Helsinki University of Technology Institute of Mathematics Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2003 A461

VECTOR-VALUED WAVELETS AND THE HARDY SPACE $H^1(\mathbb{R}^n; X)$

Tuomas Hytönen

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

Helsinki University of Technology Institute of Mathematics Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2003 A461

VECTOR-VALUED WAVELETS AND THE HARDY SPACE $H^1(\mathbb{R}^n; X)$

Tuomas Hytönen

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

Tuomas Hytönen: *Vector-valued wavelets and the Hardy space $H^1(\mathbb{R}^n; X)$*; Helsinki University of Technology Institute of Mathematics Research Reports A461 (2003).

Abstract: *We prove an analogue of* Y. Meyer*'s wavelet characterization of the Hardy space $H^1(\mathbb{R}^n)$ for the space $H^1(\mathbb{R}^n; X)$ of X-valued functions. Here X is a Banach space with the UMD property. The proof uses results of* T. Figiel *on generalized Calderón–Zygmund operators on Bôchner spaces and some new local estimates.*

AMS subject classifications: 42B30, 42C40, 46E40

Keywords: wavelet basis, atomic decomposition, generalized Calderón–Zygmund operators, UMD-space

tuomas.hytonen@hut.fi

ISBN 951-22-6511-7
ISSN 0784-3143
Espoo, 2003

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, FIN-02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1. Introduction

A wavelet basis of $L^2(\mathbb{R}^n)$ is an orthonormal basis of the form $(\psi_\lambda)_{\lambda\in\Lambda}$, where Λ is the set of dyadic n-vectors of the form $\lambda = k2^{-j} + \eta 2^{-j-1}$ ($j \in \mathbb{Z}, k \in \mathbb{Z}^n, \eta \in \{0,1\}^n \setminus \{0\}$), and $\psi_\lambda(x) = 2^{jn/2}\psi^\eta(2^j x - k)$, where $\psi^\eta \in L^2(\mathbb{R}^n)$, $\eta \in \{0,1\}^n \setminus \{0\}$, are the $2^n - 1$ mother wavelets. The basis is called r-regular if $|\partial^\alpha \psi^\eta(x)| \le C_m(1+|x|)^{-m}$ and $\int x^\alpha \psi^\eta(x)\,\mathrm{d}x = 0$ for all $|\alpha| \le r$, all $m \in \mathbb{N}$ and all $\eta \in \{0,1\}^n \setminus \{0\}$.

Y. Meyer [5] has proved the following characterization of the Hardy space $H^1(\mathbb{R}^n)$ in terms of wavelets:

Theorem 1.1 ([5]). *Let $(\psi_\lambda)_{\lambda\in\Lambda}$ be a 1-regular wavelet basis of $L^2(\mathbb{R}^n)$. The following conditions are equivalent for the distribution*

$$f(x) := \sum_{\lambda\in\Lambda} \alpha_\lambda \psi_\lambda(x) :$$

$$f \in H^1(\mathbb{R}^n), \tag{1.2}$$

$$\sup_{F\subset\Lambda} \sup_{\varepsilon\in\{\pm1\}^\Lambda} \left\| \sum_{\lambda\in F} \varepsilon_\lambda \alpha_\lambda \psi_\lambda(\cdot) \right\|_{L^1(\mathbb{R}^n)} < \infty, \tag{1.3}$$

$$\left(\sum_{\lambda\in\Lambda} |\alpha_\lambda|^2 \, |\psi_\lambda(\cdot)|^2 \right)^{1/2} \in L^1(\mathbb{R}^n), \tag{1.4}$$

$$\left(\sum_{\lambda\in\Lambda} |\alpha_\lambda|^2 \, |Q(\lambda)|^{-1} \, 1_{Q(\lambda)}(\cdot) \right)^{1/2} \in L^1(\mathbb{R}^n), \tag{1.5}$$

$$\left(\sum_{\lambda\in\Lambda} |\alpha(\lambda)|^2 \, |Q(\lambda)|^{-1} \, 1_{R(\lambda)}(\cdot) \right)^{1/2} \in L^1(\mathbb{R}^n), \tag{1.6}$$

where

- *the first supremum in* (1.3) *is taken over all finite subsets F of Λ,*
- *$Q(\lambda) := 2^{-j}([0,1[^n + k)$ for $\lambda = k2^{-j} + \eta 2^{-j-1}$, and*
- *$R(\lambda) := 2^{-j}(A^\eta + k)$, where A^η is any non-degenerate cube.*

Our purpose is to give an analogue of this result in the context of the Hardy space $H^1(\mathbb{R}^n; X)$ of X-valued functions, where X is a Banach space with the so called UMD property (unconditionality of martingale differences), a UMD-space for short.

The properties of the wavelet bases in the L^p spaces ($p \in]1,\infty[$) of UMD-valued functions have been studied by T. Figiel [2] already in the 80's. His methods are based on the unconditionality of the Haar system on $L^p([0,1]; X)$, and of its analogues on $L^p(\mathbb{R}^n; X)$, which could actually be taken as the definition of the space X being UMD. This approach does not apply to the Hardy space

$H^1(\mathbb{R}^n;X)$, since the Haar system is not a basis of $H^1(\mathbb{R}^n)$, even in the scalar-valued setting. Instead, the Haar system spans a smaller dyadic Hardy space, which is useful for certain purposes but a little less "natural" than the usual Hardy space. It would be of interest also to understand the wavelet expansions on the usual $H^1(\mathbb{R}^n;X)$ space, and this is the task taken up here.

It is well-known that the "right" substitute in general Banach spaces for the quadratic estimates as in (1.4) through (1.6) (which work well for Hilbert spaces) is in terms of Rademacher averages. We denote by ε_λ independent random variables on some probability space Ω with distribution $\mathbb{P}(\varepsilon_\lambda = +1) = \mathbb{P}(\varepsilon_\lambda = -1) = 1/2$. $\mathbb{E}_\varepsilon$ denotes the corresponding expectation. Then we have:

Theorem 1.7. *Let X be a UMD-space, $(\psi_\lambda)_{\lambda\in\Lambda}$ a 1-regular wavelet basis of $L^2(\mathbb{R}^n)$, and $\alpha \in X^\Lambda$. The following conditions are equivalent for the X-valued distribution*

$$f(x) := \sum_{\lambda\in\Lambda} \alpha_\lambda \psi_\lambda(x) :$$

$$f \in H^1(\mathbb{R}^n;X), \tag{1.8}$$

$$\sup_{F\subset\Lambda} \sup_{\varepsilon\in\{\pm1\}^\Lambda} \Big\| \sum_{\lambda\in F} \varepsilon_\lambda \alpha_\lambda \psi_\lambda(\cdot) \Big\|_{L^1(\mathbb{R}^n;X)} < \infty, \tag{1.9}$$

$$\int_{\mathbb{R}^n} \mathbb{E}_\varepsilon \Big| \sum_{\lambda\in\Lambda} \varepsilon_\lambda \alpha_\lambda \psi_\lambda(x) \Big|_X \,\mathrm{d}x < \infty, \tag{1.10}$$

$$\int_{\mathbb{R}^n} \mathbb{E}_\varepsilon \Big| \sum_{\lambda\in\Lambda} \varepsilon_\lambda \alpha_\lambda \left|Q(\lambda)\right|^{-1/2} 1_{Q(\lambda)}(x) \Big|_X \,\mathrm{d}x < \infty, \tag{1.11}$$

$$\int_{\mathbb{R}^n} \mathbb{E}_\varepsilon \Big| \sum_{\lambda\in\Lambda} \varepsilon_\lambda \alpha_\lambda \left|Q(\lambda)\right|^{-1/2} 1_{R(\lambda)}(x) \Big|_X \,\mathrm{d}x < \infty, \tag{1.12}$$

where F, λ, $Q(\lambda)$ and $R(\lambda) = 2^{-j}(A^\eta + k)$ have the same meaning as in Theorem 1.1.

Moreover, each of the expressions (1.9) *through* (1.12) *define equivalent norms of $H^1(\mathbb{R}^n;X)$. Consequently, the wavelet series of f converges unconditionally to f in the $H^1(\mathbb{R}^n;X)$-norm.*

Note that the condition (1.12) *a priori* depends on the choice of the cubes A^η defining the $R(\lambda)$'s. However, the proof of the Theorem will show that the validity of this condition for any one choice of the A^η's already implies it for all possible choices.

To simplify the matters, note that it suffices to establish the equivalence of the different norms in the case of $(\alpha_\lambda)_{\lambda\in\Lambda}$ finitely non-zero. The general case then follows by standard arguments, using the density in $H^1(\mathbb{R}^n;X)$ of such functions.

The definition of the Hardy space $H^1(\mathbb{R}^n;X)$, which we use, is in terms of atoms: We have, by definition, $f \in H^1(\mathbb{R}^n;X)$ if and only if f has an expansion of the form

$$f(x) = \sum_{i=1}^{\infty} a_i(x), \qquad \operatorname{supp} a_i \subset \bar{B}_i, \qquad \int a_i(x)\,\mathrm{d}x = 0,$$

where the $\bar{B}_i$ are balls in $\mathbb{R}^n$, and we have

$$\sum_{i=1}^{\infty} \|a_i\|_{L^p(\mathbb{R}^n;X)} \left|\bar{B}_i\right|^{1/p'} < \infty, \tag{1.13}$$

where some value of $p \in \left]1,\infty\right[$ is fixed, and p' denotes the conjugate exponent, $1/p + 1/p' = 1$. The norm $\|f\|_{H^1(\mathbb{R}^n;X)}$ is defined as the infimum of the above series taken over all such decompositions. It depends, of course, on the choice of $p \in \left]1,\infty\right[$, but it is well-known that each $p \in \left]1,\infty\right[$ (actually also $p = \infty$) gives the same space $H^1(\mathbb{R}^n;X)$ with an equivalent norm. This will also follow from our theorem and its proof, since the conditions (1.9) through (1.12) do not contain any explicit or implicit reference to the parameter p.

