak{M},\mu)$) s.t. $\mathfrak{M}$ is a σ-*algebra* on Q (i.e., $\mathfrak{M}$ is a collection of subsets of Q s.t. $Q \in \mathfrak{M}$ and $\mathfrak{M}$ is closed under complements and countable unions), and μ is *countably additive*, i.e., $\mu(\cup_{k=0}^{\infty} E_k) = \sum_{k=0}^{\infty} \mu(E_k)$ whenever $(E_k)_{k=0}^{\infty} \subset \mathfrak{M}$ is disjoint, and $\mu(\emptyset) = 0$. (In this chapter, we only treat scalar-valued measures. See Lemma D.1.12 and Section 2.6 for vector-valued measures.)

We call μ (or Q) σ-*finite* if $Q = \cup_{k \in \mathbf{N}} Q_k$, where $Q_k \in \mathfrak{M}$ and $\mu(Q_k) < \infty$ for all $k \in \mathbf{N}$. We call μ *complete* if all subsets of null sets are measurable.

When we assume $Q \subset \mathbf{R}^n$ (or $Q \subset \partial\mathbf{D}$, where we identify $\partial\mathbf{D}$ with $[0, 2\pi)$) and omit μ, we tacitly assume that $\mu = m$, the *Lebesgue measure* (see Theorem 2.20 of [Rud86]).

A *null set* is a measurable set N with $\mu(N) = 0$. A property (e.g. $f = g$ for functions $f, g : Q \to B$) is said to hold *almost everywhere (a.e.) on* Q if it holds on $N^c := Q \setminus N$ for some null set N. Analogously, we can say that $f(q) = g(q)$ holds for *almost every (a.e.)* $q \in Q$.

Even though we have allowed a general (Q,μ) for completeness, we shall need the results only for 1. the *counting measure* μ, for which every $A \subset Q$ is measurable and $\mu(A)$ is the cardinality of elements in A, and for 2. (J,μ), where $J \subset \mathbf{R}$ is an interval (i.e., a connected subset of $\mathbf{R}$) and $\mu = m$ or $d\mu = \mathrm{e}^{-2\omega t}\, dm$ for some $\omega \in \mathbf{R}$, and even so most results given below are used only for some technical details. Note that these both are complete positive measures, and the latter measure is σ-finite (so is the former too for countable Q).

For readers interested in the control of the systems with finite-dimensional input and output spaces only (this is very restrictive in cases where the output equals the state), it suffices to consider the (componentwise) Lebesgue measurability and integral; the Bochner measurability and integral are just the infinite-dimensional counterparts with $\|\cdot\|_B$ in place of $|\cdot|$.

We remark that almost all results given for Banach spaces in this appendix are valid for Fréchet spaces, mutatis mutandis (letting B to be an arbitrary Fréchet space would make $\mathrm{L}^p(Q;B)$ spaces Fréchet spaces).

B.1 The Lebesgue integral and $\mathrm{L}^p(\mathbf{R};[0,+\infty])$ spaces

The subspace W inherits the other 8 properties of V. And there aren't even any property taxes.

— J. MacKay, Mathematics 134b

Here we shortly recall the Lebesgue integral and L^p spaces from [Rud86].

Let $R = \mathbf{C}$, or let R be a connected subset of $[-\infty,+\infty]$ (open subsets of $[-\infty,+\infty]$ are arbitrary unions of sets of form (a,b), $[-\infty,b)$, $(a,+\infty]$ with $a,b \in [-\infty,+\infty]$; note that $\mathbf{R}$ inherits is usual (metric) topology as a subset of $[-\infty,+\infty]$). A function $f : Q \to R$ is called *(Lebesgue) measurable* iff $f^{-1}[G]$ is measurable for each open $G \subset R$. (By Lemma B.2.5(b3), Lebesgue measurability is a special case of Bochner measurability for $R = \mathbf{K}$.)

If $f,g : Q \to R$ are Lebesgue measurable, then so is $\max(f,g)$, by Theorem 1.14(b) of [Rud86]; in particular, so are $f^\pm$, where $f^+ := \max(f,0)$, $f^- := \max(0,-f)$.

If $(\mu,\mathfrak{M})$ is any positive measure on Q, and $\mathfrak{M}'$ is the collection of sets $E \subset Q$ s.t. $A \subset E \subset A'$ and $\mu(A' \setminus A) = 0$ for some $A,A' \in \mathfrak{M}$ and we set $\mu'(E) := \mu(A)$ for such E, then the *completion* $(\mu',\mathfrak{M}')$ of μ is a (well-defined) complete positive measure on Q, by Theorem 1.36 of [Rud86]. Note also that $(r\mu,\mathfrak{M})$ is also a positive measure on Q for any $r \in \mathbf{R}_+$.

The *Borel (measurable) sets* of a topological space Q are the members of the minimal σ-algebra containing the open sets of Q. A function $f : Q \to \mathbf{K}$ is a *Borel (measurable) function* if $f^{-1}[V]$ is Borel measurable for all open $V \subset \mathbf{K}$. A *Borel measure* on Q is a measure $(\mu,\mathfrak{M})$ s.t. $\mathfrak{M}$ contains the Borel sets (equivalently, s.t. $\mathfrak{M}$ contains the open sets). The Lebesgue measure m is the completion of a measure whose domain is the collection of Borel sets, hence a Borel measure.

For $f : Q \to [-\infty,+\infty]$ we set $\operatorname{ess\,sup} f := \inf\{r \in [-\infty,+\infty] \mid f \le r \text{ a.e.}\}$, $\operatorname{ess\,inf} f := -\operatorname{ess\,sup} -f$. We also set $r/+\infty := 0$ $(0 \le r < +\infty)$, $r \cdot +\infty := +\infty$, $r/0 := +\infty$ $(0 < r < +\infty)$ and $0 \cdot +\infty := 0$. The function

$$\chi_E(q) := \begin{cases} 1, & q \in E; \\ 0, & q \notin E \end{cases} \tag{B.1}$$

is called the *characteristic function* of the set E.

If $n \in \mathbf{N}$, $E_0,\ldots,E_n$ are disjoint and measurable, and $\alpha_0,\ldots,\alpha_n \in [0,+\infty]$ or $\in B$, then $s := \sum_{k=0}^n \alpha_k \chi_{E_k}$ is a *simple measurable function* and the *Bochner integral* of such s is given by

$$\int_Q s\,d\mu := \sum_{k=0}^n \alpha_k \mu(E_k). \tag{B.2}$$

For general measurable $f : X \to [0,+\infty]$ we set $\int_Q f\,d\mu := \sup \int_Q s\,d\mu$, the supremum being taken over all simple measurable functions s s.t. $0 \le s \le f$. This integral is usually called the *Lebesgue integral*, and the term Bochner integral is reserved for functions whose values are not scalar (see Definition B.4.1).

If $Q' \subset Q$ is measurable, then we set $\int_{Q'} f\,d\mu := \int_Q \chi_{Q'} f\,d\mu$.

Let $1 \le p < \infty$. A measurable function $f : Q \to [0,+\infty]$ belongs to $\mathrm{L}^p(Q;[0,+\infty])$ if $\|f\|_p := \left(\int_Q f^p \, d\mu\right)^{1/p} < \infty$, and to $\mathrm{L}^\infty(Q;[0,+\infty])$ if $\|f\|_\infty := \operatorname{ess\,sup}_Q |f| < \infty$ (we sometimes write $\|f\|_{\mathrm{L}^p}$ or $\|f\|_{\mathrm{L}^p(Q;[0,+\infty])}$ instead of $\|f\|_p$).

To be exact, the L^p spaces ($1 \le p \le \infty$) are quotient spaces over with the set of functions that are 0 a.e. (i.e., L^p is the space of equivalence classes $[f]$, where $g \in [f]$ iff $g = f$ a.e.). Thus, $\mathrm{L}^p(Q;[0,+\infty])$ becomes "a normed space with scalar field $[0,+\infty]$". (One easily verifies that $\mathrm{L}^p(Q;[0,+\infty])$ is a vector space with scalar field $[0,+\infty]$ and that the axioms of a normed space are satisfied w.r.t. this scalar field; the latter requires the *Minkovski Inequality*, Theorem 3.19 of [Rud86], which says that $\|f+g\|_p \le \|f\|_p + \|g\|_p$ for all measurable $f,g : Q \to [0,+\infty]$.)

We also note the *Hölder Inequality*: $\|fg\|_1 \le \|f\|_p \|g\|_q$ for $p,q \in [1,+\infty]$ s.t. $1/p + 1/q = 1$. The reader will later note that these two inequalities extend to vector-valued functions, because for measurable (see Definition B.2.1) $f : Q \to B$ we shall define $\|f\|_p := \|\|f\|_B\|_p$.

Lebesgue's Monotone Convergence Theorem says that if $\{f_n\}$ are measurable, $0 \le f_1 \le f_2 \le \cdots \le \infty$ a.e. on Q and $f_n(q) \to f(q)$ for a.e. $q \in Q$, then f is measurable and $\int_Q f_n \, d\mu \to \int_Q f_n \, d\mu$.

Lemma B.1.1 *Let $Q \subset \mathbf{R}^n$ be measurable. For any measurable sets $\{E_n\}$ s.t. $m(E_n) > 0$ for all $n \in \mathbf{N}$, there are disjoint measurable $\{E_n'\}$ s.t. $E_n' \subset E_n$ and $\infty > m(E_n') > 0$ for each $n \in \mathbf{N}$.*

Proof: Choose $F_0 \subset E_0$ s.t. $\infty > m(F_0) > 0$. For each $n \in \mathbf{N}+1$, choose $F_n \subset E_n$ s.t. $0 < m(F_n) < 2^{-n} \max_{k=0,\ldots,n-1} m(F_k)$ and set $E_n' := F_n \setminus \cup_{k=n+1}^{\infty} F_k$. Then $m(\cup_{k=n+1}^{\infty} F_k) < \sum_{k=n+1} 2^{-k} m(F_n) \le m(F_n)$, hence $m(E_n') > 0$, for any $n \in \mathbf{N}$. $\square$

(See the notes on p. 947.)

B.2 Bochner measurability ($f \in \mathrm{L}(Q;B)$)

You cannot have a science without measurement.
— R. W. Hamming

In this section we treat Bochner measurability in order to be able to define L^p spaces and the Bochner integral in the next two sections. For most readers, it suffices to just have a look at Definition B.2.1 and Lemma B.2.5 so as to be convinced that Bochner measurability is analogous to Lebesgue measurability.

A function $s : Q \to B$ is called a *countably-valued measurable function* iff it can be written as $s = \sum_{k\in\mathbf{N}} x_k \chi_{E_k}$, where the sets $E_k \subset Q$ are measurable and $x_k \in B$ ($k \in \mathbf{N}$). In general, Bochner measurability is defined as follows:

Definition B.2.1 (Bochner measurability) *A function $f : Q \to B$ is* (Bochner) measurable[1], *denoted by $f \in \mathrm{L}(Q;B)$, if some sequence of countably-valued measurable functions converges to f a.e.*

A function $f : Q \to B$ is called almost separably-valued *if it can be redefined on a* null set *(a set of measure zero) so that $f[Q]$ becomes separable.*

In Lemma B.2.5(g) we shall show that a function $f : Q \to \mathbf{K}$ (or $f : Q \to [-\infty,+\infty]$) is measurable iff it is Lebesgue measurable.

We follow the standard convention to identify functions equal a.e. as members of L. Thus, the elements of L are actually equivalence classes. We sometimes write "$[f]$" instead of "f", when it would otherwise not be obvious that we refer to the class, not to the function.

Let $f \in \mathrm{L}(Q;B)$ and $F \in \mathcal{C}(B;B_3)$. Then $F \circ f$ is measurable (because $F \circ s_n \to F \circ f$ a.e.). If $f = g$ a.e., then $F \circ f = F \circ g$ a.e. Thus, we can follow the standard convention to define $F \circ [f] := [F \circ f]$. Most common examples of this are the definitions $\alpha[f] := [\alpha f]$ and $[f]+[g] := [f+g]$ for $f, g \in \mathrm{L}(Q;B)$, $\alpha \in \mathbf{K}$. We also extend to classes any other operations that can be well defined through representatives.

For separability results we shall need the following:

Lemma B.2.2 *Let the sets A_k be at most countable ($k \in \mathbf{N}$), and let $n \in \mathbf{N}$. Then the sets $A_0 \times A_1 \times \cdots \times A_n$, $\cup_{k\in\mathbf{N}} A_k$, $\{B \subset A_0 \,\big|\, B \text{ has } n \text{ elements}\}$ and $\{B \subset A_0 \,\big|\, B$ is finite$\}$ are at most countable.*

For any set A, $\operatorname{card} A < \operatorname{card} 2^A$ (recall that 2^A is the set of all subsets of A). If A and B are nonempty sets, (at least) one of which is infinite, then $\operatorname{card}(A \times B) = \max\{\operatorname{card} A, \operatorname{card} B\} = \operatorname{card}(A \cup B)$. $\square$

(The countability claims are easily deduced from Theorems 2.12 and 2.13 of [Rud76]; the general cardinality claims are also well known and follow from, e.g., Theorems 176 and 180, pp. 276 and 280 of [Kelley].)

Next we list a few separability results in order to be able to prove Lemma B.2.5.

Lemma B.2.3 (Separability) *Let $E_n \subset B$ be separable for all $n \in \mathbf{N}$. Then*

[1] The term "strongly measurable" is sometimes used, but we reserve it for Definition F.1.1.

(a) The closure and span of E_0 are separable.

(b) If $E \subset E_0$, then E is separable.

(c) The union $\cup_n E_n$ is separable.

(d) If $E \subset B$ is weakly separable, then it is separable.

(e) Subsets of $\mathbf{R}^n$ are separable.

(e) If Q is separable and $f \in \mathcal{C}(Q,Q')$, then $f[Q']$ is separable.

Proof: Let $S_n \subset E_n$ be dense in E_n and countable, for each n.

(a) The set S_0 is dense in $\bar{E}_0$ and the set of finite linear combinations of elements of S_0 with coefficients of form $r+iq$ ($q,r \in \mathbf{Q}$) is dense in the span of E_0.

(b) Let $S_0 = \{x_k\}_{k\in\mathbf{N}}$. For each $n,k \in \mathbf{N}$, choose an element $x_k^n \in E_k^n := \{y \in E \,\big|\, \|x_k - y\| < 1/(n+1)$, if such an element exists. Then the union S of these elements is dense in E, because if $x \in E$ and $N > 0$, we can choose x_k s.t. $|x - x_k| < 1/2N$, so that E_k^n is nonempty for $n < 2N$, hence $\|x - x_k^{2N}\| \le \|x - x_k\| + \|x_k - x_k^{2N}\| < 1/N$. Thus, S is dense in E.

(c) The union $\cup_n S_n$ is dense in $\cup_n E_n$.

(d) Let $S \subset E$ be countable and dense in the weak topology of B. Let M be the closure of the span of E. Then M is weakly separable, by (a) (whose proof is valid for any topological vector space), hence M is separable, by Theorem 3.12 of [Rud73], hence E is separable, by (b).

(e) The countable set $\mathbf{Q}^n$ is dense in $\mathbf{R}^n$, hence (e) follows from (b).

(f) If $S \subset Q$ is dense, then $f[S] \subset Q'$ is obviously dense. □

Lemma B.2.4 *Let B be separable and $\Omega \subset \mathbf{C}$. If $f \in \mathcal{C}(\Omega;\mathcal{B}(B,B_2))$, then $f \in \mathcal{C}(\Omega;\mathcal{B}(B,B_2'))$ for some separable closed subspace $B_2' \subset B_2$.*

Proof: Let $\Omega' \subset \Omega$ and $Q \subset B$ be dense and countable. Then $B_2'' := f[\Omega'][Q] \subset f[\Omega][B]$ is dense and countable, by continuity ($x_k \to x$ & $s_k \to s \Rightarrow f(s_k)x_k \to f(s)x$). Consequently, the closed span B_2' of B_2'' is a separable Banach space, and it contains $f[\Omega][B]$, i.e., $f(s)[B] \subset B_2'$ (i.e., $f(s) \in \mathcal{B}(B,B_2')$) for any $s \in \Omega$. □

Now we are ready to list the standard properties of Bochner measurability:

Lemma B.2.5 (Bochner Measurability) *Let $f_n, g, h : Q \to B$ be measurable ($n \in \mathbf{N}$) and $\alpha, \beta \in \mathbf{K}$.*

(a1) The function $\alpha g + \beta h$ is measurable.

(a2) If $T \in \mathcal{B}(B,B_2)$ (or $T \in \mathcal{C}(B,B_2)$), then Tg is measurable.

(a3) If $f : Q \to \mathcal{B}(B,B_2)$ is measurable, then so is f^*.

(a4) If also $f : Q \to B_2$ is measurable, then so is $(f,g) : Q \to B \times B_2$.

Thus, then $B(f,g) : Q \to B_3$ is measurable for any continuous $B : B \times B_2 \to B_3$.

(b1) A function $f : Q \to B$ is measurable iff Λf is measurable for all $\Lambda \in B^*$ (or for all Λ in a norming subset of B^*) and f is almost separably-valued.

(b2) A function $f : Q \to B$ is measurable iff some sequence of countably-valued measurable functions converges to f uniformly outside some null set.

(b3) Let $f : Q \to B$. Then (i)⇒(ii)⇒(iii); if B is separable, then (i)–(iii) are equivalent:

(i) f is measurable;

(ii) the set $f^{-1}[V] := \{q \in Q \,|\, f(q) \in V\}$ is measurable for each open $V \subset B$;

(iii) Λf is measurable for all $\Lambda \in B^*$.

This holds also with $[-\infty, +\infty]$ in place of B.

(c) If $f_n : Q \to B$ ($n \in \mathbf{N}$) are measurable and $f_n(t) \to f(t)$ for a.e. $t \in Q$, then f is measurable.

(d1) Let $Q_n \subset Q$ be measurable for all $n \in \mathbf{N}$ and $Q = \cup_{n \in \mathbf{N}} Q_n$. Then $f : Q \to B$ is measurable iff $f|_{Q_n}$ is measurable $Q_n \to Q$ for all $n \in \mathbf{N}$.

(d2) If B_2 is a closed subspace of B and $fQ \subset B_2$, then f is measurable $Q \to B_2$ iff f is measurable $Q \to B$.

(d3) A function $f : Q \to B_1 \times B_2 \times \cdots \times B_k$ is measurable iff f_j is measurable for $j = 1, \ldots, k$.

(e) Any continuous function $Q \to B$ is measurable if Q is separable and μ is a Borel measure (e.g., $\mu = m$ and $Q \subset \mathbf{R}^n$ is measurable).

(f) If $B = \mathbf{K}$, then $(\mathrm{Re}\, g)^{\pm}$, $(\mathrm{Im}\, g)^{\pm}$ are measurable.

(g) Let $n \in \mathbf{N} + 1$. A function $f : Q \to \mathbf{K}^n$ is Bochner measurable iff each of its components is Lebesgue measurable.

In (a4) the map B can be multiplication, inner product, a continuous bilinear map, or similar. By (a2), $\|g\|_B : Q \to [0, \infty)$ is measurable.

Note for (d1) that the restriction of μ to some measurable $Q' \subset Q$ is also a complete positive measure. Note also that condition (ii) in (b3) depends on $\mathfrak{M}$ (and B) only, whereas conditions (i) and (iii) depend on the measure too.

See Lemma B.4.10 for the measurability of $f \circ \phi$ with $\phi \in \mathcal{C}^1$ increasing.

By (d1), piecewise continuous functions are Borel-measurable (hence Lebesgue-measurable if $Q \subset \mathbf{R}^n$ with $\mathbf{R}^n$'s topology).

Proof of Lemma B.2.5: (a)&(d) Choose $g_n \to g$ and $h_n \to h$ (a.e.) as in the definition of measurability. Then $\alpha g_n + \beta h_n$ is countably-valued and measurable for all $n \in \mathbf{N}$ and converges to $\alpha g + \beta h$ a.e. The proofs of (a2), (a3), (d1), (d2) and (d3) are more or less analogous.

Note for (d1) that $\mathfrak{M}_{Q'} := \{E \in \mathfrak{M} \,|\, E \subset Q'\}$ is a σ-algebra on Q', and $\mu|_{\mathfrak{M}_{Q'}}$ is a complete, positive measure on $\mu_{Q'}$.

(Claim (a3) is a special case of (a2), it holds to both Hilbert space adjoint $f^* : Q \to \mathcal{B}(B_2, B)$ (in case that B and B_2 are Hilbert spaces) and to the Banach space adjoint $f^* : Q \to \mathcal{B}(B_2^*, B^*)$ (and to any other continuous involution operator). Claim (a2) holds even when T is only continuous from B to the weak topology of B_2 (use (b1) and Lemma B.2.3(d)).)

(b1) Without our reference to norming sets, the proof on pp. 72–73 of [HP] applies. Assume then that f is almost separably-valued and Λf is measurable for all $\Lambda \in C$, where $C \subset B^*$ is a norming set.

Redefine f on a null set so that $f[Q]$ is separable, and replace B by the closed span of $f[Q]$, which is separable. Let $S \subset B$ be dense, and choose countable $C' \subset C$ s.t. $\|x\|_B = \sup_{\Lambda \in C'} |\Lambda x|$ for all $x \in S$ Then $\|x\|_B = \sup_{\Lambda \in C'} |\Lambda x|$ for all $x \in B$. It follows that $\|f(t)\|_B = \sup_{\Lambda \in C'} |\Lambda f(t)|$ for all $t \in Q$, therefore, $\|f\|_B$ is measurable; analogously, so is $\|f - x\|_B$ for any $x \in B$. Thus, the rest of the proof on pp. 72–73 of [HP] is valid.

(b2) This is Corollary 1 on p. 73 of [HP].

(b3) The second claim follows from the first and (b1), so we only need to prove the first claim.

1° *(i)⇔(ii)⇔(iii) for $B = \mathbf{K}$ and for $B = [-\infty, +\infty]$:* This can be deduced from Theorems 1.14 and 1.17 of [Rud86] (note that (i)⇔(iii) is trivial, because $\mathbf{K}^* = \mathbf{K}$).

2° *(ii)⇒(iii):* If f satisfies (ii), $\Lambda \in B^*$, and $V \subset \mathbf{K}$ is open, then $(\Lambda \circ f)[V] = f^{-1}[\Lambda^{-1}[V]]$ is measurable, because $\Lambda^{-1}[V]$ is open. Thus, Λf is measurable, by 1°.

3° *(i)⇒(ii):* Assume (i), so that some sequence $\{s_n\}$ of countably-valued, measurable functions converge to f everywhere (redefine s_n and f to be zero on a null set, if necessary). Let $V \subset B$ be open, and define $F \in \mathcal{C}(B; \mathbf{K})$ by $F(x) := d(x, V^c) := \inf_{y \in V_c} \|x - y\|_B$. Then $F \circ s_n \to F \circ f$ everywhere, hence $F \circ f \in \mathrm{L}(Q)$, because $F \circ s_n$ is countably-valued and measurable for all $n \in \mathbf{N}$.

Therefore, the set $f^{-1}[V] = f^{-1}[F^{-1}[\mathbf{K} \setminus \{0\}]] = (F \circ f)^{-1}[\mathbf{K} \setminus \{0\}]$ is open, because $F \circ f$ satisfies (ii), by the implication "(i)⇒(ii)" of 1°.

4° *(iii)⇒(i) for separable B:* This follows from (b1).

(Note that $I : B \to B$ satisfies (ii) but not (i) if B is an unseparable Banach space with the counting measure.)

(c) By Theorem 1.14 of [Rud86], Λh is measurable, so we only have to show that h is almost separably-valued.

Choose a null set $N \subset Q$ s.t. $h_n(t) \to h(t)$ for $t \in N^c$. For each n, choose a null set N_n s.t. $h_n[N_n^c]$ is separable. Let $N' := N \cup \cup_n N_n$, $Q' := Q \setminus N'$. Then $E := \cup_n h_n[Q']$ is separable, hence so is the closed subspace M of B spanned by E. But $h(t) = \lim_{n \to +\infty} h_n(t) \in M$ for each $t \in Q'$, hence h is almost separably-valued.

(e) By Lemma B.2.3(e), $f[Q]$ is separable. Because $f^{-1}[V]$ is open, hence measurable for each open $V \subset B$, f is measurable, by (b3).

(f) By (a2), $\mathrm{Re}\, g$ and $\mathrm{Im}\, g$ are measurable; by Section B.1, so are $(\mathrm{Re}\, g)^{\pm}$, $(\mathrm{Im}\, g)^{\pm}$.

(g) This follows from (b3). □

The rest of this section consists of less important results, hence the reader might wish to skip them.

We will use the following lemma to generalize several scalar results to the vector-valued case.

Lemma B.2.6 *Let f be Bochner measurable. Then $f = 0$ a.e. iff $\Lambda f = 0$ a.e. for all $\Lambda \in B^*$ (or for all Λ in a norming subset of B^*).*

If f is only "weakly vector measurable", i.e., Λf is measurable for all $\Lambda \in B^*$, then we may have $\Lambda f = 0$ a.e. for all $\Lambda \in B^*$ even though $f \neq 0$ everywhere (set $f(t) := e_t$, where $\{e_t\}_{t\in\mathbf{R}}$ is the natural base of $B := \ell^2(\mathbf{R})$ (i.e., $e_t := \chi_{\{t\}}$), so that for any $y := \sum_k \alpha_k e_{t_k} \in \ell^2(\mathbf{R})$ we have $\langle f(t), y\rangle_B = 0$ for $t \notin \cup_k\{t_k\}$).

Proof of Lemma B.2.6: The necessity is clear, so we assume that $\|f\| \geq \varepsilon > 0$ on $E \subset \mathbf{R}$ and $m(E) > 0$, and find $\Lambda \in B^*$ s.t. $\Lambda f \neq 0$ on a set of positive measure.

W.l.o.g. we assume that $f[E]$ has a dense countable subset $\{b_k\}_{k\in\mathbf{N}}$. For some $k \in \mathbf{N}$, we have $m(A_k) > 0$, where $A_k := \{r \in E \,|\, \|f(r) - b_k\| < \varepsilon/3\}$. If $|\Lambda b_k| > \|b_k\|/2$ and $\|\Lambda\| \leq 1$, then $|\Lambda f(r)| > |\Lambda b_k| - \varepsilon/3 =: M > \|b_k\|/2 - \varepsilon/2 > 0$ for $r \in A_k$. Thus, $\|\Lambda f\|_\infty \geq M \gg 0$. □

For $x \in B$ and $r > 0$ we set $D_r(x) := \{x' \in B \,|\, \|x - x'\| < r\}$.

Lemma B.2.7 (Ess range(f)) *Let $f : Q \to B$ be measurable. Then*

$$\operatorname{ess\,range}(f) := \{x \in B \,|\, r > 0 \Rightarrow \mu(f^{-1}[D_r(x)]) > 0\} \tag{B.3}$$

is closed and $\mu(f^{-1}[\operatorname{ess\,range}(f)^c]) = 0$. Moreover, $\operatorname{ess\,range}(f)$ is the smallest set with these properties.

Proof: One easily verifies that $E_f := \operatorname{ess\,range}(f)$ is closed. We assume w.l.o.g. (see Lemma B.2.5(b1)) that B is separable. Let $\{x_k\}$ be dense in E_f^c. For each k, set $r_k := \sup\{r > 0 \,|\, \mu(f^{-1}[D_r(x_k)]) = 0\}$. Obviously, $E_f^c \subset V := \cup_k D_{r_k/2}(x_k)$; but $\mu(f^{-1}[V]) = 0$, because $\mu(f^{-1}[D_{r_k/2}(x_k)]) = 0$ for all k; hence $\mu(f^{-1}[E_f^c]) = 0$.

By the definition of E_f, the set E_f^c contains any open set $G \subset B$ s.t. $\mu(f^{-1}[G]) = 0$, i.e., E_f^c is the biggest of such sets. □

Lemma B.2.8 *Let $f : Q \to B$ be measurable.*

(a) If $\mu(Q) > 0$, then there is $a_0 \in Q$ s.t. for each $\varepsilon > 0$ we have $\mu(A_\varepsilon) > 0$, where $A_\varepsilon := \{a \in Q \,|\, \|f(a) - f(a_0)\|_B < \varepsilon\}$.

(b) If(f) f is not zero a.e., then there are $A \subset Q$ and $\Lambda \subset B^*$ s.t. $\mu(A) > 0$ and $\operatorname{Re}\Lambda f > 1$ on A.

If, in addition, B_0 is a closed subspace of B and $\mu(E) > 0$, where $E := \{t \in Q \,|\, f(t) \notin B_0\}$, then we can choose Λ and A so that $\Lambda = 0$ on B_0.

Proof: (a) Choose any $a_0 \in f^{-1}[\operatorname{ess\,range}(f)]$.

(b) If B_0 has not been given, set $B_0 = \{0\}$. Choose $n \in \{1, 2, 3, \ldots\}$ s.t. $A_n := \{q \in Q \,|\, d(f(q), B_0) > 1/n\}$ has a positive measure. Choose then $a_0 \in A_n$

for $f|_{A_n}$ and $\varepsilon := 1/2n$ as in (a). Then $V := \{b \in B \,|\, \|b - f(a_0)\|_B < 1/2n\}$ is convex, open and nonempty, $V \cap B_0 = \emptyset$, $A_0' := f^{-1}[V]$ has a positive measure (by (a), because $A_0 = A_0' \cap A_n$), and also B_0 is convex and nonempty, hence there are $\Lambda \in B^*$ and $\gamma \in \mathbf{R}$ s.t.

$$\operatorname{Re} \Lambda x < \gamma \leq \operatorname{Re} \Lambda y \quad \text{for all } \ x \in V, \ y \in B_0, \tag{B.4}$$

by Theorem 3.4 of [Rud73]. Thus, $\Lambda[B_0]$ is a proper subspace of $\mathbf{K}$, hence $\Lambda[B_0] = \{0\}$. Find $k \in \{1, 2, 3, \ldots\}$ s.t. $A := \{a \in A_0' \,|\, \operatorname{Re} \Lambda f(a) < -1/k\}$ has a positive measure, and then replace Λ by $-k\Lambda$. □

The following result can be used for convolutions:

Lemma B.2.9 *Let $f : \mathbf{R}^n \to B$ be measurable. Then $(r, s) \mapsto f(r - s)$ is measurable $\mathbf{R}^n \times \mathbf{R}^n \to B$.*

Proof: Let $s_n(r) \to f(r)$, as $n \to +\infty$, for all $r \in \mathbf{R}^n \setminus N$, where $m(N) = 0$ and s_n is countably-valued and measurable ($n \in \mathbf{N}$) (see Definition B.2.1). We may and do require that $s_n = \sum_{k=0}^{\infty} x_k^n \chi_{E_k^n}$ where E_k^n is Borel measurable for each k, n (by redefining each s_n on a null set).

Set $N_2 := \{(r, s) \,|\, r - s \in N\}$. Then (see Theorems 8.11 and 8.12 of [Rud86])

$$m(N_2) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \chi_{N_2}(r, s) \, dr \, ds = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \chi_{s+N}(r) \, dr \, ds = 0. \tag{B.5}$$

But $s_n(r - s) \to f(r - s)$ for all $(r, s) \in N_2^c$, and $(r, s) \mapsto s_n(r - s)$ is countably-valued and Borel measurable, because $(r, s) \mapsto r - s$ is a Borel function. □

(See the notes on p. 947.)

B.3 Lebesgue spaces ($\mathrm{L}^p(Q,\mu;B)$)

If God is perfect, why did He create discontinuous functions?

In this section, we first define the L^p spaces ("Lebesgue spaces") and then go on to prove several technical results, some of which are considered to be well-known although not easily found in the literature.

The continuity of $\|\cdot\|_B : B \to [0,+\infty]$ implies that if f is measurable, then so is $\|f\|_B$. Thus we can make the following definition:

Definition B.3.1 ($\mathrm{L}^p(Q;B)$) *Let* $1 \le p \le \infty$. $\mathrm{L}^p(Q;B) := \mathrm{L}^p(Q,\mu;B)$ *is the space of (equivalence classes of) measurable functions* $f : Q \to B$ *whose norm* $\|f\|_p := \|\,\|f(\cdot)\|_B\,\|_{\mathrm{L}^p(Q)}$ *is finite. We set* $\mathrm{L}^p(Q,\mu) := \mathrm{L}^p(Q,\mu;\mathbf{K})$.

If B is a Hilbert space, then we set $\langle f,g\rangle_{\mathrm{L}^2} := \int_Q \langle f,g\rangle_B \, d\mu$. *By* $\ell^p(Q;B)$ *we mean* $\mathrm{L}^p(Q;B)$ *with the counting measure.*

If compact subsets of Q are measurable, then we set

$$\mathrm{L}^p_{\mathrm{loc}}(Q,\mu;B) := \{f \in \mathrm{L}(Q;B) \,\big|\, \|f\|_{\mathrm{L}^p(K,\mu;B)} < \infty \text{ for all compact } K \subset Q\}. \quad \text{(B.6)}$$

If, in addition, Q is σ-compact, then we equip $\mathrm{L}^p_{\mathrm{loc}}$ *with the topology induced by the seminorms* $\|f\|_{\mathrm{L}^p(K,\mu;B)}$ *(see Theorem 1.37 of [Rud73]); in particular, then* $f_n \to f$ *in* $\mathrm{L}^p_{\mathrm{loc}}$ *if* $\|f_n - f\|_{\mathrm{L}^p(K,\mu;B)} \to 0$ *for all compact* $K \subset Q$ *(and* $f_n, f \in \mathrm{L}^p_{\mathrm{loc}}$ *for all n).*

Note that $\|f\|_p = 0$ iff $f = 0$ a.e., i.e., if $[f] = [0]$. Thus, $\|\cdot\|_p$ becomes a norm on L^p (with equivalence classes as elements). We define the "quasinorms" $\|f\|_p := \big(\int_Q \|f\|_B^p \, d\mu\big)^{1/p}$ for $p \in (0,1)$ too (but the vector spaces L^p, $p \in (0,1)$ are not normed spaces, cf. [Rud73]). The topology of $\mathrm{L}^p_{\mathrm{loc}}$ is only rarely needed, hence the reader may well skip it; the resulting convergence condition given at the end of the above definition is only slightly more useful.

Obviously, $\mathrm{L}^p(Q;B) \subset \mathrm{L}^p_{\mathrm{loc}}(Q;B)$ $(p \in [1,\infty])$. If $\mu(K) < \infty$ for all compact subsets of Q, then $\mathrm{L}^p_{\mathrm{loc}}(Q;B) \subset \mathrm{L}^r_{\mathrm{loc}}(Q;B)$ $(\infty \ge p \ge r \ge 1)$, by the Hölder inequality; if, in addition, Q or B is separable and μ is a Borel-measure, then $\mathcal{C}(Q;B) \subset \mathrm{L}^p_{\mathrm{loc}}(Q;B)$. One usually equips $\mathcal{C}(Q;B)$ with $\mathrm{L}^\infty_{\mathrm{loc}}(Q;B)$ topology, but we do not need this.

Theorem B.3.2 *The space* $\mathrm{L}^p(Q;B)$ *is a Banach space (a Hilbert space if $p = 2$ and B is a Hilbert space). If* $f_n \to f$ *in* L^p*, then some subsequence of* $\{f_n\}$ *converges to f a.e.*

If $\Omega \subset \mathbf{R}^n$ *is open and μ is a Borel measure on Ω, then* $\mathrm{L}^p_{\mathrm{loc}}(\Omega,\mu;B)$ *is a Fréchet space (hence a complete metric TVS).*

Note that if $f_n \to f$ in L^p, $f_n \to g$ in L^q and $f_n \to h$ a.e. pointwise, as $n \to \infty$, then $f = g = h$ a.e. (take a subsequence of $\{f_n\}$ converging pointwise a.e. to f and g).

Proof: 1° The proof of the first paragraph is identical to the scalar case (e.g., [Rud86, Theorems 3.11 & 3.12]), and hence omitted.

2° *Claims on* $\mathrm{L}^p_{\mathrm{loc}}$*:* Let $\{K_k\}_{k\in\mathbf{N}}$ be as in Lemma A.2.3. One easily verifies that the norms $\|\cdot\|_{\mathrm{L}^p(K_k;B)}$ generate the topology of $\mathrm{L}^p_{\mathrm{loc}}(\Omega;B)$; in particular, $\mathrm{L}^p_{\mathrm{loc}}$ is metrizable, by Remark 1.38(c) of [Rud73].

Let $\{f_n\}$ be a Cauchy-sequence in $\mathrm{L}^p_{\mathrm{loc}}$. Then $\{f_n\}$ has a limit f^k in each $\mathrm{L}^p(K_k,\mu;B)$. Set $f := \sum_k \chi_{K_k\setminus K_{k-1}} f^k$. Obviously, $f = f^k$ a.e. on each K_k and $f \in \mathrm{L}(\Omega;B)$, hence $f_n \to f$ in each $\mathrm{L}^p(K_k;B)$, so that $f_n \to f$ in $\mathrm{L}^p_{\mathrm{loc}}$. Because $\{f_n\}$ was arbitrary, $\mathrm{L}^p_{\mathrm{loc}}$ is complete. By Theorem 1.37 of [Rud73], $\mathrm{L}^p_{\mathrm{loc}}$ is a Fréchet space (i.e., a complete locally convex metrizable TVS). □

Also most other standard analysis results extend to the vector-valued case; in particular, standard properties of $\mathrm{L}^p(\mathbf{K})$ spaces (e.g., reflexivity (when $1 < p < \infty$) and separability (when $1 \le p < \infty$)) are inherited by $\mathrm{L}^p(\mathbf{R}^n;B)$ if(f) also B possesses the same property; see [DU]. The most important exceptions (fortunately not much needed in this monograph) are that the dual of $\mathrm{L}^p(Q;B)$ need not be $\mathrm{L}^q(Q;B^*)$ (see [DU, Theorem IV.1.1, p. 98]; cf. Lemma B.4.15), and that an absolutely continuous measures (and functions) with values in B need not be differentiable, unless B is a *Radon–Nikodym space* (and μ is σ-finite and $1 \le p < \infty$). For most purposes, it suffices to know that Hilbert spaces and other reflexive spaces are Radon–Nikodym spaces [DU, p. 61, p. 98 & p. 82].

Derivatives are defined as in the scalar case:

Definition B.3.3 ($\frac{df}{dt}$) *Let J be an interval of* $\mathbf{R}$*. The* derivative *of a function $f : J \to B$ at $t \in J$ is*

$$f'(t) := \lim_{h\to 0,\ t+h\in J} \frac{f(t+h)-f(t)}{h}. \tag{B.7}$$

Let $\Omega \subset \mathbf{R}^n$ be open, $n \in \mathbf{N}+1$, $q \in \Omega$, $j \in \{1,2,\dots,n\}$. Then the jth partial derivative $(D_j g)(q) := g_j(q)$ *of $g : \Omega \to B$ at q is the derivative of $h \mapsto g(q+he_j)$ at* 0.

If g has all n partial derivatives at q [and they are continuous], then g is [continuously] differentiable at q*. If g is [continuously] differentiable at each point of Ω, then g is* [continuously] differentiable *(on Ω).*

The partial derivatives of g of order $k \in \mathbf{N}$ *are the functions $D_1^{\alpha_1}\cdots D_n^{\alpha_n} g$ for which $\alpha \in \mathbf{N}^n$, $\sum_{j=1}^n \alpha_j = k$. If all partial derivatives of g of order k exist [at q], then g is* k times differentiable [at q].

(Here e_j is the jth unit vector ($e_1 := (1,0,0,\dots,0)$, $e_2 := (0,1,0,0,\dots,0)$, ...).)

Obviously, the definition of $f'(t)$ implies that $(Tf)'(t) = T(f'(t))$ for $T \in \mathcal{B}(B,B_2)$ whenever $f'(t)$ exists. Note that we allow one-sided derivatives at the endpoints of J (if they belong to J).

Let Q be a topological space. The space $\mathcal{C}(Q;B)$ is the vector space of continuous functions $f : \Omega \to B$. We equip the following subspaces of $\mathcal{C}(Q;B)$ (see p. 1045 for details) with supremum norm $f \mapsto \sup_{q\in Q}\|f(q)\|_B$: $\mathcal{C}_{\mathrm{b}}$ (f bounded), $\mathcal{C}_{\mathrm{bu}}$ (f bounded and uniformly continuous; this requires that Q is metric), $\mathcal{C}_0 := \{f \in \mathcal{C}_{\mathrm{b}} \,|\, \text{for any } \varepsilon > 0, \text{ there is compact } K \subset Q \text{ s.t. } \|f\| < \varepsilon \text{ on } Q \setminus K\}$ (often called as “the functions vanishing at infinity”), and $\mathcal{C}_{\mathrm{c}} := \{f \in \mathcal{C} \,|\, \operatorname{supp} f \text{ is compact}\}$. Obviously, $\mathcal{C}_{\mathrm{c}} \subset \mathcal{C}_0 \subset \mathcal{C}_{\mathrm{b}} \subset \mathcal{C}$.

Let $\Omega \in \mathbf{R}^n$ be open (or an interval). Let $\mathcal{V}$ be one of the symbols $\mathcal{C}$, $\mathcal{C}_{\mathrm{b}}$, $\mathcal{C}_{\mathrm{bu}}$, $\mathcal{C}_0$, $\mathcal{C}_{\mathrm{c}}$. Then we set $\mathcal{V}^0 := \mathcal{V}$, $\mathcal{V}^{k+1}(\Omega;B) := \{f \in \mathcal{V}(\Omega;B) \,\big|\, D_j f \in \mathcal{V}^k(\Omega;B)$ for all $j \in \{1,2,\dots,n\}\}$ (when $k \in \mathbf{N}$), $\mathcal{V}^\infty(\Omega;B) := \cap_{k\in\mathbf{N}} \mathcal{V}^k(\Omega;B)$.

Lemma B.3.4 ($\mathcal{C}_{\mathrm{c}} \subset \mathcal{C}_0 \subset \mathcal{C}_{\mathrm{bu}} \subset \mathcal{C}_{\mathrm{b}}$) *Let Ω be a metric space. Then $\mathcal{C}_{\mathrm{c}}(\Omega;B) \subset \mathcal{C}_0(\Omega;B) \subset \mathcal{C}_{\mathrm{bu}}(\Omega;B) \subset \mathcal{C}_{\mathrm{b}}(\Omega;B)$; the spaces $\mathcal{C}_0$, $\mathcal{C}_{\mathrm{bu}}$ and $\mathcal{C}_{\mathrm{b}}$ are Banach spaces (under the supremum norm), and $\mathcal{C}_{\mathrm{c}}$ is dense in $\mathcal{C}_0$.*

Proof: 1° *Claims $\mathcal{C}_{\mathrm{c}}(\Omega;B) \subset \mathcal{C}_0(\Omega;B) \subset \mathcal{C}_{\mathrm{bu}}(\Omega;B) \subset \mathcal{C}_{\mathrm{b}}(\Omega;B)$:* These are obvious except the uniform continuity of function $f \in \mathcal{C}_0(\Omega;B)$. Given $f \in \mathcal{C}_0(\Omega;B)$ and any $\varepsilon > 0$, set $K := \{q \in \Omega \,\big|\, \|f(q)\| \ge \varepsilon/3\}$, so that K is a closed subset of a compact set, hence compact. Analogously, $K' := \{q \in \Omega \,\big|\, \|f(q)\| \ge \varepsilon/2\}$ is compact. Since $K' \subset V := \{q \in \Omega \,\big|\, \|f(q)\| > \varepsilon/3\} \subset K$ and V is open, we have $\delta' := d(K',K^c) \ge d(K',V^c) > 0$, by Lemma A.2.1(c).

Choose $\delta > 0$ s.t. $\|f(q) - f(q')\|_B < \varepsilon$ for all $q,q' \in K$ s.t. $d(q,q') < \delta$. Then, given $q,q' \in \Omega$ s.t. $d(q,q') < \min(\delta,\delta')$, we have $\|f(q) - f(q')\|_B < \varepsilon$.

Indeed, if $q \notin K$, then $q' \notin K'$, hence then $\|f(q) - f(q')\|_B < \varepsilon/3 + \varepsilon/2 < \varepsilon$; the case with $q' \notin K$ is analogous, and the case $q,q' \in K$ follows from the definition of δ.

2° *The completeness of $\mathcal{C}_0$, $\mathcal{C}_{\mathrm{bu}}$ and $\mathcal{C}_{\mathrm{b}}$:* Let $\{f_n\}$ be a $\mathcal{C}_{\mathrm{b}}$-Cauchy-sequence. Then so is $\{f_n(q)\}$, hence $f(q) := \lim_{n\to\infty} f_n(q) \in B$ exists, for each $q \in Q$. Given $\varepsilon > 0$, there is $N \in \mathbf{N}$ s.t. $\|f_n - f_m\|_{\mathcal{C}_{\mathrm{b}}} := \sup_{q\in Q} \|f_n(q) - f_m(q)\|_B < \varepsilon/2$ for all $n,m \ge N$, so that $\sup_{q\in Q} \|\lim_{n\to\infty} f_n(q) - f_m(q)\|_B \le \varepsilon/2 < \varepsilon$ for all $m \ge N$. Consequently, $f_m \to f$ uniformly as $m \to \infty$. As one easily verifies, it follows that f is continuous and bounded. hence $f_n \to f$ in $\mathcal{C}_{\mathrm{b}}$.

If $f_n \in \mathcal{C}_0$ for all n, then, for each $\varepsilon > 0$ we can choose N s.t. $\|f_N - f\| < \varepsilon/2$ and K s.t. $\|f_N\| < \varepsilon/2$ on K^c, so that $\|f\| < \varepsilon/2 + \varepsilon/2$ on K^c; thus, then $f \in \mathcal{C}_0$.

Finally, assume that $f_n \in \mathcal{C}_{\mathrm{bu}}$ for each n. Given $\varepsilon > 0$, choose N s.t. $\|f_N - f\| < \varepsilon/3$, and choose then $\delta > 0$ s.t. $\|f_N(q) - f_N(q')\| < \varepsilon/3$ whenever $q,q' \in \Omega$, $d(q,q') < \delta$. Then $\|f(q) - f(q')\| \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3$ whenever $q,q' \in \Omega$, $d(q,q') < \delta$. Thus, then $f \in \mathcal{C}_{\mathrm{bu}}$.

3° *$\mathcal{C}_{\mathrm{c}}$ is dense in $\mathcal{C}_0$:* (This holds also when Ω is a locally compact or normal (possibly non-metrizable) Hausdorff space.)

Let $f \in \mathcal{C}_0(\Omega;B)$ and $\varepsilon > 0$. Choose K, K', δ' as in 1°. Set $g(q) := 1$ in K' and $g(q) := 0$ in K^c, so that $g \in \mathcal{C}(K' \cup K^c,[0,1])$. By Tietze's Extension Theorem [Kelley], g has an extension $h \in \mathcal{C}(\Omega;[0,1])$. But $\operatorname{supp} h \subset K$, hence $h \in \mathcal{C}_{\mathrm{c}}(\Omega)$. It follows that $hf \in \mathcal{C}_{\mathrm{c}}(\Omega;B)$ and $\|hf - f\| \le \varepsilon/2$ (since $hf = f$ on K' and $\|hf - f\| \le \|f\| < \varepsilon/2$ on $(K')^c$). Since $f \in \mathcal{C}_0(\Omega;B)$ and $\varepsilon > 0$ were arbitrary, $\mathcal{C}_{\mathrm{c}}$ is dense in $\mathcal{C}_0$. $\square$

The sequence spaces c_{c} (of finite sequences) and c_{o} (of vanishing sequences) are sometimes needed; we briefly introduce them below as special cases of $\mathcal{C}_{\mathrm{c}}$ and $\mathcal{C}_0$.

For any set Q, the function $d(q,q') := \begin{cases} 1, & \text{when } q \ne q'; \\ 0, & \text{when } q = q' \end{cases}$ is a metric, called the *discrete metric* of Q. In the corresponding *discrete topology* every subset of Q is open, hence every function $Q \to B$ is continuous. A subset of Q is compact iff it is finite. We set $\mathrm{c_o}(Q;B) := \mathcal{C}_0(Q;B)$ and $\mathrm{c_c}(Q;B) := \mathcal{C}_c(Q;B) = \{f : Q \to B \,\big|\, f(q) = 0$ except for finitely many $q \in Q\}$, where Q is equipped by its discrete topology. (Recall that empty set is finite, i.e., $0 \in \mathrm{c_c}(Q;B)$ (for $Q \neq \emptyset$).) Thus, $\mathrm{c_o}(Q;B)$ is the closure of $\mathrm{c_c}(Q;B)$ under the supremum norm (i.e., in $\ell^\infty(Q;B)$, the set of bounded functions $Q \to B$), by Lemma B.3.4.

The following two lemmas contain important facts:

Lemma B.3.5 (f' is measurable) *Let $J \subset \mathbf{R}$ be an interval and let $f : J \to B$ be Lebesgue measurable. If $f'(t)$ exists for a.e. $t \in J$, then f' is Lebesgue measurable.* □

(This follows from Lemma B.2.5(c).)

Lemma B.3.6 *Let $p \in [1,\infty]$. If $f \in \mathrm{L}^p(Q;B)$ and $T \in \mathcal{B}(B,B_2)$, then $Tf \in \mathrm{L}^p(Q;B_2)$. If $F \in \mathrm{L}^p(Q;\mathcal{B}(B,B_2))$, then $F^* \in \mathrm{L}^p(Q;\mathcal{B}(B_2^*,B^*))$.* □

(This follows easily from Lemma B.2.5(a2)&(a3). This implies that the Bochner integral is a special case of so called Pettis integral.)

A casual reader might wish skip the rest this section except Lemma B.3.9 and Theorem B.3.11(a1)&(b1). Most other results below are rather technical and less often needed.

Most "L^p mass" of a L^p function lies on a compact set (unless $p = \infty$):

Lemma B.3.7 *Let $f \in \mathrm{L}^p(\mathbf{R}^n;B)$ and $p \in (0,\infty)$, or $f \in \mathcal{C}_0(\mathbf{R}^n;B)$ and $p = \infty$. Then $\|\chi_{\{q\in\mathbf{R}^n \,\big|\, |q|>R\}} f\|_p \to 0$ as $R \to \infty$. In particular, for any $\varepsilon, r > 0$, there is $R > 0$ s.t. $\|\chi_{\{q\in\mathbf{R}^n \,\big|\, |q|\le r\}} \tau^s f\|_p < \varepsilon$ for $s \in \mathbf{R}^n$ s.t. $|s| > R$.* □

(This follows from the (scalar) Monotone Convergence Theorem (note that $|q+s| > R-r$ when $|q| \le r$), except in the (trivial) $\mathcal{C}_0$ case.)

Corollary B.3.8 ($\pi_+\tau^t f \to 0$) *Let $f \in \mathrm{L}^p(\mathbf{R};B)$ and $1 \le p < \infty$.*

Then $\tau^t f \to \tau^T f$ in L^p, as $t \to T \in \mathbf{R}$, and $\pi_{[-T,t)} f \to f$, $\pi_{[0,t)} f \to \pi_+ f$, $\pi_{[0,t^{-1})} f \to 0$ and $\pi_+\tau(t) f \to 0$ in L^p, as $t, T \to +\infty$.

If $g \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R};B)$, then $\|\pi_+\tau^{-t} g\|_p \to \|g\|_p$ as $t \to +\infty$. □

(One obtains this easily from Lemma B.3.7.)

Lemma B.3.9 *Let $p \in [1,\infty)$ and $f \in \mathrm{L}^p(\mathbf{R}^n;B)$. Then*

$$\|f - \tau(h) f\|_p \to 0 \quad \text{as } h \to 0 \text{ in } \mathbf{R}^n. \tag{B.8}$$

Proof: By uniform continuity, (B.8) holds for $f \in \mathcal{C}_c(\mathbf{R}^n;B)$. For general $f \in \mathrm{L}^p$ and $\varepsilon > 0$, choose $\phi \in \mathcal{C}_c$ s.t. $\|f - \phi\|_p < \varepsilon/3$, and then choose $\delta > 0$ s.t. $\|\phi - \tau(h)\phi\|_p < \varepsilon/3$ for $|h| < \delta$. Then $\|f - \tau(h) f\|_p < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$ for $|h| < \delta$. □

Characteristic functions can be approximated by smooth functions:

Lemma B.3.10 *If $K \subset \Omega \subset \mathbf{R}^n$, K is compact, and Ω is open, there is $\phi \in C_c^\infty(\Omega)$ s.t. $\chi_K \le \phi \le \chi_\Omega$.*

If $\Omega' \subset \mathbf{R}^n$ is open, $E \subset \Omega' \subset \mathbf{R}^n$, $m(E) < \infty$, $p \in [1,\infty)$, [and $w \ge 0$ is L^1 on a neighborhood of E], and $\varepsilon > 0$, then we can choose K and Ω so that $K \subset E \subset \Omega \subset \Omega'$ and $m(\Omega \setminus K) < \varepsilon^p$ [$\int_{\Omega\setminus K} w\,dm < \varepsilon^p$] to obtain $\|\chi_E - \phi\|_p < \varepsilon$ [$\|\chi_E - \phi\|_{\mathrm{L}^p(\Omega,w\,dm)} < \varepsilon$].

Note that ϕ is infinitely differentiable and with a compact support $\subset \Omega$, $\phi = 1$ on K, and $0 \le \phi \le 1$. Here w is a nonnegative weight function, i.e., we refer to the measure $E \mapsto \int_E w\,dm$.

Proof: (We only sketch the proof; see [Adams, Section 2.17] for details on mollifiers.)

1° *$w \equiv 1$:* Because $r := d(K,\Omega^c) > 0$, we can set $K' := \{x \in \Omega \,|\, d(x,K) \le r/2\}$, and then take $\phi := \phi_k * \chi_{K'}$ for some large k, where $\{\phi_k\} \subset C_c^\infty$ converges to the delta distribution δ_0.

The last claim follows from the regularity of m [Rud86, Theorem 2.20].

2° *The general case:* Let $w \in \mathrm{L}^1(\Omega'')$, where $E \subset \Omega'' \subset \Omega$ and Ω'' is open. By [Rud86, Exercise 1.12], $\int_{\Omega\setminus K} w\,dm < \varepsilon^p$, when $m(\Omega \setminus K)$ is small enough. Consequently,

$$\int_\Omega |\chi_E - \phi|^p w\,dm \le \int_\Omega w\,dm < \varepsilon^p. \tag{B.9}$$

□

The space of simple L^p functions (as well as C_c^∞) is dense in L^p, even in $\mathrm{L}^{p_1} \cap \mathrm{L}^{p_2} \cap \cdots \cap \mathrm{L}^{p_n}$ (and even with different weight functions), when $p < \infty$:

Theorem B.3.11 (C_c^∞ is dense in L^p) *Let $\Omega \subset \mathbf{R}^n$ be open and $1 \le p < \infty$.*

(a1) Simple L^p functions are dense in $\mathrm{L}^p(Q;B)$, and countably-valued L^∞ functions are dense in $\mathrm{L}^\infty(Q;B)$.

(a2) If we are given $n \in \mathbf{N}+1$, exponents $p_1,\dots,p_n \in [1,\infty)$ and $\varepsilon > 0$, then, for any $f \in \cap_{k=1}^n \mathrm{L}^{p_k}(Q;B)$, there is a simple function $s \in \mathrm{L}^p$ s.t. $\|f - s\|_{\mathrm{L}^{p_k}(Q;B)} < \varepsilon$.

(a3) At least in (a1)–(b2) and (d), given a dense subspace B_0 of B, we may choose the (dense set of) functions so that they have their values in B_0.

(b1) Finite-dimensional $C_c^\infty(\Omega;B)$ functions are dense in $\mathrm{L}^p(\Omega;B)$.

(b2) If we are given $n \in \{1,2,\dots\}$, exponents $p_1,\dots,p_n \in [1,\infty)$, nonnegative (weight) functions $w_1,\dots,w_n \in \mathrm{L}^1_{\mathrm{loc}}(\Omega;B)$, and $\varepsilon > 0$, then, for any $f \in \cap_{k=1}^n \mathrm{L}^{p_k}(\Omega, w_k\,dm;B)$, there are a simple function $s \in \mathrm{L}^p$ and a finite-dimensional $C_c^\infty(\Omega;B)$ function ϕ s.t.

$$\|f-s\|_{\mathrm{L}^{p_k}(\Omega,w_k\,dm;B)} < \varepsilon \ \text{ and } \ \|f-\phi\|_{\mathrm{L}^{p_k}(\Omega,w_k\,dm;B)} < \varepsilon \ \text{ for all } k \in \{1,\dots,n\}. \tag{B.10}$$

(b3) $\{\phi \in \mathcal{S}(\mathbf{R};B) \,|\, \widehat{\phi} \in C_c^\infty(i\mathbf{R};B)\}$ is dense in $\mathrm{L}^p(\mathbf{R};B)$.

(c) The closure of $C_c(Q;B)$ in L^∞ is $C_0(Q;B)$ when Q is metrizable.

(d) The closure of simple measurable functions in $\mathrm{L}^\infty(Q;B)$ *equals*

$$\mathrm{L}^\infty_{\mathrm{K}}(Q;B) := \{f \in \mathrm{L}^\infty(Q;B) \,\big|\, \text{there is a compact } K \subset B \text{ s.t. } f(q) \in K \text{ for a.e. } q \in Q\}. \tag{B.11}$$

We have $C_0(Q;B) \subset \mathrm{L}^\infty_{\mathrm{K}}(Q;B)$ *when* Q *is metrizable. Moreover,* $\mathrm{L}^\infty_{\mathrm{K}}(\Omega;B) = \mathrm{L}^\infty(\Omega;B)$ *iff* $\dim B < \infty$, *and* $\mathrm{L}^\infty_{\mathrm{K}}(Q;B) = \mathrm{L}^\infty(Q;B)$ *if* $\dim B < \infty$.

(By using mollifiers one could obtain further density results; cf. pp. 29–52 of [Adams].)

Recall that simple means finite-valued, hence s is a simple $\mathrm{L}^p(Q;B)$ function iff $s = \sum_{k=0}^n \chi_{E_k} x_k$ for some $n \in \mathbf{N}$, $\{x_k\} \subset B$, $E_k \subset \mathfrak{M}$, and $\|s\|_p < \infty$.

Proof: (a1) 1° *Case* $p = \infty$*:* This follows from Lemma B.2.5(b2).

2° *Case* $p < \infty$*:* Let $f \in \mathrm{L}^p(Q;B)$ and $\varepsilon > 0$. By Theorem 3.13 of [Rud86], there is a simple function $s : Q \to \mathbf{R}_+$ s.t. $\|\|\|f\|_B - s\|_p < \varepsilon/3$. Set $K := \{q \in Q \,\big|\, s(q) \neq 0\}$, so that $m(K) < \infty$ and $\|f - \widetilde{f}\| < \varepsilon/3$, where $\widetilde{f} := f\chi_K \in \mathrm{L}^p$. By the (scalar) Monotone Convergence Theorem, we have $\widetilde{f}\chi_{\{\|f\| \leq n\}} \to \widetilde{f}$ in L^p, as $n \to +\infty$, hence $g := f\chi_K\chi_{\{\|f\| \leq M\}}$ satisfies $\|f - g\| < \varepsilon/4$ for M big enough. Note that $\|g\|_\infty \leq M$.

By 1°, there is a (countably-valued measurable) function $h = \sum_{k=1}^\infty b_k \chi_{E_k} : K \to B$ ($b_k \in B$, $E_k \subset K$ measurable and disjoint, χ_{E_k} its characteristic function for each k) s.t. $\|g - h\|_\infty$ is arbitrarily small. Because $m(K) < \infty$, it follows that we can take $\|g - h\|_p$ arbitrarily small, say $< \min(\varepsilon/4, 1)$ (and, simultaneously $\|h\|_\infty \leq M + 1 < \infty$).

By applying the (scalar) dominated convergence theorem to $\|h\|_B^p$ (with the constant function $M + 1 \in \mathrm{L}^1(K)$ as the majorant), we see that $\int_K \|h - \sum_{k=1}^n b_k \chi_{E_k}\|_B^p \, dm < (\varepsilon/4)^p$ for some n. Thus, $\|f - \sum_{k=1}^n b_k \chi_{E_k}\|_p < 3\varepsilon/4 < \varepsilon$.

(b1) By Lemma B.3.10, we may approximate each χ_{E_k} above by some $\phi_k \in C_c^\infty(\Omega)$ to get

$$\|\sum_{k=1}^n b_k(\chi_{E_k} - \phi_k)\|_p \leq \sum_{k=1}^n \|b_k\|_B \|\chi_{E_k} - \phi_k\|_p < \varepsilon/4 \tag{B.12}$$

(by the Minkovski inequality), hence $\|f - \sum_{k=1}^n b_k \phi_k\|_p < 4\varepsilon/4 = \varepsilon$.

(a2) Work as in (a1) for each p_k. Let K be the union of "K's" and let M be the maximum of "M's". Require $\|g - h\|_\infty$ to be small enough for each p_k. Let n be the maximum of "n's".

(b2) Write $\Omega = \cup_l K_l$, where $K_1 \subset K_2 \subset \cdots$ are compact sets (e.g., $K_l := \{x \in \Omega \,\big|\, |x| \leq l \wedge d(x, \Omega^c) \geq 1/l\}$), then, for some $M \in \{1, 2, \ldots\}$, we have

$$\int_\Omega \|f - f\chi_{K_M}\chi_{\{\|f\|_B \leq M\}}\|_B^{p_k} w_k \, dm < \varepsilon^{p_k} \quad (k = 1, \ldots, n), \tag{B.13}$$

by the dominated convergence theorem, because $\|f\|^{p_k} w_k \in \mathrm{L}^1$ ($k = 1, \ldots, n$), by the assumption on f; this gives the function $g := f\chi_K\chi_{\{\|f\|_B \leq M\}}$, where $K := K_M$.

By taking $\|g - h\|_\infty < \min(\delta, 1)$, where $\delta^{p_k} \int_K w_k \, dm < \varepsilon/4$ for all k, we get a suitable countably-valued function h, and obtain then s as some partial sum of h, as in the proof of (a1). The rest goes approximately as in (b1) (when applying Lemma B.3.10, use unions (over $k \in \{1,2,\ldots,n\}$) of compact sets and intersections of open sets).

(a3) This is obvious in the sense of simple (or countably-valued) functions. The procedure in (b1) obviously keeps the values in B_0, so does that in (b2) too.

(b3) This is given in Lemma 2.3 of [Zimmermann] and its proof (which shows that we may additionally require that $0 \notin \operatorname{supp}\widehat{\phi}$ if $p > 1$).

(See Appendix D for $\mathcal{S}$ and the Fourier transform $\widehat{\phi}$. Note that the claim follows easily from (b1) if $p = 2$ and B is a Hilbert space.)

(c) This follows from Lemma B.3.4.

(d) (Note that $\mathrm{L}^\infty_\mathrm{K}$ equals the space L_∞ used in the interpolation theory of [BL].)

1° Obviously, $\mathcal{C}_\mathrm{c} \subset \mathrm{L}^\infty_\mathrm{K}$, hence $\mathcal{C}_0 \subset \mathrm{L}^\infty_\mathrm{K}$ when Q is metrizable, by (c).

2° SMF *is a dense subset of* $\mathrm{L}^\infty_\mathrm{K}$*:* Let SMF denote the set of simple measurable functions (obviously, SMF $\subset \mathrm{L}^\infty_\mathrm{K}$).

Let $\varepsilon > 0$ and $f \in \mathrm{L}^\infty_\mathrm{K}$ be arbitrary. Choose K for f as in the definition of $\mathrm{L}^\infty_\mathrm{K}$. Choose $n \in \mathbf{N}$, $x_0,\ldots,x_n \in K$ s.t. $K \subset \cup_{k=0}^n D(x_k,\varepsilon)$.

Set $E_k' := f^{-1}[D(x_k,\varepsilon)]$, $E_0 := E_0'$, $E_{k+1} := E_{k+1}' \setminus \cup_{j=0}^k E_j'$ ($k = 0,\ldots,n-1$), $s := \sum_{k=0}^n x_k \chi_{E_k}$ to obtain that $\|s - f\|_B < \varepsilon$ a.e., hence $\|f - s\|_\infty \le \varepsilon$. Since $\varepsilon > 0$ and $f \in \mathrm{L}^\infty_\mathrm{K}$ were arbitrary, we observe that SMF is dense in $\mathrm{L}^\infty_\mathrm{K}$.

3° $\mathrm{L}^\infty_\mathrm{K}$ *is the closure of* SMF*:* By 2°, we only need to show that $f \in \mathrm{L}^\infty_\mathrm{K}$ assuming that $\{s_n\} \subset$ SMF and $\|s_n - f\|_\infty < 1/n$ for all $n \in \mathbf{N}+1$ (so that $\mathrm{L}^\infty_\mathrm{K}$ is closed).

For each $n \ge 1$, choose a null set N_n s.t. $\|s_n - f\| < 1/n$ on N_n^c. Set $N := \cup_{n\ge 1} N_n$, $A := f[N^c] \subset B$. Given $\varepsilon > 0$, choose $n > 1/\varepsilon$, so that $\|s_n - f\| < \varepsilon$ on N^c. Write s_n as $s_n = \sum_{k=1}^n x_k \chi_{E_k}$ with $E_k \cap E_j = \emptyset$ for $k \ne j$.

Then $A \subset D(0,\varepsilon) \cup \big(\cup_{k=1}^n D(x_k,\varepsilon)\big)$, because $\|f - x_k\| < \varepsilon$ on E_k and $\|f\| < \varepsilon$ on $(\cup_k E_k)^c \setminus N$. Because $\varepsilon > 0$ was arbitrary, the set A is totally bounded (i.e., precompact), hence so is $K = \overline{A}$ (use 2ε in place of ε), hence K is compact (use, e.g., Theorem 9.4 of [Bredon] and the completeness of B). Moreover, $f(q) = \lim_n s_n(q) \in K$ for $q \in N^c$, hence a.e. Therefore, $f \in \mathrm{L}^\infty_\mathrm{K}$.

4° $\mathrm{L}^\infty_\mathrm{K}(Q;B) = \mathrm{L}^\infty(Q;B)$ *if* $\dim B < \infty$*:* If $\dim B < \infty$, then we can take $K := D(0,\|f\|_\infty)$ for any $f \in \mathrm{L}^\infty$ to observe that $f \in \mathrm{L}^\infty_\mathrm{K}$.

5° $\mathrm{L}^\infty_\mathrm{K}(\Omega;B) = \mathrm{L}^\infty(\Omega;B)$ *iff* $\dim B < \infty$*:* Assume that $\dim B = \infty$. Let $\{E_k\} \subset \Omega$ are disjoint sets of positive measure (in fact, we need not have $\Omega \subset \mathbf{R}^n$ as long as this property is satisfied).

The unit ball D_1 of B is not compact, by Theorem 1.23 of [Rud73], hence there is a sequence $\{x_k\} \subset D_1$ without limit points (by Exercise 2.26 and Theorem 2.37 of [Rud76], this is equivalent to noncompactness in any metric space). Set $f := \sum_{n\in\mathbf{N}} x_k \chi_{E_k} \in \mathrm{L}^\infty(\Omega;B)$ to obtain that $f[N^c] = \{x_k\}$ whenever N is a null set; in particular, $f[N^c]$ is not contained in any compact subset of B, hence $f \notin \mathrm{L}^\infty_\mathrm{K}$.

(N.B. the above example also shows that $\mathrm{L}^1 \cap \mathrm{L}^\infty(\Omega;B) \not\subset \mathrm{L}^\infty_\mathrm{K}(\Omega;B)$; e.g., choose $\{E_k\}$ s.t. $\sum_k m(E_k) < \infty$ to have $f \in \mathrm{L}^1(\Omega;B)$ too.) □

If μ is the completion of another measure μ', then the simple functions constructed above can be chosen to be "μ'-measurable":

Lemma B.3.12 *Let $X = B$ or $X = [-\infty,\infty]$. Let $s = \sum_{k\in\mathbf{N}} x_k\chi_{E_k}$, where $x_k \in X$ and $E_k \in \mathfrak{M}$ for all $k \in \mathbf{N}$. If $\mathfrak{M}$ is the completion of another σ-algebra $\mathfrak{M}'$, then there are sets $\{E'_k\} \subset \mathfrak{M}'$ s.t. $\sum_{k\in\mathbf{N}} x_k\chi_{E'_k} = s$ a.e.* □

(We omit the trivial proof.)

We generalize the Hölder inequality to the case $r > 1$:

Lemma B.3.13 ($\|fg\|_r \le \|f\|_p\|g\|_q$) *Let $f \in \mathrm{L}^p(Q;B)$, $g \in \mathrm{L}^q(Q;B_2)$, $p,q \in (0,\infty]$ and $\|bb_2\|_{B_3} \le \|b\|_B\|b_2\|_{B_2}$ ($b \in B$, $b_2 \in B_2$). Then $\|fg\|_r \le \|f\|_p\|g\|_q$, where $r^{-1} := p^{-1} + q^{-1}$.*

Moreover, if $Q = \mathbf{R}^n$, $\mu = m$, $p,q < \infty$ and $\varepsilon > 0$, then there is $R > 0$ s.t. $\|f\tau^t g\|_r < \varepsilon$ for all $t \in \mathbf{R}^n$ s.t. $|t| > R$ (if $f \in C_0$, then we can allow $p = \infty$; if $g \in C_0$, then we can allow $q = \infty$).

Thus, we may have, e.g., $B = \mathbf{K}$ ($B_3 = B_2$), $B = B_2^*$ ($B_3 = \mathbf{K}$) or $B = \mathcal{B}(B_2,B_3)$. Note that we may have $r < 1$, but $\min\{p,q\}/2 \le r \le \min\{p,q\}$.

Proof: 1° *Inequality $\|fg\|_r \le \|f\|_p\|g\|_q$:* If $p = \infty$ or $q = \infty$, then this is obvious (and $r = \min\{p,q\}$), so we assume that $p,q \in [1,\infty)$. Set $F := \|f\|_B$, $G := \|g\|_{B_2}$, $t := (p+q)/q$, $t' := (p+q)/p$, so that $rt = p$, $rt' = q$ and $t^{-1} + t'^{-1} = 1$. Then, by the Hölder inequality, we have

$$\|fg\|_r^r = \int_Q F^r G^r \, d\mu \le \|F^r\|_t \|G^r\|_{t'} \le \|F\|_{rt}^r \|G\|_{t'r}^r \le \|F\|_p^r \|G\|_q^r, \tag{B.14}$$

hence $\|fg\|_r \le \|f\|_p\|g\|_q$.

2° *Finding R:* Assume that $\|f\|_p \le 1$ and $\|g\|_q \le 1$. By Lemma B.3.7, there is $R > 0$ s.t. $\|\chi_{D_R^c} f\|_p < \varepsilon/2$ and $\|\chi_{D_R}\tau^t g\|_q < \varepsilon/2$ for $t \in \mathbf{R}^n$ s.t. $|t| > q$. Consequently, $\|f\tau^t g\| < \varepsilon$ for such t. □

If $p \in [p_0, p_1]$, then $\mathrm{L}^p \subset \mathrm{L}^{p_0} \cap \mathrm{L}^{p_1}$:

Lemma B.3.14 ($\|f\|_p \le \max\{\|f\|_{p_0}, \|f\|_{p_1}\}$) *Let $f \in \mathrm{L}^{p_0}(Q;B) \cap \mathrm{L}^{p_1}(Q;B)$, $1 \le p_0 \le p \le p_1 \le \infty$. Then*

$$\|f\|_p \le \|f\|_{p_0}^{1-\theta}\|f\|_{p_1}^{\theta} \le \max\{\|f\|_{p_0}, \|f\|_{p_1}\}, \tag{B.15}$$

where $\theta := \frac{p^{-1} - p_0^{-1}}{p_1^{-1} - p_0^{-1}}$.

Proof: The scalar case is Theorem 5.1.1 of [BL]; see also Theorem 4.1.2 and p. 27 of [BL]. (The definition of L^∞ in [BL] coincides with the standard one in the scalar case, by Theorem 1.17 of [Rud86].) Apply the scalar case to $\|f\|_B \in \mathrm{L}^{p_0}(Q)$ to obtain the general case. □

Next we note that, roughly speaking, L^p is separable iff $p < \infty$ and B is separable:

Lemma B.3.15 *Let B be separable and $1 \le p < \infty$. Then $\ell^p(Q;B)$ is separable iff Q is at most countable. If $Q \subset \mathbf{R}^n$ is measurable and $d\mu = |f|\,dm$ for some $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n)$, then $\mathrm{L}^p(Q;B)$ is separable.*

If μ is as above, $\mu(Q) \neq 0$ and $B_2 \neq \{0\}$, then $\mathrm{L}^\infty(Q,\mu;B_2)$ is unseparable.

If $0 < \mu(E) < \infty$ for some measurable $E \subset Q$, then $\mathrm{L}^p(Q;B_2)$ and $\mathrm{L}^\infty(Q;B_2)$ are unseparable for unseparable B_2.

Proof: 1° *Preparations:* For each $k \in \mathbf{N}$, we let P_k be the set of points in $\mathbf{R}^n$ whose coordinates are integral multiples of 2^{-k}, and V_k the collection of $2^{-k} \times \cdots 2^{-k}$ boxes with corners at points of P_k. Set $V := \cup_k V_k$, so that V is countable and any open $G \subset \mathbf{R}^n$ is the union of disjoint elements of V (see [Rud86, 2.19] for details). Let $S \subset B$ be dense and countable.

