Helsinki University of Technology Institute of Mathematics Research Reports

Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja

Espoo 2003 A463

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Marko Huhtanen, Jan von Pfaler: *The real linear eigenvalue problem in* $\mathbb{C}^n$; Helsinki University of Technology Institute of Mathematics Research Reports A463 (2003).

Abstract: *In this paper we study the real linear eigenvalue problem in* $\mathbb{C}^n$. *We present results concerning the location of the eigenvalues of a real linear operator together with structured problems for which we can find the spectrum numerically reliably. We consider ways to achieve savings in computational complexity. Various classes of real linear operators are introduced among which the structure of the spectrum can be regarded, at least partially, as understood. Continuation techniques are implemented for locating components and subsets of the spectrum once an eigenvalue is available.*

AMS subject classifications: 14Q05, 15A04, 65F15

Keywords: $\mathbb{R}$–linear operator in $\mathbb{C}^n$, spectrum, characteristic bivariate polynomial, real algebraic plane curve, path following techniques

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ISBN 951-22-6801-9
ISSN 0784-3143

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1 Introduction

The purpose of this paper is to complement our work on real linear operators by studying the $\mathbb{R}$–linear eigenvalue problem in $\mathbb{C}^n$ in detail. For the background of the study, see [6]. See also [20] and [21] for an operator theoretic approach motivated by applications to planar elasticity.

A real linear operator $\mathcal{M}$ acts on $\mathbb{C}^n$ according to

$$z \mapsto \mathcal{M}(z) = Mz + M_{\#}\overline{z}, \tag{1}$$

for a pair of matrices $M, M_{\#} \in \mathbb{C}^{n \times n}$ called the linear and anti–linear parts of $\mathcal{M}$. The spectrum of $\mathcal{M}$ consists of those points $\lambda \in \mathbb{C} \simeq \mathbb{R}^2$ for which $\lambda I - \mathcal{M}$ is not invertible giving rise to a bounded (possibly empty) real algebraic plane curve of degree $2n$. Belonging in this sense to the realm of real algebraic geometry we have, as opposed to the standard eigenvalue computation corresponding to the special case of having $M_{\#} = 0$, a mildly nonlinear and numerically very challenging eigenvalue problem. A point of departure from the usual real algebro–geometric setting is that, for practical reasons, it is not realistic to assume having the bivariate polynomial determining the spectrum available. This is an analogy of the standard eigenvalue problem where the characteristic polynomial is hardly ever available before the spectrum is. In our setting this is a more serious obstacle, e.g., due to the lack of the fundamental theorem of algebra in the real analytic case.

In this paper we introduce classes of real linear operators that allow us either to solve the eigenvalue problem numerically reliably or lead to significant savings in computational complexity. We consider families of real linear operators for which the structure of the spectrum can be regarded, at least partially, as understood. For instance, polynomials in an anti–linear operator belong to this category. To give another example, we introduce real linear eigenvalue problems possessing various symmetry properties. The members of the most interesting class thus obtained we call symmetric real linear operators. We also suggest numerical methods for locating components of the spectrum for systems of moderate size and outline techniques for finding eigenvalues of large scale problems.

For computational purposes it seems natural to divide the problem into three categories, after the possible structure of the problem has been identified. The first task is to find the whole spectrum when the dimension n of the system is moderate. Some ideas to this end were proposed in [6]. In the second category we are concerned with finding the component to which a given eigenvalue of a real linear operator belongs. Alternatively, we look for all the components of the spectrum intersecting a prescribed line. In this paper we apply continuation techniques to this end. This latter setting is particularly natural when dealing with the symmetric real linear eigenvalue problem. The third task is to find a portion of the spectrum, possibly located inside a prescribed region, when n is assumed to be large.

The paper is organized as follows. In section 2 we consider properties of the spectrum. Since the spectral mapping theorem does not hold for real linear operators, we study under which circumstances it can be partially reestablished. We present bounds on the degree of a given component of the spectrum. In section 3 symmetry properties of the spectrum are considered. We introduce the field of values of a real linear operator. Using this we identify two extremes of real linear eigenvalue problems: in the first one we can expect the spectrum to have many components while in the second one very few (and small), if any. We also give an illustration of a structured eigenvalue problem with a real linear circulant operator. In section 4 we describe our continuation techniques for computing eigenvalues and components of the spectrum. Three numerical examples are presented.

2 Properties of the spectrum of an $\mathbb{R}$–linear operator in $\mathbb{C}^n$

For an $\mathbb{R}$–linear operator $\mathcal{M}(z) = Mz + M_\# \overline{z}$ in $\mathbb{C}^n$ the spectrum $\sigma(\mathcal{M})$ consists of those complex numbers $\lambda = \alpha + i\beta$ for which the matrix $A(\alpha, \beta) \equiv \alpha I - \beta J - A$ is not invertible. Here

$$A = \begin{bmatrix} \mathrm{Re}(M + M_\#) & -\mathrm{Im}(M - M_\#) \\ \mathrm{Im}(M + M_\#) & \mathrm{Re}(M - M_\#) \end{bmatrix} \in \mathbb{R}^{2n \times 2n} \tag{2}$$

denotes the real form of $\mathcal{M}$ and $J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$. For a more detailed exposition, see [6]. We call $\det A(\alpha, \beta)$ the characteristic bivariate polynomial of $\mathcal{M}$. It is of degree $2n$, its zero set is a real algebraic plane curve in $\mathbb{R}^2 \simeq \mathbb{C}$ giving the spectrum of $\mathcal{M}$, and it is monic in the following sense.

Proposition 1 *For any $A \in \mathbb{R}^{2n \times 2n}$ we have*

$$\det A(\alpha, \beta) = (\alpha^2 + \beta^2)^n + \text{ lower order terms }. \tag{3}$$

Proof. Expanding $\det A(\alpha, \beta)$ along the first row we can infer that only those terms in the expansion which are the product of the entries involving the variables α or β each can give rise to monomials of degree $2n$. These terms correspond to paths running from the first row down to the last row, by visiting every column exactly once, through entries involving variables α or β. These are located on the diagonals in the 2-by-2 block structure of $A(\alpha, \beta)$. Only on the north-eastern diagonal $-\beta$ appears while elsewhere we have either either $+\alpha$ or $+\beta$. Each time $-\beta$ term is encountered we have a unique transposition. Hence, all the monomials of degree $2n$ have the coefficient $+1$.

It remains to count the arising monomials. The case α^{2n} is clear so let us consider the coefficient of $\alpha^{2(n-1)}\beta^2$. To this end we need those terms in the expansion of $\det A(\alpha, \beta)$ whose jth factor is $-\beta - a_{j,n+j}$ for $1 \leq j \leq n$. This enforces the $(n - j + 1)$th factor to be $\beta - a_{n+j,j}$. Since j can be chosen in n different ways, we obtain $\binom{n}{1}$. The other binomial coefficients follow by the same reasoning. □

This proposition is of interest since the leading term of $\det A(\alpha, \beta)$ can be used to bound the number of connected components of the spectrum of $\mathcal{M}$. For recent bounds, see [22]. Recall that Harnack's theorem is classical for bounding the number of components of a real nonsingular *projective* plane curve; see, e.g., [2, Chapter 11.6]. For an elementary introduction and references, see [16].

In view of (3), a natural classification problem is to characterize those bounded algebraic plane curves that can appear as the spectrum of an $\mathbb{R}$–linear operator. For $n = 1$ it is readily verified that we can only have a circle due to the fact that then the part consisting of the lower order terms in (3) is a linear bivariate polynomial (and any such a polynomial can appear by choosing an appropriate $\mathbb{R}$–linear operator).

In what follows we will occasionally present bounds on the degree of a component of the spectrum, or find regions of the complex plane where the spectrum is located. This kind of results are of use due to the following consequence of Poincaré's formula [3].

Theorem 2 *Suppose Γ is an algebraic plane curve of degree at most k and D is a disk of radius r. Then the length of $\Gamma \cap D$ is at most $2\pi kr$.*

Since the eigenvalue problem considered is real algebraic, complex analytic techniques are seldom applicable. This is best illustrated by the fact that the spectrum of a real linear operator can be empty. Therefore most of the classical tools for $\mathbb{C}$–linear operators, i.e., for matrices, have to be questioned and reformulated, if possible. To start with, for a real linear operator $\mathcal{M}$ we do not have a spectral mapping theorem so that to find the eigenvalues of a polynomial p in $\mathcal{M}$, the spectrum may need to be computed each time anew. Being enormously elaborate, any results where $\sigma(\mathcal{M})$ can somehow be benefited from are of use. To give an example, in case $\mathcal{M}$ is similar to an upper (lower) triangular $\mathbb{R}$–linear operator under a $\mathbb{C}$–linear similarity transformation, we can determine the spectrum of $p(\mathcal{M})$ in terms of $\sigma(\mathcal{M})$; see [6]. (The spectrum is preserved in a $\mathbb{C}$–linear similarity transformation; in an $\mathbb{R}$–linear similarity transformation the real eigenvalues are preserved [6].)

Remark 1 By a polynomial p in an $\mathbb{R}$–linear operator $\mathcal{M}$ in $\mathbb{C}^n$ we mean $p(\mathcal{M}) = \sum_{j=0}^{k} \alpha_j \mathcal{M}^j$ for $\alpha_j \in \mathbb{C}$ with $0 \leq j \leq k$. Note that factoring p and computing the corresponding real linear operator by performing repeated compositions (after fixing an order for the zeros) differs from $p(\mathcal{M})$ in general. Moreover, polynomials in $\mathcal{M}$ is vector space over $\mathbb{C}$ but not an algebra unless $\mathcal{M}$ is, e.g., $\mathbb{C}$–linear. (To get an algebra, we should, expressing it in terms of the real form A of $\mathcal{M}$, take the algebra generated by A and J over $\mathbb{R}$.)

For the spectral mapping theorem the simplest option is to take a linear polynomial in $\mathcal{M}$. By a pre–multiplication we mean $\mathcal{M} \circ (\mu I)$ for $\mu \in \mathbb{C}$ while the post–multiplication is defined as $\mu I \circ \mathcal{M} = \mu \mathcal{M}$. It is clear how the spectrum behaves in translations and post–multiplications. It is less obvious that this is also so in pre–multiplications:

Proposition 3 *The spectrum of $\mathcal{M} \circ (\mu I)$ equals $\mu\,\sigma(\mathcal{M})$.*

Proof. The case $\mu = 0$ is clear so let $\mu \neq 0$. Then we have $\mathcal{M}(\mu z) = \mu(Mz + \frac{\overline{\mu}}{\mu} M_{\#}\overline{z})$, i.e., the pre-multiplication can be expressed as a post–multiplication. But $z \mapsto Mz + \frac{\overline{\mu}}{\mu} M_{\#}\overline{z}$ is similar to $\mathcal{M}$ under the $\mathbb{C}$–linear similarity transformation $z \mapsto \mu z$. The claim follows since for the post–multiplication the claim is true. $\square$

Second degree polynomials suffice to give an example such that $p(\sigma(\mathcal{M}))$ can not be the spectrum of $p(\mathcal{M})$. For instance, with $n = 1$ let $\mathcal{M}(z) = 2z + \overline{z}$ so that $\sigma(\mathcal{M})$ is the unit circle centered at 2. Take $p(z) = z^2$. Then $p(\sigma(\mathcal{M}))$ is not a circle, which it should in order to be the spectrum of a real linear operator acting on $\mathbb{C}$.

Take a polynomial p. It is not true in general that $|p(\lambda)| \leq ||p(\mathcal{M})||$ for every $\lambda \in \sigma(\mathcal{M})$, which is true for $\mathbb{C}$–linear operators (used, e.g., in [19, Chapter 2.10]). To see this, consider $\mathcal{M}(z) = \frac{-i}{2} z + \frac{i}{2}\overline{z}$ acting on $\mathbb{C}$. This is a nilpotent operator, i.e., we have $\mathcal{M}^2 = 0$, while its spectrum equals the circle of radius $\frac{1}{2}$ centered at $\frac{-i}{2}$.

Regardless of these discouraging observations, the real eigenvalues behave expectedly in the following sense.

Theorem 4 *Let λ be an eigenvalue of $\mathcal{M}$ and p a polynomial. If $\lambda \in \mathbb{R}$, or the kernel of $\lambda I - \mathcal{M}$ contains a $\mathbb{C}$–linear subspace, then $p(\lambda)$ is an eigenvalue of $p(\mathcal{M})$.*

Proof. If $\lambda \in \mathbb{R}$ an eigenvalue and z is the corresponding eigenvector of $\mathcal{M}$, then forming $p(\mathcal{M})z$ gives us $\sum_{j=0}^{k} \alpha_j \lambda^j z$ by the fact that $\mathcal{M}$ is real linear. The second part of the claim follows by choosing z from the $\mathbb{C}$–linear subspace corresponding to the eigenvalue $\lambda \in \mathbb{C}$. $\square$

This implies, in particular, that with the images of the real elements of $\sigma(\mathcal{M})$ we can use path following techniques to locate the corresponding components of $\sigma(p(\mathcal{M}))$; see section 4.