The main arguments which show that (1.8) implies the other conditions are based on results concerning generalized Calderón–Zygmund operators on UMD-Bôchner spaces, due to T. Figiel [3]. The reverse direction involves some essentially pointwise estimates.

Acknowledgments. I wish to thank Dr. Hans-Olav Tylli who brought the results of T. Figiel to my knowledge, and Prof. Tadeusz Figiel himself, who kindly supplied me with further pieces of his work.

I acknowledge financial support from the Magnus Ehrnrooth Foundation.

2. Implications using Calderón–Zygmund operators

In proving Theorem 1.7, we will need to apply several transformations of the wavelet series. All these transformations will have the generic form of an integral operator

$$Tf(x) = \int_{\mathbb{R}^n} k(x,y) f(y)\,\mathrm{d}y,$$

where the kernel k is actually bounded and integrable. What is important is to obtain appropriate uniform bounds for operator norms of different operators T of this kind.

T. Figiel [3] has generalized the famous $T(1)$ theorem of G. David and J.-L. Journé to the setting of X-valued L^p spaces. (See also [4], where an intermediate estimate omitted in [3] is proved in detail.) A rather general formulation of this result is given in [3]; for our purposes, the following version is sufficiet:

Proposition 2.1 ([3])**.** *Let $k(x,y) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ satisfy the standard estimates*

$$|k(x,y)| \le \kappa\, |x-y|^{-n}, \qquad |\nabla_x k(x,y)| + |\nabla_y k(x,y)| \le \kappa\, |x-y|^{-n-1}.$$

Assume, moreover, that T is bounded on $L^2(\mathbb{R}^n)$ with operator norm at most κ. Then T is also bounded on $L^p(\mathbb{R}^n;X)$, where X is any UMD space, with norm $\leq C_p(X)\kappa$, for all $p \in \,]1,\infty[$, and it is bounded from $H^1(\mathbb{R}^n;X)$ to $L^1(\mathbb{R}^n;X)$ with norm $\leq C_1(X)\kappa$. If, in addition,

$$[T'(1)](y) := \int_{\mathbb{R}^n} k(x,y)\,\mathrm{d}x \equiv 0,$$

then T is bounded on $H^1(\mathbb{R}^n;X)$ with norm $\leq C_0(X)\kappa$.

This proposition is essentially a statement of the fact that for an operator defined in terms of a kernel which verifies the standard estimates, the conditions of the $T(1)$ theorem are necessary and sufficient: Since T is bounded on $L^2(\mathbb{R}^n)$, it satisfies these conditions, but then the vector-valued version applies to give the boundedness on $L^p(\mathbb{R}^n;X)$. For our purposes, we would actually only need a special $T(1)$ theorem, i.e., the case $T(1) = 0 = T'(1)$.

It is a well-known fact, in which the vector-valued situation brings no complications, that an integral operator satisfying the standard estimates and bounded on $L^p(\mathbb{R}^n;X)$ is also bounded from $H^1(\mathbb{R}^n;X)$ to $L^1(\mathbb{R}^n;X)$. Concerning the $H^1(\mathbb{R}^n;X)$-boundedness under the additional assumption, see Y. Meyer and R. Coifman [6], Th. 3 of Ch. 7. (This is also an extension argument, which goes through in the vector-valued setting without modifications.)

Corollary 2.2. *Let $(a_\lambda)_{\lambda\in\Lambda}$, $(b_\lambda)_{\lambda\in\Lambda}$ be orthogonal sets in $L^2(\mathbb{R}^n)$ satisfying*

$$|a_\lambda(x)| \leq C_m \frac{2^{nj/2}}{(1+|2^j x - k|)^m}, \qquad |\nabla a_\lambda(x)| \leq C_m \frac{2^{nj/2+j}}{(1+|2^j x - k|)^m}$$

for all $\lambda = k2^{-j} + \eta 2^{-j-1}$ and all $m \in \mathbb{N}$, with similar estimates for the $(b_\lambda)_{\lambda\in\Lambda}$. Consider the integral operators with kernels given by

$$k(x,y) = \sum_{\lambda\in F} \nu_\lambda a_\lambda(x) b_\lambda(y),$$

where $F \subset \Lambda$ is any finite set and $\nu_\lambda \in \mathbb{C}$, $|\nu_\lambda| \leq 1$.

These are uniformly bounded on $L^p(\mathbb{R}^n;X)$, and from $H^1(\mathbb{R}^n;X)$ to $L^1(\mathbb{R}^n;X)$, with the operator norms depending only on $p \in \,]1,\infty[$, the UMD-constant of the space X, and the quantities C_m, $m \in \mathbb{N}$. If the a_λ's have vanishing integral, then we also have boundedness on $H^1(\mathbb{R}^n;X)$ with a similar estimate for the norm.

Proof. From the assumed pointwise estimates, it easily follows that $\|a_\lambda\|_2 \leq C$, which depends only on the C_m's, and similarly $\|b_\lambda\|_2 \leq C$. Then a bound depending only on the C_m's is easily derived for the operator norm of $f \mapsto \sum_{\lambda\in F} \nu_\lambda a_\lambda \langle b_\lambda, f\rangle$ on $L^2(\mathbb{R}^n)$, using the orthogonality of the two sets (a_λ) and (b_λ).

It is also a routine exercise to verify the standard estimates for the kernel k, with the constant only depending on the C_m's. Then the assertion follows from Prop. 2.1. $\square$

Now the first steps in our main theorem follow at once:

Proof of (1.8) $\Rightarrow$ (1.9) $\Rightarrow$ (1.10). The first implication is immediate from the fact that, for any $F \subset \Lambda$, $\varepsilon \in \{\pm 1\}^\Lambda$,

$$\sum_{\lambda \in F} \varepsilon_\lambda \psi_\lambda(x) \bar{\psi}_\lambda(y)$$

are kernels of the kind considered in Cor. 2.2. Clearly the integral operator with the kernel given above maps f to $\sum_{\lambda \in F} \varepsilon_\lambda \alpha_\lambda \psi_\lambda(\cdot)$.

The second implication is obvious, since the L^1 norm on the probability space Ω is dominated by the L^∞ norm. □

For the proof of further implications, we will need regular wavelet bases with the mother wavelet *non*-vanishing at a preassigned point. This is a somewhat untypical need, since usually it is the cancellation and vanishing properties of the wavelets which are desired.

Lemma 2.3. *For every $x \in \mathbb{R}$, there exists an infinitely regular wavelet ψ on $\mathbb{R}$ such that $\psi(x) \neq 0$.*

Proof. The proof is based on a modification of Meyer's construction of the Littlewood–Paley multiresolution analysis ([5], §2.2), and the related wavelet ([5], §3.2). In that construction, one starts with an even, non-negative function $\theta \in \mathcal{D}(\mathbb{R})$, such that $\theta(\xi) = 1$ for $|\xi| \leq 2\pi/3$, $\theta(\xi) = 0$ for $|\xi| \geq 4\pi/3$, and $\theta^2(\xi) + \theta^2(2\pi - \xi) = 1$ for $\xi \in [0, 2\pi]$. Our modification consists of choosing an $\eta \in C^\infty(\mathbb{R})$, which is required to be 0 on $[-2\pi/3, 2\pi/3]$ but otherwise arbitrary, and taking $\vartheta(\xi) := \theta(\xi) e^{\mathbf{i}\eta(\xi)}$. We set $\phi := \check{\vartheta}$, the inverse Fourier transform.

It follows, for $m(\xi) := \sum c_k e^{\mathbf{i}k\xi}$, that

$$\left\| \sum c_k \phi(x-k) \right\|_2^2 = \frac{1}{2\pi} \left\| m(\xi) \vartheta(\xi) \right\|_2^2 = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \int_0^{2\pi} |m(\xi)\vartheta(\xi + 2\pi j)|^2 \,\mathrm{d}\xi$$
$$= \frac{1}{2\pi} \int_0^{2\pi} |m(\xi)|^2 \,\mathrm{d}\xi = \sum |c_k|^2,$$

since $\sum |\vartheta(\xi + 2\pi j)|^2 \equiv 1$, as is easily verified, and so $\phi(\cdot - k)$, $k \in \mathbb{Z}$, are the orthonormal basis of a closed subspace V_0 of $L^2(\mathbb{R})$, which gives rise to a multiresolution analysis of $L^2(\mathbb{R})$.

We then pass to the construction of the corresponding wavelet ψ. Following [5], §3.2, we compute the auxiliary coefficients

$$\alpha_k = \int_{-\infty}^{\infty} \frac{1}{2}\phi\left(\frac{x}{2}\right) \bar{\phi}(x+k) \,\mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \vartheta(2\xi) \bar{\vartheta}(\xi) e^{\mathbf{i}k\xi} \,\mathrm{d}\xi = \frac{1}{2}\phi\left(\frac{k}{2}\right),$$

since $\vartheta(\xi) = 1$ on the support of $\vartheta(2\xi)$.