2° *The closed span of $\mathcal{F} := \{x\chi_G \,\big|\, x \in S,\ G \in V\}$ is dense in $\mathrm{L}^p(\mathbf{R}^n,\mu;B)$:* Let $A \subset \mathbf{R}^n$ have a finite measure and $\varepsilon > 0$. By the Monotone Convergence Theorem, there is $R > 0$ s.t. $A_R := \{q \in A \,\big|\, |q| < R\}$ satisfies $\|\chi_A - \chi_{A_R}\|_1 < \varepsilon/4$. Choose open $G \subset \mathbf{R}^n$ s.t. $\|\chi_{A_R} - \chi_G\|_1 < \varepsilon/4$ (see (B.28) and use the fact that μ is absolutely continuous w.r.t. m on $\{|q| < R\}$).

Write G as a disjoint union of elements $G_0, G_1, G_2, \ldots \in V$, so that $khi_G = \sum_{k\in\mathbf{N}} \chi_{G_k}$, and use the Monotone Convergence Theorem to choose $k \in \mathbf{N}$ s.t. $\|\chi_G - \sum_{j=0}^k \chi_{G_j}\|_1 < \varepsilon/4$. Then, $\|\chi_A - \sum_{j=0}^k \chi_{G_j}\|_1 < 3\varepsilon/4$.

Given $x \in B$, we have $\|x\chi_A - x' \sum_{j=0}^k \chi_{G_j}\|_1 < \varepsilon$ for some $x' \in S$. By linearity, it follows that the closed span of $\mathcal{F}$ contains that of simple measurable functions, i.e., it is equal to $\mathrm{L}^p(\mathbf{R}^n;B)$.

3° *The closed span of $\mathcal{F} := \{x\chi_G\chi_Q \,\big|\, x \in S,\ G \in V\}$ is dense in $\mathrm{L}^p(Q,\mu;B)$:* Apply 2° to the zero extensions of f and μ onto $\mathbf{R}^n$.

4° *$\ell^p(Q;B)$:* The closed span of $\{\chi_{\{q\}}x \,\big|\, x \in S,\ q \in Q\}$ is obviously dense in $\ell^p(Q,X)$ (e.g., use the proof of 2° with $V := \{\{q\} \,\big|\, q \in Q\}$).

5° *$\ell^\infty(Q;B_2)$ is unseparable for uncountable Q and $B_2 \neq \{0\}$:* Let $\varepsilon < 1/2$. Choose $x \in B_2$ s.t. $\|x\| = 1$. Then the ε-neighborhoods of no countable subset of ℓ^∞ can contain every $x\chi_q$, $q \in Q$, because $\|x\chi_q - x\chi_{q'}\|_\infty = 1$ for $q' \neq q$.

6° *$\mathrm{L}^\infty(Q,\mu;B_2)$ is unseparable:* Let $Q_0 := Q$, $r_0 := \infty$. Given (Q_k, r_k), choose $r_{k+1} \in (0, r_k)$ s.t. $0 < \mu(Q_{k+1}) < \mu(Q_k)$, where $Q_{k+1} := \{q \in Q_k \,\big|\, |q| < r_{k+1}\}$, and set $Q'_k := Q_k \setminus Q_{k+1}$. Then $Q = \cup_{k\in\mathbf{N}} Q'_k$, and the sets Q'_k are disjoint and of positive (or infinite) measure.

Choose $x \in B_2$ s.t. $\|x\|_{B_2} = 1$. For each $E \subset \mathbf{N}$, set $f_E := x\chi_{\cup_{n\in E} Q'_n}$. Then $\|f_E - f_{E'}\|_\infty = 1$ whenever $E \neq E'$, so that the ε-neighborhoods of no countable set can contain every f_E for $\varepsilon < 1/2$. Thus, $\mathrm{L}^\infty(Q,\mu;B)$ is unseparable.

7° *$\mathrm{L}^p(Q;B_2)$ is unseparable for unseparable B_2 if $0 < \mu(E) < \infty$ for some measurable $E \subset Q$:* The subspace $\{x\chi_E \,\big|\, x \in B_2\}$ of L^p is isometrically isomorphic to B_2, hence unseparable. Therefore, so is L^p. □

Recall from Lemma A.3.1(a1)&(a2) that $\dim H$ means the cardinality of an arbitrary orthonormal basis of H. We have $\dim \mathrm{L}^2(Q;H) = \dim H$ for infinite-dimensional H's and most Q's:

Lemma B.3.16 ($\dim \mathrm{L}^2(Q,\mu;H) = \dim H$) *Assume that H is a Hilbert space. Then $\dim \mathrm{L}^2(Q;H) = \dim \mathrm{L}^2(Q) \times \dim H$; in particular, $\dim \ell^2(Q;H) = \operatorname{card} Q \times \dim H$ ($= \dim H$ whenever $Q \neq \emptyset$, Q is at most countable and H is infinite-dimensional).*

If $Q \subset \mathbf{R}^n$ or $Q \subset \partial\mathbf{D}$ is measurable, $\mu(Q) \neq 0$, and $d\mu = |f|\,dm$ for some $f \in \mathrm{L}^1_{\mathrm{loc}}(Q,m)$, then $\dim \mathrm{L}^2(Q,\mu;H) = \operatorname{card}\mathbf{N} \times \dim H$ ($= \dim H$ whenever H is infinite-dimensional).

Proof: 1° Let $\{x_a\}_{a\in A}$ be an orthonormal base of H (so that $\dim H = A$) Let F be an orthonormal base of $\mathrm{L}^2(Q,\mu)$. Then $\{fx_a\}_{f\in F, a\in A} \subset \mathrm{L}^2(Q,\mu;H)$ is an orthonormal base of $\mathrm{L}^2(Q,\mu;H)$ (its closed span is L^2, by the density of simple L^2 functions) of cardinality of $F \times A$. Thus, $\dim \mathrm{L}^2(Q;H) = \operatorname{card}(F \times A) = \dim \mathrm{L}^2(Q,\mu) \times \dim H$.

2° Since the set of simple $\ell^2(Q)$ functions is exactly $\mathrm{c_c}(Q)$, the set $\{\chi_{\{q\}}\}_{q\in Q}$ is an orthonormal base of $\ell^2(Q)$. Consequently, $\dim \ell^2(Q) = \operatorname{card} Q$, so that $\dim \ell^2(Q;H) = \operatorname{card} Q \times \dim H$, by 1°.

3° Let Q be as in the latter paragraph. By Lemma B.3.15, $\mathrm{L}^2(Q,\mu)$ is separable. It is obviously infinite-dimensional, hence $\dim \mathrm{L}^2(Q,\mu) = \operatorname{card}\mathbf{N}$. Consequently, $\dim \mathrm{L}^2(Q,\mu;H) = \operatorname{card}\mathbf{N} \times \dim H$, by 1°.

4° By Lemma B.2.2, $A \times \dim H = \dim H$ when $A \neq \emptyset$ and $A \leq \dim H \geq \operatorname{card}\mathbf{N}$. $\square$

(See the notes on p. 947.)

B.4 The Bochner integral ($\int_Q : \mathrm{L}^1(Q;B) \to B$)

Adde parvum parvo manus acervus erit.
— Ovid (43 B.C. – 17 A.D.)

In this section we define the Bochner integral $\mathrm{L}^1(Q;B) \to B$ and present the Bochner integral extensions of several more and less known Lebesgue integral results. A casual reader might wish just to have a look at Subsections B.4.1–B.4.3 and Theorem B.4.6 and then skip the rest of this section, just remembering that the Bochner integral is "the Lebesgue integral with '$\|\cdot\|$' in place of '$|\cdot|$'".

By Theorem B.3.11(a1) and Lemma A.3.10, we may use the natural definition and density to define the Bochner integral:

Definition B.4.1 (Bochner integral) *We recall that for simple functions* $s := \sum_{k=0}^n x_k \chi_{E_k}$ *($x_k \in B$, E_k measurable for all k) we have set*

$$\int_Q s\,d\mu := \sum_{k=0}^n x_k \mu(E_k) \in B. \tag{B.16}$$

The unique continuous extension of $\int_Q \cdot\, d\mu$ *onto* $\mathrm{L}^1(Q;B)$ *is called the* Bochner integral.

Let $f \in \mathrm{L}^1$. *Then f is called* Bochner integrable *and the* integrand *of* $\int_Q f\,d\mu$, *which, in turn, is said to* converge absolutely.

Obviously, $\|\int_Q s\,d\mu\|_B \le \|s\|_1$, hence the same holds for any integrable function $s : Q \to B$, by Lemma A.3.10. One easily verifies that if $f \in \mathrm{L}^1(Q;\mathbf{K})$ and $f \ge 0$, then this coincides with the (Lebesgue) integral defined in Section B.1.

Sometimes we write $\int_Q f(t)\,d\mu(t) := \int_Q f\,d\mu$ (e.g., $\int_Q t^2\,d\mu(t)$). If $-\infty \le a \le b \le +\infty$ and $\mu = m$ (or $m(E) = 0 \Rightarrow \mu(E) = 0$, i.e., μ is *absolutely continuous w.r.t. m*), then we set $\int_a^b f(t)\,dt := \int_a^b f\,dm := \int_{(a,b)} f\,dm$. For $b < a$ we set $\int_a^b := -\int_b^a$.

If $B = \mathbf{K}$ (or "$B = [0,+\infty]$"), then the Bochner integral is called the Lebesgue integral, etc. By Theorem 11.33 of [Rud76], a function $f : Q \to \mathbf{K}$ is Riemann integrable iff it is bounded and continuous a.e. It follows that Riemann integrable functions belong to L^1. Moreover, the Riemann integral coincides with the Lebesgue integral. We shall not be using the Riemann integral.

An equivalent way to define the integral is to define it for simple functions in the natural way and then use Lemma A.3.10 and Theorem B.3.11 to extend it to all L^1 functions. This definition is used in [HP] and illustrated below:

Lemma B.4.2 *The Bochner integral is in* $\mathcal{B}(\mathrm{L}^1, B)$, *its norm is* 1 *(unless* $\mathrm{L}^1 = \{0\}$*); in particular,*

$$\|\int_Q f\,d\mu\| \le \int_Q \|f\|_B\,d\mu =: \|f\|_1, \tag{B.17}$$

and the integral commutes with bounded linear transformations:

$$T\int_Q f\,d\mu = \int_Q Tf\,d\mu \quad (T \in \mathcal{B}(B,B_2)). \tag{B.18}$$

Moreover, $\int_Q f\,d\mu$ *is the unique element of* B *satisfying* $\Lambda \int_Q f\,d\mu = \int_Q \Lambda f\,d\mu$ *for all* $\Lambda \in B^*$.

Finally, if $s = \sum_{k\in\mathbf{N}} x_k \chi_{E_k} \in \mathrm{L}^1$*, with the sets* E_k *being disjoint and measurable, then* $\int_Q s\,d\mu = \sum_{k\in\mathbf{N}} x_k \mu(E_k)$.

It follows that our definitions of integrable functions and the Bochner integral are equivalent to those of [HP], Section 3.7.

Proof: The $BL(\mathrm{L}^1, B)$ claim and (B.17) follow from Definition B.4.1. If $\mathrm{L}^1 \neq \{0\}$, then there are $x \in B \setminus \{0\}$ and $E \subset Q$ s.t. $\mu(E) \in (0, +\infty)$, and we have $\|\int_Q x\chi_E\,d\mu\| = \|x\|_B \mu(E) = \|x\chi_E\|_1$.

We have $T\int = \int T$ for simple functions, hence for all L^1 functions, by continuity; (B.18). Elements $\{\Lambda x \,|\, \Lambda \in B^*\}$ determines $x \in B$ uniquely.

If s is as in the final claim, then $s_n := \sum_{k=0}^n x_k \chi_{E_k} \to s$ in L^1, by the Monotone Convergence Theorem, hence $\int_Q s_n\,d\mu \to \int_Q s\,d\mu$. $\square$

The standard results can be extended with ease:

Theorem B.4.3 (Lebesgue's Dominated Convergence Theorem) *Assume that* $1 \le p < \infty$*, that the functions* $f_n : Q \to B$ *be measurable (*$n \in \mathbf{N}$*), that the limit* $f(q) := \lim_{n\to+\infty} f_n(q)$ *exists a.e., and that there is* $g \in \mathrm{L}^p(Q; [0, +\infty])$ *s.t.* $\|f_n(q)\|_B \le g(q)$ *a.e. for each* $n \in \mathbf{N}$.

Then $f \in \mathrm{L}^p(Q; B)$ *and* $f_n \to f$ *in* L^p*. In particular, if* $p = 1$*, then* $\lim_{n\to+\infty} \int_Q f_n\,d\mu = \int_Q f\,d\mu$. $\square$

(This follows by applying the scalar LCD Theorem with $F_n := \|f - f_n\|_B^p$, $F := 0$, $\mathrm{L}^p \ni G := (2g)^p \ge F_n$. Obviously, this does not holds for $p = \infty$.)

From the above and the Monotone Convergence Theorem applied to $\sum_{n=1}^N \chi_{E_k} f$ we obtain that if the sets $E_n \subset Q$ ($n \in \mathbf{N}$) are measurable and disjoint, and $f : Q \to B$ is measurable, then $\int_{\cup_n E_n} f\,d\mu = \sum_{n\in\mathbf{N}} \int E_n f\,d\mu$ whenever either sides converges absolutely (i.e., with $\|f\|_B$ in place of f).

Next we extend the standard definition of a product measure:

Definition B.4.4 ($\overline{\mu\times\nu}$) *Assume that* $\mu : \mathfrak{M} \to [0, \infty]$ *and* $\nu : \mathfrak{M}' \to [0, \infty]$ *are* σ*-finite, positive measures on* Q *and* R*, respectively. By* $\mathfrak{M}_{\mu\times\nu}$ *we denote the smallest* σ*-algebra containing* $\{E \times E' \,|\, E \in \mathfrak{M},\ E' \in \mathfrak{M}'\}$*. The* product measure *of* μ *and* ν *is given by*

$$(\mu\times\nu)(E) := \int_Q \nu(\{r \,|\, (q,r) \in E\})\,d\mu \quad (E \in \mathfrak{M}_{\mu\times\nu}). \tag{B.19}$$

By $\overline{\mu\times\nu} : \mathfrak{M}_{\overline{\mu\times\nu}} \to [0, \infty]$ *we denote the completion of* $\mu\times\nu$*. By* $\mathrm{L}(Q\times R; B)$ *we refer to* $\overline{\mu\times\nu}$*-measurable functions* $Q\times R \to B$.

Since in the definitions and results of this chapter we have assumed μ to be complete (as in [HP]), we shall use $\overline{\mu\times\nu}$ (not $\mu\times\nu$) on $Q\times R$. The basic properties of this measure are listed below:

Lemma B.4.5 *Assume that* $\mu : \mathfrak{M} \to [0, \infty]$ *and* $\nu : \mathfrak{M}' \to [0, \infty]$ *are* σ*-finite, positive measures on* Q *and* R*, respectively.*

(a) The measure $\overline{\mu\times\nu}$ is σ-finite.

(b) We have $(\overline{\mu\times\nu})(E) := \int_R \mu(\{q \,|\, (q,r)\in E\})\,d\nu$ for all $E\in\mathfrak{M}_{\overline{\mu\times\nu}}$.

(c) Moreover, if m_k is the Lebesgue measure on $\mathbf{R}^k$ ($k\geq 1$), then $m_{n+k} = \overline{m_n\times m_k}$ ($n,k\geq 1$).

(d1) If $E\in\mathfrak{M}_{\overline{\mu\times\nu}}$, then $E_q := \{r\in R \,|\, (q,r)\in E\}$ is measurable for a.e. $q\in Q$, and $E^r := \{q\in Q \,|\, (q,r)\in E\}$ is measurable for a.e. $r\in R$.

(d2) If N is a null set, then $\nu(N_q)=0$ for a.e. $q\in Q$ and $\mu(N^r)=0$ for a.e. $r\in R$.

(e) **(“cl($\mathfrak{M}\times\mathfrak{M}'$) = $\mathfrak{M}_{\overline{\mu\times\nu}}$”)** *Let $\mathcal{E}$ be the collection of finite unions of elements of $\mathfrak{M}\times\mathfrak{M}'$. Let $E\in\mathfrak{M}_{\overline{\mu\times\nu}}$ and $\overline{\mu\times\nu}(E)<\infty$. Then, for any $\varepsilon>0$, there is $F\in\mathcal{E}$ s.t. $\|\chi_E-\chi_F\|<\varepsilon$.*

(f) If $f\in\mathrm{L}^p(Q\times R;B)$, $p\in(0,\infty)$, then simple measurable functions $\{s_n\}$ of form $s_n=\sum_{k=0}^{N_n}\chi_{E_{n,k}\times F_{n,k}}b_{n,k}$ and $E_{n,k}\times F_{n,k}\in\mathfrak{M}\times\mathfrak{M}'$, $b_{n,k}\in B$ ($n,k\in\mathbf{N}$) s.t. $\|s_n-f\|_p\to 0$ as $n\to\infty$.

Note that each s_n can be written as $\sum_{k=0}^{N'_n}\chi_{E'_{n,k}}s'_{n,k}$, where $E'_{n,k}\in\mathfrak{M}$ and $s'_{n,k}: R\to B$ is simple ($n,k\in\mathbf{N}$).

Proof: Claims (a)–(c) are proved in Chapter 8 of [Rud86]. Claims (d1) and (d2) follow from the Fubini Theorem (with $f=\chi_E$).

(e) 1° *Case $\mu(Q),\nu(R)<\infty$:* Set $\mathfrak{M}_0 := \{E\in\mathfrak{M}_{\overline{\mu\times\nu}} \,|\, E$ the claim in (e) holds for $E\}$, so that we only need to show that $\mathfrak{M}_0=\mathfrak{M}_{\overline{\mu\times\nu}}$, i.e., that $\mathfrak{M}_0$ is a σ-algebra (because, trivially, $\mathfrak{M}\times\mathfrak{M}'\subset\mathcal{E}\subset\mathfrak{M}_0$). This follows from 1.1° and 1.2°.

1.1° One easily verifies that $\|\chi_E-\chi_F\|_1=\|\chi_{E^c}-\chi_{F^c}\|_1$ and $\|\chi_{E\cup E'}-\chi_{F\cup F'}\|_1\leq\|\chi_E-\chi_F\|_1+\|\chi_{E'}-\chi_{F'}\|_1$, hence $\mathfrak{M}_0$ is closed under complements and finite unions.

1.2° Let $E_j\in\mathfrak{M}_0$ ($j\in\mathbf{N}$) and $E=\cup_{j\in\mathbf{N}}E_j$. Set $E'_{j+1}:=E_{j+1}\setminus\cup_{k=0}^{j}E_k$, so that $E=\cup_j E'_j$ and the sets E'_j are disjoint. Because $\overline{\mu\times\nu}(Q\times R)<\infty$, we have $\|\chi_E-\chi_{\cup_{j\leq N}E'_j}\|<\varepsilon/2$ for some $N\in\mathbf{N}$. But $\|\chi_{\cup_{j\leq N}E'_j}-\chi_F\|_1<\varepsilon/2$ for some $F\in\mathcal{E}$, hence $E\in\mathfrak{M}_0$.

2° *General Case:* Let $Q=\cup_{n\in\mathbf{N}}Q_n$, $R=\cup_{n\in\mathbf{N}}R_n$, where $Q_0\subset Q_1\subset\cdot$ and $R_0\subset R_1\subset\cdot$. Given $\varepsilon>0$ and $E\in\mathfrak{M}_{\overline{\mu\times\nu}}$ s.t. $\overline{\mu\times\nu}(E)<\infty$, there is $N\in\mathbf{N}$ s.t. $\|\chi_E-\chi_{E\cap(Q_n\times R_n)}\|_1<\varepsilon/2$. By 1°, there is $F\in\mathcal{E}$ s.t. $\|\chi_{E\cap(Q_n\times R_n)}-\chi_F\|_1<\varepsilon/2$, hence (e) holds.

(f) The first claim follows from (e) and Theorem B.3.11. For the second claim, given $n\in\mathbf{N}$, let $\{E'_{n,k} \,|\, k=0,\ldots,N'_n\}$ consist of the sets $\{(\cup_{k\in s}E_{n,k})\setminus\cup_{k\notin s}E_{n,k} \,|\, s\subset\{0,1,\ldots,N_n\}\}$, and note that $s_n(q,r)=s_n(q',r)=: s_{n,k}(r)$ for all $q,q'\in E'_{n,k}$ ($k\in\mathbf{N}$). □

As in the scalar case, the norm of a $\overline{\mu\times\nu}$-measurable function may be integrated in any order, and for L^1 functions the same applies to the function itself:

Theorem B.4.6 (Fubini) *Assume that μ and ν are σ-finite, complete, positive measures on Q and R, respectively. Let $f\in\mathrm{L}(Q\times R;B)$.*

(a1) We have $\int_R \|f\|_B \, d\nu \in \mathrm{L}(Q;[0,+\infty])$ *and* $\int_Q \|f\|_B \, d\mu \in \mathrm{L}(R;[0,+\infty])$.

(a2) If $\int_Q \int_R \|f\|_B \, d\nu \, d\mu < \infty$ *or* $\int_R \int_Q \|f\|_B \, d\mu \, d\nu < \infty$, *then* $f \in \mathrm{L}^1(Q \times R;B)$.

(b) If $f \in \mathrm{L}^1(Q \times R;B)$, *then* $g(q) := \int_R f(q,r) \, d\nu(r)$ *and* $h(r) := \int_Q f(q,r) \, d\mu(q)$ *are defined a.e. and satisfy* $g \in \mathrm{L}^1(Q;B)$, $h \in \mathrm{L}^1(R;B)$ *and*

$$\int_{Q\times R} f \, d\overline{\mu \times \nu} = \int_Q g \, d\mu = \int_R h \, d\nu. \tag{B.20}$$

Proof: (a1)&(a2) This follows from the classical Fubini Theorem (Theorem 8.8 of [Rud86]).

(b) This is Theorem 3.7.13 of [HP]. □

The rest of this section is rather technical.

In order to satisfy the $\overline{\mu \times \nu}$-measurability assumption of the Fubini Theorem and its several important applications, we need to show that our function is (product) measurable and study the relation between $f : Q \times R \to B$ and $f : Q \to \mathrm{L}(R;B)$. We start from basic facts:

Lemma B.4.7 *Assume that* $\mu : \mathfrak{M} \to [0,\infty]$ *and* $\nu : \mathfrak{M}' \to [0,\infty]$ *are* σ*-finite, complete, positive measures on* Q *and* R*, respectively. Let* $p \in [1,\infty]$.

(a) If $f \in \mathrm{L}(\overline{\mu \times \nu};B)$, *then* $f(q,\cdot) \in \mathrm{L}(R;B)$ *for a.e.* $q \in Q$.

(b) If $f \in \mathrm{L}(Q;B)$, $g \in \mathrm{L}(R;B_2)$, *and* $B \times B_2 \to B_3$ *is continuous, then* $fg \in \mathrm{L}(Q \times R;B_3)$.

(c) Let $f, g \in \mathrm{L}(Q \times R;B) \cap \mathrm{L}(Q;\mathrm{L}^p(R;B))$. *If* $f = g$ *a.e. on* $Q \times R$, *then* $[f] = [g] \in \mathrm{L}(Q;\mathrm{L}^p(R;B))$. *Conversely, if* $[f] = [g]$ *as elements of* $\mathrm{L}(Q;\mathrm{L}^p)$, *i.e.,* $f(q,\cdot) = g(q,\cdot)$ *a.e. on* R *for a.e.* $q \in Q$, *then* $f = g$ *a.e. on* $Q \times R$.

(d) Let $f \in \mathrm{L}(Q \times R;B)$ *and* $p \in [1,\infty)$. *Then* $f \in \mathrm{L}^p(Q;\mathrm{L}^p(R;B))$ *iff* $f \in \mathrm{L}^p(Q \times R;B)$. *If* $f \in \mathrm{L}^p(Q \times R;B)$, *then* $\|f\|_{\mathrm{L}^p(Q\times R;B)} = \|f\|_{\mathrm{L}^p(Q;\mathrm{L}^p(R;B))}$.

Proof: (a) Let $s_n \to f$ pointwise on N^c, where $N \subset Q \times R$ is a null set and $s_n = \sum_{k\in\mathbf{N}} \chi_{E_{n,k}} b_{n,k}$, where $E_{n,k} \in \mathfrak{M}_{\overline{\mu\times\nu}}$, $b_{n,k} \in B$ and $k \neq j \Rightarrow E_{n,k} \neq E_{n,j}$ for all $n,k,j \in \mathbf{N}$.

Let $n \in \mathbf{N}$. For each $q \in Q$, the function $s_n(q,\cdot) : R \to B$ is countably-valued. But there is a null set $N' \subset Q$ s.t. $(E_{n,k})_q$ is measurable for all $q \in Q \setminus N'$ and all $k \in \mathbf{N}$, by Lemma B.4.5(d1). Moreover, $\nu(N_q) = 0$ for a.e. $q \in Q$, say, for $q \in Q \setminus N''$, where N'' is a null set. Set $N''' := N' \cup N''$. Then $s_n(q,\cdot) \to f(q,\cdot)$ and $s_n(q,\cdot)$ is countably-valued and measurable for all $q \in Q \setminus N'''$. Thus, $f(q,\cdot) \in \mathrm{L}(R;B)$ for all $q \in Q \setminus N'''$, hence a.e.

(b) Let $f_n : Q \to B$ and $g_n : R \to B_2$ be countably-valued and measurable ($n \in \mathbf{N}$), and let $f_n \to f$ on $Q \setminus N_Q$, $R \setminus N_R$, where $\mu(N_Q) = 0 = \nu(N_R)$. Then $f_n g_n \to fg$ on $N_Q^c \times N_R^c$, and $\overline{\mu \times \nu}((Q \times R) \setminus (N_Q^c \times N_R^c)) = 0$.

(c) (The assumptions on f mean that $f : Q \times R \to B$ is measurable, $f(q,\cdot) \in \mathrm{L}^p(R;B)$ for a.e. $q \in Q$, and $(q \mapsto f(q,\cdot)) \in \mathrm{L}(Q;\mathrm{L}^p)$. The first claim shows that $f \in \mathrm{L}$ is independent of the representative of $[f] \in \mathrm{L}(Q \times R;B)$. However, $[f] \in \mathrm{L}(Q;\mathrm{L}^p(R;B))$ may have a representative $h : Q \to \mathrm{L}^p(R;B)$ that is not measurable $Q \times R \to B$, see Example B.4.18; but if h is measurable, then $h = f$ a.e. on $Q \times R$, by the converse claim.)

If $(f-g) = 0$ a.e., say on $Q \times R \setminus N$, where $\overline{\mu \times \nu}(N) = 0$, then $(f-g)(q) = 0$ a.e. on R for a.e. q, by Lemma B.4.5(d2), hence $[f] = [g] \in \mathrm{L}^a(Q;\mathrm{L}^p)$.

Conversely, assume that $[f] = [g] \in \mathrm{L}^a$ and set $N := \{(q,r) \in R \times Q \,|\, f(q,r) \neq g(q,r)\}$. Then $f(q) = g(q)$ a.e. on R for all $q \in Q \setminus N'$, where $\mu(N') = 0$, i.e., $\nu(N_q) = 0$ for all $q \in Q \setminus N'$. But

$$\overline{\mu \times \nu}(N) := \int_Q \nu(N_q)\, d\mu = 0, \tag{B.21}$$

i.e., $f = g$ a.e. on $Q \times R$.

(d) By (b), we can consider f also as a function $Q \to \mathrm{L}(R;B)$ (by redefining f on a null subset of Q; this affects f only on a null subset of $Q \times R$).

1° Let $f \in \mathrm{L}^p(Q \times R;B)$. Choose simple measurable functions $\{s_n\}$ as in Lemma B.4.5(f). It follows that $s_n : Q \to \mathrm{L}^p(R;B)$ is measurable ($n \in \mathbf{N}$). But

$$\|s_n - s_m\|^p_{\mathrm{L}^p(Q;\mathrm{L}^p)} = \int_Q \int_R \|s_n - s_m\|^p_B \, d\nu \, d\mu = \|s_n - s_m\|^p_{\mathrm{L}^p(Q \times R;B)} \to 0 \tag{B.22}$$

as $n,m \to \infty$, hence s_n converges in $\mathrm{L}^p(Q;\mathrm{L}^p)$ to some $g : Q \to \mathrm{L}^p(R;B)$.

Replace $\{s_n\}$ by a subsequence s.t. $s_n(q) \to g(q)$ for $q \in Q \setminus N'$ and $s_n(q,r) \to f(q,r)$ for $(q,r) \in Q \times R \setminus N$, where $\mu(N') = 0 = \overline{\mu \times \nu}(N)$. By Lemma B.4.5(d2), there is $N'' \subset Q$ s.t. $\nu(N_q) = 0$ for all $q \in Q \setminus N''$. Set $N''' := N' \cup N''$.

Let $q \in Q \setminus N'''$. Then $s_n(q,r) \to f(q,r)$ for a.e. $r \in R$ and $s_n(q) \to g(q)$ in $\mathrm{L}^p(R;B)$, hence $f(q) = g(q)$ a.e., i.e., as elements of $\mathrm{L}^p(R;B)$, Because $\mu(N''') = 0$, we have $f = g$ as elements of $\mathrm{L}(Q;\mathrm{L}^p(R;B))$; thus, $f = g \in \mathrm{L}^p(Q;\mathrm{L}^p(R;B))$.

2° Let $f \in \mathrm{L}^p(Q;\mathrm{L}^p(R;B))$. Then, by the Fubini Theorem,

$$\|f\|^p_p = \int_Q \int_R \|f\|^p_B \, d\nu \, d\mu = \int_Q \|f\|^p_{\mathrm{L}^p(R;B)} \, d\mu =: \|f\|^p_{\mathrm{L}^p(Q;\mathrm{L}^p(R;B))}, \tag{B.23}$$

hence then $\|f\|_p = \|f\|_{\mathrm{L}^p(Q;\mathrm{L}^p(R;B))} < \infty$. □

If $f(\cdot,\cdot)$ is continuous w.r.t. one argument and measurable w.r.t. the other, then f is product measurable:

Lemma B.4.8 *Assume that $\mu : \mathfrak{M} \to [0,\infty]$ and $\nu : \mathfrak{M}' \to [0,\infty]$ are σ-finite, complete, positive measures on Q and R, respectively. Assume, in addition, that Q is a separable metric space and the open subsets of Q are measurable.*

If $f : Q \times R \to B$ is s.t. $f(q,\cdot) \in \mathrm{L}(R;B)$ for a.e. q and $f(\cdot,r) \in \mathcal{C}(Q;B)$ for a.e. r, then $f \in \mathrm{L}(Q \times R;B)$.

Proof: Let $N_{-1} \subset R$ be a null set s.t. $f(\cdot,r) \in \mathcal{C}(Q;B)$ for $r \in R' := N^c_{-1}$. Let $Q' \subset Q$ be s.t. $\mu(Q \setminus Q') = 0$ and $f(q,\cdot) \in \mathrm{L}(R;B)$ for all $q \in Q'$.

Let $\{q_k\}_{k \in \mathbf{N}}$ be dense in Q'. For each $k \in \mathbf{N}$, find a sequence of countably-valued measurable functions $s_{k,n} : R \to B$ s.t. $s_{k,n}(\cdot) \to f(q_k,\cdot)$ uniformly on N^c_k for some null set $N_k \subset R$ (use Lemma B.2.5(b2)). Set $R' := R \setminus \cup_{k=-1}^{\infty} N_k$. We require that

$$\|f(q_k,r) - s_{k,n}(r)\|_B < 1/(n+1) \quad (k,n \in \mathbf{N},\ r \in N_k^c) \tag{B.24}$$

(replace $\{s_{k,n}\}_{n\in\mathbf{N}}$ by its subsequence, for each $k \in \mathbf{N}$, if necessary).

Set $B_{k,n} := \{q \in Q' \,\big|\, d(q,q_k) < 1/n)\}$, $A_{0,n} := B_{0,n}$, $A_{k+1,n} := B_{k+1,n} \setminus \cup_{j\le k} B_{j,n}$ for all $k \in \mathbf{N}$ (the sets $\{A_{k,n}\}_{k\in\mathbf{N}}$ are measurable and disjoint, their union is Q', and $d(q,q_k) < 1/n$ for all $q \in A_{k,n}$).

Then the functions

$$s_n(q,r) := \sum_{k\in\mathbf{N}} s_{k,n}(r)\chi_{A_{k,n}}(q) \tag{B.25}$$

form a sequence of countably-valued $\overline{\mu\times\nu}$-measurable functions that converge to f on $Q'\times R'$, as $n\to\infty$. Indeed, given $(q,r) \in Q'\times R'$ and $\varepsilon > 0$, choose $N > 2/\varepsilon$ s.t. $\|f(q,r) - f(q',r)\|_B < \varepsilon/2$ for $d(q,q') < 1/(N+1)$. Then, for $n > N$, we have

$$\|f(q,r) - s_n(q,r)\|_B \le \|f(q,r) - f(q_k,r)\|_B + \|f(q_k,r) - s_{k,n}(r)\|_B \tag{B.26}$$
$$< \varepsilon/2 + 1/(n+1) < \varepsilon \tag{B.27}$$

(here k is chosen s.t. $q \in A_{k,n}$, so that $d(q,q_k) < 1/n \le 1/(N+1)$ and hence $\|f(q,r) - f(q_k,r)\|_B < \varepsilon/2$; note that $s_{k,n}(r) = s_n(q,r)$).

But $(Q'\times R')^c \subset ((Q\setminus Q')\times R) \cup (Q\times(R\setminus R'))$ is a null set, hence f is measurable. □

If Q is also a topological space, then μ is called *regular* if

$$\mu(E) = \inf\{\mu(V) \,\big|\, V \supset E,\ V \text{ open }\} = \sup\{\mu(K) \,\big|\, K \subset E,\ K \text{ compact }\} \tag{B.28}$$

(μ is outer regular if the former and inner regular if the latter condition is satisfied). If $Q \subset \mathbf{R}^n$ is open, then, by Theorem 2.18 of [Rud86], any locally finite Borel-measure on Q is regular; in particular, m is regular on Q.

One often defines a measure from another one by using a weight function; the "formula $d\nu = f\,d\mu$" works also in the vector-valued case:

Lemma B.4.9 ($\boldsymbol{\nu = f\,d\mu}$) *Let $f : Q \to [0,+\infty]$ be measurable. Set*

$$\widetilde{\nu}(E) := \int_E f\,d\mu \quad (E \in \mathfrak{M}), \tag{B.29}$$

Then $\widetilde{\nu}$ is a positive measure on Q; let ν be the completion of $\widetilde{\nu}$. Then

$$\int_Q g\,d\nu = \int_Q gf\,d\mu \tag{B.30}$$

when g is measurable $Q \to [0,\infty]$ or $g \in \mathrm{L}^1(Q,\nu;B)$ (equivalently, $gf \in \mathrm{L}^1(Q,\mu;B)$).

If $Q \subset \mathbf{R}^n$ is open, $\mu = m$ and $\int_K f\,dm < \infty$ for compact $K \subset Q$, then ν is regular.

Proof: 1° *Case $g : Q \to [0,\infty]$:* The claim on $\widetilde{\nu}$ is Theorem 1.29 of [Rud86], which also contains the claim on g provided that g is $\widetilde{\nu}$-measurable. By Lemma B.3.12, general $g : Q \to [0,\infty]$ will do.

2° *Case $g : Q \to B$:* By 1°, we have $g \in \mathrm{L}^1(Q,\nu;B) \Leftrightarrow gf \in \mathrm{L}^1(Q,\mu;B)$. Apply 1° to $(\mathrm{Re}\, g)^{\pm}$ and $(\mathrm{Im}\, g)^{\pm}$ to cover the case $B = \mathbf{K}$. Replace g by Λg to obtain the general case.

3° *Regularity:* By Theorem 2.18 of [Rud86], measures ν and $\widetilde{\nu}$ are regular, □

Standard changes of variable can be applied without problems for measurable functions that nonnegative or integrable:

Lemma B.4.10 (Change of variable) *Assume that $\phi \in C^1(J_1;\mathbf{R})$ is s.t. $\phi' > 0$ on J_1 (or that $\phi' \geq 0$ and ϕ' has at most a countable number of zeros), where J_1 is an interval. Set $J_2 = \phi[J_1]$. Let $X = B$ or $X = [0,+\infty]$.*

Then a function $f : J_2 \to X$ is measurable iff $f \circ \phi : J_1 \to X$ is measurable. Moreover, $\|f\|_\infty = \|f \circ \phi\|_\infty \leq \infty$. Let $p \in [1,\infty]$. Then $f \in \mathrm{L}^p_{\mathrm{loc}} \Leftrightarrow f \circ \phi \in \mathrm{L}^p_{\mathrm{loc}}$. If $\varepsilon < \phi' < \varepsilon^{-1}$ on J_1 for some $\varepsilon > 0$, then $f \in \mathrm{L}^p \Leftrightarrow f \circ \phi \in \mathrm{L}^p$.

If $f \in \mathrm{L}^1(J_2;B)$ or f is measurable $J_2 \to [0,+\infty]$, then

$$\int_{J_2} f(t)\,dt = \int_{\phi^{-1}[J_2]} f(\phi(s))\phi'(s)\,ds \tag{B.31}$$

Here, as elsewhere, intervals are equipped with the Lebesgue measure.

By Lemma B.5.5, the functions f and $f \circ \phi$ have “same” Lebesgue points.

Proof: We shall replace J_1 by $[a,b] \subset J_1$ s.t. $\phi' \geq r > 0$ on (a,b) (note $J_1' := J_1 \setminus \{t \in J_1 \mid \phi'(t) = 0\}$ is a countable union of such intervals, and it is enough to treat J_1'). Let $[\alpha,\beta] = \phi[[a,b]]$.

1° Note that $\phi^{-1} \in C^1([\alpha,\beta];[a,b])$ and $\phi^{-1'} \geq 1/r$ (recall that we use one-sided derivatives at endpoints), and $\phi^{-1}[E]$ is a Borel set for each Borel set E.

2° *If $N \subset [a,b]$ a null set, then so is $\phi[N]$:* Set $R := \max \phi'$. Let $\varepsilon > 0$ and assume w.l.o.g. that $N \subset (a,b)$. Now $N \subset V$ for some open V with $m(V) < \varepsilon/R$ (see Theorem 2.20 of [Rud86]). Write V as the union of a countable number of disjoint open intervals: $V = \cup_{n\in\mathbf{N}}(a_n,b_n)$. Then $m(\phi[(a_n,b_n)]) = m((\phi(a_n),\phi(b_n))) \leq |b_n - a_n|R = m((a_n,b_n))R$, so that $m(\phi[N]) \leq m(\phi[V]) < \varepsilon$. Because $\varepsilon > 0$ was arbitrary, $m(\phi[N]) = 0$.

3° *Case $f : J_2 \to X$ is measurable:* Let $\{s_n\}$ be countably-valued and measurable and s.t. $s_n \to f$ a.e. Redefine each s_n on a null set so that they become Borel-measurable; then still $s_n \to f$ outside some null set $N \subset J_2$. Consequently, $s_n \circ \phi \to f \circ \phi$ outside $\phi^{-1}[N]$, which is a null set, by 2°. But $s_n \circ \phi$ is a countably-valued Borel-measurable function, by 1°, for each n, hence $f \circ \phi$ is measurable.

4° *Measurability: the general case:* Exchange the roles of ϕ and ϕ^{-1} to obtain a converse for 3°. (Note also that $(f \circ \phi)\phi'$ is measurable iff $f \circ \phi$ is measurable.)

5° *$\|f\|_\infty = \|f \circ \phi\|_\infty \leq \infty$:* This follows from 2°.

6° *(B.31) for $X = [0,+\infty]$:* This is contained in (15) on p. 156 of [Rud86].

7° *L^1 and $\mathrm{L}^1_{\mathrm{loc}}$ claims:* Apply 6° to $\|f\|_B$ (for $\mathrm{L}^1_{\mathrm{loc}}$ we note that $\phi[K] \subset J_2$ is compact for compact $K \subset J_1$ and $\phi^{-1}[K] \subset J_1$ is compact for compact $K \subset J_2$; moreover, ϕ', $\phi^{-1'}$ are bounded on compact sets).

8° *(B.31) for $X = B$:* If $X = \mathbf{K}$, then this follows from 6° (for $(\mathrm{Re}\, f)^\pm$, $(\mathrm{Im}\, f)^\pm$); in the general case we replace f by Λf ($\Lambda \in B^*$) and use Lemma B.4.2.

9° L^p *and* $\mathrm{L}^p_{\mathrm{loc}}$ *claims:* If $p < \infty$, this follows from 7° applied to $\|f\|_B^p$. If $p = \infty$, then this follows 4° and 5°.

10° *Case* $\phi' \not> 0$*:* Let f be measurable. Now $G := \{t \in J_1 \,\big|\, \phi'(t) > 0\}$ consists of a countable number of disjoint open intervals, by Lemma A.2.2. The image of the null set $J_1 \setminus G$ is a null set, so it follows from Lemma B.2.5(d1) that $f \circ \phi$ is measurable. The proof of the converse implication is analogous (set $G := \{t \in J_1 \,\big|\, \phi^{-1\prime}(t) < +\infty\}$ etc.).

Now for $X = [0, +\infty]$, we can write the integral as a countable sum of integrals, by the Monotone Convergence theorem; for $X = B$ the results follow as in 6°–9°. □

By Theorem B.3.2, L^2 is a Hilbert space, hence $(\mathrm{L}^2)^* = \mathrm{L}^2$. The multidimensional version of this goes as follows:

Lemma B.4.11 *Let H be a Hilbert space and $n \in \{1, 2, \ldots\}$. Then, for each $\Lambda \in \mathcal{B}(\mathrm{L}^2(Q; H), \mathbf{K}^n)$, there is a unique $F \in \mathrm{L}^2(Q; \mathcal{B}(H, \mathbf{K}^n))$ s.t.*

$$\Lambda f = \int_Q F f \, d\mu \quad (f \in \mathrm{L}^2(Q; H)). \tag{B.32}$$

Moreover, $\|\Lambda\|_{\mathcal{B}(\mathrm{L}^2, \mathbf{K}^n)} \le \|F\|_2 \le \sqrt{n} \|\Lambda\|_{\mathcal{B}(\mathrm{L}^2, \mathbf{K}^n)}$.

Conversely, by the Hölder Inequality, each $F \in \mathrm{L}^2(Q; \mathcal{B}(H, \mathbf{K}^n))$ defines $\Lambda_F \in \mathcal{B}(\mathrm{L}^2(Q; H), \mathbf{K}^n)$, by (B.32). In Example B.4.13(c), we have $\|F\|_2 = \sqrt{n}$, $\|F\|_{\mathcal{B}(\mathrm{L}^2, \mathbf{K}^2)} = 1$, hence the constant $\sqrt{n}$ is optimal. By Example B.4.13(d), we may have $\|F\|_2 = \infty$ if $\mathbf{K}^n$ is replaced by an infinite-dimensional (Hilbert) space.

Proof: Let $P_k \in \mathcal{B}(\mathbf{K}^n, \mathbf{K})$ be the projection $P_k : x \mapsto x_k$. By Theorem B.3.2, there are $F_1, \ldots, F_n \in \mathrm{L}^2(Q; H)$ s.t. $P_k \Lambda_k = \langle F_k, f \rangle_{\mathrm{L}^2}$ ($k = 1, \ldots, n$) and $\|F_k\| = \|P_k \Lambda\|$. Set $F := \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$ to obtain (B.32).

Obviously, $F \mapsto \Lambda$ is linear and $\Lambda = 0 \Rightarrow F = 0$, hence F is unique. By the Hölder Inequality, we have $\|\Lambda\| \le \|F\|$. On the other hand, $\|F_k\| \le \|\Lambda\|$, hence $\|F\|^2 \le n \|\Lambda\|^2$. □

The L^p norm of a measurable function $f : Q \to B$ is the supremum of $|\int_Q f \phi \, d\mu|$, where $\phi \in \mathrm{L}^q(Q; B^*)$, $1/p + 1/q = 1$. We can even take ϕ to be simple, and, with some extra assumptions, smooth:

Theorem B.4.12 *Let B be a Banach space, let μ be a complete σ-finite positive measure on Q, let $p, q \in [1, \infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$, and let X be a norming subspace of B^* for B. Let $f : Q \to B$ be measurable. Then the following hold:*

(a) We have $\|f\|_\infty = \sup_{\Lambda \in X_0,\ \|\Lambda\| \le 1} \|\Lambda f\|_\infty \le \infty$ *if X_0 is a norming subset of B^* for B.*

(b1) We have

$$\|f\|_p = \sup_{\phi \in \mathcal{F}} |\int_Q \phi f \, d\mu| \leq \infty, \tag{B.33}$$

where either $\mathcal{F} = \mathcal{F}_1$ or $\mathcal{F} = \mathcal{F}_2$, and $\mathcal{F}_1 := \{\phi \in \mathrm{L}^q(Q;X) \,\big|\, \|\phi\|_q \leq 1 \;\&\; \phi f \in \mathrm{L}^1\}$, $\mathcal{F}_2 := \{\sum_{k=1}^n x_k \chi_{E_k} \in \mathcal{F}_1 \,\big|\, n \in \mathbf{N},\ x_k \in X,\ m(E_k) < \infty$ for all $k\}$.

(b2) If Q is an open subset of $\mathbf{R}^n$, μ is regular, and $\int_K \|f\|_B \, d\mu < \infty$ for compact $K \subset \mathbf{R}^n$ (i.e., $f \in \mathrm{L}^1_{\mathrm{loc}}(Q;B)$), then we can take $\mathcal{F} = \mathcal{F}_3 := C_c^\infty(Q;X) \cap \mathcal{F}_1$ in (b1).

(b3) If μ is the counting measure (i.e., $\mathrm{L}^p(Q;B) = \ell^p(Q;B)$), then $\mathcal{F}_2 = \mathrm{c_c} \cap \mathcal{F}_1 = \mathrm{c_c}$ in (b1), where $\mathrm{c_c} := \{x : Q \to B \,\big|\, x_q = 0$ except for finitely many $q \in Q\}$. Moreover, then we can allow Q to be an arbitrary set (i.e., μ need not be σ-finite).

(b4) In (b1)–(b3), we may require, in addition, that $\phi = \sum_{k=1}^m \phi_k x_k$, where $m \in \mathbf{N}$, ϕ_k is scalar and $x_k \in X$, for each k.

(c) If $g : Q \to [0,\infty]$ is measurable, we can take $X = [0,\infty]$ and have

$$\|g\|_p = \sup_{\phi \in \mathcal{F} \;\&\; \phi \geq 0} \int_Q \phi g \, d\mu \leq \infty, \tag{B.34}$$

where $\mathcal{F}$ is as in any of (b1)–(b3); we can even drop the requirement $\phi g \in \mathrm{L}^1$, as well as the requirement $g \in \mathrm{L}^1_{\mathrm{loc}}$ (in (b2)).

(d) If $p = \infty$, then, in (b1)–(b3) above, we may let X be any normed space s.t. $X \times B \to B_2$ is bilinear and $\sup_{\|x\|_X \leq 1} \|xb\|_{B_2} = \|b\|_B$ for some normed space B_2.

For $p < \infty$, we must replace "$=$" by "$\geq$" in (B.33) for such a general X (see Example B.4.13), but the supremum is still nonzero, i.e., $f \neq 0 \Longrightarrow \int_Q \phi f \, d\mu \neq 0$ for some $\phi \in \mathcal{F}$.

(e) We have $f = 0$ a.e. if $f \in \mathrm{L}^1(Q;B)$ and $\int_E f \, d\mu = 0$ for all measurable $E \subset \mathbf{R}$, or $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n;B)$ and $\int_E f \, dm = 0$ for all bounded measurable $E \subset \mathbf{R}$.

(The sets L^q and C_c^∞ are defined for incomplete normed spaces exactly as for the complete ones.)

The assumption $\phi f \in \mathrm{L}^1$ is required to make the integral $\int_Q \phi f \, d\mu$ well defined; for $f \in \mathrm{L}^1_{\mathrm{loc}}$ and $\mathcal{F} = \mathcal{F}_3$ it is redundant. Clearly it is not needed for (B.34).

In (d), we may have, e.g., $X = \mathcal{B}(B, B_2)$, $B = \mathcal{B}(X, B_2)$ or $X = \mathbf{K}$.

See Examples B.4.13 and B.4.14 for "counter-examples".

Proof: (a) Clearly "$\geq$" holds, so it is enough to assume that $\|f\| > M$ on a set $E \subset Q$ of positive measure, and find $\Lambda \in X_0$ s.t. $\|\Lambda\| \leq 1$ and $|\Lambda f| > M$ on a set of positive measure.

Pick $\varepsilon > 0$ s.t. $E' := \{t \in Q \,\big|\, \|f(t)\| > M + \varepsilon\}$ has a positive measure. Choose $a_0 \in E'$, $A_\varepsilon \subset E'$ for $f_{|E'}$ and ε as in Lemma B.2.8(a). Choose $\Lambda \in X_0$ s.t. $|\Lambda f(a_0)| > \|f(a_0)\| - \varepsilon/2$. Then (recall that $\|\Lambda\| \leq 1$)

$$|\Lambda f(a)| > |\Lambda f(a_0)| - \varepsilon/2 > \|f(a_0)\| - \varepsilon > M \tag{B.35}$$

for $a \in A_\varepsilon$, hence $\|\Lambda f\|_\infty > M$.

(b1) If $f \in \mathrm{L}^p$ and $\phi \in \mathcal{F}$, then $\| \int_Q \phi f \, d\mu \| \leq \|f\|_p$ by the Hölder inequality; trivially, we have $\| \int_Q \phi f \, d\mu \| \leq \|f\|_p$ for $f \notin \mathrm{L}^p$ (i.e., $\|f\|_p = \infty$). Thus, we only need to assume that $0 < M < \|f\|_p \leq \infty$ and find $\phi \in \mathcal{F}$ s.t. $\| \int_Q \phi f \, d\mu \| > M$. We divide the proof of this fact into three parts.

Case I — $p = \infty$, $\mathcal{F} = \mathcal{F}_2$: Take $M' > M$ s.t. $\mu(E) > 0$, where $E := \{t \in Q \,\big|\, \|f(t)\| > M'\}$. Because μ is σ-finite, there is $A \subset E$ s.t. $0 < \mu(A) < \infty$. Choose $a_0 \in A$ and $A' := A_\varepsilon \subset A$ for $\varepsilon := (M' - M)/2$ as in Lemma B.2.8. Choose $x \in X$ s.t. $\|x\| = 1$ and $xf(a_0) > \|f(a_0)\| - \varepsilon > M + \varepsilon$. Then for $t \in A'$, we have $\|f(t)\|_B \in (f(a_0) - \varepsilon, f(a_0) + \varepsilon)$, and $\|xf(t) - xf(a_0)\| < \varepsilon$. Therefore, $\phi := x\chi_{A'}/\mu(A') \in \mathcal{F}_2$, and

$$\| \int_Q \phi f \, d\mu \| = \mu(A')^{-1} \| \int_{A'} xf \, d\mu \| \geq \|xf(a_0)\| - \varepsilon > M. \tag{B.36}$$

Case II — $p < \infty$, $\mathcal{F} = \mathcal{F}_2$: 1° Assume that $\mu(Q) < \infty$, $\|f\|_p = 1$, and $f = \sum_{j=1}^k b_j \chi_{Q_j}$, where the sets Q_j, $j \in \mathbf{N}$ are disjoint.

For each j choose $x_j \in X$ s.t. $\|x_j\| = 1$ and $x_j b_j > \|b_j\| - \delta$, where $\delta := (1 - M)/\sum_{j=1}^k \|b_j\|^{p-1} \mu(Q_j) < 1 - M$,

For $\phi := \sum_{j=1}^k \|b_j\|^{p-1} x_j \chi_{Q_j}$, we have (because $q(p-1) = p$) that

$$\int_Q \|\phi(t)\|^q \, d\mu = \int_Q \sum_{j=1}^k \|b_j\|^p \chi_{Q_j}(t) \, d\mu = \|f(t)\|_p^p = 1 \quad \text{for all } t \in Q, \tag{B.37}$$

for $q < \infty$, and $\|\phi\|_\infty = 1$ as well for $q = \infty$. Moreover,

$$\| \int_Q \phi f \, d\mu \| = \sum_{j=1}^k \|b_j\|^{p-1} (x_j b_j) \mu(Q_j) \geq \int \|f\|_p^p - h(\delta) = 1 - h(\delta), \tag{B.38}$$

where $h(\delta) := \sum_{j=1}^k \|b_j\|^{p-1} (\|b_j\| - x_j b_j) \mu(Q_j) < \delta \sum_{j=1}^k \|b_j\|^{p-1} \mu(Q_j) = 1 - M$, hence $\| \int_Q \phi f \, d\mu \| > M$.

2° By scaling and density (Theorem B.3.11), any $f \in \mathrm{L}^p$ will do in 1°, if $\mu(Q) < \infty$.

3° For the general case, let $Q = \cup_{j \in \mathbf{N}} Q_j$, where the sets Q_j, $j \in \mathbf{N}$ are disjoint and $\mu(Q_j) < \infty$ for all j. Set $E_j := \{t \in Q_1 \cup \cdots \cup Q_j \,\big|\, \|f(t)\|_B < j\}$.

By the monotone convergence theorem, $\int_Q \chi_{E_j} \|f\|_B^p \, d\mu > M$ for some $j \in \mathbf{N}$. Use now 2° to find $\phi = \phi \chi_{E_j} \in \mathcal{F}$ for E_j and $\chi_{E_j} f$; the same ϕ will clearly do for Q and f.

Case III — $\mathcal{F} = \mathcal{F}_1$: This follows easily from case $\mathcal{F} = \mathcal{F}_2$.

(b2) There is a nested sequence of open sets $\Omega_j \subset Q$ with compact closure s.t. $\cup_{j \in \mathbf{N}} \Omega_j = Q$. By the monotone convergence theorem, it follows that for some j, we have $\|f\|_{\mathrm{L}^p(\Omega_j, \mu; B)} > M$. Find $g = g\chi_{\Omega_j} = \sum_{i=1}^k x_i \chi_{E_i} \in \mathcal{F}_2$ s.t. $M_f := \| \int_{\Omega_j} gf \, d\mu \| > M$. Set $M_g := \max_i \|x_i\| > 0$.

Because $f \in \mathrm{L}^1(\Omega_j, \mu; B)$, we can find, for each i, a compact $K_i \subset \Omega_j$ and an open $V_i \subset \Omega_j$ s.t. $K_i \subset E_i \subset V_i$ and $\int_{V_i \setminus K_i} \|f\|_B \, d\mu < \delta := (M_f - M)/2kM_g$. By Lemma B.3.10, there is some $\phi_i \in \mathcal{C}_c^\infty(\Omega_j)$ s.t. $\chi_{K_i} \leq \phi_i \leq \chi_{V_i}$. Set $\phi :=$

$\sum_{i=1}^k x_i\phi_i \in \mathcal{F}_2$ It follows that

$$\|\int_{\Omega_j} \phi f\,d\mu\| \geq \|\int_{\Omega_j} gf\,d\mu\| - \sum_{i=1}^k \int_{V_i\setminus K_i} \|g-\phi\|_X\|f\|_B\,d\mu \geq \|\int_{\Omega_j} gf\,d\mu\| - \sum_{i=1}^k 2M_g\delta > M. \tag{B.39}$$

(b3) 1° *Case* $p=\infty$*:* If $M < \|f\|_\infty = \sup_{q\in Q}\|f\|$, then there is $q \in Q$ s.t. $\|f(q)\|_B > M$, so that $\|f(q)x\| > M$ for some $x \in X$ with $\|x\| = 1$. Thus, then we can take $\phi := \chi_{\{q\}}x$.

2° *Case* $1 \leq p < \infty$*:* Set $A := \mathrm{supp}(f) := \{q \in Q \,|\, f(q) \neq 0\}$. If A is countable, then (b1) applies; obviously, $\mathcal{F}_2 = \mathrm{c_c} \cap \mathcal{F}_1 = \mathrm{c_c}$. Assume then that A is uncountable. Choose $n \in \mathbf{N}+1$ s.t. $A_n := \{a \in A \,|\, \|f(a)\| > 1/n\}$ is uncountable (hence infinite). Choose distinct elements $\{a_k\}_{k\in\mathbf{N}} \subset A_n$. For each k, choose $x_k \in X$ s.t. $\|x_k\| = 1$ and $f(a_k)x_k > 1/n$. Then $\int_Q \phi_m f\,d\mu = \sum_{k=1}^m f(a_k)x_k > m/n$, where $\phi_m := \sum_{k=1}^m x_k\chi_{\{a_k\}} \in \mathrm{c_c}$ and $m \in \mathbf{N}$ is arbitrary, hence then (B.33)$= \infty = \|f\|_p$.

(b4) The functions ϕ constructed in the proofs of (b1)–(b3) are of this form.

(c) Finally, for $g : Q \to X$, $X = [0,\infty]$, we can choose $\phi \in \mathcal{F}$ to prove (B.34) as above. By Hölder inequality, $\phi g \in \mathrm{L}^1$ may be dropped; by the trick used in "3°" above (with $E_j := \{t \in Q \,|\, f(t) < j\}$) we may drop the assumption $f \in \mathrm{L}^1_{\mathrm{loc}}$.

(d) Case I above applies here too (mutatis mutandis). For $p < \infty$, we still have the Hölder inequality ("$\geq$"); naturally for $B_2 = \mathbf{K}$ this reduces to (b1)–(b3).

Finally, if $f \neq 0$, then some $\phi = \sum_{k=1}^m \phi_k b_k^* \in \mathcal{F}$ (here $b_k^* \in B^*$ for all k) satisfies $\int_Q \phi f\,d\mu \neq 0$, by (b4), hence then $b := \int_Q \phi_k f\,d\mu \neq 0$ for some k. Choose $x \in X$ s.t. $xb \neq 0$, and set $\phi = \phi_k x$. □

We now show that part (d) does not hold for $p < \infty$:

Example B.4.13 Let $Q = [0,2]$, $\mu = m$, $1 \leq p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

(a) $B = \mathcal{B}(\mathbf{K},\mathbf{K}^2)$, $X = \mathbf{K}$, $f = \chi_{[0,1]}\begin{bmatrix}1\\0\end{bmatrix} + \chi_{[1,2]}\begin{bmatrix}0\\1\end{bmatrix} \in \mathrm{L}^p(Q;\mathcal{B}(\mathbf{K},\mathbf{K}^2)))$, Then

$$\|f\|_p = 2^{1/p} > \sup_{\phi\in\mathrm{L}^q(Q;X),\ \|\phi\|_q=1} \|\int_Q f\phi\,d\mu\|_{\mathbf{K}^2}. \tag{B.40}$$

Thus, $\|f\|_p > \|f\|_{\mathcal{B}(\mathrm{L}^q(Q;\mathbf{K}),\mathbf{K}^2)}$.

However, $\|f\|_p = \|f^*\|_{\mathrm{L}^p(Q;\mathcal{B}(\mathbf{K}^2,\mathbf{K}))} = \|f^*\|_{\mathcal{B}(\mathrm{L}^q(Q;\mathbf{K}^2),\mathbf{K})}$, hence $\|f^*\|_{\mathcal{B}(\mathrm{L}^q(Q;\mathbf{K}^2),\mathbf{K})} > \|f\|_{\mathcal{B}(\mathrm{L}^q(Q;\mathbf{K}),\mathbf{K}^2)}$.

(b) Analogously, $\|f\|_p > \|f\|_{\mathcal{B}(\mathrm{L}^q(Q;\mathbf{K}^2),\mathbf{K}^2)}$ if we set $f = \chi_{[0,1]}\begin{bmatrix}1&0\\0&0\end{bmatrix} + \chi_{[1,2]}\begin{bmatrix}0&0\\1&0\end{bmatrix} \in \mathrm{L}^p(Q;\mathcal{B}(\mathbf{K}^2))$.

(c) Moreover, by setting $f = \sum_{k=1}^n \chi_{[n-1,n]}P_n^*$, where $P_n^* \in \mathcal{B}(\mathbf{K},\mathbf{K}^n)$ is defined by $P_n^*\alpha = \alpha e_n$, we get $\|f\|_{\mathrm{L}^2([0,n];B)} = \sqrt{n}$, $\|\int fu\,dm\| \leq 1$ when $\|u\|_2 \leq 1$, where we can take $B = \mathcal{B}(X,\mathbf{K}^n)$ for any nontrivial Banach space X (replace f by $f\Lambda$ for some $\Lambda \in X^* \setminus \{0\}$).

(d) Finally, if $f = \sum_{k=1}^\infty \chi_{[n-1,n]}P_n^*$, $B_2 := \ell^2(\mathbf{N})$ where $P_n^* \in \mathcal{B}(\mathbf{K},B_2)$ is defined by $P_n^*\alpha = \alpha e_n$, then $\|\int fu\,dm\| \leq \|u\|_{\mathrm{L}^2(\mathbf{R}_+;\mathbf{K})}$, i.e., $\|f\|_{\mathcal{B}(\mathrm{L}^2(Q;\mathbf{K}),B_2)} = 1$, although $\|f\|_{\mathrm{L}^2(\mathbf{R}_+;\mathcal{B}(\mathbf{K};B_2))} = \infty$. Note that $\|f^*\|_{\mathcal{B}(\mathrm{L}^2(Q;B_2),\mathbf{K})} = \|f^*\|_2 = \|f\|_2 = \infty$.

$\triangleleft$

Proof: (a) Let $q < \infty$. Obviously, $\|f\|_p^p = 2$, hence $\|f\|_p = 2^{1/p}$. Let $\phi \in \mathrm{L}^q(Q;X)$, $\|\phi\|_q = 1$. Set $P_1 := \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$, $P_2 := \left[\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right]$. Then

$$\|P_1 \int_Q f\phi\, dm\|_X = \| \int_0^1 P_1 \phi\, dm\|_X \le \|\phi\|_{\mathrm{L}^q([0,1];X)} =: a, \tag{B.41}$$

by the Hölder inequality. Analogously, $\|P_2 \int_Q f\phi\, dm\|_X \le \|\phi\|_{\mathrm{L}^q([1,2];X)} =: b$, hence $\|\int_Q f\phi\, dm\|_X^2 \le a^2 + b^2$. For $q \le 2$ we have $a^2 + b^2 \le a^q + b^q \le \|\phi\|_q^q = 1$ and hence $\|\int_Q f\phi\, dm\|_X^2 \le 1$. For $q > 2$, the maximum of a^2+b^2 given $a^q + b^q \le 1$ is $2^{-2/q} + 2^{-2/q} = 2^{1/p-1/q} < 2^{1/p} < (2^{1/p})^2 = \|f\|_p^2$. Finally, for $q = \infty$ we have $\|P_k \int_Q f\phi\, dm\|_X \le 1$ $(k = 1,2)$, hence $\|\int_Q f\phi\, dm\|_X \le \sqrt{2} < 2 = \|f\|_p$.

(b) This follows from (a).

(c)&(d) These can be proved as for (a) (for $p = 2 = q$ this is obvious: $\|\int f\phi\|^2 \le \sum_n \|\phi\|^2_{\mathrm{L}^2([n-1,n];X)} = \|\phi\|_2^2$, as in (B.41)).

(e) 1° For $f \in \mathrm{L}^1(Q;B)$ this follows from Lemma B.2.8(b) (since $\Lambda \int_A f\, d\mu = \int_A \Lambda f\, d\mu > 0$ for such A).

2° If $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n;B)$ and $\int_E f\, dm = 0$ for all bounded measurable $E \subset \mathbf{R}$, then $\chi_{[-R,R]^n} f = 0$ a.e. for all $R > 0$, by 1°, hence $f = 0$ a.e. $\square$

Even if f is strongly measurable, "weak L^p" differs from L^p:

Example B.4.14 ($\Lambda f \in \mathbf{L}^p$ for all $\Lambda \not\Rightarrow f \in \mathbf{L}^p$) Set $H := \ell^2(\mathbf{N})$. Define $f \in \ell^2(\mathbf{N};H)$ by $f(k) := e_k$. Then, for any $x \in H$, we have $\|\langle f, x\rangle\|_2 = \|x_\cdot\|_2 = \|x\|_H$. Thus, $\|\Lambda f\|_2 = \|\Lambda\|$ for all $\Lambda \in H^*$, although $\|f\|_2 = \infty$. (The same holds for $\widetilde{f} := \sum_{k\in\mathbf{N}} \chi_{[k,k+1)} e_k \in \mathrm{L}^2(\mathbf{R}_+;H)$.) $\triangleleft$

By The Hölder Inequality, the formula $\ell^p(Q;B) \ni f \mapsto \sum_{t\in Q} F_t f_t \in \mathbf{K}$ determines a contractive map $\ell^q(Q;B^*) \ni F \mapsto \sum F\cdot \in \ell^p(Q;B)^*$. Usually, all elements of $\ell^p(Q;B)^*$ are of this form:

Lemma B.4.15 ($\boldsymbol{\ell}^{\mathbf{p}}(\mathbf{Q};\mathbf{B})^* = \boldsymbol{\ell}^{\mathbf{q}}(\mathbf{Q};\mathbf{B}^*)$) *Let Q be a set, $1 \le p < \infty$ and $p^{-1} + q^{-1} = 1$. Then $\ell^p(Q;B)^* = \ell^q(Q;B^*)$ and $\mathrm{c_o}(Q;B)^* = \ell^1(Q;B)$ (with equal norms).*

Recall that $\ell^p(Q;B)$ refers to $\mathrm{L}^p(Q,\sigma;B)$, where σ is the counting measure. Note that it follows that $\ell^p_r(S;B)^* = \ell^q_{1/r}(S;B^*)$ for $S \subset \mathbf{Z}$, where $\ell^p_r := r^{\cdot}\ell^p$ (see (13.2)).

Proof: 0° *Remarks:* We allow B to be an arbitrary Banach space; this cannot be done in the case of, e.g., $\mathrm{L}^p([0,1];B)$; cf. [DU]. For infinite Q and $B \ne \{0\}$, the closed subspace $\mathrm{c_o}(Q;B)$ of $\ell^\infty(Q;B)$ is not dense, hence there is $F \in \ell^\infty(Q;B)^*$ s.t. $Ff = 0$ for all $f \in \mathrm{c_o}$, in particular, $F \notin \ell^1(Q;B^*)$ (a constructive example is given in Exercise 3.4 of [Rud73]).

1° *Sufficiency:* Let $1 \le p \le \infty$. Let $F \in \ell^q(Q;B^*)$. By Lemma B.4.12(b3), $F \in \ell^p(Q;B)^*$ and $\|F\|_{\ell^p(Q;B)^*} = \|F\|_{\ell^q(Q;B^*)}$ (and the same holds with $\mathrm{c_o}$ in place of ℓ^p if $p = \infty$).

2° *Necessity, case $p < \infty$:* For the converse, assume that $F \in \ell^p(Q;B)^*$ and $p < \infty$. Since $\pi^*_{\{t\}} \in \mathcal{B}(B, \ell^p(Q;B))$ (here $\pi^*_{\{t\}}x$ is x at t and zero elsewhere), we have $G_t := F\pi_{\{}t\}^* \in \mathcal{B}(B,\mathbf{K}) = B^*$.

This function $G : Q \to B^*$ satisfies $Ff = \sum_{t\in Q} G_t f_t$ for each $f \in \mathrm{c_c}(Q;B)$, by linearity. Since $|\sum_{t\in Q} G_t f_t| = |Ff| \le \|F\|\|f\|$ for all $f \in \mathrm{c_c}(Q;B)$, we have $\|G\|_{\ell^q(Q;B^*)} \le \|F\|$, by Lemma B.4.12(b3) (with $\mathcal{F} = \mathcal{F}_2 = \mathrm{c_c}$). Consequently, $G \in \ell^p(Q;B)^*$, by 1°.

It follows that $G = F$ on the closure of $\mathrm{c_c}(Q;B)$, i.e., on $\ell^p(Q;B)$. Thus, F is of the required form. By density (see Theorem B.3.11(a)), $G = F$.

3° *Necessity for* $\mathrm{c_o}(Q;B)$*:* If $F \in \mathrm{c_o}(Q;B)^*$ and $p = \infty$, then the above proof applies, mutatis mutandis, and $G = F$ on $\mathrm{c_o}(Q;B)$, by density (see Theorem B.3.11(c)), hence F is again of the required form. □

The Minkovski Integral Inequality (not to be mixed to the Minkovski inequality $\|f+g\|_p \le \|f\|_p + \|g\|_p$) says that $\|\int_R f\,d\nu\|_p \le \int_R \|f\|_p\,d\nu$; also this can be extended to vector-valued functions

Theorem B.4.16 (Minkovski Integral Inequality) *Let (Q,μ) and (R,ν) be complete, positive, σ-finite measure spaces. Equip $Q \times R$ with $\overline{\mu \times \nu}$, the completion of $\mu \times \nu$. Let $1 \le p \le \infty$.*

(a) If $f : Q \times R \to [0,\infty]$ is measurable, then

$$\| \int_R f\,d\nu \|_{\mathrm{L}^p(Q)} \le \int_R \|f\|_{\mathrm{L}^p(Q)}\,d\nu \le \infty. \tag{B.42}$$

(b) If B is a Banach space, $f : Q \times R \to B$ is measurable, and $M := \int_R \|f\|_{\mathrm{L}^p(Q)}\,d\nu < \infty$, then $g(q) := \int_R f(q,\cdot)\,d\nu \in B$ is defined a.e., $g \in \mathrm{L}^p(Q;B)$ and $\|g\|_p \le M$, i.e., (B.42) holds.

If we did not make the assumption $\int_R \|f\|_{\mathrm{L}^p(Q)}\,d\nu < \infty$ in (b), we would have to write

$$\|\chi_{f(q,\cdot)\in \mathrm{L}^1(R,X)} \int_R f\,d\nu\|_{\mathrm{L}^p(Q,X)} \le \int_R \|f\|_{\mathrm{L}^p(Q,X)} \tag{B.43}$$

or use some other trick to make the function in the left well defined (here it is taken to be zero for those q, for which $f(q,\cdot) \notin \mathrm{L}^1$). The set $\{q \in Q \,|\, \int_R \|f(q,\cdot)\|\,d\nu = \infty\}$ is measurable by the Fubini Theorem, hence so is $\chi_{f(q,\cdot)\in\mathrm{L}^1(R,X)}$, and thus (B.43) can be proved in the same way as (b).

Proof: (a) By the Fubini Theorem, the inner integrals define measurable functions; in particular, $g(q) := \int_R f(q,r)\,dr$ defines a measurable function $Q \to [0,\infty]$.

If $s \geq 0$ is a simple function and $\|s\|_{p'} \leq 1$, where $1/p+1/p'=1$, then

$$\int_Q gs\,d\mu = \int_Q \int_R f(q,r)s(q)\,d\nu(r)\,d\mu(q) \tag{B.44}$$

$$= \int_R \int_Q f(q,r)s(q)\,d\mu(q)\,d\nu(r) \leq \int_R \|f\|_{\mathrm{L}^p(Q)}\,d\nu, \tag{B.45}$$

hence $\|\int_R f\,d\nu\|_{\mathrm{L}^p(Q)} = \|g\|_{\mathrm{L}^p(Q)} \leq \int_R \|f\|_{\mathrm{L}^p(Q)}$, by Theorem B.4.12 ($p \in [1,\infty]$).

(b) Find sets $Q_j \subset Q$ with $\mu(Q_j) < \infty$ ($j \in \mathbf{N}$) s.t. $Q_1 \subset Q_2 \subset \cdots$ and $Q = \cup_{j\in\mathbf{N}} Q_j$. The assumption $\int_R \|f\|_p\,d\nu < \infty$ implies that

$$\int_Q \left| \int_R \|f\|_B\,d\nu \right|^p d\mu \leq \Big(\int_R \|f\|_{\mathrm{L}^p(Q)}\,d\nu \Big)^p < \infty, \tag{B.46}$$

by (a) (applied to $\|f\|_B$) for $p < \infty$, and $\|\int_R \|f\|_B\,d\nu\|_\infty \leq \int_R \|f\|_\infty\,d\nu < \infty$ for $p=\infty$. In particular, $\int_R \|f\|_B\,d\nu < \infty$ for a.e. $q \in Q'$, where $\mu(Q\setminus Q') = 0$, and g is defined on Q', hence a.e.

Because $\mathrm{L}^p(Q_j;B) \subset \mathrm{L}^1(Q_j;B)$ for $j \in \mathbf{N}$, we have $\int_{Q_j}\int_R \|f\|_B\,d\nu\,d\mu < \infty$, hence $g_j := \int_R f\,d\nu : Q_j \to B$ is defined a.e. and measurable, by the Fubini Theorem.

Because $g_j = g$ on Q_j, also g is measurable. By (B.46), $\|g\|_{\mathrm{L}^p(Q)} \leq \int_R \|f\|_{\mathrm{L}^p(Q)}\,d\nu < \infty$. □

The following result allows one to take Laplace and Fourier transforms of $\mathcal{C}(\mathbf{R};\mathrm{L}^p)$ functions componentwise (see Proposition D.1.13).

Lemma B.4.17 ($\int_Q f(\cdot)(r)\,d\mu = (\int_Q f\,d\mu)(r)$) *Assume that $\mu : \mathfrak{M} \to [0,\infty]$ and $\nu : \mathfrak{M}' \to [0,\infty]$ are σ-finite, complete, positive measures on Q and R, respectively.*

Let $f \in \mathrm{L}^1(Q;\mathrm{L}^p(R;B)) \cap \mathrm{L}(Q\times R;B)$, $p \in [1,\infty]$. Then $g(r) := \int_Q f(\cdot)(r)\,d\mu$ exists a.e., and $g = \int_Q f\,d\mu \in \mathrm{L}^p(R;B)$.

Thus, then $g(r) = (\int_Q f\,d\mu)(r)$ for almost every $r \in R$. See also Example B.4.18.

