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DIFFUSIVE TOMOGRAPHY METHODS: SPECIAL BOUNDARY CONDITIONS AND CHARACTERIZATION OF INCLUSIONS

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Abstract: *This thesis presents mathematical analysis of optical and electrical impedance tomography. We introduce papers* [I–III]*, which study these diffusive tomography methods in the situation where the examined object is contaminated with inclusions that have physical properties differing from the background.*

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Preface

This work has been carried out at Institute of Mathematics at Helsinki University of Technology during the years 2001–2004.

I want to express my gratitude to professor Erkki Somersalo, the supervisor and instructor of my thesis, for providing me with interesting research subjects and advice when needed. It has been a pleasure to work under his easy-going guidance. I am also grateful to my workmate Mr Juha–Matti Perkkiö for his endless enthusiasm about mathematics and eagerness to discuss any problems related to my scientific work—a favour I have not always been able return.

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I want to thank my parents, brother and sisters for their support and my friends for occasionally guiding my thoughts away from mathematics. I am truly grateful to my dear girlfriend Merja for her love and ability to absorb all my worries and bad moods. To her I dedicate my thesis.

Espoo, April 1, 2004.

Nuutti Hyvönen

The thesis consists of this overview and the following papers:

Publications

[I] N. Hyvönen. Analysis of optical tomography with non-scattering regions. *Proceedings of the Edinburgh Mathematical Society*, 45:257–276, 2002.

[II] N. Hyvönen. Complete electrode model of electrical impedance tomography: Approximation properties and characterization of inclusions. *SIAM Journal on Applied Mathematics*, 64:902-931, 2004.

[III] N. Hyvönen. Characterizing inclusions in optical tomography. *Inverse Problems*, 20:737–751, 2004.

1 Introduction

The function of medical imaging is to provide information of human organs without having to draw upon surgery. Nowadays a number of imaging methods are available to physicians. The decision which method is to be used is made by stating at least the following questions: Which physical parameters provide the most useful information about the physiological state of the patient? What is the needed resolution to make the correct diagnosis? How large a dose of ionizing radiation is one prepared to expose the patient on? Can the patient be moved? How quickly should the reconstructions be available? What are the financial limitations? In almost all situations either x-ray tomography or magnetic resonance imaging answers the first two questions in a satisfactory manner. However, they may often have difficulties fulfilling some of the latter four conditions: x-rays are ionizing, magnetic resonance imaging is expensive and neither of them can be used with portable equipment. These observations suggest that there is a need for a low cost, non-ionizing tomography method capable of on-line monitoring. For example ultra sound tomography is one good option but it is not applicable in all situations, and so there is reason to investigate other possibilities.

Over the past fifteen years there has been attempts to develop a diagnostic imaging modality based on near-infrared radiation. One way to model propagation of near-infrared light in strongly scattering biological tissue is the diffusion approximation of the radiative transfer equation, which leads to a second order elliptic partial differential equation called the diffusion equation. The inverse problem corresponding to optical tomography is to reconstruct the coefficients of the diffusion equation, known to contain information about the absorption and scattering distributions inside the investigated object, from the knowledge of flux measurements on the boundary of the object. It is well-known that this problem is severely ill-posed—far more so than inverting x-ray tomography data. In consequence, inverse mathematics plays a crucial role in the research of optical tomogarphy.

Another medical imaging method that involves similar mathematics as optical tomography is electrical impedance tomography, the idea of which is to conduct electrical current into a physical body, measure the resulting potential on the boundary and use this data to gather information about the admittance distribution inside the object. By making some physically sound assumptions, one sees that the electrical potential inside the body satisfies a simple second order elliptic partial differential equation, for which the conductivity tensor is the only coefficient and the measured pairs of currents and voltages serve as boundary data. The inverse problem of electrical impedance tomography, also called the inverse conductivity problem, has been of great interest for the mathematical community since Calderón published his influential paper [15] in 1980.

As indicated above, in both optical and electrical impedance tomography the task is to determine the coefficients of an elliptic partial differential equation from the knowledge of some boundary data. From the mathematical point of view this kind of inverse problems involve mainly the following three questions: Do the boundary measurements define the coefficients uniquely? Do the coefficients depend stably on the boundary data? How should one reconstruct the coefficients? When observing the problem from a more practical point of view, a fourth question arises: What are the correct boundary conditions for real-world measurements and how do they affect the answers to the first three questions? In this thesis we consider the third and the fourth question, for both optical and electrical impedance tomography, in the situation where the investigated object is contaminated by inclusions with physical properties differing from the background. In [I] the forward problem corresponding to optical tomography is analyzed for objects that contain transparent regions, i.e. regions with no scattering. [II] considers approximation properties of the complete electrode model of electrical impedance tomography and modifies the factorization method [25, 13] for locating inclusions so that it can be used with measured data. In [III] we apply the factorization method for localizing diffusive inclusions in optical tomography.

This text is organized as follows: Section 2 reviews the theoretical backgrounds of optical and electrical impedance tomography and in Section 3 we introduce the factorization method [25] and demonstrate how it can be used for inclusion characterization with diffusive tomography methods. The results of [I] can be found in Subsection 2.2, the material of [II] is considered in Subsections 2.1 and 3.1, and the main result of [III] is formulated in Subsection 3.2.

2 Diffusive tomography methods

For diffusive tomography methods the mathematical task is to determine the coefficients of an elliptic partial differential equation by using known boundary values of solution functions. Since diffusive phenomena are smoothing, the problem is severely ill-posed. In this thesis we examine the mathematics of optical tomography and electrical impedance tomography. To begin with, we review the theory of the latter one because it is better established and more fundamental. The theoretical background of optical tomography will be considered in Subsection 2.2.

2.1 Electrical impedance tomography

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded simply connected set. Assume that one is able to conduct alternating electrical current of frequency ω through the boundary $\partial\Omega$ and the corresponding boundary potential can be measured. The aim of electrical impedance tomography (EIT) is to use the measured pairs of current and voltage to determine the complex admittance function

$$\gamma(x) = \sigma(x) + i\omega\epsilon(x), \quad x \in \Omega, \tag{1}$$

where the symmetric matrices $\sigma, \epsilon \in \mathbb{R}^{n\times n}$ are the electrical conductivity and the electric permittivity, respectively.

Because different materials have different electrical properties, γ can be used to infer the internal structure of Ω. Due to this observation, EIT could be applied to fields such as medicine, geophysics, environmental sciences and non-destructive testing of materials. Examples of applications in medicine are detection of pulmonary emboli, monitoring of apnoea, monitoring of heart function and blood flow and detection of breast cancer. For further discussion and potential use of EIT in other fields of science, we refer to the review articles [11, 17] and the references therein.