Then

$$m_0(\xi) := \sum_{-\infty}^{\infty} \alpha_k e^{\mathbf{i}k\xi} = \sum_{-\infty}^{\infty} \vartheta(-2(\xi + 2k\pi)).$$

by POISSON's summation formula, and $\hat{\psi}(\xi) := e^{-\mathbf{i}\xi/2}\vartheta_1(\xi)$, where

$$\vartheta_1(\xi) := \bar{m}_0(\xi/2 + \pi)\vartheta(\xi/2) = \begin{cases} \vartheta(\xi/2) & \xi \in \pm[4\pi/3, 8\pi/3] \\ \bar{\vartheta}(-\xi \pm 2\pi) & \xi \in \pm[2\pi/3, 4\pi/3] \\ 0 & \text{else,} \end{cases}$$

where the last equality follows readily when taking into account the sets on which ϑ equals 1 or 0. Note that $\vartheta_1|_{\pm[2\pi/3,4\pi/3]}$ is obtained from $\vartheta_1|_{\pm[4\pi/3,8\pi/3]}$ by reflecting and scaling about the point $\pm 4\pi/3$; in fact

$$\vartheta_1(4\pi/3 - \xi) = \bar{\vartheta}(2\pi/3 + \xi), \quad \vartheta_1(4\pi/3 + 2\xi) = \vartheta(2\pi/3 + \xi) \quad \text{for } \xi \in [0, 2\pi/3],$$

and similarly on the negative axis. Thus

$$\begin{aligned}(2.4) \quad \psi(x + 1/2) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mathbf{i}\xi(x+1/2)} e^{-\mathbf{i}\xi/2} \vartheta_1(\xi)\, \mathrm{d}\xi \\ &= \int_0^{2\pi/3} \left(\bar{\vartheta}(2\pi/3 + \xi) e^{\mathbf{i}(4\pi/3 - \xi)x} + 2\vartheta(2\pi/3 + \xi) e^{\mathbf{i}(4\pi/3 + 2\xi)x} \right) \mathrm{d}\xi \\ &\qquad + \text{an integral over the negative half-line.}\end{aligned}$$

Now the phase of ϑ on $\pm[4\pi/3, 8\pi/3]$ is in our control; moreover, it can be adjusted independently on the positive and negative line segments. By symmetry, it then suffices to show that we can make the integral $\int_0^{2\pi/3} (\dots)\, \mathrm{d}\xi$ above non-vanishing with an appropriate choice of this phase. We choose this phase in such a way that

$$\mathrm{Re} \int_0^{2\pi/3} \vartheta(2\pi/3 + \xi) e^{\mathbf{i}(4\pi/3 + 2\xi)x}\, \mathrm{d}\xi \geq \frac{3}{4} \int_0^{2\pi} |\vartheta(2\pi/3 + \xi)|\, \mathrm{d}\xi;$$

then the integral in (2.4) is estimated by

$$\begin{aligned}\left| \int_0^{2\pi/3} (I(\xi) + II(\xi))\, \mathrm{d}\xi \right| &\geq \left| \int_0^{2\pi/3} II(\xi)\, \mathrm{d}\xi \right| - \int_0^{2\pi/3} |I(\xi)|\, \mathrm{d}\xi \\ &\geq \left(\frac{3}{2} - 1 \right) \int_0^{2\pi/3} |\vartheta(2\pi/3 + \xi)|\, \mathrm{d}\xi > 0.\end{aligned}$$

Thus, for an arbitrary $x \in \mathbb{R}$, we have constructed a wavelet ψ such that $\psi(x + 1/2) \neq 0$; in fact, one with $|\psi(x + 1/2)| \geq c$, where $c > 0$ does not depend on x. $\square$

The n dimensional version follows readily by a tensor product construction. Recall that the $2^n - 1$ mother wavelets in the n-dimensional setting are naturally indexed by $\eta \in \{0, 1\}^n \setminus \{0\}$. We denote by $\iota := (1, \dots, 1)$ the n-vector, all of whose entries are 1.

Corollary 2.5. *For any $x \in \mathbb{R}^n$, there exists an infinitely regular wavelet basis of $L^2(\mathbb{R}^n)$ such that $\psi^\iota(x) \neq 0$.*

Proof. Let $\psi_{i,0} := \phi_i$, $\psi_{i,1} := \psi_i$ be (infinitely regular) father, resp. mother, wavelets on $\mathbb{R}$ for $i = 1, \ldots, n$. For $\eta \in \{0,1\}^n$, $y \in \mathbb{R}^n$, denote

$$\psi^\eta(y) := \prod_{i=1}^n \psi_{i,\eta_i}(y_i).$$

Then ψ^η, $\eta \in \{0,1\}^n \setminus \{0\}$, is the set of (infinitely regular) mother wavelets for a multiresolution analysis of $L^2(\mathbb{R}^n)$. By choosing the 1-dimensional wavelets $\psi_{i,1}$ in such a way that $\psi_{i,1}(x_i) \neq 0$ for a given $x = (x_1, \ldots, x_n)$, we clearly ensure the condition $\psi^\iota(x) \neq 0$. $\square$

Proof of (1.8) $\Rightarrow \forall A^\eta : (1.12) \Rightarrow (1.11)$. Let A^η, $\eta \in \{0,1\}^n \setminus \{0\}$, be non-degenerate cubes, and denote

$$A := \bigcup_{\eta \in \{0,1\}^n \setminus \{0\}} \overline{A^\eta};$$

this is a compact set.

For every $x \in A$, we choose an infinitely regular wavelet basis $(\psi_{x,\lambda})_{\lambda \in \Lambda}$ such that $\psi^\iota_x(x) \neq 0$. By continuity of ψ^ι_x, we have $\psi^\iota_x(U_x) \not\ni 0$ for some neighbourhood U_x of x, and then by compactness we can choose finitely many, say m, infinitely regular wavelet bases $(\psi_{i,\lambda})_{\lambda \in \Lambda}$ such that $\sum_{i=1}^m |\psi^\iota_i(x)| \geq c > 0$ for all $x \in A$. Now the kernels

$$\sum_{\lambda \in F : \eta = \eta_0} \varepsilon_\lambda 2^{jn/2} \psi^\iota_i(2^j x - k) \bar{\psi}_\lambda(y)$$

satisfy the assumptions of Cor. 2.2; hence they define uniformly bounded integral operators from $H^1(\mathbb{R}^n; X)$ to $L^1(\mathbb{R}^n; X)$, and thus

$$\sum_{i=1}^m \mathbb{E}_\varepsilon \int_{\mathbb{R}^n} \left| \sum_{\lambda \in F} \varepsilon_\lambda \alpha_\lambda 2^{jn/2} \psi^\iota_i(2^j x - k) \right|_X \mathrm{d}x \leq C \, \|f\|_{H^1(\mathbb{R}^n; X)} .$$

The contraction principle permits replacing $\psi^\iota_i(2^j x - k)$ by its absolute value above, and using the fact that $\sum_{i=1}^m |\psi^\iota_i(2^j x - k)| \geq c 1_{A^\eta}(2^j x - k) = c 1_{R(\lambda)}(x)$ and the contraction principle again, we finally deduce

$$\mathbb{E}_\varepsilon \int_{\mathbb{R}^n} \left| \sum_{\lambda \in F} \varepsilon_\lambda \alpha_\lambda \, |Q(\lambda)|^{-1/2} \, 1_{R(\lambda)}(x) \right|_X \mathrm{d}x \leq C \, \|f\|_{H^1(\mathbb{R}^n; X)} .$$

The fact that (1.12) for *all* A^η implies (1.11) is evident, since (1.11) is just the special case of (1.12) with $A^\eta = [0, 1[^n$. $\square$

Proof of (1.10) $\Rightarrow \exists A^\eta : (1.12)$. It suffices to observe that necessarily $|\psi^\eta(x)| \geq c > 0$ for all x in some cube A^η; then the expression in (1.12) can be dominated by that in (1.10) according to the contraction principle. $\square$

Now we have shown that

$$\begin{aligned}&(1.8) \implies (1.9) \implies (1.10) \implies \exists A^\eta : (1.12), \qquad \text{and}\\ &(1.8) \implies \forall A^\eta : (1.12) \implies (1.11) \implies \exists A^\eta : (1.12)\end{aligned}$$

(where the last implication was not mentioned explicitly before, but it is trivial).

3. Construction of the atomic decomposition

To complete the proof of Theorem 1.7, we need to show that the condition (1.12), for any cubes A^η whatsoever, implies the existence of an atomic decomposition for f; moreover, the H^1 norm of f computed in terms of this decomposition should be controlled in terms of the expression in (1.12). Note that, without loss of generality, we may take the A^η to be dyadic cubes of side-length ≤ 1, since the expression in (1.12) decreases when the sets A^η (and hence $R(\lambda)$) decrease. When this is done, it follows that the $R(\lambda)$ are dyadic cubes as well.

To achieve the atomic decomposition, we are going to modify the construction used by Meyer [5]. Certain parts of the proof are in almost one-to-one correspondence with the scalar-valued case; however, there are also significant and essential departures from Meyer's reasoning.

Let us fix an $\eta_0 \in \{0,1\}^n \setminus \{0\}$, and consider $f = \sum_{\lambda:\eta=\eta_0} \alpha_\lambda \psi_\lambda(x)$. It clearly suffices to decompose each of the $2^n - 1$ series of this kind. Then we can use a different indexing system which is more convenient in the present context: Let $\mathcal{R}$ be the collection of all the cubes $R(\lambda) = 2^{-j}(A^\eta + k)$ such that $\eta = \eta_0$. Then, instead of Λ, we can use $\mathcal{R}$ as our index set, and we write ε_R instead of ε_λ. Moreover, write $\alpha_R := \alpha_\lambda$ for $R = R(\lambda)$ and $\eta = \eta_0$. Since $|Q(\lambda)|$ and $|R(\lambda)|$ only differ by a multiplicative constant independent of λ (as long as $\eta = \eta_0$ is fixed), we can further replace the factor $|Q(\lambda)|^{-1/2}$ in our equations by $|R|^{-1/2}$.