Corollary 5 *Let $\mathcal{N}$ be a polynomial in $\mathcal{M}$. If $\sigma(\mathcal{M}) \cap \mathbb{R} \neq \emptyset$, then $\sigma(\mathcal{N}) \neq \emptyset$.*

If $\mathcal{M}$ is invertible, then one readily verifies that $\mathcal{M}^{-1}$ is a polynomial in $\mathcal{M}$ by, e.g., considering its real form. Hence for this curious polynomial the spectrum can be deduced by knowing $\sigma(\mathcal{M})$ since for an eigenpair (λ, z) of $\mathcal{M}$ we obviously have $\mathcal{M}^{-1}(\lambda z) = \frac{1}{\lambda}\lambda z$. Consequently, we have the following:

Proposition 6 *If $\mathcal{M} : \mathbb{C}^n \to \mathbb{C}^n$ is invertible, then $1/\sigma(\mathcal{M})$ is the spectrum of $\mathcal{M}^{-1}$ and hence a real algebraic plane curve of degree $2n$ at most.*

Aside from forming polynomials in a matrix, its natural counterpart is to consider polynomials in an anti–linear operator. For an anti–linear operator

$$z \mapsto M_{\#}\overline{z} \tag{4}$$

the eigenvalue problem can be solved, at least for problems of moderate size, by finding the real non–negative eigenvalues of the matrix $M_{\#}\overline{M_{\#}}$ [10, Proposition 4.6.6]. For numerical stability it is preferable to determine the spectrum by finding the real eigenvalues of the real form of (4).

Assume thus that $\mathcal{N}(z) = Nz + N_{\#}\overline{z}$ is a polynomial in the anti–linear operator (4). Then we have

$$N = s(M_{\#}\overline{M_{\#}}) \quad \text{and} \quad N_{\#} = M_{\#}q(\overline{M_{\#}}M_{\#}) \tag{5}$$

for polynomials $s(\mu) = \sum_{j=0}^{k} \alpha_j \mu^j$ and $q(\mu) = \sum_{j=0}^{l} \beta_j \mu^j$. Conversely, if N and $N_{\#}$ are of this form, then $\mathcal{N}$ is a polynomial in (4). Even though polynomials in an anti–linear operator do not remain anti–linear, we have an analogy of the $\mathbb{C}$–linear case:

Proposition 7 *Let $\mathcal{M}$ be an anti–linear operator in $\mathbb{C}^n$. Then polynomials in $\mathcal{M}$ is an algebra over $\mathbb{C}$ of dimension $2n$, generically.*

Proof. It is clear that linear combinations of polynomials in $\mathcal{M}$ are again polynomials in $\mathcal{M}$. For a product, let $\mathcal{N}_j(z) = s_j(M_{\#}\overline{M_{\#}})z + M_{\#}q_j(\overline{M_{\#}}M_{\#})\overline{z}$, with $j = 1, 2$, be two polynomials in $\mathcal{M}$. Then, after rearranging the linear and the anti–linear parts, we readily find that $\mathcal{N}_1 \circ \mathcal{N}_2$ is of the same form. Generically $M_{\#}\overline{M_{\#}}$ is nonderogatory, in which case the dimension of the algebra is $2n$. $\square$

A necessary condition for having a polynomial in an anti–linear operator is given by:

Proposition 8 *Let $\mathcal{N}(z) = Nz + N_{\#}\overline{z}$ be a polynomial in an anti–linear operator. Then N commutes with $N_{\#}\overline{N_{\#}}$.*

Proof. Assume $\mathcal{N}$ is a polynomial in (4). Hence we have $N = s(M_{\#}\overline{M_{\#}})$ and $N_{\#} = M_{\#}q(\overline{M_{\#}}M_{\#})$ for polynomials s and q. The claim follows after noticing that the matrix $N_{\#}\overline{N_{\#}}$ is a polynomial in $M_{\#}\overline{M_{\#}}$ and thus commutes with N. $\square$

In particular, if N commutes with $N_{\#}\overline{N_{\#}}$ which is nonderogatory, then $\mathcal{N}$ is a polynomial in the anti–linear operator $z \mapsto N_{\#}\overline{z}$ by the fact that N is necessarily a polynomial in the matrix $N_{\#}\overline{N_{\#}}$ [11, Theorem 4.4.17].

For the eigenvalues of (4) we have a spectral mapping theorem as follows, where we employ the notation introduced in (5).

Proposition 9 *Let λ be an eigenvalue of (4). If $\mathcal{N}(z) = s(M_{\#}\overline{M_{\#}})z + M_{\#}q(\overline{M_{\#}}M_{\#})\overline{z}$, then $s(|\lambda|^2) + \lambda q(|\lambda|^2)$ is an eigenvalue of $\mathcal{N}$.*

Proof. If $M_{\#}\overline{M_{\#}}$ has positive real eigenvalues, then the spectrum of (4) is non–empty consisting of circles centered at the origin [11, Chapter 4.6]. Take a point $\lambda = re^{i\theta}$ belonging to such a circle and let $z \in \mathbb{C}^n$ be a corresponding eigenvector. Then $\mathcal{N}(z) = \gamma z$ with

$$\gamma = \sum_{j=0}^{k} \alpha_j |\lambda|^{2j} + \sum_{j=1}^{l} \beta_j \lambda |\lambda|^{2j} = \sum_{j=0}^{\max\{k,l\}} (\widehat{\alpha}_j(r) + \widehat{\beta}_j(r)\lambda) = A(r) + B(r)\lambda, \tag{6}$$

where the coefficients $\widehat{\alpha}_j$ and $\widehat{\beta}_j$ have an obvious relation with α_j and β_j depending only on r. $\square$

The coefficients A, B in (6) depend only on r so that, when $\theta \in [0, 2\pi)$ varies, we obtain a circle centered at $A(r)$. This circle reduces to a point in case $B(r) = 0$. Hence the spectrum of $p(\mathcal{M})$ can be very small compared with $\sigma(\mathcal{M})$, and vice versa.

Unfortunately a polynomial in an anti–linear operator can have additional eigenvalues to those given by Proposition 9.

Example 1 Assume the spectrum of $z \mapsto M_{\#}\overline{z}$ is empty (this is so if, for instance, $M_{\#} = [\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}]$) while the polynomial is such that $N_{\#} = 0$. Clearly then $\sigma(\mathcal{N}) \neq \emptyset$.

A straightforward way to determine these additional eigenvalues of a polynomial in (4) is to employ the canonical form of Youla [23]. For canonical forms of anti–linear operators, see [9]. That is, we have $M_{\#} = QYQ^T$ with a unitary matrix Q such that $Y = \left[\begin{smallmatrix} \triangle & B \\ 0 & \Omega \end{smallmatrix}\right]$ with an upper triangular matrix $\triangle$ of size $(n-2k)$-by-$(n-2k)$, and Ω is a $2k$-by-$2k$ upper block triangular matrix with 2-by-2 diagonal blocks. These blocks can be arranged to have further structure [23] but for us this suffices. This is due to the fact that we have $Q^* \circ \mathcal{N} \circ (Qz) = \widetilde{Y}z + \widetilde{Y_{\#}}\overline{z}$ with an analogous block structure. Hence, to determine the spectrum of $\mathcal{N}$ we need to solve a real linear 2-by-2 eigenvalue problem k times. To this end we only compute the corresponding blocks of $\widetilde{Y}$ and $\widetilde{Y_{\#}}$; no other entries need to be found. In particular, we have:

Proposition 10 *The spectrum of a polynomial in an anti–linear operator is the union of real algebraic plane curves of degree* 4 *at most.*

Although the computation of the Youla decomposition is beyond the scope of this paper, let us make a few remarks concerning it. If $M_{\#} = Q_1 R_1$ is the QR–factorization of $M_{\#}$, then $R_1\overline{Q_1}$ is unitarily congruent to $M_{\#}$. Repeating this, we obtain a "congruent QR–iteration" such that with $Z_1 = M_{\#}$ we compute

$$\begin{aligned} \textbf{for} \quad & k = 1, 2, ..., K \\ & Z_k = Q_k R_k \\ & Z_{k+1} = R_k \overline{Q_k} \\ \textbf{end} \quad & \end{aligned}$$

for approximating Y while $Q_1 \cdots Q_{K-1} Q_K \approx Q$. With this plain iteration the convergence is not fast enough but this idea might be used as a starting point for devising a more efficient algorithm. Based on preliminary experimenting, analogously to the standard QR–iteration, it seems advisable to run the iteration with $\widehat{M_{\#}} = Q_0^* M_{\#} \overline{Q_0}$, where Q_0 is unitary such that $\widehat{M_{\#}}$ is a Hessenberg matrix. For finding such a Q_0, see [6].

Remark 2 Any normal matrix is a polynomial in a Hermitian matrix; see, e.g., [12]. Pushing this point of view farther, any Hermitian matrix is a polynomial in an anti–linear operator (4) with ${M_\#}^T = M_\#$. Collecting polynomials in this manner can be regarded as too narrow a generalization of normality since the existence of 2-by-2 blocks in the Youla decomposition is an intrinsic property of anti–linearity. Accepting this, it seems natural to allow congruence normal matrices here. Recall that $M_\#$ is congruence normal if $M_\# \overline{M_\#}$ is normal. A more tangible equivalent characterization can be given in terms of the canonical form of Youla of $M_\#$ [7]. See also [9, Section 2.]. In summary, collect those $\mathcal{M}(z) = Mz + M_\# \overline{z}$ for which M is normal and $M_\#$ is congruent normal such that M and $M_\# \overline{M_\#}$ commute. This class provides a candidate for the concept of normality among $\mathbb{R}$–linear operators in $\mathbb{C}^n$. For example, any $\mathbb{R}$–linear operator with circulant parts belongs to this class (see Example 2 and (7) below). By using [9, (2.7) and (2.8)] together with Lemma 11 below, the spectrum of such an operator consists of circles.

In view of Theorem 4, to an eigenvalue λ of $\mathcal{M}$ there is always related an $\mathbb{R}$–linear invariant subspace of $\mathcal{M}$ arising from the solution set of $\mathcal{M}(z) - \lambda z = 0$, say, of dimension r. Let m be the dimension of the largest $\mathbb{C}$–linear subspace it contains. The resulting multiplicity index $(r/2, m)$ is of interest (see [6]) since the spectral mapping theorem holds for this eigenvalue in case $m > 0$.

Example 2 Let $M, M_\# \in \mathbb{C}^{n \times n}$ be circulant matrices both. Then $\mathcal{M}$ has at least one or two eigenvectors with real components according as n is odd or even (since the matrices M and $M_\#$ have such eigenvectors simultaneously). Hence, for the correspoding eigenvalue of $\mathcal{M}$ the spectral mapping holds.

With circulant matrices we actually have a complete understanding of the spectrum. To this end, denote by $F_n \in \mathbb{C}^{n \times n}$ the Fourier matrix (see, e.g., [4, p.32]). Assume $M = F_n D F_n^*$ and $M_\# = F_n D_\# F_n^*$ are circulant matrices both. Then we have $\mathcal{N}(z) = F_n^* \mathcal{M}(F_n z) = Dz + D_\# \Gamma \overline{z}$ with a diagonal D and

$$\Gamma = F_n^* \overline{F_n} = F_n^{2*} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \end{bmatrix}, \quad \text{so that} \quad D_\# \Gamma = \begin{bmatrix} d^\#_{1,1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & d^\#_{2,2} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 \\ 0 & d^\#_{n,n} & \cdots & 0 \end{bmatrix}. \qquad (7)$$

Clearly, span $\{e_1\}$ is a $\mathbb{C}$–linear invariant subspace of $\mathcal{N}$, and so are span $\{e_j, e_{n+2-j}\}$, for $2 \leq j \leq n$ (if n is even, then one of these latter subspaces is one dimensional as well). Namely, ignoring the first row and column of D and $D_\# \Gamma$, the linear part is diagonal while the anti–linear part is anti–diagonal, i.e., nonzero entries appear only on the diagonal joining the left lower corner with the right upper one. It is easy to find a closed form solution to the spectrum for $\mathbb{R}$–linear operators of this type since it suffices to consider 2-by-2 eigenvalue problems with the following structure (the case $d_3 = 0$ is obvious).

