Helsinki University of Technology, Institute of Mathematics, Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2004 A478

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Niko Marola: *Moser's method for minimizers on metric measure spaces*; Helsinki University of Technology, Institute of Mathematics, Research Reports A478 (2004).

Abstract: *The purpose of this note is to show that Moser's method applies in a metric measure space. The measure is required to be doubling and the space is assumed to support a Poincaré inequality.*

AMS subject classifications: 49N60, 35J20

Keywords: Sobolev space, Newtonian space, Caccioppoli estimates, Harnack's inequality

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ISBN 951-22-7386-1
ISSN 0784-3143

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1 Introduction

In the Euclidean case minimizers of the p-Dirichlet integral

$$\int_\Omega |\nabla u|^p \, dx,$$

where $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ open, are known to be locally Hölder continuous. The minimizers are weak solutions to the p-Laplace equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

In a general metric measure space it is not clear what is the counterpart of the p-Laplace equation, but the variational approach is available. The main reason for this is that Sobolev spaces can be defined without the concept of a partial derivative on a metric measure space [3, 17, 30]. Hence it is possible to study minimizers of the corresponding p-Dirichlet integral

$$\int_\Omega g_u^p \, d\mu,$$

where g_u denotes a substitute for the modulus of a gradient in a metric space and μ is a Borel regular measure. In the Euclidean case there are several ways to prove continuity of the minimizers. One possible approach is to use Moser's iteration technique (see [25, 26] and [4, 11, 15, 23, 28]) to obtain Harnack's inequality from which Hölder continuity follows. At the first sight, it may seem that there is a drawback in Moser's argument in metric measure spaces, since it is strongly based on the differential equation. There exists another approach by De Giorgi, which relies only on the minimization property, see [5] and [10, 12, 23]. De Giorgi's method first gives Hölder continuity and Harnack's inequality follows from this as in [6]. In [20] De Giorgi's method was used in the metric setting. It was shown that if the measure is doubling and the space supports a weak $(1, q)$-Poincaré inequality for some q with $1 < q < p$, then the minimizers, and even the quasiminimizers, are locally Hölder continuous and satisfy Harnack's inequality. Quasiminimizers minimize a variational integral only up to a multiplicative constant, particularly, minimizers are 1-quasiminimizers.

In this note we adapt Moser's method to metric measure spaces. We will impose slightly weaker requirements on the space than in [20]. More precisely we require that the space supports a weak $(1, p)$-Poincaré inequality instead of a weak $(1, q)$-Poincaré inequality for some q with $1 < q < p$. By a result of Keith and Zhong [18] a complete metric measure space that supports a weak $(1, p)$-Poincaré inequality, with a doubling Borel regular measure, admits a weak $(1, q)$-Poincaré inequality for some $1 < q < p$. However, we show that Moser's method applies without refering to a deep theorem of Keith and Zhong.

This work is organized as follows. In the second section we focus on the preliminary notation, definitions and concepts used throughout this work. Newtonian spaces, the Sobolev space counterpart in the metric setting are defined and we also fix the general setup. In the third section we present four lemmas which are mathematical folklore, but since they do not appear explicitly in the literature and we shall use them extensively, we present them here. In the fourth section we prove certain Caccioppoli type estimates for minimizers. In Section 5 the actual Moser's method is used in the metric setting and Harnack's inequality for minimizers is proved.

2 Preliminaries

We assume that X is a metric measure space equipped with a Borel regular measure μ. We assume that the measure of every nonempty open set is positive and that the measure of every bounded set is finite. Later we will impose further requirements on the space and the measure. Throughout the work we use the convention that $B(z,r)$ refers to an open ball centered at z and with radius $r>0$ and by tB, where $t>0$, we denote a ball concentric with B but with radius t times that of B. Constants are usually labeled as c, and their values may change even in a single line. If A is a subset of X, then χ_A denotes the characteristic function of A. If not otherwise stated, p is a real number satisfying $1 \le p < \infty$.

By a path in X we will mean any continuous mapping $\gamma : [a,b] \to X$, where $[a,b], a<b$, is an interval in $\mathbb{R}$. Its image will be denoted by $|\gamma| = \gamma([a,b])$, the length of γ is defined as

$$l(\gamma) = \sup_{a=t_0<t_1<\ldots<t_n=b} \sum_{i=0}^{n-1} d(\gamma(t_i), \gamma(t_{i+1})).$$

We say that the curve is rectifiable if $l(\gamma) < \infty$. Let Γ_{rect} be the collection of all non-constant rectifiable paths $\gamma : [a,b] \to X$. See [16, 17, 32] for the discussion of rectifiable paths and path integration.

The p-modulus of a family of paths Γ in X is the number

$$\operatorname{Mod}_p(\Gamma) = \inf_\rho \int_X \rho^p \, d\mu,$$

where the infinum is taken over all non-negative Borel measurable functions ρ so that for all rectifiable paths γ which belong to Γ we have

$$\int_\gamma \rho \, ds \ge 1.$$

It is known that the p-modulus is an outer measure on the collection of all paths in X. From the above definition it is clear that the p-modulus of the family of all non-rectifiable paths is zero, thus non-rectifiable paths are not interesting in this study. See [8, 16, 32] for additional information about p-modulus.

Upper gradients

In a metric measure space an upper gradient is a counterpart for the Sobolev gradient.

Definition 2.1. Let u be an extended real-valued function on X. We say that a non-negative Borel measurable function g is an *upper gradient* of u if for all rectifiable paths γ joining points x and y in X we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds. \tag{2.2}$$

See [3, 16, 17, 30] for a discussion of upper gradients. A property is said to hold for *p-almost all* paths, if the set of paths for which the property fails is of zero p-modulus. This set of paths is called the exceptional set. If (2.2) holds for p-almost all paths γ, then g is said to be a *p-weak upper gradient* of u. It is known that if $1 \leq p < \infty$ and u has a p-weak upper gradient in $L^p(X)$ then u has the least p-weak upper gradient g_u in $L^p(X)$. It is the smallest in the sense that if g is another p-weak upper gradient in $L^p(X)$ of u then $g \geq g_u$ μ-almost everywhere. This fact has been proved in [29]. An alternative proof is given in [3].