Following [5], we denote

$$\sigma(x) := \mathbb{E}_\varepsilon \left| \sum_{R\in\mathcal{R}} \varepsilon_R \alpha_R |R|^{-1/2} 1_R(x) \right|_X,$$

and we have $\sigma \in L^1(\mathbb{R}^n)$ by the standing assumption (1.12).

We further adopt the following notations:

$$E_k := \{x : \sigma(x) > 2^k\}, \quad \mathcal{C}_k := \{R \in \mathcal{R} : |R \cap E_k| \geq \beta |R|\}, \quad \Delta_k := \mathcal{C}_k \setminus \mathcal{C}_{k+1},$$

where we fix some $\beta \in]0,1[$. Note that, if $\alpha_R \neq 0$, then $\sigma(x) \geq |\alpha_R|_X$ for all $x \in R$. Thus $R \subset E_k$ and hence $R \in \mathcal{C}_k$ for all small enough k.

The maximal members of $\mathcal{C}_k$ will be denoted by $R(k,\ell)$, where ℓ runs over an appropriate index set, and

$$\Delta(k,\ell) := \{R \in \Delta_k : R \subset R(k,\ell)\}.$$

Note that

$$\sum_{\ell} |R(k,\ell)| \leq \sum_{\ell} \beta^{-1} |R(k,\ell) \cap E_k| \leq \beta^{-1} |E_k| \tag{3.1}$$

and

$$\sum_{-\infty}^{\infty} 2^k |E_k| \leq 2 \|\sigma\|_{L^1(\mathbb{R}^n)} . \tag{3.2}$$

We then come to a key estimate in the proof of (1.12) $\Rightarrow$ (1.8). The statement of this estimate is little more than a vector-valued analogue of the corresponding step in [5]; however, the proof is substantially longer and very different in spirit. The proof in [5] (where $p = 2$) exploits the Hilbert space structure of the scalar-valued L^2 space, which at first seems to give little hope of extending the result beyond Hilbert space framework. In view of this, it is perhaps surprising that the argument given below actually requires no geometric restrictions on the underlying Banach space X. The proof is very local in spirit; it essentially involves going through every cube $R \in \mathcal{R}$ one by one, in sharp contrast to the "global" argument in [5] in terms of the orthogonal expansions.

Lemma 3.3. *With the notation adopted above, we have the estimate*

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathbb{E}_\varepsilon \left| \sum_{R \in \Delta(k,\ell)} \varepsilon_R \alpha_R |R|^{-1/2} 1_R(x) \right|_X^p \mathrm{d}x \\
&\leq \frac{1}{1-\beta} \int_{R(k,\ell) \setminus E_{k+1}} \mathbb{E}_\varepsilon \left| \sum_{R \in \Delta(k,\ell)} \varepsilon_R \alpha_R |R|^{-1/2} 1_R(x) \right|_X^p \mathrm{d}x \leq c_p \frac{2^{(k+1)p}}{1-\beta} |R(k,\ell)| .
\end{aligned}$$

Proof. The second inequality is clear from Kahane's inequality $\mathbb{E}_\varepsilon \left|\sum \varepsilon_i x_i\right|_X^p \leq c_p \left(\mathbb{E}_\varepsilon \left|\sum \varepsilon_i x_i\right|_X\right)^p$ and the fact that $\sigma(x) \leq 2^{k+1}$ for $x \notin E_{k+1}$. We will then concentrate on the first inequality.

Observe that if $R_1 \cap R_2 \neq \emptyset$, then necessarily $R_1 \subset R_2$ or $R_2 \subset R_1$, since R_1, R_2 are dyadic cubes. If $\tilde{R} \in \Delta(k,\ell)$ is minimal, in the sense that $R \subsetneq \tilde{R} \implies R \notin \Delta(k,\ell)$, then for $x \in \tilde{R}$ we have

$$\mathbb{E}_\varepsilon \left| \sum_{R \in \Delta(k,\ell)} \varepsilon_R \alpha_R |R|^{-1/2} 1_R(x) \right|_X = \mathbb{E}_\varepsilon \left| \sum_{R \in \Delta(k,\ell), R \supset \tilde{R}} \varepsilon_R \alpha_R |R|^{-1/2} \right|_X , \tag{3.4}$$

i.e., this expression is constant for $x \in \tilde{R}$.

More generally, if $\tilde{R} \in \Delta(k,\ell)$, and

$$\tilde{R}_0 := \tilde{R} \setminus \bigcup_{\substack{R \in \Delta(k,\ell) \\ R \subsetneq \tilde{R}}} R, \tag{3.5}$$

then (3.4) holds for all $x \in \tilde{R}_0$.

It suffices to establish the assertion of the lemma in the case when only finitely many $\alpha(Q)$ are non-zero, since the general case then follows from the monotone convergence theorem. Then the summations involved are finite, and we can avoid all convergence problems in the following. Replacing $\Delta(k,\ell)$ by $\{R \in \Delta(k,\ell) : \alpha_R \neq 0\}$, if necessary, we can assume that $\Delta(k,\ell)$ is finite.

Let R be one of the maximal members of $\Delta(k,\ell)$. It clearly suffices to prove, for all such R, that

$$\begin{aligned}\int_R \mathbb{E}_\varepsilon \left| \sum_{\tilde{R}\in\Delta(k,\ell),\tilde{R}\subset R} \varepsilon(\tilde{Q})\alpha(\tilde{Q})1_{\tilde{R}}(x)\right|_X^p \mathrm{d}x \\ \leq \frac{1}{1-\beta}\int_{R\setminus E_{k+1}} \mathbb{E}_\varepsilon \left| \sum_{\tilde{R}\in\Delta(k,\ell),\tilde{R}\subset R} \varepsilon(\tilde{Q})\alpha(\tilde{Q})1_{\tilde{R}}(x)\right|_X^p \mathrm{d}x.\end{aligned} \tag{3.6}$$

To prove this inequality, we need to introduce some notation. We say that $\tilde{R}$ is a Δ-subcube of R if $\tilde{R} \subsetneq R$ and $\tilde{R} \in \Delta(k,\ell)$. We say that $\tilde{R}$ is a *first order* Δ-subcube of R if, in addition, the following property holds: there is no $\hat{R} \in \Delta(k,\ell)$ with $\tilde{R} \subsetneq \hat{R} \subsetneq R$. We label the first order Δ-subcubes of R by R_i, where i runs over an appropriate finite index set. The first order Δ-subcubes of R_i, which are labelled R_{ij}, are called the *second order* Δ-subcubes of R, and so on, in an obvious fashion. The mth order Δ-subcubes of R will be denoted by R_α, where $\alpha = \alpha_1 \ldots \alpha_m$ is a string of m indices. We further denote $R_{\alpha 0} := R_\alpha \setminus \cup R_{\alpha i}$, which is obviously equivalent to the earlier definition (3.5). For convenience, we also denote $E := E_{k+1}$.

Since the proof of the inequality (3.6) in the general situation involves a very large amount of indices, it is helpful first to consider a special case in which only first and second order Δ-subcubes of R are involved. If $S \subset R$, we denote by $I(S)$ the integral over S of the same integrand as in (3.6), and $\mu(S) := I(S)/|S|$ if $|S| > 0$, and $\mu(S) := 0$ otherwise.

Now in our special situation, the cube R is decomposed into disjoint parts as follows:

$$R = R_0 \cup \bigcup_{i\in I} R_i \cup \bigcup_{j\in J}\left(R_{j0} \cup \bigcup_{k\in K_j} R_{jk}\right), \tag{3.7}$$

where R_i, $i \in I$ are those first order Δ-subcubes of R which have no further Δ-subcubes, whereas $R_j = \cup_{k\in\{0\}\cup K_j} R_{jk}$, $j \in J$, are those first order Δ-subcubes of R which do have some further Δ-subcubes, namely the R_{jk}, $k \in K_j$.

Now

$$I(R\setminus E)=I(R_0\setminus E)+\sum_{i\in I}I(R_i\setminus E)+\sum_{j\in J}\left(I(R_j\setminus E)+\sum_{k\in K_j}I(R_{jk}\setminus E)\right)$$
$$=|R_0\setminus E|\,\mu(R_0)+\sum_{i\in I}|R_i\setminus E|\,\mu(R_i)$$
$$+\sum_{j\in J}\left(|R_{j0}\setminus E|\,\mu(R_j)+\sum_{k\in K_j}|R_{jk}\setminus E|\,\mu(R_{jk})\right),$$

since the integrand is constant on each of the sets R_0, R_i, R_{j0}, R_{jk}, as was observed above.