2.1.1 Continuum forward model

Assume that alternating current with the amplitude $f : \partial\Omega \to \mathbb{C}$ penetrates the object boundary $\partial\Omega$. Starting from Maxwell's equations and making some physically reasonable approximations [11], one readily obtains that the electrical potential inside the object satisfies the elliptic boundary value problem:

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega, \qquad \nu \cdot \gamma \nabla u = f \quad \text{on } \partial\Omega, \tag{2}$$

where $\nu = \nu(x)$ is the unit normal pointing out of Ω. It follows easily from the theory of elliptic partial differential equations that (2) has a unique weak solution $u \in H^1(\Omega)/\mathbb{C}$ for f belonging to the space

$$H_0^{-1/2}(\partial\Omega) = \{v \in H^{-1/2}(\partial\Omega) \mid \langle v, \mathbf{1}\rangle_{L^2(\partial\Omega)} = 0\}, \tag{3}$$

where $\langle \cdot, \cdot \rangle_{L^2(\partial\Omega)}$ denotes the dual pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, if it is assumed that the boundary $\partial\Omega$ is regular enough and the admittance $\gamma \in \mathbb{C}^{n\times n}$ satisfies

$$\mathrm{Re}(\gamma x \cdot \overline{x}) \geq c|x|^2, \quad |\gamma x \cdot \overline{x}| \leq C|x|^2, \quad c, C > 0, \tag{4}$$

for all $x \in \mathbb{C}^n$ almost everywhere in Ω. By looking for the solution u in a quotient space, we emphasize the fact that the ground level of the potential is something that can be chosen as one wishes.

From now on we will take equation (4) for granted and refer to problem (2) as the continuum forward model of EIT. One could also formulate the forward problem using Dirichlet data, but for our purposes the above definition is the most useful one. Before we begin to consider the corresponding inverse problem, we note that the continuum model formulated above is mathematically well-established, but it does not model real-life electrode measurements well at all [38, 18]. Consequently, some other models are needed in practice.

2.1.2 Inverse problem

The Neumann-to-Dirichlet operator that takes the applied current pattern onto the boundary value of the potential

$$\Lambda_\gamma : H_0^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)/\mathbb{C}, \quad f \mapsto u|_{\partial\Omega}$$

is a linear isomorphism. To know Λ_γ is essentially equivalent of knowing all possible pairs of boundary current and boundary potential. Hence, the inverse problem of EIT, as mathematicians tend to formulate it, is as follows: Using the known Neumann-to-Dirichlet map Λ_γ, determine γ in Ω. In what follows, this problem will be referred to as the inverse continuum problem of EIT.

Since the publication of Calderón's fundamental paper [15], the inverse continuum problem of EIT has been of great interest for mathematicians. The three most important lines or research have, arguably, been uniqueness, stability (or instability) and reconstruction. In this subsection we will briefly consider the main results in these fields; for more information we refer once again to the review article [11].

Uniqueness. In 1984 Kohn and Vogelius [27] proved that Λ_γ determines $\frac{\partial^k \gamma}{\partial \nu^k}|_{\partial\Omega}$ uniquely for all $k \geq 0$ and $\gamma \in C^\infty(\bar{\Omega})$. In [28] they continued with the first interior result showing that piecewise analytic, scalar γ is uniquely determined by Λ_γ if $\partial\Omega$ is smooth.

The unique solvability of the inverse continuum problem for non-alternating currents and scalar conductivities of a wide class was first obtained in dimensions $n \geq 3$ by Sylvester and Uhlmann in 1987 [42] and later in two dimensions by Nachman in 1996 [31]. Their regularity assumptions on the conductivity and the boundary $\partial\Omega$ have been reduced by several authors since [33, 30, 2, 12, 36]. Quite recently considerable progress was made as Astala and Päivärinta solved the problem in two dimensions for the natural regularity assumption $\gamma \in L^\infty(\Omega)$ [8].

It is worthwhile noticing that the uniqueness results can be generalized for complex admittances, i.e. for non-zero frequencies ω, in three and higher dimensions, but in two dimensions this problem is essentially open, although some progress has been made [21]. For the anisotropic case, there is no uniqueness as noted, for example, in [40].

Stability. Even if the inverse map

$$\Lambda_\gamma \mapsto \gamma, \quad \mathcal{L}(H_0^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)/\mathbb{C}) \to L^\infty(\Omega)$$

existed, it would be discontinuous as demonstrated, for example, in [1]. This makes the inverse continuum problem severely ill-posed. However, if some smoothness assumptions are placed on the admittance, logarithmic stability is obtained as shown by Alessandrini in three and higher dimensions [1]. Stability analysis in two dimensions is conducted in [29, 10].

Reconstruction. The reconstruction methods of EIT can be divided into two categories: iterative and direct algorithms. When using iterative methods, one scans some set of admissible admittances for $\tilde{\gamma}$ which in some sense minimizes the difference between the measured Neumann-to-Dirichlet map Λ_γ and the computed one $\Lambda_{\tilde{\gamma}}$ over a set of test currents $\{f_k\}$. Usually, one either minimizes the difference between the outputs $\{\Lambda_\gamma f_k\}$ and $\{\Lambda_{\tilde{\gamma}} f_k\}$ in the least squares sense by some Newton-type algorithm or uses variational methods [11]. Either way, the minimization process must be regularized due to the ill-posedness of the underlying problem.

Maybe the most fundamental of the direct reconstruction algorithms is the one by Siltanen, Mueller and Isaacson [37] since it is a numerical implementation of Nachman's constructive uniqueness proof in two dimensions [31]. Other direct methods worth mentioning are the layer stripping algorithm [39, 41], the idea of which is to recover the admittance layer by layer, and the factorization method [13, 14], which originates from inverse obstacle scattering theory [25]. The factorization method is one of the main concerns of this thesis and it will be addressed more thoroughly in Section 3.

2.1.3 Complete electrode model

To end this short survey on EIT, we will introduce the complete electrode model, which has been shown to predict real-world electrode measurements reasonably well [38]. Assume that the boundary of the investigated object Ω is partially covered with disjoint electrodes $e_m \subset \partial\Omega$, $1 \leq m \leq M$, which are identified by the parts of the surface that they cover and assumed to be ideal conductors. For simplicity it is assumed that the electrodes are open and simply connected and have smooth boundaries. The union of the electrode patches is denoted by $\Gamma_e = \cup_m e_m \subset \partial\Omega$. All electrodes are used for both current injection and voltage measurement, and the current and voltage patterns are denoted by $\{I_m\}, \{U_m\} \subset \mathbb{C}$, $1 \leq m \leq M$, respectively.