We implicitly use the following observation several times. If u is a function with $L^p(X)$-integrable p-weak upper gradient g_u. Then there is a family Γ of rectifiable paths in X so that $\mathrm{Mod}_p(\Gamma) = 0$ and for all rectifiable paths $\gamma \notin \Gamma$, connecting x and y in X, u satisfies (2.2) with p-weak upper gradient g_u for all subpaths $\gamma' \subset \gamma$. By $\gamma' \subset \gamma$ we mean that γ' is a restriction of γ to a subinterval of $[a, b]$. This is based on a simple fact of the p-modulus, namely, if Γ_0 and Γ_1 are two path families such that each curve $\gamma_1 \in \Gamma_1$ has a subpath $\gamma_0 \in \Gamma_0$, then $\mathrm{Mod}_p(\Gamma_1) \leq \mathrm{Mod}_p(\Gamma_0)$, see [32].

Newtonian spaces

Here we introduce the notion of Sobolev spaces on a metric measure space based on the concept of upper gradients. Following [30] we define the space $\widetilde{N}^{1,p}(X)$ to be the collection of all real-valued p-integrable functions u on X that have a p-integrable p-weak upper gradient g_u. We equip this space with a seminorm

$$\|u\|_{\widetilde{N}^{1,p}(X)} = \left(\|u\|^p_{L^p(X)} + \|g_u\|^p_{L^p(X)} \right)^{1/p}.$$

This seminorm partitions $\widetilde{N}^{1,p}(X)$ into equivalence classes. We say that u and v belong to the same equivalence class, or simply write $u \sim v$ if

$\|u-v\|_{\widetilde{N}^{1,p}(X)}=0$. The *Newtonian space* $N^{1,p}(X)$ is defined to be the space $\widetilde{N}^{1,p}(X)/\sim$ with the norm

$$\|u\|_{N^{1,p}(X)}=\|u\|_{\widetilde{N}^{1,p}(X)}.$$

For basic properties of Newtonian spaces we refer to [30].

Definition 2.3. Let $u:X\to\mathbb{R}$ be a given function and $\gamma\in\Gamma_{\text{rect}}$ be an arc-length parametrized path in X. We say that

(i) u is *absolutely continuous along a path* γ if $u\circ\gamma$ is absolutely continuous on $[0,l(\gamma)]$,

(ii) u is *absolutely continuous on p-almost every curve*, or simply ACC_p, if for p-almost every γ, $u\circ\gamma$ is absolutely continuous.

It is very useful to know that if u is a function in $\widetilde{N}^{1,p}(X)$, then u is ACC_p. See [30] for the proof.

The *p-capacity* of a set $E\subset X$ with respect to the space $N^{1,p}(X)$ is defined by

$$\mathrm{Cap}_p(E)=\inf_u\|u\|^p_{N^{1,p}(X)},$$

where the infinum is taken over all functions $u\in\widetilde{N}^{1,p}(X)$ whose restriction to E is bounded below by 1. Sets of zero capacity are also of measure zero, but the converse is not true. See [21] for more properties of the capacity in the metric setting.

We also need a counterpart of the Sobolev functions with zero boundary values in a metric measure space in order to be able to compare the boundary values of Sobolev functions. Let $E\subset X$ be an arbitrary set. Following the method of [19], we define the space $\widetilde{N}^{1,p}_0(E)$ to be the set of functions $\widetilde{u}\in\widetilde{N}^{1,p}(X)$ for which

$$\mathrm{Cap}_p\left(\{x\in X\setminus E\ :\ \widetilde{u}(x)\neq 0\}\right)=0.$$

The *Newtonian space with zero boundary values* $N^{1,p}_0(E)$ is then $\widetilde{N}^{1,p}_0(E)/\sim$ equipped with the norm

$$\|u\|_{N^{1,p}_0(E)}=\|\widetilde{u}\|_{\widetilde{N}^{1,p}(X)}.$$

The norm on $N^{1,p}_0(E)$ is unambiguously defined by [31] and the obtained space is a Banach space. From now on we usually identify the equivalence class with its representative.

Doubling property and Poincaré inequalities

We will impose some further requirements on the space and the measure. Namely, the measure μ is said to be *doubling* if there is a constant $c_\mu \geq 1$, called the *doubling constant* of μ, so that

$$\mu(B(z, 2r)) \leq c_\mu \mu(B(z, r)) \tag{2.4}$$

for every open ball $B(z, r)$ in X. By the doubling property, if $B(y, R)$ is a ball in X, $z \in B(y, R)$ and $0 < r \leq R < \infty$, then

$$\frac{\mu(B(z, r))}{\mu(B(y, R))} \geq c \left(\frac{r}{R}\right)^Q \tag{2.5}$$

for $c = c(c_\mu) > 0$ and $Q = \log_2 c_\mu$. The exponent Q serves as a counterpart of dimension related to the measure. A metric space X is said to be *doubling* if there exists a constant $c < \infty$ such that every ball $B(z, r)$ can be covered by at most c balls with the radii $r/2$. If X is equipped with a doubling measure, then X is doubling.

Let $1 < p < \infty$. The space X is said to *support a weak* $(1, p)$*-Poincaré inequality* if there are constants $c > 0$ and $\tau \geq 1$ such that

$$\fint_{B(z,r)} |u - u_{B(z,r)}| \, d\mu \leq cr \left(\fint_{B(z,\tau r)} g_u^p \, d\mu \right)^{1/p} \tag{2.6}$$

for all balls $B(z, r)$ in X, for all integrable functions u in $B(z, r)$ and for all p-weak upper gradients g_u of u. If $\tau = 1$, the space is said to support a $(1, p)$-Poincaré inequality. A result of [13] (see also [14]) shows that in a doubling measure space a weak $(1, p)$-Poincaré inequality implies a *Sobolev-Poincaré inequality*. More precisely, there is $c = c(p, \kappa, c_\mu) > 0$ such that

$$\left(\fint_{B(z,r)} |u - u_{B(z,r)}|^{\kappa p} \, d\mu \right)^{1/\kappa p} \leq cr \left(\fint_{B(z,5\tau r)} g_u^p \, d\mu \right)^{1/p}, \tag{2.7}$$

where $1 \leq \kappa \leq Q/(Q - p)$ if $1 < p < Q$ and $\kappa = 2$ if $p \geq Q$, for all balls $B(z, r)$ in X, for all integrable functions u in $B(z, r)$ and for all p-weak upper gradients g_u of u. We will also need an inequality for Newtonian functions with zero boundary values. If $u \in N_0^{1,p}(B(z, r))$, then there exists $c = c(p, \kappa, c_\mu) > 0$, the constant c is independent of u, such that

$$\left(\fint_{B(z,r)} |u|^{\kappa p} \, d\mu \right)^{1/\kappa p} \leq cr \left(\fint_{B(z,r)} g_u^p \, d\mu \right)^{1/p} \tag{2.8}$$

for every ball $B(z, r)$ with $0 < r < \operatorname{diam}(X)/10$. For this result we refer to [20].

