Helsinki University of Technology Institute of Mathematics Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2004 A477

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Tuomo Kuusi: *Moser's Method for a Nonlinear Parabolic Equation*; Helsinki University of Technology Institute of Mathematics Research Reports A477 (2004).

Abstract: *We show the Harnack type estimate for a weak solution of the equation*

$$div(|Du|^{p-2}Du) = \frac{\partial(u^{p-1})}{\partial t}$$

by using Moser's method.

AMS subject classifications: 35B55

Keywords: Doubly nonlinear parabolic, Caccioppoli estimates, Harnack's inequality, parabolic BMO

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ISBN 951-22-7369-1
ISSN 0784-3143

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1 Introduction

Let Ω be a domain in $\mathbb{R}^n$, $t_1 < t_2$ and $1 < p < \infty$. We define a positive weak supersolution (subsolution) of the equation

$$\operatorname{div}(|Du|^{p-2}Du) = \frac{\partial(u^{p-1})}{\partial t} \tag{1}$$

as a function in the local parabolic Sobolev space $L^p_{loc}(t_1, t_2; W^{1,p}_{loc}(\Omega))$ satisfying the equation

$$\int_{t_1}^{t_2} \int_{\Omega} (|Du|^{p-2}Du \cdot D\eta - u^{p-1}\frac{\partial \eta}{\partial t})\, dxdt \geq \ (\leq)\, 0 \tag{2}$$

for all $\eta \in C_0^\infty(\Omega \times (t_1, t_2))$, $\eta \geq 0$. A weak solution of equation (1) is both supersolution and subsolution.

As far as we know, equation (1) occured the first time in [Tru68], where the Harnack inequality for a weak solution was proved. The proof was given via parabolic BMO. The article generalized Moser's famous article [Mo64]. The main result in [Mo64] was the parabolic version of the well-known John-Nirenberg Lemma. Twenty years later, in [FaGa], the proof of this Main Lemma was further simplified. The approach via BMO, however, is technically involved. Consequently, we prefer to give a simple proof using Moser's techniques in [Mo71]. In the article he used ideas of Bombieri [Bomb], [BoGi]. Some generalizations to Moser's article were also made in chapter 5 of [SaCo].

We show that the parabolic Harnack inequality holds for a weak solution of (1). For any fixed $0 < \sigma \leq 1$, $\tau \in \mathbb{R}$ and for a ball $B(z, r) \subset \mathbb{R}^n$, $r > 0$, we define

$$\sigma U^+ = B(z, \sigma r) \times (\tau + \frac{1}{2}r^p - \frac{1}{2}(\sigma r)^p, \tau + \frac{1}{2}r^p + \frac{1}{2}(\sigma r)^p),$$

$$\sigma U^- = B(z, \sigma r) \times (\tau - \frac{1}{2}r^p - \frac{1}{2}(\sigma r)^p, \tau - \frac{1}{2}r^p + \frac{1}{2}(\sigma r)^p)$$

and

$$Q = B(z, r) \times (\tau - r^p, \tau + r^p).$$

Our result is the following theorem.

Theorem 1.1. *Let $u \geq \rho > 0$ be a weak solution of equation (1) in Q and let $0 < \sigma < 1$. Then we have*

$$\operatorname*{ess\,sup}_{\sigma U^-} u \leq C \operatorname*{ess\,inf}_{\sigma U^+} u, \tag{3}$$

where the constant C depends only on n, p and σ.

It has come to our attention that a similar question has been studied in a recent work by Gianazza and Vespri [GiVe] using a different method.

A well-known result is that the Hölder continuity of the solution in the parabolic case is a consequence of the Harnack inequality [Mo64] as $p = 2$. In general, due to the nonlinearity of the term $(u^{p-1})_t$, the same proof is not valid for equation (1). This leaves the role of the Harnack inequality open in the proof of solution's Hölder continuity.

The method used here also covers more general equations

$$\frac{\partial(u^{p-1})}{\partial t} - \mathrm{div} A(x, u, Du) = 0,$$

where the function A is only assumed to be measurable and satisfy the structural conditions (see e.g. [DiBe], [DBUV], [WZYL])

$$\begin{aligned} A(x, u, Du) \cdot Du &\geq C_0 |Du|^p, \\ |A(x, u, Du)| &\leq C_1 |Du|^{p-1}, \end{aligned}$$

where C_0 and C_1 are positive constants. We can use similar argumentation as in [Tru68] to show Caccioppoli type estimates in section 2.1. After that our method does not use any information about the equation. Therefore, for expository purposes, we only consider equation (1).

I would like to express my gratitude to Juha Kinnunen for his encouragement and valuable suggestions. I also wish to express my thanks to U. Gianazza and V. Vespri for their helpful interest.

1.1 Preliminary results

In the appendix we show some consequences of the definition of a supersolution (subsolution). Especially, we show that it is possible to substitute a test function depending on u in (2). The result is

$$\begin{aligned} 0 \ \leq (\geq) \ & (p-1) \int_{\tau_1}^{\tau_2} \int_\Omega |Du|^p u^{p-2} f''(u^{p-1}) \eta \, dx d\tau \\ + \ & \int_{\tau_1}^{\tau_2} \int_\Omega |Du|^{p-2} Du \cdot D\eta f'(u^{p-1}) \, dx d\tau \\ + \ & \left[\int_\Omega f(u^{p-1}) \eta \, dx \right]_{\tau=\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2} \int_\Omega f(u^{p-1}) \frac{\partial \eta}{\partial \tau} \, dx d\tau \end{aligned} \tag{4}$$

for almost every τ_1, τ_2, $t_1 < \tau_1 < \tau_2 < t_2$, where $f \in C^2(0, \infty)$, $f' \geq 0$ and f' is bounded on the range of u^{p-1}.

We start with an elementary lemma.

Lemma 1.1. *Suppose that $u \geq \rho > 0$ is a supersolution. Then $v = u^{-1}$ is a subsolution.*

Proof. Let $\eta \in C_0^\infty(\Omega \times (t_1, t_2))$, $\eta \geq 0$, and τ_1, τ_2 be such that $\eta(x, \tau_1) = \eta(x, \tau_2) = 0$ for every $x \in \Omega$. By substituting the function $f(s) = -s^{-1}$ in (4) we have

$$\begin{aligned} 0 &\leq -2(p-1) \int_{\tau_1}^{\tau_2} \int_\Omega |Du|^p u^{1-2p} \eta \, dx d\tau \\ &+ \int_{\tau_1}^{\tau_2} \int_\Omega |Du|^{p-2} Du \cdot D\eta \, u^{2(1-p)} \, dx d\tau + \int_{\tau_1}^{\tau_2} \int_\Omega u^{1-p} \frac{\partial \eta}{\partial \tau} \, dx d\tau \\ &\leq - \int_{\tau_1}^{\tau_2} \int_\Omega |Dv|^{p-2} Dv \cdot D\eta - v^{p-1} \frac{\partial \eta}{\partial \tau} \, dx d\tau. \end{aligned}$$

and the result follows by letting $\tau_1 \to t_1$ and $\tau_2 \to t_2$. $\square$

Next, we formulate two well known results. The first lemma is a straightforward consequence of standard Sobolev's inequality and the proof can be found from [DiBe].