Lemma 11 *For $d_1, d_2, d_3, d_4 \in \mathbb{C}$, with $d_3 \neq 0$, the spectrum of the problem*

$$\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 & d_3 \\ d_4 & 0 \end{bmatrix} \begin{bmatrix} \overline{z_1} \\ \overline{z_2} \end{bmatrix} = \lambda \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

is $\left\{\lambda \in \mathbb{C} : (\lambda - d_1)(\overline{\lambda} - \overline{d_2}) = \overline{d_3} d_4\right\}$.

Hence, in the prescribed circulant case the spectrum can be computed fast and fairly reliably since only a single (well–known) unitary similarity transformation needs to be performed. In particular, analogously to circulant matrices, we have a closed form solution of the spectrum.

A closed form solution of the spectrum is readily computable also when the linear part is anti–diagonal and the anti–linear part is diagonal.

Remark 3 In (7) we employed the fact that N is diagonal and $N_\#$ is a PD–matrix [4, Section 5.3], i.e., a product of a diagonal matrix and a permutation. With this structure, more generally, a closed form solution for the spectrum can be found, after reordering the basis together with backward substituting and conjugating the equations in the problem $Nz + N_\# \overline{z} = \lambda z$.

Lemma 11 illustrates well what sort of difficulties one can encounter when solving real linear eigenvalue problems in finite precision by using the characteristic bivariate polynomial $\det A(\alpha, \beta)$ only. Assuming $d_1 = d_2$ and $d_3 = d_4$, we clearly have a circle. However, a tiny perturbation in either d_3 or d_4 that yields a non–real $\overline{d_3} d_4$ leads to an unsolvable equation and vanishing of the spectrum. In view of this, for continuity we have:

Proposition 12 *The spectrum function $\mathcal{M} \mapsto \sigma(\mathcal{M})$ is upper semicontinuous, that is, for every open set U containing $\sigma(\mathcal{M})$ there exists $\delta > 0$ such that $||\mathcal{M} - \mathcal{N}|| < \delta$ implies $\sigma(\mathcal{N}) \subset U$.*

Proof. We proceed by contradiction by assuming there exists a sequence $\mathcal{N}_j$ converging to $\mathcal{M}$ with $\lambda_j \in \sigma(\mathcal{N}_j)$ contained in the complement of U. Since the spectra are uniformly bounded, we can assume that λ_j converges to λ. Clearly $\mathcal{M} - \lambda I$ is invertible. Since the set of invertible $\mathbb{R}$–linear operators in $\mathbb{C}^{n \times n}$ is open (identified with the set of invertible matrices in $\mathbb{R}^{2n \times 2n}$), we have a contradiction. □

Aside from associating an index to a single eigenvalue of a real linear operator $\mathcal{M}(z) = Mz + M_\# \overline{z}$, we are interested in families of eigenvalues of $\mathcal{M}$ (see Theorem 2). More precisely, to classify components (or subsets) of the spectrum, a $\mathbb{C}$–linear subspace $V \subset \mathbb{C}^n$ is said to be invariant for $\mathcal{M}$ if $\mathcal{M}(V) \subset V$. This is readily seen to hold if and only if $M(V) \subset V$ and $M_\#(\overline{V}) \subset V$ [6]. For an invariant subspace V of $\mathcal{M}$ let us denote the restriction of $\mathcal{M}$ to V by $\mathcal{M}_{|V}$.

Definition 13 *$\Gamma \subset \mathbb{C}$ is a geometric component of $\sigma(\mathcal{M})$ if $\Gamma = \sigma(\mathcal{M}_{|V})$ for a $\mathbb{C}$–linear invariant subspace V of $\mathcal{M}$.*

If $\mathcal{M}$ has a $\mathbb{C}$–linear invariant subspace of dimension k, then its spectrum has a real algebraic subvariety of degree $2k$ at most. For small k one can actually consider computing the characteristic bivariate polynomial of $\mathcal{M}_{|V}$ after first finding of order k^2 of its eigenvalues accurately. The algorithms of [15] can be used thereon.

Denote by $\deg(M)$ the degree of the minimal polynomial of $M \in \mathbb{C}^{n\times n}$.

Proposition 14 *Any $\lambda \in \sigma(\mathcal{M})$ is contained in a geometric component of $\sigma(\mathcal{M})$ of degree $2\deg(M)(\operatorname{rank}(M_\#)+1)$ at most.*

Proof. Let z be an eigenvector corresponding to an eigenvalue λ of $\mathcal{M}$. According to the proof of [6, Proposition 3.7], z belongs to an invariant subspace of $\mathcal{M}$ of dimension $\deg(M)(\operatorname{rank}(M_\#)+1)$ at most. Hence, λ belongs to the spectrum of $\mathcal{M}$ restricted to this subspace. $\square$

Recall that we always have $\deg(M) \leq \operatorname{rank}(M)+1$.

Any square matrix is consimilar to a real matrix [9]. Finding this transformation is of interest since by the same technique we have:

Proposition 15 *Let $SM_\#\overline{S}^{-1} = R_\# \in \mathbb{R}^{n\times n}$ for an invertible $S \in \mathbb{C}^{n\times n}$. Then any $\lambda \in \sigma(\mathcal{M})$ is contained in a geometric component of $\sigma(\mathcal{M})$ of degree*

$$4\deg(R_\#)(\operatorname{rank}(M)+1)$$

at most.

Proof. We can consider $S \circ \mathcal{M} \circ S^{-1}(z) = Nz + R_\#\overline{z}$. For this, let z be an eigenvector corresponding to λ. Then $z_1 = \mathcal{N}(z) = n_1 + R_\#\overline{z}$ with n_1 in the range of N. Also $\mathcal{N}(z_1) = n_2 + R_\#(\widehat{n_2} + R_\# z)$ with n_2 and $\widehat{n_2}$ in the range of N and $\overline{N}$, respectively. Continuing this inductively, we get vectors that are linear combinations of vectors belonging the block Krylov subspaces

$$\mathcal{K}(R_\#; N),\ \mathcal{K}(R_\#; \overline{N}),\ \mathcal{K}(R_\#; z) \text{ and } \mathcal{K}(R_\#; \overline{z}).$$

Since the dimension of the span of these subspaces is $2\deg(R_\#)(\operatorname{rank}(M)+1)$ at most, the claim follows. $\square$

Let us emphasize that the degree of a matrix can change in a consimilarity transformation even from n to 1.

In what follows, note that the linear and anti–linear parts of $\mathcal{N} = S \circ \mathcal{M} \circ S^{-1}$ can commute while those of $\mathcal{M}$ do not.

Theorem 16 *Assume $\mathcal{N} = S \circ \mathcal{M} \circ S^{-1}$ for an invertible $S \in \mathbb{C}^{n\times n}$ such that $N = \mu I + \kappa R$ with $R \in \mathbb{R}^{n\times n}$ and $\mu, \kappa \in \mathbb{C}$. If $NN_\# = N_\# N$, then any $\lambda \in \sigma(\mathcal{M})$ is contained in a geometric component of $\sigma(\mathcal{M})$ of degree $2\deg(M)$ at most.*

Proof. Since the spectrum is preserved in a $\mathbb{C}$–linear similarity transformation, we can consider $\mathcal{N}$. Because the spectral mapping theorem holds for the transformation $\frac{1}{\kappa} \circ (\mathcal{N} - \mu I)$, we can assume $\mu = 0$ and $\kappa = 1$, i.e., that N is real. The degree of a component of the spectrum clearly does not change in this latter transformation either.

Consider $\lambda \in \sigma(\mathcal{N})$ and assume z is a corresponding eigenvector. Set

$$\mathcal{K}_j(N; z) = \operatorname{span}\{z, Nz, \ldots, N^{j-1}z\}. \tag{8}$$

We prove by induction that $\mathcal{N}(\mathcal{K}_j(N; z)) \subset \mathcal{K}_{j+1}(N; z)$ for $j \geq 1$.

First we have $\mathcal{N}((\rho + i\psi)z) = (\rho - i\psi)\lambda z + i\,2\psi Nz \in \operatorname{span}\{z, Nz\}$ for any $\rho, \psi \in \mathbb{R}$. Hence, assume the claim is true with j. Take $p(N)z$ for a polynomial of degree $j-1$ at most. Denote by $\overline{p}$ the polynomial obtained by conjugating the coefficients of p. By the fact that N and $N_\#$ commute, we have $\mathcal{N}(p(N)z) = Np(N)z + \overline{p}(N)N_\#\overline{z}$. The first term is clearly in $\mathcal{K}_{j+1}(N; z)$. The second one is also since $N_\#\overline{z} = \lambda z - Nz$. Consequently, λ belongs to the spectrum of $\mathcal{N}$ restricted to $\mathcal{K}_n(N; z)$ which is of dimension $\deg(N) = \deg(M)$ at most. $\square$

As a special case, this yields the fact that the spectrum consists of circles for $\mathcal{M}$ whose linear part is a multiple of the identity; see [6]. Actually, as the proof demonstrates, the degree of the geometric component is less than anticipated in case the dimension of the subspace (8) is less than $\deg(M)$ for $j = n$.

By the same technique we have:

Theorem 17 *Assume $\mathcal{N} = S \circ \mathcal{M} \circ S^{-1}$ for an invertible $S \in \mathbb{C}^{n\times n}$ such that $N = \mu I + \kappa R$ and $N_\# = \mu_\# I + \kappa_\# R_\#$, with R, $R_\# \in \mathbb{R}^{n\times n}$ and μ, κ, $\mu_\#$, $\kappa_\# \in \mathbb{C}$. If $NN_\# = N_\# N$, then any $\lambda \in \sigma(\mathcal{M})$ is contained in a geometric component of $\sigma(\mathcal{M})$ of degree $4\deg(N_\#)$ at most.*

Proof. Since the spectrum is preserved in a $\mathbb{C}$–linear similarity transformation, we can again consider $\mathcal{N}$.

Let z be an eigenvector corresponding to λ. Then $\mathcal{N}((\rho + i\psi)z) = (\rho + i\psi)\lambda z - i\,2\psi N_\#\overline{z} \in \operatorname{span}\{z, N_\#\overline{z}\}$ for any $\rho, \psi \in \mathbb{R}$. Similarly we have $\mathcal{N}((\rho + i\psi)N_\#\overline{z}) \in \operatorname{span}\{N_\# z, N_\#\overline{z}, {N_\#}^2 z\}$ by using the commutativity of N and $N_\#$, and $\mathcal{N}(z) = \lambda z$ together with the fact that $N = \mu I + \kappa R$ and $N_\# = \mu_\# I + \kappa_\# R_\#$, where R, $R_\# \in \mathbb{R}^{n\times n}$. By continuing this inductively, we can deduce that

$$V = \operatorname{span}_{j,k\geq 0}\{{N_\#}^j z, {N_\#}^k \overline{z}\} \tag{9}$$

is an invariant subspace of $\mathcal{N}$ containing z. Hence $\mathcal{N}_{|V}$ has λ as its eigenvalue and because the dimension of V is at most $2\deg(N_\#)$, we have the claim. $\square$

For a sharper bound, the degree is determined by the dimension of (9).

If the dimension n is large and the spectrum of $\mathcal{M}$ has a geometric component of moderate degree, then an attractive option is to employ iterative methods to locate it. Real linear operators characterized by the preceding results are ideal in this respect although it is not clear how the iteration should be realized in practice.

The above theorem also shows that the effect of an *anti–translation* $M_\# = \kappa I$ to the spectrum of a standard $\mathbb{C}$–linear eigenvalue problem $Mz = \lambda z$ is not too severe in case M is a translation and a rotation of a real matrix. Actually, if M is a general complex matrix and $\lambda \in \sigma(M)$ is such that the intersection of the null spaces of $\lambda I - M$ and $\overline{\lambda} I - \overline{M}$ is nontrivial, then the spectrum of $\mathcal{M}(z) = Mz + \kappa\overline{z}$ contains a circle of radius $|\kappa|$ centered at λ (use [6, Proposition 2.10]). Hence, if M is real, then under an anti–translation all its real eigenvalues extend to be circles.

Let $\mathcal{M}$ be invertible. Plainly, V is an invariant subspace over $\mathbb{C}$ for $\mathcal{M}$ if and only V is an invariant subspace over $\mathbb{C}$ for $\mathcal{M}^{-1}$. Consequently, if any of the above conditions hold for $\mathcal{M}^{-1}$ instead, then we can make the same conclusion regarding $\mathcal{M}$, and vice versa. Let us illustrate this with the following:

Example 3 Consider the inverse of $\mathcal{M}(z) = Mz + M_{\#}\overline{z}$ with $M = cI$, for a nonzero $c \in \mathbb{C}$ and $M_{\#}$ of rank k. Then we have $\mathcal{M}^{-1}(z) = Nz + N_{\#}\overline{z}$ with $N_{\#}$ of rank k and $N = \frac{1}{c}I + F$, where F is of rank k as well [6]. Hence Proposition 14 gives a pessimistic bound when used with $\mathcal{M}^{-1}$.