Proof: (By using the Fubini Theorem, one easily verifies that $f=0$ as an element of $\mathrm{L}^1(Q;\mathrm{L}^p(R;B))$ iff $f=0$ as an element of $\mathrm{L}(Q\times R;B)$, hence $\mathrm{L}^1(Q;\mathrm{L}^p(R;B)) \cap \mathrm{L}(Q\times R;B)$ is well-defined (we used the assumption that $f \in \mathrm{L}(Q\times R;B)$; by Counter-example 8.9(c) of [Rud86], there is $f : [0,1]\times[0,1] \to \mathbf{R}$ s.t. f is not measurable, but $[f] \in \mathrm{L}^1([0,1];\mathrm{L}^p([0,1];B))$ has a representative that is measurable, because $f(t) = 1$ a.e. for every $t \in [0,1]$, i.e., $[f] = [1]$.)

By Theorem B.4.16, $g(r)$ exists for a.e. $r \in R$, $g \in \mathrm{L}^p(R;B)$ and $\|g\|_p \leq \|f\|_1$. Set $F := \int_Q f\,d\mu \in \mathrm{L}^p(R;B)$.

1° *Case $f = \chi_E b$, $b \in \mathrm{L}^p$, $E \in \mathfrak{M}$:* Now $F = \mu(E)b$, $g(r) = \mu(E)b(r)$ ($r \in R$), hence $F = g$.

2° *Case f is simple:* This follows easily from 1°, by linearity.

3° *General case:* Let $f_n \to f$ in L^1, as $n \to \infty$, and let f_n be simple and measurable for each $n \in \mathbf{N}$. Set $F_n := \int_Q f_n\,d\mu$, $g_n(r) := \int_Q f_n(\cdot)(r),d\mu$ ($r \in R$), so that $F_n = g_n$, by 2°.

Because $\int$ is continuous, we have $F_n \to F$ in L^p, as $n \to \infty$. By Theorem B.4.16,

$$\|g_n - g\|_p = \|\int_Q (f_n - f)(\cdot)(r)\,d\mu\|_p \le \int_Q \|f_n - f\|\,d\mu = \|f_n - f\|_1 \to 0, \tag{B.47}$$

as $n \to \infty$. Therefore, some subsequence of $\{g_n\}$ converges to g and F a.e., hence $g = F$ a.e. $\square$

The assumption that $f \in \mathrm{L}(Q \times R;B)$ is necessary in Lemma B.4.17:

Example B.4.18 [$\boldsymbol{\int_Q f(\cdot)(r)\,d\mu \neq (\int_Q f\,d\mu)(r)}$] Define $f : [0,1] \times [0,1] \to \mathbf{R}$ as in Counter-example 8.9(c) of [Rud86] (which uses the Continuum Hypothesis), and define $h : [0,1] \times [0,1] \to \mathbf{R}$ by $h \equiv 1$. Let $p, a \in [1,\infty]$.

Then f is not measurable $[0,1] \times [0,1] \to \mathbf{R}$, but for each $q \in Q := [0,1]$, we have $f(q) = 1 = h(q)$ a.e. on R, in particular, $f(q) \in \mathrm{L}^p([0,1];B)$. Thus, $[f] = [h]$ as elements of $\mathrm{L}^a([0,1];\mathrm{L}^p([0,1];B))$, even as elements of $\mathcal{C}([0,1];\mathrm{L}^p([0,1];B))$.

Even worse, for each $r \in [0,1]$, we have $f(q,r) = 0$ for a.e. $q \in Q$. Thus, $g(r) := \int_Q f(q,r)\,dq = 0$ for each $r \in [0,1]$, although $\int_Q f\,dm = \int_Q 1\,dm = 1 \in \mathrm{L}^p([0,1];B)$, hence $g \neq \int_Q f\,dm$ (cf. Lemma B.4.17). $\triangleleft$

If $f \in \mathrm{L}^a(Q;L^p)$, then there is $h \in \mathrm{L}(Q \times R)$ s.t. $h(q) = f(q)$ a.e. on R (hence as elements of L^p) for a.e. $q \in Q$ (we omit the nontrivial proof). Thus, then Lemma B.4.17 can be applied to h, but not to f: the value of $\int_Q f(q)(r)\,d\mu(q)$ may differ everywhere from

$$(\int_Q f\,d\mu)(r) = (\int_Q h\,d\mu)(r) = \int_Q h(q)(r)\,d\mu(q) \tag{B.48}$$

(this equality holds for a.e. $r \in R$), as shown in the above example.

If a continuous function has a L^1 limit function on the left boundary, then this function is L^1 over the whole rectangle and also sideways:

Lemma B.4.19 *Let $f \in \mathcal{C}((r,s] \times (a,b);B)$ and $f_{|\{r\}\times(a,b)} \in \mathrm{L}^1(\{r\} \times (a,b);B)$, $a,b,r,s \in \mathbf{R}$, $r < s$, $a < b$.*

If $f(t,\cdot) \to f(r,\cdot)$ in $\mathrm{L}^1((a,b);B)$, as $t \to r+$, then $f \in \mathrm{L}^1((r,s] \times (a,b);B)$ and $f(\cdot,c) \in \mathrm{L}^1((r,s);B)$ for a.e. $c \in (a,b)$.

Proof: Being continuous, f is Borel-measurable on $Q := (r,s] \times (a,b)$, hence Lebesgue-measurable on Q. Due to the continuity and convergence, $M := \sup_{t \in [r,s]} \|f(t,\cdot)\|_{\mathrm{L}^1((a,b);B)} < \infty$. By The Fubini Theorem,

$$(s-r)M \ge \int_Q \|f\|_B\,dm = \int_r^s \int_a^b \|f\|_B\,dm = \int_a^b \int_r^s \|f\|_B\,dm; \tag{B.49}$$

in particular, $\int_r^s \|f\|_B\,dm < \infty$ for a.e. $c \in (a,b)$ and $f \in \mathrm{L}^1(Q;B)$. $\square$

(See the notes on p. 947.)

B.5 Differentiation of integrals ($\frac{\mathrm{d}}{\mathrm{dt}}\int$)

"Chesire-Puss," she began, "would you tell me, please, which way I ought to go from here?"
"That depends a good deal on where you want to get to," said the Cat.
"I don't care much where–" said Alice.
"Then it doesn't matter which way you go," said the Cat.
— Lewis Carroll (1832–98)

In this section we present Lebesgue points and a few results on differentiation of integrals.

Also for vector-valued functions, almost all points are Lebesgue points:

Theorem B.5.1 (Lebesgue points) *Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n;B)$. Then, for almost all $t \in \mathbf{R}^n$, we have*

$$\lim_{r\to 0+} m(D_r)^{-1}\int_{D_r(t)}\|f(s)-f(t)\|_B\,ds = 0 \quad \textit{and} \quad f(t) = \lim_{r\to 0+} m(D_r)^{-1}\int_{D_r(t)} f(s)\,ds. \tag{B.50}$$

Such t are called the Lebesgue points *of f, and* $\mathrm{Leb}(f) \subset \mathbf{R}^n$ *is the set of such t.*

(Here $D_r(t) := \{t' \in \mathbf{R}^n \mid |t'-t| < r, D_r := D_r(0).$) From (B.50) it follows that $\|f(t)\| \le \|f\|_\infty \le \infty$ for all $t \in \mathrm{Leb}(f)$.

Note that if f is continuous at t, then $t \in \mathrm{Leb}(f)$. Obviously, $\mathrm{Leb}(f) \cap \mathrm{Leb}(g) \subset \mathrm{Leb}(\alpha f + \beta g)$ for any $f, g \in \mathrm{L}^1_{\mathrm{loc}}$, $\alpha, \beta \in \mathbf{K}$.

Proof: The first claim follows from Theorem 3.8.5 of [HP] for $t \in D_k$, if we replace f by $\chi_{D_k} f$ ($k \in \mathbf{N}$); the second claim follows from the first for (at least) same t:

$$\|f(t) - m(D_r)^{-1}\int_{D_r(t)} f(s)\,ds\|_B = \|m(D_r)^{-1}\int_{D_r(t)}(f(t)-f(s))\,ds\|_B \to 0. \tag{B.51}$$

Let $N_k \subset D_k$ be the null set where the first limit in (B.50) is nonzero or does not converge. Then (B.50) holds for $t \in \mathbf{R}^n \setminus \cup_{k\in\mathbf{N}} N_k$, hence a.e. (To be exact, [HP] gives the result for cubes, but it can be easily generalized to any nicely shrinking sets, as in Section 7 of [Rud86].) □

From the above theorem we obtain

Corollary B.5.2 *Let $J \subset \mathbf{R}$ be an interval and $f \in \mathrm{L}^1_{\mathrm{loc}}(J;B)$. Then, for almost all $t \in J$,*

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s)\,dt = f(t) \quad \textit{and} \quad \lim_{h\to 0}\frac{1}{h}\int_t^{t+h}\|f(s)-f(t)\|\,ds = 0. \tag{B.52}$$

In particular, if $a \in J$ and $F(t) = \int_a^t f(s)\,ds$, then $F \in \mathcal{C}(J;B)$ and $F' = f$ a.e.

□

(Of course, we have set $\int_b^a := -\int_a^b$.)

One easily verifies that a function F of the above form is locally absolutely continuous (see (scalar) Definition 7.17 of [Rud86]). However, unlike in the finite-dimensional (or Hilbert) case, there are absolutely continuous functions that are nowhere differentiable (however, this is not the case for reflexive spaces, a fortiori not in Hilbert spaces; see, e.g., [DU] or p. 125 of [KOS] for details).

There is a method for uniquely choosing representatives for $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n;B)$ "functions" so that these representative have all possible Lebesgue points:

Lemma B.5.3 (Lebesgue representative) *Let $[f] \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n;B)$. For each $t \in \mathbf{R}^n$ s.t.*

$$\lim_{r\to 0+} m(D_r)^{-1}\int_{D_r(t)} |f - x_t|\,dm = 0 \tag{B.53}$$

for some $x_t \in B$, we set $(Lf)(t) := x_t$; for other values of t, we set $(Lf)(t) := 0$.

It follows that $(Lf)(t) = f(t)$ for all $t \in \mathrm{Leb}(f)$, hence $Lf = f$ a.e. and $\mathrm{Leb}(f) \subset \mathrm{Leb}(Lf)$. Moreover, $\|Lf\| \leq \|f\|_\infty$ everywhere and Lf depends on $[f]$ only.

Moreover, for all $[f],[g] \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R};Y)$, $\Lambda \in Y^*$, $\psi \in \mathcal{GC}(\mathbf{R})$, and every $t \in \mathbf{R}$ *we have*

$$(L\psi f)(t) = (\psi Lf)(t), \qquad \mathrm{Leb}(\psi f) = \mathrm{Leb}(f), \tag{B.54}$$

$$\mathrm{Leb}(f) \subset \mathrm{Leb}(\Lambda f), \qquad \|(\Lambda Lf)(t)\| \leq \|(L\Lambda f)(t)\|, \tag{B.55}$$

$$t \in \mathrm{Leb}(f)\cap\mathrm{Leb}(g) \implies t \in \mathrm{Leb}(\alpha f + \beta g), \quad (L(\alpha f + \beta g))(t) = \alpha Lf(t) + \beta Lg(t). \tag{B.56}$$

Finally, for each linear $F : B \to \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}^n;B_2)$ and $t \in \mathbf{R}^n$, the space

$$B_t := \{x \in B \,\big|\, t \in \mathrm{Leb}(LFx)\} \tag{B.57}$$

is a subspace of B. □

(We omit the simple proof.) Note that $\mathrm{Leb}(Lf)$ becomes the union of the Lebesgue sets of all representatives of $[f]$, and that $f \mapsto Lf$ and $[f] \mapsto Lf$ are not linear, neither $f \mapsto (Lf)(t)$ for any $t \in \mathbf{R}^n$ (but $f \mapsto [Lf]$ and $[f] \mapsto [Lf]$ are linear, since $[Lf] = [f]$).

The standard differentiation formula for integrals can easily be extended:

Lemma B.5.4 *Let $J, J' \subset \mathbf{R}$ be intervals, $a, b \in \mathcal{C}^1(J;J')$. Let $f : J \times J' \ni (t,s) \to f(t,s) \in B$ be s.t. $f, f_t \in \mathcal{C}(J\times J';B)$. Then $F(t) := \int_{a(t)}^{b(t)} f(t,s)\,ds$ is in $\mathcal{C}^1(J;B)$, and*

$$F'(t) := \int_{a(t)}^{b(t)} f_t(t,s)\,ds + b'(t)f(t,b(t)) - a'(t)f(t,a(t)) \text{ for all } t \in J. \tag{B.58}$$

Recall that we allow J and J' be non-open (the derivatives at the endpoints are, of course, one-sided).

Proof: Because $\int_{a(t)}^{b(t)} = \int_c^{b(t)} - \int_c^{a(t)}$, where $c \in J'$ is arbitrary, we may assume that a is a constant. Now

$$F(t+h)-F(t)+\int_{b(t)}^{b(t+h)}(f(t,b(t)))\,ds \tag{B.59}$$

From the first term we compute

$$\begin{aligned}&\frac{1}{h}\left(\int_a^{b(t)}(f(t+h,s)-f(t,s))\,ds-\int_a^{b(t)}f_t(t,s)\,ds\right)\\&=\frac{1}{h}\int_a^{b(t)}\int_t^{t+h}(f_t(r,s)-f_t(t,s))\,dr\,ds=\frac{1}{h}\int_t^{t+h}\int_a^{b(t)}\to 0,\end{aligned} \tag{B.60}$$

by the continuity of f_t. (By the Fubini Theorem, we were allowed to interchange the order of integration.)

One easily verifies that the second and third terms of (B.59) multiplied by $1/h$ converge to 0 and $b'(t)f(t,b(t))$, respectively. The continuity of F' follows analogously. $\square$

A smooth change of variable preserves Lebesgue points:

Lemma B.5.5 *Let* $-\infty \le a < b \le +\infty$, $\phi \in C^1((a,b);\mathbf{R})$, $\phi' > 0$ *and* $f \in \mathrm{L}^1_{\mathrm{loc}}((\alpha,\beta);B)$, *where* $\alpha := \phi(a)$, $\beta := \phi(b)$. *Then* $\mathrm{Leb}(f) = \phi[\mathrm{Leb}(f\circ\phi)]$.

Here Leb refers to the zero extensions (or any other $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R};B)$ extensions) of f and $f\circ\phi$.

Proof: Let $T \in (a,b)$. Set $g := f\circ\phi$, $t := \phi(T)$. Choose $R > 0$ s.t. $[t-R,t+R] \subset (\alpha,\beta)$. Set $M := \max_{[t-R,t+R]}\|\phi'\|$, $M' := \max_{\phi[t-R,t+R]}\|\phi^{-1\prime}\|$. By Lemma B.4.10, we have

$$\frac{1}{2r}\int_{t-r}^{t+r}\|f(s)-f(t)\|_B\,ds = \frac{1}{2r}\int_{\phi^{-1}(t-r)}^{\phi^{-1}(t+r)}\|g(s)-g(T)\|_B\phi'(s)\,ds \tag{B.61}$$

$$= \frac{MM'}{2rM'}\int_{T-M'r}^{T+M'r}\|g(s)-g(T)\|_B\,ds. \tag{B.62}$$

But this converges to zero whenever $T \in \mathrm{Leb}(g)$, hence then $t \in \mathrm{Leb}(f)$. Because $T \in (a,b)$ was arbitrary, we have $\phi[\mathrm{Leb}(f\circ\phi)] \subset \mathrm{Leb}(f)$. Exchange the roles of f and g (and ϕ and ϕ^{-1}) to obtain that $\phi^{-1}[\mathrm{Leb}(f)] \subset \mathrm{Leb}(f\circ\phi)$, i.e., $\mathrm{Leb}(f) \subset \phi[\mathrm{Leb}(f\circ\phi)]$. $\square$

The Mean Value Theorem is only true for $\mathbf{R}$-valued functions (not even for $\mathbf{C}$-valued or $\mathbf{R}^2$-valued; e.g., set $f(t) := e^{it}$, $[a,b] := [0,2\pi]$); in the multidimensional case it becomes a mere inequality:

Lemma B.5.6 (Mean Value Inequality) *Let* $a < b$, *and let* $f \in C([a,b];B)$ *be differentiable on* (a,b). *Then there is* $\xi \in (a,b)$ *s.t.*

$$\|f(b)-f(a)\|_B \le (b-a)\|f'(\xi)\| \le (b-a)\sup_{t\in(a,b)}\|f'(x)\|. \tag{B.63}$$

If, in addition, $f'(a)$ *exists, then there is* $\widetilde{\xi} \in (a,b)$ *s.t.*

$$\|\frac{f(b)-f(a)}{b-a}-f'(a)\|_B \le \|f'(\widetilde{\xi})-f'(a)\|_B. \tag{B.64}$$

Proof: Let $\Lambda \in X^*$ be s.t. $\|\Lambda\| \leq 1$ and $\Lambda(f(b)-f(a)) = \|f(b)-f(a)\|_B$. Obviously, $(\mathrm{Re}\,\Lambda f)' = \mathrm{Re}\,\Lambda f'$, hence

$$\|f(b)-f(a)\|_B = \mathrm{Re}\,\Lambda f(b) - \mathrm{Re}\,\Lambda f(a) = (b-a)(\mathrm{Re}\,\Lambda f')(\xi) \leq (b-a)\|f'(\xi)\|_B, \tag{B.65}$$

by the classical Mean Value Theorem. The second inequality follows from the first applied to $F(t) := f(t) - f(a) - tf'(a)$ (note that $F'(t) = f'(t) - f'(a)$). □

Lemma B.5.7 *Let $J \subset \mathbf{R}$ be an interval and $n \in \mathbf{N}$. Then the following are equivalent for $F : J \to B$:*

(i) $F \in C^{n+1}(J;B)$;

(ii) $F = \int f$ for some $f \in C^n(J;B)$.

Moreover, if (ii) holds, then $F' = f$.

By $F = \int f$ we mean that $F(t) = F(a) + \int_a^t f(s)\,ds$ for all $t \in J$ and some (hence all) $a \in J$. See Lemma B.7.6 for an analogous result for absolutely continuous functions.

Proof: One easily verifies that (ii) implies (i). Given (i), fix $a \in J$ and set $f := F'$, $G(t) := F(a) + \int_a^t f$. Then $(F-G)' = 0$ on J, hence $(\Lambda F - \Lambda G)' = 0$ on J and $(\Lambda F - \Lambda G)(a) = 0$, hence $\Lambda F = \Lambda G$ on J; this holds for all $\Lambda \in B^*$, hence $F = G$. Therefore, (i) implies (ii). □

If $f(\cdot,q) \in C^1(\Omega;B)$ for fixed $q \in Q$, and $f(z,\cdot)$ is measurable and has a common L^1 majorant for all z, then $\int_Q f(z,\cdot)\,d\mu \in C^1(\Omega;B)$ with derivative $\int_Q f_z(z,\cdot)\,d\mu \in C(\Omega;B)$, by (c)&(d) below:

Lemma B.5.8 *Let Ω be a metric space, let Q be σ-finite, let $1 \leq p \leq \infty$, let $f : \Omega \times Q \to B$, and set $F(z) := f(z,\cdot)$. Then we have the following:*

(a) We have $F \in C(\Omega;\mathrm{L}^p(Q;B))$, if $p < \infty$ and (1.)–(3.) hold, where

(1.) $f(\cdot,q) \in C(\Omega;B)$ for a.e. $q \in Q$;
(2.) $f(z,\cdot) \in \mathrm{L}(Q;B)$ for all $z \in \Omega$;
(3.) there is $g \in \mathrm{L}^p(Q;[0,\infty])$ s.t. $\|f(z,\cdot)\|_B \leq g$ a.e. for all $z \in \Omega$.

(b) If $F \in C(\Omega;\mathrm{L}^1(Q;B))$, then $\widetilde{F} \in C(\Omega;B)$, where $\widetilde{F}(z) := \int_Q f(z,\cdot)\,d\mu$.

(c) Claims (a) and (b) also hold with C^1 in place of C if $\Omega \subset \mathbf{R}$ is an interval and in (a) we also require that $\|f_z(z,\cdot)\|_B \leq g$ a.e. for all $z \in \Omega$.

In case (a) we then also have $F'(z) = f_z(z,\cdot)$, hence then (d) applies.

(d) Assume that $\Omega \subset \mathbf{R}$ is an interval, $f(\cdot,q) \in C^1(\Omega;B)$ for a.e. $q \in Q$, and $F, G \in C(\Omega;\mathrm{L}^p)$, where $G(z) = f_z(z,\cdot)$.

Then $F \in C^1(\Omega;\mathrm{L}^p)$ and $F' = G$ (and $\widetilde{F} \in C^1(\Omega;B)$ and $\widetilde{F}'(z) = \int_Q f_z(z,\cdot)\,d\mu$ if $p = 1$) for all $z \in \Omega$.

(e) Claims (c) and (d) also hold with H *in place of* $\mathcal{C}$ *if* $\Omega \subset \mathbf{C}$ *is open.*

See Definition D.1.3 for $\mathrm{H}(\Omega; B)$.

Proof: (a) If $z_n \to z$, then $F(z_n) \to F(z)$, by Theorem B.4.3. Thus, $F \in \mathcal{C}(\Omega; \mathrm{L}^p)$.

(b) This follows from (a) Lemma B.4.2.

(d) By Lemma B.5.7, we have $f(z,q) - f(a,q) = \int_a^z f_z(s,q)\,ds$ $(z, a \in \Omega)$.

Moreover, Ω is σ-finite, separable and metric, and open subsets of Ω are m-measurable. Thus, f_z is $m \times \mu$-measurable, by Lemma B.4.8.

Fix $z \in \Omega$. Given now $h \in \Omega$, we have (here $\|\cdot\|_p$ refers to the $\mathrm{L}^p(Q; B)$ norm and $q \in Q$ is the corresponding dummy (i.e., dependent) variable)

$$\|h^{-1}\big(F(z+h) - F(z)\big) - G(z)\|_p = \|h^{-1} \int_z^{z+h} \big(f_z(s,q) - f_z(z,q)\big)\,ds\|_p \tag{B.66}$$

$$\leq h^{-1} \int_z^{z+h} \|f_z(s,q) - f_z(z,q)\|_p\,ds \to 0, \tag{B.67}$$

(the inequality is from Theorem B.4.16(b)) as $h \to 0$, by Theorem B.4.3, because $\|f_z(s,q) - f_z(z,q)\|_p = \|G(s) - G(z)\|_p \to 0$ as $s \to z$, and $\|G(s) - G(z)\|_p$ is bounded ($\leq M < \infty$) near z, by continuity, and $M \in \mathrm{L}^1([z, z+h])$.

Therefore, $F' = G$ exists in L^p. Consequently, $F \in \mathcal{C}^1(\Omega; \mathrm{L}^p)$.

(c) By (a), we have $F, G \in \mathcal{C}(\Omega; \mathrm{L}^p)$, where $G(z) := f_z(z, \cdot)$ (note that $f_z(z,q) = \lim_{n\to\infty} n^{-1}(f(z+1/n, q) - f(z,q))$ for a.e. $q \in Q$, hence also f_z satisfies condition (2.)). Therefore, we get the conclusions from (d).

(e) This is analogous to the $\mathcal{C}^1$ case (use a path integral and (b5) (and (b1) for $p = 1$) of Lemma D.1.2 instead of Lemma B.5.7). $\square$

By (B.50), The averages of any $f \in \mathrm{L}^p(\mathbf{R}; B)$ converge to f pointwise a.e.; we also have the convergence in L^p:

Lemma B.5.9 *Let* $1 \leq p < \infty$ *and* $f \in \mathrm{L}^p(\mathbf{R}; B)$. *Then*

$$\|\frac{1}{r} \int_0^r f(t+s)\,ds - f(t)\|_p \to 0, \quad \text{as} \;\; r \to 0 \tag{B.68}$$

(as functions of t).

Proof: By the Minkovski Integral Inequality, we have (here the L^p norm refers to the variable t)

$$\frac{1}{|r|}\|\int_{s=0}^r [f(t+s) - f(t)]\,ds\|_p \leq \frac{1}{r}\int_{s=0}^r \|f(t+s) - f(t)\|_p\,ds \leq \sup_{s\in[0,r]} \|\tau^s f - f\|_p \to 0, \tag{B.69}$$

as $r \to 0$, by Lemma B.3.9. $\square$

We finish this section by a technical lemma:

Lemma B.5.10 *Let $T>0$. If 0 is a Lebesgue point of $f \in \mathrm{L}^1([0,T);B)$, then*

$$\int_0^T s\mathrm{e}^{-st} f(t)\,dt \to f(0), \quad \textit{as} \ \ s \to +\infty. \tag{B.70}$$

Proof: Because $\int_0^T s\mathrm{e}^{-st}\,dt \to 1$, we have (B.70) for constant functions f. Therefore, we may assume that $f(0)=0$, i.e., that

$$g(t) := t\int_0^t \|f(r)\|_B\,dr \to 0 \quad \text{as} \ \ t \to 0+\,. \tag{B.71}$$

Set $F(t) := \int_0^t f\,dm \in \mathcal{C}([0,T];B)$. Then $F(0)=0$, $\|\int_0^\varepsilon s^2\mathrm{e}^{-st}F(t)\,dt\|_B \le g(\varepsilon)$ and $\int_\varepsilon^T s^2\mathrm{e}^{-st}F(t)\,dt \to 0$, as $s \to +\infty$, for any $\varepsilon \in (0,T)$. Using these three facts and partial integration, one easily obtains (B.70). □

Notes for Sections B.1–B.5

As indicated in the proofs, many of the above results are known at least to some extent or in the scalar case. A further treatment on Bochner measurability, Bochner integral and vector-valued L^p spaces is given in, e.g., Sections 3.5–3.9 of [HP] and in [KOS], [DU] and [Yosida]; the monograph [Dinculeanu] treats same concepts from the Bourbaki point of view. The scalar case (the Lebesgue integral and measurability and L^p and $\mathcal{C}$ spaces) is contained in most books on real analysis, such as [Rud86].

B.6 Vector-valued distributions $\mathcal{D}'(\Omega;B)$

No, my friend, the way to have good and safe government, is not to trust it all to one, but to divide it among the many, distributing to every one exactly the functions he is competent to. It is by dividing and subdividing these republics from the national one down through all its subordinations, until it ends in the administration of every man's farm by himself; by placing under every one what his own eye may superintend, that all will be done for the best.

— Thomas Jefferson (1743–1826), to Joseph Cabell, 1816

Here we briefly present straightforward vector-valued generalizations of some basic scalar distribution results. We have written the other sections so that the reader may skip this section, but its contents give a deeper insight to some concepts needed in, e.g., Section B.7.

Throughout this section, B is a Banach space and Ω is an open subset of $\mathbf{R}^n$.

The space $\mathcal{D}(\Omega)$ is not equal to $C_c^\infty(\Omega)$, although rather close. One traditionally uses a rather complicated topology; fortunately, for most applications one does not need to know this topology, just some of its basic implications:

Definition B.6.1 *The* test function space $\mathcal{D} := \mathcal{D}(\Omega)$ *is the set of functions* $\phi \in C^\infty(\mathbf{R})$, *whose support* $\operatorname{supp}\phi := \overline{\{x \in \mathbf{R}^n \mid \phi(x) \neq 0\}}$ *lies in* Ω, *equipped with the standard (locally convex, complete, non metrizable) test function topology [Rud73, 6.3–6.5], where a sequence* $\{\phi_k\}$ *converges to* $\phi \in \mathcal{D}$ *iff there is a compact* $K \subset \Omega$ *s.t.* $\operatorname{supp}\phi_k \subset K$ *for all* k *and* $D^\alpha\phi_k \to D^\alpha\phi$ *uniformly for all* $\alpha \in \mathbf{N}^n$.

The elements of $\mathcal{D}' := \mathcal{D}'(\Omega;B) := \mathcal{B}(\mathcal{D}(\Omega);B)$ *are called* (B-)distributions.

As in the proof of [Rud73, Theorem 6.6], one can show that a linear mapping $T : \mathcal{D}(\Omega) \to B$ is continuous iff it is sequentially continuous, i.e., iff $\phi_k \to \phi$ (in $\mathcal{D}(\Omega)$) implies $T\phi_k \to T\phi$ (it is enough to verify this for $\phi = 0$). Also most other results of [Rud73, Section 6] can easily be generalized.

We define the α*th weak derivative* (or αth distributional derivative) $\partial^\alpha T \in \mathcal{D}'$ of $T \in \mathcal{D}'$ by

$$\partial^\alpha T(\phi) := (-1)^{|\alpha|} T(D^\alpha \phi) \quad (\phi \in \mathcal{D}(\Omega)), \tag{B.72}$$

in particular, $\partial T(\phi) := -T(\phi')$ if $n = 1$. Here we have used the standard *multi-index* notation: $\alpha \in \mathbf{N}^n$, $|\alpha| := \sum_{j=1}^n \alpha_j$, $(x_1, \ldots, x_n)^\alpha := x_1^{\alpha_1} + \cdots + x_n^{\alpha_n}$, $D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}$; here $D_j := \frac{d}{dx_j}$ and ∂_j is the corresponding weak derivative.

A function $f \in L^1_{loc}(\Omega;B)$ (i.e., $f : \Omega \to B$ is s.t. $f \in L^1(K;B)$ for each compact $K \subset \Omega$; note that $L^p \subset L^1_{loc}$ for $p \in [1,\infty]$). is identified with the distribution $\phi \mapsto \int_\Omega f\phi\, dm$. in $\mathcal{D}'(\Omega;B)$. The inclusion $L^1_{loc}(\Omega;B) \subset \mathcal{D}'(\Omega;B)$ is linear and injective. Similarly, a constant (function) $b \in B$ is identified with the $\phi \mapsto \int_\Omega b\phi\, dm = b\int_\Omega \phi\, dm$.

A distribution with zero partial derivatives is a constant:

Lemma B.6.2 *Let* Ω *be connected and* $T \in \mathcal{D}'(\Omega;B)$. *If* $\delta_j T = 0$ *for all* j, *then* $T \in B$.

Proof: If $\Lambda \in B^*$, then $\Lambda T \in \mathcal{D}'$ and $\partial_j \Lambda T = \Lambda \partial_j T = 0$ for all j, hence $\Lambda T = \alpha_\Lambda \in \mathbf{K}$, by [Rauch, p. 256]. Choose $\phi_1 \in \mathcal{D}$ s.t. $\int_\Omega \phi_1 \, dm = 1$, and set $b_T := T\phi_1$.

Then $\Lambda b_T = \Lambda T \phi_1 = \alpha_\Lambda \int_\Omega \phi_1 \, dm = \alpha_\Lambda$ for all $\Lambda \in B^*$, hence $\Lambda T \phi = \alpha_\Lambda \int_\Omega \phi \, dm = \Lambda b_T \int_\Omega \phi \, dm$ for all $\phi \in \mathcal{D}(\Omega)$, $\Lambda \in B^*$, i.e., $T\phi = b_T \int \phi = \int_\Omega b_T \phi$ for all ϕ. Thus, $T = b_T \in B$. $\square$

By induction, we see that if the kth partial derivatives of T are zero, then T is a polynomial of degree k:

Corollary B.6.3 *Let Ω be connected and $T \in \mathcal{D}'(\Omega;B)$. If $|\alpha| = k \Rightarrow \partial^\alpha T = 0$, then $T = \sum_{|\beta| \le k-1} q^\beta b_\beta$, where $b_\beta \in B$ for all β.* $\square$

Corollary B.6.4 *If $J \subset \mathbf{R}$ is an open interval, $T \in \mathcal{D}'(J;B)$, and $\partial T = f \in \mathrm{L}^1_{\mathrm{loc}}(J;B)$, then there is a locally absolutely continuous function $F : J \to B$ s.t. $T = F$ and $F(t) = F(a) + \int_a^t f \, dm$ when $a \in J$; in particular $F' = f$ a.e.* $\square$

(This follows by defining $G(t) := \int_a^t f \, dm$ and then noting that $\partial(T - G) = 0$.)

Similarly, for any $F, f \in \mathcal{C}(\Omega;B)$ s.t. $\partial_j F = f$, the derivative $D_j F$ exists and equals f, as one can easily show by using mollifiers.

Notes

The contents of this section are well known, although it may be difficult to find any references; some other results on vector-valued distributions are given in [Treves]. See, e.g., [Rud73] or [Rauch] for the scalar case; most scalar results also hold in our setting with same proofs, mutatis mutandis (some existence results require the Radon–Nikodym property).

B.7 Sobolev spaces $\mathrm{W}^{k,p}(\Omega;B)$

The reason that every major university maintains a department of mathematics is that it's cheaper than institutionalizing all those people.

In this section, we briefly generalize some facts about scalar Sobolev spaces to their vector-valued counterparts. Most of the time we follow the scalar representations in [Adams]. A casual reader might skip the definitions and other results and just read Lemma B.7.6 and Theorem B.7.4, since they suffice for most applications. Also other readers might wish to read first Lemma B.7.6 to get some intuition to $\mathrm{W}^{k,p}$ spaces.

Throughout this section, B is a Banach space, $1 \le p \le \infty$, $k \in \mathbf{N}$, $n \in \mathbf{N}+1$, $\Omega \subset \mathbf{R}^n$ is open, and m is the Lebesgue measure on $\mathbf{R}^n$.

Definition B.7.1 ($\partial^\alpha g$) *Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega;B)$ and $\alpha \in \mathbf{N}^n$. We call $g \in \mathrm{L}^1_{\mathrm{loc}}(\Omega;B)$ the* αth weak derivative *of f (on Ω) and we write $\partial^\alpha f = g$ if*

$$\int_\Omega g\phi\, dm = (-1)^{|\alpha|}\int_\Omega f D^\alpha \phi\, dm \quad (\phi \in C_c^\infty(\Omega)). \tag{B.73}$$

(By using linearity, projections, mollifiers and a partition of unity, one could in fact show that (B.73) for $\phi \in C_c^\infty(\Omega)$ implies (B.73) for all $\phi \in C_c^{|\alpha|}(\Omega;X)$, where X is as in Theorem B.4.12(d). We omit the proof.)

Here we have used the standard *multi-index* notation: $\alpha \subset \mathbf{N}^n$, $|\alpha| := \sum_{j=1}^n \alpha_j$, $(x_1,\dots,x_n)^\alpha := x_1^{\alpha_1}+\dots+x_n^{\alpha_n}$, $D^\alpha := D_1^{\alpha_1}\cdots D_n^{\alpha_n}$; here $D_j := \frac{\mathrm{d}}{\mathrm{d}x_j}$ is the (classical) jth partial derivative (use Definition B.3.3 with the other coordinates fixed). If $n=1$, we write $\partial := \partial^1$. (Outside this and previous section, we use the same notation for weak and ordinary derivatives.)

Recall that $f \in \mathrm{L}^1_{\mathrm{loc}}(\Omega;B)$ means that $f:\Omega \to B$ is s.t. $f \in \mathrm{L}^1(K;B)$ for each compact $K \subset \Omega$. The weak derivate is unique (as an element of $\mathrm{L}^1_{\mathrm{loc}}$, that is, a.e.), by Theorem B.4.12(d).

If $f \in C^k(\Omega;B)$ and $|\alpha| \le k$, then $\partial^\alpha f = D^\alpha f$ on Ω, by partial integration. See also Theorem B.7.4 and Lemma B.7.6.

It is obvious that if $\partial^\alpha f = g$ on Ω, then $\partial^\alpha \pi_{\Omega'} f = \pi_{\Omega'} g$ on Ω' for any open $\Omega' \subset \Omega$.

Definition B.7.2 ($\mathbf{W}^{k,p}$) *The* Sobolev space $\mathrm{W}^{k,p}$ *is defined by*

$$\mathrm{W}^{k,p}(\Omega;B) := \{f \in \mathrm{L}^p(\Omega;B) \,\big|\, \partial^\alpha f \in \mathrm{L}^p \ \textit{when } |\alpha| \le k\} \tag{B.74}$$

for $p \in [1,\infty]$ and $k \in \mathbf{N}$, with norm

$$\|f\|_{k,p} := \Big[\sum_{|\alpha|\le k} \|\partial^\alpha f\|_p^p\Big]^{1/p} \ (1 \le p < \infty), \qquad \|f\|_{k,\infty} := \max_{|\alpha|\le k} \|\partial^\alpha f\|_\infty \tag{B.75}$$

We denote closure of $C_c^\infty(\Omega;B)$ in $\mathrm{W}^{k,p}(\Omega;B)$ by $\mathrm{W}_0^{k,p}(\Omega;B)$. When $\Omega \subset \mathbf{R}^1$, we set $f \in \mathrm{W}^{k,p}_{\mathrm{loc}}(\Omega;B)$ if $f \in \mathrm{W}^{k,p}(J;B)$ for each bounded open interval $J \subset \Omega$.

In particular, $\mathrm{W}^{0,p} = \mathrm{L}^p$. Let $\Omega' \subset \Omega$ be open. Obviously, $f \in \mathrm{W}^{k,p}(\Omega;B) \Rightarrow \pi_{\Omega'} f \in \mathrm{W}^{k,p}(\Omega';B)$; in particular $\mathrm{W}^{k,p} \subset \mathrm{W}^{k,p}_{\mathrm{loc}}$.

Let $\Omega \subset \Omega''$, $f \in \mathrm{W}^{k,p}_0(\Omega;B)$, $|\alpha| \le k$ and $\partial^\alpha f = g$. Then $\pi_{\Omega''} f \in \mathrm{W}^{k,p}(\Omega'';B)$ with the same norm, and $\partial^\alpha \pi_{\Omega''} f = \pi_{\Omega''} g$ (for $f \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega;B)$ the latter claim follows by integration by parts; for general $f \in \mathrm{W}^{k,p}_0(\Omega;B)$ by continuity; the former claim follows from this).

Theorem B.7.3 *The spaces* $\mathrm{W}^{k,p}(\Omega;B)$ *and* $\mathrm{W}^{k,p}_0$ *are Banach spaces.*

If $p < \infty$, then $\mathcal{C}^\infty(\Omega;B) \cap \mathrm{W}^{k,p}(\Omega;B)$ is dense in $\mathrm{W}^{k,p}(\Omega;B)$, and $\mathrm{W}^{k,p}_0(\mathbf{R}^n;B) = \mathrm{W}^{k,p}(\mathbf{R}^n;B)$.

However, $\mathcal{C}^1((0,1);B)$ is not dense in $\mathrm{W}^{1,\infty}((0,1);B)$ when $B \neq \{0\}$ (take an absolutely continuous function whose derivative has a jump discontinuity).

The space $\mathrm{W}^{k,p}$ is dense in L^p ($p < \infty$), because $\mathcal{C}^\infty_{\mathrm{c}} \subset \mathrm{W}^{k,p}$. However, $\mathrm{W}^{1,\infty}(\mathbf{R};B)$-functions are continuous, by Lemma B.7.6, hence not dense in L^∞.

Proof: 1° *The completeness of* $\mathrm{W}^{k,p}$*:* Let $\{f_n\}$ be a Cauchy sequence in $\mathrm{W}^{k,p}$ and let $|\alpha| \le n$. Then $\{\partial^\alpha f_n\}$ is a Cauchy sequence in L^p, hence there is $f_\alpha \in \mathrm{L}^p$ s.t. $\partial^\alpha f_n \to f_\alpha$ in L^p. Let $\phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega)$. Because $D^\alpha \phi \in \mathcal{C}^\infty_{\mathrm{c}}(\Omega) \subset \mathrm{L}^q(\Omega)$, the Hölder inequality implies that

$$\langle f_\alpha, \phi\rangle \leftarrow \langle \partial^\alpha f_n, \phi\rangle := (-1)^{|\alpha|} \int D^\alpha \phi f_n \, dm \to (-1)^{|\alpha|} \int D^\alpha \phi f_0 \, dm =: \langle \partial^\alpha f, \phi\rangle. \tag{B.76}$$

Because α and ϕ were arbitrary, we have $f_0 \in \mathrm{W}^{k,p}$ and $\partial^\alpha f_0 = f_\alpha$ for $|\alpha| \le k$. Clearly $f_n \to f_0$ in $\mathrm{W}^{k,p}$.

2° Being a closed subspace of $\mathrm{W}^{k,p}$, also $\mathrm{W}^{k,p}_0$ is a Banach space. The last sentence of the theorem follows from the straightforward generalizations of [Adams, 3.15–3.19] (note that on p. 53 of [Adams] the condition ‘contained in “U_k”’ should be ’contained in U_k but not in U_{k-1}’). □

One can interpret $\mathrm{W}^{k,p}$ as a closed subspace of $\prod_{1\le j\le N_{n,k}} \mathrm{L}^p$, hence it is separable if B is (cf. [Adams, 3.4]). Similarly, $\mathrm{W}^{k,2}$ is a Hilbert space if B is.

If $\psi \in \mathcal{C}^k_{\mathrm{b}}(\Omega)$, then $\psi \in \mathcal{B}(\mathrm{W}^{k,p})$ and $\|\psi\|_{\mathcal{B}} \le M_k \|\psi\|_{\mathcal{C}^k_{\mathrm{b}}}$ (as a multiplication operator on $\mathrm{W}^{k,p}$) and ψf can be weakly differentiated by the Leibniz’ rule. For $\Lambda \in \mathcal{B}(B,B_2)$ we have $\|\Lambda\|_{\mathcal{B}(\mathrm{W}^{k,p}(\Omega,B);\mathrm{W}^{k,p}(\Omega,B_2))} = \|\Lambda\|_{\mathcal{B}(B,B_2)}$ (unless $\Omega = \emptyset$), moreover, $\partial^\alpha \Lambda f = \Lambda \partial^\alpha f$ for $f \in \mathrm{W}^{k,p}$.

An open $\Omega \subset \mathbf{R}$ has the *cone property*, if there is a finite cone C s.t. each point $x \in \Omega$ is the vertex of a finite cone $C_x \subset \Omega$ congruent to C. In particular, any ball or cube or a product of such will do.

Theorem B.7.4 (Sobolev Imbedding Theorem) *Let B be a Banach space, let the open set $\Omega \subset \mathbf{R}^n$ have the cone property, and let $1 \le p < \infty$, $mp > n$, and $j \in \mathbf{N}$.*

Then $\mathrm{W}^{j+m,p}(\Omega;B) \subset \mathcal{C}^j_{\mathrm{b}}(\Omega;B)$, and $\mathrm{W}^{j+m,p}_0(\Omega;B) \subset \mathcal{C}^j_0(\Omega;B)$, and these imbeddings are continuous.

In particular, $\mathrm{W}^{j+m,p}(\mathbf{R}^n;B) \subset C_0^j(\mathbf{R}^n;B)$. Of course, $f \in C_0$ means that there is $\widetilde{f} \in C_0$ s.t. $f = \widetilde{f}$ a.e. See also Corollary B.7.7 (which allows $p = \infty$).

Proof: (We use the norm $\|f\|_{C_b^j} := \sum_{|\alpha|\le j} \sup\|D^\alpha f\|$ in C_b^j and C_0^j.) By [Adams, 5.4C], there is $c = c_{j,m,p,n,\Omega}$ s.t. for $F \in W^{j+m,p}(\Omega)$ we have $\|\partial^\alpha F\|_\infty \le c\|F\|_{j+m,p}$ when $|\alpha| \le j$.

1° Case $f \in C^k \cap \mathrm{W}^{j+m,p}$: Let $|\alpha| \le j$. If we had $\|(\partial^\alpha f)(t)\| > c\|f\|_{j+m,p} =: M$, then there were $\Lambda \in X^*$ s.t. $\|\Lambda\| \le 1$ and $\Lambda\partial^\alpha f(t) > M$. But $\|\partial^\alpha \Lambda f\|_\infty \le c\|\Lambda f\|_{j+m,p} \le c\|f\|_{j+m,p} < (\partial^\alpha \Lambda f)(t)$ were a contradiction, hence $\|(\partial^\alpha f)(t)\| \le c\|f\|_{j+m,p}$.

2° Case $f \in \mathrm{W}^{j+m,p}$: By Theorem B.7.3, some sequence $\{f_n\} \subset C^k \cap \mathrm{W}^{j+m,p}$ converges to f. By 1°, $\{f_n\}$ is a C_b^j Cauchy sequence, hence it converges in C_b^j to a function g. Because a subsequence converges to f a.e. (generalize the corresponding scalar result, e.g., [Rud86, Theorem 3.12]), we have $g = f$ a.e. Moreover (because of the convergence in C_b^j), we have

$$\|\partial^\alpha f\|_\infty = \lim_{n\to\infty} \|D^\alpha f_n\|_\infty \le \lim_{n\to\infty} c\|f_n\|_{j+m,p} = c\|f\|_{j+m,p}. \tag{B.77}$$

3° $\mathrm{W}_0^{j+m,p}$: Here we can take $\{f_n\} \subset C_c^\infty(\mathbf{R}^n;X)$ and obtain that $g\ (= f)$ belongs to the closure of C_c^∞ in C_b^j, i.e., to C_0^j.

4° The continuity of the imbeddings follows from the bound c. □

If first (resp. kth) weak partial derivatives of f are zero, then f is a constant (resp. a polynomial of order$< k$):

Lemma B.7.5 *Let Ω be connected and $f \in \mathrm{W}^{1,p}(\Omega;B)$. If $\delta_j f = 0$ for all j, then $f \in B$.*

Let, in addition, $f \in \mathrm{W}^{k,p}(\Omega;B)$. If $|\alpha| = k \Rightarrow \partial^\alpha f = 0$, then $f = \sum_{|\beta|\le k-1} x^\beta b_\beta$, where $b_\beta \in B$ for all β.

Proof: If $\Lambda \in B^*$, then $\Lambda f \in \mathcal{D}'$ and $\partial_j \Lambda f = \Lambda \partial_j f = 0$ for all j, hence $\Lambda f = \alpha_\Lambda \in \mathbf{K}$, by corresponding scalar result (see p. 256 of [Rauch]). Choose $\phi_1 \in \mathcal{D}$ s.t. $\int_\Omega \phi_1\, dm = 1$, and set $b_f := f\phi_1$.

Then $\Lambda b_f = \Lambda f \phi_1 = \alpha_\Lambda \int_\Omega \phi_1\, dm = \alpha_\Lambda$ for all $\Lambda \in B^*$, hence $\Lambda f \phi = \alpha_\Lambda \int_\Omega \phi\, dm = \Lambda b_f \int_\Omega \phi\, dm$ for all ϕ, $\Lambda \in B^*$, i.e., $f\phi = b_f \int \phi = \int_\Omega b_f \phi$ for all ϕ. Thus, $f = b_f \in B$. The $\mathrm{W}^{k,p}$ claim follows by induction. □

Being a $\mathrm{W}^{1,1}$ function on an interval is equivalent to absolute continuity:

Lemma B.7.6 ($\mathbf{W^{1,p} = \int L^p}$) *Let $J \subset \mathbf{R}$ be an open interval and $f \in \mathrm{L}^p(J;B)$. Then the following are equivalent:*

(i) $f \in \mathrm{W}^{1,p}(J;B)$;

(ii) there is $g \in \mathrm{L}^p$ s.t. $f(t) = \int_a^t g\, dm + f(a)$ $(t \in J)$ for some $a \in J$.

Assume (ii). Then $g = f'$ a.e., $\partial f = g$, f is locally absolutely continuous, and (ii) holds for any $a \in J$. If $p = 1$, then f is absolutely continuous and has one-sided limits at the endpoints of J.

Thus, if $f, \partial f \in \mathrm{L}^p$, then the weak derivative ∂f is also a pointwise derivative of f a.e. However, if f is the Cantor function of Theorem 7.16 of [Rud86], then f' exists a.e. and $f, f' \in \mathrm{L}^p([0,1])$ for any $p \in [1,+\infty]$, but ∂f does not exist (in $\mathrm{L}^1_{\mathrm{loc}}$), because $f' = 0$ and $f \neq f(0) + \int_0^t 0\,dm$.

Proof: 1° "(ii)⇒(i)": Assume (ii). By using the Fubini theorem, one easily verifies that $\partial f = g$, hence $f \in \mathrm{W}^{1,p}$.

2° "(i)⇒(ii)": Let $f \in \mathrm{W}^{1,p}$ and $g := \partial f \in \mathrm{L}^p$. Define $F \in \mathcal{C}$ by $F(t) := \int_a^t g\,dt + f(a)$ for some $a \in J$. By 1°, we have $\partial F = g$, hence $F = f + b$ for some $b \in B$, by Lemma B.7.5, and $b = F(a) - f(a) = 0$. Thus, $f = F$.

3° Assume (ii). By Corollary B.5.2, $f' = g$ a.e. and f is locally absolutely continuous. The rest follows from 1°–2° (recall that ∂f is unique, by Theorem B.4.12(d)), except the $p = 1$ claims, which follow by choosing for $\varepsilon > 0$ a simple function $s \in \mathrm{L}^1$ s.t. $0 \le s \le \|g\|$ and $\int_J \|g\| - s\,dm < \varepsilon/2$, and setting $\delta_\varepsilon := \varepsilon/2\max s$ (or $\delta_\varepsilon = 1$ if $s = 0$) — then $m(E) < \delta_\varepsilon$ implies that $\int_E \|g\|\,dm < \varepsilon$; because $\varepsilon > 0$ was arbitrary, f is absolutely continuous (see, e.g., Definition 7.17 of [Rud86]), hence it obviously is continuous on $\bar{J}$. □

In case $n = 1$, we can slightly improve Theorem B.7.4:

Corollary B.7.7 ($\mathbf{W}^{k+1,p}(J;B) \subset \mathcal{C}^k(J;B)$) *Let $J \subset \mathbf{R}$ be open. Then $\mathrm{W}^{k+1,p}(J;B) \subset \mathcal{C}^k(J;B)$. The mapping $\mathrm{W}^{k+1,p}(J;B) \to \mathcal{C}^k(K;B)$ is continuous for each compact interval $K \subset J$.*

Proof: Let $f \in \mathrm{W}^{k+1,p}(J;B)$. Let $J' \subset J$ be an interval. Then $f \in \mathrm{W}^{k+1,p}(J';B)$, hence $f \in \mathcal{C}^k(J';B)$, by Lemma B.7.6 and induction. Because J was arbitrary, we have $f \in \mathcal{C}^k(J;B)$.

By Lemma A.3.6, $\mathrm{W}^{k+1,p}(J;B) \to \mathcal{C}^k(K;B)$ is continuous (because $\mathcal{C}^k(K;B) \subset \mathrm{L}^p(K;B)$, continuously). □

For $J = \mathbf{R}$, the above weak derivatives are, in fact, L^p derivatives (and vice versa):

Lemma B.7.8 *Let $f \in \mathrm{L}^p(\mathbf{R};B)$. Then the following are equivalent:*

(i) $f \in \mathrm{W}^{1,p}(\mathbf{R};B)$;

(ii) there is $g \in \mathrm{L}^p$ s.t. $f(t) = \int_0^t g\,dm + f(0)$ $(t \in \mathbf{R})$.

(iii) there is $\widetilde{g} \in \mathrm{L}^p$ s.t. $h^{-1}[\tau(h) - I]f \to \widetilde{g}$ in L^p, as $h \to 0$;

If (i)–(iii) hold, then $g = \widetilde{g} = f'$ a.e., $\partial f = g$, and f is locally absolutely continuous.

Proof: The equivalence (i)⇔(ii) follows from Lemma B.7.6.

1° "(iii)⇒(i)": Set $D_h := h^{-1}[\tau(h) - I]$. Let $\phi \in \mathcal{C}_c^\infty(\mathbf{R})$ be arbitrary. We have $D_h\phi \to \phi'$ uniformly, as $h \to 0$, hence $D_h\phi \to \phi'$ in L^q, where $1/p + 1/q = 1$. Therefore, by the Hölder inequality, we have the two convergences

$$\int_{\mathbf{R}} \widetilde{g}\phi\,dm \leftarrow \int_{\mathbf{R}} (D_h f)\phi\,dm = -\int_{\mathbf{R}} f(D_{-h}\phi)\,dm \to -\int_{\mathbf{R}} f\phi'\,dm, \tag{B.78}$$

as $h \to 0$. Therefore $\partial f = \widetilde{g}$; in particular, $f \in \mathrm{W}^{1,p}$.

2° "(ii)⇒(iii)": Assume (ii). Set $F_h(x) := [f(x+h) - f(x)]/h = h^{-1}\int_0^h g(x+t)\,dt$, so that $F_h(x) \to g(x)$ for a.e. x. Using the Minkovski integral inequality (Theorem B.4.16), we get for $\varepsilon > 0$ that (here L^p norm is taken w.r.t. x)

$$\|F_h(x) - g(x)\|_p := \|h^{-1}\int_0^h [g(x+t) - g(x)]\,dt\|_p \tag{B.79}$$

$$\leq h^{-1}\int_0^h \|g(x+t) - g(x)\|_p\,dt < \varepsilon, \tag{B.80}$$

when $|h| < \delta_\varepsilon$, by Lemma B.3.9. Therefore $F_h \to g$ in L^p, as required.

3° Assume (ii). The identity $g = \widetilde{g} = f'$ follows from 1°–3° (recall that ∂f is unique, by Theorem B.4.12(d)), the rest from Lemma B.7.6. □

The subset $\mathrm{W}_0^{1,p}$ of $\mathrm{W}^{1,p}$ refers to the elements that are "zero on the boundary" in some sense:

Lemma B.7.9 ($\mathbf{W_0^{1,p}}$) *Let $J \subset \mathbf{R}$ be an open interval and $p < \infty$. Then $\mathrm{W}^{1,p}(J;B) \subset \mathcal{C}_0(\bar{J};B)$, and $\mathrm{W}_0^{1,p}(J;B) = \{f \in \mathrm{W}^{1,p}(J;B) \mid f(\inf J) = 0 = f(\sup J)\}$.*

Thus, a $\mathrm{W}^{1,p}(J;B)$ function f has limits at endpoints of J; we have $f \in \mathrm{W}_0^{1,p}$ iff these limits are zero (they are necessarily zero at endpoints $\pm\infty$, if any).

Proof: 0° Let $\frac{1}{p} + \frac{1}{q} = 1$. Choose $a, b \in [-\infty, +\infty]$ s.t. $J = (a,b)$. We shall assume that $a = 0 < b$. By translation, this then extends to any $a \in (-\infty, b)$. Using reflection, we can then cover intervals of form $(a, +\infty)$. The case $J = \mathbf{R}$ follows from by Theorem B.7.4,

1° Let $f \in \mathrm{W}^{1,p}(J;B)$. For $n \in \mathbf{N}+1$, choose $b_n \in J$ s.t. $\mathrm{L}^p((b_n,b);B)$ norm of f and f' is less than $1/n$. By this, Lemma B.7.6 and the Hölder Inequality, we have

$$\|f(s) - f(t)\| \leq \int_s^t \|f'\|_B\,dm \leq n^{-1}|t-s|^{1/q} \quad (s,t \in (b_n,b)). \tag{B.81}$$

2° Assume that $b = \infty$. By (B.81) and the Hölder Inequality we have

$$\|f(t)\|_B - 1/n \leq \|\int_t^{t+1} f\|_B \leq n^{-1}\cdot 1, \tag{B.82}$$

i.e., $f(t) < 2/n$. Because $t > b_n$ was arbitrary, we have $f(t) \to 0$ as $t \to +\infty$.

3° Assume that $b < \infty$. By the Hölder Inequality, we have $f, f' \in \mathrm{L}^1(J;B)$, in particular, f has a limit at b and f is absolutely continuous on J.

4° By 2° and 3°, we have $\mathrm{W}^{1,p}(J;B) \subset \mathcal{C}_0(\bar{J};B)$.

5° Because $f \mapsto f(b)$ is continuous, by Theorem B.7.4, we have $0 = f(b)$ for $f \in \mathrm{W}_0^{1,p}$ for $b < \infty$; for $b = \infty$ this was shown in 2°. Analogously, $f(0) = 0$.

6° Now only the converse for 5° remains: We assume that $f \in \mathrm{W}^{1,p}(J;B)$ is s.t. $f(0) = 0 = f(b)$, and prove that $f \in \mathrm{W}_0^{1,p}$. By Theorem B.7.3, we may assume that $f \in \mathcal{C}^\infty(\Omega;B) \cap \mathrm{W}^{1,p}(J;B)$.

Let $\varepsilon \in (0,1)$. We shall construct $g \in W^{1,p}$ with $\operatorname{supp} g \subset \Omega$ and $\|g - f\|_{W^{1,p}} < \varepsilon$; then one can find $\phi \in C_c^\infty$ s.t. $\|\phi - g\|_{W^{1,p}} < \varepsilon$ by mollifying g (see [Adams]).

$6\frac{1}{3}^\circ$ *The case where $f = 0$ on $[b', b)$ for some $b' < b$:* Choose $\delta \in (0, \min(\varepsilon/5, b/2))$ s.t. $\|f\|_p, \|f'\|_p, \|f\|_\infty < \varepsilon/5$ over $(0,\delta)$. It follows that $\|f(\delta)\| \le \delta^{1/q}\varepsilon/5 \le \varepsilon/5 < 1$ (cf. (B.81). Choose $r \in (0,1)$ s.t. $r^{-1/q} < 2$. Set

$$g(t) := \begin{cases} 0, & t \in [0, (1-r)\delta]; \\ (1 - \frac{1}{r} + \frac{t}{r\delta}) f(\delta), & (t \in ((1-r)\delta, \delta)); \\ f(t), & t \in [\delta, b). \end{cases} \tag{B.83}$$

It follows that $g \in W^{1,p}$ and $g' = f(\delta)/r\delta$ on $((1-r)\delta, \delta))$ and $g' = f'$ on (δ, b). Moreover,

$$\|g'\|^p_{L^p((0,\delta);B)} < r\delta(\delta^{1/q}\varepsilon/5r\delta)^p = r^{1-p}\varepsilon^p/5^p, \tag{B.84}$$

hence $\|g'\|_{L^p((0,\delta);B)} < 2\varepsilon/5$. Obviously, $\|g\| \le \|f(\delta)\| < \varepsilon/5$ on $(0,\delta)$, hence $\|g\|_{L^p((0,\delta);B)} < \varepsilon/5$. It follows that $\|f - g\|_{W^{1,p}} < \varepsilon/5 + \varepsilon/5 + 2\varepsilon/5 + \varepsilon/5 = \varepsilon$, as required.

$6\frac{2}{3}^\circ$ *The general case:* If $b < \infty$, we can define g on $(b - \delta, b)$ in the same way as above, so that $\|f - g\|_{W^{1,p}} < 2\varepsilon$. Let $b = \infty$. Then we can take $\delta := \varepsilon/6$, replace "$b - \delta$" above by some $b' \in (1,\infty)$ s.t. $\|f\|_p, \|f'\|_p, \|f\|_\infty < \delta^{1/q}\varepsilon/5$ over (b', ∞), and go on as above. □

We set $W^{1,p}_\omega := \{f \in L^p_\omega \mid \partial f \in L^p_\omega\}$ for $\omega \in \mathbf{R}$ (the meaning of W_0 is apparent from the context), and $W^{1,p}_{0,\omega}$ denotes the closure of C_c^∞ in $W^{1,p}_\omega$.

Lemma B.7.10 ($\mathbf{W^{1,p}_\omega}$) *The mapping $T_\alpha : f \mapsto e^{\alpha\cdot} f$ is a Banach isorphism of $W^{k,p}_\omega$ onto $W^{k,p}_{\omega+\alpha}$ and of $W^{k,p}_{0,\omega}$ onto $W^{k,p}_{0,\omega+\alpha}$, and it is a bijection of C_c^∞ onto C_c^∞ and C^∞ onto C^∞.*

Moreover, Theorem B.7.3, Corollary B.7.7 and Lemmas B.7.5, B.7.6, B.7.8 and B.7.9 hold with replacements $L^p \mapsto L^p_\omega$, $W^{1,p} \mapsto W^{1,p}_\omega$, and $W^{1,p}_0 \mapsto W^{1,p}_{0,\omega}$ (except that if $f \in W^{1,1}_\omega(J;B)$, then $T_{-\omega} f$ is absolutely continuous, not necessarily f).

Naturally, if $f \in W^{j+m,p}_\omega$, then $e^{-\omega\cdot} f \in C^j_b$ as in Theorem B.7.4, etc. (but f itself need not be bounded).

Recall that by a (Banach) isomorphism $T : X \to Y$ we mean that $T \in \mathcal{GB}(X,Y)$; the above mapping does not map the derivatives of f to those of its image $T_\alpha f$.

Proof: 1° *Bijections:* Because $(e^{\omega\cdot} f)' = \omega e^{\omega\cdot} f + e^{\omega\cdot} f'$, we have $\|T_\alpha f\| \le (|\alpha| + 1)\|f\|$ and $(T_\alpha)^{-1} = T_{-\alpha}$, hence the $W^{1,p}_\omega$ claim holds. The claim on C_c^∞ and C^∞ is obvious, and the claim on $W^{1,p}_{0,\omega}$ follows from these two. Use induction for general k.

2° *Theorem B.7.3 and Lemma B.7.9:* These follow directly from 1°.

3° *Lemma B.7.5:* Let $f \in W^{k,p}_\omega(\Omega;B)$. Then $f \in W^{k,p}(\Omega';B)$ for each open, bounded $\Omega' \subset \Omega$, so the claim holds for $k = 1$; use induction for general k.

4° *Lemma B.7.6:* If $f \in \mathrm{L}^p_\omega(J;B)$, then $f \in \mathrm{L}^p((-T,T) \cap J;B)$ for each $T > 0$, hence Lemma B.7.6 holds (we first get $g \in \mathrm{L}^p_{\mathrm{loc}}$, but we must have $g = \partial f$, hence $f \in \mathrm{W}^{1,p}$ iff $g \in \mathrm{L}^p$).

5° *Lemma B.7.8 and Corollary B.7.7:* Modify the original proof accordingly. □

The shift is a continuous operation on $\mathrm{W}^{n,p}_\omega$:

Lemma B.7.11 *If $f \in \mathrm{W}^{n,p}_\omega(\mathbf{R};B)$, then $\tau f \in C^{n-j}(\mathbf{R};\mathrm{W}^{j,p}_\omega)$ $(j = 0,1,\ldots,n)$.* □

(This follows from Corollary B.3.8, Lemma B.7.8 and induction.)

In some examples, we shall use following semigroups:

Proposition B.7.12 *Let $p < \infty$ and $-\infty \leq a < b$, and set $J = (a,b)$. If $b = \infty$ (resp. $b < \infty$), then $\pi_J \tau \pi_J$ is a bounded C_0-semigroup on $\mathrm{L}^p_\omega(J;B)$, and its generator is the weak differentiation operator ∂ with domain $\mathrm{W}^{1,p}_\omega(J;B)$ (resp. with domain $\{f \in \mathrm{W}^{1,p}_\omega(J;B) \mid f(b) = 0\}$) and its resolvent $(\lambda - \partial)^{-1}$ $(\mathrm{Re}\,\lambda > \omega)$ maps $f \in \mathrm{L}^p_\omega(J;B)$ into the element $J \ni t \mapsto \int_t^b e^{\lambda(t-s)} f(s)\,ds$ of this domain.*

In particular, for $J = \mathbf{R}_-$ the domain is $\mathrm{W}^{1,p}_{0,\omega}(\mathbf{R}_-;B)$. □

See, e.g., Examples 3.2.3 and 3.3.2 of [Sbook] for the proof (except for the last claim, which follows from Lemmas B.7.9 and B.7.10). By using **Я** one obtains the dual results for $\pi_J \tau^* \pi_J$.

Notes

The contents of this section are well known, although it may be difficult to find any references, particularly for the vector-valued case.

Popular references for Sobolev spaces include [Adams] and [Ziemer] in the scalar case. Most of their results also hold in the vector-valued case with same proofs, mutatis mutandis.

Appendix C

Almost Periodic Functions (AP)

An aphorism is never exactly true; it is either a half-truth or one-and-a-half truths.
— Karl Kraus

In this appendix we briefly present a few facts about vector-valued almost periodic functions.

Because the Fourier transform of a discrete measure (e.g., of an I/O map consisting purely of delays) is an almost periodic function (by Lemma C.1.2(h2)), this has important applications in control theory. We shall use this theory to combine the $\mathrm{MTI_d}$ and $\mathrm{MTIC}^{\mathrm{L}^1}$ spectral factorization results for ones for MTI in Section 5.2 by using the results of this section.

Definition C.1.1 (AP) *Let B be a Banach space, and let $f : \mathbf{R} \to B$ be continuous. If $\varepsilon > 0$, then a number $T \in \mathbf{R}$ is called an* ε-almost period *of f if*

$$\sup_{t\in\mathbf{R}} \|f(t+T) - f(t)\| \leq \varepsilon. \tag{C.1}$$

The function f is called almost periodic *($f \in \mathrm{AP}(\mathbf{R};B)$) if for each $\varepsilon > 0$ there is $R > 0$ s.t. each interval $(r, r+R) \subset \mathbf{R}$ ($r \in \mathbf{R}$) contains at least one ε-almost period of f.*

Lemma C.1.2 (AP) *Let B and $B_1, \ldots, B_n$ be Banach spaces. Then we have the following:*

- *(a) The set $\mathrm{AP}(\mathbf{R};B)$ is a closed subspace of $\mathcal{C}_{\mathrm{bu}}(\mathbf{R};B)$ (with the norm $\|f\| := \sup_{\mathbf{R}} \|f\|_B$). If B is a Banach algebra, then so is $\mathrm{AP}(\mathbf{R};B)$.*
- *(b)* **Bochner's criterion** *A function $f \in \mathcal{C}_{\mathrm{b}}(\mathbf{R};B)$ is AP iff $\{f(\cdot - h)\}_{h\in\mathbf{R}}$ is compact in $\mathcal{C}_{\mathrm{b}}(\mathbf{R};B)$.*
- *(c) If $f_k \in \mathrm{AP}(\mathbf{R};B_k)$ for $k = 1, \ldots, n$, then $(f_k)_{k=1}^n \in \mathrm{AP}(\mathbf{R};B_1 \times \cdots \times B_n)$; if, in addition, $\phi \in \mathcal{C}(B_1 \times \cdots \times B_n;B)$, then $\phi(f_1, f_2, \ldots, f_n) \in \mathrm{AP}(\mathbf{R};B)$.*
- *(d1) If $f \in \mathrm{AP}(\mathbf{R};B)$, then $K := \overline{f[\mathbf{R}]}$ is compact.*
- *(d2) If f and K are as in (d1), and $\phi \in \mathcal{C}(K;B_1)$, then $\phi \circ f \in \mathrm{AP}(\mathbf{R};B_1)$.*

(e) **(AP is inverse-closed in $\mathcal{C}_b$:)** *If $f \in \mathrm{AP}(\mathbf{R};\mathcal{B}(B_1,B_2))$ and f is pointwise invertible on $\mathbf{R}$ with f^{-1} bounded (i.e., $f \in \mathcal{GC}_b$), then $f^{-1} \in \mathrm{AP}(\mathbf{R};\mathcal{B}(B_2,B_1))$ (i.e., $f \in \mathcal{G}\mathrm{AP}$).*

(f1) **(Bohr transformation)** *Let $f \in \mathrm{AP}(\mathbf{R};B)$. For each $\lambda \in \mathbf{R}$, the limit*

$$f_\lambda := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\mathrm{e}^{-i\lambda t}\,dt \in B \tag{C.2}$$

exists, the values f_λ are bounded by $\sup_{\mathbf{R}} \|f\|_B$ and determine $f \in \mathrm{AP}$ uniquely, at most countably many of them are nonzero, and 0 is the only limit point of the set $\{f_\lambda\}_{\lambda\in\mathbf{R}}$ (called the AP-spectrum *of f).*

We formally denote $f \sim \sum_{\lambda\in\mathbf{R}} f_\lambda \mathrm{e}^{i\lambda t}$.

(f2) The function $f \mapsto f_\lambda$ is linear and of norm 1.

(f3) If $f \in \mathrm{AP}(\mathbf{R};B_1)$, $g \in \mathrm{AP}(\mathbf{R};B_2)$, $\phi \in \mathcal{B}(B_1 \times B_2, B)$, and either $\sum_{\lambda\in\mathbf{R}} \|f_\lambda\|_{B_1} < \infty$ or $\sum_{\lambda\in\mathbf{R}} \|g_\lambda\|_{B_2} < \infty$ (in particular, when either AP-spectrum is finite), then for $\lambda \in \mathbf{R}$ we have

$$\phi(f,g)_\lambda = \sum_{\mu\in\mathbf{R}} \phi(f_\mu, g_{\lambda-\mu}), \tag{C.3}$$

In particular, if $B = B_1 = B_2$ is a Banach algebra, then $(fg)_\lambda = \sum_{\mu\in\mathbf{R}} f_\mu g_{\lambda-\mu}$ for $\lambda \in \mathbf{R}$.

(f4) If H is a Hilbert space, then $\|f\|_{\overline{\mathrm{AP}}} := \sum_\lambda \|f_\lambda\|_H^2 = (\|f\|_H^2)_0 \le \sup_{\mathbf{R}} \|f\|_H^2$, and $(f,g) := (\langle f,g\rangle_H)_0$ makes $\mathrm{AP}(\mathbf{R};H)$ an inner product space (i.e., a pre-Hilbert space), whose completion "$\overline{\mathrm{AP}}$" is isomorphic to $\ell^2(\mathbf{R};H)$. (Note that $\|f\| := \|f\|_\infty$, and that we write $\|f\|_{\overline{\mathrm{AP}}}$ when we refer to this Besicovitch space norm.*)*

(f5) Let $F \in \mathrm{AP}(\mathbf{R};\mathcal{B}(B,B_2))$. Then, in $\overline{\mathrm{AP}}$, $f \mapsto Ff$ has the norm $\|F\|_{\mathcal{B}(\overline{\mathrm{AP}}(\mathbf{R};B),\overline{\mathrm{AP}}(\mathbf{R};B_2))} \le \|F\|_\infty$.

(g) If $f \in \mathrm{AP}(\mathbf{R};B)$ and $\varepsilon > 0$, then there are $n \in \mathbf{N}$, $b_k \in B$, $\lambda_k \in \mathbf{R}$ $(k = 1,\ldots,n)$ s.t. $\|f - \sum_{k=1}^n b_k \mathrm{e}^{i\lambda_k \cdot}\|_\infty < \varepsilon$ and $f_{\lambda_k} \neq 0$ for $k = 1,\ldots,n$.

(h1) Periodic functions are almost periodic.

(h2) Let $(a_\lambda)_{\lambda\in\mathbf{R}} \in \ell^1(\mathbf{R};B)$. Then $f := \sum_{\lambda\in\mathbf{R}} a_\lambda \mathrm{e}^{i\lambda t}$ is in $\mathrm{AP}(\mathbf{R};B)$ and $f_\lambda = a_\lambda$ for all $\lambda \in \mathbf{R}$ (this characterizes f uniquely, by (f1)).

(i) The class of treated in (h2) is inverse-closed AP *and in* $\mathcal{C}_b$.

Proof: (a)–(d2) We have $\mathrm{AP} \subset \mathcal{C}_{bu}$, by Property 2 of [LZ, Section 1.1]. By Property 6, AP is a vector space, and by Property 3 it is closed. By Properties 7, 4 and 6, claims (c), (d1) and (d2) hold, respectively; by Theorem 1.2.1, (b) holds.

By Property 7, the first claim in (c) holds; combine this with (d2) to observe that $\phi \circ (f_1, f_2, \ldots, f_n) \in \mathrm{AP}(\mathbf{R};B)$, i.e., that the rest of (c) holds too.

Now (c) implies that the rest of (a) holds.

(e) This follows from (d2), because $K \subset \mathcal{GB}$ by Lemma A.3.3(A3).

Part (f1) follows from [LZ, pp. 22–24]; part (f2) is obvious.

(f3) Clearly $[\phi(ae^{i\mu t}, be^{i\nu t})]_\lambda = \phi(a,b)(e^{i(\mu+\nu)t})_\lambda = \phi(a,b)\chi_{\{\lambda\}}(\mu+\nu)$, when $\lambda, \mu, \nu \in \mathbf{R},\ a \in B_1,\ b \in B_2$.

Therefore, (C.3) holds for f, g with a finite AP-spectrum. For such a fixed g, both sides of (C.3) are continuous functions of f from AP to B (because the sum on the right is finite) for each $\lambda \in \mathbf{R}$, hence (C.3) holds whenever the AP-spectrum of g is finite, by (g).

If $\sum_{\lambda\in\mathbf{R}} \|f_\lambda\| < \infty$, then (C.3) holds for each g with a finite AP-spectrum and both sides of (C.3) are a continuous function of g, hence (C.3) holds for any g.

The case for $\sum_{\lambda\in\mathbf{R}} \|g_\lambda\| < \infty$ follows by exchanging the roles of f and g.

(We have no need to study whether these ℓ^1 conditions are necessary; at least they are not when B is a Hilbert space: Choose $P_n \to f$ as in (g). By (f4), $(g_\lambda)_{\lambda\in\mathbf{R}} \in \ell^2$ and $((P_n)_\lambda - f_\lambda)_{\lambda\in\mathbf{R}} = (((P_n - f)_\lambda))_{\lambda\in\mathbf{R}} \to 0$ in ℓ^2, hence also the right side of equation (C.3) with P_n in place of f converges as $n \to \infty$ (because $\|\phi(f_\mu, g_{\lambda-\mu})\|_B \le M\|f_\mu\|\|g_{\lambda-\mu}\|$ for some $M < \infty$).)

(f4) The first claim is given on p. 31 of [LZ] and it implies the isomorphism to $\ell^2(\mathbf{R};B)$ (the trigonometric polynomials map naturally to a dense subspace of $\ell^2(\mathbf{R};B)$); the rest is obvious (note that $\langle f, g\rangle \in$ AP by (c) and (d2)).

(f5) Let $f \in$ AP. By (c) and (d2), we have $Ff \in \overline{\text{AP}}$. Obviously, $(\|Ff\|_H^2)_0 \le \|F\|_\infty^2(\|f\|_H)_0^2$. Since $f \in$ AP was arbitrary, we conclude from (f4) that $\|F\|_{\mathcal{B}(\overline{\text{AP}}(\mathbf{R};B),\overline{\text{AP}}(\mathbf{R};B_2))} \le \|F\|_\infty$.