Minimizers.

Let us now define the minimization problem for the p-Dirichlet integral in a metric setting. A subset A of Ω is *compactly contained* in Ω, abbreviated $A \subset\subset \Omega$, if the closure of A is a compact subset of Ω. We say that u belongs to the *local Newtonian space* $N^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in N^{1,p}(A)$ for every measurable set $A \subset\subset \Omega$.

From now on we assume that $1 < p < \infty$ and $\Omega \subset X$ is open.

Definition 2.9. Let $\vartheta \in N^{1,p}(\Omega)$. A function $u \in N^{1,p}(\Omega)$ such that $u - \vartheta \in N^{1,p}_0(\Omega)$ is a *p-minimizer with boundary values ϑ in* Ω, if

$$\int_\Omega g_u^p \, d\mu \le \int_\Omega g_v^p \, d\mu \tag{2.10}$$

for every $v \in N^{1,p}(\Omega)$ such that $v - \vartheta \in N^{1,p}_0(\Omega)$. Here g_u and g_v are the minimal p-weak upper gradients of u and v in Ω, respectively.

Definition 2.11. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is called a *p-minimizer* in Ω if (2.10) holds in every open set $\Omega' \subset\subset \Omega$ for all v such that $v - u \in N^{1,p}_0(\Omega')$.

Definition 2.12. A function $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ is called a *p-superminimizer* in Ω if (2.10) holds in every open set $\Omega' \subset\subset \Omega$ for all v such that $v - u \in N^{1,p}_0(\Omega'), v \ge u$ μ-almost everywhere in Ω'. We say that a function u is a *p-subminimizer* in Ω if $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and (2.10) holds in every open set $\Omega' \subset\subset \Omega$ for all v such that $v - u \in N^{1,p}_0(\Omega'), v \le u$ μ-almost everywhere in Ω'.

It is easy to see that when u is a p-superminimizer, then $\alpha u + \beta$ is also a p-superminimizer when $\alpha \ge 0$ and $\beta \in \mathbb{R}$. This is true also for p-subminimizers. We will show later that u is a p-minimizer if and only if u is both a p-superminimizer and a p-subminimizer in Ω. In the Euclidean case minimizers correspond to solutions, subminimizers and superminimizers correspond to sub- and supersolutions, respectively, of the p-Laplace equation.

The following theorem implies that p-minimizers are locally bounded.

Theorem 2.13. *Let u be a p-minimizer in Ω and $B(z, 2r) \subset \Omega$. Then u is locally bounded and satisfies the inequality*

$$\operatorname*{ess\,sup}_{B(z,r)} |u| \le c \left(\fint_{B(z,2r)} |u|^p \, d\mu \right)^{1/p},$$

where the constant c is independent of the ball $B(z, r)$.

The theorem was proved by Kinnunen and Shanmugalingam in [20, Theorem 4.3]. In [20] it was assumed that the space supports a weak $(1, q)$-Poincaré inequality for some q with $1 < q < p$. However, the assumption comes in when proving Hölder continuity and is not needed in the proof of Theorem 2.13. By this theorem the assumption that the p-minimizers are locally bounded is not restrictive. We will need this result only in the proof of Lemma 4.1.

General setup

From now on we assume that the complete metric measure space X is equipped with a doubling Borel regular measure for which the measure of every nonempty open set is positive and the measure of every bounded set is finite. Furthermore we assume that the space supports a weak $(1,p)$-Poincaré inequality.

3 Lemmas

The following four lemmas are mathematical folklore, but since they do not appear explicitly in the literature, we present them here. The first two lemmas give us a way, in some sense, to calculate weak upper gradients. Finally, we prove that a function u is a p-minimizer if and only if it is both a p-subminimizer and p-superminimizer.

Lemma 3.1. *Suppose that u is p-integrable and that there is a p-integrable Borel measurable function g such that for p-almost every path γ in X the function $h : s \mapsto u(\gamma(s))$ is absolutely continuous on $[0, l(\gamma)]$ and*

$$|h'(s)| \le g(\gamma(s)) \tag{3.2}$$

almost everywhere on $[0, l(\gamma)]$. Then $u \in \widetilde{N}^{1,p}(X)$.

Proof. Let $\gamma \in \Gamma_{\text{rect}}$ be a path connecting x and y in X such that h is absolutely continuous on $[0, l(\gamma)]$ and $\Gamma \subset \Gamma_{\text{rect}}$ be the collection of paths on which h is not absolutely continuous. Then the p-modulus of Γ is zero. It follows that g is a p-integrable p-weak upper gradient of u because

$$|u(x) - u(y)| \le \int_0^{l(\gamma)} |h'(s)|\, ds \le \int_0^{l(\gamma)} g(\gamma(s))\, ds.$$

This completes the proof. □

Lemma 3.3. *Suppose that a function u has a p-integrable p-weak upper gradient g. Then for p-almost every path γ in X,*

$$|h'(s)| \le g(\gamma(s)) \tag{3.4}$$

almost everywhere on $[0, l(\gamma)]$, where $h(s) = u(\gamma(s))$, $s \in [0, l(\gamma)]$.