3 The symmetric and other structured $\mathbb{R}$–linear eigenvalue problems in $\mathbb{C}^n$

In what follows we will consider various cases where we have symmetries in the location of the spectrum of a real linear operator. This type of results are of interest, for example, for reducing computational complexity of finding eigenvalues numerically.

Analogously to the $\mathbb{C}$–linear case, the spectrum is symmetrically located with respect to the real axis when we are dealing with real matrices.

Proposition 18 *Assume $\mathcal{N} = S \circ \mathcal{M} \circ S^{-1}$ for an invertible $S \in \mathbb{C}^{n\times n}$. If $N, N_{\#} \in \mathbb{R}^{n\times n}$, then $\sigma(\mathcal{M})$ is symmetrically located relative to the real axis.*

Proof. Since N and $N_{\#}$ are real, we have by conjugating

$$Nz + N_{\#}\overline{z} = \lambda z \iff N\overline{z} + N_{\#}z = \overline{\lambda}\overline{z},$$

so that the claim follows by the fact that $\sigma(\mathcal{M}) = \sigma(\mathcal{N})$. □

Similarly, if N and $N_{\#}$ are pure imaginary, then

$$\lambda \in \sigma(\mathcal{M}) \iff -\overline{\lambda} \in \sigma(\mathcal{M}).$$

Example 4 For a curious case of symmetry relative to the real axis, let $M \in \mathbb{C}^{n\times n}$ be Hermitian and consider the "real part" of the $\mathbb{C}$–linear map $z \mapsto Mz$, i.e., $\mathcal{M}(z) = \frac{1}{2}Mz + \frac{1}{2}\overline{M}\overline{z}$. Then $\sigma(\mathcal{M})$ is symmetrically located with respect to the real axis. To see this, the real condition for an eigenpair gives $A\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \alpha\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] - \beta J\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. The assumptions force the $(2,1)$-block and $(2,2)$-block of A to be zero matrices. This implies $y = \frac{-\beta}{\alpha}x$, so that $(A_{11} - \frac{\beta}{\alpha}A_{12})x = (\alpha + \frac{\beta^2}{\alpha})x$, where A_{11} and A_{12} are the $(1,1)$-block and $(1,2)$-block of A, respectively. Now A_{11} is symmetric while A_{12} is skew–symmetric. Therefore $\alpha + \frac{\beta^2}{\alpha}$ is also an eigenvalue of $(A_{11} - \frac{\beta}{\alpha}A_{12})^T = A_{11} + \frac{\beta}{\alpha}A_{12}$, which yields the claimed symmetry of the spectrum.

The above proposition involved a $\mathbb{C}$–linear similarity transformation since the spectrum is preserved under it. In an anti–linear similarity transformation the spectrum is reflected across the real axis as follows.

Proposition 19 *Let $\mathcal{N} = \mathcal{S} \circ \mathcal{M} \circ \mathcal{S}^{-1}$ with an invertible $\mathcal{S}(z) = S_\# \overline{z}$. Then $\sigma(\mathcal{N}) = \overline{\sigma(\mathcal{M})}$.*

Proof. We have $\mathcal{S}^{-1}(z) = \overline{S_\#}^{-1}\overline{z}$. Hence

$$\mathcal{S} \circ \mathcal{M} \circ \mathcal{S}^{-1}(z) = \overline{S_\#}^{-1}\overline{M S_\#} z + \overline{S_\#}^{-1}\overline{M_\#} S_\# \overline{z}$$

so that $\mathcal{N}$ is similar to $z \mapsto \overline{M} z + \overline{M_\#}\overline{z}$ under the $\mathbb{C}$–linear similarity transformation $z \mapsto \overline{S_\#} z$. $\square$

In what follows we will drop the $\mathbb{C}$–linear similarity transformation to simplify the statements. Hence, all the results concerning the location of the spectrum hold more generally for $\mathcal{N} = S \circ \mathcal{M} \circ S^{-1}$ with an invertible $S \in \mathbb{C}^{n\times n}$.

Let M be normal with the property that there exists a function $p : \mathbb{C} \to \mathbb{C}$ with

$$p(z, \overline{z}) = z - \alpha\overline{z} - \beta, \quad \text{where} \quad \alpha, \beta \in \mathbb{C}, \tag{10}$$

annihilating M ($\overline{z}$ corresponds to taking the adjoint). Such normal matrices are denoted by $\mathcal{N}_1$ since their so–called *minimal polyanalytic polynomial* is of degree one; see [13, 14]. Equivalently, their spectrum is located on a line.

Example 5 If $M = e^{i\theta} H + \mu I$ for a Hermitian matrix H, $\theta \in [0, 2\pi)$ and $\mu \in \mathbb{C}$, then $M \in \mathcal{N}_1$ with $\alpha = e^{i2\theta}$ and $\beta = \mu - e^{i2\mu}\overline{\mu}$.

Using this notation, we have a very "scarce" spectrum in the following case.

Proposition 20 *Assume $M \in \mathcal{N}_1$ and ${M_\#}^T = -M_\#$. Then $\sigma(\mathcal{M})$ is finite and located on the line $y = \frac{\sin\theta}{\cos\theta} x + \operatorname{Re}\mu$.*

Proof. For any $\mu_1, \mu_2 \in \mathbb{C}$ the eigenvalues of $\mu_1 \mathcal{M} + \mu_2 I$ are obtained from the eigenvalues of $\mathcal{M}$ under the transformation $z \mapsto \mu_1 z + \mu_2$ and vice versa. Hence, we can assume that M is skew–Hermitian and ${M_\#}^T = -M_\#$. The claim follows from the fact that then the spectrum of $\mathcal{M}$ is located in the imaginary axis because the real form of $\mathcal{M}$ is skew–symmetric. It must be finite since the imaginary axis does not contain any other algebraic curves as a proper subset (it must be proper because the spectrum is bounded). $\square$

In the proof we used the fact that if the spectrum belongs to an unbounded component of a real algebraic plane curve, then it must be finite. Knowing finiteness of the spectrum a priori is important for computational purposes since algorithms for solving $\mathbb{R}$–linear eigenvalue problems in $\mathbb{C}^n$ can be expected to have serious difficulties in finding zero dimensional or tiny one dimensional components of the spectrum. In particular, a real linear eigenvalue problem with the structure of Proposition 20 can be solved reliably since the curve on which the eigenvalues are located is known in advance. For $\mathbb{R}$–linear operators of this type we typically have less than $2n$ eigenvalues (see Corollary 28 below).

According to Proposition 20, if M is Hermitian and ${M_\#}^T = -M_\#$, then the spectrum is located on the real axis. This structure can be regarded as an extension of the standard Hermitian one. Namely then $z^* M_\# \overline{z} = 0$ for any $z \in \mathbb{C}^n$, so that the quadratic form

$$z \mapsto z^* \mathcal{M}(z) = z^* M z + z^* M_\# \overline{z}$$

coincides with the Hermitian quadratic form $z \mapsto z^* M z$ attaining thereby only real values. (See also Corollary 27 below.)

Assume $\mathcal{M}(z) = Mz + M_\# \overline{z}$ such that $M \in \mathcal{N}_1$ while ${M_\#}^T = M_\#$ instead. In particular, if M is Hermitian (skew–Hermitian), then the real form of $\mathcal{M}$ is symmetric (Hamiltonian). First, the arising structure is preserved not only in translations and anti–translations of $\mathcal{M}$ but also in pre– and post–multiplications by a scalar. Moreover, we have:

Proposition 21 *Assume $M \in \mathcal{N}_1$ and ${M_\#}^T = M_\#$. Then $\sigma(\mathcal{M})$ is nonempty and symmetrically located relative to the line $y = \frac{\sin\theta}{\cos\theta} x + \operatorname{Im}\mu$.*

Proof. After scaling and a translation, we can assume that M is Hermitian and ${M_\#}^T = M_\#$. If A denotes the corresponding real form, then $A^T = A$ holds. Hence A has real eigenvalues and thereby $\sigma(\mathcal{M}) \neq \emptyset$. Moreover, since $(\beta J + A)^T = -\beta J + A$, the eigenvalues of $\mathcal{M}$ appear in pairs $\alpha \pm i\beta$. $\square$

This provides another natural way of extending the standard Hermitian eigenproblem once we set:

Definition 22 *For $\mathcal{M}(z) = Mz + M_\# \overline{z}$ let $\widetilde{\mathcal{M}}(z) = M^* z + {M_\#}^T \overline{z}$*

For this operation $\sigma(\widetilde{\mathcal{M}}) = \overline{\sigma(\mathcal{M})}$ holds [6].

We call a real linear operator symmetric if $\widetilde{\mathcal{M}} = \mathcal{M}$. Then the spectrum is not only symmetrically located with respect to the real axis but it is necessarily nonempty. We have at least one real eigenvalue, and $2n$ generically, due to the fact that the real form $A \in \mathbb{R}^{2n \times 2n}$ of $\mathcal{M}$ is a symmetric matrix. These particular eigenvalues yield natural starting points to locate the corresponding components of the spectrum with path following techniques; see section 4. Having as many as $2n$ real eigenvalues can be regarded as quite exceptional among real linear operators in $\mathbb{C}^n$. For comparison, the expected number of real eigenvalues of a random matrix $A \in \mathbb{R}^{2n \times 2n}$ with independent standard normal entries is $\sqrt{4n/\pi}$ as $n \to \infty$; see [5].

Example 6 Since a symmetric $\mathbb{R}$–linear operator in $\mathbb{C}^n$ has $2n$ real eigenvalues generically, the operator $\mathcal{M} : \mathbb{C}^2 \mapsto \mathbb{C}^2$ of Lemma 11 is non–generic in the symmetric case by having at most 2 real eigenvalues. Also a tiny structure preserving perturbation changes the spectrum from a circle of radius $|d_3|$ centered at d_1 (have $d_1 = d_2$ and $d_3 = d_4 \neq 0$) to two distinct points on the real axis (have $d_1 \neq d_2$). Generically we can expect a pair of connected eigenvalues to remain connected in small perturbations by regarding the spectrum as a level curve of a bivariate polynomial.

Based on the diagonal case and on numerical experimenting with n of moderate size, initially we expected all the components of $\sigma(\mathcal{M})$ to intersect the real axis in the symmetric case. Eventually we found a numerical example showing that this is not true. However, since the spectrum of a symmetric real operator in $\mathbb{C}^n$ is particularly "rich" in components (see Example 10), having at least n of them generically, their respective positioning can be used with Bézout's theorem to exclude regions where non–intersecting components can not appear. A simple example of this is:

Proposition 23 *Let $\mathcal{M} : \mathbb{C}^n \to \mathbb{C}^n$ be symmetric such that $\sigma(\mathcal{M})$ has n distinct components crossing the real axis. Assume that these components can be ordered such that the jth component encloses $j-1$ components, for $j = n, \ldots, 2$. Then $\sigma(\mathcal{M})$ does not have any other components.*

Proof. Take a point on the real axis that is enclosed by the first component. Then any straight line intersects the n components in $2n$ points. Hence by Bézout's theorem the line cannot intersect any other components of $\sigma(\mathcal{M})$. □

More generally, we can exclude sectors of the complex plane as soon as we manage to intersect $2n$ eigenvalues with a pencil of straight lines. Although beyond the scope of this paper, a fast construction of these pencils is an interesting problem in computational geometry.

