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Abstract: *It is well-known (and can be proved in a number of ways) that a densely defined, closed operator A generates a bounded C_0-semigroup if (and only if) the Hille–Yoshida resolvent condition*

$$\|(s_j - A)^{-k}\| \leq \frac{M}{s_j^k} \tag{1}$$

holds for some positive and unbounded sequence $\{s_j\}_{j\geq 1}$. We give a novel and short "frequency domain" proof for the observation that the resolvent condition (1), *indeed, is* only *required for such sequences $\{s_j\}_{j\geq 1}$. The proof is based on studying the analytic function $s \mapsto (I - A/s)^{-1}$ whose values are power bounded operators.*

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1 Introduction

Let X be a Banach space. Let $A : \mathrm{dom}\,(A) \to X$ be a generator of a bounded C_0-semigroup $\{T(t)\}_{t\geq 0}$, satisfying $\sup_{t>0} \|T(t)\| \leq M < \infty$. Such operators are precisely the closed, densely defined operators that satisfy the Hille – Yoshida resolvent condition

$$\|(s - A)^{-k}\| \leq \frac{M}{s^k} \quad \text{for all } s > 0 \text{ and } k \geq 1. \tag{2}$$

A classical references to this result are, of course, [4] (K. Yoshida) and [2] (E. Hille and R. S. Phillips). Both of these references give the stronger version of this result, as quoted in the abstract of this paper. The purpose of this paper is to give a short "frequency domain", "complex analysis" proof for the following theorem:

Theorem 1. *Let A be a densely defined (closed) operator with $s_0 \in \mathbb{R}_+ \cap \rho(A)$, and let $M < \infty$. If for $s = s_0$ we have*

$$\|\, (I - A/s)^{-k} \,\| \leq M \quad \textit{for all } k \geq 1, \tag{3}$$

then $(0, s_0] \subset \rho(A)$ and (3) holds for all $s \in (0, s_0]$.

Indeed, suppose that the resolvent condition (2) is known *only* for all $s \in \{s_j\}_{j\geq 1}$, where $\lim_{j\to\infty} s_j = +\infty$. Then (2) holds for *all* $s > 0$ as a direct consequence of Theorem 1.

Note that Theorem 1 has a flavor of the Maximum Modulus Theorem. All other proofs of Theorem 1 (that we know of) are carried out by using "time domain" techniques. It is rather unusual in harmonic analysis to have two precise characterizations of a same phenomenenon, one on "each side" of the Fourier transform[1]. This is the main motivation for writing this paper.

2 Resolvent condition for power-bounded operators

The discrete semigroups are generated by power bounded operators T. For such operators, a resolvent characterization has been published in [1] (A. Gibson), and it was independently rediscovered in [3, Theorem 2.7.1] (O. Nevanlinna).

Proposition 1. *Let $T \in \mathcal{L}(X)$ and $C < \infty$. Then the following are equivalent:*

(i) $\sup_{j\geq 0} \|T^j\| \leq C$,

(ii) for all $x > 1$ and $k \geq 1$

$$\|T^k (x - 1)^k (x - T)^{-k}\| \leq C, \tag{4}$$

and

[1] Note that the Parseval's identity is a positive example of this.

(iii) there exists a (monotone increasing) sequence $\{x_j\}_{j\geq 1} \subset (1,\infty) \cap \rho(T)$, such that $x_j \to \infty$ and the estimates (4) hold for $x = x_j$ for all $j \geq 1$ and $k \geq 1$.

Proof. Assume (i). Then for all $x > 1$, we have, by the nonnegativity of all scalar terms in sums

$$
\begin{aligned}
&\|T^k(x-1)^k(x-T)^{-k}\| = \left(1-\frac{1}{x}\right)^k \|T^k\left(I-\frac{T}{x}\right)^{-k}\| \\
&= \left(1-\frac{1}{x}\right)^k \|T^k \sum_{j\geq 0} \binom{k+j-1}{j} \left(\frac{T}{x}\right)^j \| \\
&\leq \left(1-\frac{1}{x}\right)^k \sup_{j\geq 0} \|T^j\| \sum_{j\geq 0} \binom{k+j-1}{j} \left(\frac{1}{x}\right)^j \\
&= C\left(1-\frac{1}{x}\right)^k \left(1-\frac{1}{x}\right)^{-k} = C.
\end{aligned}
$$

So the resolvent condition in claim (ii) follows. The implication (ii) $\Rightarrow$ (iii) is trivial. The final implication (iii) $\Rightarrow$ (i) just by taking the limit as $x_j \to \infty$ in the resolvent condition. $\square$

There is a slight generalization of this results, and we give it here even though it will not be needed in the proof of Theorem 1.