Lemma 1.2. *Suppose that $u \in L^p((t_1, t_2); W_0^{1,p}(\Omega))$, where $\kappa > 1$ Then there exists a constant $C = C(n, p, \kappa)$ such that*

$$\int_{t_1}^{t_2} \int_\Omega |u|^{\kappa p} \, dx dt \leq C \int_{t_1}^{t_2} \int_\Omega |Du|^p \, dx dt \cdot \left(\operatorname*{ess\,sup}_{t_1 < t < t_2} \int_\Omega |u|^{(\kappa - 1)n} \, dx \right)^{\frac{p}{n}}.$$

Following [SaCo], we call the next result as an Abstract Lemma. The original idea is due to Bombieri [Bomb].

Lemma 1.3. *Fix $0 < \delta < 1$. Let γ, C be positive constants and $0 < \alpha_0 \leq \infty$. Let f be a positive measurable function on $U_1 = U$ which satisfies the reverse Hölder type of inequality*

$$\|f\|_{\alpha_0, U_{\sigma'}} \leq \left(\frac{C}{(\sigma - \sigma')^\gamma} \mu(U)^{-1} \right)^{1/\alpha - 1/\alpha_0} \|f\|_{\alpha, U_\sigma},$$

where $U_{\sigma'} \subset U_\sigma \subset \mathbb{R}^n$ for all σ, σ', α such that $0 < \delta \leq \sigma' < \sigma \leq 1$ and $0 < \alpha \leq \min\{1, \alpha_0/2\}$. Assume further that f satisfies

$$\mu(\{x \in U \mid \log f > \lambda\}) \leq C \mu(U) \lambda^{-1}$$

for all $\lambda > 0$. Then

$$\|f\|_{\alpha_0, U_\delta} \leq A \mu(U)^{1/\alpha_0}$$

where A depends only on δ, γ, C and the lower bound on α_0.

Remark The assumption that $\log f$ belongs to weak $L^1(U)$, can be relaxed. It is sufficient to assume that $\log f$ belongs to weak $L^\eta(U)$ with any positive η. One can check this easily (Moser's proof in [Mo71]). Actually, in the proof, the only thing where we use the term λ^{-1} is that $\log(\lambda)/\lambda$ tends to zero as λ tends to infinity. Naturally, this is also true for $\log(\lambda)/\lambda^\eta$.

2 Estimates for super- and subsolutions

2.1 Caccioppoli estimates

Results stated in following three lemmata are essentially consequences of the inequality (4) proved in the appendix for a supersolution (subsolution).

Lemma 2.1. *Suppose that $u \geq \rho > 0$ is a supersolution and let $0 < \varepsilon < p-1$. Then there exists a constant $C = C(\varepsilon, p)$ such that*

$$\int_{t_1}^{t_2} \int_\Omega |Du|^p u^{-\varepsilon-1} \varphi^p \, dxdt + \operatorname*{ess\,sup}_{t_1<t<t_2} \int_\Omega u^{p-1-\varepsilon} \varphi^p \, dx$$

$$\leq C \int_{t_1}^{t_2} \int_\Omega u^{p-\varepsilon-1} |D\varphi|^p \, dxdt + C \int_{t_1}^{t_2} \int_\Omega u^{p-\varepsilon-1} \varphi^{p-1} \left| \frac{\partial \varphi}{\partial t} \right| \, dxdt$$

holds for every $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ with $\varphi \geq 0$.

Proof. We choose the function f so that

$$f(u^{p-1}) = \frac{p-1}{p-1-\varepsilon} u^{p-1-\varepsilon}, \quad f'(u^{p-1}) = u^{-\varepsilon}, \quad f''(u^{p-1}) = -\frac{\varepsilon}{p-1} u^{1-p-\varepsilon}$$

and $\eta = \varphi^p$, where $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ and $\varphi \geq 0$. Substitution of f and η in (4) gives

$$\begin{aligned}
0 \quad \leq \quad & -\varepsilon \int_{\tau_1}^{\tau_2} \int_\Omega |Du|^p u^{-\varepsilon-1} \varphi^p \, dxdt \\
& + p \int_{\tau_1}^{\tau_2} \int_\Omega |Du|^{p-1} \varphi^{p-1} |D\varphi| \, u^{-\varepsilon} \, dxdt \\
& + \frac{p(p-1)}{p-\varepsilon-1} \int_{\tau_1}^{\tau_2} \int_\Omega u^{p-\varepsilon-1} \left| \frac{\partial \varphi}{\partial t} \right| \varphi^{p-1} \, dxdt \\
& + \frac{p-1}{p-\varepsilon-1} \left[\int_\Omega u^{p-\varepsilon-1} \varphi^p \, dx \right]_{t=\tau_1}^{\tau_2} \\
= \quad & -\varepsilon I_1 + p I_2 + \frac{p(p-1)}{p-\varepsilon-1} I_3 + \frac{p-1}{p-\varepsilon-1} I_4.
\end{aligned}$$

Young's inequality yields

$$\begin{aligned}
I_2 \quad = \quad & \int_{\tau_1}^{\tau_2} \int_\Omega \left(|Du| \varphi u^{-\frac{\varepsilon+1}{p}} \right)^{p-1} \left(|D\varphi| u^{-\varepsilon + (\varepsilon+1)\frac{p-1}{p}} \right) dxdt \\
\leq \quad & \gamma I_1 + c(\gamma) \int_{\tau_1}^{\tau_2} \int_\Omega |D\varphi|^p u^{-\varepsilon p + (\varepsilon+1)(p-1)} \, dxdt \\
= \quad & \gamma I_1 + c(\gamma) \int_{\tau_1}^{\tau_2} \int_\Omega |D\varphi|^p u^{p-\varepsilon-1} \, dxdt,
\end{aligned}$$

where $\gamma > 0$. Thus, we have

$$I_1 - \frac{2p(p-1)}{\varepsilon(p-\varepsilon-1)} I_3 \leq \frac{2p\, c(\varepsilon/2p)}{\varepsilon} \int_{\tau_1}^{\tau_2} \int_\Omega |D\varphi|^p u^{p-\varepsilon-1} \, dxdt + \frac{2(p-1)}{\varepsilon(p-\varepsilon-1)} I_4,$$

where we have chosen $\gamma = \varepsilon/2p$. Furthermore, by choosing $\tau_2 < t_2$ and $\tau_1 > t_1$ such that

$$\int_\Omega u^{p-1-\varepsilon}(x,\tau_1)\varphi^p(x,\tau_1)\,dx \geq \frac{1}{2}\operatorname*{ess\,sup}_{t_1<t<t_2}\int_\Omega u^{p-1-\varepsilon}\varphi^p\,dx$$

and $\varphi(x,\tau_2) = 0$ for every $x \in \Omega$, we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_1<t<t_2}\int_\Omega u^{p-1-\varepsilon}\varphi^p\,dx \\
\leq& C\int_{\tau_1}^{t_2}\int_\Omega |D\varphi|^p u^{p-\varepsilon-1}\,dxdtx + C\int_{\tau_1}^{t_2}\int_\Omega u^{p-1-\varepsilon}\left|\frac{\partial\varphi}{\partial t}\right|\varphi^{p-1}\,dxdt \\
\leq& C\int_{t_1}^{t_2}\int_\Omega |D\varphi|^p u^{p-\varepsilon-1}\,dxdt + C\int_{t_1}^{t_2}\int_\Omega u^{p-1-\varepsilon}\left|\frac{\partial\varphi}{\partial t}\right|\varphi^{p-1}\,dxdt
\end{aligned}$$

for the parabolic term. The result follows now easily with the constant C depending on ε and p and having singularities at $\varepsilon = 0$ and $\varepsilon = p-1$. $\square$