For a more linear algebraic scheme to exclude sets of the complex plane where eigenvalues can not be located, set

$$C_l(M, M_\#) = \left\{\lambda \in \mathbb{C} : |m_{l,l} - \lambda| = |m^\#_{l,l}|\right\}$$

for an $\mathbb{R}$–linear operator $\mathcal{M}(z) = Mz + M_\# \overline{z}$. Then define

$$\mathbf{C}_l(M, M_\#) = \left\{ \lambda \in \mathbb{C} : \operatorname{dist}\bigl(\lambda, C_l(M, M_\#)\bigr) \leq \sum_{j=1, j\neq l}^{n} (|m_{l,j}| + |m^\#_{l,j}|) \right\} \quad (11)$$

for $1 \leq l \leq n$. With these "Geršgorin's annuli" we can guarantee the following:

Theorem 24 *Let $\mathcal{M}$ be real linear in $\mathbb{C}^n$. Then $\sigma(\mathcal{M}) \subset \cup_{j=1}^n \mathbf{C}_j(M, M_\#)$.*

Proof. Let $z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ be an eigenvector corresponding to λ. Take a unitary diagonal matrix $S \in \mathbb{C}^{n\times n}$ such that all the entries of $w = Sz$ are real and consider $\mathcal{N} = S \circ \mathcal{M} \circ S^{-1}$. First, each entry of N and $N_\#$ has the same modulus as those of M and $M_\#$, respectively. Moreover, the diagonal entries of M and N equal. Assume $|w_l| = \max_{1\leq j\leq n} |w_j|$ and consider the lth row of $\lambda w - \mathcal{N}(w) = 0$. Since $w \in \mathbb{R}^n$, this yields $|\lambda - m_{l,l} - n^\#_{l,l}| \leq \sum_{j=1,j\neq l}^n (|m_{l,j}| + |m^\#_{l,j}|)$ after dividing by $|w_l|$ and using the triangle inequality as in the proof of Geršgorin's theorem. (For Geršgorin's theorem, see, e.g., [11, Chapter 6].) □

For a symmetric $\mathcal{M}$ it can be worthwhile to perform a $\mathbb{C}$–linear similarity transformation, e.g., to diagonalize the linear part of $\mathcal{M}$ to exclude regions where components can not occur before executing algorithms for finding the spectrum. This is due to the fact that a single diagonalization of a Hermitian matrix is inexpensive compared with the total cost of finding the spectrum of a real linear operator. Moreover, the theorem is sharp in case $\mathcal{M}$ gets diagonalized.

Another linear algebraic idea to identify regions where the eigenvalues appear is the field of values of $\mathcal{M}$ defined via its corresponding quadratic form as

$$F(\mathcal{M}) = \{\lambda \in \mathbb{C} \,|\, \lambda = z^*\mathcal{M}(z), \text{ with } ||z|| = 1\}.$$

Clearly, we obtain a compact set such that $\sigma(\mathcal{M}) \subset F(\mathcal{M})$. Considering, for example, a symmetric $\mathcal{M}$ we see that $F(\mathcal{M})$ differs from the field of values of its real form. A number of other basic properties are immediate, such as $F(\mathcal{M}+\mathcal{N}) \subset F(\mathcal{M}) + F(\mathcal{N})$ holds for any real linear operators $\mathcal{M}$ and $\mathcal{N}$. Also we have $F(\mu\mathcal{M}) = \mu\, F(\mathcal{M})$ for $\mu \in \mathbb{C}$. To approximate the field of values from a subspace, if $U \in \mathbb{C}^{n\times m}$ has orthonormal columns, then for $\mathcal{M}_U = U^* \circ \mathcal{M} \circ U$ acting on the span of the columns of U we have $F(\mathcal{M}_U) \subset F(\mathcal{M})$ with equality in case $m = n$. However, $F(\mathcal{M})$ need not be convex even if $\mathcal{M}$ is symmetric. To see this, consider $\mathcal{M}(z) = \frac{1}{2}z + \frac{1}{2}\overline{z}$ acting on $\mathbb{C}$. Therefore we do not have, at the moment, an efficient method to compute this set. Regardless of this, $F(\mathcal{M})$ provides a useful tool for making preliminary remarks on the spectrum before numerical computations.

With the field of values we can generalize Proposition 20 as follows, where we decompose $M_\#$ into its symmetric and skew–symmetric parts, that is, $S_\# = \frac{1}{2}(M_\# + {M_\#}^T)$ and $T_\# = \frac{1}{2}(M_\# - {M_\#}^T)$.

Theorem 25 *Let $\mathcal{M}(z) = Mz + S_\#\overline{z} + T_\#\overline{z}$ with ${S_\#}^T = S_\#$ and ${T_\#}^T = -T_\#$. Then $F(\mathcal{M}) = F(\widehat{\mathcal{M}})$, where $\widehat{\mathcal{M}}(z) = Mz + S_\#\overline{z}$.*

Proof. This follows by the fact that we have $z^*T_\#\overline{z} = 0$ for every $z \in \mathbb{C}^n$. Consequently, all the points of $F(\mathcal{M})$ are given by the part $\widehat{\mathcal{M}}$. □

Recall that for a matrix $M \in \mathbb{C}^{n\times n}$ the field of values is a point if and only if $M = \lambda I$ for $\lambda \in \mathbb{C}$. For real linear operators with the smallest possible field of values we have:

Corollary 26 *For $\mathcal{M}(z) = Mz + M_\#\overline{z}$ we have $F(\mathcal{M}) = \{\lambda\}$ if and only if $M = \lambda I$ and ${M_\#}^T = -M_\#$*

Proof. After performing a translation of $\mathcal{M}$ by λI, we can assume that $\lambda = 0$. Hence, since the converse is clear, assume $F(\mathcal{M}) = \{0\}$. Then we have $z^*Mz = -z^*M_\#\overline{z}$ for any $z \in \mathbb{C}^n$. Therefore with $e^{i\theta}z$, for $\theta \in \mathbb{R}$, we obtain $z^*Mz = -e^{-i2\theta}z^*M_\#\overline{z}$. This forces $z^*Mz = -z^*M_\#\overline{z} = 0$. Since this is true for any $z \in \mathbb{C}^n$, it follows that $M = 0$. To see that ${M_\#}^T = -M_\#$, take two standard basis vectors e_j and e_k. Then

$$0 = (e_j - e_k)^* M_\# \overline{(e_j - e_k)} = -e_j^* M_\# \overline{e_k} - e_k^* M_\# \overline{e_j} = -e_j^* M_\# e_k - e_k^* M_\# e_j,$$

from which the claim follows. □

Similarly we have:

Corollary 27 *$F(\mathcal{M}) \subset \mathbb{R}$ if and only if $M^* = M$ and $S_\# = 0$.*

Proof. We can ignore the effect of $z \mapsto T_\# \overline{z}$ to the field of values.

Take $z \in \mathbb{C}^n$, so that with $e^{i\theta} z$ we obtain real $z^* M z - e^{-i2\theta} z^* S_\# \overline{z}$ for any $\theta \in \mathbb{R}$. Hence the field of values of M must be real and therefore M is Hermitian. Also, since $S_\#$ is unitarily consimilar to a diagonal matrix, this forces $S_\#$ to equal zero. □

Hence the part $z \mapsto T_\# \overline{z}$ does not contribute to the field of values. In particular, perturbing $\mathcal{M}$ with such an anti–linear operator can not move eigenvalues beyond $F(\mathcal{M})$. If this part is dominating in the following sense, we do not have any eigenvalues.

Corollary 28 *If $||{T_\#}^{-1}|| \, \left|\left|\widehat{\mathcal{M}}\right|\right| < 1$, then $\sigma(\mathcal{M}) = \emptyset$.*

Proof. Assume $\mathcal{M}(z) = \lambda z$ with $z \in \mathbb{C}^n$ of unit length. By the fact that z and $T_\# \overline{z}$ are orthogonal, we have by the Pythagorean theorem

$$|\lambda|^2 = \left|\left|\widehat{\mathcal{M}}(z)\right|\right|^2 - ||T_\# \overline{z}||^2 \le \left|\left|\widehat{\mathcal{M}}\right|\right|^2 - \left(\frac{1}{||{T_\#}^{-1}||}\right)^2,$$

from which the claim follows. □

Example 7 Consider the isometry $\mathcal{M}(z) = M_\# \overline{z}$ with $M_\# = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. (A real linear operator $\mathcal{M} : \mathbb{C}^n \to \mathbb{C}^n$ is an isometry if $||\mathcal{M}(z)|| = ||z||$ for every $z \in \mathbb{C}^n$. Equivalently, the real form of $\mathcal{M}$ is an orthogonal matrix.) By Corollary 28, the spectrum of $\mathcal{M}$ is empty.

Remark 4 Based on numerical experimenting, there is a significant difference between the number and the size of the components of the spectrum depending on whether $S_\#$ or $T_\#$ dominates in the splitting $M_\# = S_\# + T_\#$. Corollary 28 suggests that if $T_\# = 0$, then the spectrum of $\mathcal{M}$ is nonempty (which is true if, e.g., $M^* = M$ additionally). More generally, we conjecture that if $M_\#$ is condiagonalizable, then $\sigma(\mathcal{M}) \neq \emptyset$. For condiagonalizability, see [10, Theorem 4.6.11].

We have an analogy of the Bendixson–Hirsch theorem as follows, where H and K denote the Hermitian and skew–Hermitian parts of M, that is, $H = \frac{1}{2}(M + M^*)$ and $K = \frac{1}{2i}(M - M^*)$.

Corollary 29 *For $\mathcal{M}$ set $\mathcal{H}(z) = Hz + S_\#\overline{z}$ and $\mathcal{K}(z) = Kz + iS_\#\overline{z}$, and denote the corresponding real forms by A_0 and A_i. Then*

$$\min\{\lambda \in \sigma(A_0)\} \leq \min_{\lambda \in F(\mathcal{M})} \operatorname{Re}\lambda \text{ and } \max_{\lambda \in F(\mathcal{M})} \operatorname{Re}\lambda \leq \max\{\lambda \in \sigma(A_0)\},$$

and

$$\min\{\lambda \in \sigma(A_i)\} \leq \min_{\lambda \in F(\mathcal{M})} \operatorname{Im}\lambda \text{ and } \max_{\lambda \in F(\mathcal{M})} \operatorname{Im}\lambda \leq \max\{\lambda \in \sigma(A_i)\}.$$

Proof. We can consider $\widehat{M}$ since $z \mapsto T_\#\overline{z}$ does not contribute to the field of values. We prove the claim for the first pair of inequalities since the proof is analogous for the second pair by the fact that $F(\mu\mathcal{M}) = \mu F(\mathcal{M})$ for any $\mu \in \mathbb{C}$.

Because the field of values of $z \mapsto Kz$ is imaginary, all the real parts of $F(\widehat{\mathcal{M}})$ are among the real parts of the field of values of $\mathcal{H}$. Since the real form A_0 of $\mathcal{H}$ is a symmetric matrix, the smallest and the largest real part of $F(\mathcal{H})$ are given by the extreme eigenvalues of A_0. In fact, let λ_1 and λ_{2n} be the smallest and largest real eigenvalues of $\mathcal{H}$. Without loss of generality, assume $\lambda_{2n} \geq 0$. The norm of $\mathcal{H}$ is thus λ_{2n} and therefore we have

$$\operatorname{Re} z^*\mathcal{H}(z) \leq ||\mathcal{H}(z)|| = \lambda_{2n} \tag{12}$$

for any $z \in \mathbb{C}^n$ of unit length. Similar arguments apply to the smallest real part. $\square$

Note that $\mathcal{H}$ and $\mathcal{K}$ are symmetric both. In particular, if $\mathcal{M}$ is symmetric to start with, then we have

$$\max_{\lambda \in \sigma(\mathcal{M})} \operatorname{Im}\lambda = -\min_{\lambda \in \sigma(\mathcal{M})} \operatorname{Im}\lambda \leq ||M_\#||,$$

which is of use in the numerical computation of the spectrum.

Remark 5 From (12) we obtain $\max_{||z||=1} \operatorname{Re} z^*\mathcal{H}(z) = \max_{||z||=1} z^*\mathcal{H}(z) = \lambda_{2n}$ by choosing z to be an eigenvector corresponding to λ_{2n}. Hence, analogously to the $\mathbb{C}$–linear case, we can approximate the convex hull of $F(\mathcal{M})$ by finding the largest (real) eigenvalue of $z \mapsto H_\theta z + e^{i\theta}S_\#\overline{z}$ while $\theta \in [0, 2\pi)$ varies. Here H_θ denotes the Hermitian part of $e^{i\theta}M$. For approximating the field of values of a matrix, see, e.g., [11, p.33].

Example 8 Consider an anti–linear operator $\mathcal{M}(z) = M_\#\overline{z}$ in $\mathbb{C}^n$, with $n \geq 2$. For finding $F(\mathcal{M})$ it suffices to consider $z \mapsto S_\#\overline{z}$. Since $S_\#$ is symmetric, it can be unitarily condiagonalized, i.e., we have $U^*S_\#\overline{U} = D_\# = \operatorname{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n\times n}$ for a unitary matrix U. Since $F(\mathcal{M})$ is unitarily invariant, we can consider $z \mapsto z^*D_\#\overline{z} = \sum_{j=1}^n r_j^2 e^{-i2\theta_j} d_j$, where $z = (r_1e^{i\theta_1}, \ldots, r_ne^{i\theta_n})$ is of unit length. Hence, $F(\mathcal{M})$ equals the disk of radius $||S_\#||$ centered at the origin.