We want to show that the above displayed quantity is at least $(1-\beta)I(R)=:tI(R)$, denoting $t:=1-\beta$. To see this, observe that $\left|\bar{R}\cap E\right|=\left|\bar{R}\cap E_{k+1}\right|<\beta\left|\bar{R}\right|$, hence $\left|\bar{R}\setminus E\right|>(1-\beta)\left|\bar{R}\right|$ for all $\bar{R}\in\Delta_k\subset\mathcal{C}_{k+1}^c$ by the definition of $\mathcal{C}_{k+1}$. Now

$$tI(R)=$$
$$t\,|R_0|\,\mu(R_0)+\sum_{i\in I}t\,|R_i|\,\mu(R_i)+\sum_{j\in J}\left(t\,|R_{j0}|\,\mu(R_{j0})+\sum_{k\in K_j}t\,|R_{jk}|\,\mu(R_{jk})\right),$$

and hence

$$I(R\setminus E)-tI(R)=(|R_0\setminus E|-t\,|R_0|)\,\mu(R_0)+\sum_{i\in I}(|R_i\setminus E|-t\,|R_i|)\,\mu(R_i)$$
$$+\sum_{j\in J}\left((|R_{j0}\setminus E|-t\,|R_{j0}|)\,\mu(R_{j0})+\sum_{k\in K_j}(|R_{jk}\setminus E|-t\,|R_{jk}|)\,\mu(R_{jk})\right),$$

and denoting $\tau(S):=|S\setminus E|-t\,|E|$ (whence $\tau(\bar{R})>0$ for all $\bar{R}\in\Delta_k$), this can be further written as

$$=\left[\tau(R_0)+\sum_{i\in I}\tau(R_i)+\sum_{j\in J}\sum_{k\in\{0\}\cup K_j}\tau(R_{jk})\right]\mu(R_0)$$
$$+\sum_{i\in I}\tau(R_i)\,(\mu(R_i)-\mu(R_0))+\sum_{j\in J}\left(\left\{\sum_{k\in\{0\}\cup K_j}\tau(R_{jk})\right\}(\mu(R_{j0})-\mu(R_0))\right.$$
$$\left.+\sum_{k\in K_j}\tau(R_{jk})\,(\mu(R_{jk})-\mu(R_{j0}))\right).$$

Noting that the quantity in brackets $[\cdots]$ is simply $\tau(R)$, whereas that in the braces $\{\cdots\}$ is $\tau(R_j)$, we find that all the terms appearing above are non-negative, and hence $I(R\setminus E)\geq tI(R)$, which we wanted to prove.

The special case treated above already contains the essence of the matter, and it is essentially the notation which is more difficult in the general case where Δ-subcubes of higher orders are allowed. Now R is disjointly decomposed as

$$R = R_0 \cup \bigcup_{\alpha}\left(\bigcup_{i} R_{\alpha i} \cup \bigcup_{j} R_{\alpha j0}\right), \tag{3.8}$$

where α runs over an appropriate set of strings of indices, and i and j over appropriate sets (possibly different for different α) of single indices. Note that the possibility of α being the empty string is allowed. The decomposition (3.8) should be compared with the special case in (3.7).

We have

$$\begin{aligned}
&I(R\setminus E) - tI(R) = \left(|R_0\setminus E| - t\,|R_0|\right)\mu(R_0)\\
&+\sum_{\alpha}\left(\sum_{i}\left(|R_{\alpha i}\setminus E| - t\,|R_{\alpha i}|\right)\mu(R_{\alpha i}) + \sum_{j}\left(|R_{\alpha j0}\setminus E| - t\,|R_{\alpha j0}|\right)\mu(R_{\alpha j0})\right)\\
&\qquad\qquad = \tau(R_0)\mu(R_0) + \sum_{\alpha}\left(\sum_{i}\tau(R_{\alpha i})\mu(R_{\alpha i}) + \sum_{j}\tau(R_{\alpha j0})\mu(R_{\alpha j0})\right)
\end{aligned}$$

We claim that this is equal to

$$\begin{aligned}
&\left\{\tau(R_0) + \sum_{\alpha}\left(\sum_{i}\tau(R_{\alpha i}) + \sum_{j}\tau(R_{\alpha j0})\right)\right\}\mu(R_0)\\
&\qquad\qquad + \sum_{\alpha}\sum_{i}\tau(R_{\alpha i})\left(\mu(R_{\alpha i}) - \mu(R_{\alpha 0})\right)\\
&+\sum_{\alpha}\sum_{j}\left[\tau(R_{\alpha j0}) + \sum_{\beta}\left(\sum_{k}\tau(R_{\alpha j\beta k}) + \sum_{\ell}\tau(R_{\alpha j\beta\ell 0})\right)\right]\left(\mu(R_{\alpha j0}) - \mu(R_{\alpha 0})\right).
\end{aligned}$$

In the expression above, the quantity in the braces $\{\cdots\}$ is $\tau(R)\geq 0$ and that in the brackets $[\cdots]$ is $\tau(R_{\alpha j})\geq 0$, so that all the terms appearing above are non-negative. Hence it suffices to verify the claimed equality, i.e., the vanishing of the expression

(3.9)
$$\sum_{\alpha,i}\tau(R_{\alpha i})\mu(R_0)+\sum_{\alpha,j}\tau(R_{\alpha j0})\mu(R_0)-\sum_{\alpha,i}\tau(R_{\alpha i})\mu(R_{\alpha 0})-\sum_{\alpha,j}\tau(R_{\alpha j0})\mu(R_{\alpha 0})$$
$$+\sum_{\alpha,j,\beta}\left(\sum_k\tau(R_{\alpha j\beta k})+\sum_\ell\tau(R_{\alpha j\beta\ell 0})\right)\mu(R_{\alpha j0})$$
$$-\sum_{\alpha,j,\beta}\left(\sum_k\tau(R_{\alpha j\beta k})+\sum_\ell\tau(R_{\alpha j\beta\ell 0})\right)\mu(R_{\alpha 0})$$

When α runs over all strings, and j over all single indices, αj clearly runs over all strings except for the empty string. Hence the second-to-last term in (3.9) is equal to

$$\sum_{\alpha,\beta}\left[\sum_k\tau(R_{\alpha\beta k})+\sum_\ell\tau(R_{\alpha\beta\ell 0})\right]\mu(R_{\alpha 0})-\sum_{\beta,k}\tau(R_{\beta k})\mu(R_0)-\sum_{\beta,\ell}\tau(R_{\beta\ell 0})\mu(R_0)$$

Similarly, replacing the pair (j,β) by β alone in the last term of (3.9), we find that this last terms is equal to

$$-\sum_{\alpha,\beta}\left[\sum_k\tau(R_{\alpha\beta k})+\sum_\ell\tau(R_{\alpha\beta\ell 0})\right]\mu(R_{\alpha 0})+\sum_{\alpha,k}\tau(R_{\alpha k})\mu(R_{\alpha 0})$$
$$+\sum_{\alpha,\ell}\tau(R_{\alpha\ell 0})\mu(R_{\alpha 0}).$$

Now it is clear that the different terms in (3.9) cancel each other, so our claim, and hence the assertion of the lemma, is verified. $\square$

Now we denote

$$A_{k,\ell}(x,\varepsilon):=\sum_{R\in\Delta(k,\ell)}\varepsilon_R\alpha_R\,|R|^{-1/2}\,1_R(x);$$

note that

$$\sum_{k=-\infty}^{\infty}\sum_\ell A_{k,\ell}(x,\varepsilon)=\sum_{R\in\mathcal{R}}\varepsilon_R\alpha_R\,|R|^{-1/2}\,1_R(x).$$

A modification of this series will give us the required atomic decomposition of f. Observe that $\operatorname{supp}A_{k,\ell}(\cdot,\varepsilon)\subset R(k,\ell)$ by the definition of $\Delta(k,\ell)$.

Moreover, by Lemma 3.3, we have

$$
\begin{aligned}
(3.10)\quad \sum_{k,\ell} \|A_{k,\ell}\|_{L^p(\Omega\times\mathbb{R}^n;X)} |R(k,\ell)|^{1/p'} & \\
&\le \sum_{k,\ell} c_p^{1/p}(1-\beta)^{-1/p}2^{k+1}|R(k,\ell)|^{1/p}|R(k,\ell)|^{1/p'} \\
\le 2c_p^{1/p}(1-\beta)^{-1/p}\sum_k 2^k \sum_\ell |R(k,\ell)| &\overset{(3.1)}{\le} 2c_p^{1/p}(1-\beta)^{-1/p}\beta^{-1}\sum_k 2^k|E_k| \\
&\overset{(3.2)}{\le} 4c_p^{1/p}(1-\beta)^{-1/p}\beta^{-1}\|\sigma\|_{L^1(\mathbb{R}^n)}.
\end{aligned}
$$

The quantity on the left of this estimate should be compared with the definition of the H^1 norm in (1.13).

Now we are ready to finish the proof of Theorem 1.7.

Conclusion of the proof of (1.12) $\Rightarrow$ (1.8). Now we construct the atomic decomposition of f, or more precisely, of each of the subseries

$$
f_{\eta_0}(x) := \sum_{\lambda\in\Lambda:\eta=\eta_0} \alpha_\lambda\psi_\lambda(x) = \sum_{R\in\mathcal{R}} \alpha_R\psi_{\lambda(R)}(x)
$$

where $\lambda(R) := 2^{-j}k + 2^{-j-1}\eta_0$ for $R = 2^{-j}(A^{\eta_0}+k)$.

Consider a basis $(\Psi_\lambda)_{\lambda\in\Lambda}$ of compactly supported, 1-regular wavelets. The existence of such wavelet bases is well-known, see [5]. Now that A^{η_0} is a non-degenerate cube, we have $\operatorname{supp}\Psi_{2^{-j_0}k_0+2^{-j_0-1}\eta_0} = \operatorname{supp} 2^{j_0 n/2}\Psi^{\eta_0}(2^{j_0}\cdot - k_0) \subset A^{\eta_0}$ for some suitable $j_0\ge 0$ and $k_0\in\mathbb{Z}^n$.