Next, we want to show a corresponding result for a subsolution. Observe that in the following lemma we may have quantities which are a priori not necessarily finite. Nevertheless, we can make our calculations with (12) instead of (4). After we have control of the quantities, we obtain results by letting k tend to infinity by the dominated convergence theorem. In fact, this also justifies formal calculations in the proof of Lemma 2.5.

Lemma 2.2. *Suppose that $u \geq \rho > 0$ is a subsolution and let $\varepsilon > 0$. Then there exists a constant $C = C(\varepsilon, p)$, which is such that*

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega |Du|^p u^{\varepsilon-1}\varphi^p\,dxdt + \operatorname*{ess\,sup}_{t_1<t<t_2}\int_\Omega u^{p-1+\varepsilon}\varphi^p\,dx \\
\leq& C\int_{t_1}^{t_2}\int_\Omega u^{p-1+\varepsilon}|D\varphi|^p\,dxdt + C\int_{t_1}^{t_2}\int_\Omega u^{p-1+\varepsilon}\varphi^{p-1}\left|\frac{\partial\varphi}{\partial t}\right|\,dxdt
\end{aligned}$$

for every $\varphi \in C_0^\infty(\Omega\times(t_1,t_2))$ with $\varphi \geq 0$.

Proof. This time we choose the function f so that

$$f(u^{p-1}) = \frac{p-1}{p-1+\varepsilon}u^{p-1+\varepsilon}, \quad f'(u^{p-1}) = u^\varepsilon, \quad f''(u^{p-1}) = \frac{\varepsilon}{p-1}u^{\varepsilon-p+1}$$

and τ_1 and τ_2 such that

$$\int_\Omega u^{p-1-\varepsilon}(x,\tau_2)\varphi^p(x,\tau_2)\,dx \geq \frac{1}{2}\operatorname*{ess\,sup}_{t_1<t<t_2}\int_\Omega u^{p-1-\varepsilon}\varphi^p\,dx$$

holds. The assertion follows as in the proof of Lemma 2.1 and the constant C has a singularity point at $\varepsilon = 0$. $\square$

Finally, we show a Caccioppoli type estimate for the logarithm of a supersolution.

Lemma 2.3. *Suppose that $u \geq \rho > 0$ is a supersolution. Then there exists a constant $C = C(p)$ such that*

$$\int_{t_1}^{t_2} \int_\Omega |D(\log u)|^p \varphi^p \, dx dt + \operatorname*{ess\,sup}_{t_1<t<t_2} \int_\Omega |\log u| \, \varphi^p \, dx$$

$$\leq C \int_{t_1}^{t_2} \int_\Omega |D\varphi|^p \, dx dt + C \int_{t_1}^{t_2} \int_\Omega |\log u| \, \varphi^{p-1} \left| \frac{\partial \varphi}{\partial t} \right| \, dx dt$$

for every $\varphi \in C_0^\infty(\Omega \times (0,T))$ with $\varphi \geq 0$.

Proof. We choose the function $f(s) = \log s$ and $\eta = \varphi^p$, where $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$ and $\varphi \geq 0$. By denoting $v = \log u$ we have from (4) that

$$\begin{aligned} 0 \quad \leq \quad & -(p-1) \int_{\tau_1}^{\tau_2} \int_\Omega |Dv|^p \varphi^p \, dx dt \\ & + p \int_{\tau_1}^{\tau_2} \int_\Omega |Dv|^{p-1} |D\varphi| \, \varphi^{p-1} \, dx dt \\ & + \int_{\tau_1}^{\tau_2} \int_\Omega v \varphi^p \, dx dt \\ & - p \int_{\tau_1}^{\tau_2} \int_\Omega v \varphi^{p-1} \frac{\partial \varphi}{\partial t} \, dx dt \end{aligned}$$

where $t_1 < \tau_1 < \tau_2 < t_2$. Young's inequality for the second term yields

$$\begin{aligned} & \int_{\tau_1}^{\tau_2} \int_\Omega (|Dv|\varphi)^{p-1} |D\varphi| \, dx dt \\ \leq \quad & \frac{p-1}{2p} \int_{\tau_1}^{\tau_2} \int_\Omega |Dv|^p \varphi^p \, dx dt + c(p) \int_{\tau_1}^{\tau_2} \int_\Omega |D\varphi|^p \, dx dt, \end{aligned}$$

where $c = c(p)$ is a constant depending only on p. Consequently, we have

$$\begin{aligned} & \int_{\tau_1}^{\tau_2} \int_\Omega |Dv|^p \varphi^p \, dx dt - \left[\int_\Omega v \varphi^p \, dx \right]_{t=\tau_1}^{\tau_2} \\ & \leq C_1(p) \int_{\tau_1}^{\tau_2} \int_\Omega |D\varphi|^p \, dx dt - C_2(p) \int_{\tau_1}^{\tau_2} \int_\Omega v \varphi^{p-1} \frac{\partial \varphi}{\partial t} \, dx dt. \end{aligned} \tag{5}$$

The claim follows now in the standard way as in the proof of Lemma 2.1. □

Remark. In (5), the test function φ does not need to have a compact support in time. We will use this fact in the future.

2.2 Reverse Hölder inequality for a supersolution

For any $0 < \sigma \leq 1$, $\tau \in \mathbb{R}$, $\eta \geq 1/2$ and $B(z,r) \subset \mathbb{R}^n$ define

$$\sigma Q = \sigma Q(z, r, \tau, \eta) = Q(z, \sigma r, \tau, \eta) = B(z, \sigma r) \times (\tau - \eta(\sigma r)^p, \tau + \eta(\sigma r)^p).$$

In the following lemma the goal is to achieve a constant which is independent of s. In the standard approach from Moser [Mo64] we only need to iterate finite many times so we do not need to control the asymptotic behaviour of the constant. In our approach the number of iterations can be indefinably large and we have to make a certain geometrically convergent partition of the cylinder Q in order to achieve an uniform bound for the constant.