When conducting measurements with electrodes, a thin highly resistive layer is formed at the electrode-object interface [38]. It is characterized by the contact impedance $z : \partial\Omega \to \mathbb{C}$ that in our framework is assumed to be bounded and strictly positive in the real part. Note that the value of z between the electrodes indicates the fictitious value of the contact impedance, i.e. the value of the contact impedance if an electrode were present.

Traditionally, the electrode currents and potentials are handled as elements of $\mathbb{C}^M$ [38]. However, in our case it is more useful to use the space

$$T = \{V \in L^2(\Omega) \mid V = \sum_{m=1}^{M} \chi_{e_m} V_m, \ V_m \in \mathbb{C}, \ 1 \leq m \leq M\}, \tag{5}$$

and its subspace

$$T_0 = \{V \in T \mid \int_{\partial\Omega} V dS = 0\} \subset L^2_0(\partial\Omega), \tag{6}$$

which contains the admissible electrode currents. Here and in what follows, $L^2_0(\partial\Omega) \subset L^2(\partial\Omega)$ denotes the subspace consisting of the elements that integrate to zero over the boundary $\partial\Omega$.

With this convention the forward problem corresponding to the complete electrode model is as follows: For the electrode current $I = \sum \chi_{e_m} I_m \in T_0$

find $(u^e, U^e) \in (H^1(\Omega) \oplus T)/\mathbb{C}$ that satisfies weakly

$$\begin{aligned}
&\nabla \cdot \gamma \nabla u^e = 0 && \text{in } \Omega, \\
&\nu \cdot \gamma \nabla u^e = 0 && \text{on } \partial\Omega \setminus \bar{\Gamma}_e, \\
&u^e + z\nu \cdot \gamma \nabla u^e = U^e && \text{on } \Gamma_e, \\
&\tfrac{1}{|e_m|}\int_{e_m} \nu \cdot \gamma \nabla u^e dS = I_m, && 1 \leq m \leq M,
\end{aligned} \tag{7}$$

where $\nu = \nu(x)$ is the exterior unit normal of $\partial\Omega$. Note that in (7) the first part of the solution is the potential inside the object whereas the second part corresponds to the voltages measured on the electrodes. For more extensive physical justification of the above model we refer to [18].

As shown in [38], problem (7) has a unique solution. The associated current-to-voltage map takes the electrode currents onto the corresponding electrode potentials:

$$R_\gamma : T_0 \to T/\mathbb{C}, \quad I \mapsto U^e.$$

Since R_γ can be obtained through electrode measurements, the inverse problem corresponding to the complete electrode model of EIT is as follows: Given the finite-dimensional linear operator R_γ, find as much information about γ as possible. Note that in this definition we have given up the hope of determining γ uniquely: Without extremely strong a priori assumptions this real-life inverse problem is not uniquely solvable.

Although the complete electrode forward model has been shown to be in a fairly good agreement with measurements [38] and it has also been used to obtain reconstructions with real-world data [43], it remains virtually unstudied as far as the mathematics is concerned. Since the inverse continuum problem addressed in Subsection 2.1.2 is, on the other hand, quite well-established, the least one could hope for is that the electrode problem (7) were an approximation for the continuum problem (2). One of the main results of [II] reveals that this is in fact the case:

Theorem 2.1. *Let $f \in L^2_0(\partial\Omega)$ be a given input current and let $u \in H^1(\Omega)/\mathbb{C}$ be the corresponding solution of (2). Further, let $(u^e, U^e) \in (H^1(\Omega) \oplus T)/\mathbb{C}$ be the unique solution of (7) with the electrode current $I = Pf$, where $P : L^2_0(\partial\Omega) \to T_0$ is an orthogonal projection. Then it holds that*

$$||u - u^e||_{H^1(\Omega)/\mathbb{C}} \leq C\{\frac{1}{|\Gamma_e|^{1/2}}||f||_{H^{-1/2}(\partial\Omega\setminus\bar{\Gamma}_e)} + \inf_{V\in T}||(u + zf)|_{\Gamma_e} - V||_{L^2(\Gamma_e)}\}, \tag{8}$$

where $C = C(\gamma, z, \Omega)$.

If one does not try to conduct any current through the regions between the electrodes, i.e. $f|_{\partial\Omega\setminus\bar{\Gamma}_e} = 0$, the first term on the right hand side of (8) vanishes and the estimate gets better. However, if one wants to approximate, for example, sinusoidal current patterns with the electrode inputs, the first term on the right hand side of (8) is also relevant.

2.2 Optical tomography

In optical tomography a physical body is illuminated with a flux of near-infrared (NIR) photons and the outcoming flux is measured on the surface of the body. The idea is to reconstruct the optical properties, such as absorption and scatter, inside the body by using the measured pairs of input and output fluxes. NIR tomography has a few possible clinical applications, the most important of which are, arguably, screening for breast cancer and the development of a cerebral imaging modality for mapping structure and function in newborn infants, and possibly adults too. For more medical and instrumental details we refer to the articles [4, 6, 23, 5].

Propagation of electromagnetic radiation in medium is governed by Maxwell's equations. Particularly, this holds for the case of our interest, namely, NIR light travelling through some biological tissue. However, since the radiation within a strongly scattering medium is completely incoherent and the wavelength of NIR light is small compared to the characteristic distances of human tissue, the exact models are totally useless. Therefore, we will model light propagation by using approximations of the radiative transfer equation, also known as the Boltzmann equation. Because the human brain consists of strongly scattering tissue with weakly scattering cavities filled with cerebrospinal fluid [5, 35], our aim is to treat these two extremes separately and then bundle the models together to obtain the so-called radiosity-diffusion forward problem [5, 20, 19].