The Kronecker product defined as follows yields another real linear structure of interest. Like the standard Kronecker product for matrix equations [11, Chapter 4.3], it can be employed in representing $\mathbb{R}$–linear operator equations.

Definition 30 *Let A and B denote the real forms of $\mathcal{M} : \mathbb{C}^j \to \mathbb{C}^j$ and $\mathcal{N} : \mathbb{C}^k \to \mathbb{C}^k$. The Kronecker product $\mathcal{M}\widehat{\otimes}\mathcal{N}$ is the complex form of $A \otimes B$.*

Note that since $A \otimes B \in \mathbb{R}^{4jk \times 4jk}$, $\mathcal{M}\widehat{\otimes}\mathcal{N}$ acts on $\mathbb{C}^{2jk}$.

Remark 6 If $\mathcal{M}$ and $\mathcal{N}$ are both $\mathbb{C}$–linear, then it is easy to see that $\mathcal{M}\widehat{\otimes}\mathcal{N}$ is $\mathbb{C}$–linear as well. Since then $\sigma(A) = \sigma(\mathcal{M}) \cup \overline{\sigma(\mathcal{M})}$ and $\sigma(B) = \sigma(\mathcal{N}) \cup \overline{\sigma(\mathcal{N})}$, by the basic properties of the Kronecker product [10], it is immediate what is the spectrum of $\mathcal{M}\widehat{\otimes}\mathcal{N}$.

With square matrices we have $\sigma(A \otimes B) = \sigma(B \otimes A)$. However, in general we have $\sigma(\mathcal{M}\widehat{\otimes}\mathcal{N}) \neq \sigma(\mathcal{N}\widehat{\otimes}\mathcal{M})$ for $\mathbb{R}$–linear operators. The obstacle here is that $(\alpha\mathcal{M})\widehat{\otimes}\mathcal{N} \neq \mathcal{M}\widehat{\otimes}(\alpha\mathcal{N})$ in case $\alpha \in \mathbb{C}$ is not real.

Theorem 31 *Assume $\mathcal{M}$ and $\mathcal{N}$ are real linear operators in $\mathbb{C}^n$ and let μ be a real eigenvalue of $\mathcal{N}$. Then $\mu\sigma(\mathcal{M}) \subset \sigma(\mathcal{M}\widehat{\otimes}\mathcal{N})$.*

Proof. Consider the real forms A and B of $\mathcal{M}$ and $\mathcal{N}$, respectively. Let $\lambda = \alpha_1 + i\beta_1$ and $\mu = \alpha_2 + i\beta_2$ be eigenvalues of $\mathcal{M}$ and $\mathcal{N}$ with the corresponding eigenvectors $v_1 \in \mathbb{R}^{2j}$ and $v_2 \in \mathbb{R}^{2k}$ represented in the real form. Then $Av_1 = \alpha_1 v_1 - \beta_1 J v_1$ and $Bv_2 = \alpha_2 v_2 - \beta_2 J v_2$, so that

$$A \otimes B(v_1 \otimes v_2) = Av_1 \otimes Bv_2 = ((\alpha_1 - \beta_1 J)v_1) \otimes ((\alpha_2 - \beta_2 J)v_2)$$

by the basic properties of the Kronecker product.

Now $(\alpha_1 - \beta_1 J)v_1$ corresponds to multiplying the complex form of v_1 by λ. If $\mu \in \mathbb{R}$, then we have $((\alpha_1 - \beta_1 J)v_1) \otimes (\alpha_2 v_2) = (\alpha_2(\alpha_1 - \beta_1 J)v_1) \otimes v_2 = \alpha_2(\alpha_1 - \beta_1 J)(v_1 \otimes v_2)$, where J is of conforming size. Hence $\lambda\mu \in \sigma(\mathcal{M}\widehat{\otimes}\mathcal{N})$. $\square$

Corollary 32 *Assume λ and μ are real eigenvalues of $\mathcal{M}$ and $\mathcal{N}$, respectively. Then $\lambda\mu \in \sigma(\mathcal{M}\widehat{\otimes}\mathcal{N}) \cap \sigma(\mathcal{N}\widehat{\otimes}\mathcal{M})$.*

Remark 7 Analogously to the Kronecker product for matrices, our Kronecker product inherits the properties of its factors, like symmetry or isometry.

By repeating the arguments of the proof of [10, Theorem 4.2.16], we also have $F(\mathcal{M})F(\mathcal{N}) \subset F(\mathcal{M}\widehat{\otimes}\mathcal{N})$ for a pair of real linear operators $\mathcal{M}$ and $\mathcal{N}$.

In connection with the symmetric $\mathbb{R}$–linear eigenvalue problem it is natural to set:

Definition 33 *$\mathcal{M}$ and $\mathcal{N}$ are congruent if $\mathcal{N} = \mathcal{F} \circ \mathcal{M} \circ \widetilde{\mathcal{F}}$ for an invertible real linear operator $\mathcal{F}(z) = Fz + F_{\#}\overline{z}$.*

If $\mathcal{M}$ is symmetric, then so is $\mathcal{N} = \mathcal{F} \circ \mathcal{M} \circ \widetilde{\mathcal{F}}$. Then $\mathcal{M}$ and $\mathcal{N}$ have the same *real inertia* as well, i.e., the number of negative, positive and zero eigenvalues, counting multiplicities, on the real axis. These claims follow readily by considering the real forms of $\mathcal{M}$ and $\mathcal{N}$.

Remark 8 Although the real inertia is preserved for a symmetric $\mathcal{M}$, the topology of the spectrum can change even in a $\mathbb{C}$–linear congruence, i.e., when $F_{\#} = 0$ for $\mathcal{F}$. To see this, consider $\mathcal{M}$ of Lemma 11 with $d_1 = d_2 \in \mathbb{R}$ and $d_3 = d_4 \neq 0$ so that $\widetilde{\mathcal{M}} = \mathcal{M}$ and $\sigma(\mathcal{M})$ is a circle. Then $F = \operatorname{diag}(c_1, c_2)$ with $c_1 \neq c_2$ gives $\mathcal{N}$ with the spectrum consisting of two distinct points.

Example 9 With congruence the angular field of values of a real linear operator $\mathcal{M}$ defined as

$$F'(\mathcal{M}) = \{\lambda \in \mathbb{C} \,|\, \lambda = z^* \mathcal{M}(z),\ z \neq 0\}$$

is natural. (For the angular field of values, see [10].) Then it is straightforward that we have $F'(\mathcal{M}) = F'(\mathcal{F} \circ \mathcal{M} \circ \widetilde{\mathcal{F}})$ whenever $\mathcal{F}$ is $\mathbb{C}$–linear.

To end this section, we give an example of a Toeplitz–like structured $\mathbb{R}$–linear eigenvalue problem arising naturally as follows (see [20, 21]). Denote by $\mathbf{T}$ the unit circle and let $L^2 \equiv L^2(\mathbf{T})$ be the set of square integrable functions defined on $\mathbf{T}$. As a generalization of a ($\mathbb{C}$–linear) Laurent operator, two bounded measurable functions ϕ and ψ induce a real linear multiplication operator on L^2 via

$$\mathcal{L}(f) = \phi f + \psi \overline{f}, \tag{13}$$

for every f in L^2. This gives rise to a real linear *Toeplitz–Friedrichs* operator $\mathcal{T}$ as follows. Let $P : L^2 \mapsto H^2$ be the orthogonal projector onto the Hardy space $H^2 \equiv H^2(\mathbf{T})$. Define $\mathcal{T}$ on H^2 via

$$\mathcal{T}(f) = P\mathcal{L}(f) = P(\phi f + \psi \overline{f}). \tag{14}$$

Compressing this to $\operatorname{span}_{0 \le j \le n-1}\{z^j\}$, by using the standard orthonormal set $\{z^j\}_{j=0}^{n-1}$ of H^2, we obtain a Toeplitz matrix M while $M_{\#}$ is readily seen to be a Hankel matrix. We will demonstrate how circulant matrices have a natural analogue in this setting.

Let $K \in \mathbb{C}^{n \times n}$ denote the backward identity [10], i.e., the permutation matrix with ones on the diagonal joining the left lower corner with the right upper corner. Consider $\mathcal{M}(z) = Mz + M_{\#}\overline{z}$ with the property that M and $KM_{\#}$ are circulant matrices both. Hence $M_{\#}$ is a Hankel matrix with the corresponding "periodic" structure with respect to its anti–diagonals. We call such an $\mathcal{M}$ a real linear circulant operator.

The arising structure is independent on the order of the multiplication by K in the sense that $M_{\#}K$ is circulant matrix if and only if $M_{\#}$ belongs to this category of periodic Hankel matrices. This guarantees that we preserve a standard property of $\mathbb{C}$–linear circulants as follows.

Proposition 34 *The set of real linear circulant operators is an algebra over $\mathbb{C}$.*

Proof. It is clear that we have a vector space, so let us consider the product. Recall that $K^2 = I$. This implies that we have

$$\mathcal{M} \circ \mathcal{N}(z) = (MN + M_\# \overline{N_\#})z + (MN_\# + M_\# \overline{N})\overline{z}$$

such that $M_\# \overline{N_\#} = M_\# K K \overline{N_\#}$ is the product of two circulant matrices, hence a circulant matrix like the product MN. Similarly, $MN_\# = (MN_\# K)K$, where $MN_\# K$ is a circulant matrix, and $M_\# \overline{N} = K(KM_\# \overline{N})$, where $KM_\# \overline{N}$ is a circulant matrix. □

Therefore, under sufficient assumptions on invertibility (which are generically satisfied)

$$\mathcal{M}^{-1}(z) = (M - M_\# \overline{M}^{-1} \overline{M_\#})^{-1} z + (\overline{M_\#} - \overline{M} M_\#{}^{-1} M)^{-1} \overline{z}$$

has the same structure as $\mathcal{M}$. Moreover, $\mathcal{M}^{-1}$ can be computed by using the FFT techniques by inserting K in appropriate places, although a simpler way to perform the inversion is by diagonalizing $\mathcal{M}$ as follows. Denote again by $F_n \in \mathbb{C}^{n \times n}$ the Fourier matrix.

Theorem 35 *Let $\mathcal{M}$ be a real linear circulant operator in $\mathbb{C}^n$. Then $F_n^* \circ \mathcal{M} \circ F_n$ is diagonal.*

Proof. Since it is clear that $F_n^* M F_n$ is diagonal, let us consider

$$F_n^* M_\# \overline{F_n} = F_n^* K K M_\# F_n^* = F_n^* K F_n^* F_n K M_\# F_n^*.$$

By the fact that $KM_\#$ is circulant, $F_n K M_\# F_n^*$ is diagonal. The claim follows since $F_n^* K F_n^*$ turns out to be a (unitary) diagonal matrix as well. □

This also implies that the spectrum of a real linear circulant operator consists of circles and is thereby readily computable in a closed form.

4 Computational techniques for finding eigenvalues of an $\mathbb{R}$–linear operator

In this section we consider numerical methods for finding eigenvalues of a real linear operator $\mathcal{M}$ in $\mathbb{C}^n$. A method to generate a discrete approximation to the spectrum was proposed in [6] for the case of a fairly moderate dimension n. A problem with this approach is that, it overlooks the one dimensional topological structure of the spectrum. While we only compute discrete points of the spectrum we have no way of telling how eigenvalues actually are connected by the spectrum, in particular when two components nearly intersect. Below we remedy this problem with path following techniques. These methods are also well suited for locating single components of the spectrum, a realistic approach to deal with large scale problems to find a small portion of the eigenvalues. As an advantage, these computations are clearly very parallelizable.

As earlier, we denote the real form of $\mathcal{M}$ by $A \in \mathbb{R}^{n \times n}$ and its characteristic bivariate polynomial by $\det A(\alpha, \beta)$.

4.1 Path following techniques for computing a connected component of the spectrum

Assume having computed a single eigenvalue $\lambda_0 \simeq (\alpha_0, \beta_0) \in \mathbb{R}^2$ of $\mathcal{M}$. Our goal in the sequel is to device a method that produces a numerical approximation to the component of $\sigma(\mathcal{M})$ containing (α_0, β_0). We will not trace this component using directly the formal definition of the spectrum

$$\left\{(\alpha, \beta) \in \mathbb{R}^2 \mid \det A(\alpha, \beta) = 0\right\}, \tag{15}$$

that relies on the bivariate characteristic polynomial of $\mathcal{M}$. For numerical stability it is preferable to recast computation of the determinant into solving a linear algebraic problem. More precisely, we use below bordering techniques for determinants; see [17] for a good reference.