We will use the notation

$$\fint_\Omega f\,dxdt = \frac{1}{|\Omega|}\int_\Omega f\,dxdt.$$

Lemma 2.4. *Suppose that $u \geq \rho > 0$ is a supersolution in Q. Fix $0 < \delta < 1$. Then there exists positive constants $C = C(n,p,q,\delta,\eta)$ and $\gamma = \gamma(n,p)$ such that*

$$\left(\fint_{\sigma' Q} u^q dxdt\right)^{\frac{1}{q}} \leq \left(\frac{C}{(\sigma-\sigma')^{-\gamma}}\right)^{\frac{1}{s}}\left(\fint_{\sigma Q} u^s dxdt\right)^{\frac{1}{s}}$$

for all $0 < \delta \leq \sigma' < \sigma \leq 1$ *and for all* $0 < s < q < (p-1)(1+\frac{p}{n})$.

Proof. The starting point for the proof is the successive use of Sobolev's inequality and Caccioppoli's estimate for supersolutions. Without loosing the generality, we can assume that $\eta = 1$. We choose a function $v = \varphi u^{\alpha/p}$. Sobolev's inequality stated in Lemma 1.2 for this function gives

$$\begin{aligned}\int_Q v^{\kappa p}\,dxdt &\leq C\int_Q |D(\varphi u^{\alpha/p})|^p\,dxdt\cdot\left(\operatorname*{ess\,sup}_{t_1<t<t_2}\int_B (\varphi u^{\alpha/p})^p\,dx\right)^{\frac{p}{n}}\\ &\leq C\left(\int_Q |D(\varphi u^{\alpha/p})|^p\,dxdt + \operatorname*{ess\,sup}_{t_1<t<t_2}\int_B \varphi^p u^\alpha\,dx\right)^\kappa\\ &= (I_1+I_2)^\kappa,\end{aligned}$$

where we have denoted

$$\kappa = 1+\frac{p}{n},\quad \alpha = p-1-\varepsilon,\quad 0<\varepsilon<p-1,\quad t_1=\tau-r^p,\quad t_2=\tau+r^p.$$

Furthermore, a simple calculation yields that

$$\begin{aligned}I_1 &\leq C\int_Q u^\alpha|D\varphi|^p\,dxdt + C\alpha^p\int_Q u^{\alpha-p}|Du|^p\varphi^p\,dxdt\\ &= I_3+I_4.\end{aligned}$$

This together with Caccioppoli's estimate 2.1 for the terms I_2 and I_4 gives

$$\int_Q |\varphi u^{\alpha/p}|^{\kappa p}\,dxdt \leq C\left(\int_Q u^\alpha|D\varphi|^p\,dxdt + \int_Q u^\alpha\varphi^{p-1}\left|\frac{\partial\varphi}{\partial t}\right|\,dxdt\right)^\kappa.$$

Thus, we obtain

$$\left(\int_Q \varphi^{\kappa p}u^{\kappa\alpha}\,dxdt\right)^{\frac{1}{\kappa\alpha}}$$

$$\leq C^{\frac{p}{\alpha}} \left(\int_Q u^\alpha |D\varphi|^p \, dxdt + \int_Q u^\alpha \varphi^{p-1} \left| \frac{\partial \varphi}{\partial t} \right| \, dxdt \right)^{\frac{1}{\alpha}} . \tag{6}$$

The following step in the proof is to iterate the inequality above. We fix σ and σ'. We divide the interval (σ', σ) into k parts so that

$$\sigma_0 = \sigma, \quad \sigma_k = \sigma', \quad \sigma_j = \sigma_{j-1} - c(k) \frac{\sigma - \sigma'}{\kappa^j}, \; j = 1, \ldots, k.$$

Observe, that the constant $c(k)$ is uniformly bounded on k. More precisely, we have

$$\frac{\kappa - 1}{\kappa} \leq c(k) \leq 1$$

for every k. We have chosen such a partition because, as a result of the iteration, we need to have a constant independent of k. Next, we choose test functions which have following properties

$$\mathrm{spt}\,(\varphi_j) \subset \sigma_{j-1} Q,$$

$$0 \leq \varphi_j \leq 1 \;\text{ in }\; \sigma_{j-1} Q, \quad \varphi_j = 1 \;\text{ in }\; \sigma_j Q,$$

$$|D\varphi_j| \leq C \frac{\kappa^j}{r(\sigma - \sigma')}, \quad \left| \frac{\partial \varphi_j}{\partial t} \right| \leq C \left(\frac{\kappa^j}{r(\sigma - \sigma')} \right)^p \;\text{ in }\; \sigma_j Q.$$

Substituting test functions in (6) we get inequalities

$$\left(\int_{\sigma_j Q} u^{\kappa\alpha} \, dxdt \right)^{\frac{1}{\kappa\alpha}} \leq \left(\frac{C\kappa^j}{r(\sigma - \sigma')} \right)^{\frac{p}{\alpha}} \left(\int_{\sigma_{j-1} Q} u^\alpha \, dxdt \right)^{\frac{1}{\alpha}}$$

for $j = 1, \ldots, k$. We can write inequalities above equivalently

$$\left(\fint_{\sigma_j Q} u^{\kappa\alpha} \, dxdt \right)^{\frac{1}{\kappa\alpha}} \leq \left(\frac{C\kappa^j}{(\sigma - \sigma')} \right)^{\frac{p}{\alpha}} d_j(r) \left(\fint_{\sigma_{j-1} Q} u^\alpha \, dxdt \right)^{\frac{1}{\alpha}}, \tag{7}$$

where

$$d_j(r) = \frac{(\sigma_{j-1} r)^{\frac{p+n}{\alpha}}}{(\sigma_j r)^{\frac{p+n}{\kappa\alpha}}} r^{-\frac{p}{\alpha}} = \left(\frac{\sigma_{j-1}}{\sigma_j^{1/\kappa}} \right)^{\frac{p+n}{\alpha}}$$

since

$$-p + p + n - \frac{p+n}{1 + \frac{p}{n}} = 0.$$

Observe that (7) holds only when $0 < \alpha < p - 1$. This condition yields the upper bound on q.

For the iteration, we fix q and s, $q > s$, and choose k such that $s\kappa^{k-1} \leq$

$q \leq s\kappa^k$. Let ρ_0 such that $\rho_0 \leq s$ and $q = \kappa^k \rho_0$. Denote $\rho_j = \kappa^j \rho_0$ for $j = 0, \dots, k$. Then we have

$$\begin{aligned}
\left(\fint_{\sigma' Q} u^q \, dxdt \right)^{\frac{1}{q}} &\leq \left(\frac{C\kappa^k}{\sigma - \sigma'} \right)^{\frac{p}{\rho_{k-1}}} \frac{\sigma_{k-1}^{(p+n)/\rho_{k-1}}}{\sigma_k^{(p+n)/\rho_k}} \left(\fint_{\sigma_{k-1} Q} u^{\rho_{k-1}} \, dxdt \right)^{\frac{1}{\rho_{k-1}}} \\
&\leq \vdots \\
&\leq \left(\frac{c_{prod}(k)}{(\sigma - \sigma')^{\gamma^*}} \frac{\sigma^{p+n}}{\sigma'^{(p+n)/\kappa^k}} \right)^{\frac{1}{\rho_0}} \left(\fint_{\sigma Q} u^{\rho_0} \, dxdt \right)^{\frac{1}{\rho_0}}
\end{aligned}$$

where

$$c_{prod}(k) = C^{\gamma^*} \prod_{j=0}^{k-1} \left(\kappa^{j+1} \right)^{p\kappa^{-j}}, \quad \gamma^* = p \sum_{j=0}^{k-1} \kappa^{-j} = \frac{p\kappa}{\kappa - 1} (1 - \kappa^{-k}) \leq p + n.$$