We begin our work by a short glance at transport theory. Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded body with a connected complement, and let $\hat{\theta} \in S^{n-1}$ be a direction vector. The radiation flux density at $x \in \Omega$ at time $t \in \mathbb{R}$ to the infinitesimal solid angle ds in direction $\hat{\theta}$ is written as

$$d\vec{J}(x,t,\hat{\theta}) = I(x,t,\hat{\theta})\hat{\theta}ds(\hat{\theta}),$$

where the amplitude $I(x,t,\hat{\theta})$ is called the radiance. In the framework of transport theory, this scalar function satisfies the radiative transfer equation,

$$\begin{aligned} &\frac{1}{c}I_t(x,t,\hat{\theta}) + \hat{\theta}\cdot\nabla I(x,t,\hat{\theta}) + (\mu_a(x)+\mu_s(x))I(x,t,\hat{\theta}) \\ &\qquad - \mu_s(x)\int_{S^{n-1}} f(x,\hat{\theta},\hat{\omega})I(x,t,\hat{\omega})ds(\hat{\omega}) = q(x,t,\hat{\theta}), \end{aligned} \tag{9}$$

where c is the speed of light (assumed to be constant), the positive scalar functions μ_a and μ_s are the absorption and scattering coefficients, respectively, and q denotes the source term which is assumed to vanish in this discussion. The kernel f is the scattering phase function, satisfying the following three conditions:

$$\begin{aligned} \int_{S^{n-1}} f(x,\hat{\theta},\hat{\omega})ds(\hat{\theta}) &= \int_{S^{n-1}} f(x,\hat{\theta},\hat{\omega})ds(\hat{\omega}) = 1, \\ f(x,\hat{\theta},\hat{\omega}) &\geq 0, \qquad x \in \mathbb{R}^n, \quad \hat{\theta},\hat{\omega} \in S^{n-1}, \\ f(x,\hat{\theta},\hat{\omega}) &= f(x,-\hat{\omega},-\hat{\theta}), \quad \hat{\theta},\hat{\omega} \in S^{n-1}. \end{aligned}$$

Due to the first two conditions, for fixed x, f may be regarded as a probability distribution on S^{n-1} with respect to either of the variables $\hat{\theta}$ and $\hat{\omega}$. For more transport theory consult, for example, [16].

Given the radiation flux density, the total flux through an infinitesimal oriented surface patch νdS is obtained by integrating the flux density over all radiation directions,

$$d\Phi(x,t) = (\int_{S^{n-1}} d\vec{J}(x,t,\hat{\theta})) \cdot \nu dS = \vec{J}(x,t) \cdot \nu dS,$$

where the vector field $\vec{J}$ is the energy current density. We also define the scalar function

$$\varphi(x,t) = \int_{S^{n-1}} I(x,t,\hat{\theta}) ds(\hat{\theta})$$

called the energy fluency. Note that $\varphi(x,t)$ and $\vec{J}(x,t)$ are essentially the coefficients of the zeroth and first order terms for the linearization of $I(x,t,\hat{\theta})$ with respect to $\hat{\theta}$.

2.2.1 Strong scattering

Being an integrodifferential equation, the radiative transfer equation, as discussed above, leads easily to numerical problems of prohibitive size if no simplifications are made. The commonly used simplification is called the diffusion approximation, which has been shown to be justified for materials that are much more scattering than absorbing [4].

Let $P : L^2(S^{n-1}) \to \operatorname{span}\{1, \theta_1, ..., \theta_n\}$ be an orthogonal projection, which linearizes the dependence on the scattering direction. Denoting the integro-differential operator induced by the left hand side of (9) by $\mathcal{B}$, we define the diffusion approximation of the radiative transfer equation as

$$P\mathcal{B}PI = 0, \tag{10}$$

where I denotes the radiance. Due to the way that the projection P is defined, one should be able to write the diffusion approximation using only the energy fluency φ and the energy current density $\vec{J}$ defined above. Indeed, by a straightforward calculation [24, 5], one sees that equation (10) is equivalent to the coupled system

$$\frac{1}{c}\varphi_t = -\nabla \cdot \vec{J} - \mu_a \varphi \tag{11}$$

$$\frac{1}{c}\vec{J}_t = -\frac{1}{n}\nabla\varphi - (\mu_a + (I - B)\mu_s)\vec{J}, \tag{12}$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix and the symmetric matrix $B \in \mathbb{R}^{n\times n}$ is defined by

$$B_{jk} = \frac{n}{|S^{n-1}|} \int_{S^{n-1}} \int_{S^{n-1}} \theta_j \omega_k f(x, \hat{\theta}, \hat{\omega}) ds(\hat{\theta}) ds(\hat{\omega}).$$

In order to be able to handle the boundary conditions corresponding to the diffusion approximation, we need to write out the total flux inwards $(-)$ and outwards $(+)$ on the boundary $\partial\Omega$ when the dependence on the scattering direction is linearized [24]:

$$\Phi_{\pm} = (\pm\gamma\varphi + \frac{1}{2}\nu \cdot \vec{J})|_{\partial\Omega}, \tag{13}$$

where $\nu = \nu(x)$ is the exterior unit normal of $\partial\Omega$, in two dimensions $\gamma = \frac{1}{\pi}$ and in three dimension $\gamma = \frac{1}{4}$. Note that the expression for the fluxes $\Phi_{\pm}$ in (13) differs somewhat from the one given in most references. However, it is carefully conducted from the mathematical model described above and so it is one reasonable choice. Notice that here the outward flux is treated as a positive and the inward flux as a negative quantity, meaning that the net flux through the boundary is obtained by summing the two fluxes, the positive direction being outwards.

2.2.2 Non-scattering regions

In weakly scattering regions the diffusion approximation is no more valid [19, 20] and so we will have to come up with something new. Let $\Omega_{\mathrm{n}} \subset \Omega$ be a non-scattering region with a C^2-boundary. Because in a non-scattering region all the radiation is in the forward direction, we easily see that in Ω_{n} equation (9) yields the relation

$$\frac{1}{c}I_t(x,t,\hat{\theta}) + \hat{\theta} \cdot \nabla I(x,t,\hat{\theta}) + \tilde{\mu}_a I(x,t,\hat{\theta}) = 0,$$

where $\tilde{\mu}_a > 0$ is the absorption coefficient that is assumed to be constant in Ω_{n}. For the time-harmonic case, $I(x,t,\hat{\theta}) = \hat{I}(x,\hat{\theta})e^{-i\omega t}$, we have

$$(\tilde{\mu}_a - ik)\hat{I} + \hat{\theta} \cdot \nabla\hat{I} = 0,$$

yielding an attenuated plane wave solution

$$\hat{I} \sim e^{-(\tilde{\mu}_a - ik)\hat{\theta}\cdot x},$$

where $k = \omega/c$.