Let us denote $\Psi^\eta_{j,k} := \Psi_\lambda$ for $\lambda = 2^{-j}k + 2^{-j-1}\eta$, and set $\phi := \Psi^{\eta_0}_{j_0,k_0}$, and

$$
\phi_{j,k} := 2^{nj/2}\phi(2^j\cdot - k) = 2^{n(j+j_0)/2}\Psi^{\eta_0}(2^{j_0}(2^j\cdot - k) - k_0) = \Psi^{\eta_0}_{j+j_0,2^{j_0}k+k_0}.
$$

Since $j_0\ge 0$, we see that $(\phi_{j,k})_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ is a subset of $(\psi_\lambda)_{\lambda\in\Lambda}$, thus orthonormal (but not complete, of course) in $L^2(\mathbb{R}^n)$.

Now that ϕ is bounded and supported on A^{η_0}, we have

$$
|\phi(x)| \le C\,|A^{\eta_0}|^{-1/2}\,1_{A^{\eta_0}}(x),
$$

where $C = \|\phi\|_\infty |A^{\eta_0}|^{1/2}$, and then by scaling

$$
|\phi_R(x)| := |\phi_{j,k}(x)| \le C\,|R|^{-1/2}\,1_R(x)
$$

for $R = 2^{-j}(A^{\eta_0}+k)$. Then the contraction principle gives

$$
\int_{\mathbb{R}^n}\mathbb{E}_\varepsilon\left|\sum_{R\in\Delta(k,\ell)}\varepsilon_R\alpha_R\phi_R(x)\right|_X^p \mathrm{d}x \le C\int_{\mathbb{R}^n}\mathbb{E}_\varepsilon\left|\sum_{R\in\Delta(k,\ell)}\varepsilon_R\alpha_R|R|^{-1/2}1_R(x)\right|_X^p \mathrm{d}x.
$$

Now we apply Cor. 2.2 with

$$\sum_{\lambda\in\Lambda:\eta=\eta_0} \varepsilon_{R(\lambda)}\psi_\lambda(x)\bar{\phi}_{R(\lambda)}(y)$$

to the result

$$\int_{\mathbb{R}^n} \left| \sum_{\substack{\lambda\in\Lambda:\eta=\eta_0,\\ R(\lambda)\in\Delta(k,\ell)}} \alpha_\lambda\psi_\lambda(x)\right|_X^p \mathrm{d}x \le \int_{\mathbb{R}^n} \left| \sum_{R\in\Delta(k,\ell)} \varepsilon_R\alpha_R \left|R\right|^{-1/2} 1_R(x)\right|_X^p \mathrm{d}x$$

Taking the expectation $\mathbb{E}_\varepsilon$ of the right-hand side and combining this with the previous inequality, we have shown, for

$$a_{k,\ell}(x) := \sum_{\substack{\lambda\in\Lambda:\eta=\eta_0,\\ R(\lambda)\in\Delta(k,\ell)}} \alpha_\lambda\psi_\lambda(x),$$

the estimate

$$\left\|a_{k,\ell}\right\|_{L^p(\mathbb{R}^n;X)} \le C\left\|A_{k,\ell}\right\|_{L^p(\Omega\times\mathbb{R}^n;X)}. \tag{3.11}$$

Since each of the wavelets ψ_λ has a vanishing integral, so does $a_{k,\ell}$. Consider two cases:

The case of compactly supported wavelets. Since A^η is a non-degenerate cube and ψ^η has compact support, we have $\operatorname{supp}\psi^\eta \subset (A^\eta)^*$ where Q^* denotes the cube concentric with Q and having g times the side length of Q, where g is a sufficiently large constant. Then $\psi^\eta_{j,k} = \psi_{2^{-j}k+2^{-j-1}\eta} = 2^{jn/2}\psi^\eta(2^j\cdot - k)$ satisfies $\operatorname{supp}\psi^\eta_{j,k} = 2^{-j}(\operatorname{supp}\psi^\eta + k) \subset 2^{-j}((A^\eta)^*+k) = (2^{-j}(A^\eta+k))^*$, i.e., $\operatorname{supp}\psi_\lambda \subset R(\lambda)^*$.

Thus, if $R(\lambda)\in\Delta(k,\ell)$, hence $R(\lambda)\subset R(k,\ell)$, we have $\operatorname{supp}\psi_\lambda\subset R(k,\ell)^*$. This means that $\operatorname{supp} a_{k,\ell}\subset R(k,\ell)^*$, and then

$$\begin{aligned}\left\|f_{\eta_0}\right\|_{H^1(\mathbb{R}^n;X)} &\le \sum_{k,\ell}\left\|a_{k,\ell}\right\|_{L^p(\mathbb{R}^n;X)}\left|R(k,\ell)^*\right|^{1/p'}\\ &\overset{(3.11)}{\le} C\sum_{k,\ell}\left\|A_{k,\ell}\right\|_{L^p(\Omega\times\mathbb{R}^n;X)}\left|R(k,\ell)\right|^{1/p'} \overset{(3.10)}{\le} C\left\|\sigma\right\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Thus we obtain a norm estimate for f_{η_0}, and then for $f=\sum_{\eta\in\{0,1\}^n\setminus\{0\}} f_\eta$, of the desired form.

The general case. By the special case considered above, we obtain

$$\left\|\sum_{\lambda\in\Lambda}\alpha_\lambda\Psi_\lambda\right\|_{H^1(\mathbb{R}^n;X)} \le C\left\|\sigma\right\|_{L^1(\mathbb{R}^n)},$$

where $(\Psi_\lambda)_{\lambda\in\Lambda}$ is a compactly supported 1-regular wavelet basis, as above. Then it suffices to apply the $H^1(\mathbb{R}^n;X)$-boundedness assertion of Cor. 2.2 to

$$\sum \psi_\lambda(x)\bar{\Psi}_\lambda(y)$$

to conclude the desired norm estimate for $f = \sum \alpha_\lambda\psi_\lambda$, where $(\psi_\lambda)_{\lambda\in\Lambda}$ is any 1-regular wavelet basis.

This completes the proof of the implication (1.12) $\Rightarrow$ (1.8), and with it the proof of Theorem 1.7. ☐

4. On BMO($\mathbb{R}^n;X$) and duality

One can also generalize the wavelet characterization of the space BMO($\mathbb{R}^n$) from [5] to the UMD-valued situation. This generalization is not as exciting as that of the characterization of $H^1(\mathbb{R}^n)$: In essence, we just need to replace classical L^2 estimates used in [5] by the application of Cor. 2.2, but otherwise the proof follows the same lines as in [5].

Proposition 4.1. *Let X be a UMD-space and $(\psi_\lambda)_{\lambda\in\Lambda}$ a 1-regular wavelet basis. If $b \in \mathrm{BMO}(\mathbb{R}^n;X)$ and $\alpha_\lambda := \langle b, \bar{\psi}_\lambda\rangle$, then*

$$\int_{\mathbb{R}^n} \mathbb{E}_\varepsilon \left|\sum_{\lambda\in F} \varepsilon_\lambda\alpha_\lambda\psi_\lambda(x)\right|_X^p \mathrm{d}x \leq \kappa^p\,|Q| \qquad \forall F \subset \{\lambda\in\Lambda : Q(\lambda)\subset Q\}, \tag{4.2}$$

where $\kappa \leq C_p \|b\|_{\mathrm{BMO}(\mathbb{R}^n;X)}$, and $p \in]1,\infty[$.

Conversely, if (4.2) holds for some set of coefficients $(\alpha_\lambda)_{\lambda\in\Lambda} \subset X$ and all finite sets F as above, then the series

$$\sum_{\lambda\in\Lambda} \alpha_\lambda\psi_\lambda(x)$$

converges unconditionally in $L^p_{\mathrm{loc}}(\mathbb{R}^n;X)/X$, to a function in $\mathrm{BMO}(\mathbb{R}^n;X)$ with norm at most $C_p\kappa$.

By convergence in $L^p_{\mathrm{loc}}(\mathbb{R}^n;X)/X$ we mean the following: For every compact $K\subset\mathbb{R}^n$, the exist "renormalization constants" $c_\lambda \in X$ such that $\sum_{\lambda\in\Lambda}(\alpha_\lambda\psi_\lambda(\cdot)+c_\lambda)$ converges in $L^p(K;X)$.

Proof. We may assume that $(\psi_\lambda)_{\lambda\in\Lambda}$ are compactly supported wavelets, since otherwise we can apply $L^p(\mathbb{R}^n;X)$-bounded integral transformations with kernels of the form $\sum \Psi_\lambda(x)\bar{\psi}_\lambda(y)$ (Cor. 2.2) to reduce the matters to this situation. Then, as we saw in the conclusion of the proof of Theorem 1.7, we have $\operatorname{supp}\psi_\lambda \subset Q(\lambda)^*$.

Necessity of (4.2). Writing $b := (b - b_{Q^*})1_{Q^*} + (b - b_{Q^*})1_{(Q^*)^c} + b_{Q^*} =: b_1 + b_2 + b_3$, where $b_{Q^*} := |Q^*|^{-1} \int_{Q^*} b(x)\,\mathrm{d}x$, we find that $\langle b_2, \bar{\psi}_\lambda \rangle = 0$ if $Q(\lambda) \subset Q$ (since then $\operatorname{supp} \psi_\lambda \subset Q^*$), and $\langle b_3, \bar{\psi}_\lambda \rangle = 0$ for all $\lambda \in \Lambda$, since $\int \psi_\lambda(x)\,\mathrm{d}x = 0$. Thus, when $Q(\lambda) \subset Q$, we have

$$\alpha_\lambda = \langle b, \bar{\psi}_\lambda \rangle = \langle (b - b_{Q^*})1_{Q^*}, \bar{\psi}_\lambda \rangle ,$$

and so

$$\begin{aligned} \int_{\mathbb{R}^n} \mathbb{E}_\varepsilon \left| \sum_{Q(\lambda) \subset Q} \varepsilon_\lambda \alpha_\lambda \psi_\lambda(x) \right|_X^p \mathrm{d}x &\leq C \left\| (b - b_{Q^*})1_{Q^*} \right\|^p_{L^p(\mathbb{R}^n;X)} \\ &\leq C \left| Q^* \right| \|b\|^p_{\mathrm{BMO}(\mathbb{R}^n;X)} . \end{aligned}$$

This completes the first half of the proof.