The constant C depends on q since in Lemma 2.1 the constant had a singularity point at $\varepsilon = 0$. Obviously $c_{prod}(k)$ is uniformly bounded on k. From Hölder's inequality we obtain

$$\left(\fint_{\sigma' Q} u^q \, dxdt \right)^{\frac{1}{q}} \leq \left(\frac{C}{(\sigma - \sigma')^{p+n}} \right)^{\frac{1}{\rho_0}} \left(\fint_{\sigma Q} u^s \, dxdt \right)^{\frac{1}{s}}.$$

Furthermore, since $s\kappa^{k-1} \leq \rho_0 \kappa^k$, we have $\rho_0 \geq s/\kappa$ and consequently

$$\left(\fint_{\sigma' Q} u^q \, dxdt \right)^{\frac{1}{q}} \leq \left(\frac{C}{(\sigma - \sigma')^{\gamma}} \right)^{\frac{1}{s}} \left(\fint_{\sigma Q} u^s \, dxdt \right)^{\frac{1}{s}},$$

where $\gamma = (p+n)^2/n$. $\square$

It follows from the result that one of the assumptions made in Lemma 1.3 is fulfilled for the supersolution. We state this as a corollary.

Corollary 2.1. *In addition to assumptions of the previous lemma, require that* $0 < s < \min(1, q/2)$. *Then we have constants* $C = C(n, p, q, \delta, \eta)$ *and* $\gamma = \gamma(n, p)$ *such that*

$$\left(\int_{\sigma' Q} u^q dxdt \right)^{\frac{1}{q}} \leq \left(\frac{C}{(\sigma - \sigma')^{\gamma}} \mu(Q)^{-1} \right)^{\frac{1}{s} - \frac{1}{q}} \left(\int_{\sigma Q} u^s dxdt \right)^{\frac{1}{s}}$$

for all $0 < \delta \leq \sigma' < \sigma \leq 1$ *and for all* $0 < s < q < (p-1)(1 + \frac{p}{n})$, $0 < s < \min(1, q/2)$.

Proof. We obtain easily the right power for the measure of Q. Because σ/σ' is bounded and

$$\frac{1}{s} - \frac{1}{q} \geq \frac{1}{2s},$$

the result follows with $\gamma = (p+n)^4/n^2$. $\square$

2.3 Estimate for the essential supremum of a subsolution

Lemma 2.5. *Suppose that $u \geq \rho > 0$ is a subsolution in Q. Then there exists a positive constant $C = C(n,p,s,\sigma,\eta)$ such that*

$$\operatorname*{ess\,sup}_{\sigma' Q} u \leq \left(\frac{C}{(\sigma-\sigma')^{p+n}}\right)^{\frac{1}{s}} \left(\fint_{\sigma Q} u^s dxdt\right)^{\frac{1}{s}}$$

for all $0 < \sigma' < \sigma \leq 1$ and for all $s > p-1$.

Proof. As in the proof of Lemma 2.4 we obtain from the Sobolev's inequality and from Lemma 2.2 that

$$\left(\int_Q \varphi^{\chi p} u^{\chi\alpha}\, dxdt\right)^{\frac{1}{\chi\alpha}}$$

$$\leq (C\alpha)^{\frac{p}{\alpha}} \left(\int_Q u^\alpha |D\varphi|^p\, dxdt + \int_Q u^\alpha \varphi^{p-1} \left|\frac{\partial\varphi}{\partial t}\right|\, dxdt\right)^{\frac{1}{\alpha}}, \tag{8}$$

where

$$\chi = 1 + \frac{p}{n}, \quad \alpha = p - 1 + \varepsilon, \quad \varepsilon > 0.$$

As we can see, this time α can be as large as we want. In particular, α must be larger than $p-1$. This yields the condition for the starting index in the iteration as well as the condition in lemma's assumptions. We also observe that the constant C in Lemma 2.2 has a singularity point at $\varepsilon = 0$ so the constant C in (8) depends on the lower bound on s.

Let the choices of the test function and σ_j be as in the proof of Lemma 2.4 with an obvious exception that $\sigma_k \to \sigma'$ as k tends to infinity. Moreover, we fix $s > p-1$ and choose $\rho_0 = s$ and $\rho_j = \rho_0 \chi^j$, $j = 0, 1, \ldots$. From (8) we have

$$\left(\frac{\sigma_j^{1/\kappa}}{\sigma_{j-1}}\right)^{\frac{p+n}{\alpha}} \left(\frac{C\alpha}{(\sigma-\sigma')}\right)^{-\frac{p}{\alpha}} \left(\fint_{\sigma_j Q} u^{\chi\alpha}\, dxdt\right)^{\frac{1}{\chi\alpha}} \leq \left(\fint_{\sigma_{j-1} Q} u^\alpha\, dxdt\right)^{\frac{1}{\alpha}}.$$

Consequently, Moser's iteration yields

$$\begin{aligned}
\left(\fint_{\sigma Q} u^s\, dxdt\right)^{\frac{1}{s}} &\geq \left(\frac{\sigma_1^{1/\kappa}}{\sigma_0}\right)^{\frac{p+n}{\alpha}} \left(\frac{C}{(\sigma-\sigma')}\rho_0\right)^{-\frac{p}{\rho_0}} \left(\fint_{\sigma_1 Q} u^{\rho_1}\, dxdt\right)^{\frac{1}{\rho_1}} \\
&\geq \vdots \\
&\geq \frac{c_{prod}}{\sigma^{(p+n)/s}} (\sigma-\sigma')^\gamma \operatorname*{ess\,sup}_{\sigma' Q} u,
\end{aligned}$$

where

$$\gamma = p \sum_{j=0}^{\infty} \chi^{-j} = p + n, \quad c_{prod} = \prod_{j=0}^{\infty} (C\rho_j)^{-\frac{p}{\rho_j}}.$$

It is easy to see that the constant c_{prod} is finite. $\square$

The exponent s in the previous lemma may be replaced by any exponent t with $0 < t < s$.

Corollary 2.2. *The statement of Lemma 2.5 holds true for* $0 < s < \infty$.

Proof. Choose $s = p$ and $0 < q < p$. From lemma 2.5 we have

$$\begin{aligned}
\operatorname*{ess\,sup}_{\sigma' Q} u &\le \left(\frac{C}{(\sigma-\sigma')^{p+n}}\right)^{\frac{1}{p}} \left(\fint_{\sigma Q} u^p\,dxdt\right)^{\frac{1}{p}} \\
&\le \operatorname*{ess\,sup}_{\sigma Q} u^{\frac{q}{p}} \left(\frac{C}{(\sigma-\sigma')^{p+n}}\right)^{\frac{1}{p}} \left(\fint_{\sigma Q} u^{p-q}\,dxdt\right)^{\frac{1}{p}} \\
&\le \varepsilon \operatorname*{ess\,sup}_{\sigma Q} u + c(\varepsilon) \left(\frac{C}{(\sigma-\sigma')^{p+n}}\right)^{\frac{1}{p-q}} \left(\fint_{\sigma Q} u^{p-q}\,dxdt\right)^{\frac{1}{p-q}},
\end{aligned}$$

where we used Young's inequality with $\varepsilon > 0$. By a standard argument (see e.g. [Giaq] Lemma 5.1) we obtain the result. $\square$

2.4 Logarithmic estimate for a supersolution

We already have reverse Hölder inequalities for both super- and subsolutions. Next, we will show the condition for the logarithm in the assumptions of Abstract Lemma (1.3).