Let $x \in \partial\Omega_{\mathrm{n}}$ be a boundary point of the non-scattering region. Denote the unit normal vector of $\partial\Omega_{\mathrm{n}}$ pointing into Ω_{n} by $\nu = \nu(x)$ and let $\hat{\theta} \in S^{n-1}$ be a direction vector with $\hat{\theta} \cdot \nu(x) < 0$. Let $y(\hat{\theta}) \in \partial\Omega_{\mathrm{n}}$ be the first boundary point where the line emanating from x into direction $-\hat{\theta}$ hits the boundary $\partial\Omega_{\mathrm{n}}$. Since the radiation propagates with no scattering, the contribution to the radiation flux density at x, originating from direction $-\hat{\theta}$, is

$$d\vec{J}(x,\hat{\theta}) = \hat{\theta}\hat{I}(y,\hat{\theta})e^{-(\tilde{\mu}_a - ik)\hat{\theta}\cdot(x-y)}ds(\hat{\theta}).$$

Using the above expression and assuming that the total flux into the non-scattering region distributes uniformly to all directions, i.e. $\hat{I}(y,\hat{\theta}) = \hat{I}_0(y)$, through a straightforward geometrical consideration one obtains the following dependence between the total fluxes in (+) and out (−) of the non-scattering region (for details, see [I]):

$$\begin{aligned}\Phi_-(x) &= \frac{n-1}{|S^{n-2}|}\int_{\partial\Omega_{\rm n}} v(x,y)\frac{(\nu(x)\cdot(x-y))(\nu(y)\cdot(x-y))}{|x-y|^{n+1}}\times \\ &\quad e^{-(\tilde\mu_a-ik)|x-y|}\Phi_+(y)dS(y) \\ &= \mathcal{G}\Phi_+(x), \end{aligned} \tag{14}$$

where $v(x,y)$ is a visibility function,

$$v(x,y)=\begin{cases}1, & \text{if } tx+(1-t)y\in\Omega_{\rm n} \text{ for } 0<t<1,\\ 0, & \text{otherwise.}\end{cases}$$

Thus, we have obtained a relation between the inward and outward fluxes on the boundary of the non-scattering region, which gives us the means to handle the transparent regions using non-local boundary conditions.

2.2.3 Forward problem

Let us consider the time-harmonic radiosity-diffusion forward problem of optical tomography in a bounded domain Ω consisting of a non-scattering open region $\Omega_{\rm n}$, with $\partial\Omega\cap\partial\Omega_{\rm n}=\emptyset$, and a strongly scattering region $\Omega\setminus\bar\Omega_{\rm n}$. Assume that the time-harmonic flux $\Phi_{\rm in}(x)e^{-i\omega t}$ is conducted through the object boundary $\partial\Omega$. Solving equation (12) for the time-harmonic amplitude of the energy current density and substituting in (11), we see that the time-harmonic amplitude of the energy fluency (also denoted by φ) satisfies

$$\nabla\cdot K\nabla\varphi+(ik-\mu_a)\varphi=0 \quad \text{in } \Omega\setminus\bar\Omega_{\rm n}, \tag{15}$$

where $k=\omega/c$ and

$$K=\frac{1}{n}((\mu_a-ik)I+(I-B)\mu_s)^{-1}.$$

Further, by using identity (13), together with (12) and (14), one obtains the outer boundary condition

$$\gamma\varphi+\frac{1}{2}\nu\cdot K\nabla\varphi=\Phi_{\rm in} \quad \text{on } \partial\Omega, \tag{16}$$

where the sign of $\Phi_{\rm in}$ is inverted for convenience, and the non-local inner boundary condition

$$\nu\cdot K\nabla\varphi=-2\gamma(I-\mathcal{G})^{-1}(I+\mathcal{G})\varphi \quad \text{on } \partial\Omega_{\rm n}. \tag{17}$$

In these formulae the normal vectors point out of the strongly scattering region $\Omega\setminus\bar\Omega_{\rm n}$.

The main result of [I] tells that the radiosity-diffusion forward problem, obtained as a combination of (15), (16) and (17), is well-defined and has a unique weak solution under physically reasonable conditions:

Theorem 2.2. *Assume that $0 < c_a < \mu_a < C_a$ and $0 < \mu_s < C_s$. Then for any input flux $\Phi_{\rm in} \in H^{-1/2}(\partial\Omega)$ the time-harmonic radiosity-diffusion forward problem has a unique weak solution $\varphi \in H^1(\Omega \setminus \bar{\Omega}_{\rm n})$.*

Note that here we have formulated the uniqueness result for $H^{-1/2}$-inputs although in [I] it was assumed that the used fluxes are square integrable. However, by replacing some of the inner products of $L^2(\partial\Omega)$ by dual evaluations between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ when proving the unique existence in [I], it is easy to see that the result holds also for this milder regularity assumption.

During the pre-examination process of this thesis I was advised that there are connections between [I] and earlier work that I was not aware of at the time of writing [I]. In [I] I should have cited, e.g., [9], [22] and some of the references therein. I apologize for the inconvenience and thank the pre-examiners for pointing out my mistake.

I would also like to note that the proof of Lemma 3.6 in [I] is slightly incorrect since the chain of inequalities is not valid for $u = 0$. Furthermore, the claim of the lemma does not follow straight away from the inequality presented in the proof. However, by making the estimations of the proof in a slightly different order, it is easily deduced that

$$||\mathcal{G}u||_{L^1(\partial\Omega_{\rm n})} \leq c\,||u||_{L^1(\partial\Omega_{\rm n})}, \quad 0 < c < 1,$$

for all $u \in L^1(\partial\Omega_{\rm n})$, which validates Lemma 3.6 of [I]. These imperfections do not affect the results of the work as a whole.

2.2.4 Inverse problem

Because the introduction of non-scattering regions to the diffusion model is a somewhat new concept [4, 5], we will consider the inverse problem of optical tomography in the simplified situation where the whole object Ω is strongly scattering, i.e. $\Omega_{\rm n} = \emptyset$ and the inner boundary condition (17) may be forgotten. According to the above derived mathematical model, to know all pairs of inward and outward photon fluxes on the object boundary $\partial\Omega$ is equivalent of knowing the Robin-to-Robin boundary operator

$$\Upsilon : \Phi_{\rm in} \mapsto (\gamma\varphi - \frac{1}{2}\nu \cdot K\nabla\varphi)|_{\partial\Omega}, \tag{18}$$

where φ is the solution of (15) with boundary condition (16). If the assumptions of Theorem 2.2 hold, it is easy to see that Υ is a bounded linear operator from $H^{-1/2}(\partial\Omega)$ to itself [III]. We call the task of determining μ_a and K from the knowledge of Υ the inverse problem of optical tomography.

The inverse problem of optical tomography is not as well-established as the one corresponding to electrical impedance tomography due to mainly two reasons: The approximations that lead to the diffusion model are more controversial than the ones leading to the conductivity equation of EIT [4, 5, 3], and the inverse problem of optical tomography is less fundamental since there are two parameters that one would like to reconstruct. However, if one overcomes the first problem by believing in the above derived model, there are a number of mathematical results that can be generalized from electrical impedance tomography to the optical counterpart.