Corollary 1. *Let $\alpha \in [0,1)$ and $T \in \mathcal{L}(X)$. Then the powers of $T_\alpha := \alpha + (1-\alpha)T$ are bounded by constant C if and only if there exists a (monotone increasing) sequence $\{y_j\}_{j\geq 1} \subset (1,\infty) \cap \rho(T)$, such that $y_j \to \infty$ and the estimates*

$$\|T_\alpha^k (y_j - 1)^k (y_j - T)^{-k}\| \leq C \tag{5}$$

hold for all $k \geq 1$.

Moreover, an operator $V \in \mathcal{L}(X)$ is power bounded by constant C if and only if there exists $\alpha \in [0,1)$ and a (monotone increasing) sequence $\{y_j\}_{j\geq 1} \subset (1,\infty) \cap \rho(V_\alpha)$, such that $y_j \to \infty$ and the estimates

$$\|V^k (y_j - 1)^k \left(y_j - V_\alpha\right)^{-k}\| \leq C \tag{6}$$

hold for all $k \geq 1$, where $V_\alpha := (V-\alpha)/(1-\alpha)$.

Proof. For all $\alpha \neq 1$ and $x \in (1,\infty) \cap \rho(T_\alpha)$ we have

$$
\begin{aligned}
&(x-1)(x-T_\alpha)^{-1} = (x-1)(x-\alpha-(1-\alpha)T)^{-1} \\
&= \frac{x-1}{1-\alpha}\left(\frac{x-\alpha}{1-\alpha} - T\right)^{-1} = (y-1)(y-T)^{-1},
\end{aligned}
$$

where $y = y(x) := (x-\alpha)(1-\alpha)^{-1}$ or, equivalently, $x = x(y) = \alpha + (1-\alpha)y$.

Assume that T_α is power-bounded by C. Then by implication (i) $\Rightarrow$ (ii) of Proposition 1, we have for all $x > 1$ (and hence, because $\alpha \in [0,1)$, for all $y > 1$)

$$\|T_\alpha^k (y-1)^k (y-T)^{-k}\| = \|T_\alpha^k (x-1)^k (x-T_\alpha)^{-k}\| \leq C$$

where $k \geq 1$ is arbitrary. This estimate holds in particular for any sequence $\{y_j\}_{j\geq 1}$ converging to ∞, and the one direction of the first equivalence is now proved.

Conversely, assume that estimate (5) holds for all $k \geq 1$ and some sequence $\{y_j\}_{j\geq 1}$, having the stated properties. Define another sequence $\{x_j\}_{j\geq 1}$, by setting $x_j := \alpha + (1-\alpha)y_j$. Because $\alpha < 1$, this new sequence satisfies the same conditions that have been imposed on $\{y_j\}_{j\geq 1}$. Now, for all $j \geq 1$, we have the estimates

$$\|T_\alpha^k (x_j-1)^k (x_j-T_\alpha)^{-k}\| = \|T_\alpha^k (y_j-1)^k (y_j-T)^{-k}\| \leq C$$

where $k \geq 1$ is arbitrary. Now implication (iii) $\Rightarrow$ (i) of Proposition 1 gives the power-boundedness of T_α.

Let us proceed to prove the second equivalence. Fix $\alpha \in [0,1)$ arbitrarily. Define $T := (V-\alpha)/(1-\alpha)$. Then $T_\alpha = V$ and the power-boundedness of V is seen to be equivalent to the resolvent condition (6), by the first part of this corollary. $\square$

3 Proof of Theorem 1

Now begins the real fun, and we give the promised proof of Theorem 1.

Define for all $s \in \rho(A)$ the operator-valued function $T(s) := (I - A/s)^{-1}$. By the assumption of Theorem 1, $\sup_{k>1} \|T(s_0)^k\| =: M < \infty$. Applying Proposition 1 shows that for all $x > 1$ and integers $k > 1$

$$\|T(s_0)^k \, (x-1)^k \, (x - T(s_0))^{-k} \, \| \leq M; \tag{7}$$

in particular such $x \in \rho(T(s_0))$. But now for all $x > 1$

$$T(s_0)\,(x-1)\,(x-T(s_0))^{-1} = (x-1)\left(I - \frac{A}{s_0}\right)^{-1}\left(x - \left(I - \frac{A}{s_0}\right)^{-1}\right)^{-1}$$
$$= (x-1)\left(x\left(I - \frac{A}{s_0}\right) - I\right)^{-1} = \left(I - \frac{A}{(1-1/x)s_0}\right)^{-1}.$$

Denoting $s = (1-1/x)s_0$ we see from (7) that $\|\,(I - A/s)^{-k}\,\| \leq M$ for all such s. Because $x > 1$ was arbitrary, this estimate holds for all $s \in (0, s_0)$, thus proving Theorem 1.

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