Let $0 < \sigma \le 1$, $\tau \in \mathbb{R}$, $\eta > 0$ and $B(z,r) \subset \mathbb{R}^n$. We define

$$\sigma Q^+(\eta) = \sigma Q^+(z,r,\tau,\eta) = B(z,\sigma r) \times (\tau, \tau + \eta r^p),$$

$$\sigma Q^-(\eta) = \sigma Q^-(z,r,\tau,\eta) = B(z,\sigma r) \times (\tau - \eta r^p, \tau)$$

and Q as

$$\sigma Q(\eta) = B(z,\sigma r) \times (\tau - \eta r^p, \tau + \eta r^p).$$

In the case $p = 2$ we know that if u is a solution, then $\log u$ is a subsolution. However, for general p, $\log u$ is not a subsolution of equation (1). More precisely, one can show that $\log u$ is a subsolution for p-parabolic equation.

Lemma 2.6. *Suppose that $u \ge \rho > 0$ is a supersolution in $Q(\eta)$. Furthermore, suppose that $\varphi \in C_0^\infty(Q(\eta))$ depends only on the spatial variable x in $Q(\eta')$ where $0 < \eta' < \eta$. Moreover, φ is radially non-increasing and for $0 < \sigma < 1$ we have*

$$0 \le \varphi^p \le \frac{A}{r^n}, \quad \varphi^p(\sigma Q(\eta)) = \frac{A}{r^n}, \quad |D\varphi(x,t)|^p \le \frac{A'}{r^{n+p}}, \quad \int_{B(z,r)} \varphi(x,t)^p\,dx = 1,$$

where $(x,t) \in Q(\eta')$ and $A = A(n,\sigma)$, $A' = A'(n,\sigma)$ are constants. Let

$$\beta = \int_{B(z,r)} \varphi(x)^p \log u(x,\tau)\,dx.$$

Then there exist constants $C = C(n, p, \sigma, \eta')$ and $C' = C'(n, p, \eta')$ such that

$$\left|\{(x,t) \in \sigma Q^-(\eta') \mid \log u > \lambda + \beta + C'\}\right| \le \frac{C}{\lambda^{p-1}} |\sigma Q^-(\eta')|,$$

$$\left|\{(x,t) \in \sigma Q^+(\eta') \mid \log u < -\lambda + \beta - C'\}\right| \le \frac{C}{\lambda^{p-1}} |\sigma Q^+(\eta')|.$$

for every $\lambda > 0$.

Proof. We can assume without loosing the generality that $\eta' = 1$. We simplify the notation by denoting $Q = Q(\eta') = Q(1)$ and also

$$v(x,t) = \log u(x,t) - \beta, \quad V(t) = \int_{B(z,r)} \varphi(x)^p v(x,t)\, dx,$$

when we have $V(\tau) = 0$. From (5) we obtain

$$\int_{t_1}^{t_2} \int_{B(z,r)} |Dv|^p \varphi^p \, dxdt - \left[\int_{B(z,r)} v\varphi^p \, dx \right]_{t=t_1}^{t_2} \le C(p) \int_{t_1}^{t_2} \int_{B(z,r)} |D\varphi|^p \, dxdt,$$

where $\tau - r^p < t_1 < t_2 < \tau + r^p$, since φ depends only on the spatial variable in Q. Furthermore, modified Poincaré's inequality (see [Lieb] p.113) yields

$$\begin{aligned} \int_{B(z,r)} |Dv|^p \varphi^p \, dx \quad &\ge \quad \frac{C(n,p)}{\sup(\varphi^p)\ r^{n+p}} \int_{B(z,r)} |v - V(t)|^p \varphi^p \, dx \\ &\ge \quad \frac{C}{r^{n+p}} \int_{\sigma B(z,r)} |v - V(t)|^p \, dx \end{aligned}$$

where the constant C depends only on n and p. It follows that

$$\frac{C}{r^{p+n}} \int_{t_1}^{t_2} \int_{\sigma B(z,r)} |v - V(t)|^p \, dxdt + V(t_1) - V(t_2) \le \frac{C'(n,p,\sigma)(t_2 - t_1)}{r^p}.$$

By denoting

$$w(x,t) = v(x,t) + \frac{C'(t-\tau)}{r^p}, \quad W(t) = V(t) + \frac{C'(t-\tau)}{r^p}, \quad W(\tau) = 0,$$

we obtain

$$\frac{C}{r^{n+p}} \int_{t_1}^{t_2} \int_{\sigma B(z,r)} |w - W(t)|^p \, dxdt + W(t_1) - W(t_2) \le 0$$

whereby $W(t_2) \ge W(t_1)$ for all $\tau + r^p \ge t_2 \ge t_1 \ge \tau - r^p$. Since W is a monotonic function it is differentiable almost everywhere. As a consequence we have

$$\frac{C}{r^{n+p}} \int_{\sigma B(z,r)} |w - W(t)|^p \, dx - W'(t) \le 0 \tag{9}$$

for almost every $t \in (t_1, t_2)$. Next, choose $t_1 = \tau - r^p$, $t_2 = \tau$ and let