Uniqueness and stability. It has been demonstrated by Arridge and Lionheart [7] that the knowledge of Υ for static measurements, $k = 0$, does not allow the separable unique reconstruction of μ_a and K. However, for the time-modulated case, $k \neq 0$, in three or higher dimensions one should in principle be able to extend the uniqueness proof of Sylvester and Uhlmann [42] to determine smooth enough scalar μ_a and K uniquely—for further discussion see [5, 34, 30]. On the other hand, the two-dimensional uniqueness result of Nachman [31] cannot be generalized for optical tomography in a straightforward manner. The possibility of utilizing the uniqueness proof of Astala and Päivärinta [8] in optical tomography has not been investigated yet.

Even if the data determined the optical parameters inside the object of interest uniquely, one can safely say that the reconstruction process in optical tomography is at least as ill-posed as in electrical impedance tomography.

Reconstruction. Most of the reconstruction methods used in optical tomography are either based on a straightforward linearization of the highly non-linear inverse problem or they are Newton-type algorithms aiming at minimizing the difference between the measured and the computed data in the least squares sense [4, 5, 6]. In [III] we examine the possibility of introducing the non-iterative factorization method [25, 13] to the context of optical tomography.

3 Characterizing inclusions

In various practically important situations the background material parameters of the examined object are known but the object is contaminated with inclusions that one wants to locate. For example, detection of cracks and air bubbles in some building material and distinguishing cancerous tissue from healthy background fall into this category of problems. The factorization method, which was introduced and justified for inverse scattering by Kirsch [25] and later for electrical impedance tomography by Brühl and Hanke [13, 14], provides a tool that can be applied to this kind of situations. In what follows, we will present the factorization method in a fairly general form that is useful for our purposes.

Let $\Omega \in \mathbb{R}^n$, $n = 2, 3$, be our object of interest with known background properties and let $D \subset \Omega$ be an inhomogeneity that has unknown material parameters. Assume that A is the operator that maps the controlled input through the object boundary $\partial\Omega$ onto the measured output on $\partial\Omega$ (here the input may be flux of photons, electrical current etc.). Further, we denote by A_0 the input-to-output operator corresponding to the object without the inclusion. Note that A can be obtained through measurements and A_0 through computations. Finally, we assume that $A - A_0 \in \mathcal{L}(H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega))$.

The most essential ingredient of the factorization method is to prove that the difference of the two boundary maps can be factorized as

$$A - A_0 = LFL', \tag{19}$$

where $L \in \mathcal{L}(H^{-1/2}(\partial D), H^{1/2}(\partial\Omega))$, L' is the adjoint operator of L and $F \in \mathcal{L}(H^{1/2}(\partial D), H^{-1/2}(\partial D))$. Moreover, one needs to show that L and L' depend only on the shape of the inclusion D, not on the material parameters inside D, and that F is self-adjoint, positive definite and bijective. In order to make sure that the latter two conditions on F hold true, one must usually set some suitable conditions on the material parameters inside D. For completeness, it should be mentioned that the positivity and self-adjointness conditions of F can be relaxed as indicated in [26] and [32].

If the above introduced conditions are valid, it can be shown that the ranges of L and the square root of the restricted operator

$$\{(A - A_0)|_{L^2(\partial\Omega)}\}^{1/2} : L^2(\partial\Omega) \to L^2(\partial\Omega)$$

coincide [13]. In other words, the range $\mathcal{R}(\{(A - A_0)|_{L^2(\partial\Omega)}\}^{1/2})$, which can in principle be obtained by boundary measurements, depends only on the known background material parameters of Ω and the shape of D, not on the material parameters inside D. By comparing outputs corresponding to carefully chosen singular solutions with the range $\mathcal{R}(\{(A-A_0)|_{L^2(\partial\Omega)}\}^{1/2})$ in a clever way, one may try to extract the information about the shape of D from the measurements and characterize the inclusion explicitly—in particular, this works for electrical impedance tomography and optical tomography as we shall see in the next two subsections.

Note that the above reasoning holds true for $A_0 - A$, instead of $A - A_0$, if the inner boundary operator F happens to be negative definite. Furthermore, notice that the factorization method also works if the Sobolev trace spaces used above are replaced by the subspaces $H_0^{-1/2}(\partial\Omega)$, $H_0^{1/2}(\partial\Omega)$, $H_0^{-1/2}(\partial D)$, $H_0^{1/2}(\partial D)$ and $L_0^2(\partial\Omega)$, which are defined in the spirit of (3).

3.1 Factorization method in EIT

Let us assume that the conductivity inside the examined object Ω is of the simple form

$$\sigma = \begin{cases} \kappa & \text{in } D, \\ 1 & \text{in } \Omega \setminus \bar{D}, \end{cases}$$

where $\kappa < 1$ is a positive constant and $D \subset \Omega$ is an open connected set with connected complement and a smooth boundary $\partial D \cap \partial\Omega = \emptyset$. In these considerations we will assume that the input current is static in time, meaning that $\omega = 0$ and $\gamma = \sigma$ in (1). In what follows, we will denote by Λ_σ the Neumann-to-Dirichlet map corresponding to the above defined conductivity and by $\Lambda_{\mathbf{1}}$ the one corresponding to the unit conductivity.

Martin Brühl demonstrated in [13] that under the above assumptions the difference of the two Neumann-to-Dirichlet maps has the required factorization

$$\Lambda_\sigma - \Lambda_{\mathbf{1}} = \tilde{L}\tilde{F}\tilde{L}',$$

with $\tilde{F} : H_0^{1/2}(\partial D) \to H_0^{-1/2}(\partial D)$ self-adjoint, positive definite and isomorphic, and $\tilde{L} : H_0^{-1/2}(\partial D) \to H_0^{1/2}(\partial\Omega)$, $\tilde{L}' : H_0^{-1/2}(\partial\Omega) \to H_0^{1/2}(\partial D)$ bounded and dual to each other. Moreover, $\tilde{L}$ is defined by

$$\tilde{L} : \phi \mapsto v|_{\partial\Omega},$$

where $v \in H^1(\Omega \setminus \bar{D})$ integrates to zero over $\partial\Omega$ and satisfies

$$\Delta v = 0 \quad \text{in } \Omega \setminus \bar{D}, \quad \frac{\partial v}{\partial \nu} = \phi \quad \text{on } \partial D, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

It should be noted that Nachman obtained a factorization of the same kind already in [31].