Proof. Let Γ_{rect} be the family of paths on which u is absolutely continuous and on which

$$|u(x') - u(y')| \le \int_{\gamma'} g\, ds.$$

holds for every subpath γ' of $\gamma \in \Gamma_{\text{rect}}$, where γ' connects points x' and y' in X, whereas γ connects x and y in X. By the definition, the family of rectifiable paths in X for which the above inequality fails is of zero p-modulus. Now if $s_0 \in (0, l(\gamma))$ we have

$$\begin{aligned} |h'(s)| &= \lim_{s_0 \to s} \left| \frac{h(s_0) - h(s)}{s_0 - s} \right| \\ &= \lim_{s_0 \to s} \left| \frac{u(\gamma(s_0)) - u(\gamma(s))}{s_0 - s} \right| \\ &\leq \lim_{s_0 \to s} \frac{1}{|s_0 - s|} \left| \int_s^{s_0} g(\gamma(t))\, dt \right| = g(\gamma(s)) \end{aligned}$$

by Lebesgue's theorem for $\mathcal{L}^1$-almost every $s \in [0, l(\gamma)]$ and the assertion follows from this. □

The next lemma is of importance, since we often have to truncate Newtonian functions and it is useful to know that the obtained function still belongs to the Newtonian space.

Lemma 3.5. *Suppose that $u_1, u_2 \in \widetilde{N}^{1,p}(X)$ and let $A = \{x \in X \ : \ u_1(x) < u_2(x)\}$. Then $u = \min(u_1, u_2) \in \widetilde{N}^{1,p}(X)$ and u has a p-weak upper gradient g_u such that*

$$g_u(x) = \begin{cases} g_{u_1}(x), & \mu\text{-a.e. in } A \\ g_{u_2}(x), & \mu\text{-a.e. in } X \setminus A \end{cases} \tag{3.6}$$

Proof. We may assume that u_1, u_2 are the ACC_p representatives of u_1, u_2, respectively. Let $\widetilde{g}(x) = g_{u_1}(x),\ x \in A$ and $\widetilde{g}(x) = g_{u_2}(x),\ x \in X \setminus A$. Since, a priori, A is only a measurable set, we need to modify the function $\widetilde{g}(x)$. Choose a Borel measurable function g such that $g = \widetilde{g}$ μ-almost everywhere and $g \geq \widetilde{g}$ [27]. We claim that g is a p-weak upper gradient of u.

To this end let γ be rectifiable path in X, connecting x and y in X, such that u_1 and u_2 are absolutely continuous on γ and that g_{u_1} and g_{u_2} are the p-weak upper gradients of u_1 and u_2 on γ such that γ and none of its subpaths are exceptional for g_{u_1} and g_{u_2}. Since $s \mapsto u_i(\gamma(s)), i = 1, 2$, are absolutely continuous on $[0, l(\gamma)]$, the function $s \mapsto u(\gamma(s))$ is absolutely continuous and for $\mathcal{L}^1$-almost every $s \in [0, l(\gamma)]$

$$|(u \circ \gamma)'(s)| \leq |(u_1 \circ \gamma)'(s)| \chi_I(s) + |(u_2 \circ \gamma)'(s)| \chi_{[0,l(\gamma)] \setminus I}(s),$$

where

$$I = \{s \in [0, l(\gamma)] \ : \ (u_1 \circ \gamma)(s) < (u_2 \circ \gamma)(s)\}.$$

Since $s \in [0, l(\gamma)]$ belongs to I if and only if $\gamma(s) \in A$ and since $|(u_1 \circ \gamma)'(s)| \leq g_{u_1}(\gamma(s))$ and $|(u_2 \circ \gamma)'(s)| \leq g_{u_2}(\gamma(s))$ for $\mathcal{L}^1$-almost every $s \in [0, l(\gamma)]$ we obtain

$$|(u \circ \gamma)'(s)| \leq g_{u_1}(\gamma(s)) \chi_I(s) + g_{u_2}(\gamma(s)) \chi_{[0,l(\gamma)] \setminus I}(s) \leq g(\gamma(s)).$$

The minimality remains to be shown here. To be more precise, we prove that $g = g_u$ is a minimal p-weak upper gradient of u. As in [1], we apply this lemma (without the minimality part) with the roles of u and u_1 interchanged, we see that $g_u\chi_A + g_{u_1}\chi_{X\setminus A}$ is a p-weak upper gradient of u_1. Since g_u and g_{u_1} are minimal we have that

$$g_{u_1} \le g_u \le g = g_{u_1}$$

μ-almost everywhere in A. Hence $g_u = g$ μ-almost everywhere on A. If we apply this lemma with the roles of u and u_2 interchanged, we obtain $g_u = g$ μ-almost everywhere on $X \setminus A$. This finishes the proof. $\square$

Remarks 3.7. (1) The lemma remains true if A is replaced by the set $\{x \in X \,:\, u_1(x) \le u_2(x)\}$.
(2) If A is a Borel set, then the modification of $\widetilde{g}$ is not needed.

If $A \subset X$ is a Borel set and $u \in \widetilde{N}^{1,p}(X)$ is a constant μ-almost everywhere in $X \setminus A$ we will use the following observation: If g is an upper gradient of u, then $g_u\chi_A$ is a p-weak upper gradient of u and hence we may assume that $g_u = 0$ μ-almost everywhere on $X \setminus A$. For open sets A this has been proven in [30]. The general claim follows from the fact that the measure of a Borel set can be approximated with arbitrary accuracy by measures of open sets containing the set. A different proof of Lemma 3.5 can be found in [1].

The following lemma shows that Newtonian spaces posses the same useful properties as first order Sobolev spaces.

Lemma 3.8. *If $u, v \in \widetilde{N}^{1,p}(X)$ then the functions*

(a) $\min(u, \lambda), \lambda \in \mathbb{R}$,

(b) $|u|$,

(c) $\max(u, v)$

all belong to $\widetilde{N}^{1,p}(X)$ (and thus in $N^{1,p}(X)$).

Proof. The claims follow from similar considerations as in the proof of Lemma 3.5. $\square$

Let us go back to the minimizers. The following lemma shows the correspondence between minimizers and sub- and superminimizers.