Sufficiency of (4.2). Let $\bar{B}$ be a ball of radius r. We investigate separately the two series

$$\sum_{|Q(\lambda)| \leq |\bar{B}|} \alpha_\lambda \psi_\lambda(x) \qquad \text{and} \qquad \sum_{|Q(\lambda)| > |\bar{B}|} \alpha_\lambda \psi_\lambda(x).$$

Concerning the first series, if $x \in \bar{B}$ and $x \in \operatorname{supp} \psi_\lambda \subset Q(\lambda)^*$ for some x, then $\bar{B} \cap Q(\lambda)^* \neq \emptyset$, and from the size assumption $|Q(\lambda)| \leq |\bar{B}|$ it follows that $Q(\lambda) \subset \bar{B}^\star$, where the $\star$ designates expansion about the same centre by a sufficiently large factor which only depends on the expansion factor implicit in the notation $Q(\lambda)^*$. Thus

$$\begin{aligned} \int_{\mathbb{R}^n} \mathbb{E}_\varepsilon \left| \sum_{\substack{\lambda \in F : |Q(\lambda)| \leq |\bar{B}|, \\ \bar{B} \cap \operatorname{supp} \psi_\lambda \neq \emptyset}} \varepsilon_\lambda \alpha_\lambda \psi_\lambda(x) \right|_X^p \mathrm{d}x &\leq \int_{\mathbb{R}^n} \mathbb{E}_\varepsilon \left| \sum_{\lambda \in F : Q(\lambda) \subset \bar{B}^\star} \cdots \right|_X^p \mathrm{d}x \\ &\leq c\kappa^p \left| \bar{B} \right| . \end{aligned} \tag{4.3}$$

From this estimate, which is uniform for finite sets $F \subset \Lambda$, and the fact that $c_0 \not\subset X$ for X UMD, it follows that the series $\sum \varepsilon_\lambda \alpha_\lambda \psi_\lambda(\cdot)$ (summation over $\lambda \in \Lambda : |Q(\lambda)| \leq |\bar{B}|, \bar{B} \cap \operatorname{supp} \psi_\lambda \neq \emptyset$) converges almost surely (with respect to the ε_λ's) in $L^p(\mathbb{R}^n; X)$. But due to the $L^p(\mathbb{R}^n; X)$-boundedness of the integral transformations with kernels $\sum \varepsilon_\lambda \psi_\lambda(x) \bar{\psi}_\lambda(y)$, it actually converges surely, i.e., $\sum \alpha_\lambda \psi_\lambda(x)$ (summation restricted as above) converges unconditionally. For $x \in \bar{B}$, this series agrees with

$$\sum_{\lambda \in \Lambda, |Q(\lambda)| \leq |\bar{B}|} \alpha_\lambda \psi_\lambda(x),$$

which hence converges unconditionally in $L^p(\bar{B}; X)$.

We then consider summation over $|Q(\lambda)| > |\bar{B}|$. For each fixed size $2^{-jn} = |Q(\lambda)|$, there are at most a bounded number, say m, of dyadic cubes $Q(\lambda)$ such that $Q(\lambda)^* \cap \bar{B} \neq \emptyset$. Moreover, denoting by x_0 the centre of $\bar{B}$, we have for $x \in \bar{B}$

$$|\psi_\lambda(x) - \psi_\lambda(x_0)| \leq |(x - x_0) \cdot \nabla \psi_\lambda(\xi)| \leq C 2^{nj/2+j} r,$$

where r is the radius of $\bar{B}$ and $\lambda = 2^{-j}k + 2^{-j-1}\eta$. From (4.2) it follows that $|\alpha_\lambda|_X \leq C\kappa 2^{-nj/2}$. Combining these observations, it follows that

$$\sum_{|Q(\lambda)|>|\bar{B}|, Q(\lambda)^* \cap \bar{B} \neq \emptyset} |\alpha_\lambda|_X \, |\psi_\lambda(x) - \psi_\lambda(x_0)| \leq \sum_{2^{-jn} > |\bar{B}|} m\kappa 2^{-nj/2} C 2^{nj/2+j} r \leq c\kappa \sum_{2^j < r^{-1}} 2^j r \leq c\kappa, \tag{4.4}$$

and this shows that $\sum_{|Q(\lambda)|>|\bar{B}|} \alpha_\lambda(\psi_\lambda(x) - \psi_\lambda(x_0))$ converges absolutely in X, uniformly on $\bar{B}$; thus $\sum_{|Q(\lambda)|>|\bar{B}|} \alpha_\lambda \psi_\lambda(x)$ converges unconditionally on $L^p(\bar{B}; X)/X$.

The asserted convergence of $\sum \alpha_\lambda \psi_\lambda(x)$ has now been established. Moreover, the estimates (4.3) and (4.4) combined give

$$\int_{\bar{B}} \left| \sum_{|Q(\lambda)| \leq |\bar{B}|} \alpha_\lambda \psi_\lambda(x) + \sum_{|Q(\lambda)| > |\bar{B}|} \alpha_\lambda \left(\psi_\lambda(x) - \psi_\lambda(x_0)\right) \right|_X^p \mathrm{d}x \leq C\kappa^p \left|\bar{B}\right|,$$

which shows the membership of the limit element in $\mathrm{BMO}(\mathbb{R}^n; X)$, and the asserted norm estimate. $\square$

Finally, we wish to exploit the wavelet framework to give a new point-of-view to the H^1-BMO duality in the UMD-valued situation. It should be noted that Fefferman's duality theorem holds in the vector-valued situation under much milder geometric assumptions (see O. Blasco [1]), but requires a different approach then.

Proposition 4.5. *Let X (and then also X') be a UMD-space and $(\psi_\lambda)_{\lambda \in \Lambda}$ (and then also $(\bar{\psi}_\lambda)_{\lambda \in \Lambda}$) a 1-regular wavelet basis of $L^2(\mathbb{R}^n)$. Let*

$$b(x) = \sum_{\lambda \in \Lambda} \alpha'_\lambda \bar{\psi}_\lambda(x) \in \mathrm{BMO}(\mathbb{R}^n; X'), \qquad \alpha'_\lambda = \langle b, \psi_\lambda \rangle \in X',$$

where the convergence is unconditional in $L^p_{\mathrm{loc}}(\mathbb{R}^n; X')/X'$. Then

$$A(f) = A\left(\sum_{\lambda \in \Lambda} \alpha_\lambda \psi_\lambda\right) := \sum_{\lambda \in \Lambda} \alpha'_\lambda(\alpha_\lambda) \tag{4.6}$$

converges unconditionally for every $f = \sum_{\lambda \in \Lambda} \alpha_\lambda \psi_\lambda \in H^1(\mathbb{R}^n; X)$, and defines an element of $H^1(\mathbb{R}^n; X)'$ with $\|A\|_{H^1(\mathbb{R}^n;X)'} \leq \|b\|_{\mathrm{BMO}(\mathbb{R}^n;X')}$.

Conversely, every $A \in H^1(\mathbb{R}^n;X)'$ *is of the form* (4.6)*, where* $\sum_{\lambda\in\Lambda} \alpha'_\lambda \bar{\psi}_\lambda$ *converges in* $L^p_{\mathrm{loc}}(\mathbb{R}^n;X')/X'$ *to* $b \in \mathrm{BMO}(\mathbb{R}^n;X)$*, which satisfies* $\|b\|_{\mathrm{BMO}(\mathbb{R}^n;X')} \le C\,\|A\|_{H^1(\mathbb{R}^n;X)'}$.

Proof. Let $F \subset \Lambda$ be finite. Then

$$\sum_{\lambda\in F} \alpha'_\lambda(\alpha_\lambda) = \int_{\mathbb{R}^n} \left\langle \sum_{\lambda\in F} \alpha'_\lambda \bar{\psi}_\lambda(x), \sum_{\mu\in F} \alpha_\mu \psi_\mu(x) \right\rangle \mathrm{d}x. \tag{4.7}$$

According to Prop. 4.1, the $\mathrm{BMO}(\mathbb{R}^n;X)$-norms of $b_F := \sum_{\lambda\in F} \alpha'_\lambda \bar{\psi}_\lambda$ are bounded by $C\,\|b\|_{\mathrm{BMO}(\mathbb{R}^n;X)}$ for all $F \subset \Lambda$. On the other hand, from Theorem 1.7 it follows that the $H^1(\mathbb{R}^n;X)$-norms of $f_F := \sum_{\mu\in F} \alpha_\mu \psi_\mu$ are uniformly bounded, and also that $\|f_F\|_{H^1(\mathbb{R}^n;X)}$ can be made smaller than any positive ϵ as soon as $F \subset F_\epsilon^c$, where F_ϵ is a sufficiently large set.