$$E_\lambda^-(t) = \{(x,t) \in \sigma Q^- \mid w(x,t) > \lambda\}.$$

We find that
$$\int_{\sigma B} |w - W(t)|^p \, dx \geq |E_\lambda^-(t)|(\lambda - W(t))^p \geq |E_\lambda^-(t)|\lambda^p,$$
because $W(t) \leq W(\tau) = 0$ as $\tau > t > t - r^p$. Thus, we have
$$-\frac{W'(t)}{(\lambda - W(t))^p} + C\frac{|E_\lambda^-(t)|}{|Q^-|} \leq 0$$
for almost every $\tau > t > t - r^p$. We integrate this over $(\tau - r^p, \tau)$ and obtain
$$\frac{|E_\lambda^-|}{|\sigma Q^-|} \leq C\left[(\lambda - W(t))^{-(p-1)}\right]_{t=\tau-r^p}^{\tau} \leq \frac{C(n,p,\sigma)}{\lambda^{p-1}}.$$
This yields
$$\left|\{(x,t) \in \sigma Q^- \mid \log u > \lambda + \beta + C'\}\right| \leq |E_\lambda^-| \leq \frac{C}{\lambda^{p-1}}|\sigma Q^-|.$$
Now, choose $t_1 = \tau$, $t_2 = \tau + r^p$ and let
$$E_\lambda^+(t) = \{(x,t) \in \sigma Q^+ \mid w(x,t) < -\lambda\}.$$
Similarly to the case of Q^- we conclude that
$$\int_{\sigma B(z,r)} |w - W(t)|^p \, dx \geq |E_\lambda^+(t)|(\lambda + W(t))^p \geq |E_\lambda^-(t)|\lambda^p,$$
because $W(t) \geq W(\tau) = 0$ as $\tau < t < t + r^p$. Thus, from (9), we have
$$-\frac{W'(t)}{(\lambda + W(t))^p} + C\frac{|E_\lambda^+(t)|}{|Q^+|} \leq 0, \quad \tau < t < t + r^p$$
for almost every $\tau < t < t + r^p$. An integration over $(\tau, \tau + r^p)$ gives
$$\frac{|E_\lambda^+|}{|\sigma Q^+|} \leq -C\left[(\lambda + W(t))^{-(p-1)}\right]_{t=\tau}^{\tau+r^p} \leq \frac{C(n,p,\sigma)}{\lambda^{p-1}}.$$
This yields
$$\left|\{(x,t) \in \sigma Q^+ \mid \log u < -\lambda + \beta - C' \}\right| \leq |E_\lambda^+| \leq \frac{C}{\lambda^{p-1}}|\sigma Q^+|$$
and the claim follows. □

3 Harnack's inequality

For any fixed $0 < \sigma \leq 1$, $\tau \in \mathbb{R}$ and $B(z,r) \subset \mathbb{R}^n$ we define
$$\sigma U^+ = B(z,\sigma r) \times (\tau + \frac{1}{2}r^p - \frac{1}{2}(\sigma r)^p, \tau + \frac{1}{2}r^p + \frac{1}{2}(\sigma r)^p),$$
$$\sigma U^- = B(z,\sigma r) \times (\tau - \frac{1}{2}r^p - \frac{1}{2}(\sigma r)^p, \tau - \frac{1}{2}r^p + \frac{1}{2}(\sigma r)^p).$$
We define Q as
$$Q = B(z,r) \times (\tau - r^p, \tau + r^p).$$
We have the weak Harnack inequality.

Lemma 3.1. *Let $u \geq \rho > 0$ be a supersolution in Q and q be fixed with $0 < q < (p-1)(1+\frac{p}{n})$. Then there exists a constant C depending on n, p, δ and q such that*

$$\left(\fint_{\delta U^-} u^q dx dt \right)^{\frac{1}{q}} \leq C \operatorname*{ess\,inf}_{\delta U^+} u,$$

where $0 < \delta < 1$.

Proof. We fix $0 < \delta < 1$. Let φ be as in the assumptions of Lemma 2.6. Let β and C' be the corresponding constants depending on u. We define $v^+ = u^{-1} e^{\beta - C'}$ and $v^- = u\, e^{-\beta - C'}$. We apply Lemma 2.6 for the function u and have

$$\left| \{(x,t) \in \frac{1+\delta}{2} U^+ \mid \log(v^+) > \lambda\} \right| \leq \frac{C}{\lambda^{p-1}} \left| \frac{1+\delta}{2} U^+ \right|,$$

$$\left| \{(x,t) \in \frac{1+\delta}{2} U^- \mid \log(v^-) > \lambda\} \right| \leq \frac{C}{\lambda^{p-1}} \left| \frac{1+\delta}{2} U^- \right|,$$

with the constant β, which depends on u, and the constant C, which depends only on n, p and δ. Here we have chosen $\eta' = ((1+\delta)/2)^p$ in the assumptions of Lemma 2.6. From Lemma 1.1 we obtain that v^+ is a subsolution in Q. Consequently, Lemma 2.5 yields

$$\operatorname*{ess\,sup}_{\sigma' U^+} v^+ \leq \frac{C}{(\sigma - \sigma')^{(p+n)/s}} \left(\fint_{\sigma U^+} |v^+|^s dx dt \right)^{\frac{1}{s}}$$

for all $\delta \leq \sigma' < \sigma \leq (1+\delta)/2$ and for all $s > 0$ by Corollary 2.2. We use Lemma 1.3 and obtain

$$\operatorname*{ess\,sup}_{\delta U^+} v^+ \leq C^+(n, p, \delta). \tag{10}$$

Furthermore, we have from the corollary of Lemma 2.4 for v^- that

$$\left(\int_{\sigma' U^-} |v^-|^q dx dt \right)^{\frac{1}{q}} \leq \left(\frac{C}{(\sigma - \sigma')^\gamma} |U^-|^{-1} \right)^{\frac{1}{s} - \frac{1}{q}} \left(\int_{\sigma U^-} |v^-|^s dx dt \right)^{\frac{1}{s}}$$

for every $\delta \leq \sigma' < \sigma \leq (1+\delta)/2$, $0 < s < q < (p-1)(1+\frac{p}{n})$ and $0 < s < \min(1, q/2)$. From Lemma 1.3 we again obtain

$$\left(\int_{\delta U^-} |v^-|^q dx dt \right)^{\frac{1}{q}} \leq C^- \, |U^-|^{\frac{1}{q}},$$

where C^- depends on n, p, δ and the lower bound on q. Multiplying this with (10) gives

$$\left(\fint_{\delta U^-} |u|^q dx dt \right)^{\frac{1}{q}} \leq C \operatorname*{ess\,inf}_{\delta U^+} u,$$

where C depends only on n, p, δ and the lower bound on q and the result follows. $\square$

To proof Harnack's inequality we simply collect the results of previous lemmata.

Proof of theorem 1.1. We apply Lemma 3.1 with $\delta = (1+\sigma)/2$. The result follows now from Lemma 2.5. $\square$

4 Appendix

To justify formal calculations in the text we show some elementary properties which are consequences of the definition. These results are standard for the case $p=2$ and also for the parabolic p-Laplace equation ([DiBe],[WZYL]).

Let u be a solution (supersolution, subsolution). For u, it is equivalent to write (2) as follows

$$\int_{\tau_1}^{\tau_2}\int_\Omega |Du|^{p-2}Du\cdot D\eta - u^{p-1}\frac{\partial\eta}{\partial t}\,dx dt + \left[\int_\Omega u^{p-1}\eta\,dx\right]_{t=\tau_1}^{\tau_2} \geq (\leq)\, 0 \quad (11)$$

for almost every τ_1, τ_2, $t_1 < \tau_1 < \tau_2 < t_2$. To show this we let j_ε be a standard mollifier in one dimension and define

$$\phi_\varepsilon(t) = \int_{t-\tau_2}^{t-\tau_1} j_\varepsilon(s)ds.$$

It is noteworthy that $\phi_\varepsilon(t) \to 0$, $t \notin (\tau_1,\tau_2)$, and $\phi_\varepsilon(t) \to 1$, $t \in (\tau_1,\tau_2)$, as $\varepsilon \to 0$. As a test function we choose $\zeta(x,t) = \phi_\varepsilon(t)\eta(x,t)$ where $\eta \in C_0^\infty(\Omega\times(t_1,t_2))$ so that also $\zeta \in C_0^\infty(\Omega\times(t_1,t_2))$. We substitute the test function in the left hand side of (2) and obtain

$$\int_{t_1}^{t_2}\int_\Omega \left(|Du|^{p-2}Du\cdot D\eta - u^{p-1}\frac{\partial\eta}{\partial t}\right)\phi_\varepsilon\,dx dt + \left[j_\varepsilon * \int_\Omega u^{p-1}\eta\,dx\right]_{t=\tau_1}^{\tau_2}.$$

We let ε tend to zero, which yields the result. More precisely, we obtain the pointwise convergence in Lebesgue points of $\int_\Omega u^{p-1}\eta\,dx$.