(g) See pages 17–18 and 24 of [LZ].

(h1) This is obvious (take $R = 2T$ in Definition C.1.1).

(h2) By (c), we have $a_k e^{it_k\cdot} \in \text{AP}(\mathbf{R};B)$ for all k, hence $\sum_{k\in\mathbf{N}} a_k e^{it_k\cdot} \in \text{AP}(\mathbf{R};B)$, by (a).

(i) Combine Wiener's Lemma (11.6 of [Rud73] or Theorem 4.1.1(a) for MTI_d) with (c). □

Notes

All results in this section are well known. A canonical reference on almost periodic functions is [LZ]. Further questions have been treated in [Basit] and [Zhang] and in the articles that they refer to.

Appendix D

Laplace and Fourier Transforms ($\mathcal{L}u = \widehat{u}$)

The meeting of two personalities is like the contact of two chemical substances: if there is any reaction, both are transformed.

— Carl Jung (1875–1961)

In this appendix, we mainly study holomorphic vector-valued functions. This includes H^p spaces, Laplace and Fourier transforms and Poisson integral formulae. We also present some results on convolutions.

Throughout this chapter, B, B_2 and B_3 are complex Banach spaces, U, H and Y are complex Hilbert spaces, $\mathbf{K} = \mathbf{C}$, and $\Omega \subset \mathbf{C}$ is open.

A function $f : \Omega \to B$ is *holomorphic* ($f \in \mathrm{H}(\Omega; B)$) if the (complex) *derivative*

$$f'(s) := \lim_{h \to 0} \frac{f(s+h) - f(s)}{h} \tag{D.1}$$

of f exists at each $s \in \Omega$. By $f^{(k)}$ we denote the kth derivative of f. The Banach space $\mathrm{H}^\infty(\Omega; B)$ of bounded holomorphic funcitons is defined by

$$\mathrm{H}^\infty(\Omega; B) := \{ f \in \mathrm{H}(\Omega; B) \bigm| \|f\|_{\mathrm{H}^\infty} := \sup_{s \in \Omega} \|f(s)\|_B < \infty \}. \tag{D.2}$$

We start with the basic properties of holomorphic functions:

Lemma D.1.1 (Weakly holomorphic ⇒ holomorphic) *Let $\Omega \subset \mathbf{R}$ be open, $f : \Omega \to B$, and $F : \Omega \to \mathcal{B}(B, B_2)$.*

(a) If $\Lambda f \in \mathrm{H}(\Omega)$ for all $\Lambda \in B^*$, then $f \in \mathrm{H}(\Omega; B)$.

We may replace B^* by any $A \subset B^*$ satisfying $\|x\|_B = \sup\{|\Lambda x| \bigm| \Lambda \in \mathrm{span}(A),\ \|\Lambda\| \le 1\}$ for all $x \in B$.

(b) If $\Lambda F(\cdot)b \in \mathrm{H}(\Omega)$ for all $b \in B$, $\Lambda \in B_2^*$, then $F \in \mathrm{H}(\Omega; \mathcal{B}(B, B_2))$.

(c) If $\Lambda F(\cdot)b \in \mathrm{H}^\infty(\Omega)$ for all $b \in B$, $\Lambda \in B_2^*$, then $F \in \mathrm{H}^\infty(\Omega; \mathcal{B}(B, B_2))$.

(d) If $G \in \mathcal{B}(B, \mathcal{B}(B_2^*, \mathrm{H}^\infty(\Omega)))$ (resp. $G \in \mathcal{B}(B, \mathrm{H}^\infty(\Omega; B_2))$) and $\Lambda G(\cdot)b \in \mathrm{H}^\infty(\Omega)$ for all $b \in B$, $\Lambda \in B_2^*$, then $G \in \mathrm{H}^\infty(\Omega; \mathcal{B}(B, B_2^{**}))$ (resp. $G \in \mathrm{H}^\infty(\Omega; \mathcal{B}(B, B_2))$).

Thus, if f is weakly differentiable on Ω, then it is differentiable on Ω.

Proof: (a)&(b) These follow from [HP, Theorem 3.10.1]; note that, by (a), we can replace B and B_2^* in (b) by $A \subset B$ and $A_2 \subset B_2^*$ as in (a). (Span means the set of linear combinations.)

(c) If $\Lambda F(\cdot)b \in \mathrm{H}^\infty(\Omega)$ for all $b \in B$, $\Lambda \in B_2^*$, then F is uniformly bounded, by the uniform boundedness theorem (fix first Λ, then b).

(d) 1° Assume that $G \in \mathcal{B}(B, \mathcal{B}(B_2^*, \mathrm{H}^\infty(\Omega)))$. Define $g(\cdot) \in \mathcal{B}(B, B_2^{**})$ by $(g(s)b)\Lambda := (Gb\Lambda)(s) \in \mathbf{C}$. Then $\sup_\Omega \|g\| = \|G\|_{\mathcal{B}}$ (obviously) and $g \in \mathrm{H}^\infty$, by (c). Naturally, we can and will identify G and g.

2° Assume that $G \in \mathcal{B}(B, \mathrm{H}^\infty(\Omega; B_2))$. Then $G \in \mathrm{H}^\infty(\Omega; \mathcal{B}(B, B_2^{**}))$, by 1°. But $G(s)b\Lambda = \Lambda(Gb(s))$ for all b, Λ, s, hence $G(s)b = Gb(s) \in B_2$, for any $s \in \Omega$, $b \in B$. Thus, $G \in \mathrm{H}^\infty(\Omega; \mathcal{B}(B, B_2))$.

(Note analogous altered versions of (a) and (b) also hold with, e.g., $f \in \mathcal{B}(B, \mathcal{B}(B_2^*, \mathrm{H}(\Omega)))$ in (a), where $\mathrm{H}(\Omega)$ equipped with the topology of uniform convergence on compact subsets.) $\square$

Next we extend the standard properties of scalar holomorphic functions to the vector-valued case:

Lemma D.1.2 (Holomorphic functions) *Let $s_0 \in \Omega$. For $r \in [0, \infty]$ we set $D_r(s_0) := \{s \in \mathbf{C} \,\big|\, |s - s_0| < r\}$.*

Let $f, g \in \mathrm{H}(\Omega; B)$. Then $f' \in \mathrm{H}(\Omega; B)$, $f \in C^\infty(\Omega, B)$, and the Cauchy integral formula applies; in particular, if Γ is a closed path in Ω, and the index of Γ around $s_0 \in \Omega$ is 1 (see [Rud73, p. 79]), then

$$\int_\Gamma f(s)\,ds = 0 \quad \textit{and} \quad f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\Gamma \frac{f(z)\,dz}{(z - z_0)^{k+1}} \in H \ \textit{for all } k \in \mathbf{N}_0. \quad \text{(D.3)}$$

Moreover, we have the following:

(a) If $\{h_n\} \subset \mathrm{H}(\Omega; B)$ and $h_n \to h$ uniformly on compact subsets of Ω, then $h \in \mathrm{H}(\Omega; B)$ and, for each $k \in \mathbf{N}$, $h_n^{(k)} \to h^{(k)}$ uoc on Ω.

(b1) If $T \in \mathcal{B}(B, B_2)$, then $Tf \in \mathrm{H}(\Omega; B_2)$ and $(Tf)'(s) = T(f'(s))$ for all $s \in \Omega$.

(b2) If $F \in \mathrm{H}(\Omega; \mathcal{B}(B, B_2))$ and $F(s) \in \mathcal{GB}(B, B_2)$ for some $s \in \Omega$, then F^{-1} is analytic on a neighborhood of s, and $(F^{-1})'(s) = -F(s)^{-1}F'(s)F(s)^{-1}$.

(b3) If $F \in \mathrm{H}(\Omega; \mathcal{B}(B_2, B_3))$, then $Ff \in \mathrm{H}(\Omega; B_3)$ and $(Ff)' = F'f + Ff'$.

(b4) If $\phi \in \mathrm{H}(\Omega', \Omega)$, then $f \circ \phi \in \mathrm{H}(\Omega', B)$.

(b5) Let $F, G \in C(\Omega; B)$. We have $F \in \mathrm{H}(\Omega; B)$ and $F' = G$ iff $F(z) - F(a) = \int_{[a,z]} G\,dm$ whenever $[a, z] \subset \Omega$ (here $[a, z] := \{(1-t)a + tz \,\big|\, t \in [0, 1]\}$).

(b6) If $F \in \mathrm{H}(\Omega; \mathcal{B}(U, Y))$, then $F(\bar{\cdot})^* \in \mathrm{H}(\Omega; \mathcal{B}(Y, U))$.

(c) **(Liouville)** *If $\Omega = \mathbf{C}$ and Λf is bounded for each $\Lambda \in B^*$, then f is a constant (in B).*

(d) **(Morera)** *If(f) $F \in C(\Omega; B)$ and $\int_\gamma F(s)\,ds = 0$ whenever $\gamma = \partial([x_1, x_2] \times [y_1, y_2])$ (a rectangle whose sides are parallel to the coordinate axes), then $F \in \mathrm{H}(\Omega; B)$.*

(e) **(Analytic continuation)** *If Ω is connected, and $f = g$ on a set $A \subset \Omega$ having a limit point in Ω, then $f = g$ on Ω.*

(f) **(Maximum modulus principle)** *If Ω is bounded and $f \in \mathcal{C}(\bar{\Omega};B) \cap \mathrm{H}(\Omega;B)$, then $\|f\|_{\mathrm{H}^\infty} = \sup_{\partial\Omega} \|f(\cdot)\|_B$.*

Moreover, if this is the case, then the values of f on Ω can be obtained by the Poisson integral formula.

(g1) Let $\{b_n\} \subset B$, and set $\rho := \left(\limsup_{n\in\mathbf{N}} \|b_n\|^{1/n}\right)^{-1}$. Then $\sum_{n\in\mathbf{N}} b_n(s - s_0)^n =: F(s)$ converges absolutely and uoc to a function $F \in \mathrm{H}(D_\rho(s_0);B)$. Moreover, $F^{(k)}(s_0) = k!a_k$ for all k.

(g2) **(Taylor series)** *Let $F \in H(D_R(z_0),B)$ and $r < R$. Then $\|F^{(k)}(s_0)\| \le k!Mr^{-k}$, where $M := \sup_{D_r(z_0)} \|F(\cdot)\|$, and*

$$F(s) = \sum_{k\in\mathbf{N}} \frac{F^{(k)}(s_0)}{k!}(s-s_0)^k \quad \textit{when } |s-s_0| < R; \tag{D.4}$$

this presentation is unique.

(h) If $h \in \mathrm{H}(\Omega \setminus \{s_0\};B)$ is bounded on a neighborhood of s_0, then $b := \lim_{s\to s_0} h(s)$ exists and $h \in \mathrm{H}(\Omega;B)$ if we set $h(s_0) := b$ (such points are often called removable singularities).

(i) Let $B \subset B_2$ continuously. Then $\mathrm{H}(\Omega;B) = \mathcal{C}(\Omega;B) \cap \mathrm{H}(\Omega;B_2)$.

(j) If $F \in \mathrm{H}(\Omega;B)$ and $F(s_0) = 0$, then $F/(s-s_0) \in \mathrm{H}(\Omega;B)$.

The fact (h) is equivalent to $\mathrm{H}^\infty(\Omega \setminus \{s_0\};B) = \mathrm{H}^\infty(\Omega;B)$ (for arbitrary open sets Ω).

Proof: The first claims follow from [HP, Theorem 3.10.1], [Rud73, Theorem 3.31] and induction (alternatively, by applying the corresponding scalar claims to Λf, $\Lambda \in B^*$).

Claims (a), (e), (g1) and (g2) follow from [HP, pp. 96–100].

Claims (b1) and (b3) are obvious, claims (b4), (c), (d) and (f) follow from corresponding scalar claims (in (c) we also need uniform boundedness theorem, in (d) also Lemma D.1.1).

(b2) (N.B. This proof applies also if $\mathcal{B}(B,B_2)$ is replaced by a Banach algebra.) By Lemma A.3.3(A2), F^{-1} exists on a neighborhood of s. Therefore,

$$F(s+h)\Big(\frac{F(s+h)^{-1} - F(s)^{-1}}{h} + F(s)^{-1}F'(s)F(s)^{-1}\Big)F(s) \tag{D.5}$$

$$= \frac{F(s) - F(s+h)}{h} + F(s+h)F(s)^{-1}F'(s) \to 0, \tag{D.6}$$

as $h \to 0$, because $F(s+h)F(s)^{-1}F'(s) \to F'(s)$, by continuity, and $\frac{F(s)-F(s+h)}{h} \to -F'(s)$. To remove the outer terms from (D.5), multiply it by $F(s+h)^{-1}$ to the left and by $F(s)^{-1}$ to the right, and use Lemma A.3.1(j3) (see Lemma A.3.4(F1)).

(b5) 1° *"If":* Let $a \in \Omega$. Choose $\delta > 0$ s.t. $|z-a| < \delta \Rightarrow z \in \Omega$ & $\|G(z) - G(a)\|_B < \varepsilon$. Then $(z-a)^{-1}(F(z) - F(a)) - G(a) < \varepsilon$, for $|z-a| < \delta_\varepsilon$, hence $F'(a) = G(a)$. Because $a \in \Omega$ was arbitrary, we have $F \in \mathrm{H}$ and $F' = G$.

2° *"Only if":* This follows from the scalar case.

(b6) Now $\frac{F(\bar{s})^*-F(\bar{s}_0)^*}{s-s_0} = \big(\frac{F(\bar{s})-F(\bar{s}_0)}{\bar{s}-\bar{s}_0}\big)^* \to F'(\bar{s}_0)^*$, as $s \to s_0$.

(h) By the scalar case, Λh extends to $\mathrm{H}(\Omega;\mathbf{C})$ for all $\Lambda \in B^*$. The operator $h(s_0) : \Lambda \mapsto (\Lambda h)(s_0)$ is linear and bounded, hence $h \in \mathrm{H}(\Omega;B^{**})$, by Lemma D.1.1(b). By continuity, $h(s_0) \in B$.

(i) We have $\mathrm{H}(\Omega;B) \subset \mathcal{C}(\Omega;B) \cap \mathrm{H}(\Omega;B_2)$, by (b1). Conversely, if $F \in \mathcal{C}(\Omega;B) \cap \mathrm{H}(\Omega;B_2)$ then $\int_\gamma F(s)\,ds = 0$ in B_2, hence in B, by (B.18) applied to $I : B \to B_2$, when γ is as in (d). Therefore $F \in \mathrm{H}(\Omega;B)$.

(j) We have $\Lambda F/(s-s_0) \in \mathrm{H}(\Omega;B)$ for all $\Lambda \in B^*$, by Theorem 10.18 of [Rud86], hence $F/(s-s_0) \in \mathrm{H}(\Omega;B)$, by Lemma D.1.1. □

We will use the following notation:

Definition D.1.3 ($\mathbf{H}^p$, $\mathbf{L}^p_r$, $\mathbf{H}^p_r$, $\mathbf{C}_J$, $\mathbf{C}^\pm_r$, $\mathbf{C}_{a,b}$) *Let $1 \le p \le \infty$, $-\infty \le a < b \le \infty$, and $r \in \mathbf{R}$, let $J \subset \mathbf{R}$ be an interval and let $\Omega \subset \mathbf{R}$ be open. Set $\mathbf{C}^\pm_r := \pm\{s \in \mathbf{C} \mid \operatorname{Re} s > r\}$, $\mathbf{C}_{a,b} := \{s \in \mathbf{C} \mid a < \operatorname{Re} s < b\}$,*

$$\mathrm{L}^p_r(J;B) := \mathrm{e}^{r\cdot}\mathrm{L}^p(J;B) = \{f : J \to B \mid \mathrm{e}^{-r\cdot}f(\cdot) \in \mathrm{L}^p(J;B)\}, \tag{D.7}$$

$$\mathrm{H}^p_r(B) := \mathrm{H}^p(\mathbf{C}^+_r;B) := \{g \in \mathrm{H}(\mathbf{C}^+_r;B) \mid \|g\|_{\mathrm{H}^p_r(B)} < \infty\}, \textit{ where} \tag{D.8}$$

$$\|g\|_{\mathrm{H}^p_r(B)} := \sup_{r'>r} \|g(r'+i\cdot)\|_{\mathrm{L}^p(\mathbf{R};B)} = \lim_{r'\to r+} \|g(r'+i\cdot)\|_{\mathrm{L}^p(\mathbf{R};B)}. \tag{D.9}$$

We also set $\mathrm{H}^p := \mathrm{H}^p_0$, $\mathrm{H}^p_\infty := \cup_\omega \mathrm{H}^p_\omega$, $\mathbf{C}^\pm := \mathbf{C}^\pm_0$, $\|f\|_{\mathrm{L}^p_r} := \|\mathrm{e}^{-r\cdot}f\|_p$. If $B = U$ is a Hilbert space, then, in H^2_r, we define the inner product

$$\langle f,g\rangle_{\mathrm{H}^2_r} := \lim_{r'\to r+} \langle f(r'+i\cdot), g(r'+i\cdot)\rangle_{\mathrm{L}^2(\mathbf{R};U)}. \tag{D.10}$$

The spaces $\mathrm{H}^p(r\mathbf{D};B)$ ($r > 0$) are defined analogously, with $\|g\|_{\mathrm{H}^p(\mathbf{D}_r;B)} := \sup_{r'<r} \|g(r'\mathrm{e}^{i\cdot})\|_{\mathrm{L}^p([0,2\pi);B)}$.

Finally, $\|g\|_{\mathrm{H}^p(\mathbf{C}_J;B)} := \sup_{r\in J} \|g(r+i\cdot)\|_{\mathrm{L}^p(\mathbf{R};B)}$ defines (and norms) a subspace of $\mathrm{H}(\mathbf{C}_J;B)$, where $\mathbf{C}_J := \{s \in \mathbf{C} \mid \operatorname{Re} s \in J\}$, if J is open.

Recall that $r\mathbf{D} = \mathbf{D}_r := \{z \in \mathbf{C} \mid |z| < r\}$. Note that $\mathbf{C}_{\mathbf{R}_\pm} = \mathbf{C}^\pm$, $\mathbf{C}_{(a,b)} = \mathbf{C}_{a,b}$, and that $F \in \mathcal{G}\mathrm{H}^p_\infty$ iff there is $\omega \in \mathbf{R}$ s.t. $F \in \mathcal{G}\mathrm{H}^p_\omega$.

With the aid of Section 6.2 of [HP], one easily verifies that Definition D.1.3 is correct, that the above inner product induces the original $\|\cdot\|_{\mathrm{H}^2}$ norm, and that the above definitions of L^p_0 and H^∞_r coincide with their previous definitions. Note also that we use the (one-dimensional) Lebesgue measure on $i\mathbf{R}$ and $[0,2\pi)$ (in particular, we have no 2π-normalization).

Note that each $\mathrm{H}^p(\mathbf{C}^+_\omega)$ result has a "mirror image result" for $\mathrm{H}^p(\mathbf{C}^-_{-\omega})$, because $f \mapsto f(-\cdot)$ is an isometric isomorphism between these two spaces ($p \in [1,\infty]$, $\omega \in \mathbf{R}$).

Lemma D.1.4 ($\mathbf{L}^p_r$, $\mathbf{H}^p_r$) *Let $1 \le p_1 \le p \le p_2 \le \infty$, and $-\infty < r < r' < \infty$, and let $\Omega \subset \mathbf{C}$ be open. Then the following holds:*

(a) The spaces L^p_r, $\mathrm{H}^\infty(\Omega;B)$ and H^p_r are Banach spaces; in particular, $\mathrm{H}^2(\mathbf{C}^+_r;U)$ is a Hilbert space.

By Theorem 3.3.1(b), we can use the boundary functions of f and g to write $\langle f,g\rangle_{\mathrm{H}^2_r} = \langle f,g\rangle_{\mathrm{L}^2(r+i\mathbf{R};U)}$.

(b1) The mapping $f \mapsto \mathrm{e}^{a\cdot} f$ *is an isometric isomorphism of* L^p_r *onto* L^p_{r+a}*. The mapping* $g \mapsto g(\cdot - a)$ *is an isometric isomorphism of* H^p_r *onto* H^p_{r+a}*.*

(b2) If J is bounded, then $\mathrm{L}^p_r = \mathrm{L}^p = \mathrm{L}^p_{r'}$ *with equivalent norms. If* $p < \infty$*, then* $\mathrm{L}^p_r(J;B) = \mathrm{L}^p(J,\mu_r;B)$ *with equal norms, where* $d\mu_r = \mathrm{e}^{-rp\cdot}\,dm$*, for* $p < \infty$*.*

(b3) If $f_n \to f$ *in* L^p_r *and* $f_n \to g$ *in* $\mathrm{L}^{p_2}_\omega$ *(or* $f_n \to g$ *pointwise a.e.) for some functions f and g and some* $\omega \in \mathbf{R}$*, then* $f = g$ *a.e.*

(b4) We have $\mathrm{L}^{p_2}_r(\mathbf{R}_+;B) \underset{c}{\subset} \mathrm{L}^p_{r'}(\mathbf{R}_+;B)$*, indeed,* $\|f\|_{\mathrm{L}^p_{r'}} \le M_{p,p_2,r'-r}\|f\|_{\mathrm{L}^{p_2}_r}$ *for all* $f \in \mathrm{L}^{p_2}_r(\mathbf{R}_+;B)$*.*

(c) If $g \in \mathrm{H}^\infty(\mathbf{C}^+_r;B)$ *is continuous to the boundary* $r + i\mathbf{R}$*, then* $\|g\|_{\mathrm{H}^\infty_r} = \sup_{r+i\mathbf{R}}\|g(\cdot)\|_B$*.*

If $B = U$ *or* $B = \mathcal{B}(U,Y)$*, then Theorem 3.3.1(a2)&(c1)&(c2)provide analogous results for an arbitrary* $g \in \mathrm{H}^\infty_r$*.*

(d) We have $\mathrm{H}^p_r(B) \underset{c}{\subset} \mathrm{H}^{p_2}_{r'}(B)$ *and* $\mathrm{H}^{p_1}_r(B) \cap \mathrm{H}^{p_2}_r(B) \underset{c}{\subset} \mathrm{H}^p_r(B)$*.*

(e) $(\mathbf{H}^{\boldsymbol{p}}(\boldsymbol{r}\mathbf{D};\boldsymbol{B}))$ *Results analogous to (a), (c) and (d) hold for* $\mathrm{H}^p(r\mathbf{D};B)$ *too, and the mapping* $g \mapsto g(t\cdot)$ *is an isometric isomorphism of* $\mathrm{H}^p(r\mathbf{D};B)$ *onto* $\mathrm{H}^p(tr\mathbf{D};B)$*.*

(f) Let $f \in \mathrm{H}^p(\mathbf{C}^+_\omega;B)$*,* $1 \le p < \infty$*,* $\omega \in \mathbf{R}$*,* $\varepsilon > 0$*. Then* $\sup_{|\theta|\le\pi/2}\|f(\omega+\varepsilon+r\mathrm{e}^{i\theta})\|_B \to 0$ *as* $r \to \infty$*.*

See also Lemma F.3.2.

Proof: (a) For L^p_r this follows from (b1). A H^∞-Cauchy sequence is a pointwise uniformly a Cauchy-sequence, hence it converges uniformly, and the limit is holomorphic, by Lemma D.1.2(a). Thus $\mathrm{H}^\infty(\Omega;B)$ is complete.

A H^p_r-Cauchy sequence $\{f_n\}$ converges to some function f in each $\mathrm{L}^p(t+i\mathbf{R};B)$ $(t > r)$. By (6.4.3) of [HP] (applied to some $\sigma \in (r,t)$; cf. Theorem 3.3.1(a3)), this convergence is uniform on $\mathbf{C}^+_t$ for each $t > r$), hence f is holomorphic. Obviously, $f_n \to f$ in H^p_r.

(b1)&(b2) These are obvious.

(b3) By Theorem B.3.2, there are $n_1 < n_2 < n_3 < \cdots$ s.t. $f_{n_k} \to f$ and $f_{n_k} \to g$ pointwise a.e., hence $f = g$ a.e.

(b4) By (b1), we may assume that $r' = 0$. Assume that $r' = 0 > r$. Then

$$\|f\|_p \le \|\mathrm{e}^{-r\cdot}f\|_{p_2}\|\mathrm{e}^{r\cdot}\|_q \le M_{p,p_2,r}\|f\|_{\mathrm{L}^{p_2}_r} \tag{D.11}$$

by Lemma B.3.13, where $M_{p,p_2,r} := \|\mathrm{e}^{r\cdot}\|_q < \infty$, $q^{-1} := p^{-1} - p_2{}^{-1} \in [1,\infty]^{-1}$.

(c) This follows from the scalar case (alternatively, from the Maximum modulus principle by using a conformal map of $\mathbf{C}^+_r \to \mathbf{D}_{1+\varepsilon}$).

(d) The latter claim follows easily from the fact that $\mathrm{L}^{p_1} \cap \mathrm{L}^{p_2} \underset{c}{\subset} \mathrm{L}^p$, by Lemma E.1.1.

We have a continuous embedding $\mathrm{H}^p_r(B) \subset \mathrm{H}^\infty_{r'}(B)$, by Theorem 6.4.2 of [HP] (shift it by $(r'-r)/2$); obviously also inclusion $\mathrm{H}^p_r(B) \subset \mathrm{H}^p_{r'}(B)$ is continuous, hence

$$\mathrm{H}^p_r(B) \underset{c}{\subset} \mathrm{H}^p_{r'}(B) \cap \mathrm{H}^\infty_{r'}(B) \underset{c}{\subset} \mathrm{H}^{p_2}_{r'}(B). \tag{D.12}$$

(e) The above proofs apply mutatis mutandis (see also Theorem 3.3.1).

(f) This follows from Theorem 6.4.2 of [HP] (replace ω by $\omega + \varepsilon/2$ and ε by $\varepsilon/2$ to overcome the nonstandard assumption (iii) of Definition 6.4.1 of [HP]; cf. Theorem 3.3.1(a3)). □

The norm of a holomorphic function is greatest on the boundary (cf. Lemma D.1.2(f)):

Lemma D.1.5 (Hadamard Three Line Theorem) *Let* $f \in \mathcal{C}_\mathrm{b}(\overline{\mathbf{C}_{a,b}};B) \cap \mathrm{H}(\mathbf{C}_{a,b};B)$*. Set* $M_r := \sup \|f(r+i\mathbf{R})\|_B$ $(r \in [a,b])$*. Then*

$$M_r \le M_a^{1-\theta_r} M_b^{\theta_r} \le \max\{M_a, M_b\} \quad (r \in [a,b]), \tag{D.13}$$

where $\theta_r := (r-a)/(b-a)$.

Use the mapping $s \mapsto \mathrm{e}^s$ to obtain the Hadamard Three Circle Theorem (cf. p. 264 of [Rud86]).

Proof: For $B = \mathbf{C}$, $a = 0$ and $b = 1$, this is Lemma 1.1.2 of [BL]. By setting $g(s) := f((b-a)s + a)$ we obtain the scalar version of the theorem. For a general f, we then have $\|\Lambda f(r+it)\| \le M_a^{1-\theta_r} M_b^{\theta_r}$ for all $r + it \in \overline{\mathbf{C}_{a,b}}$ and $\Lambda \in B^*$ s.t. $\|\Lambda\| \le 1$; the general claim follows from this. □

Definition D.1.6 (Convolution) *Let* $B \times B_2 \to B_3$ *be continuous and bilinear. The* convolution $f * g$ *of* $f : \mathbf{R}^n \to B$ *and* $g : \mathbf{R}^n \to B_2$ *is defined by*

$$(f*g)(t) := \int_{\mathbf{R}^n} f(t-r)g(r)\,dr = \int_{\mathbf{R}^n} f(r)g(t-r)\,dr \in B_3, \tag{D.14}$$

for those $t \in \mathbf{R}^n$ *for which* $f(t-\cdot)g(\cdot) \in \mathrm{L}^1(\mathbf{R}^n;B_3)$.

If, e.g., f (resp. g) is defined on $\Omega \subset \mathbf{R}^n$ only, we define the above convolution by declaring $f = 0$ (resp. $g = 0$) outside Ω.

Here $B = B_2 = B_3$ might be a Banach algebra, but even more common is the case where $B = \mathcal{B}(B_2, B_3)$ (e.g., $B = \mathbf{C}$ and $B_3 = B_2$, or $B = B_2^*$ and $B_3 = \mathbf{C}$); in fact the latter contains the former.

Standard convolution results hold also for vector-valued functions:

Lemma D.1.7 ($\|f*h\|_q \le M\|f\|_p\|h\|_r$) *Let* $p,q,r \in [1,\infty]$*,* $f \in \mathrm{L}^p(\mathbf{R}^n;B)$ *and* $g \in \mathrm{L}^q(\mathbf{R}^n;B_2)$*, where* $B \times B_2 \to B_3$ *is bilinear and* $\|bb_2\|_{B_3} \le M\|b\|_B\|b_2\|_{B_2}$.

If $1/p + 1/q = 1$*, then* $f*g$ *exists on the whole* $\mathbf{R}^n$*,* $f*g \in \mathcal{C}_\mathrm{bu}(\mathbf{R}^n;B_3)$ *and* $\|f*g\|_\infty \le M\|f\|_p\|g\|_q$ *(and* $f*g \in \mathcal{C}_0(\mathbf{R}^n;B_3)$ *if* $1 < p < \infty$*).*

If $h \in \mathrm{L}^1(\mathbf{R}^n;B_2)$*, then* $f*h$ *exists a.e. and* $\|f*h\|_p \le M\|f\|_p\|h\|_1$*. If* $h \in \mathrm{L}^r(\mathbf{R}^n;B_2)$ *and* $p^{-1} + r^{-1} = 1 + q^{-1}$*, then* $f*h$ *exists a.e. and* $\|f*h\|_q \le M\|f\|_p\|h\|_r$.

Moreover, in all combinations listed above (e.g., L^1*,* L^1 *and* L^p *or* L^1*,* L^p*,* L^q*, in any order), the laws* $\tau^T(G*F) = (\tau^T G)*F = G*\tau^T F$ $(T \in \mathbf{R}^n)$ *(time-invariance),* $(G*F)*H = G*(F*H)$ *(commutativity) are valid, and for* $n=1$ *we have* $\pi_- F = 0 = \pi_- G \Rightarrow \pi_-(F*G) = 0$ & $(F*G)(t) = (\pi_{[0,t)}F * \pi_{[0,t)}G)(t)$ *(causality).*

All above claims also hold with $\mathrm{L}^{\cdot}_{\omega}$ *in place of* $\mathrm{L}^{\cdot}$*. All above claims hold with* $\mathbf{Z}^n$ *in place of* $\mathbf{R}^n$*.*

If $B \times B_2 \to B_3$ is bilinear and continuous, then $\|bb_2\|_{B_3} \le M\|b\|_B\|b_2\|_{B_2}$ for some $M < \infty$, by Lemma A.3.4(J1). In most cases, one has $M = 1$ (e.g, when $B_2 = \mathcal{B}(B, B_3)$).

We also note that if $f \in \mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}_+; B)$ and $g \in \mathrm{L}^q_{\mathrm{loc}}(R_+; B_2)$, then $f*g \in \mathcal{C}(\mathbf{R}_+; B_3)$ (apply the lemma to $\pi_{[0,T)}f$ and $\pi_{[0,T)}g$ for each $T \in \mathbf{R}_+$, and use causality to obtain continuity on $[0,T]$).

Proof of Lemma D.1.7: The proof of the third inequality (Young's Inequality) is analogous to that of Theorem E.1.7 and hence omitted; see Theorem 1.2.2 of [BL] for the proof (the $\mathbf{Z}^n$ case is analogous). The proofs of the other claims are analogous to that in the finite-dimensional case (for $\mathbf{R}^n$); see, e.g., pp. 39 & 63–64 of [GLS] and use Lemma B.3.9 for uniform continuity and Corollary B.3.8 for the $\mathcal{C}_0$ property. Note that because $\|(f*h)(t)\|_{B_3} \le M(\|f\|_B * \|h\|_{B_2})(t) < \infty$ for a.e. t, the convolution integrand is in L^1 for a.e. t.

We note that the commutativity claims follow easily from the Fubini Theorem.

Let $f \in \mathrm{L}^p_\omega$ and $h \in \mathrm{L}^r_\omega$, and set then $\widetilde{f} = e^{-\omega\cdot} f \in \mathrm{L}^p$, $\widetilde{h} = e^{-\omega\cdot} h$. Then, obviously, $f*h = e^{\omega\cdot}(\widetilde{f}*\widetilde{h})$, hence $\|f*h\|_{\mathrm{L}^q_\omega} \le M\|f\|_{\mathrm{L}^p_\omega}\|h\|_{\mathrm{L}^r_\omega}$ as above, etc. □

Let $\Omega = \mathbf{C}^+$ or $\Omega = \mathbf{D}$. Let $\{z_n\} \in \Omega$ and $z_n \to z_\infty$, where $z_\infty \in \partial\Omega$. We say that $z_n \to z_\infty$ *nontangentially* if there is an open cone $C \subset \Omega$ with vertex at z_∞ s.t. $\{z_n\} \subset C$ (see [Hoffman], [Rud86] or [Garnett] for equivalent definitions). In particular, $z_n \to 0$ nontangentially on $\mathbf{C}^+$, as $n \to +\infty$, iff $z_n \in \mathbf{C}$ for all n, $z_n \to 0$, as $n \to +\infty$, and there is $M < \infty$ s.t. $\operatorname{Im} z_n / \operatorname{Re} z_n < M$ for all n.

The Poisson integral $P_r * f$ of a function f converges to f in many ways:

Lemma D.1.8 (Poisson integral formula) *Let* $P_r(t) := \frac{r}{\pi(r^2+t^2)}$.

(a1) The operator $f \mapsto P_r * f$ *is a stable* C_0*-semigroup on* $\mathcal{C}_{\mathrm{bu}}(\mathbf{R}; B)$ *and on* $\mathrm{L}^p(\mathbf{R}; B)$ *when* $1 \le p < \infty$*.*

(a2) If $f \in \mathcal{C}(i\mathbf{R} \cup \{\infty\}; B)$*, then* $r + it \mapsto (P_r * f)(t)$ *is in* $\mathcal{C}(\overline{\mathbf{C}^+} \cup \{\infty\}; B)$*. In particular,* $f \in \mathcal{C}_0(\mathbf{R}; B)$ *implies that* $(P_r * f)(t) \to 0$ *as* $|r+it| \to \infty$ *and* $r + it \in \overline{\mathbf{C}^+}$*.*

(a3) Let $f \in \mathrm{L}^p(\mathbf{R}; B)$*,* $1 \le p \le \infty$*. Then* $f \mapsto P_r * f$ *is a semigroup (*$P_r * P_{r'} = P_{r+r'}$ *for* $r, r' > 0$*), although not necessarily strongly continuous (in case* $p = \infty$*),* $\|P_r * f\|_p \le \|f\|_p$ $(r > 0)$*,* $\|P_r * f\|_p \to \|f\|_p$*, and* $\int_{-R}^R g P_r * f\, dm \to \int_{-R}^R g f\, dm$*, as* $r \to 0+$*, for any* $R > 0$ *and* $g \in \mathrm{L}^\infty(\mathbf{R}; \mathcal{B}(B, B_2))$*. Moreover,* $(P_r * f)(t) \to f(t)$ *as* $r \to 0+$ *nontangentially, for every Lebesgue point* t *of* f*, hence a.e.*

(a4) If $f \in \mathrm{L}^\infty(\mathbf{R};B)$ *and* $g \in \mathrm{L}^1(\mathbf{R};\mathcal{B}(B,B_2))$*, then* $\int_{\mathbf{R}} g \cdot (P_r * f)\,dm \to \int_{\mathbf{R}} gf\,dm$*, as* $r \to 0+$.

(b) Let $F \in \mathcal{C}(\overline{\mathbf{C}^+};B) \cap \mathrm{H}^\infty(\mathbf{C}^+;B)$*. Then* F *is given on* $\mathbf{C}^+$ *by the Poisson integral formula*

$$F(r+it) = \frac{r}{\pi}\int_{\mathbf{R}} \frac{F(iy)\,dy}{r^2+(t-y)^2} = (P_r * F(i\cdot))(t) \quad (r>0,\ t\in\mathbf{R}). \tag{D.15}$$

(c) If $F \in \mathrm{H}^\infty(\mathbf{C}^+;B)$*, then* $F \in \mathcal{C}_{\mathrm{bu}}(r+i\mathbf{R};B)$ *for any* $r>0$*, and* $r \mapsto F(r+i\cdot)$ *is continuous* $(0,\infty) \to \mathcal{C}_{\mathrm{bu}}(\mathbf{R};B)$.

(d) The above results hold mutatis mutandis for $\mathbf{D}$ *in place of* $\mathbf{C}^+$ *and* $\partial\mathbf{D}$ *in place of* $i\mathbf{R}$*, with* $P_r(t) := \frac{1-r^2}{1-2r\cos t+r^2}$ $(0 \le r < 1,\ t\in\mathbf{R})$.

(We have set here $P_0 := \delta$, i.e., $P_0 * f := f$.) Note that in the above results, the imaginary axis ($i\mathbf{R}$) can be shifted right or left, whichever is needed. If B is a Hilbert space, then an arbitrary H^p function F has an L^p boundary function (whose Poisson integral is F, by the scalar result; see also [HP, p. 227]), for any $p \in [1,\infty]$; see Theorem 3.3.1(a2)&(a1).

Proof: (a1) The semigroup claim can be proved as in the scalar case [Garnett, pp. 12–17]; note that $\int_{\mathbf{R}} P_r\,dm = 1$.

(a2) For $\mathbf{D}$ in place of $\mathbf{C}_+$, this follows from [Rud86, Theorem 11.8] if g is finite-dimensional; in the general case one can approximate g by a finite-dimensional continuous function (e.g., by a function that is "piecewise linear on $[0,2\pi]$") and deduce the continuity as in the proof of [Rud86, Theorem 11.8]). The original ("$\mathbf{C}^+$") claim follows through Cayley transform.

(a3) 1° *Basic properties:* By p. 13–14 of [Garnett], we have $P_r * P_{r'} = P_{r+r'}$ and $\int_{\mathbf{R}} P_r\,dm = 1$, hence P_r* is a semigroup and $\|P_r * f\|_p \le \|f\|_p$.

2° *Nontangential limits:* For $f \in \mathrm{L}^p(\mathbf{D};B)$ (cf. (d)), the existence of a nontangential limit at each Lebesgue point follows exactly as in the proof of (scalar) Theorem 11.23 of [Rud86]. Therefore, the same holds for $\mathrm{L}^p(i\mathbf{R};B)$, because the Cayley transform maps $\mathrm{L}^p(i\mathbf{R};B)$ one-to-one into $\mathrm{L}^p(\partial\mathbf{D};B)$, and preserves Lebesgue points and nontangential limits, by Lemma 13.2.1(e). (Recall that $r \to 0+$ nontangentially means that $r \to 0$ on a subset of $\mathbf{C}^+$ where $\operatorname{Im} r/\operatorname{Re} r$ is bounded (e.g., on $(0,+\infty)$).)

3° $\|P_r * f\|_p \to \|f\|_p$ *as* $r \to 0+$*:* For $p<\infty$ this follows from Theorem 6.4.3(ii) of [HP]. For $p=\infty$ this follows 2° and the inequality $\|P_r * f\|_p \le \|f\|_p$.

4° $\int_{-R}^R gP_r * f\,dm \to \int_{-R}^R gf\,dm$*:* For $p=\infty$ this follows from (a4); for $p<\infty$ this follows from the fact that $\chi_{[-R,R]}g \in \mathrm{L}^q$ and $P_r * f \to f$ in L^p, where $p^{-1}+q^{-1}=1$.

(a4) $\int_{\mathbf{R}} g(t)\int_{\mathbf{R}} P_r(t-s)f(s)\,ds - g(t)f(t)\,dt = \int_{\mathbf{R}}\int_{\mathbf{R}} g(t)P_r(t-s)\,dt\,f(s)\,ds \to \int_{\mathbf{R}} g(s)f(s)\,ds \to 0$, by The Fubini Theorem (and The Hölder Inequality), as $r \to 0+$, since $\int_{\mathbf{R}} g(t)P_r(t-s)\,dt \to g$ in L^1, by (a1) (note that $P_r(t-s) = P_r(s-t)$).

(b) $\Lambda F = \Lambda(P_r * F(i\cdot))$ on $\mathbf{C}^+$ for all $\Lambda \in B^*$, by [Garnett, Lemma 3.4], hence (D.15) holds.

(c) By shifting F to the left by $r/2$, we see that we can assume that $\mathcal{C}(\overline{\mathbf{C}^+};B) \cap \mathrm{H}^\infty(\mathbf{C}^+;B)$. Because $P_r \in \mathrm{L}^1$ and $F(i\cdot) \in \mathcal{C}_\mathrm{b} \subset \mathrm{L}^\infty$, we have $P_r * F(i\cdot) \in \mathcal{C}_\mathrm{bu}(\mathbf{R};B)$, by Lemma D.1.7.

From (a1) it follows that $t \mapsto P_t * F(i\cdot) = P_{t-r} * P_r * F(i\cdot) \in \mathcal{C}_\mathrm{bu}$ is continuous on (r,∞); because $r > 0$ was arbitrary, the claim follows.

(d) The proof are analogous (see the same references). □

We use the standard definition of the Laplace transfrom; it follows that the Fourier transform becomes its restriction to $i\mathbf{R}$. (Thus, this is the standard Fourier transform rotated clockwise by $\pi/2$.)

Definition D.1.9 (Laplace and Fourier transforms) *Let $u : \mathbf{R} \to B$ be measurable. The* Laplace transform *of u is the function*

$$\mathcal{L}u := \widehat{u}(s) := \int_\mathbf{R} \mathrm{e}^{-st} u(t)\, dt, \tag{D.16}$$

defined on those $s \in \mathbf{C}$ for which the integral converges (i.e., for which $\mathrm{e}^{-s\cdot} u \in \mathrm{L}^1(\mathbf{R};B)$). If $u \in \mathrm{L}^1(\mathbf{R};B)$, we call the restriction $\widehat{u}_{|i\mathbf{R}} : i\mathbf{R} \to B$ the Fourier transform *of u.*

Naturally, the Laplace and Fourier transforms inherit the properties of the Bochner integral; in particular, they are linear and commute with bounded linear operators.

It is shown in Lemma D.1.11 that $\widehat{u} \in \mathrm{L}^1(\mathbf{R};B) \Longrightarrow \widehat{u}(i\cdot) \in \mathcal{C}_0(i\mathbf{R};B)$, and that $\widehat{u} \in \mathrm{L}^1(\mathbf{R}_+;B) \Longrightarrow \widehat{u}(i\cdot) \in \mathrm{H}^\infty(\mathbf{C}^+;B)$.

Next we list some basic properties of Laplace transforms:

Lemma D.1.10 ($\widehat{f}$) *With the notation of Definition D.1.3, we have the following:*

(a1) If $f \in \mathrm{L}^1_r(\mathbf{R}_+;B)$, then $\|\widehat{f}(s)\|_B \le \|f\|_{\mathrm{L}^1_r}$ for $s \in \bar{\mathbf{C}}^+_r$,

(a2) If $f \in \mathrm{L}^p_r(\mathbf{R}_+;B)$ and $t > r$, then $\widehat{f} \in \mathrm{H}^\infty(\mathbf{C}^+_t;B)$, and $\|\widehat{f}\|_B \le \|\mathrm{e}^{-(t-r)\cdot}\|_q \|f\|_{\mathrm{L}^p_r} < \infty$ on $\mathbf{C}^+_t$.

(a3) Let $a < b$. If $\|f\|_{\mathrm{L}^p_r(\mathbf{R};B)} \le M$ for $r \in (a,b)$, then $f \in \mathrm{L}^p_a \cap \mathrm{L}^p_b$ and $\|f\|_{\mathrm{L}^p_r} \in \mathcal{C}([a,b])$.

(a4) If $f \in \mathrm{L}^p_a \cap \mathrm{L}^p_b$, then $\|f\|_{\mathrm{L}^p_r(\mathbf{R};B)} \le M := \max\{\|f\|_{\mathrm{L}^p_a}, \|f\|_{\mathrm{L}^p_b}\}$ for all $r \in [a,b]$ (hence (a3) applies).

(b) **(Uniqueness)** *Let $u, v : \mathbf{R}_+ \to B$ be measurable and let $\mathrm{e}^{r\cdot} u, \mathrm{e}^{r\cdot} v \in \mathrm{L}^1$ for some $r \in \mathbf{R}$. Then $u = v$ a.e. iff $\widehat{u} = \widehat{v}$ on $\mathbf{C}^+_r$ (iff $\widehat{u} = \widehat{v}$ on $r + i\mathbf{R}$).*

See also Theorem 3.3.1.

Proof: (a1) This is obvious.

(a2) Choose q s.t. $1/p + 1/q = 1$. Because $\|\mathrm{e}^{-s\cdot}\mathrm{e}^{r\cdot}\|_q \le \|\mathrm{e}^{-(t-r)\cdot}\|_q < \infty$ for $s \in \mathbf{C}^+_t$, the function $\mathrm{e}^{-s\cdot} f$ is in L^1 and we get the norm bound (by Hölder's inequality). By the dominated convergence theorem, we may differentiate under the integral sign, thus we see f is holomorphic. See, e.g., [Sbook] for details.

(a3) 1° *Case $p < \infty$:* Let $g \in \mathrm{L}^p(\mathbf{R}_+;B)$ and $\|g\|_{\mathrm{L}^p_r} \le M$ for $r \in (a,b)$. Then

$$\|g\|^p_{\mathrm{L}^p_r} := \int_{\mathbf{R}} \mathrm{e}^{-rpt}\|g(t)\|^p_B\,dt \to \int_{\mathbf{R}} \mathrm{e}^{-bpt}\|g(t)\|^p_B\,dt =: \|g\|^p_{\mathrm{L}^p_b}, \tag{D.17}$$

as $t \to b-$, by the Monotone Convergence Theorem, hence $g \in \mathrm{L}^p_b$ and $\|g\|_{\mathrm{L}^p_b} = \lim_{r\to b-}\|g\|_{\mathrm{L}^p_r}$.

The same holds for any $g \in \mathrm{L}^p(\mathbf{R}_-;B)$, by the Dominated Convergence Theorem; consequently, an arbitrary $g \in \mathrm{L}^1(\mathbf{R};B)$ will do. Apply this to $\mathbf{Я}g$ (and $(-b,-a)$) to obtain the same at a.

Set $F(t) := \|f\|_{\mathrm{L}^p_t}$. By the above, $F(t) = \lim_{s\to t} F(s)$ for $t \in \{a,b\}$; apply this for $[a,t]$ and $[t,b]$ to see that F is continuous at every $t \in [a,b]$.

2° *Case $p = \infty$:* Let $g \in \mathrm{L}^\infty_c(\mathbf{R}_+)$ for some $c < 0$. Set $M_r := \|g\|_{\mathrm{L}^\infty_r} := \|\mathrm{e}^{-r\cdot}g\|_\infty$ for $r \in \mathbf{R}$. Then there is $T > 0$ s.t. $\mathrm{e}^{c/2}|g| < M_0$ for a.e. $t > T$. It follows that $M_r \le M\mathrm{e}^{-rT}$ for $r > c/2$, hence $M_0 \ge \limsup_{r\to 0-} M_r$. Therefore, $M_0 = \lim_{r\to 0-} M_r$ (because M_r is obviously decreasing).

If $g \in \mathrm{L}^\infty(\mathbf{R}_-)$, then M_r is increasing and, so M is, obviously, again continuous at 0 from the left. The same holds for any $g \in \mathrm{L}^\infty(\mathbf{R})$, because $\max : \mathbf{R}^2 \to \mathbf{R}$ is continuous.

Apply the above for $g := \|\mathrm{e}^{-b\cdot}f\|_B$ to obtain continuity at b. The rest follows as in 1°.

(a4) W.l.o.g. we assume that $a = 0$, $B = \mathbf{C}$, $f \ge 0$, $f \ne 0$ and $p < \infty$ (case $p = \infty$ is obvious). Set

$$g(r) := \|f\|^p_{\mathrm{L}^p_r} = \int_{\mathbf{R}} \mathrm{e}^{-prt}f(t)\,dt \quad (r \in [0,b]). \tag{D.18}$$

Obviously, $g'(r) = -\int_{\mathbf{R}} pt\mathrm{e}^{-prt}f(t)\,dt$, $g''(r) = -pg'(r) + p^2\int_{\mathbf{R}} t^2\mathrm{e}^{-prt}f(t)\,dt > -pg'(r)$ for all $r \in (0,b)$. Thus, if $g'(r_0) = 0$ for some $r_0 \in (0,b)$, then $g''(r_0) > 0$, hence g has no maximum on $(0,b)$, hence (a4) holds.

(b) For separable U, this result is contained in [CZ, Theorem A.6.19], and we may always replace U by the closed span of $f[\mathbf{C}^+]$, which is a separable Hilbert space (because $f[\mathbf{C}^+]$ is a continuous image of a separable set, hence separable).

(c) This follows from the scalar case (or from Lemma D.1.11(a3')) (and of Lemma D.1.11(a3). □

As in the scalar case, the Fourier transform maps $\mathrm{L}^1(\mathbf{R};B)$ to continuous functions on $i\mathbf{R}$, vanishing at infinity:

Lemma D.1.11 ($\mathbf{L^1}$ Fourier transform) *Let B be a Banach space, $f \in \mathrm{L}^1(\mathbf{R};B)$ and $\omega \in \mathbf{R}$. Then we have the following:*

(a1) $\widehat{f} \in \mathcal{C}_0(i\mathbf{R};B)$, and $\|\widehat{f}(ir)\|_B \le \|f\|_{\mathrm{L}^1}$ for $r \in \mathbf{R}$.

(a2) The Fourier transform $f \mapsto \widehat{f}$ is linear and continuous, i.e., in $\mathcal{B}(\mathrm{L}^1, \mathcal{C}_0)$.

(a3) **(Uniqueness)** *If $\widehat{f} = 0$ on $i\mathbf{R}$, then $f = 0$ (in L^1, that is, a.e.).*

(b) **(Riemann–Lebesgue)** *We have $\|\widehat{f}(ir)\|_B \to 0$ as $|r| \to \infty$ and $r \in \mathbf{R}$.*

(c) $\widehat{f * g} = \widehat{f}\widehat{g}$ *on* $i\mathbf{R}$ *when* $g \in \mathrm{L}^p(\mathbf{R}; B_2)$ *(*$p \in \{1,2\}$*) and* $B \times B_2 \to B_3$ *is bilinear and continuous. See also Lemma D.1.12(c).*

(d) If $f, f' \in \mathrm{L}^1$*, then* $\widehat{f'}(ir) = ir\widehat{f}(ir) - f(0)$.

(e1) **(Inverse Fourier transform)** *If* $\widehat{f} \in \mathrm{L}^1(i\mathbf{R}; B)$*, then* $f(\cdot) = \frac{1}{2\pi}\int_{\mathbf{R}} \mathrm{e}^{ix\cdot}\widehat{f}(ix)\,dx \in \mathcal{C}_0(\mathbf{R}; B)$ *a.e. (and in all points t of continuity of f).*

(e2) If $f, f', f'' \in \mathrm{L}^1(\mathbf{R}; B)$ *(e.g., if* $f \in \mathcal{C}_c^2(\mathbf{R}; B)$*), then* $\widehat{f} \in \mathrm{L}^p(i\mathbf{R}; B)$ *for all* $p \in [1, \infty]$.

(f1) $\widehat{\tau(t)f}(s) = \mathrm{e}^{st} f(s)$ *(*$s \in i\mathbf{R}$*) (for* $s \in \overline{\mathbf{C}^+}$ *if* $f = \pi_+ f$*).*

(f2) If $f, \mathrm{e}^{\omega\cdot} f \in \mathrm{L}^1$*, then* $\widehat{\mathrm{e}^{\omega\cdot} f}(s) = \widehat{f}(s - \omega)$ *(*$s \in i\mathbf{R}$*) (for* $s \in \overline{\mathbf{C}^+}$ *if* $f = \pi_+ f$*).*

Assume, in addition, that $f = \pi_+ f$*. Then we have the following:*

(a1') $f \in \mathrm{H}^\infty(\mathbf{C}^+; B) \cap \mathcal{C}_0(\overline{\mathbf{C}^+}; B)$*, and* $\sup_{\mathbf{C}^+} \|\widehat{f}\| = \sup_{i\mathbf{R}} \|\widehat{f}\| \le \|f\|_{\mathrm{L}^1}$.

(a2') The Laplace transform $f \mapsto \widehat{f}$ *is linear and continuous, i.e., in* $\mathcal{B}(\pi_+\mathrm{L}^1, \mathrm{H}^\infty)$.

(a3') **(Uniqueness)** *If* $\widehat{f} = 0$ *on* $\mathbf{C}^+$*, then* $f = 0$ *(in* L^1*, that is, a.e.).*

(b') We have $\|\widehat{f}(s)\|_B \to 0$ *as* $|s| \to \infty$ *and* $s \in \overline{\mathbf{C}^+}$.

(c') $\widehat{f * g} = \widehat{f}\widehat{g}$ *on* $\mathbf{C}^+$ *if the assumptions of (c) are satisfied and* $\pi_- g = 0$.

(d') One can obtain $\widehat{f}(r + it)$ *from the* Poisson integral formula

$$\widehat{f}(r+it) = \frac{r}{\pi}\int_{\mathbf{R}} \frac{\widehat{f}(iy)\,dy}{r^2 + (t-y)^2} \quad (r > 0,\ t \in \mathbf{R}). \tag{D.19}$$

(d") If $f, f' \in \mathrm{L}^1$*, then* $\widehat{f'}(s) = s\widehat{f}(s) - f(0)$ *for* $s \in \mathbf{C}^+$.

As above, we do not distinguish between the Fourier and Laplace transforms unless it is necessary.

If one would define the Fourier transform as $\mathcal{F} f := \widehat{f}(i\cdot)$, then one would obtain from (e1) that $\mathcal{F}^{-1} = \frac{1}{2\pi}\mathbf{Я}\mathcal{F} = \frac{1}{2\pi}\mathcal{F}\mathbf{Я}$. However, we consider the Fourier transform of f in the Laplace sense, hence its domain is $i\mathbf{R}$, not $\mathbf{R}$. Despite this rotation by $\pi/2$, the properties of the Fourier transform are shared by its inverse, e.g., $\widehat{f} \in \mathrm{L}^1$ implies that $f \in \mathcal{C}_0$.

Proof: (a1) The uniform continuity is most easily obtained from (b); the rest is obvious.

(a1') The holomorphicity is noted in [HP, pp. 215–216]. The continuity is obtained from the Dominated Convergence Theorem. The norm claim follows from the scalar case (alternatively, from the Poisson formula). The uniform continuity is most easily obtained from (b');

(a2)&(a2') These are obvious.

(a3)&(a3')&(d)&(d')&(d") These follow from the scalar case (cf. Lemma B.2.6) (the Poisson kernel is an L^1 function; pages 229 and 227 of [HP] give an alternative proof of (g')). (In (d) and (d"), the derivative f' need not exist everywhere, it is enough that $f \in \mathrm{W}^{1,1}$; analogously, in (e2) it suffices that $f \in \mathrm{W}^{2,1}$.)

(b') Let $\varepsilon > 0$. Choose $\phi \in \mathcal{C}_c^\infty((0,+\infty);B)$ s.t. $\|F-\phi\|_1 < \varepsilon/2$. Because $\|\widehat{\phi}(s)\|_B = \|\widehat{\phi'}/s\|_B \le \|\phi'\|_1/|s|$ on $\mathbf{C}^+$, we have $\|\widehat{\phi}(s)\|_B < \varepsilon/2$, hence $\|F(s)\|_B < \varepsilon$, for $|s|$ large enough (by continuity (see (a1')), $\|F(s)\| \le \varepsilon$ for $s \in i\mathbf{R}$ too).

(b) If $f \in \mathrm{L}^1(\mathbf{R};B)$, then $\widehat{f} = \widehat{\pi_+ f} + \mathbf{Я}\widehat{\mathbf{Я}\pi_- f}$ vanishes at infinity, by (b).

(c) This can be proved as in the scalar case [Rud86, Theorem 9.2(c)].

(c') This is contained in [HP, Theorem 6.2.4].

(e1) Use the scalar case (and (a1)&(b) for $\mathcal{C}_0$).

(e2) By (d) and (a1), $ir\widehat{f}(ir) - f(0)$ and $(ir)^2\widehat{f}(ir) - (ir+1)f(0)$ are bounded, hence so is $r^2\widehat{f}(ir)$. Because $\widehat{f} \in \mathcal{C}_0$, it follows that $\widehat{f} \in \mathrm{L}^1 \cap \mathrm{L}^\infty$, hence in each L^p.

(f) These are obvious. □

Many of the properties treated in the previous lemma hold for general vector-valued measures. To make things simple, we only treat "MTI", which corresponds to certain (all if $\dim B < \infty$) measures having only a discrete part plus an absolutely continuous part:

Lemma D.1.12 (MTI Fourier transform) *Let B be a Banach space and set*

$$\mathrm{MTI}_B := \{\mu = \sum_{j\in\mathbf{Z}} a_j\delta_{t_j} + f \mid \|\mu\|_{\mathrm{MTI}} := \sum_j \|a_j\|_B + \|f\|_{\mathrm{L}^1(\mathbf{R};B)} < \infty\}, \quad \text{(D.20)}$$

$$\mathrm{MTIC}_B := \{\mu = \sum_j a_j\delta_{t_j} + f \in \mathrm{MTI}_B \mid f \in \mathrm{L}^1(\mathbf{R}_+;B)\ \&\ t_j \ge 0 \textit{ for all } j\}. \quad \text{(D.21)}$$

Let $\mu = \sum_{j\in\mathbf{Z}} a_j\delta_{t_j} + f \in \mathrm{MTI}_B$.

We use the standard definitions of the Fourier transform of $\mu \in \mathrm{MTI}_B$ and the Laplace transform of $\mu \in \mathrm{MTIC}_B$:

$$\widehat{\mu}(s) := \sum_{j\in\mathbf{Z}} a_j e^{-st_j} + \widehat{f}(s), \quad \text{(D.22)}$$

for $s \in i\mathbf{R}$ (for $s \in \overline{\mathbf{C}^+}$, if $\mu \in \mathrm{MTIC}_B$).

We have the following:

(a1) $\widehat{\mu} \in \mathcal{C}_{\mathrm{bu}}(i\mathbf{R};B)$, and $\|\widehat{\mu}(ir)\|_B \le \|\mu\|_{\mathrm{MTI}}$ for $r \in \mathbf{R}$.

(a2) The Fourier transform $\mu \mapsto \widehat{\mu}$ is linear and continuous, i.e., in $\mathcal{B}(\mathrm{MTI}_B;\mathcal{C}_{\mathrm{bu}})$.

(a3) **(Uniqueness)** *If $\widehat{\mu} = 0$ on $i\mathbf{R}$, then $\mu = 0$.*

(b1) We have $a_j = \lim_{T\to+\infty} \int_{-T}^T \widehat{\mu}(ir)e^{-it_jr}\,dr$ for all $j \in \mathbf{Z}$.

(b2) If $f = 0$, then $\widehat{\mu} \in \mathrm{AP}(i\mathbf{R};B)$ (i.e., $\widehat{\mu}(i\cdot) \in \mathrm{AP}(\mathbf{R};B)$).

(c) Let $\nu \in \mathrm{MTI}_{B_2}$, and let $B \times B_2 \to B_3$ be bilinear and $\|bb_2\|_{B_3} \le \|b\|_B\|b_2\|_{B_2}$. Then (c1)–(c3) hold.

(c1) $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$ on $i\mathbf{R}$ and $\|\mu*\nu\|_{\mathrm{MTI}} \le \|\mu\|_{\mathrm{MTI}}\|\nu\|_{\mathrm{MTI}}$, where $\delta_t * \delta_r := \delta_{r+t}$ and $\delta_t * f := \tau(-t)f$.

(c2) For all $1 \le p \le \infty$ *and* $g \in \mathrm{L}^p(\mathbf{R};B_2)$*, the convolution* $\mu * g$ *exists a.e. and* $\|\mu * g\|_{\mathrm{L}^p(\mathbf{R};B_3)} \le \|\mu\|_{\mathrm{MTI}}\|g\|_p$.

(c3) We have $(\mu * \nu) * \lambda = \mu * (\nu * \lambda)$ *and* $(\mu * \nu) * g = \mu * (\nu * g)$ *when* $\lambda \in \mathrm{MTI}_{\mathcal{B}(B_3,B_4)}$ *and* $g \in \mathrm{L}^p(\mathbf{R};\mathcal{B}(B_3,B_4))$.

(d) Assume that $B = \mathcal{B}(U,Y)$*, where* U *and* Y *are Hilbert spaces. Then* $\|\mu *\|_{\mathrm{TI}(U,Y)} \le \|\mu\|_{\mathrm{MTI}}$, $(\mu *)^* = (\mathbf{Я}\mu^*) * \in \mathrm{TI}(Y,U)$, $(\mu *)^{\mathrm{d}} = \mu^* * \in \mathrm{TI}(Y,U)$, $\mathbf{Я}(\mu *)\mathbf{Я} = (\mathbf{Я}\mu) * \in \mathrm{TI}(U,Y)$*, and* $\mathrm{e}^{\omega\cdot}(\mu *)\mathrm{e}^{-\omega\cdot} = (\mathrm{e}^{\omega\cdot}\mu) * \in \mathrm{TI}_\omega(U,Y)$*. In particular, then* $\mu * \in \mathrm{MTIC}_B \Leftrightarrow (\mu *)^{\mathrm{d}} \in \mathrm{MTIC}_B$.

Moreover, then $\widehat{\mu * g} = \widehat{\mu}\widehat{g}$ *a.e. on* $i\mathbf{R}$ *for all* $g \in \mathrm{L}^1(\mathbf{R}_+;U) \cup \mathrm{L}^2(\mathbf{R};U)$ *(and on* $\mathbf{C}^+$ *if* $\pi_- g = 0 = \pi_-\mu$*). Therefore,* $\widehat{\mu}$ *coincides with* $\widehat{\mu *}$ *of of Theorem 3.1.3 (and with that of Theorem 6.2.1 if* $\pi_-\mu = 0$*).*

Assume, in addition, that $\mu, \nu \in \mathrm{MTIC}_B$*. Then we have the following:*

(a1') $\widehat{\mu} \in \mathrm{H}^\infty(\mathbf{C}^+;B) \cap \mathcal{C}_{\mathrm{bu}}(\overline{\mathbf{C}^+};B)$*, and* $\sup_{\mathbf{C}^+}\|\widehat{\mu}\| = \sup_{i\mathbf{R}}\|\widehat{\mu}\| \le \|\mu\|_{\mathrm{MTI}}$.

(a2') The Laplace transform $\mu \mapsto \widehat{\mu}$ *is linear and continuous, i.e., in* $\mathcal{B}(\mathrm{MTIC}_B;\mathrm{H}^\infty)$.

(a3') **(Uniqueness)** *If* $\widehat{\mu} = 0$ *on* $\mathbf{C}^+$*, then* $\mu = 0$.

(b') $\|\widehat{\mu}(s) - M\|_B \to 0$ *as* $\operatorname{Re} s \to +\infty$ *and* $s \in \mathbf{C}^+$*, where* $M \in B$ *is the coefficient of* δ_0.

(c') We have $\widehat{\mu * \nu} = \widehat{\mu}\widehat{\nu}$ *on* $\overline{\mathbf{C}^+}$ *if* $\nu \in \mathrm{MTIC}_{B_2}$ *and the assumptions of (c) hold.*

(d') For $r + it \in \mathbf{C}^+$ *one can obtain* $\widehat{\mu}(r+it)$ *from the* Poisson integral formulaPoisson integral formula!for measures

$$\widehat{\mu}(r+it) = \frac{r}{\pi}\int_{\mathbf{R}} \frac{\widehat{\mu}(iy)\,dy}{r^2 + (t-y)^2}. \tag{D.23}$$

Because B is not necessarily an algebra, we have defined MTI_B to consist of "measures", not of the corresponding convolution operators (we identify L^1 functions and corresponding absolutely continuous measures $E \mapsto \int_E f\,dm$).

If we had $\|bb_2\|_{B_3} \le M\|b\|_B\|b_2\|_{B_2}$, $M > 0$, in (c), then the conlusions of (c) would still be valid, provided that we added M also to the other inequalities in (c).

To clarify (d), we note that $\mathrm{e}^{\omega\cdot}\delta_{t_k} = \mathrm{e}^{\omega t_k}\delta_{t_k}$, hence

$$\mathrm{e}^{\omega\cdot}(F + \sum_{k\in\mathbf{N}} T_k\delta_{t_k}) = \mathrm{e}^{\omega\cdot}F(\cdot) + \sum_{k\in\mathbf{N}} \mathrm{e}^{\omega t_k}T_k\delta_{t_k}. \tag{D.24}$$

Thus, the "stability" of μ can be shifted (cf. Remark 6.1.9). Most of (d) is valid for general B, B_2 and B_3 too. Here "$(\mathbf{Я}\mu)$" refers to $\mathbf{Я}$ operating μ (on MTI), not on L^2.

The functions in the space generated by the sums of AP functions and continuous functions having limits at infinity (which contains the Fourier transforms of MTI_B functions) are called *semi-almost periodic functions* (see, e.g., [Sarason] for more on such functions).

Proof: The parts (a1), (a2), (a1') and (a2') follows easily from those of Lemma D.1.11.

(a3) This follows from (b1) and Lemma D.1.11(a3).

(a3') This follows from (a1') and (a3).

(b1)&(b2) These follow from Lemma C.1.2(h2), because f has no effect on the limit in (b1), by Lemma D.1.11(b).

(b') W.l.o.g. we assume that $M = 0$ and $f = 0$ (by Lemma D.1.11(b')).

Given $\varepsilon > 0$, take first $N \in \mathbf{N}$ s.t. $\sum_{|k|>N} \|a_k\| < \varepsilon/3$. Because $e^{-st_k} \to 0$, as $\operatorname{Re} s \to +\infty$, for each k, there is $R > 0$ s.t. $\sum_{|k|\le N} \|a_k\| < \varepsilon/3$ for $\operatorname{Re} s > R$.

(c1)–(c2) These are not hard to verify (the hardest part is contained in Lemma D.1.11(c); use formulae analogous to (2.55) and Theorem 2.6.4(d) for $\|\mu * \nu\| \le \|\mu\|\|\nu\|$). Note that we use the standard (associative and distributive) definition of $*$, which coincides with

$$(\mu * f)(t) := \int_{\mathbf{R}} d\mu(\cdot) \cdot f(t-\cdot) \tag{D.25}$$

(if the integral is defined in some reasonable sense; cf. [Dobrakov]; we do not need this).

(c3) Decompose μ, ν, λ and verify the claims for distributed parts (Obviously, the convolution is distributive).

(Note that one can take the values of λ or g on X and $B_3 = \mathcal{B}(X, B_4)$ for some Banach space X, so that X is isometrically isomorphic to a closed subspace of $\mathcal{B}(B_3, B_4)$. More generally, we could as well assume that $B \times B_2 \times X \to B_4$ is "trilinear" and continuous.)

(d) Clearly $\mu* \in \mathrm{TI}$. By (c2) we have $\|\mu *\|_{\mathrm{TI}(U,Y)} \le \|\mu\|_{\mathrm{MTI}}$. The second claim follows from equations (here $f, g \in \mathrm{L}^2$, $F \in \mathrm{L}^1(\mathbf{R}; B)$, $T_k \in B$, $t_k \in \mathbf{R}$)

$$\langle T_k \tau(-t_k) f, g\rangle = \langle f, T_k^* \tau(t_k) g\rangle \ \text{ and} \tag{D.26}$$

$$\int_{\mathbf{R}} \langle \int_{\mathbf{R}} F(t-r) f(r)\, dr, g(t)\rangle\, dt = \int_{\mathbf{R}} \langle f(r), \int_{\mathbf{R}} (\boldsymbol{\mathcal{R}} F^*)(r-t) g(t)\, dt\rangle\, dr, \tag{D.27}$$

(use the Fubini Theorem). Note that $(\boldsymbol{\mathcal{R}}\mu)^* = \boldsymbol{\mathcal{R}}(\mu^*)$, so the order of these operations does not matter. Equations

$$(F * \boldsymbol{\mathcal{R}} f)(-s) = \int_{\mathbf{R}} F(-t-s) f(-s)\, ds = \int_{\mathbf{R}} F(-t+s) f(s)\, ds = ((\boldsymbol{\mathcal{R}} F) * f)(s) \tag{D.28}$$

and $\boldsymbol{\mathcal{R}}(\tau(t_k)\boldsymbol{\mathcal{R}} f) = \tau(-t_k) f$ imply that $\boldsymbol{\mathcal{R}}(\mu * \boldsymbol{\mathcal{R}} f) = (\boldsymbol{\mathcal{R}}\mu) * f$. By combining the two identities proved above we obtain that $(\mu*)^{\mathrm{d}} = \mu^* *$, which implies the final claim.

The $e^{-\omega\cdot}$ claim is most easily obtained from the equation $\mathcal{L}(e^{\omega\cdot}\mu) = \widehat{\mu}(s-\omega)$ (cf. Remark 2.1.6).

If $g \in \mathrm{L}^1$, then the claim on $\widehat{\mu * g}$ is contained in (c1) and (c'). If $g \in \mathrm{L}^2 \setminus \mathrm{L}^1$, then this follows from the L^1 case by density: choose $\{g_n\} \subset \mathrm{L}^1 \cap \mathrm{L}^2$ s.t. $g_n \to g$ in L^2 and use (c2) and Lemma D.1.15. The claims on TI and TIC follow from this.

(c')&(d') These can be proved as parts (c') and (d') of Lemma D.1.11. □

See, e.g., [DU, pp. 1–6] and [Dinculeanu] for more general vector-valued measures.

If $f \in \mathrm{L}^1(\mathbf{R};\mathrm{L}^p)$, then $\widehat{f} \in \mathcal{C}_0(i\mathbf{R};\mathrm{L}^p)$, by Lemma D.1.11(a1). Therefore, $\widehat{f}(ir) \in \mathrm{L}^p$, so that $\widehat{f}(ir)$ is "well-defined a.e.", i.e., $(\widehat{f}(ir))(\cdot)$ makes sense as an equivalent class. But can we fix some point t and then transform $(f(\cdot))(t)$? The answer is positive for a.e. t:

Proposition D.1.13 ($(\mathcal{L}f)(t) = \mathcal{L}(f(\cdot)(t))$) *Let $(Q,\mathfrak{M},\mu)$ be σ-finite. Let $\omega \in \mathbf{R}$, $p,q \in [1,\infty]$.*

If $f \in \mathrm{L}^1_\omega(\mathbf{R};\mathrm{L}^p(Q;B)) \cap \mathrm{L}(\mathbf{R}\times Q;B)$ and $s \in \omega + i\mathbf{R}$, then $\int_{\mathbf{R}} \mathrm{e}^{-sr} f(r)(t)\,dr$ exists for a.e. $t \in Q$, and

$$\widehat{f}(s)(t) = \int_{\mathbf{R}} \mathrm{e}^{-sr} f(r)(t)\,dr \in B \quad \text{for a.e. } t \in Q. \tag{D.29}$$

In particular, if $f \in \mathrm{L}^q_\omega(\mathbf{R}_+;\mathrm{L}^p(Q;B)) \cap \mathcal{C}(\mathbf{R}_+;\mathrm{L}^p(Q;B))$, then, for each $s \in \mathbf{C}^+_\omega$, $\widehat{f}(s)(t) = \int_{\mathbf{R}_+} \mathrm{e}^{-sr} f(r)(t)\,dr \in B$ for a.e. $t \in Q$; (the integrand in in $\mathrm{L}^1(Q;B)$ for a.e. $t \in Q$).

Note that this is result is nontrivial, although it is unfortunaly widely used without reference (we do not know any).

Proof: Let $s \in \omega + i\mathbf{R}$, so that $\mathrm{e}^{-s\cdot} f \in \mathrm{L}^1(\mathbf{R};\mathrm{L}^p(Q;B)) \cap \mathrm{L}(\mathbf{R}\times Q;B)$. By Lemma B.4.17, $g_s(t) := \int_{\mathbf{R}} \mathrm{e}^{-sr} f(r)(t)\,dr$ exists a.e., and $g_s = \int_{\mathbf{R}} \mathrm{e}^{-sr} f(r)\,dr \in \mathrm{L}^p(Q;B)$. In the second claim we have $\mathrm{e}^{-s\cdot} f \in\in \mathrm{L}^1(\mathbf{R};\mathrm{L}^p(Q;B))$ for $\operatorname{Re} s > \omega$ (cf. the proof of Lemma D.1.10(a2)). □

The set of bounded complex Borel measures on a locally compact Hausdorff space Q equals $\mathcal{C}_0(Q)^*$, by the (a) Riesz representation theorem [Rud86, Theorem 6.19]. Therefore, one often defines measures as the set $\mathcal{C}_0^*$ (see, e.g., [Dinculeanu]). We shall need some basic results on even more general "measures":

Lemma D.1.14 (M) *Let H be a Hilbert space, and let $\Omega \subset \mathbf{R}^n$ be open. Set $M := \mathcal{B}(\mathcal{C}_0(\Omega;B);H)$.*

(a) For each $\mu \in M$ and $\varepsilon > 0$, there is a compact $K_\varepsilon \subset \Omega$ s.t.

$$\|\mu\phi\|_\infty \le \varepsilon\|\phi\|_\infty + \|\mu\| \sup_{K_\varepsilon} \|\phi\|_B \quad \text{for all } \phi \in \mathcal{C}_0(\Omega;B). \tag{D.30}$$

(b) Each $\mu \in M$ has a unique norm-preserving extension $\mu \in \mathcal{B}(\mathcal{C}_\mathrm{b}(\Omega;B);H)$.