Let $\Phi_y \in C^\infty(\Omega \setminus \{y\})$ be the singular solution of the homogeneous Neumann problem

$$\Delta\Phi(x) = \hat{\alpha} \cdot \nabla\delta(x-y) \quad \text{in } \Omega, \quad \frac{\partial \Phi}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \quad \int_{\partial\Omega} \Phi dS = 0, \qquad (20)$$

where $y \in \Omega$ is a parameter, δ is the delta functional and $\hat{\alpha} \in \mathbb{R}^n$ is a unit vector. By using the fact $\mathcal{R}(\{(\Lambda_\sigma - \Lambda_{\mathbf{1}})|_{L^2(\partial\Omega)}\}^{1/2}) = \mathcal{R}(\tilde{L})$, Brühl proved the following theorem that gives an explicit characterization for the inclusion D. Note that he also generalized the theorem for the case of multiple inclusions and non-constant conductivities [13]. It is also worth noticing that the case $\kappa > 1$ can be handled by considering $\Lambda_{\mathbf{1}} - \Lambda_\sigma$ instead of $\Lambda_\sigma - \Lambda_{\mathbf{1}}$.

Theorem 3.1. *The Dirichlet boundary value of the singular solution to* (20), *$\Phi_y|_{\partial\Omega}$, belongs to the range of $\{(\Lambda_\sigma - \Lambda_{\mathbf{1}})|_{L^2(\partial\Omega)}\}^{1/2}$ if and only if $y \in D$.*

Since the continuum forward model of electrical impedance tomography is not compatible with measurements, the above theorem does not straight away induce any algorithm that could be used with real data. However, in [II] the factorization method is considered in the framework of the complete electrode model and it is demonstrated that the method works in a limit sense:

Assume once again that the boundary of the object Ω is partially covered with electrodes $e_m \subset \partial\Omega$, $1 \le m \le M$, which are identified by the parts of the surface that they cover and used for both current injection and voltage measurement. Furthermore, assume that the measurements are modelled with the complete electrode forward model introduced in Subsection 2.1.3 and let us also adopt the notations used in that subsection. In particular, let $R_\sigma, R_{\mathbf{1}} : T_0 \to T_0$ denote the electrode current to electrode potential maps corresponding to the piecewise constant conductivity σ and the unit conductivity, respectively. Here we have used the fact that $T/\mathbb{C} \sim T_0 \sim \mathbb{C}^{M-1}$ and chosen the ground level of the potential in the correct way.

In [II] it is shown that the following factorization is valid:

$$R_\sigma - R_{\mathbf{1}} = LFL', \tag{21}$$

where $F : H_0^{1/2}(\partial D) \to H_0^{-1/2}(\partial D)$ is self-adjoint, positive definite and bijective and $L : H_0^{-1/2}(\partial D) \to T_0$, $L' : T_0 \to H_0^{1/2}(\partial D)$ are bounded and dual to each other. In particular,

$$L : \phi \mapsto V,$$

where V is the electrode potential part of the solution $(v, V) \in H^1(\Omega \setminus \bar{D}) \oplus T_0$ to

$$\begin{aligned}
&\Delta v = 0 && \text{in } \Omega \setminus \bar{D},\\
&\tfrac{\partial v}{\partial \nu} = \phi && \text{on } \partial D,\\
&\tfrac{\partial v}{\partial \nu} = 0 && \text{on } \partial\Omega \setminus \bar{\Gamma}_e\\
&v + z\tfrac{\partial v}{\partial \nu} = V && \text{on } \Gamma_e,\\
&\tfrac{1}{|e_m|}\int_{e_m} \tfrac{\partial v}{\partial \nu} dS = 0, && 1 \le m \le M.
\end{aligned}$$

Although factorization (21) resembles quite a bit the one given in (19), there is no hope of characterizing D without having to draw upon some kind of limit process since $R_\sigma - R_{\mathbf{1}}$ is a finite-dimensional operator. In consequence, we consider a sequence of electrode configurations $\{\mathcal{T}_M\}$ defined by

$$\mathcal{T}_M = \{e_1^M, \ldots, e_M^M \subset \partial\Omega \mid e_l^M \cap e_m^M = \emptyset \text{ if } l \neq m\}, \quad \Gamma_M = \cup_{m=1}^M e_m^M,$$

for $M \in \mathbb{N}$. We assume that the electrodes cover the boundary $\partial\Omega$ in a well-organized manner as M goes to infinity:

$$|\partial\Omega \setminus \Gamma_M|, \ \max_{1 \le m \le M} d(e_m^M) \to 0 \quad \text{when } M \to \infty,$$

where $d(e_m^M)$ is the diameter of e_m^M, i.e. $d(e_m^M) = \sup_{x,y \in e_m^M} |x - y|$. The spaces T^M and T_0^M, corresponding to the electrode configuration $\mathcal{T}_M$, are defined in accordance with (5) and (6), respectively. We will also use similar index notation for operators depending on the electrode configuration.

Suppose that the above assumptions on the electrode configurations $\{\mathcal{T}_M\}$ are valid and let $\{\alpha_M\} \subset \mathbb{R}_+$ be a sequence of regularization parameters. Consider the minimizers $\{I^M\} \subset L_0^2(\partial\Omega)$, $I^M \in T_0^M$, of the Tikhonov functionals

$$\left|\left|(R_\sigma^M - R_{\mathbf{1}}^M)^{1/2} I - \Phi_y\right|\right|_{L^2(\partial\Omega)}^2 + \alpha_M \left|\left|I\right|\right|_{L^2(\partial\Omega)}^2, \quad I \in T_0^M, \ M \in \mathbb{N}, \tag{22}$$

where $(R_\sigma^M - R_{\mathbf{1}}^M)^{1/2} : T_0^M \to T_0^M$ is the unique, positive, self-adjoint square root of $R_\sigma^M - R_{\mathbf{1}}^M$. Since $R_\sigma^M - R_{\mathbf{1}}^M$ can be obtained through electrode measurements, so can $(R_\sigma^M - R_{\mathbf{1}}^M)^{1/2}$. Hence, the behaviour of the sequence $\{I^M\}$ is in principle something that can be observed by non-invasive methods. The following electrode counterpart of Theorem 3.1 is the second main result of [II].

Theorem 3.2. *Assume that the contact impedance z is smooth. Let $\{I^M\} \subset L_0^2(\partial\Omega)$, $I^M \in T_0^M$, be the minimizing sequence for the functionals* (22) *and assume that $\{\alpha_M\} \subset \mathbb{R}_+$ converges to zero but is such that the sequence*

$$\left\{ \frac{\inf_{V \in T^M} ||\Phi_y - V||^2_{L^2(\partial\Omega)}}{\alpha_M} \right\}$$

is bounded. Then $y \in D$ if and only if the sequence $\{I^M\}$ is bounded in $L_0^2(\partial\Omega)$.