To this end, consider $(\alpha, \beta, u, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ satisfying

$$A(\alpha, \beta)u = A(\alpha, \beta)^T v = 0 \;\text{ and }\; \|u\| = \|v\| = 1. \tag{16}$$

Clearly the projection of the solution set of (16) to the first two components is exactly the set (15).

Now, let $u^1, u^2, v^1, v^2 \in \mathbb{R}^n$ correspond to the splittings $u = \left[\begin{smallmatrix} u^1 \\ u^2 \end{smallmatrix}\right]$ and $v = \left[\begin{smallmatrix} v^1 \\ v^2 \end{smallmatrix}\right]$. Hence $\alpha + i\beta$ and $u^1 + iu^2$ is an eigenpair of $\mathcal{M}$ while $\alpha - i\beta$ and $v^1 + iv^2$ is an eigenpair of $\widetilde{\mathcal{M}}$. Define two functions F and F_* according to

$$F(u, v, \alpha, \beta, x, \delta) = \begin{bmatrix} A(\alpha, \beta) & v \\ u^T & 0 \end{bmatrix} \begin{bmatrix} x \\ \delta \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$F_*(u, v, \alpha, \beta, y, \delta) = \begin{bmatrix} A(\alpha, \beta)^T & u \\ v^T & 0 \end{bmatrix} \begin{bmatrix} y \\ \delta \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $x, y \in \mathbb{R}^{2n}$ and $\delta \in \mathbb{R}$. Clearly for any solution of (16) we have

$$F(u, v, \alpha, \beta, u, 0) = F_*(u, v, \alpha, \beta, v, 0) = 0. \tag{17}$$

Conversely, if $\alpha, \beta, u, v, x, y$ are such that

$$F(u, v, \alpha, \beta, x, 0) = F_*(u, v, \alpha, \beta, y, 0) = 0 \tag{18}$$

hold, then necessarily $x, y, u, v \neq 0$. Then also for $u' = x/\|x\|$ and $v' = y/\|y\|$ the pair of equations (17) is satisfied with u', v' in place of u, v. Hence locally we may choose to continue the curve (α, β, x, y) satisfying (18) for fixed u and v, instead of the curve (15).

In what follows we describe our scheme in more detail in four parts: 1. Description of the continuation step, 2. Choosing the step length, 3. Stopping criteria, and 4. Conditions that cause the algorithm to break down.

1. Continuation step. Assume $A(\alpha_0, \beta_0)$ has one dimensional kernel. Fix u_0 and v_0 satisfying (16) in the sense that

$$A(\alpha_0, \beta_0)u_0 = A(\alpha_0, \beta_0)^T v_0 = 0 \;\text{ and }\; \|u_0\| = \|v_0\| = 1.$$

Then, implicitly, the equation $F(u_0, v_0, \alpha, \beta, x, \delta) = 0$ defines a map $\Psi : (\alpha, \beta) \mapsto (x, \delta)$ uniquely in some neighbourhood U of (α_0, β_0). That is, an evaluation of Ψ corresponds to solving the respective linear system. Similarly $F_*(u_0, v_0, \alpha, \beta, y, \delta) = 0$ defines $\Psi_* : (\alpha, \beta) \mapsto (y, \delta)$. By implicit differentiation we obtain linear equations for the first and second derivatives of Ψ as

$$\begin{bmatrix} A(\alpha,\beta) & u_0 \\ v_0^T & 0 \end{bmatrix} \begin{bmatrix} \partial_\alpha x & \partial_\beta x \\ \partial_\alpha \delta & \partial_\beta \delta \end{bmatrix} = \begin{bmatrix} -x & Jx \\ 0 & 0 \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} A(\alpha,\beta) & u_0 \\ v_0^T & 0 \end{bmatrix} \begin{bmatrix} \partial_\alpha^2 x & \partial_\alpha\partial_\beta x & \partial_\beta^2 x \\ \partial_\alpha^2 \delta & \partial_\alpha\partial_\beta \delta & \partial_\beta^2 \delta \end{bmatrix} = \begin{bmatrix} -2\partial_\alpha x & J\partial_\alpha x - \partial_\beta x & 2J\partial_\beta x \\ 0 & 0 & 0 \end{bmatrix} \tag{20}$$

Let us denote

$$\delta_0 = \delta(\alpha_0, \beta_0), \quad \delta_0' = \begin{bmatrix} \frac{\partial \delta_0}{\partial \alpha} \\ \frac{\partial \delta_0}{\partial \beta} \end{bmatrix}, \quad \text{and} \quad \delta_0'' = \begin{bmatrix} \frac{\partial \delta_0}{\partial \alpha^2} & \frac{\partial \delta_0}{\partial \alpha \beta} \\ \frac{\partial \delta_0}{\partial \beta \alpha} & \frac{\partial \delta_0}{\partial \beta^2} \end{bmatrix}. \tag{21}$$

Correspondingly, we will use notations x', x'' and Ψ' with obvious definitions.

Consider a point $(\alpha^{(0)}, \beta^{(0)}) \in U$. We project $(\alpha^{(0)}, \beta^{(0)})$ back to the curve (15) by applying the following quasi–Newton iteration to the equation $\delta = 0$, i.e., we solve repeatedly the under–determined equation

$$\delta'(\alpha_0, \beta_0) \begin{bmatrix} \alpha^{(k)} - \alpha^{(k+1)} \\ \beta^{(k)} - \beta^{(k+1)} \end{bmatrix} = \delta(\alpha^{(k)}, \beta^{(k)})$$

for $(\alpha^{(k+1)}, \beta^{(k+1)})$. Denote the computed numerical approximation of the limes of this iteration by (α_1, β_1). Since an evaluation of δ implies an evaluation of Ψ, we obtain the vector x_1 as a by–product. For $u_1 = x_1/\|x_1\|$ we then have

$$F(u_1, v_0, \alpha_1, \beta_1, u_1, 0) = 0.$$

Letting $(y_1, \delta_1) = \Psi_*(\alpha_1, \beta_1)$ and $v_1 = y_1/\|y_1\|$, we obtain $(\alpha_1, \beta_1, u_1, v_1)$. This satisfies (16) and thus (α_1, β_1) belongs to the spectrum of $\mathcal{M}$.

Assuming the kernel of $A(\alpha_1, \beta_1)$ is one dimensional, we then repeat the continuation step by using (α_1, β_1) as a starting point.

2. Choosing the step length. The choice of $(\alpha^{(0)}, \beta^{(0)})$, given (α_0, β_0), can be based on the local second order approximation

$$0 = \delta_0 + \delta_0'^T \sigma + \frac{1}{2}\sigma^T \delta_0'' \sigma \tag{22}$$

of the curve (15), where the step $\sigma = h([\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}]\, \delta_0' + \frac{g}{h}\, \delta_0')/\|\delta_0'\|$, with $h, g \in \mathbb{R}$, is a linear combination of the tangential direction $[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}]\, \delta'$ and the gradient direction δ'. Note that $A(\alpha, \beta)$ is real analytic, so that Ψ is real analytic too, in some neighbourhood of (α_0, β_0).

The equation (22) defines g as a function of h in the neighbourhood of the point $h = 0$. Take $g(h)$ to be the root with the smaller absolute value. For small enough h we then have $\|(\alpha^{(0)}, \beta^{(0)}) - (\alpha^{(k)}, \beta^{(k)})\| = O(h^3)$ for any $k \geq 1$. The parameter h is chosen to satisfy the following criteria:

C1 The relative distance of $(\alpha^{(0)}, \beta^{(0)})$ from the tangent at (α_0, β_0) should not be too large, i.e., $\frac{g}{h} < \gamma_{\text{angle}}$ for some choice of $\gamma_{\text{angle}} \in [0, 1]$. In (22) this is guaranteed by imposing

$$h \leq \frac{2\gamma_{\text{angle}}}{(1 + \gamma_{\text{angle}}^2)\|\delta_0'\|\|\delta_0''\|}.$$

C2 The step length should be small enough, to guarantee that the Ψ stays well defined, that is, $u^T x > 0$ and $v^T y > 0$ in the region. Since $u^T x(\alpha, \beta) = 1$, i.e., $u^T x_0' = 0$, we require that $\|x - x_0\| < \gamma_{\text{u}}\|x\| = \gamma_{\text{u}}$ for some $\gamma_{\text{u}} \in [0, 1]$. Using the first degree approximation of $x(\alpha, \beta)$ around (α_0, β_0) we get the bound $h\|x_0'\| < \gamma_{\text{u}}$.

C3 The step length should not more than double from the previous step. This is to guarantee a nice looking curve, and to be a safe guard for the first two criteria C1 and C2.

The following criteria are used to determine whether to accept or reject (α_1, β_1) once computed. If the point is rejected, we recompute it with step length $h/4$.

C4 Repeating the computations in criteria C1 and C2 above at (α_1, β_1) we require that the new step length thus obtained for the next step is not less than $h/3$. The aim of this criterion is, on the one hand, to work as a safe guard in addition to the other criteria, and on the other, to guarantee a nice looking curve.

C5 The angle between the line passing through the points (α_0, β_0) and (α_1, β_1), the tangent at (α_0, β_0), and the tangent at (α_1, β_1) should be small enough, that is,

$$(\beta_1 - \beta_0, \alpha_0 - \alpha_1)^T \delta_1' > \sqrt{1 - \gamma_{\text{angle}}^2}\|\delta_1'\|\|(\alpha_1 - \alpha_0, \beta_1 - \beta_0)\|, \text{ and}$$
$$(\beta_1 - \beta_0, \alpha_0 - \alpha_1)^T \delta_0' > \sqrt{1 - \gamma_{\text{angle}}^2}\|\delta_0'\|\|(\alpha_1 - \alpha_0, \beta_1 - \beta_0)\|.$$

Furthermore, note that if the continuation jumps to a nearby component, then typically δ_0' and δ_1' point roughly in opposite directions. Whereas for small enough γ_{angle} the criterion above guarantees that $\delta_0'^T \delta_1' \geq 0$, immediately detecting this anomalia.

C6 To guarantee the validity of the approximation (22) we require

$$\|(\alpha_1 - \alpha^{(0)}, \beta_1 - \beta^{(0)})\| < g/2.$$

In the numerical examples of Section 4.3 we used values $\gamma_{\text{angle}} = \tan \pi/8$ and $\gamma_{\text{u}} = 1/5$.

3. Stopping criteria. Because the components of the spectrum are closed curves of finite length the continuation will run interminably unless explicitly stopped. To detect returning of the continuation back to the starting point we fix a line L transversal to the tangent direction at the starting point, and track crossings of the line. At each crossing we compute the intersection of the continued curve with L. If the intersection is the starting point with a given tolerance, we return the point as the end point. Otherwise we continue the usual way neglecting the computed intersection point.

Since the intersection of the spectrum with a line (e.g. the real axis) is easy to compute, we might well start off by knowing intersections of the spectrum with a line. Then the components of the spectrum intersecting the line are obtained by continuing each known intersection until the line is reached anew. Note that in a typical case the component of the spectrum is a closed curve without self–intersections, and then it is enough to continue only in the direction of $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \sigma'$ and only until the line is encountered anew for the first time.

4. Breaking down. There are several conditions causing the algorithm to break down. We have not designed any recovery procedures for them, but only list four conditions here.

If at same point both $\delta' = 0$ and $x' = 0$ the step length choice fails.

Since $\|\delta'\| \sim \| \left[\begin{smallmatrix} A(\alpha,\beta) & u \\ v^T & 0 \end{smallmatrix}\right]^{-2} \|$ and $\|\delta''\| \sim \| \left[\begin{smallmatrix} A(\alpha,\beta) & u \\ v^T & 0 \end{smallmatrix}\right]^{-3} \|$, the continuation immediately breaks down if[1] $\dim N(A(\alpha,\beta)) > 1$. Numerically this may happen even if the next to the smallest singular value approaches zero.

If the step size is too large, it may happen that $u \in R(A(\alpha,\beta))$ or $v \in R(A(\alpha,\beta)^T)$ so that $\left[\begin{smallmatrix} A(\alpha,\beta) & u \\ v^T & 0 \end{smallmatrix}\right]$ becomes singular. The criterion C2 above is designed to prevent this.

If the continuation step jumps over to another nearby lying component of the spectrum, the stopping criterion is not necessarily met at all. Again the criterion C5 will make it less likely.

Remark 9 The fact that we know the derivatives δ' and δ'' at every point (α_k, β_k) allows us to consider a spline approximation for the curve. We store δ' along with the points (α_k, β_k) and use cubic splines in our examples.

4.2 Remarks on finding eigenvalues of large scale problems

Since for finding the spectrum of $\mathcal{M}$ we only need the real eigenvalues and the corresponding eigenvectors of $-\beta J - A$ while β varies, an obvious idea to reduce the computational burden is to have a method that ignores the complex eigenvalues of a real matrix. This can result in a significant reduction in the computational complexity by the fact that for certain matrices only a small fraction of the eigenvalues are real; see [5] for the random matrix case.