(c) $\widetilde{\mu} : t \mapsto \mu(\mathrm{e}^{it\cdot}) \in H$ satisfies $\widetilde{\mu} \in \mathcal{C}_\mathrm{bu}(\mathbf{R}^n;\mathcal{B}(B,H))$ for all $\mu \in \mathcal{B}(\mathcal{C}_0(\mathbf{R}^n;B);H)$.

(d) $\mu\tau^t\phi \to 0$ in H as $|t| \to \infty$, $t \in \mathbf{R}^n$, for all $\mu \in M$ and $\phi \in \mathcal{C}_0(\mathbf{R}^n;B)$.

(e) This lemma also holds with $\mathbf{Z}^n$ in place of $\mathbf{R}^n$.

By (b), $\mathcal{C}_\mathrm{b}(\Omega;B)$ is a closed subspace of $\mathcal{B}(M,H)$.

Proof: (a) 1° *For each $T \in M$ and $\varepsilon > 0$, there is a compact $K'_\varepsilon \subset \Omega$ s.t. $\|T\phi\| < \varepsilon\|\phi\|$ for $\phi \in \mathcal{C}_0$ s.t. $\phi = 0$ on K'_ε:* (This does not hold if, e.g., $H = \mathcal{C}_0$ (a Banach space) and $T = I$.)

Let $\|T\| = 1$, w.l.o.g. Find $\psi \in \mathcal{C}_0$ s.t. $\|\psi\| \leq 1$ and $\|T\psi\|^2 > 1 - \varepsilon^2$. Set $K'_\varepsilon := \mathrm{supp}(\psi)$. If $\phi = 0$ on K'_ε and $\|\phi\| \leq 1$, then $\|\alpha\phi + \psi\| \leq 1$ for $|\alpha| \leq 1$, hence

$$1 \geq \sup_{|\alpha| \leq 1} \|\alpha T\phi + T\psi\|^2 \geq \|T\phi\|^2 + \|T\psi\|^2 > \|T\phi\|^2 + 1 - \varepsilon^2. \tag{D.31}$$

Thus, $\|T\phi\| < \varepsilon$.

2° *The rest of (a):* Choose a compact $K_\varepsilon \subset \Omega$ s.t. $K'_\varepsilon \subset K^o_\varepsilon$ (e.g., use Lemma A.2.3). Let $\phi \in \mathcal{C}_0(\Omega; B)$ be arbitrary.

By Lemma B.3.10, there is $\psi_k \in \mathcal{C}^\infty_\mathrm{c}(\Omega)$ s.t. $\chi_{K'_\varepsilon} \leq \psi \leq \chi_{K^o_\varepsilon}$. By 1°, we have $\|T(1-\psi)\phi\|_H \leq \varepsilon\|(1-\psi)\phi\|_\infty \leq \varepsilon\|\phi\|_\infty$, hence

$$\|T\phi\| \leq \|T(1-\psi)\phi\| + \|T\psi\phi\| \leq \varepsilon\|\phi\|_\infty + \|T\| \sup_{K_\varepsilon} \|\phi\|_B. \tag{D.32}$$

(b) 1° *Isometric extension:* Let $\{K_k\} \subset \Omega$ be as in Lemma A.2.3. For each $k \in \mathbf{N}+1$, there is $\psi_k \in \mathcal{C}^\infty_\mathrm{c}(\Omega)$ s.t. $\chi_{K_k} \leq \psi_k \leq \chi_{K^o_{k+1}}$, by Lemma B.3.10.

Let $\phi \in \mathcal{C}_\mathrm{b}(\Omega; B)$. Set $\phi_k := \phi\psi_k \in \mathcal{C}_\mathrm{c}(\Omega; B)$. For each $T \in M$, the sequence $\{T\phi_k\}$ is a Cauchy-sequence in H, by (a)1° (because $\phi_k - \phi_{k+j} = 0$ on K_k and $\|\phi_k\| \leq \|\phi\|$ for $k, j \in \mathbf{N}+1$). Let $\phi T \in H$ be the limit of this sequence.

Then $\phi : M \to H$ becomes linear and $\|\phi T\| \leq \|T\|\|\phi\|$. On the other hand, $\|\phi\|_{\mathcal{B}(M,H)} \geq \|\phi\|_{\mathcal{C}_\mathrm{b}}$, because given $\varepsilon > 0$, we can choose $q \in \Omega$ and $S \in \mathcal{B}(B, H)$ s.t. $\|S\| \leq 1$ and $\|S\phi(q)\|_H \geq \|\phi\| - \varepsilon$ (note that $S\delta_q : \widetilde{\phi} \mapsto S\widetilde{\phi}(q)$ is in M and $\phi S\delta_q = S\phi(q)$). Thus, $\|\phi\|_{\mathcal{B}(M,H)} = \|\phi\|_{\mathcal{C}_\mathrm{b}}$.

2° *Uniqueness:* Assume w.l.o.g. that $\mu \in M$ is extended as above, and $\|\mu\| = 1 = \|\mu'\|$, where also μ' is a continuous extension of μ.

Assume $\mu' \neq \mu$ to obtain a contradiction. Then $\varepsilon := \|\mu'(\phi) - \mu(\phi)\|_H > 0$ for some $\phi \in \mathcal{C}_\mathrm{b}$ with $\|\phi\| = 1$.

Choose $r \in (0,2)$ s.t. $(1-\varepsilon/2r)^2 + (\varepsilon/2)^2 > 1$. Choose $k \in \mathbf{N}+2$ s.t. $K_{\varepsilon/r} \subset K^o_{k-1}$. Then $\phi_k - \phi = 0$ on K_k, hence $\mu(\phi_k - \phi) < \varepsilon/r < \varepsilon/2$, hence $\mu'(\phi_k - \phi) > \varepsilon/2$.

There is $\widetilde{\psi} \in \mathcal{C}_0$ s.t. $\|\widetilde{\psi}\| = 1$ and $\mu(\widetilde{\psi}) > 1 - \varepsilon/r$. Set $\psi := \psi_{k-1}\widetilde{\psi}$, so that $|\mu(\psi)| > 1 - \varepsilon/2r$. Now $\psi = 0$ on K^c_k and $\phi_k - \phi = 0$ on K_k, hence $\widetilde{\phi}_\alpha := \alpha\psi + \phi_k - \phi \in \mathcal{C}_\mathrm{b}$ satisfies $\|\widetilde{\phi}\| \leq 1$ when $|\alpha| = 1$. But

$$\sup_{|\alpha|=1} |\mu'(\widetilde{\phi}_\alpha)| \geq (1-\varepsilon/2r)^2 + (\varepsilon/2)^2 > 1, \tag{D.33}$$

hence $\|\mu'\| > 1$, a contradiction.

(c) The vector $\mu(\mathrm{e}^{it\cdot})$ in the lemma refers to the map $\mu_0 \in \mathcal{B}(\mathcal{C}_0(\mathbf{R}^n); \mathcal{B}(B,H))$ induced by $\mu \in M$ through $\mu_0(\phi)x := \mu(\phi x)$ for $\phi \in \mathcal{C}_0(\mathbf{R}^n)$, $x \in B$. Obviously, $\|\mu_0\|_\mathcal{B} \leq \|\mu\|_M$.

Given $\varepsilon > 0$, choose $k \in \mathbf{N}+1$ s.t. $K_{\varepsilon/4} \subset K_k$. Choose $\delta > 0$ s.t. $|\mathrm{e}^{ih\cdot} - 1| \leq \varepsilon/2\|\mu\|$ on K_{k+1} when $|h| < \delta$. We have $|\mathrm{e}^{i(t+h)q} - \mathrm{e}^{itq}| = |\mathrm{e}^{ihq} - 1|$ ($q \in \mathbf{R}^n$), hence

$$\|\widetilde{\mu}(t+h) - \widetilde{\mu}(t)\|_H := \|\mu_0(\mathrm{e}^{i(t+h)\cdot}) - \mu_0(\mathrm{e}^{it\cdot})\| \le \|\mu_0(\phi_k \mathrm{e}^{it\cdot}(\mathrm{e}^{ih\cdot} - 1))\| \tag{D.34}$$

$$+\|\mu_0((1-\phi_k)\mathrm{e}^{it\cdot}(\mathrm{e}^{ih\cdot} - 1))\| < \|\mu\|\varepsilon/2\|\mu\| + 2\varepsilon/4 = \varepsilon. \ \ (t \in \mathbf{R}^n,\ |h|_{\mathbf{R}^n} < \delta). \tag{D.35}$$

Thus, $\widetilde{\mu} \in \mathcal{C}_{\mathrm{bu}}$.

(d) This follows from (a).

(e) This analogous but easier. □

Next we present the Fourier–Plancherel and Paley–Wiener Theorems. Functions in $\mathrm{L}^1 \cap \mathrm{L}^2$ can be Fourier transformed with ease, the transforms being continuous, by Lemma D.1.11(a1). Fortunately, in a Hilbert space setting, the L^2 norm of the function is preserved modulo the factor $\sqrt{2\pi}$ (see (D.36)), hence the Fourier transform $\mathrm{L}^1 \to \mathcal{C}_0$ can be extended to a transform (isomorphism) $\mathrm{L}^2 \to \mathrm{L}^2$, by Lemma A.3.10. Thus, for $f \in \mathrm{L}^2(\mathbf{R};H)$, the transform is defined by $\lim \widehat{f_n}$ (the limit being taken in L^2) for any sequence $\{f_n\} \subset \mathrm{L}^1 \cap \mathrm{L}^2$ converging to f in L^2 (note that the integral (D.16) need not converge).

Lemma D.1.15 (Fourier–Plancherel transform) *Let H be a Hilbert space. If $f : \mathbf{R} \to H$ is in $\mathrm{L}^1 \cap \mathrm{L}^2$, then $\|\widehat{f}\|_2 = \sqrt{2\pi}\|f\|_2$. Therefore, the Fourier(–Plancherel) transform can be extended to an (isometric times $\sqrt{2\pi}$) isomorphism of $\mathrm{L}^2(\mathbf{R};H)$ onto $\mathrm{L}^2(i\mathbf{R};H)$.*

An analogous result holds on L^2_ω ($\omega \in \mathbf{R}$), hence, for all $F, G \in \mathrm{L}^2_\omega(\mathbf{R}_+;H)$ and $f, g \in \mathrm{L}^2(\mathbf{R};H)$, we have that

$$\langle \hat{f}, \hat{g} \rangle_{\mathrm{L}^2} = 2\pi \langle f, g \rangle_{\mathrm{L}^2}, \quad \|\hat{f}\|_{\mathrm{L}^2} = \sqrt{2\pi}\|f\|_{\mathrm{L}^2}, \quad \langle \hat{F}, \hat{G} \rangle_{\mathrm{H}^2_\omega} = 2\pi \langle F, G \rangle_{\mathrm{L}^2_\omega}. \tag{D.36}$$

Similarly, the mapping of $a = \sum_{k\in\mathbf{Z}} a_k \in \ell^2(\mathbf{Z};H)$ to $\widehat{a}(z) := \sum_{k\in\mathbf{Z}} a_k z^k \in \mathrm{L}^2(\partial\mathbf{D};H)$ is an (isometric times $\sqrt{2\pi}$, as above) isomorphism of $\ell^2(\mathbf{Z};H)$ onto $\mathrm{L}^2(\partial\mathbf{D};H)$, and it maps $\ell^2(\mathbf{N};H)$ onto $\mathrm{H}^2(\mathbf{D};H)$.

Here the norm on $\mathrm{L}^2(r\partial\mathbf{D};H)$ is given by $\|\widehat{f}\|_2^2 := \int_0^{2\pi} \|\widehat{f}(r\mathrm{e}^{it})\|_H^2\, dt$ (cf. [Rud86], p. 89 & 337), hence $\|1\|_2 = \sqrt{2\pi}$, and

$$\|\widehat{f}\|_{\mathrm{H}^2(r\mathbf{D};H)} := \sup_{0<t<r} \|\widehat{f}\|_{\mathrm{L}^2(t\partial\mathbf{D};H)} = \lim_{t\to r-} \|\widehat{f}\|_{\mathrm{L}^2(t\partial\mathbf{D};H)} = \|\widehat{f}\|_{\mathrm{L}^2(r\partial\mathbf{D};H)}. \tag{D.37}$$

□

Thus, $\int_0^{2\pi} \|\widehat{f}(\mathrm{e}^{i\theta})\|_H^2\, d\theta = 2\pi \sum_{n\in\mathbf{Z}} \|f_n\|_H^2 < \infty$ for all $f \in \ell^2(\mathbf{Z};H)$. (Recall that we have defined the Lebesgue measure of $\partial r\mathbf{D}$ to be 2π.)

The above facts can be verified as in the scalar case (see e.g., [Rud86]), and they are presented (in varying generalities; note that H can be assumed to be separable w.l.o.g.) in [RR], [Nikolsky], [HP] and [CZ]. We omit the details. The last equation in (D.37) refers to the boundary function of $\widehat{f}$, cf. Theorem 3.3.1.

By "isometric times $\sqrt{2\pi}$" we mean "$\|\hat{f}\|_{\mathrm{L}^2} = \sqrt{2\pi}\|f\|_{\mathrm{L}^2}$" (recall that isometric means norm-preserving).

If H is an arbitrary Banach space (contrary to standing assumptions of this section), even $\mathrm{L}^1 \cap \mathrm{L}^2(\mathbf{R};H)$ functions are not in general mapped into L^2, as illustrated in Example 3.3.4 (see its second remark); however, if $f \in \mathrm{L}^2$ is finite-dimensional, then its range is isomorphic to some Hilbert space $\mathbf{C}^n$, hence then $\widehat{f} \in \mathrm{L}^2$ is well-defined (and coincides with $\widehat{f} \in \mathcal{C}_0$ a.e. if $f \in \mathrm{L}^1 \cap \mathrm{L}^2$).

We call $\mathcal{S}(\mathbf{R};B) := \{f \in \mathcal{C}^\infty(\mathbf{R};B) \mid \|f\|_{k,n} := \|x^k f^{(n)}\|_\infty < \infty \text{ for all } k,n \in \mathbf{N}\}$, equipped with the (Fréchet space) topology induced by the $\|\cdot\|_{k,n}$ seminorms (cf. [Rud73, Theorem 1.37]), the space of *rapidly decreasing functions*.

Lemma D.1.16 ($\mathcal{S}(\mathbf{R};B)$) *The space $\mathcal{S} := \mathcal{S}(\mathbf{R};B)$ is a complete topological vector space (Fréchet space), and $\mathcal{C}_c^\infty$ is a dense subset of $\mathcal{S}$. Moreover, $\mathcal{S} \subset \mathrm{L}^p$ for all $p \in [1,\infty]$.*

Set $\mathcal{F} f := (\mathcal{L}f)(-i\cdot)$ for $f \in \mathrm{L}^1(\mathbf{R};B)$. Then $\mathcal{F}[\mathcal{S}] = \mathcal{S}$, $\mathcal{F}$ is an isomorphism (linear continuous bijection) on $\mathcal{S}$, and $\mathcal{F}^2 f = \text{Я} f$ for all $f \in \mathcal{S}$. Moreover, $\text{Я} \in \mathcal{B}(\mathcal{S})$ and $\tau\phi \in \mathcal{C}(\mathbf{R};\mathcal{S})$ for all $\phi \in \mathcal{S}$. □

(The proof for the scalar case is given in, e.g., Sections 7.3–7.10 of [Rud73], and in Section 2 of [Rauch]. Those proofs cover also the vector-valued case, mutatis mutandis, so we omit the proof.)

Here we extend the typical tool for Cauchy integrals:

Lemma D.1.17 *Assume that γ is a σ-finite, complete, positive measure space, $\Omega \subset \mathbf{C}$ is open, $g \in \mathrm{L}^1(\gamma;B)$, $f : \gamma \times \Omega \to \mathcal{B}(B,B_2)$, $f(t,\cdot) \in \mathrm{H}(\Omega;*)$ for each $t \in \gamma$, $f(\cdot,z) \in \mathrm{L}^\infty(\gamma;*)$ and $\sup_{z\in\Omega} \|f(\cdot,z)\|_\infty < \infty$. Then $F \in \mathrm{H}^\infty(\Omega;B_2)$, where*

$$F(z) := \int_\gamma f(t,z) g(t)\,dt. \tag{D.38}$$

By taking $f(t,z) := (2\pi i(t-z))^{-1}$ and letting γ be curve in $\mathbf{C}$, we get a Cauchy integral.

Proof: Set $M := \sup_{z\in\Omega} \|f(\cdot,z)\|_\infty$. Since $\|f(\cdot,z)g\|_1 \le M\|g_1\|_1 =: M'$ ($z \in \Omega$), we have $\|F\|_{B_2} \le M'$ on Ω. By The Dominated Converge Theorem (with majorant $M\|g\|_B \in \mathrm{L}^1(\gamma)$), F is continuous on Ω. For every rectangle R in Ω, we have, by The Fubini Theorem, that

$$\int_R F(z)\,dz = \int_\gamma \int_R f(t,z)\,dz g(t)\,dt = \int_\gamma g(t) 0\,dt = 0, \tag{D.39}$$

(note that $\int_R \int_\gamma \|f(t,z)g(t)\|\,dt\,dz \le \int_R M\|g\|_1\,dz < \infty$ and that $f \in \mathrm{L}(\gamma \times \Omega; \mathcal{B}(B,B_2)$, by Lemma B.4.8). Therefore, $F \in \mathrm{H}(\Omega;B_2)$, by The Morera Theorem. Since $\|F\|_\infty \le M\|g\|_1$, we have $F \in \mathrm{H}^\infty(\Omega;B_2)$. □

If f is holomorphic on both sides of $i\mathbf{R}$ and weakly $\mathrm{L}^1_{\mathrm{loc}}$-continuous to $i\mathbf{R}$, then f is holomorphic on $i\mathbf{R}$ too:

Proposition D.1.18 ($\mathrm{H}(\mathbf{C}_{a,b}) \cap \mathrm{H}(\mathbf{C}_{b,c}) = \mathrm{H}(\mathbf{C}_{a,c})$) *Let $\Omega \subset \mathbf{C}$ be open. Assume that $f : \Omega \to B$ is in $\mathrm{H}(\Omega \setminus (b + i\mathbf{R});B)$, and that $\Lambda f(t + i\cdot) \to \Lambda f(b + i\cdot)$ in $\mathrm{L}^1([u,v])$, as $t \to b$, for all $\Lambda \in B^*$ and $u,v \in \mathbf{R}$ s.t. $b + [u,v]i \subset \Omega$.*

Then f can be redefined on a null subset of $b + i\mathbf{R}$ so that $f \in \mathrm{H}(\Omega;B)$.

Naturally, we can replace $b + i\mathbf{R}$ by any other straight line (rotate the plane). An analogous result holds for $\partial r\mathbf{D}$ in place of $b + i\mathbf{R}$.

(Recall that $L^p([u,v]) \subset_c L^1([u,v])$ for $p \in [1,\infty]$ and that $\mathbf{C}_{a,b} := \{z \in \mathbf{C} \,\big|\, a < \operatorname{Re} z < b\}$.)

In particular, we actually have $f(t+i\cdot) \to f(b+i\cdot)$ uniformly on each such $[-u,v]$ (only outside a null set if we do not redefine f), hence in $L^p([u,v])$ for any $p \in [1,\infty]$.

Proof: We take $b = 0$ w.l.o.g.

1° *Case* $B = \mathbf{C}$*:*

1.1° *Choosing* u, v, ε, a, b*:* Let $u_0, v_0 \in \mathbf{R}$ be s.t. $u_0 < v_0$ and $[u_0, v_0]i \subset \Omega$. By Lemma A.2.1(c), we have $\varepsilon := d([u_0, v_0]i, \Omega^c) > 0$. By Lemma B.4.19, there are $u \in [u_0, u_0 - \varepsilon)$ and $v \in [v_0, v_0 + \varepsilon)$ s.t. $f(\cdot + iu), f(\cdot + iv) \in L^1((-\varepsilon, \varepsilon))$. Set $a := -\varepsilon/2,\ c := \varepsilon/2$, so that $[a,c] \times [u,v] \subset \Omega$. Set $A := (a,c) \times (u,v)$. Then $f \in L^1(\gamma)$, where $\gamma := \partial A$ (with the $\mathbf{R}^1$– Lebesgue measure).

1.2° *Defining* $F \in H^\infty(A)$*:* By Lemma D.1.17 (applied to each $(a+\delta, c'-\delta) \times (u+\delta, v-\delta) \subset A$, $\delta > 0$, to keep $(s-z)^{-1}$ bounded), we have $F \in H^\infty(A)$, where

$$F(z) := \int_\gamma h(s,z) f(s)\, ds, \quad h := (2\pi i(s-z))^{-1} \tag{D.40}$$

1.3° *Showing that* $f = F$ *on* $A \setminus i\mathbf{R}$*:* For any $z \in A_c^\delta$, we have $\int_{\gamma_c^\delta} h(s,z) f(s)\, ds = f(z)$ and $\int_{\gamma_a^\delta} h(s,z) f(s)\, ds = 0$, where $\gamma_c^\delta = \partial A_c^\delta$, $A_c^\delta := (\delta, c) \times (u,v) \subset A$, $\gamma_a^\delta = \partial A_a^\delta$, $A_a^\delta := (a, -\delta) \times (u,v) \subset A$, $\delta \in (0,c)$, by the Cauchy Formula ([Rud86, 10.15]). An analogous claim holds for $z \in A_a^\delta$.

Let $z \in A_c^0$. Since $h(\cdot, z)$ is continuous and bounded near $i\mathbf{R}$ and $f(\delta + \cdot) \to f(\cdot)$ in $L^1([u,v])$, as $\delta \to 0$, we have $h(\delta + \cdot, z) f(\delta + \cdot) \to h(\cdot, z) f(\cdot)$ in $L^1([u,v])$ too.

Consequently, $\int_{\gamma_c^\delta} h(s,z)\, ds = f(z)$ and $\int_{\gamma_a^\delta} h(s,z)\, ds = 0$ hold also for $\delta = 0$, by continuity. An analogous claim holds for $z \in A_a^0$. But $\int_{\gamma_a^0 + \gamma_c^0} = \int_\gamma$, hence $F(z) = \int_\gamma h = f(z)$ for every $z \in A_a^0 \cup A_c^0 = A \setminus i\mathbf{R}$.

1.4° $f = F$ *a.e. on* $A \cap i\mathbf{R}$*:* This follows from 1.3°, since $F(\delta + \cdot) - f(\delta + \cdot) \to 0$ in $L^1([u,v])$, as $\delta \to 0$ (due to the continuity of F), so that $F - f = 0$ as an element of $L^1([u,v])$.

1.5° *Case where* $\Gamma := i\mathbf{R} \cap \Omega$ *is connected:* Set $\widetilde{u} := \inf \Gamma/i$, $\widetilde{v} := \sup \Gamma_i$, and choose sequences $\{u_n\}, \{v_n\} \subset \Gamma/i$ s.t. $u_n \to \widetilde{u}$ and $v_n \to \widetilde{v}$. Then by applying 1.1°–1.4° to each $[u_n, v_n]i$, we see that $f(it) := \lim_{\Omega \setminus i\mathbf{R} \ni z \to it} f(z)$ coincides with the original f a.e. (on each $[u_n, v_n]i$, hence a.e. on $(\widetilde{u}, \widetilde{v})i$) and makes f holomorphic on Ω.

1.6° *The original claim for* $B = \mathbf{C}$*:* Set $\Gamma := i\mathbf{R} \cap \Omega$. By Lemma A.2.2, $\Gamma = \cup_{n \in \mathbf{N}} \Gamma_n$, where the sets $\Gamma_n \subset i\mathbf{R}$ are disjoint open intervals. Redefine f by continuity as in 1.5°. For each n and each $z \in \Gamma_n$, we have $d(z, \cup_{k \neq n} \Gamma_k) > 0$, hence (the new) f is holomorphic at z; consequently (the new) f is holomorphic on the whole Ω. Since $\mathbf{N}$ is countable, f becomes redefined on a null set only.

2° *General* B*:*

2.1° $G \in H(\Omega; B)$ *s.t.* $G = f$ *on* $\Omega \setminus i\mathbf{R}$*:* Let $ir \in \Omega$. Then $G(ir)\Lambda := \lim_{t \to 0} \Lambda f(ir + t) \in \mathbf{C}$ exists for all $\Lambda \in B^*$. By Theorem 2.8 of [Rud73], it follows that $G(ir) \in \mathcal{B}(B^*, \mathbf{C}) =: B^{**}$.

Set $G := f$ on $\Omega \setminus i\mathbf{R}$ and define G on $i\mathbf{R} \cap \Omega$ as above. By 1.6°, $G\Lambda \in \mathrm{H}(\Omega)$ for all $\Lambda \in B^*$. By Lemma D.1.1(b), we have $G \in \mathrm{H}(\Omega; B^{**})$. By continuity, $G(z) \in B$ for $z \in i\mathbf{R} \cap \Omega$ too, hence $G \in \mathrm{H}(\Omega; B)$.

2.2° *$G = f$ a.e. on $\Omega \cap i\mathbf{R}$:* Because $f[\Omega] \subset B$ is separable (since Ω is separable), we can w.l.o.g. assume that B is separable. By Lemma A.3.9, there is a countable norming $\{\Lambda_k\}_{k \in \mathbf{N}} \subset B^*$. By the methods of 1.5°–1.6°, it suffices to consider an arbirary $[u, v]i \subset \Omega$ only. We shall use the diagonal argument.

Since $\Lambda_k f(t + i\cdot) \to \Lambda_k f(i\cdot)$ in $\mathrm{L}^1([u, v])$ for all $k \in \mathbf{N}$, there is a null set $N_0 \subset [u, v]$ and a sequence $t_j^0 \to 0+$ s.t. $\Lambda_0 f(t_j^0 + ir) \to \Lambda_0 f(ir)$, as $j \to \infty$, for all $r \in [u, v] \setminus N_0$, by Theorem B.3.2.

Analogously, there is a subsequence $\{t_j^1\}$ of $\{t_j^0\}$ and a null set $N_1 \subset [u, v]$ s.t. $\Lambda_1 f(t_j^1 + ir) \to \Lambda_1 f(ir)$ for all $r \in [u, v] \setminus N_1$. Given $N_0, \ldots, N_k$ and $\{t_j^0\}, \cdots, \{t_j^k\}$ as above, choose a subsequence $\{t_j^{k+1}\}$ of $\{t_j^k\}$ and a null set $N_{k+1} \subset [u, v]$ s.t. $\Lambda_{k+1} f(t_j^{k+1} + ir) \to \Lambda_{k+1} f(ir)$ for all $r \in [u, v] \setminus N_{k+1}$.

Consequently, $\Lambda_k f(t_j^j + ir) \to \Lambda_k f(ir)$, as $j \to \infty$, for all $r \in [u, v] \setminus N$, where $N := \cup_k N_k$. Thus, for such r (hence a.e.) we have $\Lambda_k[f(ir) - G(ir)] = 0$ for all k, hence $f(ir) = G(ir)$.

3° *The final claims:* If we rotate $i\mathbf{R}$, then we just have to use the Lebesgue measure on this oblique line.

If $f : \Omega \to B$ is in $\mathrm{H}(\Omega \setminus \partial b\mathbf{D}; B)$, and $\Lambda f(r\mathrm{e}^{i\cdot}) \to \Lambda f(b\mathrm{e}^{i\cdot})$ in $\mathrm{L}^1([u, v])$, as $t \to b$, for all $\Lambda \in B^*$ and $u, v \in \mathbf{R}$ s.t. $b\mathrm{e}^{[u,v]i} \subset \Omega$, then f can be redefined on a null subset of $\Omega \cap \partial b\mathbf{D}$ so that $f \in (\Omega; B)$.

(The above proof applies, mutatis mutandis; se polar coordinates for the application of Lemma B.4.19 in 1.1° (locally, near an arc of $\partial(r\mathbf{D})$).)

Remark: Analogously, if $f : \Omega \to B$ is in $\mathrm{H}(\Omega \setminus \Gamma; B)$, $\Gamma \subset \Omega$ is a continuous image $g[J]$ of an interval J, and any subinterval $[u, v] \subset g^{-1}[J \cap \Omega]$ has a neighborhood homeomorphic to $(-\varepsilon, \varepsilon) \times (u, v)$, with the homeomorphism satisfying $h(0, \cdot) = g$ on (u, v), and $\Lambda f(h(t, \cdot)) \to \Lambda f(h(0, \cdot))$ in $\mathrm{L}^1([u, v])$, as $t \to 0$, for all $\Lambda \in B^*$, then $f \in \mathrm{H}(\Omega; B)$ after a redefinition on a null subset of Γ. $\square$

Analogously, if f is H^p on both sides of $i\mathbf{R}$ and has same boundary function from both sides, then f is holomorphic and H^p on the whole region:

Lemma D.1.19 ($\mathbf{H}^p(\mathbf{C}_{a,b}) \cap \mathbf{H}^p(\mathbf{C}_{b,c}) = \mathbf{H}^p(\mathbf{C}_{a,c})$) *Let $a < b < c$. Assume that $f : \mathbf{C}_{a,c} \to B$ is in $\mathrm{H}^p(\mathbf{C}_{a,b}; B) \cap \mathrm{H}^p(\mathbf{C}_{b,c}; B)$ and that f is the boundary function of itself on $b + i\mathbf{R}$ from both sides, in the sense of (6.) of Theorem 3.3.1. Then f can be redefined on a null subset of $b + i\mathbf{R}$ so that we get $f \in \mathrm{H}^p(\mathbf{C}_{a,c}; B)$.*

Proof: By Proposition D.1.18, we get $f \in \mathrm{H}(\mathbf{C}_{a,c}; B)$ (since $\|\cdot\|_{\mathrm{L}^1([u,v])} \leq M\|\cdot\|_{\mathrm{L}^p(\mathbf{R})}$). By assumption (6.), $f \in \mathrm{H}^p(\mathbf{C}_{a,c}; B)$ (and $\|f\|_{\mathrm{H}^p(\mathbf{C}_{a,c};B)} = \max\{\|f\|_{\mathrm{H}^p(\mathbf{C}_{a,b};B)}, \|f\|_{\mathrm{H}^p(\mathbf{C}_{b,c};B)}\}$). $\square$

If f is in H^2 on both $\mathbf{C}^+$ and on $\mathbf{C}^-$, then f is a constant:

Proposition D.1.20 ($\mathbf{H}^p_+ \cap \mathbf{H}^p_- = \mathbf{C}$) *Let $f_\pm \in \mathrm{H}^p(\mathbf{C}^\pm_\omega; B)$, where $\omega \in \mathbf{R}$, $p \in [1, \infty]$.*

If $\Lambda f_+ = \Lambda f_-$ on $\omega + i\mathbf{R}$ for each $\Lambda \in B^*$, then $f_+ \equiv x \equiv f_-$ for some $x \in B$ ($x = 0$ if $p < \infty$).

Analogously, if $f_\pm \in \mathrm{H}^p(\mathbf{D};B)$ and $\Lambda f_+(z) = \Lambda f_-(z^{-1})$ for a.e. $z \in \partial\mathbf{D}$, for each $\Lambda \in B^*$, then $f_+ \equiv x \equiv f_-$ for some $x \in B$.

Proof: 1° *Case* $\mathrm{H}^p(\mathbf{C}^\pm_\omega;B)$*:* Naturally, by "$\Lambda f_+ = \Lambda f_-$" we mean that the $\mathrm{L}^p(\omega + i\mathbf{R})$ boundary functions are equal a.e.; such functions exist, by Theorem 3.3.1(a2).

Assume w.l.o.g. that $\omega = 0$. Let $\Lambda \in B^*$. By Lemma D.1.19, we have $\Lambda f_+ = f_\Lambda = \Lambda f_-$ for some $f_\Lambda \in \mathrm{H}^p(\mathbf{C}^+)$.

But then $f_\Lambda \in \mathrm{H}^\infty(\mathbf{C}^+_0)$, by (6.4.3) of [HP]; analogously, $f_\Lambda \in \mathrm{H}^\infty(\mathbf{C}^-_1)$, hence $f_\Lambda \in \mathrm{H}^\infty(\mathbf{C})$.

By the Liouville Theorem, $f_\Lambda \equiv x_\Lambda$ for some $x_\Lambda \in \mathbf{C}$. This holds for each $\Lambda \in B^*$, hence f_+ and f_- are equal to a single constant $x \in B$. If $p < \infty$, then, obviously, $x = 0$.

2° *Case* $\mathrm{H}^p(\mathbf{D};B)$*:* Set $f := f_+$ on $\mathbf{D}$ and $f(z) := f_-(z^{-1})$ for $z \in \overline{\mathbf{D}}^c$. As above, we see that we Λf can be extended $f_\Lambda \in \mathrm{H}(\mathbf{D})$. Since $f_\Lambda(\infty) = \Lambda f_-(0)$, f_Λ is bounded, hence f_Λ is a constant. Thus, f is a constant, as in 1°. □

If f belongs to L^p_r for two different r's, then $\widehat{f}$ is holomorphic on the corresponding strip:

Proposition D.1.21 ($\mathcal{L}[\mathrm{L}^p_a \cap \mathrm{L}^p_b] \subset \mathrm{H}(\mathbf{C}_{a,b};*)$) *Let $f \in \mathrm{L}^p_a(\mathbf{R};B) \cap \mathrm{L}^p_b(\mathbf{R};B)$, $p \in [1,\infty]$, $a < b$. Then*

(a1) We have $\widehat{f} \in \mathrm{H}^\infty(\mathbf{C}_{a',b'};B)$ whenever $a < a' < b' < b$; in particular, $\widehat{f} \in \mathrm{H}(\mathbf{C}_{a,b};B)$. Moreover, $f \in \mathrm{L}^1_r(\mathbf{R};B)$ for all $r \in (a,b)$, hence $\widehat{f}(s)$ converges absolutely on $\mathbf{C}_{a,b}$.

(a2) The mapping $\mathrm{L}^p_a(\mathbf{R};B) \cap \mathrm{L}^p_b(\mathbf{R};B) \to \mathrm{H}^\infty(\mathbf{C}_{a',b'};B)$ is continuous.

(b) If $p = 1$, then $\widehat{f} \in \mathcal{C}_{\mathrm{bu}}(\overline{\mathbf{C}_{a,b}};B) \cap \mathrm{H}^\infty(\mathbf{C}_{a,b};B)$ and $\|\widehat{f}(s)\|_B \le \|\pi_+ f\|_{\mathrm{L}^p_a} + \|\pi_- f\|_{\mathrm{L}^p_b}$ for all $s \in \overline{\mathbf{C}_{a,b}}$.

(c) **($\mathcal{L}[\mathrm{L}^2_a \cap \mathrm{L}^2_b] = \mathrm{H}^2(\mathbf{C}_{a,b};H)$)** *Assume that $p = 2$ and that $B = H$ is a Hilbert space. Then $\widehat{f} \in \mathrm{H}^2(\mathbf{C}_{a,b};H)$, and $\widehat{f}$ has the nontangential boundary function $\widehat{f}$ (the Plancherel transform of f) on $a + i\mathbf{R}$ and on $b + i\mathbf{R}$ (in the sense of (1.), (2.) and (4.)–(6.) of Theorem 3.3.1(a)) and $(2\pi)^{-1/2}\|\widehat{f}(r + i\cdot)\|_2 \le \|\pi_+ f\|_{\mathrm{L}^2_a} + \|\pi_- f\|_{\mathrm{L}^2_b}$ ($r \in [a,b]$).*

Conversely, if $\widehat{g} \in \mathrm{H}^2(\mathbf{C}_{a,b};H)$, then there is $g \in \mathrm{L}^2_a(\mathbf{R};H) \cap \mathrm{L}^2_b(\mathbf{R};H)$ s.t. $\widehat{g} = \mathcal{L}g$.

Recall that $\mathrm{H}^p(\mathbf{C}_{a,b};B) := \{f \in \mathrm{H}(\mathbf{C}_{a,b};B) \mid \sup_{r \in (a,b)} \|f(r + i\cdot)\|_p < \infty\}$.

Proof: Note that $f \in \pi_+ \mathrm{L}^p_a(\mathbf{R};B) \cap \pi_- \mathrm{L}^p_b(\mathbf{R};B)$.

(a1) By Lemma D.1.10(a1)&(a2), we have $\widehat{\pi_+ f} \in \mathrm{H}^\infty(\mathbf{C}^+_{a'};B)$ and $\widehat{\pi_- f} \in \mathrm{H}^\infty(\mathbf{C}^-_{b'};B)$, hence $\widehat{f} \in \mathrm{H}^\infty(\mathbf{C}_{a',b'};B)$. The second claim follows from the proof of Lemma D.1.10(a2).

(a2) See the proof of (a1) and that of Lemma D.1.10(a2) for a bound of form $\|\widehat{f}\|_{\mathrm{H}^\infty} \le M(\|f\|_{\mathrm{L}^p_a} + \|f\|_{\mathrm{L}^p_b}$.

(b) Cf. the proof of (a) (use Lemma D.1.11(a1')).

(c) 1° *Properties of $\widehat{f}$:* Note that $\mathcal{L}\pi_+ f \in \mathrm{H}^2(\mathbf{C}^+_a; B)$ and $\mathcal{L}\pi_- f \in \mathrm{H}^2(\mathbf{C}^-_b; B)$, by Lemma D.1.15. By Theorem 3.3.1(a2), $\mathcal{L}\pi_+ f$ has a boundary function on $a + i\mathbf{R}$; by continuity, so does $\mathcal{L}\pi_- f$ too (in the sense of (1.) and (2.) of Theorem 3.3.1(a); claim (6.) (which implies (5.) and (4.)) follows from the strong continuity of Lemma D.1.8(a1)).

The "mirror image" boundary function on $b + i\mathbf{R}$ (from the left) is obtained analogously. The norm estimate is obvious.

2° *The converse claim:* For each $r \in (a,b)$, there is $g_r \in \mathrm{L}^2_r(\mathbf{R}; H)$ s.t. $\widehat{g} = \mathcal{L}g_r$ a.e. on $r + i\mathbf{R}$, by the Fourier–Plancherel Theorem.

Let $d \in (a,b)$. Set $F := \pi_+ g_d$, $G := \pi_- g_d$. Then $\mathcal{L}G \in \mathrm{H}^2(\mathbf{C}^-_d; H)$ is the boundary function of itself, by Theorem 3.3.1(b) (whose proof is obviously independent on this lemma, relying on [RR] and the part of Appendices preceding this lemma), and so is $\widehat{g} \in \mathrm{H}^2(\mathbf{C}_{a,d}; H)$ on $d + i\mathbf{R}$, hence $\widehat{g} - \mathcal{L}G = \mathcal{L}F$ is the boundary function of $\widehat{h}$ (a.e.) on $d + i\mathbf{R}$. But $\mathcal{L}F \in \mathrm{H}^2(\mathbf{C}^+_d; H)$, hence $\widehat{h}$ extends to a function $\widehat{h} \in \mathrm{H}^2(\mathbf{C}^+_a; H)$, by Lemma D.1.19.

It follows that $\widehat{h} = \mathcal{L}h$ on $\bar{\mathbf{C}}^+_a$ for some $h \in \mathrm{L}^2_a(\mathbf{R}_+; H)$. But then $F = h$. Analogously, we can show that $G \in \mathrm{L}^2_b(\mathbf{R}_-; H)$. It follows that $g_d = F + G \in \mathrm{L}^2_a \cap \mathrm{L}^2_b$. Because $\widehat{g} = \mathcal{L}g_d$ on $d + i\mathbf{R}$, we have $\widehat{g} = \mathcal{L}g_d$ on the whole $\mathbf{C}_{a,b}$, by Lemma D.1.2(e). □

If $f \in \mathrm{L}^\infty(\mathbf{R}; B)$, then $\|f\chi_A\|_1 \le \|f\|_\infty m(A) < \infty$ whenever $m(A) < \infty$. Thus, we can then apply the following test:

Lemma D.1.22 *Let B_0 be a closed subspace of the Banach space B, and $f \in \mathrm{L}^\infty(\mathbf{R}; B)$. If $\mathcal{L}f\chi_A \in \mathcal{C}_0(\mathbf{R}; B_0)$ whenever $m(A) < \infty$, then $f \in \mathrm{L}^\infty(\mathbf{R}; B_0)$ i.e., $f(t) \in B_0$ for a.e. $t \in \mathbf{R}$.*

Note that always $\mathcal{L}f\chi_A \in \mathcal{C}_0(\mathbf{R}; B)$, because $f\chi_A \in \mathrm{L}^1$.

Proof: (We could have as well assume that $f \in \mathrm{L}^1_{\mathrm{loc}}$ and that A is bounded and measurable.)

Set $E := f^{-1}[B \setminus B_0]$. We assume that $m(E) > 0$, derive a contradiction, and deduce that $f(t) \in B_0$ for a.e. $t \in \mathbf{R}$.

Because $f_{|E}$ is nowhere zero, Lemma B.2.8(b) provides us $A \subset E$ and $\mathrm{L} \in B^*$ s.t. $0 < m(A)$, $\Lambda = 0$ on B_0, and $\operatorname{Re}\Lambda > 1$ on A. Choose $A' \subset A$ s.t. $0 < m(A') < \infty$. Then $0 \ne \Lambda f\chi_{A'} \in \mathrm{L}^1(\mathbf{R}; B)$, hence $0 \ne \mathcal{L}\Lambda f\chi_{A'} = \Lambda\mathcal{L}f\chi_{A'} \in \mathcal{C}_0(\mathbf{R})$, by Lemma D.1.11(a3). But $\mathcal{L}f\chi_{A'}$ is B_0-valued, hence $\Lambda\mathcal{L}f\chi_{A'} = 0$, a contradiction. □

If a function is holomorphic around $\partial\mathbf{D}$, then it is the inverse Cayley transform of some $\mathrm{MTI}^{\mathrm{L}^1}$ operator.

Lemma D.1.23 *Assume that $\Omega \subset \mathbf{C}$ is open and s.t. $\partial\mathbf{D} \subset \Omega$, and that $g \in \mathrm{H}(\Omega; B)$. Then there are $b \in B$ and $f \in \mathrm{L}^1(\mathbf{R}; B)$ s.t. $g = b + \widehat{f} \circ \phi^{-1}_{\mathrm{Cayley}}$.*

Proof: 1° $b_0 = b_1 = b_2 = b_3 = 0$ *w.l.o.g.:* Write g as $\sum_{k=0}^{\infty} b_k(z+1)^k$ on a neighborhood of $K := \{z \in \mathbf{C} \,\big|\, |z+1| \le \varepsilon\} \subset \Omega$ (for some $\varepsilon > 0$). Because

$$(\cdot + 1) = \widehat{2\pi_+ e^{-\cdot}} \circ \phi_{\text{Cayley}}^{-1} = \widehat{F} \circ \phi_{\text{Cayley}}^{-1}, \tag{D.41}$$

where $\widehat{F} = \widehat{2\pi_+ e^{-\cdot}} \in \mathcal{L}[\mathrm{L}^1(\mathbf{R}_+; B)]$, we can take $\widehat{F_3} := \sum_{k=0}^{3} b_k \widehat{F}^k$, to obtain that $g - \widehat{F_3} \circ \phi_{\text{Cayley}}^{-1} = \sum_{k=4}^{\infty} b_k(z+1)^k$. By Lemma D.1.12(c'), $\widehat{F_3} = b' + \widehat{f_3}$ for some $b' \in B$ and $f_3 \in \mathrm{L}^1(\mathbf{R}_+; B)$ (since $\pi_+ e^{-\cdot} \in \mathrm{L}^1(\mathbf{R}_+; B)$). Thus, we have reduced the problem to the case where $b_0 = b_1 = b_2 = b_3 = 0$.

2° Set $h := g \circ \phi_{\text{Cayley}}$. Assuming $b_0 = b_1 = b_2 = b_3 = 0$, it follows that $z \mapsto g(z)/z^4$ is holomorphic around K; let $M := \max_{z \in K} \|g(z)/z^4\|_B$. Then $\|g(z)\|_B \le |z+1|^4 M$ for $z \in K$, i.e., $\|h(s)\|_B \le |2/(1+s)|^4 M$ when $|2/(1+s)| < \varepsilon$; in particular $h_{|i\mathbf{R}} \in \mathrm{L}^1(i\mathbf{R}; B)$, because h is continuous on $i\mathbf{R}$.

3° Because ϕ_{Cayley}' and ϕ_{Cayley}'' are bounded on $i\mathbf{R}$, it follows (as in 2°) from the chain rule that also h' and h'' are in $\mathrm{L}^1(i\mathbf{R}; B)$, hence

$$f := \frac{1}{2\pi} \int_{\mathbf{R}} e^{ix\cdot} h(ix)\, dx = \frac{1}{2\pi} \widehat{h(i\cdot)}(-i\cdot) \in \mathrm{L}^1(\mathbf{R}; B) \tag{D.42}$$

and $h = \widehat{f}$, by Lemma D.1.11(e2)&(e1). $\square$

Sometimes the properties of a transfer function $\widehat{\mathbb{D}} : i\mathbf{R} \to B$ can be recovered by multiplying it with the following function:

Lemma D.1.24 *Let $\varepsilon > 0$, $p \in [1, \infty]$. For each $t > 0$ and $r \in \mathbf{R}$, we set*

$$f_{t,r}(x) := \pi_+ 2t^{3/2} x e^{-(t-ir)x}, \quad \textit{i.e.,} \quad \widehat{f}_{t,r}(s) = \widehat{f}_{t,0}(s - ir) = 2t^{3/2}(s + t - ir)^{-2}. \tag{D.43}$$

We have $f_{t,r} \in \mathrm{L}^2_{-t/2}(\mathbf{R}_+)$, $\|f_{t,r}\|_2 = 1$, and $\widehat{f} \in \mathrm{L}^p(\omega + i\mathbf{R}) \cap \mathcal{C}_0(\omega + i\mathbf{R})$ ($\omega \ge 0$). Moreover, the following hold:

(a) There is $\delta > 0$ s.t.

$$\sup_{s \in \mathbf{C},\, |s - ir| \ge \varepsilon} |\widehat{f}_{t,r}(s)| < \varepsilon \ \ \& \ \ \|\widehat{f}_{t,r}\|_{\mathrm{L}^p(\{i\rho \,|\, |\rho - r| \ge \varepsilon\})} < \varepsilon \ \ \& \ \ \|\widehat{f}_{t,r}\|_{\mathrm{L}^p(\omega + i\mathbf{R})} < \varepsilon \tag{D.44}$$

whenever $0 < t \le \delta$, $r \in \mathbf{R}$, $\omega \ge \varepsilon$.

(b) If $E \subset i\mathbf{R}$ is measurable and $p \in [1, 2]$, then, for a.e. $r \in E$, there is $\delta = \delta_{f,E,r,\varepsilon} > 0$ s.t.

$$\|\widehat{f}_{t,r}\|_{\mathrm{L}^p(i\mathbf{R} \setminus E_{\varepsilon,r})} < \varepsilon \quad (0 < t \le \delta), \tag{D.45}$$

where $E_{\varepsilon,r} := \{i\rho \in E \,\big|\, |\rho - r| < \varepsilon\}$.

(c) For any $g \in \mathrm{L}^p(i\mathbf{R}; B)$ and $p \in [1, \infty)$ (or $g \in \mathcal{C}_0(i\mathbf{R}; B)$ and $p = \infty$) and $t > 0$, there is $R \in \mathbf{R}$ s.t. $\int_{i\mathbf{R}} \|\widehat{f}_{t,r} g\|_B \, dm < \varepsilon$ and $\int_{i\mathbf{R}} |\widehat{f}_{t,r}|^2 \|g\|_B \, dm < \varepsilon$ whenever $r \in \mathbf{R}$, $|r| \ge R$.

Proof: The claims at the beginning of the lemma follow from straightforward computations (which we omit).

(a) We shall assume that $r = 0$ (use translation), w.l.o.g.

1° $\sup_{s\in\mathbf{C},\,|s|\geq\varepsilon}|\widehat{f}_{t,0}(s)| < \varepsilon$: Set $\delta_1 := \min\{\varepsilon/2, (\varepsilon^2/9)^{2/3}\}$. Then, for all $s \in \mathbf{C}$ and $t \in (0,\delta_1]$, we have

$$|s| \geq \varepsilon \implies |\widehat{f}(s)| \leq \frac{2t^{3/2}}{(|s|-t)^2} \leq \frac{2\varepsilon^2/9}{(\varepsilon-\varepsilon/2)^2} = \frac{2\varepsilon^2/9}{\varepsilon^2/4} < \varepsilon, \tag{D.46}$$

which provides the first inequality in (D.45).

2° $\|\widehat{f}_{t,0}\|_{\mathrm{L}^p(\omega+i\mathbf{R})} \to 0$: For $p = \infty$ this follows from 1°. For $p < \infty$, we have

$$\|\widehat{f}_{t,0}(\omega+i\cdot)\|_p^p \leq \|2^p|\varepsilon+i\cdot|^{-2p}t^{3p/2}\|_1 \to 0, \tag{D.47}$$

as $t \to 0+$, for any $\omega \geq \varepsilon$, because $|\varepsilon+i\cdot|^{-2p} = |\varepsilon^2+\cdot^2|^{-p} \in \mathrm{L}^1$. Therefore, there is $\delta_3 > 0$ s.t. third inequality in (D.45) is achieved for all $t \in (0,\delta_3]$.

3° $\|\widehat{f}_{t,0}\|_{\mathrm{L}^p(\{i\rho\,\big|\,|\rho|\geq\varepsilon\})} < \varepsilon$: We have $|\widehat{f}_{t,0}(i\rho)| = 2t^{3/2}/(\rho^2+t^2) \leq 2t^{3/2}/\varepsilon^2 \to 0$, as $t \to 0+$, whenever $|\rho| \geq \varepsilon$. Case $p = \infty$ follows directly from this; case $p < \infty$ follows from this and the Dominated Convergence Theorem, because $\widehat{f}_{t,0} \in \mathrm{L}^p$, as noted in 2°.

(b) 1° *Assumptions:* By Section 7.11 of [Rud86], we have

$$\lim_{\varepsilon\to 0+}\frac{m(E^c\cap(r-\varepsilon,r+\varepsilon))}{m(r-\varepsilon,r+\varepsilon)} = 0 \tag{D.48}$$

for a.e. $r \in E$. We assume that r is such and $\varepsilon > 0$, and find $\delta > 0$ s.t. (D.45) holds. W.l.o.g. we assume that $r = 0$ (use translation of E and f).

2° $\|\widehat{f}_{t,0}\|_{\mathrm{L}^p(i\mathbf{R}\setminus E_{\varepsilon,r})} < \varepsilon$: Choose $R > 0$ s.t. $2^{p+1}\int_R^\infty |y^2+1|^{-p}\,dy < \varepsilon^p/4^p$. Choose $\eta \in (0,\varepsilon)$ s.t. $m(E^c\cap(-\gamma,\gamma))/2\gamma < \varepsilon_1 := \varepsilon^p/3^p2^{p+1}R$ for all $\gamma \in (0,\eta]$. Set $\delta := \min\{1, \eta/R, \varepsilon^4/8, \varepsilon/2, \varepsilon^3\}$. Then, for any $t \in (0,\delta]$, we have

$$\int_{|y|\geq Rt}|\widehat{f}_{t,0}(iy)|^p\,dy = 2^{p+1}t^{3p/2}\int_{y\geq Rt}|y^2+t^2|^{-p}\,dy = 2^{p+1}t^{3p/2}\int_{u\geq R}|u^2t^2+t^2|^{-p}t\,du \tag{D.49}$$

$$= 2^pt^{3p/2-2p+1}\int_{u\geq R}|u^2+1|^{-p}\,du < 2^pt^{1-p/2}\varepsilon^p/4^p \leq \varepsilon^p/2^p, \tag{D.50}$$

because $1-p/2 \geq 0$. Moreover, $Rt < \eta$, hence

$$\int_{\{|y|<Rt,\,y\in E^c\}}|\widehat{f}_{t,0}(iy)|^p\,dy \leq \|\widehat{f}_{t,0}\|_\infty^p\cdot 2Rt\cdot\varepsilon_1 \leq \left(\frac{2t^{3/2}}{t^2}\right)^p\cdot 2tR\varepsilon_1 \tag{D.51}$$

$$\leq 2^{p+1}t^{-p/2+1}R\varepsilon_1 = \varepsilon^p/3^p \tag{D.52}$$

Because $\varepsilon^p/2^p+\varepsilon^p/3^p < \varepsilon^p$, we have established (b).

(c) Because $\widehat{f}, |\widehat{f}|^2 \in \mathrm{L}^{p/(p-1)}(i\mathbf{R})\cap\mathcal{C}_0(i\mathbf{R})$, this follows from Lemma B.3.13. $\square$

The following function is handy when dealing with Fourier transforms:

Lemma D.1.25 *Define* $\phi(t) := \mathrm{e}^{-t^2/2}$. *Then* $\widehat{\phi}$ *and* $\widehat{\phi}^{-1}$ *are entire functions and* $\widehat{\phi}(s) = \sqrt{2\pi}\mathrm{e}^{s^2/2} \neq 0$ *for* $s \in \mathbf{C}$; *in particular,* $\widehat{\phi}(ir) = \sqrt{2\pi}\mathrm{e}^{-r^2/2} > 0$ *for* $r \in \mathbf{R}$.

Moreover, $\phi \in \mathcal{S}(\mathbf{R}) \cap \mathrm{L}^p_\omega$ *for all* $p \in [1,\infty]$ *and* $\omega \in \mathbf{R}$, *and the functions of the form* $\sum_{k=1}^n \tau(t_k)\phi b_k$, *where* $n \in \mathbf{N}+1$, *and* $t_k \in \mathbf{R}$ *and* $b_k \in B$ *for all* $k \in \mathbf{N}$, *are dense in* $\mathrm{L}^1(\mathbf{R};B)$ *as well as in* $\mathrm{L}^2(\mathbf{R};B)$.

Proof: By the dominated convergence theorem, we can exchange the order of integration and differentiation to obtain that $\widehat{\phi}$ is holomorphic everywhere ($\widehat{\phi} \in \mathrm{H}(\mathbf{C})$). By [Rauch, pp. 64–65], $\widehat{\phi}(s) = \sqrt{2\pi}\mathrm{e}^{s^2/2}$ holds for $s \in i\mathbf{R}$, hence it holds everywhere, by Lemma D.1.2(e). One easily verifies that $\phi \in \mathcal{S}(\mathbf{R}) \cap \mathrm{L}^p_\omega(\mathbf{R})$ for all p, ω.

Because the Fourier transform of ϕ is nowhere zero, the density claim holds for $B = \mathbf{C}$, by, e.g., Theorem 9.5 of [Rud73] (the L^1 case) and p. 145 of [Katzn] (the L^2 case); the general case follows from the density of finite-dimensional functions in L^p (Theorem B.3.11). □

We finish this section by presenting one more vector-valued extension of a standard result:

Lemma D.1.26 *The span of* $\{\pi_-\mathrm{e}^{s\cdot}u_0 \,|\, s \in (\omega,\omega+1),\ u_0 \in U\}$ *is dense in* $\mathrm{L}^2_\omega(\mathbf{R}_-;U)$.

Proof: We take $\omega = 0$ w.l.o.g. Simple functions are dense in $\mathrm{L}^2(\mathbf{R}_-;U)$, by Theorem B.3.11, hence so is the span of $\{\phi u_0 \,|\, \phi \in \mathrm{L}^2(\mathbf{R}_-),\ u_0 \in U\}$. Therefore, we may and will assume that $U = \mathbf{C}$ w.l.o.g.

Let $u \in \mathrm{L}^2(\mathbf{R}_-)$. If $u \in \{\pi_-\mathrm{e}^{s\cdot} \,|\, s \in (0,1)\}^\perp$, i.e.,

$$0 = \langle \mathrm{e}^{s\cdot}, u\rangle_{\mathrm{L}^2(\mathbf{R}_-)} = \int_{-\infty}^0 \mathrm{e}^{st}u(t)\,dt = \int_0^\infty \mathrm{e}^{-st}u(-t)\,dt = \widehat{\text{Я}u}(s), \tag{D.53}$$

for all $s \in (0,1)$, then $\widehat{\text{Я}u} = 0$ on $\mathbf{C}^+$, hence then $u = 0$, by Lemma D.1.2(e). Therefore, $\{\mathrm{e}^{s\cdot} \,|\, s \in (0,1)\}$ is dense in $\mathrm{L}^2(\mathbf{R}_-)$. □

Notes

As obvious from the proofs, many of the results of this appendix are well known at least in the scalar case. Lemma D.1.23 is due to O. Staffans.

Further results on holomorphic vector-valued functions are given in, e.g., [HP]. General vector-valued measures (with MTIC as a special case; cf. Lemma D.1.12 and Section 2.6) are treated in [DU], [Dinculeanu], [Dobrakov], [Park] and in references therein.

Appendix E

Interpolation Theorems

Mathematicians have announced the existence of a new whole number which lies between 27 and 28. "We don't know why it's there or what it does," says Cambridge mathematician, Dr. Hilliard Haliard, "we only know that it doesn't behave properly when put into equations, and that it is divisible by six, though only once."

— On The Hour

Here we present the Riesz–Thorin Interpolation Theorem, the Hausdorff–Young Theorem and similar results. In these results a function mapping $X_1 \to Y_1$ and $X_2 \to Y_2$, continuously, is shown to map "$X_r \to Y_r$ when $1 < r < 2$" when X_r, Y_r are suitable spaces (of vector-valued functions).

We use the assumptions of Chapter B in this chapter; in particular, the scalar field **K** may be either **C** or **R**.

E.1 Interpolation theorems ($\mathrm{L}^{p_1} + \mathrm{L}^{p_2} \to \mathrm{L}^{q_1} + \mathrm{L}^{q_2}$)

We start by a few auxiliary lemmas and definitions.

Lemma E.1.1 ($\mathbf{L}^{p} + \mathbf{L}^{q}$) *Let* $p \in [1,\infty]$. *The space* $\mathrm{L}^1(Q;B) + \mathrm{L}^\infty(Q;B)$ *is a Banach space. For all measurable* $f : Q \to B$, *we have*

$$\frac{1}{2}\|f\|_{\mathrm{L}^1\cap\mathrm{L}^\infty} \le \|f\|_p \le \|f\|_1 + \|f\|_\infty. \tag{E.1}$$

Finally, $\mathrm{L}^q \cap \mathrm{L}^r \underset{c}{\subset} \mathrm{L}^p \underset{c}{\subset} \mathrm{L}^q + \mathrm{L}^r$, *when* $1 \le q \le p \le r \le \infty$.

Thus, L^q and L^r are sum-compatible. See Lemmas A.3.17 and A.3.18 for the norms of $X+Y$ and $X \cap Y$.

Proof: 1° $\mathrm{L}^1(Q;B) + \mathrm{L}^\infty(Q;B)$ *is a Banach space:* Let $0 \ne f \in \mathrm{L}^1 + \mathrm{L}^\infty$. Find $\varepsilon > 0$ s.t. $\mu(E_\varepsilon) > 0$, where $E_\varepsilon := \{q \in Q \,\big|\, \|f(q)\| > \varepsilon\}$. If $f = g + h$ and $\|h\|_\infty < \varepsilon$, then $\|g\|_1 > \int_{E_\varepsilon} \varepsilon \ge \varepsilon\mu(E_\varepsilon)$. Therefore $\|f\|_{\mathrm{L}^1+\mathrm{L}^\infty} \ge \min\{\varepsilon, \mu(E_\varepsilon)\varepsilon\} > 0$. Thus, $f \ne 0 \Rightarrow \|f\| > 0$.

For sums we have

$$\|f+g\|_+ \leq \inf_{f=f_1+f_\infty,\ g=g_1+g_\infty} (\|f_1+g_1\|_1 + \|f_\infty + g_\infty\|_\infty) \tag{E.2}$$

$$\leq \inf_{f=f_1+f_\infty,\ g=g_1+g_\infty} (\|f_1\|_1 + \|g_1\|_1 + \|f_\infty\|_\infty + \|g_\infty\|_\infty) \leq \|f\|_+ + \|g\|_+. \tag{E.3}$$

Obviously, $\|\alpha f\|_+ = |\alpha|\|f\|_+$ for $\alpha \in \mathbf{K}$. Thus, we can apply by Lemma A.3.17 (set $Z := \mathrm{L}^1 + \mathrm{L}^\infty$), and deduce that $\mathrm{L}^1 + \mathrm{L}^\infty$ is complete.

2° $\|f\|_{\mathrm{L}^1+\mathrm{L}^\infty} \leq 2\|f\|_p$*:* W.l.o.g., we assume that $1 < p < \infty$ and $\|f\|_p = 1$. Set $g := f\chi_{E_1}$, $h := f - g$. Then $\|h\|_\infty \leq 1$ and

$$\mu(E_1) \leq \int_{E_1} \|f\|_B^p \, d\mu \leq \|f\|_p^p = 1, \tag{E.4}$$

hence $\|g\|_1 \leq \|f\|_p \|\chi_{E_1}\|_q \leq 1 \cdot 1 = \|f\|_p$, where $p^{-1} + q^{-1} = 1$. Thus, $\|f\| \leq \|g\|_1 + \|h\|_1 \leq 2$.

3° $\|f\|_p \leq \|f\|_1 + \|f\|_\infty$*:* This follows from Lemma B.3.14.

4° *Embeddings:* By (E.1), the embeddings are continuous for $q = 1$, $r = \infty$. Thus, we can apply Lemmas A.3.17 and A.3.18 to obtain those claims in the general case. □

Given $p \in [1, \infty]$ and a measurable $f : Q \to [0, +\infty)$ s.t. $f > 0$ a.e., we define $\mathrm{L}^{p,f}(Q; B)$ to be the space of (equivalence classes of) measurable functions $g : Q \to B$ s.t.

$$\|g\|_{p,f} := \|fg\|_p < \infty. \tag{E.5}$$

Obviously, $g \mapsto fg$ is an isometric isomorphism of $\mathrm{L}^{p,f}$ onto L^p; in particular, $\mathrm{L}^{p,f}$ is a Banach space. Note also that the zero elements of all such spaces are equal. Of these spaces, we are mainly interested in spaces $\mathrm{L}^p_r := \mathrm{L}^{p,e^{-r\cdot}}$. Such spaces are sum-compatible for all p's and r's:

Lemma E.1.2 ($\mathbf{L}^p_r + \mathbf{L}^q_{r'}$) *Let $1 \leq p_1 \leq p_2 \leq p_3 \leq \infty$, and let $f_k : Q \to [0, +\infty)$ be s.t. $f_k > 0$ a.e. ($k = 1, 2, 3$). Then*

Then L^{p_1,f_1} and L^{p_3,f_3} are sum-compatible. Moreover,

(a) L^{p_1,f_1} and L^{p_3,f_3} are sum-compatible.

(b1) if $f_k \geq f_2$ ($k = 1, 3$), then $\mathrm{L}^{p_1,f_1} \cap \mathrm{L}^{p_3,f_3} \underset{c}{\subset} \mathrm{L}^{p_2,f_2}$.

(b2) if $f_k \leq f_2$ ($k = 1, 3$), then $\mathrm{L}^{p_2,f_2} \underset{c}{\subset} \mathrm{L}^{p_1,f_1} + \mathrm{L}^{p_3,f_3}$.

In particular, if $J \subset \mathbf{R}$ is an interval and $\omega_k \in \mathbf{R}$ ($k = 1, 3$), then $\mathrm{L}^{p_1}_{\omega_1}$ and $\mathrm{L}^{p_3}_{\omega_3}$ are sum-compatible.

Analogously, $\ell^{p_1}_{r_1}$ and $\ell^{p_3}_{r_3}$ are sum-compatible for $r_1, r_3 > 0$ (see (13.2)).

Proof: (a) 1° *If $f_2 \geq f_1$, then $\mathrm{L}^{p,f_2} \underset{c}{\subset} \mathrm{L}^{p,f_1}$:* Obviously, $\|\cdot\|_{p,f_1} \leq \|\cdot\|_{p,f_2}$.

2° *Case $p_1 = p_3$:* By 1°, we have $\mathrm{L}^{p_1,f_k} \underset{c}{\subset} \mathrm{L}^{p_1,f}$ ($k = 1, 3$), where $f := \max\{f_1, f_3\}$. Thus, L^{p_1,f_1} and L^{p_1,f_3} are sum-compatible.

3° *Case $f_1 = f_3$:* Use Lemma E.1.1 and the isometric isomorphism $f_1\cdot : \mathrm{L}^{p,f_1}\to\mathrm{L}^p$ to see that $\|\cdot\|_{\mathrm{L}^{p_1,f_1}+\mathrm{L}^{p_3,f_1}}$ is a norm. Thus, $\mathrm{L}^{p_1,f_1}+\mathrm{L}^{p_3,f_1}$ is a Banach space, hence L^{p_1,f_1} and L^{p_1,f_3} are sum-compatible.

4° *General case:* Set $f := \max\{f_1,f_3\}$. By 3°, $Z := \mathrm{L}^{p_1,f}+\mathrm{L}^{p_3,f}$ is a Banach space. By 1°, $\mathrm{L}^{p_k,f}\underset{c}{\subset} Z$ $(k=1,3)$, hence $\mathrm{L}^{p_1,f}$ and $\mathrm{L}^{p_3,f}$ are sum-compatible.

(b1) Let $f_k\ge f_2$ $(k=1,3)$. Then $\mathrm{L}^{p_k,f_k}\underset{c}{\subset}\mathrm{L}^{p_k,f_2}$ $(k=1,3)$, by 1°. Therefore, $\cap_k\mathrm{L}^{p_k,f_k}\underset{c}{\subset}\cap_k\mathrm{L}^{p_k,f_2}$ $(k=1,3)$, by Lemma A.3.19(c1)&(c2). But $\cap_k\mathrm{L}^{p_k,f_2}\underset{c}{\subset}\mathrm{L}^{p_2,f_2}$, by Lemma E.1.1 (and the isomorphism, cf. 3°), hence (b1) holds.

(b2) The proof of (b2) is analogous to that of (b1) and hence omitted.

The final claim: Now $\mathrm{L}^{p_k}_{\omega_k} := \mathrm{L}^{p_k,\mathrm{e}^{-\omega_k\cdot}}\underset{c}{\subset}\mathrm{L}^{p_1,f}+\mathrm{L}^{p_3,f}$ $(k=1,3)$, by (b2), where $f := \min_{k=1,3}\mathrm{e}^{-\omega_k\cdot}$. □

The following concept is required for some interpolation results:

Definition E.1.3 ($\mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$) *Let (X_1,X_2) and (Y_1,Y_2) be sum-compatible pairs of normed spaces. Then we write $T\in\mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$ if $T\in\mathrm{Hom}(X_1+X_2,Y_1+Y_2)$ is s.t. $T\in\mathcal{B}(X_1,Y_1)$, $T\in\mathcal{B}(X_2,Y_2)$. We set*

$$\|T\|_{\mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)} := \max\{\|T\|_{\mathcal{B}(X_1,Y_1)},\|T\|_{\mathcal{B}(X_1,Y_1)}\}. \tag{E.6}$$

Here we do not distinguish between T and its restrictions. The above requirements force T to be continuous $X_1+X_2\to Y_1+Y_2$ (with norm$\le\max_{k=1,2}\|T\|_{\mathcal{B}(X_k,Y_k)}$), by 1° of the proof of Lemma E.1.4.

Note that Lemma A.3.18 provides an alternative definition for $\mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$; obviously, this coincides with the one above, up to the norm (we do not consider $\mathcal{B}(X_k,Y_k)$ being a vector subspace of some vector space Z $(k=1,2)$ if $T\in\mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$ does not imply $T(x_1+x_2)$ being defined and equal to Tx_1+Tx_2 for $x_k\in X_k$ $(k=1,2)$; the latter condition implies that $T\in\mathrm{Hom}(X_1+X_2,Y_1+Y_2)$).

We obviously have $\mathcal{B}(X,Y_1)\cap\mathcal{B}(X,Y_2)=\mathcal{B}(X,Y_1\cap Y_2)$, isometrically.

We now give four equivalent definitions of $\mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$ (the fourth on X_0):

Lemma E.1.4 *Let (X_1,X_2) be a sum-compatible pair of normed spaces, and let (Y_1,Y_2) a sum-compatible pair of Banach spaces. Then (i)⇔(ii)⇔(iii), where*

(i) $T\in\mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$

(ii) $T\in\mathcal{B}(X_1+X_2,Y_1+Y_2)$, $T\in\mathcal{B}(X_1,Y_1)$ and $T\in\mathcal{B}(X_2,Y_2)$;

(iii) $T\in\mathcal{B}(X_1+X_2,Y_1+Y_2)$, $T[X_1]\subset Y_1$ and $T[X_2]\subset Y_2$.

Let X_0 be a dense subspace of X_1, X_2 and $X_1\cap X_2$. If $T_0\in\mathrm{Hom}(X_0,Y_1+Y_2)$ and there is $M<\infty$ s.t.

$$\|T_0x\|_{Y_k}\le M\|x\|_{X_k}\ \ (x\in X_0,\ k=1,2), \tag{E.7}$$

then $T_0=T_{|X_0}$ for a unique $T\in\mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$. Moreover, T_0 has unique continuous extensions in $\mathcal{B}(X_1\cap X_2,Y_1\cap Y_2)$, $\mathcal{B}(X_1,Y_1)$, $\mathcal{B}(X_2,Y_2)$ and

$\mathcal{B}(X_1+X_2, Y_1+Y_2)$, and these are of norm $\le M$ and coincide with T on their domains.

Conversely, if T satisfies any (hence all) of (i)–(iii), then $T_0 := T_{|X_0}$ is as above.

If $X_1 = \mathrm{L}^p_\omega(J;B)$ and $X_2 = L^q_{\omega'}(J;B)$ for some $p,q \in [1,\infty)$, $\omega,\omega' \in \mathbf{R}$ (see Definition D.1.3), then we may take $X_0 := X_1 \cap X_2$, or $X_0 := C_c^\infty$ or let X_0 be the set of simple L^1 functions (or any set between these), by Theorem B.3.11.

Proof: 1° *(i)–(iii):* Trivially, (ii)⇒(i)&(iii). By Lemma A.3.6, we have (iii)⇒(ii). Obviously, $\|T\|_{\mathcal{B}(X_1+X_2,Y_1+Y_2)} \le \max_{k=1,2}\|T\|_{\mathcal{B}(X_k,Y_K)}$, so that (i) implies (ii).

2° *$T_0 \leftrightarrow T$:* If T satisfies (ii), then $T_0 := T_{|X_0}$ satisfies (E.7) for $M := \max_{k=1,2}\|T\|_{\mathcal{B}(X_k,Y_k)}$.

For the converse, let T_0 be as in the lemma. By Lemma A.3.10 and (E.7), T_0 has unique continuous extensions $T_0' \in \mathcal{B}(X_1\cap X_2, Y_1\cap Y_2)$ and $T_k \in \mathcal{B}(X_k,Y_k)$ ($k=1,2$), with norm $\le \|M\|$. If $\{x_n\} \subset X_0$ and $x_n \to x \in X_1\cap X_2$, then $T_0'x_n \to T_0'x$ in $X_1 \cap X_2$, hence in X_1 and in X_2 too, so that $T_0'x$, T_1x and T_2x must all be equal to this limit. Thus, T_0', T_1 and T_2 coincide on $X_1\cap X_2$.

By Lemma A.3.19, X_0 is dense in X_1+X_2. One easily verifies that $\|T_0x\|_{Y_1+Y_2} \le M\|x\|_{X_1+X_2}$, so that there is a unique continuous extension $T \in \mathcal{B}(X_1+X_2, Y_1+Y_2)$ too, again with norm $\le M$, by 1°. If $X_0 \ni x_n \to x$ in X_k, then $Tx_n \to T_kx$ in Y_k and $Tx_n \to Tx$ in $Y_1\cap Y_2$, hence $T_kx = Tx$ ($x \in X_k$, $k=1,2$). Thus T coincides with T_0', T_1 and T_2. In particular, (iii) (hence (i)–(iii)) is satisfied. □

Next we give equivalent conditions for $T \in \mathcal{B}(X,Y)$:

Lemma E.1.5 ($T \in \mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$ & $T[X]\subset Y \Rightarrow T \in \mathcal{B}(X,Y)$) *Let X_1, X_2, Y_1, Y_2 be normed spaces. Let X and Y be Banach spaces. Let (X_1,X_2) and (Y_1,Y_2) be sum-compatible pairs. Let $X \underset{c}{\subset} X_1+X_2$ and $Y \underset{c}{\subset} Y_1+Y_2$. Let $T \in \mathcal{B}(X_1,Y_1)\cap\mathcal{B}(X_2,Y_2)$.*

Then the following are equivalent:

(i) $T_{|X} \in \mathcal{B}(X,Y)$;

(ii) $T[X] \subset Y$;

(iii) there are $M<\infty$ and a dense subspace $\widetilde{X} \subset X$ s.t.

$$\|\widetilde{x}\|_X = 1 \Rightarrow \|T\widetilde{x}\|_Y \le M \quad (\widetilde{x} \in \widetilde{X}). \tag{E.8}$$

Moreover, if (iii) holds, then $\|T\|_{\mathcal{B}(X,Y)} \le M$.

Recall that $\|y\|_Y = \infty$ for $y \notin Y$.

Proof: (As the proof shows, we can allow X to be incomplete if we give up the implication (ii)⇒(i).)

Recall first that $T \in \mathcal{B}(X_1+X_2, Y_1+Y_2)$, hence $T \in \mathcal{B}(X, Y_1+Y_2)$ (because $X \underset{c}{\subset} X_1+X_2$).

1° *(i)⇒(ii):* This is obvious.

2° *(ii)⇒(i):* Now (ii) implies (i), by Lemma A.3.6, because $T\in\mathcal{B}(X,Y_1+Y_2)$.