Lemma 3.9. *A function u is a p-minimizer if and only if it is both a p-subminimizer and a p-superminimizer.*

Proof. The assertion follows easily. If u is a p-minimizer, then (2.10) is clearly satisfied in every open $\Omega' \subset\subset \Omega$ for all v such that $v - u \in N_0^{1,p}(\Omega')$, $v \le u$ and $v \ge u$ μ-almost everywhere in Ω'. Thus u is a p-subminimizer and a p-superminimizer. Conversely, let u be both a p-subminimizer and a

p-superminimizer in Ω, $\widetilde{v} \in N^{1,p}(\Omega)$ so that $\widetilde{v} - u \in N_0^{1,p}(\Omega')$ and $\Omega' \subset\subset \Omega$ open. Then by the definition, $u \in N^{1,p}_{\mathrm{loc}}(\Omega)$,

$$\int_{\Omega'} g_u^p \, d\mu \leq \int_{\Omega'} g_{v_1}^p \, d\mu$$

holds for all v_1 so that $v_1 - u \in N_0^{1,p}(\Omega'), v_1 \leq u$ μ-almost everywhere in Ω' and

$$\int_{\Omega'} g_u^p \, d\mu \leq \int_{\Omega'} g_{v_2}^p \, d\mu$$

is valid for all v_2 so that $v_2 - u \in N_0^{1,p}(\Omega'), v_2 \geq u$ μ-almost everywhere in Ω'. We define

$$A = \{x \in \Omega' \; : \; \widetilde{v}(x) \leq u(x)\}$$

and set $v = v_1 \chi_A + v_2 \chi_{\Omega' \setminus A}$, where $v_1 = \min(\widetilde{v}, u)$ whereas $v_2 = \max(\widetilde{v}, u)$. As in Lemma 3.5,

$$g_v(x) = \begin{cases} g_{v_1}(x), & \mu\text{-a.e. in } A \\ g_{v_2}(x), & \mu\text{-a.e. in } \Omega' \setminus A, \end{cases}$$

is a p-weak upper gradient of v which implies that (2.10) is valid in Ω' for all $v \in N^{1,p}(\Omega)$. In addition, $v - u \in N_0^{1,p}(\Omega')$. Since $\widetilde{v} \in N^{1,p}(\Omega)$ was arbitrary, u is a p-minimizer. □

4 Caccioppoli type inequalities

We will show that Caccioppoli type estimates can be obtained for $p-$ subminimizers and p-superminimizers by using a convexity argument. See e.g. [15, 23] for the corresponding estimates for subsolutions and supersolutions in the Euclidean case.

Lemma 4.1. *Suppose u is a locally bounded p-subminimizer in Ω so that* $\operatorname{ess\,inf}_\Omega u > 0$ *and let $\varepsilon > 0$. Let η be a compactly supported Lipschitz continuous function in Ω such that $0 \leq \eta \leq 1$. Then*

$$\int_\Omega g_u^p u^{\varepsilon - 1} \eta^p \, d\mu \leq c \int_\Omega u^{p+\varepsilon-1} g_\eta^p \, d\mu, \tag{4.2}$$

where $c = (p/\varepsilon)^p$.

Proof. Note that, a priori, it is not known whether the right-hand side in (4.2) is finite or infinite. Since the assertion is trivial in the latter case, we may assume that the right-hand side is finite. Choose an open set Ω' so that $\Omega' \subset\subset \Omega$ and $\operatorname{spt}(\eta) \subset \Omega'$. Fix $0 < \alpha < 1$ small enough so that $\varepsilon \alpha^\varepsilon u^{\varepsilon - 1} \leq 1$. Let $w = u - \eta^p (\alpha u)^\varepsilon$, then $w \leq u$.

Let $\Gamma_{\rm rect}$ denote the family of all rectifiable paths $\gamma : [0,1] \to X$. Let the family $\Gamma \subset \Gamma_{\rm rect}$ be such that $\mathrm{Mod}_p(\Gamma) = 0$ and γ be the arc-length parametrization of the path in $\Gamma_{\rm rect} \setminus \Gamma$ on which the function u is absolutely continuous. Since η is Lipschitz continuous, it is absolutely continuous on γ. We define $h : [0, l(\gamma)] \to [0,\infty)$,

$$h(s) = (u \circ \gamma)(s) - (\eta \circ \gamma)(s)^p (\alpha u \circ \gamma)(s)^{\varepsilon}.$$

Then h is absolutely continuous and for $\mathcal{L}^1$-almost every $s \in [0, l(\gamma)]$ we have

$$\begin{aligned} h'(s) &= (u \circ \gamma)'(s) - p(\eta \circ \gamma)(s)^{p-1} (\eta \circ \gamma)'(s) (\alpha u \circ \gamma)(s)^{\varepsilon} \\ &\quad - \varepsilon (\eta \circ \gamma)(s)^p (\alpha u \circ \gamma)(s)^{\varepsilon - 1} \alpha (u \circ \gamma)'(s) \\ &= (1 - \varepsilon\alpha(\eta \circ \gamma)(s)^p (\alpha u \circ \gamma)(s)^{\varepsilon-1}) (u \circ \gamma)'(s) \\ &\quad - p(\eta \circ \gamma)(s)^{p-1} (\eta \circ \gamma)'(s) (\alpha u \circ \gamma)(s)^{\varepsilon}. \end{aligned}$$

Since $|(u \circ \gamma)'(s)| \le g_u(\gamma(s))$ and $|(\eta \circ \gamma)'(s)| \le g_\eta(\gamma(s))$ for $\mathcal{L}^1$-almost every $s \in [0, l(\gamma)]$, we obtain

$$\begin{aligned} |(w \circ \gamma)'(s)| = |h'(s)| &\le (1 - \varepsilon\alpha\eta(\gamma(s))^p (\alpha u(\gamma(s)))^{\varepsilon-1}) g_u(\gamma(s)) \\ &\quad + p\eta(\gamma(s))^{p-1} (\alpha u(\gamma(s)))^{\varepsilon} g_\eta(\gamma(s)) \end{aligned}$$

for $\mathcal{L}^1$-almost every $s \in [0, l(\gamma)]$. Thus we have

$$g_w \le \left(1 - \varepsilon \alpha^{\varepsilon} \eta^p u^{\varepsilon - 1}\right) g_u + p\eta^{p-1} (\alpha u)^{\varepsilon} g_\eta$$

μ-almost everywhere in Ω.
Since $0 \le \varepsilon\alpha^{\varepsilon}\eta^p u^{\varepsilon-1} \le 1$, we may exploit the convexity of the function $t \mapsto t^p$ to obtain

$$g_w^p \le \left(1 - \varepsilon\alpha^{\varepsilon}\eta^p u^{\varepsilon-1}\right) g_u^p + \varepsilon^{1-p} \alpha^{\varepsilon} p^p u^{p+\varepsilon-1} g_\eta^p.$$