The proof of Theorem 3.2 is based on factorization (21) and known mapping properties of $\tilde{L}$ together with the fact that the operator L approximates $\tilde{L}$ better and better as the electrodes get smaller and smaller and cover the boundary $\partial\Omega$ more and more tightly. The proof is quite laborious; the complete version can be found in [II]. As in the case of the continuum model, the situation $\kappa > 1$ can be handled by considering $R_{\mathbf{1}} - R_\sigma$ instead of $R_\sigma - R_{\mathbf{1}}$.

In real life one is, naturally, not able to construct a sequence of electrode configurations with the properties given above. However, when conducting measurements with a fixed setting of electrodes that are relatively small and cover a large portion of the object boundary, Theorem 3.2 gives a reason to believe that the electrode currents needed for minimizing functional (22), with a fixed small $\alpha > 0$, are larger when $y \in \Omega \setminus D$ than when $y \in D$. This observation leads to a possibility of numerical implementation.

3.2 Diffusive inclusions in optical tomography

Let us return to the inverse problem of optical tomography and consider a strongly scattering physical body $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. In what follows, we will use the notation introduced in Subsection 2.2. In addition, we will assume that the used input fluxes are static in time, meaning that $k = 0$ in the formulae of Subsection 2.2.

Assume that the diffusion matrix and the absorption coefficient inside our object of interest Ω are of the form

$$K = \begin{cases} K_0 + \kappa & \text{in } D, \\ K_0 & \text{in } \Omega \setminus \bar{D}, \end{cases} \qquad \mu = \begin{cases} \mu_0 + \delta & \text{in } D, \\ \mu_0 & \text{in } \Omega \setminus \bar{D}, \end{cases} \tag{23}$$

where $K_0, \mu_0 \in C^\infty(\Omega)$ are the known background diffusion tensor and absorption coefficient, respectively, D is an open connected subset of Ω with connected complement and a smooth boundary $\partial D \cap \partial\Omega = \emptyset$ and $\kappa, \delta \in C^\infty(\bar{D})$ are the perturbations corresponding to D. In what follows, we will denote the Robin-to-Robin boundary map corresponding to (K, μ) by Υ and the map corresponding to (K_0, μ_0) by Υ_0. The definitions of $\Upsilon, \Upsilon_0 : H^{-1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ follow in a natural way from (18).

Although the operators Υ and Υ_0 are not smoothening as such, their difference maps $H^{-1/2}(\partial\Omega)$ to $H^{1/2}(\partial\Omega)$ (see [III]). Thus, we may try to apply the factorization method, as described in the beginning of this section, to the characterization of the inclusion D. In fact, one can quite easily demonstrate that the following formula holds [III]:

$$\Upsilon - \Upsilon_0 = LFL',$$

where $F : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ is bounded and self-adjoint, and $L : H^{-1/2}(\partial D) \to H^{1/2}(\partial\Omega)$, $L' : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial D)$ are bounded and dual to each other. Moreover, L is defined through

$$L : \Psi \mapsto (\gamma v - \frac{1}{2}\nu \cdot K_0 \nabla v)|_{\partial\Omega},$$

where $v \in H^1(\Omega \setminus \bar{D})$ is the unique solution of

$$\begin{aligned} \nabla \cdot K_0 \nabla v - \mu_0 v = 0 \qquad & \text{in } \Omega \setminus \bar{D}, \\ \gamma v + \tfrac{1}{2}\nu \cdot K_0 \nabla v = 0 \qquad & \text{on } \partial\Omega, \\ \gamma v + \tfrac{1}{2}\nu \cdot K_0 \nabla v = \Psi \qquad & \text{on } \partial D. \end{aligned}$$

Here the unit normals point out of $\Omega \setminus \bar{D}$.

In order to be able to proceed further, one needs to make sure that the intermediate operator F is positive definite and bijective. As shown in [III], a sufficient condition for this to be valid is the following: K_0 and κ are scalar functions, $\kappa|_{\partial D} < 0$ and

$$\kappa \leq 0 \quad \text{and} \quad \delta \leq 0 \quad \text{in } D. \tag{24}$$

As discussed in the beginning of this section, if these conditions hold, the ranges of $\{(\Upsilon - \Upsilon_0)|_{L^2(\partial\Omega)}\}^{1/2}$ and L coincide.

Before we can formulate the main result of [III] concerning the characterization of the inclusion D, we still need to introduce a suitable singular solution for scanning the object Ω. Let $h_y \in C^\infty(\Omega \setminus \{y\})$ be the solution of the following homogeneous Robin problem:

$$\begin{aligned} \nabla \cdot K_0 \nabla h(x) - \mu_0 h(x) = \delta(x - y) \qquad & \text{in } \Omega, \\ \gamma h + \tfrac{1}{2}\nu \cdot K_0 \nabla h = 0 \qquad & \text{on } \partial\Omega, \end{aligned} \tag{25}$$

where $y \in \Omega$ is a parameter and δ is the delta functional. Because h_y is singular at y, the following result is obtained by comparing the range of L with the outward flux corresponding to h_y.

Theorem 3.3. *Let $K_0, \mu_0 \in C^\infty(\Omega)$ and $\kappa, \delta \in C^\infty(\bar{D})$ be scalar functions, $\kappa|_{\partial D} < 0$ and assume that* (24) *holds. Then the outward flux $(\gamma h_y - \frac{1}{2}\nu \cdot K_0 \nabla h_y)|_{\partial\Omega}$, corresponding to the singular solution of* (25), *belongs to the range of $\{(\Upsilon - \Upsilon_0)|_{L^2(\partial\Omega)}\}^{1/2}$ if and only if $y \in D$.*

When proving the above result in [III], a part of the analysis is done with anisotropic diffusion coefficients. This is somewhat misleading and mostly in vain since the final result is for the isotropic case only.

There are a few things worth of noticing. First, if the inequality conditions on the perturbations κ and δ are turned the other way around, then Theorem 3.3 holds for $\Upsilon_0 - \Upsilon$. Second, the theorem can easily be generalized for the case of multiple inclusions by following the guidelines suggested in [III]. Third, in Theorem 3.3 the behaviour of the diffusion tensor is more important than the behaviour of the absorption coefficient since a strict inequality is posed only on κ. In consequence, the factorization method is applicable to the characterization of purely diffusive inclusions whereas there is no guarantee that it would work for purely absorbing inhomogeneities.

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