3° *(i)⇒(iii):* Take $\widetilde{X}=X$, $M:=\|T_{|X}\|$.

4° *(iii)⇒(i):* Let $T_0\in\mathcal{B}(X,Y)$ be the unique continuous extension of T_0. If $\widetilde{X}\ni x_n\to x$ in X, then $Tx_n\to Tx$ in Y_1+Y_2 and $Tx_n=T_0x_n\to T_0x$ in Y, hence in Y_1+Y_2 too, so that $T_0x=Tx$. This holds for all $x\in X$, hence $T_0=T_{|X}$. □

Now that the preparations are done, we can give four interpolation results. We start with the vector-valued forms of two celebrated theorems:

Theorem E.1.6 (Riesz–Thorin Interpolation Theorem) *Let $p_k,q_k\in[1,\infty]$ ($k=0,1$). Let μ and μ' be complete positive measures on sets Q and Q', respectively. Let B and B' be complex Banach spaces.*

If $T\in\cap_{k=0,1}\mathcal{B}(\mathrm{L}^{p_k}(Q;B),\mathrm{L}^{q_k}(Q';B'))$, then $T\in\mathcal{B}(\mathrm{L}^{p}(Q;B),\mathrm{L}^{q}(Q';B'))$ with norm

$$M_\theta\le M_0^{1-\theta}M_1^{\theta}\le\max\{M_0,M_1\},\tag{E.9}$$

provided that $0\le\theta\le1$, $M_k:=\|T\|_{\mathcal{B}(\mathrm{L}^{p_k}(Q;B),\mathrm{L}^{q_k}(Q';B'))}$ *($k=0,1$),*

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\quad\textit{and}\quad\frac{1}{q}=\frac{1-\theta}{q_0}+\frac{\theta}{q_1}.\tag{E.10}$$

Thus, $\log M_\theta$ is convex.

Proof: (This is an extended and rigorous version of the (scalar case) proof of Theorem 1.1.1 of [BL]. The theorem also holds for real B with $2M_0^{1-\theta}M_1^{\theta}$ in place of $M_0^{1-\theta}M_1^{\theta}$ (note that $\|T\|_{\mathcal{B}(\mathrm{L}^p(Q;B+iB),\mathrm{L}^q(Q';B'+iB'))}\le 2\|T\|_{\mathcal{B}(\mathrm{L}^p(Q;B),\mathrm{L}^q(Q';B'))}$, where $B+iB$ is the complexification of B (allow complex scalars with natural operations and, e.g., $\|x+iy\|_{B+iB}:=(\|x\|_B^2+\|y\|_B^2)^{1/2}$).)

1° *Only $|\int_{Q'}gTf\,d\mu'|\le M$ needs to be shown:* W.l.o.g., we assume that $0<\theta<1$, $1\le p_0<p_1\le\infty$ and $q_0\ne q_1$. Let $q^{-1}+q'^{-1}=1$, $q_k^{-1}+q_k'^{-1}=1$ ($k=1,2$). By Lemma E.1.5 and Theorem B.3.11, we only have to show that (E.8) is satisfied by the number M_θ and the set

$$\widetilde{X}:=\mathrm{SMFF}(Q;B):=\{\sum_{k=1}^{n}x_k\chi_{E_k}\,\big|\,n\in\mathbf{N},\ \mu(E_k)<\infty,\ x_k\in B\ (k=1,\dots,n)\}.\tag{E.11}$$

By Theorem B.4.12, it suffices that given $f\in\mathrm{SMFF}(Q;B)$, $g\in\mathrm{SMFF}(Q;B)$ s.t. $\|f\|_p=1=\|g\|_{q'}$, we have $|\int_{Q'}gTf\,d\mu'|\le M$ (note that $Tf\in\mathrm{L}^{q_0}$ and $g\in\mathrm{L}^{q_0'}$, hence $gTf\in\mathrm{L}^1$, by the Hölder Inequality).

2° *We show it:* Set $1/p(z):=p_0^{-1}+z(p_1^{-1}-p_0^{-1})$, $1/q'(z):=q_0'^{-1}+z(q_1'^{-1}-q_0'^{-1})$, as in (E.10), so that $p,q'\in\mathrm{H}(\mathbf{C})$, and $\operatorname{Re}p(z)^{-1}\in(p_1^{-1},p_0^{-1})$, $\operatorname{Re}q'(z)\in(q_1'^{-1},q_0'^{-1})$ when $\operatorname{Re}z\in(0,1)$. Set

$$\phi(z,t):=\|f(t)\|_B^{p/p(z)}f(t)/\|f(t)\|_B,\quad\psi(z,t'):=\|g(t')\|_B^{q'/q'(z)}g(t')/\|g(t')\|_B\tag{E.12}$$

for $z \in \bar{\Omega}$, $t \in Q$, $t' \in Q'$ (we set $\phi(z,t) = 0$ if $f(t) = 0$, and $\psi(z,t') = 0$ if $g(t') = 0$).

Obviously, $\phi(\cdot,q), \psi(\cdot,q) \in \mathrm{H}(\Omega;B)$ and $\operatorname{Re} p/p(z) \in (p/p_1, p/p_0) \subset (0, p/p_0)$ ($\operatorname{Re} z \in (0,1)$) (analogously for q'). It follows that $\phi, \psi, \phi_z, \psi_z$ (which are in SMFF) have a majorant in $g \in$ SMFF, independent of $z \in \bar{\Omega}$.

Therefore, $\phi \in \mathcal{C}(\bar{\Omega};\mathrm{L}^{p_0}) \cap \mathrm{H}(\Omega;\mathrm{L}^{p_0})$, by Lemma B.5.8. Analogously, $\psi \in \mathcal{C}(\bar{\Omega};\mathrm{L}^{q_0'}) \cap \mathrm{H}(\Omega;\mathrm{L}^{q_0'})$ (if $q_0' = \infty$, just note that $\psi, \psi_z \in \mathcal{C}(\bar{\Omega};\mathrm{L}^\infty)$, and use (d)&(e) of Lemma B.5.8).

By Lemma D.1.2(b1)&(b3) and the Hölder Inequality, we have $F \in \mathcal{C}(\bar{\Omega}) \cap \mathrm{H}(\Omega)$, where $F(z) := \int_Q \psi(z) T\phi(z)\, d\mu$. For any $t \in \mathbf{R}$, we have

$$\|\phi(it)\|_{p_0} = \|\,|f|^{p/p_0}\,\|_{p_0} = \|f\|_p^{p/p_0} = 1, \tag{E.13}$$

and, similarly, $\|\phi(1+it)\|_{p_1} = 1 = \|\psi(it)\|_{q_0'} = \|\psi(it)\|_{q_1'}$. Therefore, $|F(it)| \le M_0$ and $|F(1+it)| \le M_1$ for $t \in \mathbf{R}$, hence $|F(\theta)| \le M_0^{(1-\theta)} M_1^\theta$, by Lemma D.1.5. But $\phi(\theta) = f$ and $\psi(\theta) = g$, hence $F(\theta) = \int_Q gTf\, d\mu$, so we have reached our aim. □

We already know that the Fourier transform maps $\mathrm{L}^1(\mathbf{R};H) \to \mathcal{C}_0(i\mathbf{R};H) \subset \mathrm{L}^\infty(i\mathbf{R};H)$ and $\mathrm{L}^2(\mathbf{R};H) \to \mathrm{L}^2(i\mathbf{R};H)$. From the above theorem we obtain corresponding interpolation results for $\mathrm{L}^p(\mathbf{R};H)$, $1 < p < 2$:

Theorem E.1.7 (Hausdorff–Young) *Let H be a complex Hilbert space, $\omega \in \mathbf{R}$, $p \in [1,2]$ and $p^{-1} + q^{-1} = 1$. Then the Fourier transform $\mathcal{F}$ maps $\mathrm{L}^p_\omega(\mathbf{R};H)$ to $\mathrm{L}^q(\omega + i\mathbf{R};H)$ and $\mathrm{L}^p_\omega(\mathbf{R}_+;H)$ to $\mathrm{H}^q(\mathbf{C}^+_\omega;H)$ with*

$$\|\mathcal{F} f\|_q \le (2\pi)^{1/q} \|f\|_{\mathrm{L}^p_\omega} \quad (f \in \mathrm{L}^p_\omega(\mathbf{R};H)) \tag{E.14}$$

(with equality for $p = 2$).

Moreover, if $f \in \mathrm{L}^p_\omega(\mathbf{R}_+;H)$, then $\widehat{f} \in \mathrm{L}^q_\omega(\omega + i\mathbf{R};H)$ is the boundary function of $\widehat{f} \in \mathrm{H}^q(\mathbf{C}^+_\omega;H)$, and $\widehat{f * g} = \widehat{f}\widehat{g}$ on $\mathbf{C}^+_\omega$ and a.e. on $\omega + i\mathbf{R}$ for all $g \in \mathrm{L}^r_\omega(\omega + i\mathbf{R};H)$, where $r^{-1} \in [3/2 - p^{-1}, 1]$.

This means that the restriction of $\mathcal{F}$ to $\mathrm{L}^1_\omega \cap \mathrm{L}^p_\omega$ satisfies (E.14) and has hence a unique extension onto L^p. (By Lemma E.1.4, this coincides with the Plancherel Transform on $\mathrm{L}^p_\omega \cap \mathrm{L}^2_\omega$.)

Unfortunately, if H is a general Banach space, then we know this for $p = 1$ only; in particular, we cannot interpolate. E.g., $\mathcal{F}$ is not bounded $\mathcal{C}_c(\mathbf{R}^n; c_0) \to \mathrm{L}^q$ w.r.t. the L^p norm, where $p \in (1,2]$ and $c_0 := \mathcal{C}_0(\mathbf{N})$ is the space of sequences $\mathbf{N} \to \mathbf{C}$ converging to zero with the sup-norm (let $0 \ne \phi \in \mathcal{S}(i\mathbf{R})$, $\phi = \pi_{[0,1)}\phi$, $N \in \mathbf{N}$, so that $\psi := \mathcal{F}^{-1}\phi \in \mathcal{S}(\mathbf{R}) \subset \mathrm{L}^p$, and set $f_k := e^{ik\cdot}\phi$ so that $\widehat{f_k}(ir) = \phi(ir - ik)$ for $k \le N$; it follows that $\|\widehat{g}\|_p^p = N\|\phi\|_p^p$).

Proof of Theorem E.1.7: (We take $\omega = 0$ w.l.o.g.)

1° *$\widehat{f} \in \mathrm{L}^q$:* We already know this for $p = 1$ and for $p = 2$. By Lemma E.1.4, $\mathcal{F}|_{\mathrm{L}^1 \cap \mathrm{L}^2}$ has continuous extensions to $\mathcal{B}(\mathrm{L}^1, \mathrm{L}^\infty)$ and $\mathcal{B}(\mathrm{L}^2, \mathrm{L}^2)$ (by uniqueness these extensions are equal to $\mathcal{F}$) and $\mathcal{F}$ extends to $\mathcal{B}(\mathrm{L}^1 + \mathrm{L}^2, \mathrm{L}^\infty + \mathrm{L}^2)$, hence $\mathcal{F} \in \mathcal{B}(\mathrm{L}^1, \mathrm{L}^\infty) \cap \mathcal{B}(\mathrm{L}^2, \mathrm{L}^2)$. Thus, we can apply Theorem E.1.6 to obtain (E.14) by a direct computation.

2° $\widehat{f}\in\mathrm{H}^q$*:* Let $f\in\mathrm{L}^p(\mathbf{R}_+;B)$. We have $\|f\|_{\mathrm{L}^p_\alpha}\le\|f\|_p$ for all $\alpha\ge 0$, hence $\|\widehat{f}\|_{\mathrm{H}^q}\le\|f\|_p$.

Set $h:=f*g$, so that $\|h\|_v\le\|f\|_p\|g\|_r$, where $v^{-1}=p^{-1}+r^{-1}-1$, by Lemma D.1.7. Because $v\in[p,2]$, we have $\|\widehat{h}\|_{v'}\le(2\pi)^{1/v'}\|f\|_p\|g\|_r$, where $v^{-1}+v'^{-1}=1$.

If $f,g\in\mathcal{C}_c$, then $\widehat{h}=\widehat{f}\widehat{g}$ on $\overline{\mathbf{C}^+}\cup\{\infty\}$, by Lemma D.1.11(c'). In general, if $f_n,g_n\in\mathcal{C}_c$ and $f_n\to f$ in L^p, $g_n\to g$ in L^r, then $\widehat{h}_n\to\widehat{h}$ in $\mathrm{L}^{v'}$, hence a.e., and $\widehat{h}_n=\widehat{f}_n\widehat{g}_n\to\widehat{f}\widehat{g}$ a.e. (here we have replace $\{(f_n,g_n)\}$ by a suitable subsequence), so that $\widehat{h}=\widehat{f}\widehat{g}$ a.e. By Lemma D.1.11(c'), $\widehat{h}=\widehat{f}\widehat{g}$ on $\mathbf{C}^+_\alpha$ for each $\alpha>0$ (because $f,g\in\mathrm{L}^1_\alpha$), hence on $\mathbf{C}^+$.

Let $g:=\chi_{[0,1)}$, so that $h:=f*g\in\mathrm{L}^p\cap\mathrm{L}^2(\mathbf{R}_+;H)$. By Theorem 3.3.1(a2) (or by the corresponding scalar result), $\widehat{h}$ is the boundary function of itself a.e. (in the sense of condition (1.) of Theorem 3.3.1(a1)), hence so is $\widehat{f}$, because $\widehat{g}(s)=(1-\mathrm{e}^{-s})/s$ is invertible a.e. on $i\mathbf{R}$. □

By the Riesz–Thorin theorem, we can interpolate between L^p_r and L^q_r. Next we show that we can also interpolate between L^p_r and $\mathrm{L}^p_{r'}$:

Proposition E.1.8 ($\mathbf{L}^{\boldsymbol{p}}_{\boldsymbol{r}}$ interpolation w.r.t. $\boldsymbol{r}$) *Let $p,q\in[1,\infty)$, $-\infty<a<b<\infty$. Let $J\subset\mathbf{R}$ be an open interval. Let $\mathcal{F}$ be the set of simple functions $J\to B$, or $\mathcal{F}=\mathrm{L}^p_a(J;B)\cap\mathrm{L}^p_b(J;B)$ or $\mathcal{F}=\mathcal{C}^\infty_c(J;B)$.*

Let $\mathbb{E}:\mathcal{F}\to\mathrm{L}^q_a(J;B)\cap\mathrm{L}^q_b(J;B)$ be linear and s.t. $M_a,M_b<\infty$, where $M_r:=\sup\{\|\mathbb{E}f\|_{\mathrm{L}^q_r}\mid f\in\mathcal{F},\ \|f\|_{\mathrm{L}^p_r}\le 1\}$ ($r\in[a,b]$).

Then there is a unique extension $\widetilde{\mathbb{E}}$ of $\mathbb{E}$ to $\cup_{r\in[a,b]}\mathrm{L}^p_r(J;B)$ s.t. $\widetilde{\mathbb{E}}\in\mathcal{B}(\mathrm{L}^p_r,\mathrm{L}^q_r)$ for all $r\in[a,b]$. Moreover,

$$\|\widetilde{\mathbb{E}}\|_{\mathcal{B}(\mathrm{L}^p_r,\mathrm{L}^q_r)}=M_r\le M_a^{1-\theta_r}M_b^{\theta_r}\le\max\{M_a,M_b\}\quad(r\in[a,b]),\tag{E.15}$$

where $\theta_r:=(r-a)/(b-a)$.

Note that, under the above conditions, $\mathbb{E}:\mathcal{F}\to\mathrm{L}^q_a(J;B)\cap\mathrm{L}^q_b(J;B)$ has a unique continuous extension to $\mathbb{E}_r\in\mathcal{B}(\mathrm{L}^p_r,\mathrm{L}^q_r)$, because $\mathcal{C}^\infty_c\subset\mathcal{F}$ is dense in L^p_r, for any $r\in[a,b]$.

Proof: (This proof is based on that of Riesz–Thorin Interpolation Theorem.)

1° *We can take $\mathcal{F}=\mathrm{L}^p_a\cap\mathrm{L}^p_b$ w.l.o.g.:* By density (see Lemma A.3.10 and Theorem B.3.11), $\mathbb{E}$ has a unique extension $\mathbb{E}_r\in\mathcal{B}(\mathrm{L}^p_r,\mathrm{L}^q_r)$ with $\|\mathbb{E}_r\|\le M_r$ for $r=a$ and $r=b$. Let $f\in\mathrm{L}^p_a\cap\mathrm{L}^p_b$. Then, by Theorem B.3.11, there are $\{f_n\}\subset\mathcal{F}$ s.t. $f_n\to f$ in both L^p_a and L^p_b. Consequently, $\mathbb{E}_af=\lim_n\mathbb{E}f_n=\mathbb{E}_bf$ a.e., by Theorem B.3.2. Therefore, we may assume that $\mathcal{F}=\mathrm{L}^p_a\cap\mathrm{L}^p_b$.

2° *f,g,ϕ_z,ψ_z,q':* Let $f\in\mathcal{C}^\infty_c(J;B)$ be s.t. $\|f\|_{\mathrm{L}^p}=1$. Let $g\in\mathcal{C}^\infty_c(J;B^*)$ be s.t. $\|g\|_{\mathrm{L}^{q'}}=1$, where $1/q+1/q'=1$. Set $\phi_z:=\mathrm{e}^{z\cdot}f$, $\psi_z:=\mathrm{e}^{-z\cdot}g$. If follows that

$$\|\phi_z\|_{\mathrm{L}^p_{\operatorname{Re}z}}=1=\|\psi_z\|_{\mathrm{L}^{q'}_{-\operatorname{Re}z}}\quad(z\in\mathbf{C}).\tag{E.16}$$

3° *$\phi,\psi,F\in\mathcal{C}\cap\mathrm{H}$:* Set $\widetilde{J}:=(\operatorname{supp}(f)\cup\operatorname{supp}(g))^o$. Note that the spaces $\mathrm{L}^p_t(\widetilde{J};B)$ ($t\in\mathbf{R}$) are equal with equivalent norms, hence $\pi_{\widetilde{J}}\mathbb{E}\pi_{\widetilde{J}}\in\mathcal{B}(\mathrm{L}^p,\mathrm{L}^q)$.

Using the Mean Value Theorem, one easily verifies that $(z \mapsto \mathrm{e}^{z\cdot}) \in \mathrm{H}(\mathbf{C};\mathrm{L}^v(\widetilde{J}))$ for $v = \infty$, hence for any $v \in [1,\infty]$ (because $\mathrm{L}^\infty(\widetilde{J}) \subset \mathrm{L}^p(\widetilde{J})$, continuously).

Consequently, $\phi \in \mathrm{H}(\mathbf{C};\mathrm{L}^p(\widetilde{J}))$ and $\psi \in \mathrm{H}(\mathbf{C};\mathrm{L}^{q'}(\widetilde{J}))$. Therefore,

$$F(z) := F_{\phi,\psi}(z) := \int_J \psi_z(t)(\mathbb{E}\phi_z)(t)\,dt \tag{E.17}$$

satisfies $F \in \mathrm{H}(\mathbf{C})$, by Lemma D.1.2(b3).

4° $|F(r)| \le M_r'$*:* We have

$$|F(r+it)| \le \|\psi_{r+it}\|_{\mathrm{L}^{q'}_{-r}} \|\mathbb{E}\phi_{r+it}\|_{\mathrm{L}^q_r} \le 1\cdot M_r \quad (r \in [a,b],\ t \in \mathbf{R}), \tag{E.18}$$

hence $|F(r+it)| \le M_r' := M_a^{1-\theta_r} M_b^{\theta_r}$ $(r \in [a,b],\ t \in \mathbf{R})$, by Lemma D.1.5.

5° *Extending* $\mathbb{E}$*:* For all $r, r' \in [a,b]$, the following facts hold: By Theorem B.4.12,

$$\|\mathbb{E}\phi_r\|_{\mathrm{L}^q_r} := \|\mathrm{e}^{-r\cdot}\mathbb{E}\phi_r\|_q = \sup_{\widetilde{g}\in C_c^\infty,\ \|\widetilde{g}\|_{q'}=1} \int \widetilde{g}\mathrm{e}^{-r\cdot}\mathbb{E}\phi_r, \tag{E.19}$$

i.e., $\|\mathbb{E}\phi_r\|_{\mathrm{L}^q_r}$ is the supremum of functions $|F_{\phi,\psi}(r)|$, where ψ is as above. Therefore, $\|\mathbb{E}\phi_r\|_{\mathrm{L}^q_r} \le M_r'$.

But each $C_c^\infty(J;B)$ function is a scalar multiple of some ϕ_r of the above form, hence $\mathbb{E}$ extends to an operator $\mathbb{E}_r \in \mathcal{B}(\mathrm{L}^p_r, \mathrm{L}^q_r)$ with norm $\|\mathbb{E}_r\| \le M_r'$. As in 1°, we see that $\mathbb{E}_r = \mathbb{E}_{r'}$ on $\mathrm{L}^p_r \cap \mathrm{L}^p_{r'}$. □

Also a converse holds: if $\mathbb{E} \in \mathcal{B}(\mathrm{L}^p_r, \mathrm{L}^q_r)$ for all $r \in (a,b)$ with $\|\mathbb{E}\|$ having an uniform upper bound, then $\mathbb{E} \in \mathcal{B}(\mathrm{L}^p_r, \mathrm{L}^q_r)$ for all $r \in [a,b]$:

Lemma E.1.9 *Let $p,q,a,b,J,\mathcal{F},M_r$ be as in Proposition E.1.8 (we may also allow $\mathcal{F} = \mathrm{L}^p_r \cup \mathrm{L}^p_s$ for some $r,s \in [a,b]$). Let $\mathbb{E} : \mathcal{F} \to \mathrm{L}(J;B)$ be linear, and let $D \subset [a,b]$ be dense.*

Then $M := \sup_{r\in D} M_r = \sup_{r\in[a,b]} M_r = \max\{M_a, M_b\}$; in particular, $\mathbb{E}$ has a (unique) continuous extension to $\mathcal{B}(\mathrm{L}^p_r, \mathrm{L}^q_r)$ for each $r \in [a,b]$ iff $M < \infty$.

Proof: (In fact, D need not be dense; it suffices that a and b are in the closure of D.)

Part "only if" and the moreover-claim follow from Proposition E.1.8. Therefore, we assume that $\mathbb{E} \in \mathcal{B}(\mathrm{L}^p_r, \mathrm{L}^q_r)$ for all $r \in D$, hence for all $r \in (a,b)$ (by the proposition, we can use continuous extensions for r's between elements of D).

Let ϕ be a simple measurable function with $\|\phi\|_{\mathrm{L}^p_a} = 1$. Then $\|\phi\|_{\mathrm{L}^p_r} \to 1$ as $r \to a+$, by Lemma D.1.10(a3), and

$$M\|\phi\|_{\mathrm{L}^p_a} \leftarrow M\|\phi\|_{\mathrm{L}^p_r} \ge \|\mathbb{E}\phi\|_{\mathrm{L}^q_r} \to \|\mathbb{E}\phi\|_{\mathrm{L}^q_a}. \tag{E.20}$$

Because ϕ was arbitrary, we have $\mathbb{E} \in \mathcal{B}(\mathrm{L}^p_a, \mathrm{L}^q_a)$ with norm $\le M$. The same holds with b in place of a. □

Notes

It seems that Proposition E.1.8 and Lemma E.1.9 cannot be obtained as corollaries to standard interpolation theorems (see, e.g., [BL]) unless we introduce additional constants.

The Riesz–Thorin and Hausdorff–Young theorems and their proofs are from [BL], except that we have made the concepts and the proofs more rigorous and more general. See [BL] for an extensive interpolation theory, further references and historical remarks.

Appendix F

$\mathrm{L}^p_{\mathrm{strong}}$, $\mathrm{L}^p_{\mathrm{weak}}$ and Integration

Order and simplification are the first steps toward mastery of a subject — the actual enemy is the unknown.

— Thomas Mann (1875–1955)

For a function $f : Q \to \mathcal{B}(B, B_2)$, Bochner-measurability corresponds to the uniform (i.e., Banach space) topology of $\mathcal{B}(B, B_2)$. However, for several applications it suffices that, e.g., $fx : Q \to B_2$ is measurable for each $x \in B$ (i.e., that f is *strongly measurable*). We study this and the corresponding weak concept; in particular, we define and study $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathrm{L}^p_{\mathrm{weak}}$ spaces (in Section F.1; we note that $\mathrm{L}^\infty_{\mathrm{strong}}$ is usually a Banach space (Theorem F.1.9) but $\mathrm{L}^p_{\mathrm{strong}}$ is often incomplete for $p < \infty$ (Example F.1.10)).

In Section F.2, we define and study integration and convolution for strongly or weakly measurable functions. In Section F.3, we treat $\mathrm{H}^p_{\mathrm{strong}}$ and $\mathrm{H}^p_{\mathrm{weak}}$ spaces and the Laplace transform of strongly or weakly measurable functions.

In this chapter, B, B_2 and B_3 denote Banach spaces over the scalar field $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$ (we have $\mathbf{K} = \mathbf{C}$ in Section F.3), U, H, and Y denote Hilbert spaces, and μ is a complete positive measure on a set Q.

F.1 $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathrm{L}^p_{\mathrm{weak}}$

If you think you have the solution, the question was poorly phrased.

In this section, we study strong and weak (operator) measurability and corresponding L^p spaces. We start with the definitions of measurability:

Definition F.1.1 (Strong and weak measurability, L, $\mathbf{L_{strong}}$, $\mathbf{L_{weak}}$) *By $\mathrm{L}(Q;*)$ we denote the (equivalence classes of) Bochner measurable functions $Q \to *$.*

Let $F : Q \to \mathcal{B}(B,B_2)$. Then F is strongly measurable *($[F] \in \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$) if $Fx \in \mathrm{L}(Q;B_2)$ for all $x \in B$, and F is* weakly measurable *($[F] \in \mathrm{L}_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$) if $\Lambda Fx \in \mathrm{L}(Q)$ for all $x \in B$, $\Lambda \in B_2^*$.*

Elements $F,G \in \mathrm{L}$ (resp $\mathrm{L}_{\mathrm{strong}}$, $\mathrm{L}_{\mathrm{weak}}$) are identified ($[F]=[G]$, or $F \in [G]$, where $[F]$ is the equivalence class of F) if $F = G$ a.e. (resp. $Fx = Gx$ a.e. for all $x \in B$, $\Lambda Fx = \Lambda Gx$ a.e. for all $x \in B$, $\Lambda \in B_2^*$).

If $F : Q \to \mathcal{B}(B,B_2)$ and $G : Q \to \mathcal{B}(B_2^*,B^*)$ are strongly (resp. weakly) measurable and $\langle Fx,\Lambda\rangle_{\langle B_2,B_2^*\rangle} = \langle x,G\Lambda\rangle_{\langle B,B^*\rangle}$ a.e. for all $x \in B$, $\Lambda \in B_2^*$, then $[G]$ is the adjoint of $[F]]$ in $\mathrm{L}_{\mathrm{strong}}$ *(resp. $\mathrm{L}_{\mathrm{weak}}$) and we write $[F]^* = [G]$.*

If $F \in \mathrm{L}(Q;\mathcal{B}(B,B_2))$ and $G \in \mathrm{L}(Q;\mathcal{B}(B_2,B_3))$, then we define $[G][F] := [GF] \in \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_3))$.

The above definition can be generalized to situations where F is B_3-valued and there is a continuous bilinear mapping $B \times B_3 \to B_2$, but these situations can be reduced to the above by considering B_3 as a subspace of $\mathcal{B}(B,B_2)$. The definition of $[G][F]$ will be justified in the proof of Lemma F.1.3(b).

We write $[F]_{\mathrm{L}}$, $[F]_{\mathrm{L}_{\mathrm{strong}}}$ or $[F]_{\mathrm{L}_{\mathrm{weak}}}$ when there may be confusion about the sense in which an equivalence class is defined. We use the standard convention to write F in place of $[F]$ when there is no risk of confusion.

The Bochner measurability of an operator-valued function is often called *uniform measurability* (in the literature, also the term "strong measurability" is used, but we shall use that term for $\mathrm{L}_{\mathrm{strong}}$ only).

We shall interpret the definition of $\mathrm{L}_{\mathrm{strong}}$ and $\mathrm{L}_{\mathrm{weak}}$ for vector-valued functions as follows: if $f : Q \to B$, then we consider f as a function $Q \to \mathcal{B}(\mathbf{K};B)$, so that strong (operator) measurability reduces to Bochner measurability and weak (operator) measurability reduces to "weak vector measurability", i.e., to the condition that $\Lambda f \in \mathrm{L}$ for all $\Lambda \in B^*$ (for operator-valued functions, by weak measurability we refer to weak operator measurability, as defined in Definition F.1.1).

Lemma F.1.2

(a) $\mathrm{L}(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$.

(b1) If $[F] \in \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ has an adjoint in $\mathrm{L}_{\mathrm{strong}}$, then this adjoint is unique.

(b2) If $F,F^* \in \mathrm{L}_{\mathrm{strong}}$, then $[F^*]_{\mathrm{L}_{\mathrm{strong}}} = [F]^*_{\mathrm{L}_{\mathrm{strong}}}$. If $F,F^* \in \mathrm{L}_{\mathrm{weak}}$, then $[F^*]_{\mathrm{L}_{\mathrm{weak}}} = [F]^*_{\mathrm{L}_{\mathrm{weak}}}$.

(c) Let B be reflexive. If $[F] \in \mathrm{L}_{\text{weak}}(Q;\mathcal{B}(B,B_2))$, then $[F^*] \in \mathrm{L}_{\text{weak}}(Q;\mathcal{B}(B_2^*,B^*))$ and $[F^*] = [F]^*$.

In contrast to (a) and (b), in Example 3.1.4 we construct $F \in \mathrm{L}_{\text{strong}} \setminus \mathrm{L}$ s.t. $F^* \in \mathrm{L}_{\text{weak}} \setminus \mathrm{L}_{\text{strong}}$ and $\operatorname{ess\,sup} \|F^*\Lambda\|_{B^*} = \infty$ for certain $\Lambda \in B_2^*$, even though $F \in [0]_{\mathrm{L}_{\text{strong}}}$ and hence $[F]^*_{\mathrm{L}_{\text{strong}}} = [0]^*_{\mathrm{L}_{\text{strong}}} = [0]_{\mathrm{L}_{\text{strong}}}$. Thus, e.g., $[F] \in \mathrm{L}_{\text{strong}}$ may have an adjoint in $\mathrm{L}_{\text{strong}}$ even if $F^* \notin \mathrm{L}_{\text{strong}}$.

Proof: (a) This follows from (B.18) (Naturally, the inclusions should be injective. By Lemma B.2.6, this is the case (but the equivalence classes may be enlarged with "less measurable" elements and there may appear new equivalence classes as we move from L to $\mathrm{L}_{\text{strong}}$ or from $\mathrm{L}_{\text{strong}}$ to L_{weak}).)

(b2)&(c) These follow directly from the definition.

(b1) For $[F] \in \mathrm{L}_{\text{strong}}(Q;\mathcal{B}(B,B_2))$ we have, by Lemma B.2.6, that $[F] = [0] \Leftrightarrow \Lambda F x = 0$ a.e. for all $x \in B,\ \Lambda \in B_2^*$.

If $[F]^* = [G]$ and $[F]^* = [H]$, then $\langle x, (G-H)\Lambda\rangle = 0$ a.e. for all $\Lambda \in B_2^*,\ x \in B$. Therefore, hence $(G-H)\Lambda = 0$ a.e. for all $\Lambda \in B_2^*$, by Lemma B.2.6 (because $\{x \in B \,|\, \|x\|_B \leq 1\} \subset (B^*)^*$ is norming), i.e., $[G] = [H]$. □

Now we go on with further properties of strongly and weakly measurable functions:

Lemma F.1.3 *Let $[F] \in \mathrm{L}_{\text{strong}}(Q;\mathcal{B}(B,B_2))$, $[G] \in \mathrm{L}_{\text{strong}}(Q;\mathcal{B}(B_2,B_3))$, $[f] \in \mathrm{L}(Q;B)$, $[h] \in \mathrm{L}(Q)$, $H : Q \to \mathcal{B}(B,B_2)$ and $1 \leq p \leq \infty$. Then we have the following:*

(a) $Ff \in \mathrm{L}(Q,B_2)$ and $hF \in \mathrm{L}_{\text{strong}}(Q;\mathcal{B}(B,B_2))$.

(b) We have $[GF] \in \mathrm{L}_{\text{strong}}(Q;\mathcal{B}(B,B_3))$. In particular, $\mathrm{L}_{\text{strong}}(Q,\mathcal{B}(B))$ is an algebra.

(c) If $H_n \in \mathrm{L}_{\text{strong}}$ for all $n \in \mathbf{N}$ and $H_n \to H$ a.e., then $H \in \mathrm{L}_{\text{strong}}$.

(d) If $\dim B < \infty$, then $\mathrm{L}_{\text{strong}}(Q;\mathcal{B}(B,B_2)) = \mathrm{L}(Q;\mathcal{B}(B,B_2))$ (and $[F]_{\mathrm{L}_{\text{strong}}} = [F]_{\mathrm{L}}$); if B_2 is separable, then $\mathrm{L}_{\text{weak}}(Q;\mathcal{B}(B,B_2)) = \mathrm{L}_{\text{strong}}(Q;\mathcal{B}(B,B_2))$ (and $[F]_{\mathrm{L}_{\text{strong}}} = [F]_{\mathrm{L}_{\text{weak}}}$).

(e) If also R is a measure space, then $((q,r) \mapsto F(q)) \in \mathrm{L}_{\text{strong}}(Q \times R)$ and $(q,r) \mapsto F(r-q) \in \mathrm{L}_{\text{strong}}(Q \times R)$.

(f1) Assume that B is separable. Then $\|F\|$ is measurable and $\|[F]\|_{\mathrm{L}^\infty_{\text{strong}}} = \operatorname{ess\,sup} \|F\|_{\mathcal{B}(B,B_2)} \leq \infty$, in particular, $[F] = [0] \Leftrightarrow F = 0$ a.e.

Moreover, $[F] \in \mathcal{G}\mathrm{L}_{\text{strong}}$ iff $F(q) \in \mathcal{GB}$ for a.e. $q \in Q$ and $[F^{-1}] \in \mathrm{L}_{\text{strong}}$. If, B is also reflexive, then $F^* \in \mathrm{L}_{\text{strong}}$ and $[F^*] = [F]^*$.

(f2) Assume that Q is separable and $\mu(\Omega) > 0$ for open $\Omega \subset Q$.

Then $\mathcal{C}(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}_{\text{strong}}(Q;\mathcal{B}(B,B_2))$. Moreover, $\mathcal{C}_{\mathrm{b}}(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}^\infty(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}^\infty_{\text{strong}}(Q;\mathcal{B}(B,B_2))$, with equal norms.

Assume, in addition, that $F \in \mathcal{C}(Q;\mathcal{B}(B,B_2))$. *Then* $[F^*] = [F]^*$ *and* $\|[F]\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \sup_Q \|F\|_{\mathcal{B}(B,B_2)} \le \infty$; *in particular,* $[F] = [0] \Leftrightarrow F \equiv 0$.

If $F(q) \in \mathcal{GB}(B,B_2)$ *for a.e.* $q \in Q$, *then* $[F]^{-1} = [F^{-1}] \in \mathcal{G}\mathrm{L}_{\mathrm{strong}}$. *Conversely, if* $[F] \in \mathcal{G}\mathrm{L}_{\mathrm{strong}}$ *and* $[G] = [F]^{-1}$ *and* $\|G\|_{\mathcal{B}(B_2,B)} \le M$ *a.e., then* $F(q) \in \mathcal{GB}(B,B_2)$ *for all* $q \in Q$.

(g) Assume that $B_3 = B$. *Then any separable sets* $X_0 \subset B$ *and* $Y_0 \subset B_2$ *are contained, respectively, in closed separable subspaces* $X \subset B$ *and* $Y \subset B_2$, *s.t. there is a null set* $N \subset Q$ *satisfying* $F(q)x \in Y$ *and* $G(q)y \in X$ *for all* $x \in X$, $y \in Y$ *and* $q \in Q \setminus N$. *(Cf. Lemma 3.2.6.)*

(w) Replace $\mathrm{L}_{\mathrm{strong}}$ *by* $\mathrm{L}_{\mathrm{weak}}$ *everywhere above in this lemma. Then parts (c) and (e) above hold, we have* $hF \in \mathrm{L}_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$, *and* $g \in \mathrm{L}(Q;B_2^*)$ *implies that* $gFf \in \mathrm{L}(Q)$. *Parts (f1) and (f2) (provided that both* B *and* B_2 *are assumed to be separable in (f1)) also hold except possibly the claims concerning* $\mathcal{G}\mathrm{L}_{\mathrm{strong}}$ *and* $\mathcal{G}\mathrm{L}^\infty_{\mathrm{strong}}$.

Moreover, if $f \in \mathrm{L}$, *then* $Ff \in \mathrm{L}_{\mathrm{weak}}$; *if* $F \in \mathrm{L}_{\mathrm{strong}}$, *then* $GF \in \mathrm{L}_{\mathrm{weak}}$;

Note that any measurable subset of $\mathbf{R}^n$ with the Lebesgue measure (or any countable set with the counting measure) satisfies the assumptions of (f2), by Lemma B.2.3(e).

Proof: (a) The claim on hF is a special case of (b). Now $g_j := \sum_{k=1}^{j} Fx_k\chi_{E_k} \in \mathrm{L}(Q;B_2)$ for all $j \in \mathbf{N}$, $\{x_k\} \subset B$ and disjoint, measurable E_k ($k \in \mathbf{N}$). Therefore, $Ff = \lim_{j\to+\infty} g_j \in \mathrm{L}$, when $f = \sum_{k=1}^\infty x_k\chi_{E_k}$. If $f \in \mathrm{L}$ is arbitrary, $f_n \to f$ a.e., and f_n is countably-valued ($n \in \mathbf{N}$), then $\mathrm{L} \ni Ff_n \to Ff$ a.e., as $n \to +\infty$, hence then $Ff \in \mathrm{L}$, by Lemma B.2.5(c).

(b) 1° Now $Fx \in \mathrm{L}$, hence $GFx \in \mathrm{L}$, by (a), for any $x \in B$. Thus, $[GF] \in \mathrm{L}_{\mathrm{strong}}$.

2° We shall now show that $[G][F] := [GF]$ is well defined, as promised below Definition F.1.1: Let $F' \in [F]$, $G' \in [G]$ and $x \in B$. By Lemma B.2.5(b1), there is a separable subset $B_0 \subset B$ s.t. $F(q)x \in B_0$ for a.e. $q \in Q$. Choose a null set $N \subset Q$ s.t. $Fy = F'y$ on N^c for all y in a dense, countable subset of B_0, hence for all $y \in B_0$. Then $FGx = F'G'x$ a.e. on N^c, hence a.e., hence $[FG] = [F'G']$, hence multiplication is well-defined.

3° Apply 1° to $B_2 = B = B_3$ to see that $\mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B))$ is an algebra.

(c) Now $Hx = \lim_n H_nx$ a.e. hence $Hx \in \mathrm{L}_{\mathrm{strong}}$, for any $x \in B$ (thus, it were sufficient if $H_n \to H$ strongly).

(d) We assume that $\dim B < \infty$ and prove that $F \in \mathrm{L}(Q;\mathcal{B}(B,B_2))$. Take a base $e_1,\dots,e_n \subset B$, and set $f_j := Fe_j \in \mathrm{L}(Q;B_2)$ for all j. Then $F\sum_{j=1}^n \alpha_je_j = \sum_{j=1}^n \alpha_jf_j$, hence $F = \sum_j f_jP_j \in \mathrm{L}$, where $P_j \in B^*$ is the mapping $\sum_{j=1}^n \alpha_je_j \mapsto \alpha_j$. Obviously, $Fx = 0$ a.e. for all $x \in B$ iff $F = 0$ a.e., hence $[0]_{\mathrm{L}_{\mathrm{strong}}} = [0]_{\mathrm{L}}$.

If, instead B_2 is separable, then any $F \in \mathrm{L}_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$ is strongly measurable, by Lemma B.2.5(b1) (and $\Lambda Fx = 0$ a.e. for all $\Lambda \in B_2^*$ iff $Fx = 0$ a.e., hence $[0]_{\mathrm{L}_{\mathrm{strong}}} = [0]_{\mathrm{L}_{\mathrm{weak}}}$).

(e) Now $F(\cdot)x \in \mathrm{L}(Q,B_2) \subset \mathrm{L}(Q\times R,B_2)$ for all $x \in B$, hence $q \mapsto F(q)$ is in $\mathrm{L}_{\mathrm{strong}}(Q\times R)$. The second claim follows analogously, by using Lemma B.2.9.

(f1) 1° Let $\{b_k\}_{k\in\mathbf{N}}$ be dense in the unit ball of B. Then $\|F\|_{\mathcal{B}(B,B_2)} = \sup_k \|Fb_k\|_{B_2}$ is measurable and $\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \sup_k \|Fb_k\|_{\mathrm{L}^\infty} \le \operatorname{ess\,sup}\|F\|$. But if $\|F\| > M$ on E with $\mu(E) > 0$, then $\cup_k E_k = E$, where $E_k := \{q \in E \,\big|\, \|F(q)b_k\| > M\}$, hence then $\mu(E_k) > 0$ for some k, thus $\|Fb_k\|_\infty > M$. Consequently, $\sup_k \|Fb_k\|_{\mathrm{L}^\infty} \le \operatorname{ess\,sup}\|F\|$.

2° *Claim* $[F]^{-1} = [F^{-1}]$: If F^{-1} exists a.e. and $[F^{-1}] \in \mathrm{L}_{\mathrm{strong}}$, then, obviously, $[F^{-1}][F] = I$ and $[F][F^{-1}] = I$ in $\mathrm{L}_{\mathrm{strong}}$.

For the converse, assume that $B_3 = B$ and $[G] = [F]^{-1}$. Set $X_0 := B$ and apply (g) to obtain a null set $N \subset Q$ and a closed separable subspace $Y \subset B_2$ s.t. $F(q)[B] \subset Y$ for all $q \in Q \setminus N$.

Now $I_B - GF = 0$ a.e. on N^c and $(I_{B_2} - FG)_{|Y} = 0$ a.e. on N^c Therefore, $G_{|Y} = F^{-1}$ a.e. on N^c, say, on N_1^c, where N_1 is a null set. But for any $y \in B_2$ we have $FGy = y$ a.e. on N_1^c, hence $B_2 = Y$, i.e., $G = F^{-1}$ on N_1^c, hence a.e.

3° Assume now that B is also reflexive. Let $\Lambda \in B_2^*$. Then $\langle x, F^*\Lambda\rangle = \langle Fx, \Lambda\rangle$ is measurable for all $x \in B = B^{**}$, and $F^*\Lambda : Q \to B^*$ is separably-valued (B^* is separable, by Lemma A.3.4(R2)), hence $F^*\Lambda$ is measurable, by Lemma B.2.5(b1). Obviously, $[F^*] = [F]^*$.

(f2) (In fact, piecewise continuity suffices (or that $Q = \cup_{n\in\mathbf{N}} Q_n$, where, for each n, $Q_n \subset Q$ is a Borel set, $F \in \mathcal{C}(Q_n; \mathcal{B}(U,Y))$, and $\mu(\Omega) > 0$ for all open $\Omega \subset Q_n$).) Note that we implicitly assumed that Q is a topological space and that all Borel-sets are measurable.

1° By Lemma B.2.5(e), $\mathcal{C} \subset \mathrm{L}$. (Note that we have identified F and $[F]_{\mathrm{L}}$ for $F \in \mathcal{C}$; by 3°, this inclusion is injective.) Combine this with Lemma F.1.2(a) to obtain $\mathcal{C} \subset \mathrm{L} \subset \mathrm{L}_{\mathrm{strong}}$.

2° *We have* $\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \sup_Q \|F\|$: If $F \in \mathcal{C}$ and $\|F(q)x\| > M := \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ for some $x \in B$ s.t. $\|x\| \le 1$, then $\Omega := \{q \in Q \,\big|\, \|F(q)x\| > M\}$ has a positive measure, hence $\|Fx\|_\infty > M$, a contradiction, hence $\sup_Q \|F\|_{\mathcal{B}(B,B_2)} \le M$, hence $\sup_Q \|F\|_{\mathcal{B}(B,B_2)} = M$.

3° $\mathcal{C}_{\mathrm{b}} \subset \mathrm{L}^\infty \subset \mathrm{L}^\infty_{\mathrm{strong}}$ *with equal norms:* this follows from 1° and 2°.

4° Now also F^* is continuous, hence $F^* \in \mathrm{L} \subset \mathrm{L}_{\mathrm{strong}}$, by 1°. From the definition of $[F]^*$ we observe that $[F^*] = [F]^*$.

5° If $F(q) \in \mathcal{GB}(B,B_2)$ for $q \in N^c$, where N is a null set, then $F^{-1} \in \mathcal{C}(N^c; \mathcal{B}(B_2,B)) \subset \mathrm{L}_{\mathrm{strong}}(N^c; \mathcal{B}(B_2,B)) = \mathrm{L}_{\mathrm{strong}}(Q; \mathcal{B}(B_2,B))$.

6° Assume that $[F] \in \mathcal{G}\mathrm{L}_{\mathrm{strong}}$, $[G] = [F]^{-1}$ and $\|G\|_{\mathcal{B}(B_2,B)} \le M$ on N_0^c, where N_0 is a null set.

Let $x_0 \in B$ and $y_0 \in B_2$ be arbitrary. Set $X_0 := \{x_0\}$, $Y_0 := \{y_0\}$, and apply (g) to obtain closed separable subspaces X, Y and a set N s.t. $F(q)X \subset Y$ and $G(q)Y \subset X$ for all $q \in N^c$, $x_0 \in X \subset B$ and $y_0 \in Y \subset B_2$. By continuity, $F(q)X \subset Y$ for all $q \in Q$.

Since $GFx = x$ and $FGy = y$ a.e. for all $x \in B$ and $y \in B_2$, hence for all $x \in X$ and $y \in Y$, we have $[F]^{-1} = [G]$ also in $\mathrm{L}^\infty_{\mathrm{strong}}(Q; \mathcal{B}(Y,X))$. Thus, we can apply (f1) to obtain that $F(q)_{|X} \in \mathcal{GB}(X,Y)$ for a.e. $q \in Q$, say for $q \in N_{x_0,y_0}^c$, where N_{x_0,y_0} is a null set.

We now show that $F(q)_{|X} \in \mathcal{GB}(B,B_2)$ for all $q \in Q$: To obtain a contradiction, assume that $F(q_0)_{|X} \notin \mathcal{GB}(B,B_2)$ for some $q_0 \in Q$. Then there is an open $V \subset \mathcal{B}(X,Y)$ s.t. $F(q_0) \in V$ and $T \in V\ \&\ T \in \mathcal{GB}(X,Y) \Rightarrow \|T^{-1}\| > M$, by Lemma A.3.3(A4). It follows that $V' := \{q \in Q \,|\, F(q)_{|X} \in V\}$ is open and $V' \subset N_0 \cup N_{x_0,y_0}$, hence $\mu(V') = 0$, hence $V' = \emptyset$, a contradiction.

In particular, $F(q)x_0 \neq 0$ and $y_0 \in \mathrm{Ran}(F(q))$, for all $q \in Q$. Because x and y were arbitrary, we have $\mathrm{Ker}(F(q)) = \{0\}$ and $\mathrm{Ran}(F(q)) = B_2$, hence $F(q) \in \mathcal{GB}(B,B_2)$ for any $q \in Q$.

(g) Let $D_X \subset X_0$ and $D_Y \subset Y_0$ be dense and countable. For any $n \in \mathbf{N}$, we have $F(GF)^n, (GF)^nG, (GF)^n, (FG)^n \in \mathrm{L}_{\mathrm{strong}}$, by (b).

For each $x \in D_X$, there is a null set $N_0^x \subset Q$ s.t. $Y_0^x := F[Q \setminus N_0^x]x$ is separable. Set $Y_1 := \overline{\mathrm{span}(Y_0 \cup \cup_{x\in D_X} Y_0^x)}$, $N_0' := \cup_{x\in D_X} N_0^x$. It follows that $F[Q \setminus N_0']x \subset Y_1$ for all $x \in X_0$, by continuity. Moreover, $Y_1 \subset Y$ is separable, by Lemma B.2.3(a)&(c), and N_0' is a null set.

For each $k \in 1+\mathbf{N}$, given N_k' and Y_k, choose, analogously, a null set N_{k+1} and a separable subspace $X_{k+1} \subset B$ s.t. $X_k \subset X_{k+1}$ and $G[Q \setminus N_{k+1}]Y_k \subset X_{k+1}$. On the other hand, for each $k \in 1+\mathbf{N}$, given N_k and X_k, choose a null set N_k' and a separable subspace $Y_k \subset B_2$ s.t. $Y_{k-1} \subset Y_k$ and $F[Q \setminus N_k']X_k \subset Y_k$.

Set $N := \cup_k N_k \cup N_k'$, $X := \overline{\mathrm{span}(\cup_k X_k)}$, $Y := \overline{\mathrm{span}(\cup_k Y_k)}$. If $q \in Q \setminus N$, then $F(q)x \in Y$ for all $x \in \cup_k X_k$, hence for all $x \in X$, by linearity and continuity; analogously, $G(q)y \in X$ for all $y \in Y$.

(w) 1° *$Ff, GF, hF, gFf \in \mathrm{L}_{\mathrm{weak}}$:* Let $f \in \mathrm{L}$. A slight modification of the proof of (a) shows that $Ff \in \mathrm{L}_{\mathrm{weak}}$. Let now $G \in \mathrm{L}_{\mathrm{weak}}$ and $F \in \mathrm{L}_{\mathrm{strong}}$. Then $Fx \in \mathrm{L}$ for each $x \in B$, hence $GFx \in \mathrm{L}_{\mathrm{weak}}$, by the above; consequently, $GF \in \mathrm{L}_{\mathrm{weak}}$. The claims on hF and gFf follow.

2° *The other claims:* The above proofs of parts (c), (e) and (f) need only be slightly changed (in (f) we use a countable norming subset of B_2^* and a dense subset of B). □

Definition F.1.4 ($\mathbf{L}^p_{\mathbf{strong}}(Q;\mathcal{B}(B,B_2))$) *Let $1 \le p \le \infty$.*

By $\mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ we denote the space of $[F] \in \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ having a finite norm

$$\|F\|_{\mathrm{L}^p_{\mathrm{strong}}} := \sup_{\|x\|_B \le 1} \|Fx\|_{\mathrm{L}^p(Q,B_2)}. \tag{F.1}$$

By $\mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2)))$ we denote the space of $[F] \in \mathrm{L}_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$ having a finite norm

$$\|[F]\|_{\mathrm{L}^p_{\mathrm{weak}}} := \sup_{\|x\|_B, \|\Lambda\|_{B_2} \le 1} \|\Lambda Fx\|_{\mathrm{L}^p(Q)}. \tag{F.2}$$

It follows that $\mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2)) = \mathrm{L}^p(Q;\mathcal{B}(B,B_2)) \cong (\mathrm{L}^p(Q;B_2))^n$ when $n := \dim B < \infty$ (cf. Lemma A.1.1(a4)). Note also that $\|F\|_{\mathrm{L}^p_{\mathrm{weak}}} \le \|F\|_{\mathrm{L}^p_{\mathrm{strong}}} \le \|F\|_{\mathrm{L}^p}$ for $F \in \mathrm{L}^p$ and that $\|F\|_{\mathrm{L}^p_{\mathrm{strong}}}$ (resp. $\|F\|_{\mathrm{L}^p_{\mathrm{weak}}}$) is the norm of the operator $B \ni x \mapsto Fx \in \mathrm{L}^p$ (resp. the "bilinear norm" of $B \times B_2^* \ni (x,\Lambda) \mapsto \Lambda Fx \in \mathrm{L}^p(Q)$; cf. Lemma A.3.4(J1)).

One could insist that $\mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ should consist of *all* linear $F : B \mapsto \mathrm{L}_{\mathrm{strong}}(Q;B_2)$ (and analogously for $\mathrm{L}_{\mathrm{weak}}$, $\mathrm{L}^p_{\mathrm{strong}}$, $\mathrm{L}^p_{\mathrm{weak}}$), not just for those that take the form of a function (a.e.). See Theorem F.2.1(g) etc. for details. However, that broader definition would cause problems in several applications.

The spaces $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathrm{L}^p_{\mathrm{weak}}$ are normed spaces:

Lemma F.1.5 *Let* $1 \le p \le \infty$. *Then*

(a1) $\mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ *is a subspace of* $\mathcal{B}(B,\mathrm{L}^p(Q;B_2))$ *with same norm.*

(a2) $\mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$ *is a subspace of* $\mathcal{B}(B,\mathcal{B}(B_2,\mathrm{L}^p(Q)))$ *with same norm.*

(b) If $Q \subset \mathbf{R}^n$ *and* $\mu = m$, *and* B *is a Hilbert space or* $B_2 = \mathbf{K}$, *then* $\mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2)) = \mathcal{B}(B,\mathrm{L}^\infty(Q;B_2))$.

(c1) We have $\mathrm{L}^p(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2)) \subset \mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$, *continuously.*

(c2) If $F \in \mathrm{L}_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$, *then* $\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \|F\|_{\mathrm{L}^\infty_{\mathrm{weak}}}$; *if* $F \in \mathrm{L}(Q;\mathcal{B}(B,B_2))$, *then* $\|F\|_{\mathrm{L}^\infty} = \|F\|_{\mathrm{L}^\infty_{\mathrm{weak}}}$.

(d) If $F \in \mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ *and* $T \in \mathcal{B}(B_2,B_3)$, *then* $TF \in \mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_3))$ *and* $\|TF\|_{\mathrm{L}^p_{\mathrm{strong}}} \le \|T\|_{\mathcal{B}}\|F\|_{\mathrm{L}^p_{\mathrm{strong}}}$. *Also the analogous "weak" claim holds.*

(e) If $F \in \mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$ *and* B *is reflexive, then* $F^* \in \mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B_2^*,B^*))$ *and* $\|F^*\|_{\mathrm{L}^p_{\mathrm{weak}}} = \|F\|_{\mathrm{L}^p_{\mathrm{weak}}}$.

(f) $(\dim \boldsymbol{B} < \infty)$ *If* $\dim B < \infty$, *then* $\mathrm{L}^p(Q;\mathcal{B}(B,B_2)) = \mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ *(with equivalent norms). If* $\dim B_2 < \infty$, *then* $\mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2)) = \mathrm{L}^p_{\mathrm{weak}}(Q;\mathcal{B}(B,B_2))$ *(with equivalent norms).*

(g1) Assume that $p = \infty$ *or* μ *is* σ*-finite. Then* $Hg \in \mathrm{L}$ *for all* $g \in \mathrm{L}^p(Q;B)$ *iff* $H \in \mathrm{L}_{\mathrm{strong}}$.

(g2) Assume that $p = \infty$ *or* μ *is non-atomic. Then* $H\cdot \in \mathcal{B}(\mathrm{L}^p)$ *iff* $H \in \mathrm{L}^\infty_{\mathrm{strong}}$.

"Usually" $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathrm{L}^p_{\mathrm{weak}}$ are Banach spaces only for $p = \infty$; see Theorem F.1.9 and Example F.1.10 for details.

Proof: (a1)&(a2) These are obvious.

(b) Let $F \in \mathcal{B}(B,\mathrm{L}^\infty(Q;B_2))$. W.l.o.g. we assume that $Q = \mathbf{R}^n$ (replace F by $F\chi_Q$). Set $M := \|F\|_{\mathcal{B}}$. For any $q \in \mathbf{R}^n$, the set $X_q := \{x \in B \,\big|\, q \in \mathrm{Leb}(LFx)\}$ is a subspace of B, and $\|LFx\| \le \|Fx\|_\infty \le M\|x\|_B$ on Q ($x \in B$), by Lemma B.5.3.

For each $q \in Q$, the map $x \mapsto LFx(q)$ is obviously linear on X_q, hence it has a norm-preserving extension $G(q) \in \mathcal{B}(B,B_2)$, by Lemma A.3.11, so that $\|G(q)\|_{\mathcal{B}(B,B_2)} \le M$.

Let $x \in B$. Then for a.e. $q \in Q$ we have $x \in X_q$ and hence $LF(q)x = G(q)x$; but $LFx = Fx$ a.e., hence $Fx = Gx$ a.e. Consequently, $G : Q \to \mathcal{B}(B,B_2)$ is strongly measurable and $\|F - G\|_{\mathcal{B}(B,\mathrm{L}^\infty(Q;B_2))} = 0$.

Thus, we have constructed $G \in \mathrm{L}^\infty_{\mathrm{strong}}(Q;\mathcal{B}(B,B_2))$ s.t. $G = F$ as an element of $\mathcal{B}(B,\mathrm{L}^\infty(Q;B_2))$. Finally, G satisfies the additional condition $\|G(q)\|_{\mathcal{B}(B,B_2)} \le \|G\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ for every $q \in Q$.

(c1)&(d) These are obvious (and the norms of the embeddings in (c1) are at most one).

(c2) The second claim follows from Theorem B.4.12(a); the first claim follows from the second (note that the norms may be infinite).

(e) This holds because $x(F^*\Lambda) = \Lambda F x$ for all $x \in B^{**} = B$.

(f) 1° *Case* $n := \dim B < \infty$*:* By Lemma F.1.3(d), we have $L_{\text{strong}} = L$, so we only have to show that the two norms are equivalent. Let $b_1, ..., b_n$ be a vector base of B with $\|b_k\|_B = 1$ ($k = 1, \ldots, n$), and set $\varepsilon := \min_{|\alpha|_{\mathbf{K}^n}=1} \|\sum_k \alpha_k b_k\|_B > 0$. Then

$$\|T\|_{\mathcal{B}(B,B_2)} = \sup_{\|b\|=1} \|Tb\| \le \sup_{|\alpha|_{\mathbf{K}^n}=1} \|T\varepsilon^{-1} \sum_k \alpha_k b_k\|, \le \varepsilon^{-1} \sum_k \|Tb_k\|, \tag{F.3}$$

for all $T \in \mathcal{B}(B, B_2)$. Therefore, each $f \in L^p_{\text{strong}}(Q; \mathcal{B}(B, B_2))$ we have

$$\|f\|_p \le \varepsilon^{-1} \sum_k \|fb_k\|_p \le n\varepsilon^{-1} \|f\|_{L^p_{\text{strong}}}. \tag{F.4}$$

Trivially, $\|f\|_{L^p_{\text{strong}}} \le \|f\|_p$, hence the two sets are equal and have equivalent norms.

2° *Case* $n := \dim B_2 < \infty$*:* As in 1° we see that there are $\Lambda_k \in B_2^*$, $\|\Lambda_k\| = 1$ ($k = 1, \ldots, n$) s.t.

$$\|fx\|_p \le \varepsilon^{-1} \sum_{k=1}^{n} \|\Lambda_k f x\|_p \le \varepsilon^{-1} \|f\|_{L^p_{\text{weak}}} \tag{F.5}$$

for all $f \in L^p_{\text{weak}}(Q; \mathcal{B}(B, B_2))$ and $x \in B$ s.t. $\|x\| \le 1$.

(g1) "If" follows from Lemma F.1.3(a1). "Only if": If $p = \infty$, then we have $Hg \in L$ when $g \equiv x$, for each $x \in B$, hence $H \in L_{\text{strong}}$. If $p < \infty$ and $Q = \cup_{n \in \mathbf{N}} Q_n$, $\mu(Q_n) < \infty$ for all n, then we can take $g = \chi_{Q_n} x$ for each $n \in \mathbf{N}$ and $x \in B$, hence $Hx \in L$ for each $x \in B$, by Lemma B.2.5(d1).

(g2) By Theorem F.1.7(b), we have $L^\infty_{\text{strong}} \subset \mathcal{B}(L^p, L^p)$, isometrically. Conversely, if $(g \mapsto Hg) \in \mathcal{B}(L^p)$, then $H \in L_{\text{strong}}$, by (g1), and $\|H\|_{L^\infty_{\text{strong}}} = \|H \cdot\|_{\mathcal{B}(L^p)}$, by Theorem F.1.7(b). □

The following lemma makes things simpler:

Lemma F.1.6 *Let $F : Q \to \mathcal{B}(B, B_2)$. We have $F \in L^p_{\text{strong}}(Q; \mathcal{B}(B, B_2))$ iff $Fx \in L^p(Q; B_2)$ for all $x \in B$. We have $F \in L^p_{\text{weak}}(Q; \mathcal{B}(B, B_2))$ iff $\Lambda F x \in L^p(Q)$ for all $x \in B$.*

Thus, if $Fx \in L^p$ for each x, then "$Fx \in L^p$ uniformly"; the proof is based on the Closed Graph Theorem.

Proof: 1° L^p_{strong}: Let $Fx \in L^p(Q; B_2)$ for all $x \in B$. Then $x \mapsto Fx \in L^p$ is linear. Let $x_n \to 0$ in B and $Fx_n \to f$ in L^p, as $n \to \infty$. Then $Fx_{n_k} \to f$ a.e. for some subsequence, by Theorem B.3.2, hence $f = \lim_k Fx_{n_k} = F0 = 0$ a.e. Consequently, $T : x \mapsto Fx$ is bounded, by Lemma A.3.4(E1). Therefore, $\|F\|_{L^p_{\text{strong}}} = \|T\|_{\mathcal{B}(B, L^p)} < \infty$. The converse is obvious.

2° L^p_{weak}: For each $x \in B$, we haven $Fx \in L^p_{\text{strong}}(Q; \mathcal{B}(B_2^*, \mathbf{K}))$, by 1°, hence $\|\Lambda F x\|_p \le M_x \|\Lambda\|_{B_2^*}$ for some $M_x < \infty$. Thus, $Tx := Fx \in \mathcal{B}(B_2^*, L^p(Q))$. Obviously, T is linear $B \to \mathcal{B}(B_2^*, L^p)$.

Let $x_n \to 0$ in B and $Tx_n \to H$ in $\mathcal{B}(B_2^*, \mathrm{L}^p)$, as $n \to \infty$. Then, for each $\Lambda \in B_2^*$, we have $\Lambda F x_n = (Tx_n)\Lambda \to H\Lambda$ in $\mathrm{L}^p(Q)$ hence some subsequence converges pointwise a.e. But $\Lambda F x_n \to \Lambda F 0 = 0$ pointwise everywhere, hence $H\Lambda = 0$ a.e. Because $\Lambda \in B_2^*$ was arbitrary, we have $H = 0$. Consequently, $T : x \mapsto Tx$ is bounded, by Lemma A.3.4(E1). Therefore,

$$\|\Lambda F x\|_p = \|(Tx)\Lambda\|_p \le \|Tx\|_{\mathcal{B}(B_2^*,\mathrm{L}^p)} \|\Lambda\|_{B_2^*} \le \|T\|_{\mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{L}^p))} \|x\|_B \|\Lambda\|_{B_2^*}, \tag{F.6}$$

hence $\|F\|_{\mathrm{L}^p_{\mathrm{weak}}} = \|T\|_{\mathcal{B}(B,\mathrm{L}^p)} < \infty$. The converse is obvious. □

For "usual" L^p's, we have the following result with important applications:

Theorem F.1.7 ($\mathbf{L}^{\infty}_{\mathbf{strong}} \subset \mathcal{B}(\mathbf{L}^p)$) *Let $F \in \mathrm{L}_{\mathrm{strong}}(Q; \mathcal{B}(B, B_2))$, $1 \le p \le \infty$.*

(a) We have $\|Ff\|_{\mathrm{L}^p(Q;B_2)} \le \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} \|f\|_{\mathrm{L}^p(Q;B)}$ $(f \in \mathrm{L}^p(Q;B))$.

(b) Let μ be non-atomic or $p = \infty$, and $\mathrm{L}^p(Q;B) \ne \{0\}$. Then

$$\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = \sup_{f \in \mathcal{E}\setminus\{0\}} \|Ff\|_p / \|f\|_p \le \infty, \tag{F.7}$$

where $\mathcal{E}$ is $\mathrm{L}^p(Q;B)$ or $\mathcal{E} = \mathcal{E}' \cdot X$, where $\mathcal{E}' \subset \mathrm{L}^p(Q)$ and $X \subset B$ are dense.

In particular, then $\mathrm{L}^\infty_{\mathrm{strong}}$ is a subspace of $\mathcal{B}(\mathrm{L}^p)$ (with same norm).

Proof: (a) 1° Since $F \in \mathrm{L}_{\mathrm{strong}}(Q; \mathcal{B}(B, B_2))$, we have $Ff \in \mathrm{L}$ for all $f \in \mathrm{L}$, by Lemma F.1.3(a). W.l.o.g., we assume that $\|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} < \infty$ and $\|f\|_p > 0$.

2° *Case $f \in \mathrm{CVM}^p$:* Set $\mathrm{CVM}^p := \{f \in \mathrm{L}^p(Q;B) \mid f \text{ is countably-valued}\}$. Clearly $\|Ff\|_p \le \|F\|_\infty \|f\|_p$ for all $f \in \mathrm{CVM}$.

3° *Case $f \in \mathrm{L}^p$:* Let $\mathrm{CVM}^p \ni f_n \to f$ in L^p. Then $\{Ff_n\}$ is a Cauchy-sequence in $\mathrm{L}^p(Q;B_2)$, hence $Ff_n \to g$ in L^p for some $g \in \mathrm{L}^p$ with $\|g\|_p \le \|F\|_\infty \|f\|_p$, and a subsequence $\{Ff_{n_k}\}$ converges a.e. to g.

On the other hand, a subsubsequence converges to Ff, hence $Ff = g$ a.e., hence $\|Ff\|_p = \|g\|_p \le \|F\|_\infty \|f\|_p$ for all $f \in \mathrm{L}^p$.

(b) (Actually, even for $p < \infty$ it suffices that for any $E' \in \mathfrak{M}$ with $\mu(E') = \infty$, there is $E \subset E'$ s.t. $0 < \mu(E) < \mu(E')$. We show below that this condition is also necessary.) W.l.o.g., we assume that $0 < \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}} \le \infty$.

1° *A "counter-example":* Let $E' \in \mathfrak{M}$ be s.t. $\mu(E') = \infty$ and $\mu(E) \in \{0, \infty\}$ for all measurable $E \subset E'$, and let $p < \infty$ and $B \ne \{0\} \ne B_2$. Then we can take $F := \chi_{E'} T$, where $T \in \mathcal{B}(B, B_2) \setminus \{0\}$, so that $\|Ff\|_p = 0$ for all $f \in \mathrm{L}^p$.

Indeed, if $Ff \ne 0$, then $E'' := (Ff)^{-1}[B_2 \setminus \{0\}] \subset E'$ and $\mu(E'') > 0$. By Lemma B.2.8(b), there are $A \subset E''$ and $\Lambda \in B^*$ s.t. $\Lambda f > 1$ on A and $\mu(A) > 0$. Consequently, $\mu(A) = \infty$, hence $\|\Lambda f\|_p = \infty$, hence $f \notin \mathrm{L}^p$, a contradiction.

2° *The equality:* We assume that $0 \le M < \|F\|_{\mathrm{L}^i_{\mathrm{strong}} nfty}$ is arbitrary and find $\phi \in \mathcal{E} \setminus \{0\}$ s.t. $\|F\phi\|_p / \|\phi\|_p > M$. By (a), this establishes our claim.

Suppose that $0 < M < \|F\|_\infty \le \infty$. Take $x \in X$ (set $X := B$ if none is given) s.t. $\|Fx\|_\infty > M\|x\|$ and choose $E' \in \mathfrak{M}$ s.t. $\|F(q)x\| > M\|x\|$ for all $q \in E'$ & $0 < \mu(E')$ and choose $E \subset E'$ s.t. $0 < \mu(E) < \infty$.

If $p = \infty$, then $\|F\chi_E x\|_\infty > M\|x\| = M\|\chi_E x\|_\infty$; obviously the inequality is preserved if $\|\chi_E - g\|_\infty$ is small enough and χ_E is replaced by g (and x by $x' \in X$). Assume then that $p < \infty$.

For any $n \in \mathbf{N}$, there is $g \in \mathcal{E}'$ s.t. $\|g - \chi_E\|_{\mathrm{L}^p(E)} \le \|g - \chi_E\|_p < 1/n$. Set $R := \|\chi_E\|_p = \mu(E)^{1/p}$. Since $\|g\|_p / \|g\|_{\mathrm{L}^p(E)} < (R + 1/n)/(R - 1/n)$, we have

$$\|Fgx\|_p^p > M^p \|x\|^p \|g\|^p_{\mathrm{L}^p(E)} > M^p (R - 1/n)(R + 1/n)^{-1} \|gx\|_p. \tag{F.8}$$

Consequently, $\|Fgx\|_p > M\|gx\|_p$ for n big enough. Since $\|gx\|_p$ and $\|Fgx\|_p$ are continuous functions of $x \in B$, we can replace x by some $x' \in X$ close enough. □

When applying the Hölder inequality or similar results, one must make correct measurability requirements (cf. (d)):

Lemma F.1.8 *Let $1 \le p \le \infty$ & $1/p + 1/q = 1$. Let $F : Q \to \mathcal{B}(B, B_2)$, $G : Q \to \mathcal{B}(B_2, B_3)$, $f : Q \to B$. Then*

(a1) $F \in \mathrm{L}^p_{\text{strong}}$ & $G \in \mathrm{L}^\infty \Longrightarrow GF \in \mathrm{L}^p_{\text{strong}}$ & $\|GF\|_{\mathrm{L}^p_{\text{strong}}} \le \|G\|_{\mathrm{L}^\infty} \|F\|_{\mathrm{L}^p_{\text{strong}}}$.

(a2) $F \in \mathrm{L}^\infty_{\text{strong}}$ & $G \in \mathrm{L}^p \Longrightarrow GF \in \mathrm{L}^p_{\text{strong}}$ & $\|GF\|_{\mathrm{L}^p_{\text{strong}}} \le \|G\|_{\mathrm{L}^p} \|F\|_{\mathrm{L}^\infty_{\text{strong}}}$.

(a3) $F \in \mathrm{L}^p_{\text{strong}}$ & $G \in \mathrm{L}^q \Longrightarrow GF \in \mathrm{L}^1_{\text{strong}}$ & $\|GF\|_{\mathrm{L}^1_{\text{strong}}} \le \|G\|_{\mathrm{L}^q} \|F\|_{\mathrm{L}^p_{\text{strong}}}$.

(b) $F \in \mathrm{L}^\infty_{\text{strong}}$ & $f \in \mathrm{L}^p \Longrightarrow Ff \in \mathrm{L}^p$ & $\|Ff\|_{\mathrm{L}^p} \le \|F\|_{\mathrm{L}^\infty_{\text{strong}}} \|f\|_{\mathrm{L}^p}$.

(c) $G \in \mathrm{L}^\infty_{\text{strong}}$ & $F \in \mathrm{L}^p_{\text{strong}} \Longrightarrow GF \in \mathrm{L}^p_{\text{strong}}$ & $\|GF\|_{\mathrm{L}^p_{\text{strong}}} \le \|G\|_{\mathrm{L}^\infty_{\text{strong}}} \|F\|_{\mathrm{L}^p_{\text{strong}}}$.

(d) We may have $\|Ff\|_{\mathrm{L}^1} > M\|F\|_{\mathrm{L}^2_{\text{strong}}} \|f\|_{\mathrm{L}^2}$, $\|Ff\|_{\mathrm{L}^2} > M\|F\|_{\mathrm{L}^2_{\text{strong}}} \|f\|_{\mathrm{L}^\infty}$ *and* $\|GF\|_{\mathrm{L}^2_{\text{strong}}} > M\|G\|_{\mathrm{L}^2_{\text{strong}}} \|F\|_{\mathrm{L}^\infty_{\text{strong}}}$ *for any* $M > 0$, $Q = \mathbf{R}$ *and* $B = \ell^2(\mathbf{N})$ *(or* $B = \mathbf{K}^N$ *for* $N > M^2$*).*

(a') Claims (a1)–(a3) hold also with L_{weak} *in place of* $\mathrm{L}_{\text{strong}}$ *if* G *is scalar (i.e.,* $G : Q \to \mathbf{K}$*).*

(b') $F \in \mathrm{L}^\infty_{\text{weak}}$ & $f \in \mathrm{L}^p \Longrightarrow Ff \in \mathrm{L}^p_{\text{weak}}$ & $\|Ff\|_{\mathrm{L}^p_{\text{weak}}} \le \|F\|_{\mathrm{L}^\infty_{\text{weak}}} \|f\|_{\mathrm{L}^p}$.

(c') $G \in \mathrm{L}^\infty_{\text{weak}}$ & $F \in \mathrm{L}^p_{\text{strong}} \Longrightarrow GF \in \mathrm{L}^p_{\text{weak}}$ & $\|GF\|_{\mathrm{L}^p_{\text{weak}}} \le \|G\|_{\mathrm{L}^\infty_{\text{weak}}} \|F\|_{\mathrm{L}^p_{\text{strong}}}$.

(a1") $F \in \mathrm{L}^p_{\text{strong}}$ & $G^* \in \mathrm{L}^\infty_{\text{strong}} \Longrightarrow GF \in \mathrm{L}^p_{\text{weak}}$ & $\|GF\|_{\mathrm{L}^p_{\text{weak}}} \le \|G^*\|_{\mathrm{L}^\infty_{\text{strong}}} \|F\|_{\mathrm{L}^p_{\text{strong}}}$.

(a2") $F \in \mathrm{L}^\infty_{\text{strong}}$ & $G^* \in \mathrm{L}^p_{\text{strong}} \Longrightarrow GF \in \mathrm{L}^p_{\text{weak}}$ & $\|GF\|_{\mathrm{L}^p_{\text{weak}}} \le \|G^*\|_{\mathrm{L}^p_{\text{strong}}} \|F\|_{\mathrm{L}^\infty_{\text{strong}}}$.