Since u is a p-subminimizer, we have

$$\begin{aligned} \int_{\Omega'} g_u^p \, d\mu &\le \int_{\Omega'} g_w^p \, d\mu \\ &\le \int_{\Omega'} g_u^p \, d\mu - \varepsilon\alpha^{\varepsilon} \int_{\Omega'} \eta^p u^{\varepsilon-1} g_u^p \, d\mu \\ &\quad + \varepsilon^{1-p}\alpha^{\varepsilon} p^p \int_{\Omega'} u^{p+\varepsilon-1} g_\eta^p \, d\mu, \end{aligned}$$

which implies

$$\int_{\Omega'} \eta^p u^{\varepsilon-1} g_u^p \, d\mu \le \varepsilon^{-p} p^p \int_{\Omega'} u^{p+\varepsilon-1} g_\eta^p \, d\mu.$$

This is the desired estimate. $\square$

We will need a similar estimate for a p-superminimizer.

Lemma 4.3. *Suppose u is a p-superminimizer in Ω so that* $\operatorname{ess\,inf}_\Omega u > 0$ *and let $\varepsilon > 0$. Let η be a compactly supported Lipschitz continuous function in Ω such that $0 \le \eta \le 1$. Then*

$$\int_\Omega g_u^p u^{-\varepsilon-1} \eta^p \, d\mu \le c \int_\Omega u^{p-\varepsilon-1} g_\eta^p \, d\mu, \tag{4.4}$$

where $c = (p/\varepsilon)^p$.

Proof. Choose an open set Ω' so that $\Omega' \subset\subset \Omega$ and $\operatorname{spt}(\eta) \subset \Omega'$. We may assume that $u \ge \varepsilon^{1/(\varepsilon+1)}$, since otherwise we study αu for $\alpha > 0$ large enough. Let $w = u + \eta^p u^{-\varepsilon}$. Then $w \ge u$ and $w \in N^{1,p}_{\mathrm{loc}}(\Omega)$. Then as in the proof of Lemma 4.1 we have

$$g_w \le (1 - \varepsilon \eta^p u^{-\varepsilon-1}) g_u + p \eta^{p-1} u^{-\varepsilon} g_\eta$$

μ-almost everywhere in Ω.
Since $0 \le \varepsilon \eta^p u^{-\varepsilon-1} \le 1$, again by convexity we obtain

$$\begin{aligned} g_w^p &\le (1 - \varepsilon \eta^p u^{-\varepsilon-1}) g_u^p + \varepsilon \eta^p u^{-\varepsilon-1} \left(\frac{p}{\varepsilon} \frac{u}{\eta} g_\eta \right)^p \\ &= (1 - \varepsilon \eta^p u^{-\varepsilon-1}) g_u^p + \varepsilon^{1-p} p^p u^{p-\varepsilon-1} g_\eta^p. \end{aligned}$$

Since u is a p-superminimizer, we have

$$\begin{aligned} \int_{\Omega'} g_u^p \, d\mu &\le \int_{\Omega'} g_w^p \, d\mu \\ &\le \int_{\Omega'} g_u^p \, d\mu - \varepsilon \int_{\Omega'} \eta^p u^{-\varepsilon-1} g_u^p \, d\mu + \varepsilon^{1-p} p^p \int_{\Omega'} u^{p-\varepsilon-1} g_\eta^p \, d\mu. \end{aligned}$$

From this we conclude that

$$\int_\Omega \eta^p u^{-\varepsilon-1} g_u^p \, d\mu \le p^p \varepsilon^{-p} \int_\Omega u^{p-\varepsilon-1} g_\eta^p \, d\mu.$$

$\square$

This lemma was originally proved in [22].

5 Harnack's inequality

In this section we prove a weak Harnack inequality for p-subminimizers (Theorem 5.4) and p-superminimizers (Theorem 5.19). These estimates combined with Corollary 5.17 of the John–Nirenberg lemma imply Harnack's inequality for the minimizers, see Theorem 5.21.

We start with a technical lemma.

Lemma 5.1. *Let $\varphi(t)$ be a bounded nonnegative function defined on the interval $[a,b]$, where $0 \le a < b$. Suppose that for any $a \le t < s \le b$, φ satisfies*

$$\varphi(t) \le \theta\varphi(s) + \frac{A}{(s-t)^\alpha} + B, \tag{5.2}$$

where θ, A, B and α are nonnegative constants, $\theta < 1$. Then

$$\varphi(\rho) \le C\left[\frac{A}{(R-\rho)^\alpha} + B\right], \tag{5.3}$$

for all $a \le \rho < R \le b$, where $C = C(\alpha,\theta)$.

We refer to [9, Lemma 3.1, p.161] for the proof. This lemma says that under certain assumptions, we can get rid of the term $\theta\varphi(s)$.

The Moser iteration technique yields the following inequality for positive p-subminimizers.

Theorem 5.4. *Suppose that $u > 0$ is a locally bounded p-subminimizer in Ω. Then for every ball $B(z,r)$ with $B(z,2r) \subset \Omega$ and any $q > 0$ we have*

$$\operatorname*{ess\,sup}_{B(z,r)} u \le c\left(\fint_{B(z,2r)} u^q \, d\mu\right)^{1/q}, \tag{5.5}$$

where $0 < c = c(p,q,\kappa,c_\mu) < \infty$.