[1] We denote by $R(A)$ the range of A and by $N(A)$ the kernel of A.

For problems of moderate size an alternative to achieve savings is to compute a Schur decomposition of $-\beta J - A$. Then, instead of finding all the eigenvectors, one computes only those related to real eigenvalues.

With large scale problems we typically can not assume that the Schur decomposition is computable. Instead, a realistic computational task could be that of finding a component of the spectrum passing through a given region D of the complex plane. This is solvable in two steps with iterative methods as follows.

The first step is to find a real eigenvalue α of $-\beta J - A$ with $(\alpha, \beta) \in D$. For a fixed β this type of a problem gives a natural criterion for using restarted iterative eigensolvers. (For restarted eigenvalue algorithms, see [1].) Namely, those Ritz values of $-\beta J - A$ that appear to be converging to complex eigenvalues should be filtered out.

The second step, once an eigenvalue (α, β) inside D has been found, consists of employing our path following techniques just described by using (α, β) as a starting point. Since the dimension is assumed to be large, all the linear systems involved should be solved iteratively. Here an appropriate preconditioning strategy is likely to be crucial for fast computation of the component, or a portion of it.

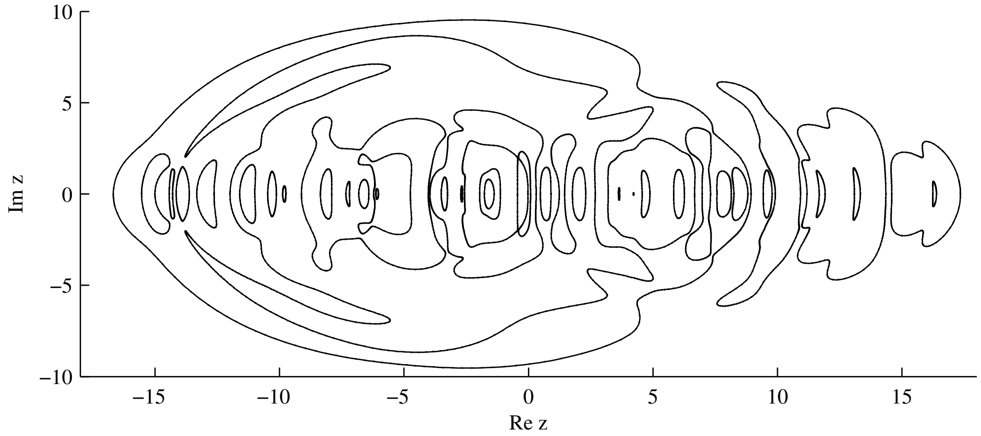
4.3 Numerical examples of computing components of the spectrum

In this subsection we illustrate the prescribed computational techniques with numerical examples. The experiments were performed with `Matlab` [18] whose syntax is used.

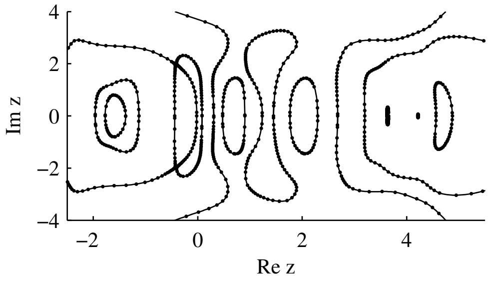
Example 10 In this experiment we consider a symmetric real linear operator $\mathcal{M}$ acting on $\mathbb{C}^n$, with $n = 40$. The task is to locate the components of the spectrum of $\mathcal{M}$ intersecting the real axis. To this end we find the eigenvalues of its real form which are then used as starting points for tracing the corresponding components with path following techniques. We take M to be the Hermitian part of a random matrix $\texttt{randn}(n, n) + i\,\texttt{randn}(n, n)$, while $M_\#$ is the symmetric part of another random matrix generated similarly. See Figure 1 for the spectrum of $\mathcal{M}$. This example illustrates well how the spectrum of a symmetric real linear operator is "rich" in components.

The effect of using variable step length in path following is illustrated by zooming in the fine details of a component of the spectrum in two steps. Circa 4200 points with tangent directions were computed for the approximation. The approximations were drawn by using cubic splines.

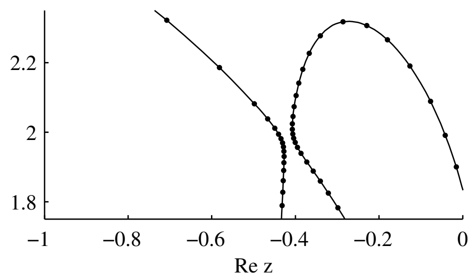
Example 11 Here we consider a real symmetric linear operator alike that of Example 10 but with $n = 10$. The purpose of the example is to illustrate the spectral mapping theorem (Theorem 4). In Figure 2 we have the spectrum of



(a) The spectrum intersecting the real axis.



(b) Magnification of (a).



(c) Magnification of (b).

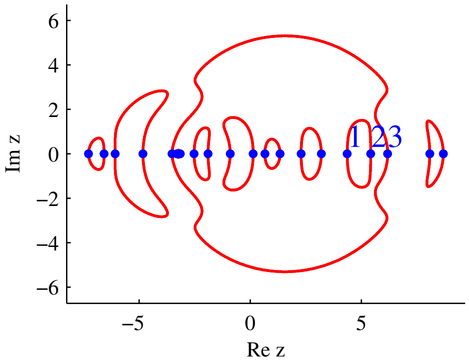
Figure 1: The spectrum intersecting the real axis of a random symmetric real linear operator in $\mathbb{C}^n$, $n = 40$, is illustrated in the top figure (a). The lower figures (b) and (c) are magnifications of the top one. The dots mark the computed eigenvalues, (with computed tangents,) the line represents the cubic spline approximation of the spectrum.

the real linear operators $\mathcal{M}$, $p_1(\mathcal{M}) := \mathcal{M}^2 + 2i\,\mathcal{M}$, and $p_2(\mathcal{M}) := \mathcal{M}^2 - 2\,\mathcal{M}$. In the sub–figure (a) we mark the intersections of the spectrum with the real axis, $\sigma(\mathcal{M}) \cap \mathbb{R}$, by a dot. In the sub–figures (b) and (c) the dots mark the points $p_k(\sigma(\mathcal{M}) \cap \mathbb{R})$, for $k = 1, 2$ respectively. The labelling of the points of $p_k(\sigma(\mathcal{M}) \cap \mathbb{R})$ corresponds to the labelling of the preimage in the sub–figure (a).

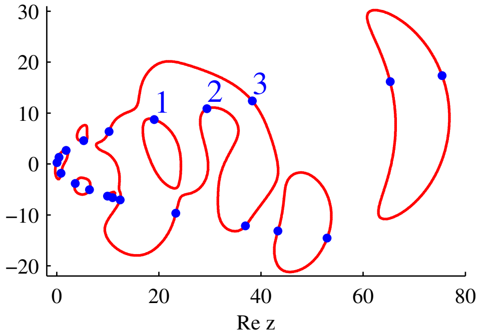
We see that that $p_k(\sigma(\mathcal{M}) \cap \mathbb{R}) \subset \sigma(p_k(\mathcal{M}))$. The topology, however, has changed. See the points labelled 1, 2 and 3. The points 1 and 2 are connected by $\sigma(\mathcal{M})$ but their images are not connected by $\sigma(p_k(\mathcal{M}))$, hence by continuity of p_k we see that $p_k(\sigma(\mathcal{M})) \neq \sigma(p_k(\mathcal{M}))$.

The symmetry of the spectrum with respect to the real axis is expectantly lost for the complex polynomial p_2 in the sub–figure (b).

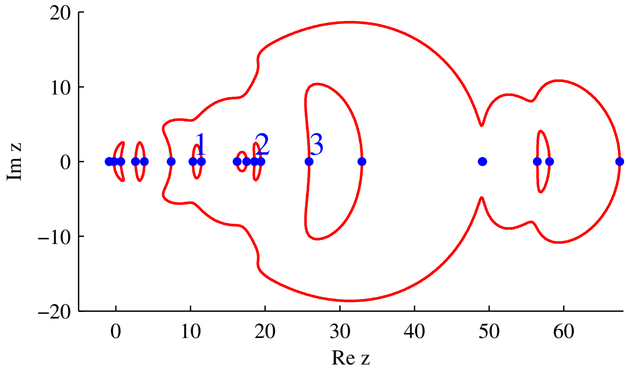
Example 12 In this example we split the anti–linear part of a real linear operator according to Theorem 25. We illustrate the effect of having in the anti–linear part $M_\# = S_\# + T_\#$ either dominating $T_\#$ or $S_\#$. We take M



(a) The spectrum of M.



(b) The spectrum of $M^2 + 2iM$.



(c) The spectrum of $M^2 - 2M$.

Figure 2: The figure illustrates the spectral mapping theorem 4; see Example 11.

and $M_\#$ to be random matrices of type $\texttt{randn}(40, 40) + i\,\texttt{randn}(40, 40)$ both. Then we set

$$\mathcal{M}_t(z) = Mz + S_\#\overline{z} + t\,T_\#\overline{z} \quad \text{with} \quad t \in \{0, 1, 2, 4, 8, 16\}.$$

By Theorem 25 the field of values $F(\mathcal{M}_t)$ is invariant of t. For $t = 0$ the anti–symmetric part $t\,T_\#$ of the anti–linear part vanishes and hence the symmetric part $S_\#$ dominates. For large values of t the anti–symmetric part of $\mathcal{M}_t$ dominates in norm (we had $\|T_\#\|/\|S_\#\| \approx 1$). In the Figure 3 we see the change in the spectrum predicted by Remark 4. For large t the spectrum resembles a circle in contrast to the very complex structure of the case $t = 0$.

5 Conclusions

In this paper we have studied the real linear eigenvalue problem in $\mathbb{C}^n$. We have considered basic properties of the spectrum, like ways to reestablish the spectral mapping theorem, bounds on the degree of the components of the spectrum, and exclusion regions for the eigenvalues. We have introduced a symmetric as well as other structured eigenvalue problems. By using the field

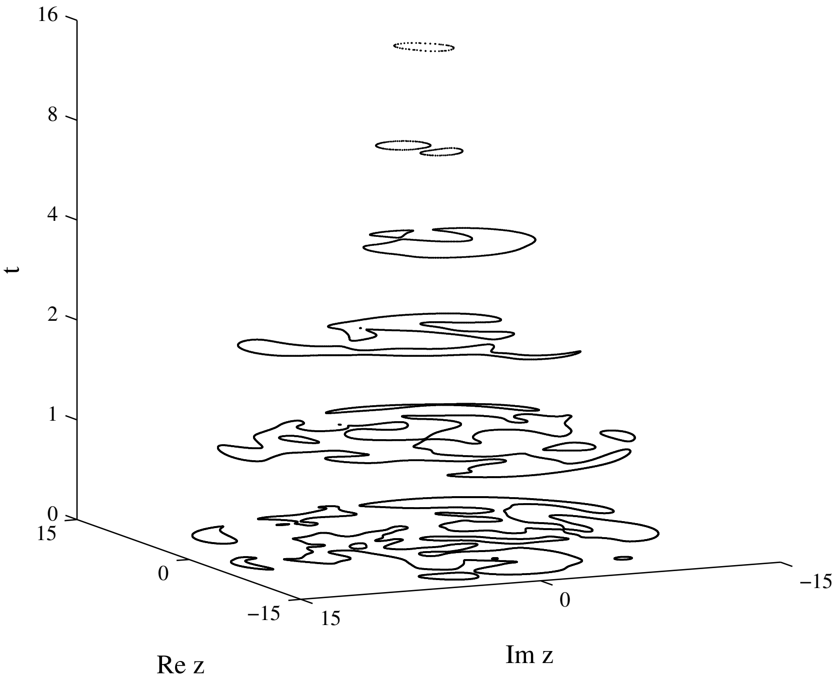


Figure 3: The figure illustrates Example 12. For every value of the parameter t on the vertical axis we plot the spectrum of $\mathcal{M}_t$ on the horizontal copy of the complex plane. The bottom picture corresponds to the case of symmetric $M_\#$. At the top $M_\#$ is highly anti–symmetric.

of values, splitting of the anti–linear part of a real linear operator into its symmetric and anti–symmetric parts yields a simple condition on having an empty spectrum.

Path continuation methods were implemented for computing components of the spectrum. Numerical examples illustrating the aspects studied were presented.

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ISBN 951-22-6801-9
ISSN 0784-3143