(a3") $F \in \mathrm{L}^p_{\text{strong}}$ & $G^* \in \mathrm{L}^q_{\text{strong}} \Longrightarrow GF \in \mathrm{L}^1_{\text{weak}}$ & $\|GF\|_{\mathrm{L}^1_{\text{weak}}} \le \|G^*\|_{\mathrm{L}^q_{\text{strong}}} \|F\|_{\mathrm{L}^p_{\text{strong}}}$.

Note that we may take $G \in \mathrm{L}(Q) := \mathrm{L}(Q;\mathbf{K})$ (and $B_3 = B_2$). Note also that $F \in \mathrm{L}^p(Q;B_2)$ can be interpreted as $F \in \mathrm{L}^p_{\mathrm{strong}}(Q;\mathcal{B}(\mathbf{K};B_2))$, so that (a1")–(a3") etc. apply.

Let $Q = \mathbf{R}^n$. Then, with the assumptions of (a3") (or (a3)), the weak convolution $G * F$ exists everywhere on $\mathbf{R}^n$ (see Theorem F.2.1(b)), hence the norm estimates of Lemma D.1.7 can be applied (to $(G^*\Lambda)$ and Fx, for each $x \in B$ and $\Lambda \in B_2^*$). However, without the assumptions of (a3") (even when, e.g., $G \in \mathrm{L}^2_{\mathrm{strong}}$, $F \in \mathrm{L}^2$), we do not know whether $G * F$ exists as a function (with values in $\mathcal{B}(B, B_3)$).

Proof of Lemma F.1.8: By Lemma F.1.3(a)&(b), we have $GF, hF \in \mathrm{L}_{\mathrm{strong}}$, $Ff \in \mathrm{L}$, hence only the claims on norms have to be shown.

(a1) Let $x \in B$. Then $\|Fx\|_{\mathrm{L}^p} \leq \|F\|_{\mathrm{L}^p_{\mathrm{strong}}}\|x\|_B$, hence $\|GFx\|_{\mathrm{L}^p} \leq \|G\|_{\mathrm{L}^\infty}\|F\|_{\mathrm{L}^p_{\mathrm{strong}}}\|x\|_B$.

(a2)&(a3) The proof is analogous to that of (a1) (use the Hölder Inequality for (a3)).

(b) 1° *Simple functions:* Let $f = \sum_{j=1}^k x_j \chi_{E_j}$ be simple with sets E_j disjoint. Then $\|Fx_j\| \leq \|F\|\|x_j\|$ $(j \leq k)$, hence $\|Ff\|_p^p \leq \|F\|^p \|f\|_p^p$.

2° *General* $f \in \mathrm{L}^p$*:* By Theorems B.3.2 and B.3.11, there are simple functions $\{f_n\} \subset \mathrm{L}^p$ s.t. $f_n \to f$ a.e. and in L^p. Then $Ff_n \to Ff$ a.e. By 1°, $\{Ff_n\}$ is an L^p-Cauchy sequence, hence $Ff_n \to g$ in L^p for some $g \in \mathrm{L}^p$ with $\|g\|_p \leq \|F\|\|f\|_p$. But a subsequence of $\{Ff_n\}$ converges a.e. to g, hence $g = Ff$ a.e., hence (b1) holds.

(c) Let $x \in B$. Then $\|Fx\|_{\mathrm{L}^p} \leq \|F\|_{\mathrm{L}^p_{\mathrm{strong}}}\|x\|_B$, hence $\|GFx\|_{\mathrm{L}^p} \leq \|G\|_{\mathrm{L}^\infty_{\mathrm{strong}}}\|F\|_{\mathrm{L}^p_{\mathrm{strong}}}\|x\|_B$, by (b).

(d) Let $B = \ell^2(\mathbf{N})$. Define $F \in \mathrm{L}^2_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(B))$ by $F = \sum_{k\in\mathbf{N}} \chi_{[k,k+1)} P_k$, where $P_k(x_j)_{j\in\mathbf{N}} := x_k e_k$, and $f \in \mathrm{L}^2$ by $f = \sum_{k=1}^N \chi_{[k,k+1)} e_k$. Then

$$\|F\sum_k \alpha_k e_k\|_2^2 = \|\sum_k \alpha_k \chi_{[k,k+1)} e_k\|_2^2 = \sum_k |\alpha_k|^2 = \|\sum_k \alpha_k e_k\|_B^2, \tag{F.9}$$

hence $\|F\|_{\mathrm{L}^2_{\mathrm{strong}}} = 1$. However, $Ff = f$, hence $\|Ff\|_2 = N^{1/2}$ and $\|Ff\|_1 = N$, although $\|f\|_\infty = 1$ and $\|f\|_2 = N^{1/2}$. (Note that we could take $f = \sum_{k=1}^\infty k^{-1}\chi_{[k,k+1)} e_k$ to obtain $\|f\|_2^2 = \sum_k k^{-2} < \infty$, $\|Ff\|_1 = \|f\|_1 = \sum_k k^{-1} = \infty$.)

Finally, set $G := \sum_{k=1}^N P_{1k}\chi_{[k,k+1)}$, where $P_{1k}(x_j)_{j\in\mathbf{N}} := x_1 e_k$. Then $\|G\|_{\mathrm{L}^\infty_{\mathrm{strong}}} = 1$, but $Ge_1 = f$, hence $\|FGe_k\|_2 = N^{1/2}$, although $\|F\|_{\mathrm{L}^2_{\mathrm{strong}}}\|G\|_{\mathrm{L}^\infty_{\mathrm{strong}}}\|e_k\|_B = 1$.

(a')–(c') The proofs of (a1)–(c) apply mutatis mutandis.

(a1")–(a3") Let $\Lambda \in B_2^*$, $x \in B$. Then $\Lambda GFx = (G^*\Lambda)Fx$, $\|G^*\Lambda\| \leq \|G^*\|\|\Lambda\|_{B_3^*}$ and $\|Fx\| \leq \|F\|\|x\|_B$ for all $x \in B$, $\Lambda \in B_3^*$, hence (a1")–(a3") hold (by corresponding scalar results). □

Now we present a lifting result (claims (s3) and (w3)) and use it to show that $\mathrm{L}^\infty_{\mathrm{strong}}$ and $\mathrm{L}^\infty_{\mathrm{weak}}$ are complete:

Theorem F.1.9 ($\mathbf{L}^\infty_{\mathbf{strong}}$ and $\mathbf{L}^\infty_{\mathbf{weak}}$ are complete) *If (1.) $Q \subset \mathbf{R}^n$ is measurable, $\mu = m$, and B is a Hilbert space or $B_2 = \mathbf{K}$, or (2.) B is separable, then the following hold:*

(s1) $\mathrm{L}^\infty_{\mathrm{strong}}(Q,\mu;\mathcal{B}(B,B_2))$ *is a Banach space.*

(s2) Any Cauchy-sequence in $\mathrm{L}^\infty_{\mathrm{strong}}$ *converges uniformly outside some null set.*

(s3) For each $[F] \in \mathrm{L}^\infty_{\mathrm{strong}}(Q,\mu;\mathcal{B}(B,B_2))$, *there is a representative* $G \in [F]_{\mathrm{L}_{\mathrm{strong}}}$ *s.t.* $\sup_{q\in Q}\|G(q)\|_{\mathcal{B}(B,B_2)} = \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$.

(s4) $\mathcal{C}_{\mathrm{b}}(Q;\mathcal{B}(B,B_2)) \cap \mathcal{G}\mathrm{L}^\infty_{\mathrm{strong}}(Q,\mu;\mathcal{B}(B,B_2)) = \mathcal{G}\mathcal{C}_{\mathrm{b}}(Q;\mathcal{B}(B,B_2))$ *if (1.) holds.*

If, instead, B and B_2^* are separable, then

(w1) $\mathrm{L}^\infty_{\mathrm{weak}}(Q,\mu;\mathcal{B}(B,B_2))$ *is a Banach space.*

(w2) Any Cauchy-sequence in $\mathrm{L}^\infty_{\mathrm{weak}}$ *converges uniformly outside some null set.*

(w3) For each $[F] \in \mathrm{L}^\infty_{\mathrm{weak}}(Q,\mu;\mathcal{B}(B,B_2))$, *there is a representative* $G \in [F]_{\mathrm{L}_{\mathrm{weak}}}$ *s.t.* $\sup_{q\in Q}\|G(q)\|_{\mathcal{B}(B,B_2)} = \|F\|_{\mathrm{L}^\infty_{\mathrm{weak}}}$.

All above assumptions are unnecessary if μ is the counting measure on a set Q (then every function is measurable and $\mathrm{L}^\infty = \mathrm{L}^\infty_{\mathrm{strong}} = \mathrm{L}^\infty_{\mathrm{weak}} = \mathcal{C}_{\mathrm{b}}$, where we use the discrete topology on Q, hence all these spaces are complete and have $[0] = \{0\}$). However, we do not know whether the theorem holds for general (Q,μ), B and B_2; the main problem is the "lifting" claim (s3)/(w3); once it is established, the completeness claim requires just the latter assumption (cf. part II below).

Recall from Lemma F.1.5(b) that we have $\mathrm{L}^\infty_{\mathrm{strong}} = \mathcal{B}(B,\mathrm{L}^\infty(Q;B_2))$ in case (1.).

Proof of Theorem F.1.9: *Part I: (s3)&(w3):* Obviously, $\sup_{q\in Q}\|G(q)\|_{\mathcal{B}(B,B_2)} \ge \|F\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ in (s3) and $\sup_{q\in Q}\|G(q)\|_{\mathcal{B}(B,B_2)} \ge \|F\|_{\mathrm{L}^\infty_{\mathrm{weak}}}$ in (w3) so we only have to show the converses.

1° *Case (1.): $Q \subset \mathbf{R}^n$:* This was shown in the proof of Lemma F.1.5(b).

2° *Case (2.): separable B:* Remove all Λ's from 3°.

3° $\mathrm{L}^\infty_{\mathrm{weak}}$*:* Let $[F] \in \mathrm{L}^\infty_{\mathrm{weak}}(Q,\mu;\mathcal{B}(B,B_2))$, and let $S \subset B$ and $S_2 \subset B_2^*$ be dense and countable. Set $M := \|F\|_{\mathrm{L}^\infty_{\mathrm{weak}}}$. Then, for each $x \in B$ and $\Lambda \in B_2^*$, we have $|\Lambda F(q)x| \le M\|x\|\|\Lambda\|$ for a.e. $q \in Q$; choose a null set $N \subset Q$ s.t. this inequality holds for all $x \in S$, $\Lambda \in S_2$ and $q \in N^c$. By density and continuity, this inequality holds for all $x \in B$, $\Lambda \in B_2^*$ and $q \in N^c$, therefore, $G := \chi_{N^c}F$ is of the required form.

Part II: (s1)&(w1): Because $\mathrm{L}^\infty_{\mathrm{strong}}$ and $\mathrm{L}^\infty_{\mathrm{weak}}$ are normed spaces, we only have to prove their completeness. (Note that our proof does not require (1.) nor (2.) explicitly, but we rely on Part I.)

1° $\mathrm{L}^\infty_{\mathrm{strong}}$ *is a Banach space:* Let $\{[F_n]\}$ be a $\mathrm{L}^\infty_{\mathrm{strong}}$-Cauchy sequence. Assume that each F_n is chosen so that $\sup\|F_n\|_{\mathcal{B}(B,B_2)} \le \|F_n\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$, as in (s3). Set $\delta_n := \sup_{k\in\mathbf{N}}\|F_n - F_{n+k}\|_{\mathrm{L}^\infty_{\mathrm{strong}}}$ ($n \in \mathbf{N}$), so that $\delta_n \to 0$ as $n \in \infty$. Set

$$M := \lim_{n\to+\infty}\|F_n\|_{\mathrm{L}^\infty_{\mathrm{strong}}} \in [0,\infty), \qquad g_x := \lim_{n\to+\infty} F_n x \in \mathrm{L}^\infty(Q;B_2) \quad (x \in B), \tag{F.10}$$

so that $\|g_x(t)\|_{B_2} \le M\|x\|_B$ for all $x \in B$ (set $g_x(t) = 0$ for t in the null set $N_x \subset N'_x$ where the limit does not exist).

Now $F_n x \to g_x$ uniformly on N_x^c, where $N_x := \cup_{n,k}\{q \in Q \,\big|\, \|F_n x - F_{n+k}x\| > \delta_n\}$.

Obviously, $X_q := \{x \in B \,\big|\, \lim_{n\to+\infty} F_n(q)x \text{ exists}\}$ is a subspace of B for all $q \in Q$. For each $q \in Q$, we let $F(q)$ be a norm-preserving (see Lemma A.3.11) extension of $(X_q \ni x \to g_x(q) \in B_2)$ so that $\|F(q)\|_{\mathcal{B}(B,B_2)} \le M$. Then, for any $x \in B$, we have for a.e. $q \in Q$ that $x \in X_q$, hence that $F(q)x = g_x(q) = \lim_n F_n(q)x$ (in particular, F is strongly measurable) and hence $\|F(q)x - F_n(q)x\| = \lim_k \|F_{n+k}(q)x - F_n(q)x\| \le \delta_n\|x\|$. Therefore, $\|F - F_n\|_{L^\infty_{\text{strong}}} \le \delta_n \to 0$, so that $F_n \to F$ in L^∞_{strong}, as $n \to +\infty$. Because $\{F_n\}$ was arbitrary, we have shown that L^∞_{strong} is complete, hence a Banach space.

2° L^∞_{weak} *is a Banach space:* Let $\{[F_n]\}$ be a Cauchy-sequence in L^∞_{weak} with each F_n chosen so that $\sup_{q\in Q}\|F_n(q)\| = \|F_n\|_{L^\infty_{\text{weak}}}$. Set $\delta_n := \sup_{k\in\mathbf{N}}\|F_n - F_{n+k}\|_{L^\infty_{\text{weak}}}$ ($n \in \mathbf{N}$), so that $\delta_n \to 0$ as $n \in \infty$.

Let $S \subset B$ and $S_2 \subset B_2$ be dense and set $N := \cup_{x\in S,\Lambda\in S_2} N_{x,\Lambda}$, where

$$N_{x,\Lambda} = \cup_{n,k\in\mathbf{N}}\{q \in Q \,\big|\, \|\Lambda F_n(q)x - \Lambda F_{n+k}(q)x\| > \delta_n\|\Lambda\|\|x\|\}, \tag{F.11}$$

so that $\mu(N) = 0$. Let $q \in N^c$. Then, for all $n,k \in \mathbf{N}$, we have $\|\Lambda F_n(q)x - \Lambda F_{n+k}(q)x\| \le \delta_n\|\Lambda\|\|x\|$ for all $\Lambda \in B_2^*$, $x \in B$, by density, hence $\|F_n(q) - F_{n+k}(q)\|_{\mathcal{B}(B,B_2)} \le \delta_n$. In particular, $\{F_n(q)\}$ is a Cauchy-sequence in $\mathcal{B}(B,B_2)$; let $F(q) \in \mathcal{B}(B,B_2)$ be its limit. Then $\|F(q) - F_n(q)\| = \lim_k \|F_{n+k}(q) - F_n(q)\| \le \delta_n$; but $q \in N^c$ was arbitrary, hence $F_n \to F$ uniformly on N^c.

Part III: (s2)&(w2): In Part II we showed that $F_n \to F$ uniformly on N^c for some null set $N \subset Q$, but we assumed $\{F_n\}$ chosen as in Part I. For general $\{F'_n\}$ we can choose the sequence $\{F_n\}$ as above, so that $F_n = F'_n$ outside some null set N_n for each $n \in \mathbf{N}$. Then $F'_n \to F$ uniformly outside the null set $N \cup \big(\cup_n N_n\big)$.

Part IV: (s4): By Lemma F.1.3(f2), $\mathcal{C}_{\text{b}}(Q;\mathcal{B}(B,B_2)) \subset L^\infty_{\text{strong}}(Q;\mathcal{B}(B,B_2))$, isometrically, hence $\mathcal{G}\mathcal{C}_{\text{b}}(Q;\mathcal{B}(B,B_2)) \subset \mathcal{G}L^\infty_{\text{strong}}(Q;\mathcal{B}(B,B_2))$. Conversely, if $F \in \mathcal{C}_{\text{b}}(Q;\mathcal{B}(B,B_2)) \cap \mathcal{G}L^\infty_{\text{strong}}(Q;\mathcal{B}(B,B_2))$, then $F(q) \in \mathcal{G}\mathcal{B}(B,B_2)$ for all $q \in Q$ and hence $[F]^{-1} = [F^{-1}]$, by two applications of Lemma F.1.3(f2) (since there is a bounded representative of $[F]^{-1}$, by (s3)), hence then $[F]^{-1} \in \mathcal{C}_{\text{b}}$. □

In contrast to the above theorem, the normed spaces L^p_{strong} and L^p_{weak} are usually not complete for $p < \infty$:

Example F.1.10 Let $Q := [0,1]$, $B := \ell^2(\mathbf{N})$. Then there is a sequence $\{F_n\} \subset \mathcal{C}(Q;\mathcal{B}(B)) \subset L^2 \subset L^2_{\text{strong}} \subset L^2_{\text{weak}}$ s.t. $\{F_n\}$ is a Cauchy-sequence in $L^2_{\text{strong}}(Q;\mathcal{B}(B))$ and hence in $L^2_{\text{weak}}(Q;\mathcal{B}(B))$ too, but $\{F_n\}$ does not converge in either of these spaces (although it does converge in $\mathcal{B}(B;L^2(Q;B))$, which is a Banach space). Moreover, $F_n(t) = F_n(t)^*$ for all $t \in [0,1]$ and $n \in \mathbf{N}$. ◁

See also Example F.3.6.

Proof: 1° *The construction of* $\{F_n\}$*:* Set $g(t) := |t|^{-1/3}$, $g_n(t) := (|t| + 1/n)^{-1/3}$ ($t \in \mathbf{R}$) so that $0 \le g_n \le n$, $g_n \in \mathcal{C}_0(\mathbf{R})$, $g \in L^2(Q)$ and $M_g :=$

$\sup_{q\in[0,1]}\|\tau^q g\|_2 < \infty$. Furthermore, $g_n(t) \to g(t)$ monotonely for each $t \in \mathbf{R}$ and $\|g - g_n\|_{L^2([-1,1])} \to 0$, by the Dominated Convergence Theorem.

Let $\{q_k\} \subset Q$ be dense. For $t \in Q$ and $n \in \mathbf{N}$, define $F_n(t) := \sum_{k\in\mathbf{N}} g_n(t-q_k)P_k$ (i.e., $F_n(t)x := \sum_{k\in\mathbf{N}} g_n(t-q_k)x_k e_k$), where P_k is the projection $P_k : \sum_{j\in\mathbf{N}} x_j e_j \mapsto x_k e_k$ ($k \in \mathbf{N}$).

Note that for $f \in L^2(Q;B)$, we have $\|f\|_2^2 = \sum_k \|f_k\|_2^2$. Obviously, $F_n(t)^* = \sum_{k\in\mathbf{N}} \overline{g_n(t-q_k)}P_k^* = \sum_{k\in\mathbf{N}} g_n(t-q_k)P_k$ for all n and t.

2° $\|F_n\|_{\mathcal{B}(B)} \le n$: This is obvious.

3° $\{F_n\} \subset C(Q;\mathcal{B}(B))$*:* Let $n \in \mathbf{N}$, $t \in Q$ and $\varepsilon > 0$. The function g_n is uniformly continuous, because $g_n \in C_0(\mathbf{R})$, hence there is $\delta > 0$ s.t. $|g_n(t') - g_n(t'')| < \varepsilon$ for $|t'-t''| < \delta$. Let $|t'-t| < \delta$ and $x \in B$. Then

$$\|(F_n(t)-F_n(t'))x\|_B^2 = \|\sum_k x_k(g_n(t-q_k)-g_n(t'-q_k))e_k\|_B^2 \le \sum_k |x_k|^2\varepsilon^2 = \varepsilon^2\|x\|_B. \tag{F.12}$$

Because x was arbitrary, we have $\|(F_n(t)-F_n(t'))\| \le \varepsilon$. Thus, F_n is continuous.

4° *$F_n \to F$ in $\mathcal{B}(B, L^2(Q;B))$:* For $t \in Q$, $n \in \mathbf{N}$, and $x \in B$, we define $F(t) := \sum_{k\in\mathbf{N}} g(t-q_k)P_k$. Then $\|Fx\|_2^2 \le \sum_k |x_k|^2 M_g^2 = M_g^2\|x\|_B^2$. Thus, $\|F\| \le M_g$. Moreover, given $\varepsilon > 0$, there is $N \in \mathbf{N}$ be s.t. $\|g - g_m\|_{L^2([-1,1))} < \varepsilon$ for all $n, m > N$, and, consequently,

$$\|F_n x - Fx\|_{L^2(Q;B)}^2 \le \sum_k |x_k|^2\|g_n(\cdot - q_k) - g(\cdot - q_k)\|_{L^2(Q)}^2 \le \varepsilon^2\|x\|_B^2 \quad (x \in B). \tag{F.13}$$

Therefore, $F_n \to F$ in $\mathcal{B}$. (In particular, $\{F_n\}$ is $L^2_{\rm strong}$-Cauchy.)

5° *$\{F_n\}$ does not converge in $L^2_{\rm weak}$:* (I.e., F does not correspond to any function $Q \to \mathcal{B}(B)$.) To obtain a contradiction, assume that $F \in L^2_{\rm weak}(Q;\mathcal{B}(B))$ is such that $\langle F_n x, y\rangle \to \langle Fx, y\rangle$ in $L^2(Q)$ for all $x, y \in B$.

We have $F_n e_k = \tau^{-q_k} g_n e_k \to \tau^{-q_k} g e_k$ in $L^2(Q;B)$ and pointwise on Q, for each $k \in \mathbf{N}$. Thus, $\langle Fe_k, e_j\rangle = \langle \tau^{-q_k} g e_k, e_j\rangle =: f_{k,j}$ a.e. for all $k, j \in \mathbf{N}$, because $\langle Fe_k, e_j\rangle$ is a.e. the pointwise limit of a subsequence of $\{\langle F_n e_k, e_j\rangle\}$, and all subsequences of the latter converge to $f_{k,j}$.

Choose a null set N s.t. $\langle Fe_k, e_j\rangle = \langle \tau^{-q_k} g e_k, e_j\rangle$ on $Q \setminus N$ for all k, j. Then $Fe_k = \tau^{-q_k} g e_k$ on $Q \setminus N$ for all $k \in \mathbf{N}$. Let $t \in Q \setminus N$ and $M > 0$. By the density of $\{q_k\}$ in Q, there is k s.t. $q_k < t$ and $(\tau^{-q_k} g)(t) = |t - q_k|^{-1/3} > M$, so that $\|F(t)e_k\|_B > M$. Consequently, $\|F(t)\|_{\mathcal{B}(B)} > M$. But this holds for all $t \in Q \setminus N$ and all $M > 0$, hence F must be unbounded almost everywhere; in particular, F is not $\mathcal{B}(B)$-valued. □

(See the notes on p. 1023.)

F.2 Strong and weak integration ($\int\!\!\!\!\!{\scriptstyle s}$, $\int\!\!\!\!\!{\scriptstyle w}$)

Whenever anyone says, "theoretically," they really mean, "not really."
— Dave Parnas

Here we define the *strong and weak (operator) integrals*; in the rest of this section we treat corresponding convolutions.

Theorem F.2.1 (Strong and weak integrals $\int\!\!\!\!\!{\scriptstyle s}$ and $\int\!\!\!\!\!{\scriptstyle w}$) *Let* $F : Q \to \mathcal{B}(B,B_2)$.

(a) **(Strong integral)** *If* $F \in \mathrm{L}^1_{\text{strong}}$*, then there is a unique* $L =: \int\!\!\!\!\!{\scriptstyle s}_Q F\,d\mu \in \mathcal{B}(B,B_2)$ *s.t.* $Lx = \int_Q F x\,d\mu$ *for all* $x \in B$*. Moreover,* $\|L\|_{\mathcal{B}(B,B_2)} \le \|F\|_{\mathrm{L}^1_{\text{strong}}}$.

(b) **(Weak integral)** *If* $F \in \mathrm{L}^1_{\text{weak}}$*, then there is a unique* $L =: \int\!\!\!\!\!{\scriptstyle w}_Q F\,d\mu \in \mathcal{B}(B,B_2^{**})$ *s.t.* $(Lx)\Lambda = \int_Q \Lambda F x\,d\mu$ *for all* $x \in B$, $\Lambda \in B_2^*$*. Moreover,* $\|L\|_{\mathcal{B}(B,B_2^{**})} \le \|F\|_{\mathrm{L}^1_{\text{weak}}}$.

(c) A Bochner integral (a uniform integral) is a strong integral (i.e., $\int_Q F\,d\mu = \int\!\!\!\!\!{\scriptstyle s}_Q F\,d\mu$ *for* $F \in \mathrm{L}^1$*), and a strong integral is a weak integral (i.e.,* $\int\!\!\!\!\!{\scriptstyle s}_Q F\,d\mu = \int\!\!\!\!\!{\scriptstyle w}_Q F\,d\mu$ *for* $F \in \mathrm{L}^1_{\text{strong}}$*).*

(d) If $F \in \mathrm{L}^1_{\text{weak}}(Q;\mathcal{B}(B,B_2))$ *and* B *is reflexive, then* $F^* \in \mathrm{L}^1_{\text{weak}}(Q;\mathcal{B}(B_2^*,B))$ *and* $\int\!\!\!\!\!{\scriptstyle w}_Q F^*\,d\mu = \left(\int\!\!\!\!\!{\scriptstyle w}_Q F\,d\mu\right)^*$.

(e) Let $T \in \mathcal{B}(B_2,B_3)$ *[and* $x \in B$*]. In (a) we have* $\int\!\!\!\!\!{\scriptstyle s}_Q TF\,d\mu = T\int\!\!\!\!\!{\scriptstyle s}_Q F\,d\mu$ *and* $\int_Q TFx\,d\mu = T(\int\!\!\!\!\!{\scriptstyle s}_Q F\,d\mu)x$*; in (b) we have* $\int\!\!\!\!\!{\scriptstyle w}_Q TF\,d\mu = T\int\!\!\!\!\!{\scriptstyle w}_Q F\,d\mu$ *and* $\int\!\!\!\!\!{\scriptstyle w}_Q TFx\,d\mu = T(\int\!\!\!\!\!{\scriptstyle w}_Q F\,d\mu)x$.

(f) Claims (a), (b), (c) and (e) also hold with replacements $\mathrm{L}^1_{\text{strong}} \mapsto \mathcal{B}(B,\mathrm{L}^1(Q;B_2))$ *and* $\mathrm{L}^1_{\text{weak}} \mapsto \mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{L}^1(Q))$.

With replacement $\mathrm{L}^1_{\text{weak}} \mapsto \mathcal{B}(B,\mathcal{B}(B_2,\mathrm{L}^1(Q;B_3)))$ *we get that* $\|\int\!\!\!\!\!{\scriptstyle w}_Q F\,d\mu\|_{\mathcal{B}(B,\mathcal{B}(B_2,B_3))} \le \|F\|_{\mathcal{B}(B,\mathcal{B}(B_2,\mathrm{L}^1(Q;B_3)))}$.

Even if B_2 *is a Hilbert space, the expression* $\mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{L}^1(Q))$ *refers to the linear dual of* B_2.

(g) Let $p \in [1,\infty]$*. Then the embedding* $\mathrm{L}^p_{\text{strong}}(Q;\mathcal{B}(B,B_2)) \to \mathcal{B}(B,\mathrm{L}^p(Q;B_2))$ *is a linear isometry to the subspace* $\mathrm{L}_{\text{strong}}(Q;\mathcal{B}(B,B_2)) \cap \mathcal{B}(B,\mathrm{L}^p(Q;B_2))$*. Analogously,*

$$\mathrm{L}^p_{\text{weak}}(Q;\mathcal{B}(B,B_2)) = \mathrm{L}_{\text{weak}}(Q;\mathcal{B}(B,B_2)) \cap \mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{L}^p(Q))) \tag{F.14}$$

isometrically.

Thus, if B_2 is reflexive (e.g., a Hilbert space) and $F \in \mathrm{L}^1_{\text{weak}}$, then $\int\!\!\!\!\!{\scriptstyle w}_Q F\,d\mu \in \mathcal{B}(B,B_2)$ and $\|\int\!\!\!\!\!{\scriptstyle w}_Q F\,d\mu\|_{\mathcal{B}(B,B_2)} \le \|F\|_{\mathrm{L}^1_{\text{weak}}}$.

If B and B_2 are Hilbert spaces, then, of course, all asterisks can be removed.

If $B = \mathbf{K}$, then $\int\!\!\!\!\!{\scriptstyle w}$ becomes the "weak*-integral" (or Gelfand Integral or Dunford Integral). The asterisk in (d) refers to Banach adjoints (since B and B_2 are only assumed to be Banach spaces); however, (d) is obviously true also for Hilbert adjoints (i.e., "F^{d}", not "F^{B}").

By Example F.1.10, the subspaces mentioned in (g) need not be closed in general.

Proof of Theorem F.2.1: The uniqueness is obvious in (a) and (b).

(a) It is obvious that $L : x \mapsto \int_Q Fx\,d\mu \in B_2$ satisfies $\|Lx\|_{B_2} \le \|Fx\|_{\mathrm{L}^1} \le \|F\|_{\mathrm{L}^1_{\mathrm{strong}}}\|x\|_B$.

(b) As in (a), we see that $L : x \mapsto (\Lambda \mapsto \int_Q \Lambda Fx\,d\mu \in \mathbf{K}) \in B_2^{**}$ satisfies $\|L\| \le \|F\|_{\mathrm{L}^1_{\mathrm{weak}}}$.

(c) This is obvious (see (B.18)).

(d) By Lemma F.1.5(e), $F^* \in \mathrm{L}^1_{\mathrm{weak}}$. Now $L^* := \left(\mathrm{w}\!\!\int_Q F\,d\mu\right)^* \in \mathcal{B}(B_2^{***}, B^*) \subset \mathcal{B}(B_2^*, B^*)$. But

$$\int_Q (F^*\Lambda)x\,d\mu = \int_Q \Lambda Fx\,d\mu = (Lx)\Lambda = (L^*\Lambda)x \quad (x \in B,\ \ \Lambda \in B_2^*), \tag{F.15}$$

hence $L^* = \mathrm{w}\!\!\int_Q F^*\,d\mu$.

(e) This follows from the definition (see (a) and (b)), because for $\Lambda \in B_3^*$ we have $\Lambda T \in B_2^*$.

(f) The above proofs will do mutatis mutandis (alternatively, use the fact that $\int$ is linear and bounded on L^1; in fact, (a) and (b) can then be deduced as special cases of (f); see (g)).

(One could also extend (d) by using the fact that $\mathcal{B}(B, \mathcal{B}(B_2, \mathrm{L}^1(Q; B_3))) = \mathcal{B}(B_2, \mathcal{B}(B, \mathrm{L}^1(Q; B_3)))$, isometrically.)

Recall from Remark A.3.22 that the linear (Banach) dual $B_2^* := B_2^{\mathrm{B}}$ is equipped with scalar multiplication $(\beta\Lambda)y := \beta(\Lambda y)$ $\alpha \in \mathbf{K},\ \Lambda \in Y^*,\ y \in Y)$, hence the isometry $B_2^{\mathrm{d}} \to B_2^*$, $y \mapsto \langle \cdot, y\rangle_{B_2}$ becomes conjugate-linear, cf. Remark A.3.22.

(g) Note first that the map $\mathrm{L}_{\mathrm{strong}}(Q; \mathcal{B}(B, B_2)) \mapsto (B \mapsto \mathrm{L}(Q; B_2))$ is well-defined (i.e., zero functions map to zero mapping), linear and one-to-one, hence an inclusion. Obviously, the $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathcal{B}$ norms are equal on $\mathrm{L}_{\mathrm{strong}}$, hence the strong claims of (g) hold. The weak claims can be proved analogously. □

The rest of this section is dedicated for $\mathcal{B}(B, \mathrm{L}^p(Q; B_2))$.

Most of the above also holds for the Banach space $\mathcal{B}(B, \mathrm{L}^p(Q; B_2))$ (and the weak claims for $\mathcal{B}(B, \mathcal{B}(B_2, \mathrm{L}^p(Q))))$. Because $\mathcal{B}(B, \mathrm{L}^p(Q; B_2))$ contains $\mathrm{L}^p_{\mathrm{strong}}(Q; \mathcal{B}(B, B_2))$ as a subspace, the following applies to $\mathrm{L}^p_{\mathrm{strong}}$ too:

Lemma F.2.2 (Strong convolutions) *Let $F \in \mathcal{B}(B, \mathrm{L}^p_\omega(\mathbf{R}^n; B_2))$ and $f \in \mathrm{L}^r_\omega(\mathbf{R}^n; B)$, $p, r \in [1, \infty]$, $n \in \mathbf{N}+1$, $\omega \in \mathbf{R}$ (we require that $\omega = 0$ if $n \ne 1$).*

(a) Assume that f is finite-dimensional and $p^{-1} + r^{-1} \ge 1$. Then

$$(F * f)(t) := \int_{\mathbf{R}^n} F(f(t-s))(s)\,ds = \int_{\mathbf{R}^n} F(f(s))(t-s)\,ds \in B_2 \tag{F.16}$$

exists for a.e. $t \in \mathbf{R}^n$, and $\|F * f\|_{\mathrm{L}^p_\omega} \le \|F\|_{\mathcal{B}}\|f\|_{\mathrm{L}^1_\omega} \le \infty$.

(b) Thus, we can (and will) extend $*$ to $\mathcal{B}(B, \mathrm{L}^p_\omega(\mathbf{R}^n; B_2)) \times \mathrm{L}^1_\omega(\mathbf{R}^n; B) \to \mathrm{L}^p_\omega(\mathbf{R}^n; B_2)$.

(c) Moreover, $\tau^T(F*f) = (\tau^T F)*f = F*\tau^T f$ $(T \in \mathbf{R}^n)$ *(time-invariance), and if* $n=1$ *and* $\pi_- F, \pi_- f = 0$*, then* $\pi_-(F*f) = 0$ *(causality).*

(d1) **($\widehat{F} \in \mathbf{H}^\infty$)** *Assume that* $n = 1 = p$*,* $\mathbf{K} = \mathbf{C}$*,* $F = \pi_+ F$ *and* $f = \pi_+ f$*. Then*

$$\|\widehat{F}\|_{\mathcal{B}(\mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))} \le \|F\|_{\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R}_+;B_2))}. \tag{F.17}$$

Moreover, if $r = 1$ *or* f *is finite-dimensional, then* $\widehat{F*f} = \widehat{F}\widehat{f}$ *on* $\mathbf{C}^+_\omega$*.*

(d2) Assume that $n = 1 = p$ *and* $\mathbf{K} = \mathbf{C}$*. Then* $\widehat{F} : \omega + i\mathbf{R} \to \mathcal{B}(B,B_2)$ *is strongly continuous and uniformly bounded:*

$$\|\widehat{F}(\omega+ir)\|_{\mathcal{B}(B,B_2)} \le \|\widehat{F}\|_{\mathcal{B}(B,\mathcal{C}_b(\omega+i\mathbf{R};B_2))} \le \|F\|_{\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R};B_2))} \quad (r \in \mathbf{R}). \tag{F.18}$$

Moreover, if $r = 1$ *[or* f *is finite-dimensional and* $r \le 2$*], then* $\widehat{F*f} = \widehat{F}\widehat{f}$ *[a.e.] on* $\omega + i\mathbf{R}$*.*

(d3) Assume that $n = 1 = p$ *and that* B *and* B_2 *are complex Hilbert spaces. Then* $\|F\|_{\mathrm{TI}(B,B_2)} \le \|F\|_{\mathcal{B}}$*. Moreover,* $F \in \mathrm{TIC} \Leftrightarrow F = \pi_+ F$*, and (if* $\mathbf{K} = \mathbf{C}$*) the function* $\widehat{F}$ *coincides with the transform* $\widehat{F*}$ *of Theorems 6.2.1 and 3.1.3.*

(e) $\mathcal{B}(B,\mathrm{L}^p_\omega(\mathbf{R}_+;B_2)) \underset{c}{\subset} \mathcal{B}(B,\mathrm{L}^{p'}_{\omega'}(\mathbf{R}_+;B_2))$ *for* $p' \in [1,p]$*,* $\omega' > \omega$*.*

(f) The convolution also extends to $\mathcal{B}(B,\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}_+;B_2)) \times \mathrm{L}^r_{\mathrm{loc}}(\mathbf{R}_+;B) \to \mathrm{L}^p_{\mathrm{loc}}(R_+;B_2)$*.*

(g) Parts (a)–(f) also hold with $\mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{L}^p_\omega(\mathbf{R}^n)))$ *in place of* $\mathcal{B}(B,\mathrm{L}^p_\omega(\mathbf{R}^n;B_2))$ *(use* $\mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{L}^p_{\mathrm{loc}}(\mathbf{R}_+)))$ *in (f),* B_2^{**} *in place of* B_2 *for the values of* $F*f$*, and* $\mathcal{B}(B,B_2^{**})$ *in place of* $\mathcal{B}(B,B_2)$ *for the values of* $\widehat{F}$*).*

In particular, $\mathcal{B}(B,\mathcal{B}(B_2^{\mathrm{B}},\mathrm{L}^1(\mathbf{R}_{[+]}))) \subset \mathrm{TI}[\mathrm{C}](B,B_2)$ *when* B *and* B_2 *are Hilbert spaces.*

(h) Parts (a)–(f) also hold with $\mathcal{B}(B,\mathrm{e}^{\omega\cdot}\mathrm{MTI}_{B_2})$ *in place of* $\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R};B_2))$ *and* $\mathrm{e}^{\omega\cdot}\mathrm{MTI}_B$ *in place of* $\mathrm{L}^r_\omega(\mathbf{R}^n;B)$ *(for* $n=1$*).*

(Also the weak (cf. (g)) and multidimensional analogies holds.)

Note that (h) corresponds to class $\mathrm{SMTI}_\omega(B,B_2)$ of Definition 2.6.3, and that $\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R};B_2))$ is its closed subspace.

Here, of course, $\mathcal{L}F := \widehat{F} \in \mathrm{H}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ is defined by

$$\widehat{F}(s)x := \int_0^\infty \mathrm{e}^{-st}(Fx)(t)\,dt \in B_2 \quad (x \in B,\ s \in \mathbf{C}^+); \tag{F.19}$$

analogously, $\widehat{F} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2^{**}))$ for $\mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{L}^1_\omega(\mathbf{R}_+)))$ (see (g)). In claim concerning $\mathcal{L}$, we assume that $\mathbf{K} = \mathbf{C}$.

We do not know whether the norm inequalities may have additional constants growing with the dimension of f (as in the proof of (a)) for $r > 1$, thus preventing the generalization to infinite dimensions for such r; at least this is not the case in (d3) (where $p = 1$ and $r = 2$ but B and B_2 are required to be Hilbert spaces). An analogous phenomenom is illustrated in Example B.4.13.

Proof of Lemma F.2.2: W.l.o.g. we assume that $\omega = 0$.

(a) 1° *$F * f$ exists:* If $f = \phi x$, $\phi \in \mathrm{L}^r(\mathbf{R}^n)$, $x \in B$, then

$$\int_{\mathbf{R}^n} F(\phi(t-s)x)(s)\,ds = \int_{\mathbf{R}^n} \phi(t-s)(Fx)(s)\,ds = (Fx * \phi)(t), \tag{F.20}$$

which exists a.e. and $\|F * \phi x\|_p \le \|F\|_{\mathcal{B}} \|\phi x\|_1$, by Lemma D.1.7. By linearity, $(F * f)(t)$ exists for a.e. $t \in \mathbf{R}^n$ whenever f is finite-dimensional.

2° *$\|F * \phi x\|_p \le \|F\|_{\mathcal{B}} \|\phi x\|_1$:* Let $E_1, \ldots, E_k \subset \mathbf{R}^n$ be measurable and disjoint, and let $x_1, \ldots, x_k \in B$, $k \in \mathbf{N}+1$. Then

$$\|F * \sum_{j=1}^k \chi_{E_j} x_j\|_p \le \sum_{j=1}^k \|F\|_{\mathcal{B}} \|\chi_{E_j} x_j\|_1 = \|F\|_{\mathcal{B}} \|\sum_{j=1}^k \chi_{E_j} x_j\|_1. \tag{F.21}$$

By density (Theorem B.3.11 and Lemma A.3.10), $F*$ has an extension $T : \mathcal{B} \times \mathrm{L}^1 \to \mathrm{L}^p$ with $\|Tf\|_p \le \|F\|_{\mathcal{B}} \|f\|_1$. But as in (F.21), one verifies that if the values of f lie in a finite-dimensional subspace B' of B, then $\|F * f\|_p \le \|F\|_{\mathcal{B}} M_{B'} \|f\|_1$. By density, the two continuous extensions to k-dimensional $f \in \mathrm{L}^1$ must coincide, hence $\|F * f\|_p \le \|F\|_{\mathcal{B}} \|f\|_1$ for finite-dimensional f.

(b) This follows from (a).

(c) This extends from case of finite-dimensional f (see Lemma D.1.7), by continuity.

(d1) The norm bound on $\widehat{F}$ follows from Lemma D.1.11(a1'). By Lemma D.1.1(c), we have $\widehat{F} \in \mathrm{H}^\infty(\mathbf{C}^+; \mathcal{B}(B, B_2))$.

Let $\phi \in \mathrm{L}^r(\mathbf{R}^n)$, $x \in B$. Then

$$\mathcal{L}(F * \phi x) = \mathcal{L}(Fx * \phi) = \widehat{Fx}\widehat{\phi} = \widehat{F}\widehat{\phi x} \tag{F.22}$$

on $\mathbf{C}^+$, by Lemma D.1.11(c'). By linearity, the same applies to any finite-dimensional f in place of ϕx. If $r = 1$, then $\widehat{F * f} = \widehat{F}\widehat{f}$ holds for general $\widehat{f}$, by continuity.

(d2) The proof is analogous to that of (d1): $\|\widehat{F}(\omega + ir)\|_{\mathcal{B}(B,B_2)} \le \|F\|_{\mathcal{B}}$ $(r \in \mathbf{R})$, by Theorem F.2.1(a). By Lemma D.1.11(a1)&(b), we have $\|\widehat{F}\|_{\mathcal{B}(B, \mathcal{C}_0(\omega + i\mathbf{R}; B_2))} \le \|F\|_{\mathcal{B}}$. As in (d1), we obtain $\widehat{F * f} = \widehat{F}\widehat{f}$ from Lemma D.1.11(c).

(d3) Theorem 3.1.3 (resp. 6.2.1) provides a unique continuous extension of $F*$ (restricted to finite-dimensional L^2 functions) to TI (resp. to TIC). Indeed, the operator defined by $\widehat{F}$ coincides with $F*$ for finite-dimensional functions, by (d2) (resp. by (d1)).

(e) This follows from Lemma D.1.4(b4).

(f) This follows from causality (as in the case of Lemma D.1.7 too).

(g) The above proofs apply mutatis mutandis. (Recall B_2^{B} from Remark A.3.22.)

(h) If $f = \sum_{j=1}^N x_j \delta_{T'_j}$, then

$$\|F * f\|_{\mathrm{MTI}} \le \sum_{j=1}^N \|Fx_k * \delta_{T'_j}\|_{\mathrm{MTI}} \le \sum_{j=1}^N \|x_k\| \|F\|_{\mathcal{B}} = \|F\|_{\mathcal{B}} \|f\|_{\mathrm{MTI}}. \tag{F.23}$$

By density, the map $F*$ (and the above inequality) has a unique continuous extension to measures of form $f = \sum_{j=1}^{\infty} x_j \delta_{T'_j}$; combine this with (a) to observe that any $f \in \mathrm{MTI}_B$ can be allowed in (a).

For (b)–(f), the proof is analogous to the original one (use Lemma D.1.12(c)&(c') in place of Lemma D.1.7), hence omitted. □

The above "strong convolution" (of functions $Q \to \mathcal{B}(B,B_2)$ and $Q \to B$) can also be defined between two operator-valued functions ($Q \to \mathcal{B}(B_2,B_3)$ and $Q \to \mathcal{B}(B,B_2)$, or, slightly more generally, as follows:

Lemma F.2.3 ($G*F$) *Let $F \in \mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R}^n;B_2))$, $G \in \mathcal{B}(B_2,\mathrm{L}^p_\omega(\mathbf{R}^n;B_3))$.*

(a) $\|G*F\|_{\mathcal{B}(B,\mathrm{L}^p_\omega(\mathbf{R}^n;B_3))} \le \|F\|_{\mathcal{B}}\|G\|_{\mathcal{B}}$, where $(G*F)(x)(t) := (G*Fx)(t)$. In particular, $\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R}^n;B))$ is a convolution Banach algebra.

(b) We have $\tau^T(G*F) = (\tau^T G)*F = G*\tau^T F$ ($T \in \mathbf{R}^n$), $(G*F)*f = G*(F*f)$, $(G*f)*g = G*(f*g)$ ($f,g \in \mathrm{L}^1_\omega(\mathbf{R}^n;*)$ with suitable $*$) and $n=1$ & $\pi_- F = 0 = \pi_- G \Rightarrow \pi_-(F*G) = 0$.

(c) $(\mathrm{e}^{s\cdot}G)*(\mathrm{e}^{s\cdot}F) = \mathrm{e}^{s\cdot}(F*G)$ for $s \in \mathbf{R}^n$.

(d1) $\widehat{G*F} = \widehat{G}\widehat{F}$ on $\mathbf{C}^+_\omega$ if $n=1$, $\mathbf{K}=\mathbf{C}$, $F = \pi_+ F$ and $G = \pi_+ G$.

(d2) $\widehat{G*F} = \widehat{G}\widehat{F}$ on $\omega + i\mathbf{R}$ if $n = 1 = p$ and $\mathbf{K} = \mathbf{C}$.

(e) If $F \in \mathrm{L}_{\text{strong}}(\mathbf{R}_+;\mathcal{B}(B,B_2))$, $\|F(\cdot)\|_{\mathcal{B}(B,B_2)} \in \mathrm{L}^1_\omega$, $G \in \mathrm{L}_{\text{strong}}(\mathbf{R}_+;\mathcal{B}(B_2,B_3))$ and $\|G(\cdot)\|_{\mathcal{B}(B_2,B_3)} \in \mathrm{L}^p_\omega$, then $F*G \in \mathrm{e}^{-\omega\cdot}\mathrm{L}^p_{\text{strong}}(\mathbf{R}_+;\mathcal{B}(B,B_3))$ and

$$\| \|G*F\|_{\mathcal{B}(B,B_3)} \|_{L^p_\omega} \le \| \|G(\cdot)\|_{\mathcal{B}(B_2,B_3)} \|_{L^p_\omega} \, \| \|F(\cdot)\|_{\mathcal{B}(B,B_2)} \|_{L^1_\omega}. \tag{F.24}$$

(h) Parts (a)–(e) also hold with $\mathcal{B}(B,\mathrm{e}^{\omega\cdot}\mathrm{MTI}_{B_2})$ in place of $\mathcal{B}(B,\mathrm{L}^1_\omega(\mathbf{R};B_2))$ (for $n=1$).

(Also the multidimensional analogies holds.)

(i) Claims (a)–(h) also hold for $F \in \mathrm{L}^1(\mathbf{R}^n;B_2)$ (take $B := \mathbf{K}$).

We are afraid that the weak analogies of the above claims do not hold.

Proof: (W.l.o.g. we assume that $\omega = 0$.)

(a) Given $x \in B$, we have $Fx \in \mathrm{L}^1(\mathbf{R}^n;B_2)$, hence $\|G*Fx\|_p \le \|G\|_{\mathcal{B}}\|Fx\|_1 \le \|G\|_{\mathcal{B}}\|F\|_{\mathcal{B}}\|x\|_B$.

(b)–(e) Analogously to (a), we obtain this from analogous claims on functions and from Lemma F.2.2 (one easily verifies that $(\mathrm{e}^{s\cdot}G)*(\mathrm{e}^{s\cdot}F) = \mathrm{e}^{s\cdot}(F*G)$ holds when F and G are functions).

E.g., by Lemma D.1.7, we have $(G*f)*g = G*(f*g)$ for one-dimensional f and g, hence for finite-dimensional, hence for all $f,g \in \mathrm{L}^1$, by density and continuity. Consequently, for $f = \phi x$, $\phi \in \mathrm{L}^1(\mathbf{R}^n)$, we have

$$(G*F)*f = (G*F)x*\phi = (G*Fx)*\phi = G*(Fx*\phi) = G*(F*f). \tag{F.25}$$

By linearity and continuity, we may again allow for any $f \in \mathrm{L}^1$.

(h) The above proofs apply mutatis mutandis (use again Lemma D.1.12(c)).
(i) Note that $\mathcal{B}(\mathbf{K}, \mathrm{L}^1(\mathbf{R}^n; B_2)) = \mathrm{L}^1(\mathbf{R}^n; B_2)$ (with equal norms). $\square$

The above "strong convolution of operator-valued functions" can be generalized to $\mathrm{L}^1_{\mathrm{loc}}$ too, if the supports of the functions are bounded to the left:

Lemma F.2.4 *Let* $F \in \mathcal{B}(B, \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+; B_2))$, $G \in \mathcal{B}(B_2, \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+; B_3))$, $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+; B)$.

The claims of Lemma F.2.3 can be generalized to this situation, and $\mathcal{B}(B, \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+; B))$ *is a convolution algebra.*

Also the extension $X := \mathcal{B}(B, B\delta_0 + \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}_+; B))$ *is a convolution algebra with unit* $I\delta_0$*, where* $\delta_0 * f := f$ *for all* f*. Moreover, if* $L \in \mathcal{B}(B)$ *and* $B_2 = B$*, then* $L\delta_0 + F$ *is invertible in* X *iff* $L \in \mathcal{GB}(B)$*.*

Proof: 1° For each $T > 0$, we have $\pi_{[0,T)}(F * G) = \pi_{[0,T)}(\pi_{[0,T)} F * \pi_{[0,T)} G)$, by causality, hence we can apply Lemma F.2.3. Obviously, X is a convolution algebra (with $(L_1\delta_0 + F_1) * (L_2\delta_0 + F_2) = L_1 L_2 \delta_0 + L_1 F_2 + F_1 L_2 + F_1 * F_2$).

2° Obviously, $L \in \mathcal{GB}(B)$ is necessary for the invertibility. Sufficiency follows as in the finite-dimensional case (see, e.g., Theorem 2.3.1 on p 42 of [GLS]). $\square$

We leave it to the reader to extend the above results for $\mathcal{B}(B, M(\mathbf{R}^n; B_2))$, where M refers to (uniform) measures; see [GLS], pp. 121–127 for the finite-dimensional case.

(See the notes on p. 1023.)

F.3 Weak Laplace transform ($\mathcal{L}_{\mathrm{w}}$)

To generalize is to be an idiot.
— William Blake (1757–1827)

In this section, we define and study $\mathrm{H}^p_{\mathrm{strong}}$ and $\mathrm{H}^p_{\mathrm{weak}}$ spaces and the Laplace transform of strongly or weakly measurable functions.

Let U and Y be complex Hilbert spaces, and B, B_2 and B_3 complex Banach spaces.

Definition F.3.1 ($\mathbf{H}^p_{\mathbf{weak}}$ and the weak Laplace transform $\widehat{F}$) *Let* $1 \leq p \leq \infty$, $\omega \in \mathbf{R}$.

By $\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2))$ *we denote the space of functions* $F : \mathbf{C}^+_\omega \to \mathcal{B}(B,B_2))$ *having a finite norm*

$$\|F\|_{\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))} := \sup_{\|\Lambda\|_{B_2^*}\leq 1,\, \|x\|_B \leq 1} \|\Lambda F x\|_{\mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))}. \tag{F.26}$$

The spaces $\mathrm{H}^p_{\mathrm{weak}}(\mathbf{D}_r; \mathcal{B}(B,B_2))$, $\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_r; \mathcal{B}(B,B_2))$ *and* $\mathrm{H}^p_{\mathrm{strong}}(\mathbf{D}_r; \mathcal{B}(B,B_2))$ *are defined analogously. We also set* $\mathrm{H}^p_{\mathrm{strong},\omega}(*) := \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega; *)$, $\mathrm{H}^p_{\mathrm{strong},\infty} := \cup_{\omega\in\mathbf{R}} \mathrm{H}^p_{\mathrm{strong},\omega}$ *($p \in [1,\infty]$).*

Let $F : \mathbf{R} \to \mathcal{B}(B,B_2)$ *and* $s \in \mathbf{C}$. *If* $\mathrm{e}^{-s\cdot} F \in \mathrm{L}^1_{\mathrm{weak}}$, *then we set* $\widehat{F}(s) := (\mathcal{L}_{\mathrm{w}} F)(s) := \not\int_{\mathbf{R}} \mathrm{e}^{-st} F(t)\, dt$. *The function* $\mathcal{L}_{\mathrm{w}} F$ *is the* (weak) Laplace transform *of* F. *The* strong Laplace transform $\mathcal{L}_{\mathrm{s}}$ *is defined analogously.*

Obviously, $\mathcal{L}_{\mathrm{w}} F$ is an extension of $\mathcal{L} F$. One easily verifies that $\|\cdot\|_{\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))} := \sup_{\|x\|_B\leq 1} \|Fx\|_{\mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))}$ and $\|\cdot\|_{\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))}$ are norms on corresponding spaces; in Lemma F.3.2(c) we shall show that $\mathrm{H}^p_{\mathrm{strong}}$ and $\mathrm{H}^p_{\mathrm{weak}}$ are Banach spaces.

Next we list the basic relations between different H^p_* spaces:

Lemma F.3.2 ($\mathbf{H}^p \subset \mathbf{H}^p_{\mathbf{strong}} \subset \mathbf{H}^p_{\mathbf{weak}} \subset \mathbf{H}^\infty$) *Let* $\omega \in \mathbf{R}$, $\varepsilon > 0$, $1 \leq p_1 \leq p \leq p_2 \leq \infty$.

(a1) $\mathrm{H}^p(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2)) \subset \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2)) \subset \mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2)) \subset \mathrm{H}^\infty(\mathbf{C}^+_{\omega+\varepsilon}; \mathcal{B}(B,B_2))$, *continuously. Moreover,* $\mathrm{H}^\infty_{\mathrm{weak}} = \mathrm{H}^\infty_{\mathrm{strong}} = \mathrm{H}^\infty$.

(a2) $\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2)) \underset{c}{\subset} \mathrm{H}^{p_2}_{\mathrm{strong}}(\mathbf{C}^+_{\omega+\varepsilon}; \mathcal{B}(B,B_2))$,
and $\mathrm{H}^{p_1}_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2)) \cap \mathrm{H}^{p_2}_{\mathrm{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2)) \subset \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2))$.
These claims also hold with "weak" or void in place of "strong".

(b) *Let* $G \in \mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2))$. *Then* $\|G(s)\|_{\mathcal{B}(B,B_2)} \leq (\pi(\mathrm{Re}\, s - \omega))^{-1/p} \|G\|_{\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))}$.

(c) The spaces $\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2))$ *and* $\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2))$ *are Banach spaces.*

(d) $\mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2)) = \mathcal{B}(B, \mathrm{H}^p(\mathbf{C}^+_\omega; B_2))$ *and* $\mathrm{H}^p_{\mathrm{weak}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2^{**})) = \mathcal{B}(B, \mathcal{B}(B_2^{\mathrm{B}}, \mathrm{H}^p(\mathbf{C}^+_\omega)))$.

(e) **($\dim B < \infty$)** *If* $\dim B < \infty$*, then* $\mathrm{H}^p(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) = \mathrm{H}^p_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ *(with equivalent norms). If* $\dim B_2 < \infty$*, then* $\mathrm{H}^p_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) = \mathrm{H}^p_{\text{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ *(with equivalent norms).*

Note from Theorem 6.2.1 that $\mathrm{TIC}_\omega(U,Y)$ operators correspond to $\mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$ functions through an isometric isomorphism onto. Thus, all $\mathrm{H}^p_{\text{weak}}$ functions over Hilbert spaces are TIC_∞ operators, by Lemma F.3.2(a).

Recall from Remark A.3.22 that $B_2^{\mathrm{B}} = B_2^*$ means the Banach dual of B_2, not the "sesquilinear" ("Hilbert") dual B_2^{d} (which is usually denoted by B_2^* if B_2 is a Hilbert space).

Proof: (a1) By Lemma D.1.1(c), we have $\mathrm{H}^\infty_{\text{weak}} = \mathrm{H}^\infty_{\text{strong}} = \mathrm{H}^\infty$. The embedding $\mathrm{H}^p \underset{c}{\subset} \mathrm{H}^p_{\text{strong}} \underset{c}{\subset} \mathrm{H}^p_{\text{weak}}$ follows from Lemma F.1.5(c1) and Lemma D.1.2(b1); the embedding $\mathrm{H}^p_{\text{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2)) \underset{c}{\subset} \mathrm{H}^\infty_{\text{weak}}(\mathbf{C}^+_{\omega+\varepsilon};\mathcal{B}(B,B_2))$ follows from Lemma D.1.4(d).

(a2) By (a1), we may replace $\mathrm{H}^{p_2}_{\text{weak}}$ by H^∞ in the second claim. By Lemma D.1.4(d), then the claims hold for void in place of "strong". Apply this for each $x_0 \in B$ [and $\Lambda \in B_2^*$] to obtain the strong [weak] claim (note that we can use same embedding bounds as for the uniform case).

(b) Case $p = \infty$ follows from (a), so we assume that $p < \infty$. If G is scalar, then this holds, by (6.4.3) of [HP] (because then the nonstandard assumption (iii) of Definition 6.4.1 of [HP] is redundant, by Theorem 3.3.1(a3)). Thus, we can replace $G(s)$ by $\Lambda G(s)x$ for any $x \in B$ and $\Lambda \in B_2^*$ with $\|x\|, \|\Lambda\| \le 1$ (because then $\|\Lambda G x\|_{\mathrm{H}^p} \le \|G\|_{\mathrm{H}^p_{\text{weak}}}$); the general inequality follows.

(c) Let $\{f_n\} \subset \mathrm{H}^p_{\text{weak}}$ be a Cauchy sequence. By (a), for each $\alpha > \omega$ there is f_α s.t. $f_n \to f_\alpha$ in $\mathrm{H}^\infty(\mathbf{C}^+_\alpha;\mathcal{B}(B,B_2))$; let $f \in \mathrm{H}(\mathbf{C}^+_\alpha;\mathcal{B}(B,B_2))$ be the pointwise limit function. Given $\Lambda \in B_2^*$ and $x \in B$, the Cauchy-sequence $\Lambda f_n x$ converges in H^p; the limit is equal its pointwise limit $\Lambda f x$, hence $f \in \mathrm{H}^p_{\text{weak}}$ and $f_n \to f$ in $\mathrm{H}^p_{\text{weak}}$.

The proof for $\mathrm{H}^p_{\text{strong}}$ is analogous, and it can be obtained from (d) too (the same applies to $\mathrm{H}^p_{\text{weak}}$, though not as obviously).

(d) The left-hand-sides are obviously (isometrically) subspaces of right-hand-sides. If $p = \infty$, then the converses follow from Lemma D.1.1(d). Assume then that $p < \infty$.

By (a), we have $\mathcal{B}(B,\mathrm{H}^p(\mathbf{C}^+_\omega;B_2)) \underset{c}{\subset} \mathcal{B}(B,\mathrm{H}^\infty(\mathbf{C}^+_\alpha;B_2)) = \mathrm{H}^\infty(\mathbf{C}^+_\alpha;\mathcal{B}(B,B_2))$, for all $\alpha > \omega$. Thus, any $F \in \mathcal{B}(B,\mathrm{H}^p(\mathbf{C}^+_\omega;B_2))$ takes form of a function $F \in \mathrm{H}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$. As noted at the beginning of the proof, the two norms on F are equal, hence $F \in \mathrm{H}^p_{\text{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$.

For $F \in \mathcal{B}(B,\mathcal{B}(B_2^*,\mathrm{H}^p(\mathbf{C}^+_\omega)))$, the proof is analogous.

(e) This follows from Lemma F.1.5(f). □

The following is rather obvious:

Lemma F.3.3 *Let* $F : \mathbf{R}_+ \to \mathcal{B}(B,B_2)$*,* $f : \Omega \to \mathcal{B}(B,B_2)$*,* $\Omega = \mathbf{C}^+_\omega$ *or* $\Omega = \partial r\mathbf{D}$*, and* $1 \le p \le \infty$*. Then*

(a1) We have $f \in \mathrm{H}^p_{\text{strong}}(\Omega;\mathcal{B}(B,B_2))$ *iff* $fx \in \mathrm{H}^p_{\text{strong}}(\Omega;B_2)$ *for all* $x \in B$*.*

(a2) We have $f \in \mathrm{H}^p_{\mathrm{weak}}(\Omega;\mathcal{B}(B,B_2))$ *iff* $\Lambda f x \in \mathrm{H}^p_{\mathrm{strong}}(\Omega;B_2)$ *for all* $x \in B$ *and* $\Lambda \in B_2^*$.

(b) If $T \in \mathcal{B}(B_2,B_3)$ *and* $x \in B$, *then* $T\mathcal{L}_{\mathrm{s}} = \mathcal{L}_{\mathrm{s}}T$, $T\mathcal{L}_{\mathrm{w}} = \mathcal{L}_{\mathrm{w}}T$, *and* $T\widehat{F}x = \widehat{TFx}$ *wherever* $\widehat{F}$ *exists.*

(c) We have $\widehat{F^{\mathrm{B}}}(s) = F(s)^{\mathrm{B}}$ *and* $\widehat{F^{\mathrm{d}}}(s) = F(\bar{s})^{\mathrm{d}}$ *for any s for which either transform is defined if B and* B_2 *are reflexive (this applies to* $\mathcal{L}_{\mathrm{w}}$ *and* $\mathcal{L}$*).*

(Recall that "F^*" refers to "F^{B}" in Banach and to "F^{d}" in Hilbert space settings.)

Proof: We get (a1)&(a2) from Lemma F.1.6, (b) from Theorem F.2.1(e), and (c) from Lemma F.2.1(d). □

Functions in $\mathrm{L}^1_{\mathrm{weak}}$ have bounded holomorphic weak Laplace transforms on the right half-plane:

Lemma F.3.4 ($\widehat{\mathbf{L}^1_{\mathbf{weak}}} \subset \mathbf{H}^\infty$) *Let* $F : \mathbf{R}_+ \to \mathcal{B}(B,B_2)$, $s_0 \in \mathbf{C}$, $r := \operatorname{Re} s_0$, $\varepsilon > 0$, $1 \le p \le \infty$, $p^{-1} + q^{-1} = 1$.

(a1) If $\mathrm{e}^{-s_0\cdot}F \in \mathrm{L}^1_{\mathrm{weak}}$, *then* $\widehat{F} \in \mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))$, *and* $\|\widehat{F}\|_{\mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))} \le \|\mathrm{e}^{-s_0\cdot}F\|_{\mathrm{L}^1_{\mathrm{weak}}}$.

(a2) If $\mathrm{e}^{-s_0\cdot}F \in \mathrm{L}^1_{\mathrm{strong}}$, *then* $\widehat{F} \in \mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(B,B_2))$, *and* $\|\widehat{F}\|_{\mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(B,B_2))} \le \|\mathrm{e}^{-s_0\cdot}F\|_{\mathrm{L}^1_{\mathrm{strong}}}$.

(b1) If $\mathrm{e}^{-s_0\cdot}F \in \mathrm{L}^p_{\mathrm{weak}}$, *then* $\widehat{F} \in \mathrm{H}(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**})) \cap \mathrm{H}^\infty(\mathbf{C}^+_{r+\varepsilon}\mathcal{B}(B,B_2^{**}))$, *and* $\|\widehat{F}\|_{\mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))} \le M_{\varepsilon,p}\|\mathrm{e}^{-s_0\cdot}F\|_{\mathrm{L}^p_{\mathrm{weak}}}$.

(b2) If $\mathrm{e}^{-s_0\cdot}F \in \mathrm{L}^p_{\mathrm{strong}}$, *then* $\widehat{F} \in \mathrm{H}(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**})) \cap \mathrm{H}^\infty(\mathbf{C}^+_{r+\varepsilon}\mathcal{B}(B,B_2^{**}))$, *and* $\|\widehat{F}\|_{\mathrm{H}^\infty(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))} \le M_{\varepsilon,p}\|\mathrm{e}^{-s_0\cdot}F\|_{\mathrm{L}^p_{\mathrm{strong}}}$.

(c1) If $p \le 2$ *and* $\mathrm{e}^{-r\cdot}F \in \mathrm{L}^p_{\mathrm{weak}}(\mathbf{R}_+;\mathcal{B}(B,B_2))$, *then* $\widehat{F} \in \mathrm{H}^q_{\mathrm{weak}}(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))$ *and* $\|\widehat{F}\|_{\mathrm{H}^q_{\mathrm{weak}}(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))} \le (2\pi)^{1/q}\|\mathrm{e}^{-r\cdot}F\|_{\mathrm{L}^p_{\mathrm{weak}}}$, *with equality for* $p = 2$.

(c2) If $p \le 2$ *and* $\mathrm{e}^{-r\cdot}F \in \mathrm{L}^p_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(B,Y))$, *then* $\widehat{F} \in \mathrm{H}^q_{\mathrm{strong}}(\mathbf{C}^+_r;\mathcal{B}(B,Y))$ *and* $\|\widehat{F}\|_{\mathrm{H}^q_{\mathrm{strong}}(\mathbf{C}^+_r;\mathcal{B}(B,Y))} \le (2\pi)^{1/q}\|\mathrm{e}^{-r\cdot}F\|_{\mathrm{L}^q_{\mathrm{strong}}}$, *with equality for* $p = 2$.

(d) The (strong/weak) Laplace transform is an isometric isomorphism modulo $\sqrt{2\pi}$ *of* $\mathrm{L}^2_\omega(\mathbf{R}^+;Y))$ *onto* $\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;Y)$, *of* $\mathcal{B}(U,\mathrm{L}^2_\omega(\mathbf{R}^+;Y))$ *onto* $\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y)) = \mathcal{B}(U,\mathrm{H}^2(\mathbf{C}^+_\omega;Y))$ *and of* $\mathcal{B}(U,\mathcal{B}(Y^{\mathrm{B}},\mathrm{L}^2_\omega(\mathbf{R}^+)))$ *onto* $\mathrm{H}^2_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y)) = \mathcal{B}(U,\mathcal{B}(Y^{\mathrm{B}},\mathrm{H}^2(\mathbf{C}^+_\omega)))$.

Recall that we may replace B_2^{**} by B_2 above if B_2 is reflexive (e.g., a Hilbert space).

From (d) we conclude that the inverse transform of $\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U,Y))$ covers in general more than $\mathrm{L}^2_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(U,Y))$, as shown in Example F.3.6.

Proof: (a2)&(b2)&(c2) Modify the proofs of (a1), (b1) and (c1) accordingly.

(a1) Let $s \in \mathbf{C}^+_r$. We have $e^{-st}F(t) = e^{-(s-s_0)t}e^{-s_0t}F(t)$, and $|e^{-(s-s_0)t}| \le 1$, hence $\|e^{-s\cdot}F\|_{\mathrm{L}^1_{\mathrm{weak}}} \le \|e^{-s_0\cdot}F\|_{\mathrm{L}^1_{\mathrm{weak}}}$. By Theorem F.2.1, it follows that $\|\widehat{F}(s)\| \le \|e^{-s_0\cdot}F\|_{\mathrm{L}^1_{\mathrm{strong}}}$. By Lemma D.1.10(a) and Lemma D.1.1(b), $\widehat{F} \in \mathrm{H}(\mathbf{C}^+_r; \mathcal{B}(B,B_2))$.

(b1) Set $M_{\varepsilon,p} := \|e^{-\varepsilon\cdot}\|_q$ and use (a1) (note that $\|\|e^{-s_0\cdot}e^{-\varepsilon\cdot}F\|_{\mathrm{L}^1_{\mathrm{weak}}}\| \le M_{\varepsilon,p}\|e^{-s_0\cdot}F\|_{\mathrm{L}^p_{\mathrm{weak}}}$).

(c1) Now $\widehat{F} \in \mathrm{H}(\mathbf{C}^+_r; \mathcal{B}(B,B_2^{**}))$, by (b1). For any $\Lambda \in B_2^*$, $x \in B$, we have $\Lambda\widehat{F}x = \mathcal{L}\Lambda F x$, by (d), and

$$\|\Lambda\widehat{F}x\|_{\mathrm{H}^q(\mathbf{C}^+_r)} \le (2\pi)^{1/q}\|\Lambda F x\|_{\mathrm{L}^p_r}, \tag{F.27}$$

by Theorem E.1.7, with equality for $p = 2$.

Taking the supremum on both sides of (F.27) over $\|\Lambda\|, \|x\| \le 1$, we obtain that $\|\widehat{F}\|_{\mathrm{H}^q_{\mathrm{weak}}(\mathbf{C}^+_r;\mathcal{B}(B,B_2^{**}))} \le (2\pi)^{1/q}\|e^{-r\cdot}\widehat{F}\|_{\mathrm{L}^p_{\mathrm{weak}}(\mathbf{R}_+;\mathcal{B}(B,B_2^{**}))}$, with equality for $p = 2$.

(Thus, $\mathcal{L}_{\mathrm{w}}$ is an isometric isomorphism of $\pi_+\mathrm{L}^2_{\mathrm{weak}}$ onto a closed subspace of $\mathrm{H}^2_{\mathrm{weak}}$. We believe that this is a proper subspace of $\mathrm{H}^2_{\mathrm{weak}}$; in particular, we believe that for any infinite-dimensional U and Y, some $\mathrm{H}^2_{\mathrm{weak}}$ functions do not have boundary functions (although they have "boundary operators" $B \times B_2^* \to \mathrm{L}^2(\omega + i\mathbf{R})$; the situation for $\mathrm{L}^2_{\mathrm{strong}}$ seems to be analogous.)

(d) The identities were given in Lemma F.3.2(d). Because $\mathcal{L}$ is an isometric isomorphism times $\sqrt{2\pi}$ of L^2_ω onto H^2_ω, by Lemma D.1.15, it is also an isometric isomorphism times $\sqrt{2\pi}$ (onto) $\mathcal{B}(U, \mathrm{L}^2_\omega(\mathbf{R}^+;Y)) \to \mathcal{B}(U, \mathrm{H}^2(\mathbf{C}^+_\omega;Y))$ and $\mathcal{B}(U, \mathcal{B}(Y^{\mathrm{B}}, \mathrm{L}^2_\omega(\mathbf{R}^+))) \to \mathcal{B}(U, \mathcal{B}(Y^{\mathrm{B}}, \mathrm{H}^2(\mathbf{C}^+_\omega)))$. □

The multiplication of elements of different H^p spaces works in the same way as that of L^p spaces:

Lemma F.3.5 ($\mathbf{H}^r_{\mathbf{weak}} \cdot \mathbf{H}^p_{\mathbf{strong}} \subset \mathbf{H}^p_{\mathbf{strong},\varepsilon}$) *Let $1 \le p \le \infty$ and $1/p + 1/q = 1$. Let $F : \mathbf{C}^+ \to \mathcal{B}(B,B_2)$, $G : \mathbf{C}^+ \to \mathcal{B}(B_2,B_3)$, $f : \mathbf{C}^+ \to B$. Then all claims (i.e., (a1)–(a3")) of Lemma F.1.8 hold with* H *in place of* L *and $G^*(\bar{\cdot})$ in place of G^*.*

Moreover, if $f \in \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_2))$ and $g \in \mathrm{H}^r_{\mathrm{weak}}(\mathbf{C}^+_\alpha; \mathcal{B}(B_2,B_3))$, where $r \in [1,\infty]$ and $\omega > \alpha$, then $gf \in \mathrm{H}^p_{\mathrm{strong}}(\mathbf{C}^+_\omega; \mathcal{B}(B,B_3))$ and $\|gf\| \le M\|g\|\|f\|$; the same holds with H^p in place of $\mathrm{H}^p_{\mathrm{strong}}$.

Naturally, by shifting one obtains an analogous $\mathbf{C}^+_\omega$ claim.

Proof: The product functions are holomorphic, by Lemma D.1.2(b3). The claims on norms follow from Lemma F.1.8.

The last claim follows from the fact that $g \in \mathrm{H}^\infty(\mathbf{C}^+_\omega; \mathcal{B}(B_2,B_3))$, by Lemma F.3.2(a). □

As noted above, the Laplace transform $\mathrm{L}^2_{\mathrm{strong}} \to \mathrm{H}^2_{\mathrm{strong}}$ is not onto in general:

Example F.3.6 [$\widehat{\mathbf{L}^2_{\mathbf{strong}}} \ne \mathbf{H}^2_{\mathbf{strong}}$] Even for $U := \ell^2(\mathbf{N})$, there are $\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+; \mathcal{B}(U))$ functions and $\mathrm{L}^2_{\mathrm{strong}}(i\mathbf{R}; \mathcal{B}(U))$ functions that are not Laplace transforms of any $\mathrm{L}^2_{\mathrm{strong}}(\mathbf{R}; \mathcal{B}(U))$ functions, not even of any $\mathrm{L}^2_{\mathrm{weak}}(\mathbf{R}; \mathcal{B}(U))$

functions (nor of any other $\mathcal{B}(U)$-valued functions, see 5° of the proof of Example F.1.10).

In fact, there is $\widehat{F} \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U)) \cap \mathrm{L}_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U))$ s.t. $\widehat{F}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(U)) \cap \mathrm{L}_{\mathrm{strong}}(i\mathbf{R};\mathcal{B}(U))$ but $\widehat{\mathbb{D}}$ is not the Laplace (or Fourier) transform of any $F : \mathbf{R}_+ \to \mathcal{B}(U)$ (nor of any $F(\mathbf{R}_+ \setminus N) \to \mathcal{B}(U)$ where N is a null set). ◁

Proof: Let $F_n \in \mathrm{L}^2_{\mathrm{strong}}([0,1);U)$ $(n \in \mathbf{N})$, $F \in \mathcal{B}(U,\mathrm{L}^2([0,1);U))$ be as in Example F.1.10.