Proof. First we assume that $q \ge p$. Write $B_l = B(z,r_l), r_l = (1+2^{-l})r$ for $l = 0,1,2,\ldots$, thus, $B_0 = B(z,2r)$ and $\bigcup_{l=0}^\infty B_l = B_0$. Let η_l be a Lipschitz continuous function such that $0 \le \eta_l \le 1$, $\eta_l = 1$ on $\overline{B}_{l+1}$, $\eta_l = 0$ in $X \setminus B_l$ and $g_{\eta_l} \le 4 \cdot 2^l/r$. Fix $1 \le t < \infty$ and let

$$w_l = \eta_l u^{1+(t-1)/p}.$$

Note that everything works fine if we fix $0 < t < \infty$, we fixed $t \ge 1$ just for convenience. As in the proof of Lemma 4.1 for μ-almost everywhere in Ω we have

$$g_{w_l} \le g_{\eta_l} u^{1+(t-1)/p} + \left(1 + \frac{t-1}{p}\right) u^{(t-1)/p} g_u \eta_l$$

and consequently

$$g_{w_l}^p \le 2^{p-1} g_{\eta_l}^p u^{p+t-1} + 2^{p-1}\left(\frac{p+t-1}{p}\right)^p u^{t-1} g_u^p \eta_l^p$$

μ-almost everywhere in Ω. By using the Caccioppoli estimate (Lemma 4.1), we obtain

$$\begin{aligned}
\left(\fint_{B_l} g_{w_l}^p \, d\mu\right)^{1/p} &\le 2^{\frac{p-1}{p}}\left(\fint_{B_l}\left(g_{\eta_l}^p u^{p+t-1} + \left(\frac{p+t-1}{p}\right)^p u^{t-1} g_u^p \eta_l^p\right) \, d\mu\right)^{1/p} \\
&\le 2 \cdot \frac{p+t-1}{t}\left(\fint_{B_l} g_{\eta_l}^p u^{p+t-1} \, d\mu\right)^{1/p} \\
&\le (p+t-1)\frac{8 \cdot 2^l}{r}\left(\fint_{B_l} u^{p+t-1} \, d\mu\right)^{1/p}.
\end{aligned}$$

The Sobolev inequality (2.8) implies

$$\begin{aligned}\left(\fint_{B_l} w_l^{\kappa p}\,d\mu\right)^{1/\kappa p} &\le c(p,c_\mu) r_l \left(\fint_{B_l} g_{w_l}^p\,d\mu\right)^{1/p}\\ &\le c(p,c_\mu)(p+t-1)(1+2^{-l}) r\frac{2^l}{r}\left(\fint_{B_l} u^{p+t-1}\,d\mu\right)^{1/p}\\ &\le c(p,c_\mu)(p+t-1)2^l\left(\fint_{B_l} u^{p+t-1}\,d\mu\right)^{1/p}\end{aligned}$$

By setting $\tau = p+t-1$ and using the doubling property of μ we have (remember that $w_l = u^{\tau/p}$ on B_{l+1})

$$\left(\fint_{B_{l+1}} (u^{\tau/p})^{\kappa p}\,d\mu\right)^{1/\kappa p} \le c(p,c_\mu)\tau 2^l \left(\fint_{B_l} u^\tau\,d\mu\right)^{1/p}.$$

Hence, we obtain

$$\left(\fint_{B_{l+1}} u^{\kappa\tau}\,d\mu\right)^{1/\kappa\tau} \le \left(c(p,c_\mu)\tau 2^l\right)^{p/\tau}\left(\fint_{B_l} u^\tau\,d\mu\right)^{1/\tau}.$$

This estimate holds for all $\tau \ge p$, we apply the estimate with $\tau = q\kappa^l$ for all $l = 0,1,2,\ldots$, we have

$$\left(\fint_{B_{l+1}} u^{q\kappa^{l+1}}\,d\mu\right)^{1/q\kappa^{l+1}} \le \left(c(p,c_\mu)(q\kappa^l)2^l\right)^{p/q\kappa^l}\left(\fint_{B_l} u^{q\kappa^l}\,d\mu\right)^{1/q\kappa^l}.$$

By iterating we obtain the desired estimate

$$\begin{aligned}\operatorname*{ess\,sup}_{B(z,r)} u &\le \left(c(p,c_\mu)^{\sum_{i=0}^\infty \kappa^{-i}}\prod_{i=0}^\infty 2^{i\kappa^{-i}}\prod_{i=0}^\infty (q\kappa^i)^{\kappa^{-i}}\right)^{p/q}\left(\fint_{B(z,2r)} u^q\,d\mu\right)^{1/q}\\ &\le \left(c(p,c_\mu)^{\frac{\kappa}{\kappa-1}}2^{\frac{\kappa}{(\kappa-1)^2}}q^{\frac{\kappa}{\kappa-1}}\kappa^{\frac{\kappa}{(\kappa-1)^2}}\right)^{p/q}\left(\fint_{B(z,2r)} u^q\,d\mu\right)^{1/q}\\ &\le c(p,q,\kappa,c_\mu)\left(\fint_{B(z,2r)} u^q\,d\mu\right)^{1/q}. \qquad (5.6)\end{aligned}$$

The theorem is proved for $q \ge p$.

By the doubling property of the measure and (2.5), it is easy to see that (5.6) can be reformulated in a bit different manner. Namely, if $0 \le \rho < \widetilde{r} \le 2r$, then

$$\operatorname*{ess\,sup}_{B(z,\rho)} u \le \frac{c}{(1-\rho/\widetilde{r})^{Q/q}}\left(\fint_{B(z,\widetilde{r})} u^q\,d\mu\right)^{1/q}, \qquad (5.7)$$

where $0 < c = c(p, q, \kappa, c_\mu) < \infty$. See Remark 4.4 in [20].

If $0 < q < p$ we want to prove that there is a postive constant c so that

$$\operatorname*{ess\,sup}_{B(z,\rho)} u \leq \frac{c}{(1-\rho/2r)^{Q/q}} \left(\fint_{B(z,2r)} u^q \, d\mu \right)^{1/q},$$

when $0 \leq \rho < 2r < \infty$. Now suppose that $0 < q < p$ and let $0 \leq \rho < \widetilde{r} \leq 2r$. We choose $q = p$ in (5.7), then

$$\begin{aligned} \operatorname*{ess\,sup}_{B(z,\rho)} u &\leq \frac{c}{(1-\rho/\widetilde{r})^{Q/p}} \left(\fint_{B(z,\tilde{r})} u^q u^{p-q} \, d\mu \right)^{1/p} \\ &\leq \frac{c}{(1-\rho/\widetilde{r})^{Q/p}} \left(\operatorname*{ess\,sup}_{B(z,\widetilde{r})} u \right)^{1-q/p} \left(\fint_{B(z,\widetilde{r})} u^q \, d\mu \right)^{1/p} \end{aligned}$$