Now f_F has an atomic decomposition $\sum a_i$, where $\operatorname{supp} a_i \subset \bar{B}_i$, $\int a_i = 0$, and $\sum \|a_i\|_{L^{p'}(\mathbb{R}^n;X)} \left|\bar{B}_i\right|^{1/p} \le (1+\epsilon)\,\|f\|_{H^1(\mathbb{R}^n;X)}$. Since the atomic series converges in $L^1(\mathbb{R}^n;X)$, and $b_F \in L^\infty(\mathbb{R}^n;X')$, we have

$$\begin{aligned} |\langle b_F, f_F\rangle| \le \sum_{i=1}^\infty |\langle b_F, a_i\rangle| &\le \sum_{i=1}^\infty \|b_F\|_{\mathrm{BMO}(\mathbb{R}^n;X')} \left|\bar{B}_i\right|^{1/p} \|a_i\|_{L^{p'}(\mathbb{R}^n;X)} \\ &\le (1+\epsilon)\,\|b\|_{\mathrm{BMO}(\mathbb{R}^n;X')}\,\|f_F\|_{H^1(\mathbb{R}^n;X)}, \end{aligned}$$

where a standard estimate for the pairing of a BMO-function and an H^1-atom was used in the second step.

From this estimate and the unconditional convergence of f_F to f in $H^1(\mathbb{R}^n;X)$ as $F \uparrow \Lambda$, it follows readily that $\sum_{\lambda\in\Lambda} \alpha'_\lambda(\alpha_\lambda)$ converges unconditionally to a complex number of absolute value at most $\|b\|_{\mathrm{BMO}(\mathbb{R}^n;X')}\,\|f\|_{H^1(\mathbb{R}^n;X)}$. This proves the first assertion.

The converse implication. Let now $A \in H^1(\mathbb{R}^n;X)'$ be arbitrary. Define $\alpha'_\lambda \in X'$ by $\alpha'_\lambda(x) := A(x\psi_\lambda)$ for $x \in X$. Since $\sum_{\lambda\in\Lambda} \alpha_\lambda\psi_\lambda$ converges unconditionally to f in $H^1(\mathbb{R}^n;X)$, we have that $\sum_{\lambda\in\Lambda} A(\alpha_\lambda\psi_\lambda) = \sum_{\lambda\in\Lambda} \alpha'_\lambda(\alpha_\lambda)$ converges unconditionally to $A(f)$. Denote $b_F := \sum_{\lambda\in F} \alpha'_\lambda\bar{\psi}_\lambda$ for finite $F \subset \Lambda$.

We estimate the $\mathrm{BMO}(\mathbb{R}^n;X)$-norm of b_F. Let $\bar{B}$ be a ball, and $f \in L^{p'}(\bar{B};X)$. Then

$$\langle b_F - (b_F)_{\bar{B}}, f\rangle = \langle b_F, f - f_{\bar{B}}1_{\bar{B}}\rangle - \langle (b_F)_{\bar{B}}1_{\bar{B}}, f\rangle + \langle b_F, f_{\bar{B}}1_{\bar{B}}\rangle,$$

and the last two terms are both equal to $\left|\bar{B}\right| \langle (b_F)_{\bar{B}}, f_{\bar{B}}\rangle$. Furthermore, note that $\langle b_F, g\rangle = \langle b_F, g_F\rangle = A(g_F)$ for any $g \in H^1(\mathbb{R}^n;X)$. Thus

$$\begin{aligned} |\langle b_F - (b_F)_{\bar{B}}, f\rangle| = |A((f - f_{\bar{B}}1_{\bar{B}})_F)| &\le \|A\|_{H^1(\mathbb{R}^n;X)'}\,\|(f - f_{\bar{B}}1_{\bar{B}})_F\|_{H^1(\mathbb{R}^n;X)} \\ &\le \|A\|_{H^1(\mathbb{R}^n;X)'}\,\|f - f_{\bar{B}}1_{\bar{B}}\|_{L^{p'}(\mathbb{R}^n;X)} \left|\bar{B}\right|^{1/p}. \end{aligned}$$

Taking the supremum over all $f \in L^{p'}(\bar{B}; X)$ of norm at most 1, and observing that the unit ball of $L^{p'}(\bar{B}; X)$ is norming for $L^p(\bar{B}; X')$, we deduce

$$\|(b_F - (b_F)_{\bar{B}})1_{\bar{B}}\|_{L^p(\mathbb{R}^n;X')} \leq 2\,\|A\|_{H^1(\mathbb{R}^n;X)'}\,\left|\bar{B}\right|^{1/p},$$

and thus $\|b_F\|_{\mathrm{BMO}(\mathbb{R}^n;X)} \leq 2\,\|A\|_{H^1(\mathbb{R}^n;X)'}$. From Prop. 4.1 it follows that this uniform estimate for b_F implies that $b_F \to b$ as $F \uparrow \Lambda$, unconditionally in the space $L^p_{\mathrm{loc}}(\mathbb{R}^n; X')/X'$, and $\|b\|_{\mathrm{BMO}(\mathbb{R}^n;X')} \leq C\,\|A\|_{H^1(\mathbb{R}^n;X)}$. Then, by the first part of the proof, b defines via duality an element $\tilde{A} \in H^1(\mathbb{R}^n; X)'$. It is clear that $\tilde{A}(f) = \langle b_F, f\rangle = A(f)$ if $f = \sum_{\lambda\in F}\alpha_\lambda\psi_\lambda$ and $F \subset \Lambda$ is finite; since such f are dense in $H^1(\mathbb{R}^n; X)$, we see that $A = \tilde{A}$, i.e., A is of the asserted form. $\square$

The previous proposition shows the fact that $H^1(\mathbb{R}^n; X)' = \mathrm{BMO}(\mathbb{R}^n; X')$ for X UMD, which, as mentioned, actually holds under more general conditions. While restricted to the UMD-setting, the present approach has the virtue of providing the explicit formula (4.6) for the evaluation of the duality pairing $\langle b, f\rangle$. Note that the wavelet coefficients α'_λ of b and α_λ of f are uniquely determined by the functions b and f, and moreover explicitly given by the formulae $\alpha'_\lambda = \langle b, \psi_\lambda\rangle$, $\alpha_\lambda = \langle f, \bar{\psi}_\lambda\rangle$. On the other hand, the atomic decomposition of f, in terms of which the H^1-BMO duality is often defined by $\langle b, f\rangle = \sum_{i=1}^{\infty}\langle b, a_i\rangle$ is far from being unique.

From the previous proof we also readily see the following, recalling that UMD spaces are reflexive:

Corollary 4.8. *Let X be a UMD space, and $(\psi_\lambda)_{\lambda\in\Lambda}$ a 1-regular wavelet basis. Then, for every $b \in \mathrm{BMO}(\mathbb{R}^n; X)$, the wavelet expansions $\sum_{\lambda\in F}\langle b, \bar{\psi}_\lambda\rangle\psi_\lambda$ converge unconditionally to b in the weak* topology $\sigma(\mathrm{BMO}(\mathbb{R}^n; X), H^1(\mathbb{R}^n; X'))$ as $F \uparrow \Lambda$.*

References

[1] O. Blasco. Hardy spaces of vector-valued functions: duality. *Trans. Amer. Math. Soc.* **308** #2 (1988), 495–507.

[2] T. Figiel. On equivalence of some bases to the Haar system in spaces of vector-valued functions. *Bull. Pol. Acad. Sci. Math.* **36** #3–4 (1988), 119–131.

[3] ______. Singular integral operators: a martingale approach. In P. F. X. Müller, W. Schachermayer (eds.), *Geometry of Banach Spaces*. Proceedings of the conference held in Strobl, Austria, 1989. London Math. Soc. Lecture Note Ser. 158, Cambridge Univ. Press, 1990.

[4] T. Figiel, P. Wojtaszczyk. Special bases in function spaces. In W. B. Johnson, J. Lindenstrauss (eds.), *Handbook of the Geometry of Banach Spaces*, Vol. I. Elsevier Science, 2001.

[5] Y. Meyer. *Wavelets and operators.* Cambridge Univ. Press, 1992.

[6] Y. Meyer, R. Coifman. *Wavelets: Calderón–Zygmund and multilinear operators.* Cambridge Univ. Press, 1997.

(continued from the back cover)

A453 Marko Huhtanen
Aspects of nonnormality for iterative methods
September 2002

A452 Kalle Mikkola
Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations
October 2002

A451 Marko Huhtanen
Combining normality with the FFT techniques
September 2002

A450 Nikolai Yu. Bakaev
Resolvent estimates of elliptic differential and finite element operators in pairs of function spaces
August 2002

A449 Juhani Pitkäranta
Mathematical and historical reflections on the lowest order finite element models for thin structures
May 2002

A448 Teijo Arponen
Numerical solution and structural analysis of differential-algebraic equations
May 2002

A447 Timo Salin
Quenching-rate estimate for a reaction diffusion equation with weakly singular reaction term
April 2002

A446 Tuomas Hytönen
R-Boundedness is Necessary for Multipliers on H^1
February 2002

A445 Philippe Clément , Stig-Olof Londen , Gieri Simonett
Quasilinear Evolutionary Equations and Continuous Interpolation Spaces
March 2002

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The list of reports is continued inside. Electronical versions of the reports are available at *http://www.math.hut.fi/reports/* .

A458 Tuomas Hytönen
Translation-invariant Operators on Spaces of Vector-valued Functions
April 2003

A457 Timo Salin
On a Refined Asymptotic Analysis for the Quenching Problem
March 2003

A456 Ville Havu , Harri Hakula , Tomi Tuominen
A benchmark study of elliptic and hyperbolic shells of revolution
January 2003

A455 Yaroslav V. Kurylev , Matti Lassas , Erkki Somersalo
Maxwell's Equations with Scalar Impedance: Direct and Inverse Problems
January 2003

A454 Timo Eirola , Marko Huhtanen , Jan von Pfaler
Solution methods for R-linear problems in C^n
October 2002

ISBN 951-22-6511-7
ISSN 0784-3143
Espoo, 2003