Because $\pi_{[0,1)}\mathrm{L}^2 \underset{c}{\subset} \mathrm{L}^2_\omega$ and $\mathcal{L}\mathrm{L}^2_\omega = \mathrm{H}^2_\omega$, we have $\widehat{F}_n, \widehat{F} \in \mathrm{H}(\mathbf{C};\mathcal{B}(U)) \cap \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U))$ for all $\omega \in \mathbf{R}$, and $\widehat{F}_n \to \widehat{F}$ in each $\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U))$. The claims on $\widehat{F}(\bar{\cdot})^*$ follow from duality and the fact that $\widehat{F}_n(\bar{\cdot})^* = \widehat{F_n^*}$ for each $n \in \mathbf{N}$.

Assume then that $\widehat{F} = \mathcal{L}G$ for some $G \in \mathrm{L}^2_{\mathrm{strong}}(\mathbf{R};\mathcal{B}(U))$ (if $G : (\mathbf{R}_+ \setminus N) \to \mathcal{B}(U)$ where N is a null set, for some reasonable sense so that $\widehat{F}u_0 = \widehat{Gu_0}$ for all $u_0 \in U$, then Gu_0 equals the inverse transform of $\widehat{F}u_0$, hence then $Gu_0 \in \mathrm{L}^2(\mathbf{R}_+;U)$; consequently, then $G \in \mathrm{L}^2_{\mathrm{strong}}(\mathbf{R}_+;\mathcal{B}(U))$).

Given u_0, we have $\widehat{F}_n u_0 \to \widehat{F}u_0$ in H^2, hence $F_n u_0 \to Gu_0$ in L^2, as $n \to \infty$, hence $F_n \to G$ in $\mathrm{L}^2_{\mathrm{strong}}(\mathbf{R};\mathcal{B}(U))$, which is a contradiction, by Example F.1.10. Thus, $\widehat{F}$ is not the transform of any $G : \mathbf{R}_+ \to \mathcal{B}(U)$. □

We finish this appendix by a more technical lemma. A main observation of the lemma is that $\widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}}$, then $\mathbb{D}$ is smoothing (see (b2) below).

Lemma F.3.7 *Let $\omega \in \mathbf{R}$, $\varepsilon > 0$, $1 \le p \le \infty$.*

(a1) **(Inverse transform of $\mathbf{H}^1$)** *Let $g \in \mathrm{H}^1(\mathbf{C}^+_\omega;B) \cap \mathrm{L}^1(\omega + i\mathbf{R};B)$ and $\gamma \ge \omega$. Then $g = \widehat{f}$, where*

$$f(t) = \frac{1}{2\pi}\mathrm{e}^{t\gamma}\int_{\mathbf{R}} \mathrm{e}^{tir} g(\gamma + ir)\,dr \in B \quad (t \in \mathbf{R}). \tag{F.28}$$

Moreover, $\mathrm{e}^{-\omega\cdot} f \in \mathcal{C}_0(\mathbf{R};B)$, $\pi_- f = 0$, and $\sup_{\mathbf{R}} \|\mathrm{e}^{-\omega\cdot} f\|_B \le \|g\|_{\mathrm{H}^1_\omega}/2\pi$.

(a1') Let $g \in \mathrm{H}^1(\mathbf{C}^+_\omega;B)$ and $\gamma > \omega$. Then $g = \widehat{f}$, where f is defined by (F.28). Moreover, $\mathrm{e}^{-\omega\cdot} f \in \mathcal{C}_{\mathrm{b}}(\mathbf{R};B)$, $\pi_- f = 0$, and $\sup_{\mathbf{R}} \|\mathrm{e}^{-\omega\cdot} f\|_B \le \|g\|_{\mathrm{H}^1_\omega}/2\pi$.

(a2) Let $G \in \mathrm{H}^1_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ and $\gamma > \omega$. Then $G = \widehat{F}$, where

$$F(t) = \frac{1}{2\pi}\mathrm{e}^{t\gamma}\,\mathrm{s}\!\!\int_{\mathbf{R}} \mathrm{e}^{tir} G(\gamma + ir)\,dr \quad (t \in \mathbf{R}). \tag{F.29}$$

Moreover, $\pi_- F = 0$, $\sup_{\mathbf{R}} \|\mathrm{e}^{-\omega\cdot} F\|_B \le \|G(\cdot - \omega)\|_{\mathrm{H}^1_{\mathrm{strong}}}/2\pi$, and $\mathrm{e}^{-\omega\cdot} fx \in \mathcal{C}_{\mathrm{b}}(\mathbf{R}, B_2)$ for all $x \in B$.

(a3) Let $G \in \mathrm{H}^1_{\mathrm{weak}}(\mathbf{C}^+_\omega;\mathcal{B}(B,B_2))$ and $\gamma > \omega$. Then $G = \widehat{F}$, where

$$F(t) = \frac{1}{2\pi}\mathrm{e}^{t\gamma}\,\mathrm{w}\!\!\int_{\mathbf{R}} \mathrm{e}^{tir} G(\gamma + ir)\,dr \quad (t \in \mathbf{R}). \tag{F.30}$$

Moreover, $\pi_- F = 0$, $\sup_{\mathbf{R}} \|e^{-\omega\cdot} F\|_B \le \|G(\cdot - \omega)\|_{\mathrm{H}^1_{\mathrm{weak}}}/2\pi$, *and* $e^{-\omega\cdot}\Lambda f x \in \mathcal{C}_0(\mathbf{R})$ *for all* $x \in B$ *and* $\Lambda \in B_2^*$.

(b1) **(Inverse transform of $\mathbf{H^2}$)** *Let* $\widehat{\mathbb{D}} \in \mathrm{H}^2(\mathbf{C}^+_\omega; \mathcal{B}(U,Y))$. *Then* $\mathbb{D} : \mathrm{L}^2_\omega(\mathbf{R}_+;U) \ni u \mapsto \mathcal{L}^{-1}\widehat{\mathbb{D}}\widehat{u}$ *satisfies* $e^{-\omega\cdot}\mathbb{D}u \in \mathcal{C}_0(\mathbf{R};Y)$, *and* $\sup_{\mathbf{R}} \|e^{-\omega\cdot}\mathbb{D}u\|_Y \le \|\widehat{\mathbb{D}}(\bar{\cdot})\|_{\mathrm{H}^2(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))}\|u\|_{\mathrm{L}^2_\omega}$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$ *(for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$ *if we extend* $\mathbb{D}$ *onto* L^2_ω*; this extension coincides with the operator* $\mathbb{D} \in \mathrm{TIC}_\omega$ *(see Theorem 6.2.1) if, in addition,* $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$*).*

(b2) Let $\widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(Y,U))$. *Then the map* $\mathbb{D} : u \mapsto \mathcal{L}_{\mathrm{w}}^{-1}\widehat{\mathbb{D}}\widehat{u}$ *satisfies* $e^{-\omega\cdot}\mathbb{D}u \in \mathcal{C}_{\mathrm{b}}(\mathbf{R};Y)$ *and* $\sup_{\mathbf{R}} \|e^{-\omega\cdot}\mathbb{D}u\|_Y \le \|\widehat{\mathbb{D}}(\bar{\cdot})^*\|_{\mathrm{H}^2_{\mathrm{strong}}(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))}\|u\|_{\mathrm{L}^2_\omega}$ *for all* $u \in \mathrm{L}^2_\omega(\mathbf{R}_+;U)$ *(for all* $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$ *if we extend* $\mathbb{D}$ *onto* L^2_ω*; this extension coincides with the operator* $\mathbb{D} \in \mathrm{TIC}_\omega$ *(see Theorem 6.2.1) if, in addition,* $\widehat{\mathbb{D}} \in \mathrm{H}^\infty(\mathbf{C}^+_\omega;\mathcal{B}(U,Y))$*).*

As stated in Theorem 3.3.1(a1), expression $g \in \mathrm{H}^1 \cap \mathrm{L}^1$ means that $g \in \mathrm{H}^1$ has a boundary function in L^1. By (a1), any element of $\mathrm{H}^1_{\mathrm{strong}}(\mathbf{C}^+;\mathcal{B}(B,B_2)$ is the (strong) Laplace transform of a bounded and weakly continuous function $\mathbf{R}_+ \to \mathcal{B}(B,B_2)$.

By (b2), an I/O map with a transfer function in "dual" $\mathrm{H}^2_{\mathrm{strong}}$ produces bounded and continuous output. This fact will be used in Theorem 6.9.1(b) to show that such maps are exactly the I/O maps with a WPLS realization with a bounded output operator.

We will see in Theorem 3.3.1(d2) that if $\widehat{\mathbb{D}}(\bar{\cdot})^* \in \mathrm{H}^2_{\mathrm{strong}} \cap \mathrm{H}^\infty$, then, actually, $\widehat{\mathbb{D}} \in \mathrm{L}^2_{\mathrm{strong}} \cap \mathrm{L}^\infty_{\mathrm{strong}}$, so that the map $u \mapsto \mathcal{L}^{-1}\widehat{\mathbb{D}}\widehat{u}$ defined in (b2) can be defined with the same formula (but with the (Fourier) transforms considered on $\omega + i\mathbf{R}$ only) for any $u \in \mathrm{L}^2_\omega(\mathbf{R};U)$.

Proof of Lemma F.3.7: We take $\omega = 0$ w.l.o.g. (replace g by $g(\omega + \cdot)$ for the general case; see also Remark 2.1.6).

(a1) By the usual contour integration argument (use (6.4.4) of [HP] and note that $e^{t(\gamma+ir)}$ is bounded for bounded γ), expression (F.28) is independent of $\gamma > 0$ for any $t \in \mathbf{R}$. But (F.28) is continuous (from the right) at $\gamma = 0$, hence we may take $\gamma = 0$ too to obtain the same f. With $\gamma = 0$ we obtain that $\|f\|_\infty \le \|g\|_{\mathrm{H}^1_\omega}/2\pi$. By Lemma D.1.11(a1)&(b), we have $f \in \mathcal{C}_0(\mathbf{R};B)$.

By p. 230 of [HP], $g = \mathcal{L}\pi_+ f$. Let $T > 0$ and set $f_T := \tau^{-T}\pi_+ f \in \mathrm{L}^\infty(\mathbf{R}_+;B)$. Then $\widehat{f_T} = e^{-T\cdot} g \in \mathrm{H}^1$, so that $\widehat{f_T} = \mathcal{L}\pi_+\widetilde{f}_T$, by the above, where

$$\widetilde{f}_T(t) = \frac{1}{2\pi}\int_{\mathbf{R}} e^{tir}\widehat{f_T}(ir)\,dr = f(t-T) \quad (t \in \mathbf{R}). \tag{F.31}$$

By uniqueness (Lemma D.1.10(b)), $f_T = \pi_+ f(\cdot - T)$, hence $\pi_{[-T,0)} f = 0$. Because $T > 0$ was arbitrary, we have $\pi_- f = 0$.

(a1') Apply (a1) with $\omega + \varepsilon$ in place of ω for some $\varepsilon > 0$.

(a2) Apply (a1') for Gx ($x \in B$).

(a3) Apply (a1) for $\Lambda G x$ ($x \in B$, $\Lambda \in B_2^*$) (note that $\Lambda G x \in \mathrm{H}^1 \cap \mathrm{L}^1$, by Theorem 3.3.1(a1)).

(b1) 1° *Case* $u \in \pi_+ \mathrm{L}^2$*:* Let $u \in \mathrm{L}^2(\mathbf{R}_+;U)$, so that $\widehat{y} := \widehat{\mathbb{D}}\widehat{u} \in \mathrm{H}^1(\mathbf{C}^+;Y)$. Then

$$\|\widehat{y}\|_{\mathrm{L}^1(i\mathbf{R};Y)} = \|\widehat{y}\|_{\mathrm{L}^1(i\mathbf{R};Y)} \le \|\widehat{\mathbb{D}}\|_{\mathrm{H}^2}\|\widehat{u}\|_{\mathrm{H}^2} \le 2\pi\|\widehat{\mathbb{D}}\|_{\mathrm{H}^2}\|u\|_2, \tag{F.32}$$

by Theorem 3.3.1(a). Thus, $\sup_{\mathbf{R}} \|y\|_Y \le \|\widehat{y}\|_{\mathrm{H}^1}/2\pi = \|\widehat{\mathbb{D}}\|_{\mathrm{H}^2}\|u\|_2$ and $y \in \mathcal{C}_0(\mathbf{R};Y)$, by (a1).

2° *Case* $u \in \mathrm{L}^2$*:* By repeating the proof of Lemma 2.1.3, we can show that each such $\mathbb{D}$ extends to a time-invariant ($\tau^t\mathbb{D} = \mathbb{D}\tau^t$ for all $t \in \mathbf{R}$) operator $\mathbb{D}_0 \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U), \mathcal{C}_0(\mathbf{R};Y))$ with the same norm. By continuity, this extension satisfies the requirements of (c1).

3° *Case* $\mathbb{D} \in \mathrm{H}^\infty$*:* Let $\widehat{\mathbb{D}} \in \mathrm{H}^\infty$ too, so that $\mathbb{D} \in \mathrm{TIC}(U,Y)$, by Theorem 6.2.1. By time-invariance, $\mathbb{D}u = \mathbb{D}_0 u$ for any $u \in \mathrm{L}^2([T,+\infty);U)$, $T \in \mathbf{R}$. If $u_n \to u$ in L^2, then $\mathbb{D}u_n$ converges in both L^2 and $\mathcal{C}_0$, so that the limits must be equal (a.e.), by Theorem B.3.2, hence $\mathbb{D} = \mathbb{D}_0$ on L^2.

(b2) By Lemma F.3.5(a3”), we have $\|\widehat{\mathbb{D}}\widehat{u}\|_{\mathrm{H}^1_{\mathrm{weak}}} \le \|\widehat{\mathbb{D}}(\bar{\cdot})^*\|_{\mathrm{H}^2_{\mathrm{strong}}}\|\widehat{u}\|_{\mathrm{H}^2}$ for $u \in \mathrm{L}^2(\mathbf{R}_+;U)$. Consequently, $\Lambda\mathbb{D}u \in \mathcal{C}_0(\mathbf{R})$ for all $\Lambda \in Y^*$, and the norm estimate holds (cf. the proof of (b1)), by (a3).

If $u \in \mathcal{C}_{\mathrm{c}}^\infty$, then $\mathbb{D}u \in \mathrm{W}^{1,2}_1 \subset \mathcal{C}$ (because $\mathbb{D} \in \mathrm{TIC}_1$), by Theorem 3.1.5 (alternatively $\mathbb{D}u \subset \mathcal{F}^{-1}[\mathrm{L}^1(1+i\mathbf{R};Y)] \subset \mathrm{e}^{1\cdot}\mathcal{C}_0 \subset \mathcal{C}$, by Lemma D.1.11(e2)&(e1)). For general u, we can take $u_n \in \mathcal{C}^\infty$ ($n \in \mathbf{N}$) s.t. $u_n \to u$ in L^2; it follows from the norm estimate that $\mathbb{D}u_n$ converges in $\mathcal{C}_{\mathrm{b}}$ and $\mathbb{D}u_n$ pointwise to $\mathbb{D}u$, hence $\mathbb{D}u \in \mathcal{C}_{\mathrm{b}}$.

The rest of the proof is analogous to 2° and 3° of (b1). (Note that we can analogously show that $\mathrm{e}^{-\omega\cdot}\Lambda\mathbb{D}u \in \mathcal{C}_0(\mathbf{R})$ for each $\Lambda \in Y^*$.) □

Notes for Appendix F

Dinculeanu [Dinculeanu], Chapter IV, has some results in this direction for functions $F : Q \to \mathcal{B}(B,B_2)$ s.t. $\|F\|$ is measurable (if $F \in \mathrm{L}^\infty_{\mathrm{strong}}$ and B is separable, then F “weakly locally integrable” in Dinculeanus terms). In [CZ], $\mathrm{L}_{\mathrm{weak}} = \mathrm{L}_{\mathrm{strong}}$ is defined for separable Hilbert spaces, but the norm $\|\|F\|_{\mathcal{B}(B,B_2)}\|_{\mathrm{L}^p}$ is used instead of the $\mathrm{L}^p_{\mathrm{strong}}$ norm; also the appendix [Sbook] will list similar results. However, for $p = \infty$ their results become special cases of those of ours, by Theorem F.1.9(s3). We do not know any studies in our generality.

For some purposes it would be more natural to define $\mathrm{L}^p_{\mathrm{strong}}$ and $\mathrm{L}^p_{\mathrm{weak}}$ as spaces of (equivalence classes of) functions, whose values are unbounded operators. Fortunately, the above setting suffices for our purposes.

See Chapter 3 for further results on $\mathrm{L}^\infty_{\mathrm{strong}}$ in the Hilbert space setting and Appendices B and D and notes therein for uniform variants of the results of this appendix.

References

An authority is a person who can tell you more about something than you really care to know.

We list below the references in the (case insensitive) alphabetical order of their abbreviations. As the main rule, the abbreviations consist of the three first letters of the author's last name (one letter of each if there are multiple authors), but short names appearing only once or twice are written in whole. We add two last digits of the publication year (and a letter) when ambiguity requires us to do so. Olof Staffans and George Weiss are referred by "S" or "W", respectively.

When we write "[KFA, p. 7]", we refer to page 7 of reference "[KFA]" below, etc. The references [IZ00], [Jacob01], [JZ00], [Mal97], [Mik97a], [Mik97b], [Mik98], [S97a], [Sbook], [SW01a] and [SW01b] (i.e., those having nothing in italics) have not yet been through a referee process (as far as we know). Whenever we refer a proof to any of these, we sketch the proof or give additional alternative references.

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This book fills a much needed gap in the literature.

— A journal referee on a monograph

Notation

Thou art a symbol and a sign
To Mortals of their fate and force;
Like thee, Man is in part divine,
A troubled stream from a pure source;
And Man in portions can foresee
His own funereal destiny.
— Lord Byron (1788–1824), "Prometheus"

We present here the notation used in this monograph. For readability, we use some redundancy. See also the standing hypotheses (if any) mentioned in the beginning of each chapter. Any alternative meanings of these symbols are specified explicitly.

For symbols we only give their continuous-time explanations, see Theorem 13.3.13 for corresponding discrete-time interpretation.

For the correspondence with the notation used by G. Weiss and M. Weiss (among others), see p. 166.

Brackets etc.

$\{x \in X \mid P(x)\}$: The set of all elements x of X for which $P(x)$ is true.

$\{x_n\}$: This refers to a sequence, usually on $\mathbf{N}$ or $\mathbf{Z}$, i.e., to $(x_n)_{n=0}^{\infty}$ or $(x_n)_{n=-\infty}^{\infty}$.

[S97b, Section 7]: A reference to Section 7 of [S97b] (see p. 1024 for the bibliography)). [Rud73, p. 131] is a reference to page 131 of [Rud73], and (1.16) is a reference to equation (1.16) of this text (see p. 29).

"Σ is [strongly/exponentially] stable iff $\widetilde{\Sigma}$ is [strongly/exponentially] stable": the conditions in brackets are optional and correspond to each other, i.e., this says that "Σ is stable iff $\widetilde{\Sigma}$ is stable", "Σ is strongly stable iff $\widetilde{\Sigma}$ is strongly stable", and "Σ is exponentially stable iff $\widetilde{\Sigma}$ is exponentially stable", as in Corollary 6.6.9. Analogously, "$a \in A$ iff $a \in B$ [and $a \in C$]" implies that both "$a \in A$ iff $a \in B$" and "$a \in A$ iff $a \in B$ and $a \in C$" hold.

$f[X], fX$: $f[X]$ is the set $\{f(x) \mid x \in X\}$, when f is a function defined (at least) on the set X; for linear f we may write fX. Special cases of this are the imaginary axis $i\mathbf{R}$ and the set $\mathrm{L}^1* := \{f* \mid f \in \mathrm{L}^1\}$; the latter means the convolution operators defined by a L^1 function, cf. Definition 2.6.3; analogously, $\mathbf{N}+1 = \{1, 2, 3, \ldots\}$.

$\left[\begin{array}{c|c} \mathbb{A} & \mathbb{B} \\ \hline \mathbb{C} & \mathbb{D} \end{array}\right]$: A WPLS (or a wpls), see Definition 6.1.1 (or 13.3.1).

$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$: The generators of a WPLS (or a wpls).

$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$: The WPLS (or wpls) generated by $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$; see Lemmas 6.1.16 and 6.2.9(a) and Definition 6.2.3 (or Lemma 13.3.3).

$[a,b]$, $[a,b)$, $(a,b]$, (a,b): The intervals $\{r \in \mathbf{R} \mid a \le r \le b\}$, $\{r \in \mathbf{R} \mid a \le r < b\}$, $\{r \in \mathbf{R} \mid a < r \le b\}$, $\{r \in \mathbf{R} \mid a < r < b\}$, respectively.

$f(t,z) > 0$ $(t \in \mathbf{R},\ z \in \mathbf{C})$: This means that $f(t,z) > 0$ for all $t \in \mathbf{R}$ and all $z \in \mathbf{C}$.

$\langle x, y\rangle_H$: The (sesquilinear) inner product of $x, y \in H$ in the Hilbert space H.

$\langle x, \Lambda\rangle_{X,X^*}, \langle x, \Lambda\rangle_{X,X^{\mathrm{d}}}, \langle x, \Lambda\rangle_{X,X^{\mathrm{B}}}$: Each of these denotes the scalar $\Lambda x := \Lambda(x)$ (here $x \in X$, $\Lambda \in X^*$). Thus, by $\langle \cdot, \cdot\rangle_{X,X^*}$ we denote the pairing $X \times X^* \to \mathbf{K}$. The subindex d or B on H specifies explicitly how the multiplication in X^* is defined, see "Superscripts", p. F.3.

More non-letter symbols

$\cdot$: The place for the argument of a function; e.g., $\mathrm{e}^{\omega \cdot}$ denotes the function $t \mapsto \mathrm{e}^{\omega t}$.

$*$: In the place of an operator the asterisk ($*$) refers to unspecified (irrelevant) entry, e.g., $\left[\begin{smallmatrix} 1 & 2 \\ 3 & * \end{smallmatrix}\right]$ may have any right bottom entry; $(J,*)$-inner means (J,S)-inner for any S. Sometimes $*$ refers to anything (possibly void) that is omitted..

$\mu * f$: The convolution of μ and f, see the index. By $\mu*$ we mean the operator $f \mapsto \mu * f$.

$f \circ g$: The composite function $t \mapsto f(g(t))$.

$A \setminus B$: The set $\{x \in A \mid x \notin B\}$ (or $A \cap B^c$).

$A := B$: A is defined to be equal to B. Equivalent to "$B =: A$".

$s \to +\infty$: s is real and s goes to $+\infty$; we often write ∞ for $+\infty$.

$x_n \rightharpoonup x$: "$x_n \to x$" weakly, i.e., $\langle x_n, y\rangle \to \langle x, y\rangle$ for all y (see Lemma A.3.1(h)–(j)).

$f : \Omega \to X$: f is a function of Ω into X; we may write, e.g., $f : \Omega \ni s \mapsto x_0 + x_3 s^3 \in X$ to specify a rule determining $f(s)$ for $s \in \Omega$.

$\cup, \cap$: Union, intersection, respectively; see Lemma A.3.17 for the norm of $X \cap Y$.

$A \subset B$, $B \supset A$: Either means that A is a subset of B ($x \in A \Rightarrow x \in B$; possibly $A = B$).

$A \underset{c}{\subset} B$: This means that $A \subset B$ and that this inclusion is continuous (see p. 890).

$A \underset{a}{\subset} B$: This means that A is an (algebraic) subclass of B (Definition 6.2.4).

$A + B$: The set $\{a + b \mid a \in A,\ b \in B\}$ (this is a special case of $f[X]$ above); see Lemma A.3.17 for corresponding norm.

$A \oplus B$: When A and B are closed subspaces of a Banach space, and $A \cap B = \{0\}$, we denote $A + B$ by $A \oplus B$ (cf. [Rud73]).

$X \times Y$: The Cartesian product $\{(x,y) \mid x \in X,\ y \in Y\}$. If X and Y are normed spaces, we use the norm $\|(x,y)\|_{X \times Y} := (\|x\|_X^2 + \|y\|_Y^2)^{1/2}$;

if X and Y are inner product spaces, we use the inner product $\langle (x,y),(x',y')\rangle_{X\times Y} := \langle x,x'\rangle_X + \langle y,y'\rangle_Y$; use induction for $\prod_{k=1}^n X_k := X_1 \times X_2 \times \cdots \times X_n$. Finally, $X^n := \prod_{k=1}^n X$. See also p. 870.

$\int_\Omega f\,dm$: The Bochner (i.e., Lebesgue, if f is $\mathbf{C}$- or $\mathbf{C}^n$-valued) integral (see p. 927) of f over Ω w.r.t. the Lebesgue measure m.

$\oint, \oint\!\!\!\!\!\!\!\nearrow$: See Theorem F.2.1, p. 1011.

$\exists$, &: "$\exists$" means "exists", "&" means "and".

$\exists A^{-1}$: "There exists a bounded inverse of the operator A" (in particular, A is one-to-one and onto).

$(s-A)$: $(s-A) := sI - A$, when $s \in \mathbf{C}$ (see Lemma A.3.3 and Section A.4).

$\partial\Omega$: $\partial\Omega := \bar{\Omega} \cap \overline{\Omega^c}$ is the boundary of the set Ω.

$f_{|A}$: The restriction of the function f on the set A (we write just f when there is no risk of confusion).

$x \perp y$: This means that $\langle x,y\rangle = 0$; if S is a set, then $x \perp S$ means that $\langle x,y\rangle = 0$ for all $y \in S$; similarly for $S \perp x$ and $S \perp S'$.

$\|\cdot\|_X$: The norm on space X (see "Function and operator spaces" on p. 1045 for most X's).

$\|A\|_{\mathcal{B}(U,Y)}$: The standard operator norm $\|A\|_{\mathcal{B}(U,Y)} := \sup_{\|u_0\|_U \le 1} \|Au\|_Y$; note that $\|A\| = \sqrt{\max \sigma(A^*A)}$, by Theorem 11.28c of [Rud73], if U and Y are Hilbert spaces.

$\|\mathbb{D}\|$: If $\mathbb{D} \in \mathrm{TI}(U,Y)$ (see below), this refers always to the $\mathrm{TI}(U,Y)$ norm $\|\mathbb{D}\|_{\mathrm{TI}} := \|\mathbb{D}\|_{\mathcal{B}(\mathrm{L}^2(\mathbf{R};U),\mathrm{L}^2(\mathbf{R};Y))}$, not to $\|\mathbb{D}\|_{\mathcal{A}}$, even if $\mathbb{D} \in \mathcal{A}$, where $\mathcal{A}$ is a subclass of TI.

$\|\cdot\|_p$: $\|\cdot\|_p := \|\cdot\|_{\mathrm{L}^p}$; in particular, $\|x\|_p := \|x\|_{\ell^p}$ when x is a sequence.

$f \ge g$: This means that $f(q) \ge g(q)$ for all q in the (common) domain of f and g, when f and g are scalar functions.

$A \ge B$: iff $B \le A$ iff $A = A^*$, $B = B^*$ and $\langle x,(A-B)x\rangle \ge 0$ for all x (if A and B are linear operators); see also p. 872.

$A > B$: iff $B < A$ iff $A = A^*$, $B = B^*$ and $\langle x,(A-B)x\rangle > 0$ for all $x \ne 0$.

$A \gg B$: iff $B \ll A$ iff for some $\varepsilon > 0$ we have $\langle (A-B)x,x\rangle \ge \varepsilon \|x\|^2$ for all x; see Lemma A.3.1(b1) for more.

$\diamondsuit, \heartsuit$: Cayley transforms (Definition 13.2.2).

Bold letters

$\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{R}_-$: $\mathbf{R} := (-\infty,\infty)$, $\mathbf{R}_+ := [0,\infty)$, and $\mathbf{R}_- := (-\infty,0]$.

$i\mathbf{R}$: $i\mathbf{R} := \{it \mid t \in \mathbf{R}\}$; we consider the differentiation, the Lebesgue measure m etc. for $f : i\mathbf{R} \to U$ as those for $f(i\cdot) : \mathbf{R} \to U$.

$\mathbf{C}$, $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{N}$: The complex/rational/integer/natural numbers, respectively; $\mathbf{N} := \{0,1,2,\ldots\}$, $\mathbf{Z}_- := \{-1,-2,-3,\ldots\}$.

$\mathbf{K}$: "The field of scalars"; outside the appendices we have $\mathbf{K} = \mathbf{C}$. In general, the results in the appendices also hold for $\mathbf{K} = \mathbf{R}$ (see the beginning of each appendix or section of an appendix for exceptions).

$\mathbf{D}$: The *unit disc* $\mathbf{D} := \{z \in \mathbf{C} \mid |z| < 1\}$, hence $\partial\mathbf{D} = \{z \in \mathbf{C} \mid |z| = 1\}$ (the *unit cirle*) and $r\mathbf{D} = \{rz \mid z \in \mathbf{D}\} = \{z \in \mathbf{C} \mid |z| < r\}$ for any $r > 0$.

$\mathbf{C}^+_\omega$, $\mathbf{C}^+$: $\mathbf{C}^+_\omega := \{s \in \mathbf{C} \mid \operatorname{Re} s > \omega\}$, $\mathbf{C}^+ := \mathbf{C}^+_0$.

$\mathbf{C}^-_\omega$, $\mathbf{C}^-$: $\mathbf{C}^-_\omega := \{s \in \mathbf{C} \mid \operatorname{Re} s < \omega\}$, $\mathbf{C}^- := \mathbf{C}^-_0$.

$\mathbf{C}_J, \mathbf{C}_{a,b}$: $\mathbf{C}_J := \{s \in \mathbf{C} \mid \operatorname{Re} s \in J\}$, $\mathbf{C}_{a,b} := \mathbf{C}_{(a,b)} = \{s \in \mathbf{C} \mid a < \operatorname{Re} s < b\}$.

$\overline{\mathbf{C}^+} \cup \{\infty\}, i\mathbf{R} \cup \{\infty\}$: See the footnote on p. 72 for corresponding topologies.

Я: Reflection around the origin: $(\text{Я}u)(t) := u(-t)$, p. 782.

Я_{-1}: The shifted (discrete-time) reflection: $(\text{Я}_{-1}x)_i := x_{-1-i}$, p. 782.

Superscripts

B^A: The set of functions $A \to B$.

f': The derivative of f; see pp. 918 and 961.

$\overline{\alpha}$: the complex conjugate of α when $\alpha \in \mathbf{C}$.

$\overline{\Omega}$: $\overline{\Omega}$ is the closure of the set Ω.

$S^\perp$: The set $\{x \mid x \perp S\}$. By Section 4.9 of [Rud86], this is a closed subspace of the underlining Hilbert space.

$\widehat{u}$: The *Laplace transform* of u, i.e., $\widehat{u}(s) := \int_{\mathbf{R}} e^{-st} u(t)\,dt$; see Definition D.1.6. If u is considered as an element of $L^1_\omega(\mathbf{R}; X)$, then $\widehat{u}$ is often considered as a function $\omega + i\mathbf{R} \to X$; this function (restriction of $\widehat{u}$ to $\omega + i\mathbf{R}$) is called the *Fourier transform* of u. In discrete time, $\widehat{\ }$ stands for the Z-transform, p. 782, for signals; see "$\widehat{\mathbb{D}}$" below for operators.

$\widehat{\mu}$: The Laplace (or Fourier) transform of the (possibly vector-valued) measure μ; see Lemma D.1.12.

$\widehat{\mathbb{D}}$: The transfer function, (symbol, either "Laplace" or "Fourier" transform) of $\widehat{\mathbb{D}}$, when $\widehat{\mathbb{D}} \in \mathrm{TI}_\infty$ (see Theorems 2.1.2 and 3.1.3). In discrete time, see Lemmas 13.1.5 and 13.1.6 instead.

$\widehat{\mathbb{A}}, \widehat{\mathbb{B}}, \widehat{\mathbb{C}}$: See Theorem 6.2.11 (or Lemma 13.3.6 in discrete time).

$\widehat{\mathcal{V}}$: The set $\{\widehat{f} \mid f \in \mathcal{V}\}$ if $\mathcal{V}$ is a set.

$\Sigma^{\frown}_{\text{ext}}, \mathbb{A}^{\frown}, \mathbb{B}^{\frown}, \mathbb{C}^{\frown}, \mathbb{D}^{\frown}, \mathbb{K}^{\frown}, \mathbb{F}^{\frown}$: The partially-closed-loop system (state feedback through first input only), pp. 614–616.

$A^{1/2} = \sqrt{A}$: The (nonnegative) square root of A (Lemma A.3.1(b4)).

A^{-1}: The inverse of A.

$\mathbb{Y}\mathbb{X}^{-1}$, $\widetilde{\mathbb{X}}^{-1}\widetilde{\mathbb{Y}}$: Sometimes a map with coprime internal loop, Definition 7.2.11.

A^{-*}, A^{-d}: $A^{-*} := (A^*)^{-1} = (A^{-1})^*$, $A^{-d} := (A^d)^{-1} = (A^{-1})^d$.

X^*, X^{d}, X^{B}, X^{H}: Adjoint, sesquilinear adjoint (often w.r.t. a pivot space; n L^2_ω contexts we use L^2 as the pivot space; in state contexts we use the state space (usually H) as the pivot space), (bilinear) Banach adjoint and (sesquilinear) Hilbert adjoint of X, respectively, when X is an operator, and the corresponding dual space, when X is a normed space. The meaning of $()^*$ depends on the context; in pivot space contexts (this is usually the case outside the appendices) it stands for the Hilbert adjoint w.r.t. the pivot space; thus, we follow the standard convention in infinite-dimensional control theory. If X is a set of operators, then $X^* = (\cdot)^*[X]$, i.e., $X^* = \{T^* \mid T \in X\}$, etc. See pp. 896–899 for details and Definition 6.1.17 for a main application. Moreover, $(\cdot)^d$ also has an alternative meaning ("causal dual") for systems and their components, see below.

$\mathbb{B}^*, \mathbb{C}^*, \mathbb{D}^*, \mathbb{E}^*$: This is explained above (Hilbert adjoint w.r.t. the pivot space L^2 (or w.r.t. ℓ^2 in discrete time)). See (6.2) and Definition 2.1.4 for details.

$\Sigma^{\mathrm{d}}, \mathbb{A}^{\mathrm{d}}, \mathbb{B}^{\mathrm{d}}, \mathbb{C}^{\mathrm{d}}, \mathbb{D}^{\mathrm{d}}$: (Causal) dual system or map. See Lemmas 6.1.4 and 3.3.8 for continuous time, and Proposition 13.3.5 and Lemma 13.1.8 for discrete time.

$\widehat{\mathbb{D}}^{\mathrm{d}}$: $\widehat{\mathbb{D}}^{\mathrm{d}}(s) := \widehat{\mathbb{D}^d}(s) = \widehat{\mathbb{D}}(\bar{s})^*$; see Lemmas 3.3.8 and 13.1.8.

$X^{n\times m}$: The set of matrices having n rows and m columns and elements from X; cf. (A.1). We set $X^n := X^{n\times 1}$.

E^c: The set of elements that do not belong to E (*complement*).

E^o: The interior of E (p. 867).

$\Sigma^o, \Sigma^o_I, \mathbb{D}^o, \mathbb{D}^o_I, \ldots$: The open and closed loop dynamic feedback systems or maps. See Definitions 7.1.1, 7.2.1 and 7.3.1..

$\begin{bmatrix} A & B \end{bmatrix}^{\mathrm{T}}$: The *transpose* $\begin{bmatrix} A^{\mathrm{T}} \\ B^{\mathrm{T}} \end{bmatrix}$ of $\begin{bmatrix} A & B \end{bmatrix}$.

$\mathbb{A}^t, \mathbb{B}^t, \mathbb{C}^t, \mathbb{D}^t$: $\mathbb{A}^t := \mathbb{A}(t)$, $\mathbb{B}^t := \mathbb{B}\tau\pi_{[0,t)}$, $\mathbb{C}^t := \pi_{[0,t)}\mathbb{C}$, $\mathbb{D}^t := \pi_{[0,t)}\mathbb{D}\pi_{[0,t)}$; see (6.5).

Further uses of superscripts are presented in "Function and operator spaces", p. 1045.

Subscripts

$\Sigma_\sharp$: An output injection closed-loop system of Σ; see Definition 6.6.21.

$\Sigma_\flat = \left[\begin{array}{c|c} \mathbb{A}_\flat & \mathbb{B}_\flat \\ \hline \mathbb{C}_\flat & \mathbb{D}_\flat \\ \mathbb{K}_\flat & \mathbb{F}_\flat \end{array}\right]$: A (state feedback) closed-loop system of Σ; see Definition 6.6.10.

$\Sigma_\circlearrowleft = \left[\begin{array}{c|c} \mathbb{A}_\circlearrowleft & \mathbb{B}_\circlearrowleft \\ \hline \mathbb{C}_\circlearrowleft & \mathbb{D}_\circlearrowleft \\ \mathbb{K}_\circlearrowleft & \mathbb{F}_\circlearrowleft \end{array}\right]$: The closed-loop system corresponding to a solution $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$ of the Riccati equation.

$\Sigma_{\mathrm{crit}} = \left[\begin{array}{c} \mathbb{A}_{\mathrm{crit}} \\ \hline \mathbb{C}_{\mathrm{crit}} \\ \mathbb{K}_{\mathrm{crit}} \end{array}\right]$: A J-critical control in WPLS form; see Theorem 8.3.9.

$\Sigma_L, \mathbb{A}_L, \mathbb{B}_L, \ldots$: Static feedback closed-loop system and maps (Propositions 6.6.2 and 6.6.18).

$H_1, H_{-1}, H_1^*, H_{-1}^*, H_B, H_C^*, H_{C,K}^*$: See Lemma 6.1.16 and Definition 6.1.17.

$C_{\mathrm{c}}, D_{\mathrm{c}}$: A compatible pair; see Definition 6.3.8.

$\mathbb{D}_{\mathrm{d}}$: $\mathbb{D}_{\mathrm{d}} := \mathbb{D}^{\mathrm{d}}$ when $\mathbb{D}$ is a one-block operator; $\begin{bmatrix} \mathbb{D}_{11} & \mathbb{D}_{12} \\ \mathbb{D}_{21} & \mathbb{D}_{22} \end{bmatrix}^{\mathrm{d}} := \begin{bmatrix} \mathbb{D}^{\mathrm{d}}_{22} & \mathbb{D}^{\mathrm{d}}_{12} \\ \mathbb{D}^{\mathrm{d}}_{21} & \mathbb{D}^{\mathrm{d}}_{11} \end{bmatrix}$ (see also p. 740).

$\Sigma_{\mathrm{d}}, \mathbb{A}_{\mathrm{d}}, \mathbb{B}_{\mathrm{d}}, \mathbb{C}_{\mathrm{d}}$: See p. 740.

$X_{\mathbf{R}}$: The complex vector space X as a real vector space (Lemma A.3.21).

$C_{\mathrm{s}}, C_{\mathrm{w}}, C_{\mathrm{L,s}}, C_{\mathrm{L,w}}$: The strong Yosida, weak Yosida, strong Lebesgue and weak Lebesgue extensions of C; e.g., $C_{\mathrm{w}}x_0 := \text{w-lim}_{s\to+\infty} s(s-A)x_0$; see Proposition 6.2.8 for details.

$\mathcal{U}_{\mathrm{u}}$: The set of admissible controls, see p. 614 (or p. 681).

$\mathcal{U}_*^*$, $\mathcal{U}_{\mathrm{out}}$, $\mathcal{U}_{\mathrm{sta}}$, $\mathcal{U}_{\mathrm{str}}$, $\mathcal{U}_{\mathrm{exp}}$: Various sets of admissible controls (see Definition 8.3.2 and Hypothesis 9.0.1).

$\left[\begin{array}{c|c} \mathbb{K}_{\mathrm{u}} & \mathbb{F}_{\mathrm{u}} \end{array}\right], \left[\begin{smallmatrix} \mathbb{H}_{\mathrm{y}} \\ \mathbb{G}_{\mathrm{y}} \end{smallmatrix}\right]$: Preliminarily stabilizing state feedback and output injection pairs, see p. 737.

$\mathbb{N}_{\mathrm{u}}, \mathbb{M}_{\mathrm{u}}, \widetilde{\mathbb{N}_{\mathrm{y}}}, \widetilde{\mathbb{M}_{\mathrm{y}}}$: Parts of a preliminary d.c.f., see p. 713 or p. 737.

$\Sigma_{\mathrm{X}}, \Sigma_{\mathrm{Y}}, \Sigma_{\mathrm{Z}}, \Sigma_{\mathbb{E}^{\mathrm{d}}}, \Sigma_{\mathbb{T}}$: See p. 744, 744, 747 (or 762), 747 or 753, respectively.

Further uses of subscripts are presented in “Function and operator spaces”, p. 1045.

Miscellaneous letters

Capital letters $(A, B, C, D, \ldots)$ often denote the generators (generating operators) of the corresponding maps ($\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$, ...), as in the third and fourth explanation below.

$\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}}_+$: In Chapter 10 (resp. Chapters 11 and 12), the symbol $\widetilde{\mathcal{A}}_+$ (resp. $\widetilde{\mathcal{A}}$) stands for MTIC or something similar; see the standing hypotheses mentioned at the beginnings of these chapters (see Theorem 8.4.9 for suitable classes).

$\mathbb{A}Bu_0 \in \mathrm{L}^1([0,1);H)$: See Section 6.8.

A, B, C; $H_1, H_{-1}, H_1^*, H_{-1}^*$; H_B, H_C^*; B^*, C^*: See Lemma 6.1.16 and Definition 6.1.17.

D: The feedthrough operator of $\mathbb{D}$. See Definition 6.2.3.

D_j: The jth partial derivative; see Definition B.3.3, p. 918.

D^α: See p. 950.

$D(x,r)$: The disc $\{y \mid d(x,y) < r\}$.

$d(x,y)$: The distance $d(x,y) := \|x-y\|$ when x and y belong to a normed space.

$d(x,A)$: The distance $\inf_{a\in A} d(x,a)$ when d is a metric (see Section A.2 for metrics).

e^x: $\mathrm{e}^x := \sum_{k=0}^\infty x^k/k!$ when x is an element of a Banach algebra (e.g., $x \in \mathbf{C}$).

e_k: The vector $e_k := \chi_{\{k\}}$; thus, $\{e_k\}_{k\in\mathbf{Z}}$ is the natural orthonormal basis of $\ell^2(\mathbf{Z})$.

$\mathcal{F}$: The Fourier transform (not always).

$\mathcal{F}_\ell(\mathbb{D}, \mathbb{Q})$: The “lower linear fractions transformation” of $\mathbb{D}$ and $\mathbb{Q}$, i.e., the map $w \mapsto z$ of Figure 7.8 or of Figure 7.10; see (7.64) or (7.98, respectively (or Definition 7.3.1 or Lemma 12.3.2).

$\mathcal{G}$: The set (group) of invertible elements (e.g., $\mathcal{GB}(X,Y)$).

$H_{\mathbb{B}}, H_{\mathbb{C}}$: The reachability and observability subspace (Definition 6.3.25).

H, U, W, Y, Z: Often complex Hilbert spaces of arbitrary dimensions.

x, u, w, y, z: Often u (resp. $\left[\begin{smallmatrix} u \\ w \end{smallmatrix}\right]$) denotes the input, x the state, x_0 the initial state and y (resp. $\left[\begin{smallmatrix} z \\ y \end{smallmatrix}\right]$) the output of a system; see Figure 6.1 and Definition 6.1.5 (resp. Figure 7.9 and Definition 7.3.1).

$\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right], \Sigma_{\text{ext}}, \Sigma_\flat$: See Definition 6.6.10 (state feedback).

$\left[\begin{smallmatrix} \mathbb{H} \\ \hline \mathbb{G} \end{smallmatrix}\right], \Sigma_\sharp, \Sigma_{\text{Total}}$: See Definition 6.6.21 (output injection).

I: The identity operator (I_X denotes the identity on X).

i: The imaginary unit ($i = \sqrt{-1}$).

J_γ: $J_\gamma := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$.

$\mathcal{J}, J$: The cost function and the cost operator, respectively. See Definitions 9.1.3, 8.3.2 and 8.1.3.

$\mathcal{L}, \mathcal{L}_{\text{s}}, \mathcal{L}_{\text{w}}$: The Laplace transform (weak, strong); see pp. 969 and F.3.1.

$P-, PB-, (P), (PB)$: See Definitions 9.1.5, 9.8.1, 9.8.4 and 14.1.1.

$\mathcal{P}$: Often a solution of the Riccati equation. Sometimes the J-critical cost operator $\mathcal{P} := \mathbb{C}_{\text{crit}}^* J \mathbb{C}_{\text{crit}}$ (Theorem 8.3.9(b1)).

$(\mathcal{P}, S, K)$, $(\mathcal{P}, S, \left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right])$: A solution of the Riccati equation. See Definitions 9.1.5, 9.8.1, 9.8.4 and 14.1.1.

$\mathbb{Q}, \mathbb{R}$: Definition 8.3.2 and Hypothesis 9.0.1 (and Hypothesis 14.0.1).

$\mathcal{T}_*$: The stability shift (Remarks 2.1.6, 6.1.9 and 13.3.9).

$\mathcal{U}_*^*$: The set of admissible controls (see Definition 8.3.2 and Hypothesis 9.0.1).

$\mathcal{U}, \mathcal{Y}$: The sets of admissible inputs and outputs, respectively; see Definitions 8.3.2 and 8.1.3.

Y^{s}, Y, Z: See Hypothesis 8.1.1 (Sections 8.1–8.2 only).

$Z^{\text{s}}, Z^{\text{u}}$: The sets of stable and unstable states (see Definition 8.3.2, Hypothesis 9.0.1 and Hypothesis 14.0.1).

$\mathcal{Z}$: The Z-transform: $(\mathcal{Z}u)(z) := \sum_{j \in \mathbf{Z}} z^j u_j$ (p. 782).

Greek letters

β, γ, ζ: See Hypothesis 9.5.1 (Sections 9.5 and 9.6 only).

γ: A fixed positive number in Chapters 11 and 12.

δ_t: The point mass at $t \in \mathbf{R}$; $\delta_t * f := f(\cdot - t) = \tau(-t)f$, when f is a function.

$\Delta^{\ell^2}, \Delta^{\text{S}}$: Discretization operators Section 13.4.

ε_+: See Hypothesis 11.2.1 (Chapter 11 only).

ϑ: See Definition 8.3.2.

κ: See Sections 10.3 and 15.2.

π_J: $(\pi_J u)(s) := u(s)$ if $s \in J$ and $(\pi_J u)(s) := 0$ if $s \notin J$, i.e., $(\pi_J u)(\cdot) := \chi_J(\cdot)u(\cdot)$. Here J is a subset of $\mathbf{R}$. This operator is used both $\mathrm{L}^2(\mathbf{R};U) \to \mathrm{L}^2(\mathbf{R};U)$ and $\mathrm{L}^2(\mathbf{R};U) \to \mathrm{L}^2(J;U)$.

π_+, π_-: $\pi_+ := \pi_{\mathbf{R}_+}$ and $\pi_- := \pi_{\mathbf{R}_-}$.

π^+, π^-: $\pi^+ := \pi_{\mathbf{N}}$, $\pi^- := I - \pi^+$ (Section 13.1).

$\widehat{\pi_\pm}, \widehat{\pm^\pm}$: $\widehat{\pi_\pm} \widehat{f} := \mathcal{L}\pi_\pm \mathcal{L}^{-1} \widehat{f}$, $\widehat{\pi^\pm} \widehat{f} := \mathcal{Z}\pi^\pm \mathcal{Z}^{-1} \widehat{f}$.

$\Pi*$: See Definition 2.6.3.

$\prod$: A product (of numbers or sets; cf. "$X \times Y$" above).

$\rho(A)$: The spectral radius $\rho(A) := \sup\{|z| \,|\, z \in \sigma(A)\} = \lim_{k\to\infty} \|A^k\|^{1/k} = \inf_{k\to\infty} \|A^k\|^{1/k} \le \|A\|$ of A (see Lemma A.3.3(r1)).

$\sigma(A)$: The spectrum $\mathbf{C} \setminus \{\lambda \in \mathbf{C} \,|\, \exists(\lambda - A)^{-1}\}$ (see pp. 871, 882 and 901).

$\Sigma = \left[\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}\right]$: A WPLS (or a wpls). See Definition 6.1.1 (or Definition 13.3.1).

$\sum$: A sum.

$\tau(t)$: The bilateral time-shift operator $\tau(t)u(s) = u(t+s)$ (this is a left-shift when $t > 0$ and a right-shift when $t < 0$).

$\phi_{\text{Cayley}}, \phi_{\text{Cayley}}^{-1}$: The Cayley function $\phi_{\text{Cayley}}(s) = \frac{1-s}{1+s} = \phi_{\text{Cayley}}^{-1}(s)$ (Lemma 13.2.1).

χ_J: The characteristic function of the set J, i.e., $\chi_J(s) := s$ if $s \in J$ and $\chi_J(s) := 0$ if $s \notin J$.

ω_A: The growth rate $\omega_A := \inf_{t>0}[t^{-1} \log \|\mathbb{A}(t)\|]$, when A is the infinitesimal generator of a semigroup $\mathbb{A}$. Note that $\omega_A \ge \sup \operatorname{Re} \sigma(A)$.

Function and operator spaces: generic notation

When $\mathcal{V}$ equals $\mathcal{B}$, $\mathcal{BC}$, $*\mathrm{TI}_*^*$ or ti_*^*, we use some or all of the following conventions:

$\mathcal{V}(X)$, $\mathcal{V}$: We set $\mathcal{V}(X) := \mathcal{V}(X,X)$. We write just $\mathcal{V}$ when we do not wish to specify X and Y; e.g., "$S,T \in \mathcal{B}$" means that S and T are linear and bounded (but they need not have same domain and range spaces).

$\mathcal{GV}$: This stands for the set of invertible (in $\mathcal{V}$) elements of $\mathcal{V}$, e.g., $\mathcal{GV}(X,Y) = \{T \in \mathcal{V}(X,Y) \,|\, ST = I_X \;\; \& \;\; TS = I_Y$ for some $S \in \mathcal{V}(Y,X)\}$.

$\mathcal{V}_\infty$: This stands for the union of $\mathcal{V}_\omega$ for all ω.

$\mathcal{V}_{\exp}$: The set of "exponentially stable functions of type $\mathcal{V}$.". We set $\mathcal{V}_{\exp} := \cup_{\omega<0} \mathcal{V}_\omega$ (resp. $\mathcal{V}_{\exp} := \cup_{\omega<1} \mathcal{V}_\omega$) if $\mathcal{V}_\omega$ is defined for all $\omega \in \mathbf{R}$ (resp. for $\omega > 0$ only).

When $\mathcal{V}$ equals $\mathcal{C}_*$, L_*^*, ℓ_*^*, H_*^* or W_*^*, we use some or all of the following conventions:

$\mathcal{V}(J;Y)$: Functions $J \to Y$ of type $\mathcal{V}$; we set $\mathcal{V}(J) := \mathcal{V}(J;\mathbf{K})$. We write just $\mathcal{V}$ when we do not wish to specify J and Y.

$\mathcal{GV}$: This stands for the set of invertible (in $\mathcal{V}$) elements of $\mathcal{V}$, e.g., $\mathcal{GV}(J;Y) = \{f \in \mathcal{V}(J;Y) \,|\, gf \equiv I \;\; \& \;\; fg \equiv I$ for some $g \in \mathcal{V}(J;*)\}$.

$\mathcal{V}_\infty$: This stands for the union of $\mathcal{V}_\omega$ for all ω.

$\mathcal{V}_{\text{strong}}$: $\mathcal{V}_{\text{strong}}(J;\mathcal{B}(X,Y)) := \{F \,|\, Fx \in \mathcal{V}(J;Y)$ for all $x \in X\}$, $\|F\|_{\mathcal{V}_{\text{strong}}} := \sup_{\|x\|_X \le 1} \|Fx\|_{\mathcal{V}}$.

$\mathcal{V}_{\text{weak}}$: $\mathcal{V}_{\text{weak}}(J;\mathcal{B}(X,Y)) := \{F \,|\, \Lambda Fx \in \mathcal{V}(J)$ for all $x \in X$, $\Lambda \in Y^*\}$, $\|F\|_{\mathcal{V}_{\text{weak}}} := \sup_{\|x\|_X \le 1,\; \|\Lambda\|_{Y^*} \le 1} \|\Lambda Fx\|_{\mathcal{V}}$.

(The strong and weak norms above become bounded in our applications due to the Uniform Boundedness Principle, Lemma A.3.4(O1).)

We often omit the subindex (if any) corresponding to the weight function if the weight function is the constant function 1 (e.g., $\ell^p = \ell^p_1$, $\mathrm{L}^p = \mathrm{L}^p_0$, $\mathrm{TI} = \mathrm{TI}_0$; note that this does not apply "WPLS" or "wpls", which equal WPLS_∞ and wpls_∞, respectively).

If $f \in \mathcal{V}_\omega$, $g \in \mathcal{V}_{\omega'}$ and $f = g$ on the intersection of their domains, then we identify f and g. (Then f and g are the unique elements of $\mathcal{V}_\omega$ and $\mathcal{V}_{\omega'}$ having that restriction on the intersection. For $*\mathrm{TI}^*_*$ (and ti^*_*, see Theorem 13.3.13) this fact is shown in Remark 2.1.9. See Lemma D.1.2(e) for the H^*_* identifications; for L^*_*, ℓ^*_* and W^*_* this is trivial.) We used this identification in the definition of $\mathcal{V}_\infty$ and $\mathcal{V}_{\exp}$.

Function and operator spaces

$\mathrm{AP}(\mathbf{R};X)$: The set of almost-periodic functions $\mathbf{R} \to X$ (see p. 957).

$\mathcal{B}(X,Y)$: The set of bounded linear operators $X \to Y$.

$\mathcal{BC}(X,Y)$: The set of compact linear operators from X into Y (p. 871).

$\mathcal{GB}+\mathcal{BC}$: See Lemma A.3.4.

$\mathcal{C}(J;X)$: The set of continuous functions $J \to X$; see p. 918 for more on $\mathcal{C}^*_*$.

$\mathcal{C}^k(J;X)$: The set of k times continuously differentiable functions $f \in \mathcal{C}(J;X)$; $\mathcal{C}^\infty := \cap_{k\in\mathbf{N}}\mathcal{C}^k$; we make similar definitions for the subspaces of $\mathcal{C}$ defined below.

$\mathcal{C}_\mathrm{b}(J;X)$: The set of bounded continuous functions $f \in \mathcal{C}(J;X)$; if J is an open or closed subset of $\mathbf{R}^n$ or an interval, then $\mathcal{C}_\mathrm{b}(J;X)$ is a Banach space with supremum norm, and $\mathcal{C}_\mathrm{bu}$ and $\mathcal{C}_0$ are closed subspaces of $\mathcal{C}_\mathrm{b}$.

$\mathcal{C}_\mathrm{bu}(J;X)$: The set of bounded and uniformly continuous functions $f \in \mathcal{C}(J;X)$.

$\mathcal{C}_0(J;X)$: The set of functions $f \in \mathcal{C}(J;X)$ vanishing at infinity (i.e., for all $\varepsilon > 0$ there is a compact $K \subset J$ s.t. $\|f(t)\|_X < \varepsilon$ for $t \in J \setminus K$); see Lemma B.3.4.

$\mathcal{C}_\mathrm{c}(J;X)$: The set of compactly supported functions $f \in \mathcal{C}(J;X)$ (note that $\mathcal{C}_\mathrm{c}(J;X) \subset \mathcal{C}_0(J;X)$).

$\mathcal{C}^\infty_\mathrm{c}(J;X)$: The set of compactly supported functions having continuous derivatives of all orders; see also Theorem B.3.11.

$\mathrm{c}_\mathrm{c}, \mathrm{c}_0$: finite and vanishing sequences, respectively. See p. 919.

$\mathrm{H}(\Omega;X)$: Holomorphic functions $\Omega \to X$ (Appendix D); note that such functions are identified with their holomorphic extensions.

$\mathrm{H}^\infty(\Omega;X)$: Bounded holomorphic functions $\Omega \to X$ with supremum norm.

$\mathrm{H}^p, \mathrm{H}^p_\omega$: See Definition D.1.3, p. 964; for discrete time (on $r\mathbf{D}$), see Lemma D.1.15, p. 977.

$\mathrm{H}^p_\mathrm{strong}, \mathrm{H}^p_{\mathrm{strong},\omega}$: See Definition F.3.1, p. 1017.

ℓ^p_r, ℓ^p: By ℓ^p we denote L^p w.r.t. the counting measure. When u is a sequence, we set $\|u\|_p := \|u\|_{\ell^p}$; thus, then $\|u\|_p^p = \sum_k \|u_k\|^p$ $(1 \le p < \infty)$, $\|u\|_\infty = \sup_k \|u_k\|$. Finally, $\|u\|_{\ell^p_r} := \|r^{-\cdot}u\|_p = \|(r^{-k}u_k)\|_p$ $(r > 0)$.

$\ell^1_\pm, \ell^1_{\mathcal{BC}}, \ell^1_{\mathcal{BC},\pm}, \mathrm{c}_\mathrm{c}$: See Section 13.1, p. 781.

$\mathrm{L}(J;X)$: The set of (equivalence classes of) (uniformly) Bochner measurable functions $J \to X$ (p. 911).

$\mathrm{L}_{\mathrm{strong}}(J;X)$; $\mathrm{L}_{\mathrm{weak}}(J;X)$: The set of (equivalence classes of) strongly; weakly measurable functions $J \to X$ (p. 998).

$\mathrm{L}^p(J;X)$: The Banach space of (equivalence classes of) X-valued L^p-functions on J. Thus, $\|f\|_{\mathrm{L}^p} := \|f\|_p$, where $\|f\|_\infty := \operatorname{ess\,sup}\|f\|_X$ and $\|f\|_p := \|f\|_{\mathrm{L}^p} = (\int_J \|f\|_X^p\,d\mu)^{1/p}$ when $1 \le p < \infty$; see also Definition B.3.1.

$\mathrm{L}^p(r\partial\mathbf{D};X)$: Here we identify $r\partial\mathbf{D}$ with $[0,2\pi)$ through $re^{i\cdot}$; cf. Lemma D.1.15.

$\mathrm{L}^p_\omega(J;X)$: The Banach space of (equivalence classes of) measurable functions $f : J \to X$ s.t. $\|f\|_{\mathrm{L}^p_\omega} := \|\mathrm{e}^{\omega\cdot}\|_p < \infty$ (we must have $J \subset \mathbf{R}$). Thus, $\mathrm{L}^p = \mathrm{L}^p_0$, and $\mathrm{e}^{\omega\cdot}$ becomes an isometric isomorphism $\mathrm{L}^p \to \mathrm{L}^p_\omega$. See Definition D.1.3.

$\mathrm{L}^p_{\mathrm{loc}}(J;X)$: The set of (equivalence classes of) functions $f : J \to X$ s.t. $f \in \mathrm{L}^p(K;X)$ whenever $K \subset J$ is compact; here $1 \le p \le \infty$. Note that $\mathrm{L}^p \subset \mathrm{L}^p_{\mathrm{loc}} \subset \mathrm{L}^1_{\mathrm{loc}}$ (and $\mathrm{L}^p_\omega \subset \mathrm{L}^p_{\mathrm{loc}}$ whenever $J \subset \mathbf{R}$). See also Definition B.3.1.

$\mathrm{L}^2_\infty(J;X)$; $\mathrm{L}^2_{+,\infty}$: The set $\cup_{\omega\in\mathbf{R}}\mathrm{L}^2_\omega(J;X)$; $\mathrm{L}^2_{+,\infty} := \mathrm{L}^2_\infty(\mathbf{R}_+;*)$.

$\mathrm{L}^p_{\mathrm{c}}(J;X)$: The set $\{u \in \mathrm{L}^p(J;X) \,|\, u = 0 \text{ (a.e.) outside } [-T,T) \text{ for some } T > 0\}$ (i.e., the $\mathrm{L}^p(J;X)$ functions with a compact essential support).

$\mathrm{L}^p_{\mathrm{strong}}, \mathrm{L}^p_{\mathrm{weak}}$: See Definition F.1.4, p. 1002.

$\mathcal{S}(\mathbf{R};X)$: The space of rapidly decreasing functions, p. 978.

$\mathrm{W}^{k,p}_\omega(J;X), \mathrm{W}^{k,p}_{0,\omega}, \ldots$: The Sobolev spaces of k times weakly differentiably $\mathrm{L}^p_\omega(J,X)$ functions. See Section B.7.

If f is a X-valued function defined on a subinterval J of $\mathbf{R}$, we often identify f with its extension $\pi_J f$ having the value zero outside J. We use the same symbol π_J both for the embedding operator from $J \to X$ to $\mathbf{R} \to X$ and for the corresponding projection operator from $\mathbf{R} \to X$ to $J \to X$. With this interpretation, $\pi_+\mathrm{L}^2(\mathbf{R};X) = \mathrm{L}^2(\mathbf{R}_+;X) \subset \mathrm{L}^2(\mathbf{R};X)$ and $\pi_-\mathrm{L}^2(\mathbf{R};X) = \mathrm{L}^2(\mathbf{R}_-;X) \subset \mathrm{L}^2(\mathbf{R};X)$, etc.

Classes of time-invariant maps

See also Chapter 2 for TI$*$, Section 13.1 for ti$*$ and Section 2.6.3 for the other classes.

$\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}}_+$: In Chapter 10 (resp. Chapters 11 and 12), the symbol $\widetilde{\mathcal{A}}_+$ (resp. $\widetilde{\mathcal{A}}$) stands for MTIC or something similar; see the standing hypotheses mentioned at the beginnings of these chapters (see Theorem 8.4.9 for suitable classes).

$\mathrm{TI}_\omega(U,Y)$: The (closed) subspace of operators $\mathbb{D} \in \mathcal{B}(\mathrm{L}^2_\omega(\mathbf{R};U);\mathrm{L}^2_\omega(\mathbf{R};Y))$ that are time-invariant (i.e., $\tau(t)\mathbb{D} = \mathbb{D}\tau(t)$ for all $t \in \mathbf{R}$). $\mathrm{TI} := \mathrm{TI}_0$, $\mathrm{TI}_\infty := \cup_{\omega\in\mathbf{R}}\mathrm{TI}_\omega$.

$\mathrm{TI}^p, \mathrm{TI}^p_*, \mathrm{TI}^{p,q}_\omega$: See Theorem 3.1.5.

$\mathrm{TI}^{\mathcal{C}_0}_\omega$, $\mathcal{C}_{0,\omega}$: See p. 92.

$\mathrm{TIC}_\omega(U,Y)$: The (closed) subspace of operators $\mathbb{D} \in \mathrm{TI}_\omega(U,Y)$ that are causal (i.e., $\pi_-\mathbb{D}\pi_+ = 0$). $\mathrm{TIC} := \mathrm{TIC}_0$, $\mathrm{TIC}_\infty := \cup_{\omega\in\mathbf{R}}\mathrm{TIC}_\omega$, $\mathrm{TIC}_{\exp} := \cup_{\omega<0}\mathrm{TIC}_\omega$.

$\mathrm{CTI}, \mathrm{CTIC}, \mathrm{CTI}^{\mathcal{BC}}, \mathrm{CTIC}^{\mathcal{BC}}$: Classes of TIC operators having continuous transforms (Definition 2.6.1).

$\mathrm{MTI}^*_*, \mathrm{MTIC}^*_*, \mathrm{SMTI}^*_*, \mathrm{SMTIC}^*_*$: Classes of TIC_∞ operators that are convolutions with certain kinds of measures (Definition 2.6.3).

$\mathrm{MTI}_X, \mathrm{MTIC}_X$: Certain kinds of measures with values in X, see Lemma D.1.12.

$\mathrm{ti}^*_*, \mathrm{tic}^*_*$: Discrete-time classes; see Definition 13.1.1, p. 783.

Abbreviated symbols

card A: The *cardinality* of A (see, e.g., Definition 151, p. 275 of [Kelley]). It suffices to know that $\operatorname{card} A \le \operatorname{card} B$ iff there is a one-to-one function of A to B (equivalently, a function of B onto A). Consequently, $\operatorname{card} A = \operatorname{card} B$ iff $\operatorname{card} A \le \operatorname{card} B$ and $\operatorname{card} B \le \operatorname{card} A$ (equivalently, there is a one-to-one map of A onto B); see, e.g., [Kelley] for details. See also Lemma B.2.2.

det A: The determinant of the matrix A.

$\operatorname{diag}(A,B,C)$: the *diagonal* matrix $\left[\begin{smallmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{smallmatrix}\right]$ with diagonal elements (or blocks) A, B and C.

$\dim(H)$: the *dimension* of the Hilbert space H = the cardinality of an arbitrary orthonormal basis of H (Lemma A.3.1(a1)).

$\mathrm{Dom}(T)$: The *domain* (of definition) of the operator T.

ess range: Essential range, Lemma B.2.7.

ess sup, ess inf: Essential supremum, essential infimum. See p. 909.

$\mathrm{Ker}(T)$: The *kernel* $\mathrm{Ker}(T) := T^{-1}[\{0\}] := \{u \in U \,|\, Tu = 0\}$, when $T : U \to Y$.

$\mathrm{Leb}(f)$: The set of Lebesgue points of f (p. 942).

log: Logarithm with base exp.

$\mathrm{Ran}(T)$: The range $T[U] := \{T(u) \,|\, u \in U\}$, when $T : U \to Y$.

Re, Im: Real part, imaginary part ($\mathrm{Re}(x+iy) = x$, $\mathrm{Im}(x+iy) = y$ for $x, y \in \mathbf{R}$).

span: $\operatorname{span} E := \{\sum_{j=0}^n \alpha_j x_j \,|\, n \in \mathbf{N},\ \alpha_j \in \mathbf{K},\ x_j \in E\ (j = 0,\ldots,n)\}$.

sup, inf, max, min: Supremum, infimum, maximum, minimum, respectively. Recall from [Rud86] that $\inf\emptyset := +\infty$, $\sup\emptyset = -\infty$.

supp f: The *support* of f = the closure of $\{t \,|\, f(t) \neq 0\}$.

$\mathrm{supp}_{\mathrm{d}}\,\mathbb{E}$: The nonzero atoms of $\mathbb{E}$ (Definition 2.6.3).

w-lim, s-lim, lim: "lim" means the *limit* in the standard topology, which for operators is the uniform (i.e., norm) topology. "s-lim" and "w-lim" refer to strong and weak limits, respectively. E.g., if $F : \mathbf{R} \to \mathcal{B}(H)$, where H is a Hilbert space, then $\text{w-lim}_{s\to+\infty} F(s) = A$ means that $F(s)x \to Ax$ weakly for all $x \in H$, as $s \to +\infty$; cf. Lemma A.3.1(h)–(j).

Glossary

We should have a great many fewer disputes in the world if only words were taken for what they are, the signs of our ideas only, and not for things themselves.

— John Locke (1632–1704)

If $T \in \mathcal{B}(H_1, H_2)$, where H_1 and H_2 are Hilbert spaces, then we use the following terms for T (partially valid also for more general functions):

one-to-one: = *injective*, i.e., $Tx = 0 \Leftrightarrow x = 0$ (for all $x \in H_1$). Equivalent to "coercive" if $\dim H_1 < \infty$.

coercive: $\|Tx\| \geq \varepsilon \|x\|$ for all $x \in H_1$, i.e., $T^*T \gg 0$ = *left-invertible* = T^* is onto = full column rank (if matrix).

onto: $T[H_1] = H_2$, i.e., $TT^* \gg 0$ = *surjective* = *right-invertible* = T^* coercive = full row rank (if matrix).

invertible: = boundedly invertible = *bijection* = one-to-one and onto.

countably infinite: A set is countably infinite if it has the same cardinality as the set $\mathbf{N}$, i.e., if there is a one-to-one function of $\mathbf{N}$ onto this set.

countable: A set is countable if it is finite or countably infinite; otherwise it is *uncountable*.

finite-dimensional vector space: A vector space spanned by a finite number of vectors.

finite-dimensional function: A vector-valued function whose values lie in a finite-dimensional subspace of the range space.

finite-dimensional system: A system whose input, state and output spaces are finite-dimensional. Recall that a transfer function has a finite-dimensional realization iff it is rational.

finite-dimensional theory: This refers to the theory of finite-dimensional systems.

classical theory: This refers usually to finite-dimensional theory (equivalently, to the theory of rational transfer functions).

time domain: This refers to $\mathbf{R}$ or $\mathbf{R}_+$ as the time horizon. The input, output and state signals in control systems are functions of time, with domain $\mathbf{R}$ or $\mathbf{R}_+$. Thus, their Laplace (or Fourier) transforms are defined on a subset of $\mathbf{C}$; by *frequency domain* we refer to such subsets or to (a part of) $\mathbf{C}$ as the domain of the argument of these transformed functions.

state-space: State-space theory refers to theory on systems (where one can also speak of the state, not just on input and output) in contrast to *I/O-theory* or *frequency-domain theory*, which ignores the internal structure of systems and treats I/O maps instead of systems (compare this to the two definitions of admissible controllers in Definition 7.1.1). Thus (due to historical reasons), "frequency-domain" has two meanings: the first refers to working with Laplace transforms (as opposed to time-domain), and the second to working with I/O maps or transfer functions (as opposed to state-space).

discrete time: This refers to time domain **Z** or **N** in place of **R** or $\mathbf{R}_+$ as opposed to *continuous time* treated above, see Part IV.

discrete part: The discrete (atomic) part of a measure is explained in Section 2.6.

map: A map means a function. However, for most of the time, we reserve the word map for the "integral operators", such as $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\mathbb{D}$ in Definition 6.1.1. See Definitions 7.2.1 and 7.2.11 and Lemma 7.2.7 for maps with internal loop.

well-posed: An I/O map is well posed (or *proper*) if it is in TIC_∞. A transfer function is well posed (or proper) if it is in H^∞_∞, i.e., if it is bounded on some right half-plane. (By Theorem 2.1.2, an I/O map is well posed iff its transfer function is well posed.) Lemma 7.2.7 shows when a map with internal loop is well posed.

stable: A function (signal) $u \in \mathrm{L}^2_{\mathrm{loc}}$ is called stable iff $u \in \mathrm{L}^2$. A map from vectors or signals to vectors or signals is called stable iff it is bounded w.r.t. to the standard norm for vectors and the L^2 norm for signals; see Definition 6.1.3 for details. In discrete-time, a function (signal; actually sequence) u is called stable iff $u \in \ell^2$; see Definition 13.3.1 for the stability of discrete-time maps

singular: A control problem is often called *singular* if the map from the control to the output is not coercive (or I-coercive over $\mathcal{U}^*_*$); otherwise it is *nonsingular*. Most control problems in the literature are nonsingular.

superfluous: An assumption is said to be superfluous if the claims are true even without the assumption ("and $n < 7$" is superfluous in "if $n > 1$ and $n < 7$, then n is positive").

redundant: As assumption is said to be redundant if it is implied by the other assumptions ("and $t = |t|$" is redundant in "if $t > 1$ and $t = |t|$, then $t^2 > t$").

greatest: If $\mathcal{A} \subset \mathcal{B}(H)$, then $A \in \mathcal{A}$ is the greatest element of $\mathcal{A}$ iff $A \geq A'$ for all $A' \in \mathcal{A}$ (iff $-A$ is the *smallest* element of $\mathcal{A}$). Recall that $A \in \mathcal{A}$ is *maximal* iff $A \leq A' \in \mathcal{A} \Rightarrow A = A'$. Obviously, a greatest element must be unique and maximal, whereas a maximal element need not be unique in general.

Abbreviations

a.e.: almost everywhere (or "almost every")

iff: if and only if

I/O: input/output ("from input to output")

p., pp.: page, pages

r.c., l.c., d.c., p.r.c., p.l.c., q.r.c., q.l.c.: See Definition 6.4.1.

r.c.f., l.c.f., d.c.f., p.r.c.f., p.l.c.f., q.r.c.f., q.l.c.f.: See Definition 6.4.4.

s.t.: such that

w.r.t.: with respect to

w.l.o.g.: without loss of generality

Acronyms

B_{w}^*-CARE: Certain simplified Riccati equation, see Definition 9.2.6

[e]CARE; [e]DARE; [e]IARE: Riccati equations. See Definition 9.8.1; 14.1.1; 9.8.4.

[e]CARI; [e]DARI; [e]IARI: Riccati inequalities; see the index.

FICP: Full-Information Control Problem; see Chapter 11 (or Section 10.4).

LQR: Linear Quadratic Regulator, see Chapter 10.

SF: State Feedback, see Definition 6.6.10 (sometimes this refers to pure state feedback, where the feedthrough term (the "F" of $\left[\begin{array}{c|c} \mathbb{K} & \mathbb{F} \end{array}\right]$) is zero, see Definition 11.1.2).

SpF: See Definition 8.4.6, p. 384

$*\mathrm{TI}_*^*$, $*\mathrm{TIC}_*^*$, ti_*^*, tic_*^*: See "Classes of time-invariant maps", p. 1046

TVS: Topological Vector Space (p. 870).

WR, WLR, WVR, WHPR, SR, SLR, SVR, SHPR, UR, ULR, UVR, UHPR: Different forms of regularity. See Definition 6.2.3; see also "regular" in the index.

WPLS_*, SOS: See Definitions 6.1.1 and 6.1.3.

wpls_*, sos: Classes of discrete-time systems. See Definition 13.3.1.

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