Furthermore, we want to show that it is possible to substitute a test function to (2) which depends on u itself and show inequality (4). We observe that by a density argument we can choose test functions from the space $W^{1,p}(t_1,t_2;W^{1,p}(\Omega))$. Let Steklov average of u be

$$u_h(x,t) = \frac{1}{h}\int_t^{t+h} u(x,s)ds.$$

We choose $\tau_1 = \tau > t_1$, $\tau_2 = \tau + h < t_2$ and an admissible test function $\zeta(x,\tau) = \min\{k, f'(v_h(x,\tau))\}\eta(x,\tau)$, $\eta \in C_0^\infty(\Omega\times(t_1,t_2))$, where $k \in \mathbb{R}^+$, $v = u^{p-1}$ and $f \in C^2(\mathbb{R})$, $f' \geq 0$, is to be defined later. We substitute η in (11) and divide the result by h. This yields

$$\int_\Omega (|Du|^{p-2}Du)_h\cdot D\zeta\,dx + \int_\Omega v_{h\tau}\zeta\,dx \geq (\leq)\, 0.$$

We denote $\Omega_{h,k}(\tau) = \{x \in \Omega : f'(v_h(x,\tau)) < k\}$. If we take $x_0 \in \Omega_{h,k}(\tau)$ we may choose h small enough so that $x_0 \in \Omega_k(\tau) = \{x \in \Omega : f'(v(x,\tau)) < k\}$ in every Lebesgue point of v in time. As a consequence, the charasteric function of $\Omega_{h,k}(\tau)$ converges pointwise to the charasteric function of $\Omega_k(\tau)$ as h tends to zero for almost every τ. By denoting $\Omega^c_{h,k}(\tau) = \Omega\backslash\Omega_{h,k}(\tau)$ and integrating from $\tau_1 > t_1$ to $\tau_2 < t_2$ we get

$$
\begin{aligned}
0 \ \leq (\geq) \quad & \int_{\tau_1}^{\tau_2}\int_{\Omega} (|Du|^{p-2}Du)_h \cdot D\zeta + v_{h\tau}\zeta \, dxd\tau \\
= \quad & \int_{\tau_1}^{\tau_2}\int_{\Omega_{h,k}(\tau)} (|Du|^{p-2}Du)_h \cdot (Dv)_h f''(v_h)\eta \, dxd\tau \\
+ \quad & \int_{\tau_1}^{\tau_2}\int_{\Omega_{h,k}(\tau)} (|Du|^{p-2}Du)_h \cdot D\eta f'(v_h) \, dxd\tau \\
+ \quad & \left[\int_{\Omega_{h,k}(\tau)} f(v_h)\eta \, dx\right]_{\tau=\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2}\int_{\Omega_{h,k}(\tau)} f(v_h)\frac{\partial\eta}{\partial\tau} \, dxd\tau \\
+ \quad & k\int_{\tau_1}^{\tau_2}\int_{\Omega^c_{h,k}(\tau)} (|Du|^{p-2}Du)_h \cdot D\eta \, dxd\tau \\
+ \quad & k\left[\int_{\Omega^c_{h,k}(\tau)} v_h\eta \, dx\right]_{\tau=\tau_1}^{\tau_2} - k\int_{\tau_1}^{\tau_2}\int_{\Omega^c_{h,k}(\tau)} v_h\frac{\partial\eta}{\partial\tau} \, dxd\tau
\end{aligned}
$$

It follows by a standard result in [DiBe] and the dominated convergence theorem that we can let h tend to zero and obtain

$$
\begin{aligned}
0 \ \leq (\geq) \quad & \int_{\tau_1}^{\tau_2}\int_{\Omega} |Du|^{p-2}Du \cdot D(\min\{f'(v),k\})\eta \, dxd\tau \\
+ \quad & \int_{\tau_1}^{\tau_2}\int_{\Omega} |Du|^{p-2}Du \cdot D\eta \min\{f'(v),k\} \, dxd\tau \\
+ \quad & \left[\int_{\Omega} F_k(v)\eta \, dx\right]_{\tau=\tau_1}^{\tau_2} - \int_{\tau_1}^{\tau_2}\int_{\Omega} F_k(v)\frac{\partial\eta}{\partial\tau} \, dxd\tau
\end{aligned}
\tag{12}
$$

for almost every $t_1 < \tau_1 < \tau_2 < t_2$, where

$$
F_k(v) = \begin{cases} f(v), & f'(v) < k \\ kv, & f'(v) > k \end{cases} .
$$

If f' is bounded on the range of v, we obtain (4) easily.

References

[Bomb] E. Bombieri, Theory of minimal surfaces and a counterexample to the bernstein conjecture in high dimension, Mimeographed Notes of Lectures Held at Courant Institute, New York University (1970)

[BoGi] E. Bombieri, E. Giusti, Harnack's inequality for elliptic differential equations on minimal surfaces, Invent. Math. 15, 24-46 (1972)

[DiBe] E. DiBenedetto, Degenerate parabolic equations, Springer-Verlag (1993)

[DBUV] E. DiBenedetto, J.M. Urbano, V. Vespri, Current issues on singular and degenerate evolution equations, Elsevier (in press)

[FaGa] E.B. Fabes, N. Garofalo, Parabolic B.M.O. and Harnack's inequality, Proc. Amer. Math. Soc. 50, no. 1, 63-69 (1985)

[GiVe] U. Gianazza, V. Vespri, A Harnack inequality for solutions of doubly nonlinear parabolic equation, Preprint (2004)

[Giaq] M. Giaquinta, Introduction to regularity theory for nonlinear elliptic systems, Birkhäuser Verlag (1993)

[Lieb] G.M. Lieberman, Second order parabolic equations, World Scientific (1996)

[Mo64] J. Moser, A Harnack inequality for parabolic differential equations, Comm. Pure Appl. Math. 17, 101-134 (1964), and correction in Comm. Pure Appl. Math. 20, 231-236 (1967)

[Mo71] J. Moser, On a pointwise estimate for parabolic equations, Comm. Pure Appl. Math. 24, 727-740 (1971)

[SaCo] L. Saloff-Coste, Aspects of Sobolev-type inequalities, London Mathematical Society Lecture Note Series 289, Cambridge University Press (2002)

[Tru68] N.S. Trudinger, Pointwise estimates and quasilinear parabolic equations, Comm. Pure Appl. Math. 21, 205-226 (1968)

[WZYL] Z. Wu, J. Zhao, J. Yin, H.Li, Nonlinear diffusion equations, World Scientific (2001)

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ISSN 0784-3143