By Young's inequality

$$\begin{aligned} \operatorname*{ess\,sup}_{B(z,\rho)} u &\leq \frac{p-q}{p} \operatorname*{ess\,sup}_{B(z,\widetilde{r})} u + \frac{c}{(1-\rho/\widetilde{r})^{Q/q}} \left(\fint_{B(z,\widetilde{r})} u^q \, d\mu \right)^{1/q} \\ &\leq \frac{p-q}{p} \operatorname*{ess\,sup}_{B(z,\widetilde{r})} u + \frac{c}{(\widetilde{r}-\rho)^{Q/q}} \left((2r)^Q \fint_{B(z,2r)} u^q \, d\mu \right)^{1/q}, \end{aligned}$$

where the doubling property (2.5) was used to obtain the last inequality. We need to get rid of the first term on the right-hand side. By Lemma 5.1 (let $\varphi(t) = \operatorname{ess\,sup}_{B(z,t)} u$) we have

$$\operatorname*{ess\,sup}_{B(z,\rho)} u \leq \frac{c}{(1-\rho/2r)^{Q/q}} \left(\fint_{B(z,2r)} u^q \, d\mu \right)^{1/q}$$

for all $0 \leq \rho < 2r$, where $0 < c = c(p, q, \kappa, c_\mu) < \infty$. If we set $\rho = r$, we obtain (5.6) for every $0 < q < p$ and the proof is complete. □

Remark 5.8. The minimizing property (2.10) was not needed in the proof of Theorem 5.4. Instead we used estimate (4.2). Therefore the statement of the theorem can be restated to hold for functions which satisfy the Caccioppoli type estimate (4.2).

Next we present a theorem which gives a lower bound for positive $p-$ superminimizers.

Theorem 5.9. *Suppose that $u > 0$ is a p-superminimizer in Ω. Then for every ball $B(z,r)$ with $B(z,2r) \subset \Omega$ and any $q > 0$ we have*

$$\operatorname*{ess\,inf}_{B(z,r)} u \geq c \left(\fint_{B(z,2r)} u^{-q} \, d\mu \right)^{-1/q}, \tag{5.10}$$

where $0 < c = c(p, q, \kappa, c_\mu) < \infty$.

Proof. As in the proof of Theorem 5.4, write $B_l = B(z, r_l), r_l = (1+2^{-l})r$ for $l = 0, 1, 2, \ldots$ Let η_l be a Lipschitz continuous function such that $0 \le \eta_l \le 1$, $\eta_l = 1$ on $\overline{B}_{l+1}$, $\eta_l = 0$ in $X \setminus B_l$ and $g_{\eta_l} \le 4 \cdot 2^l/r$. Fix $t \ge 1$ such that $p - t - 1 < 0$ and let

$$w_l = \eta_l u^{1+(-t-1)/p}.$$

Then for μ-almost everywhere in Ω we have

$$g_{w_l} \le g_{\eta_l} u^{1+(-t-1)/p} + \left(\frac{t+1-p}{p}\right) u^{(-t-1)/p} g_u \eta_l$$

and consequently

$$g_{w_l}^p \le 2^{p-1} g_{\eta_l}^p u^{p-t-1} + 2^{p-1} \left(\frac{t+1-p}{p}\right)^p u^{-t-1} g_u^p \eta_l^p$$

μ-almost everywhere in Ω. By using the Caccioppoli estimate (Lemma 4.3) for p-superminimizers, we obtain

$$\begin{aligned}
\left(\fint_{B_l} g_{w_l}^p \, d\mu\right)^{1/p} &\le 2^{\frac{p-1}{p}} \left(\fint_{B_l} \left(g_{\eta_l}^p u^{-(t+1-p)} + \left(\frac{t+1-p}{p}\right)^p u^{-t-1} g_u^p \eta_l^p\right) \, d\mu\right)^{1/p} \\
&\le 2 \cdot \frac{t+1-p}{t} \left(\fint_{B_l} g_{\eta_l}^p u^{-(t+1-p)} \, d\mu\right)^{1/p} \\
&\le (t+1-p) \frac{8 \cdot 2^l}{r} \left(\fint_{B_l} u^{-(t+1-p)} \, d\mu\right)^{1/p}.
\end{aligned}$$

The Sobolev inequality (2.8) implies

$$\begin{aligned}
\left(\fint_{B_l} w_l^{\kappa p} \, d\mu\right)^{1/\kappa p} &\le c(p, c_\mu) r_l \left(\fint_{B_l} g_{w_l}^p \, d\mu\right)^{1/p} \\
&\le c(p, c_\mu)(t+1-p)(1+2^{-l}) r \frac{2^l}{r} \left(\fint_{B_l} u^{-(t+1-p)} \, d\mu\right)^{1/p} \\
&\le c(p, c_\mu)(t+1-p) 2^l \left(\fint_{B_l} u^{-(t+1-p)} \, d\mu\right)^{1/p}
\end{aligned}$$

By setting $\tau = t + 1 - p > 0$ and using the doubling property of μ we have (notice that $w_l = u^{-\tau/p}$ on B_{l+1})

$$\left(\fint_{B_{l+1}} (u^{-\tau/p})^{\kappa p} \, d\mu\right)^{1/\kappa p} \le c(p, c_\mu) \tau 2^l \left(\fint_{B_l} u^{-\tau} \, d\mu\right)^{1/p}.$$

Hence, we obtain

$$\left(\fint_{B_{l+1}} u^{-\kappa\tau} \, d\mu\right)^{-1/\kappa\tau} \ge c(p, c_\mu)^{-p/\tau} \tau^{-p/\tau} 2^{-lp/\tau} \left(\fint_{B_l} u^{-\tau} \, d\mu\right)^{-1/\tau}.$$

This estimate holds for all $\tau > 0$, we apply the estimate with $\tau = q\kappa^l$ for all $l = 0, 1, 2, \ldots$, we have

$$\left(\fint_{B_{l+1}} u^{-q\kappa^{l+1}} \, d\mu \right)^{-1/q\kappa^{l+1}} \geq \left(c(p, c_\mu)(q\kappa^l)2^l \right)^{-p/q\kappa^l} \left(\fint_{B_l} u^{-q\kappa^l} \, d\mu \right)^{-1/q\kappa^l}.$$

By iterating as in the proof of Theorem 5.4, we obtain the desired estimate

$$\operatorname*{ess\,inf}_{B(z,r)} u \geq c(p, q, \kappa, c_\mu) \left(\fint_{B(z,2r)} u^{-q} \, d\mu \right)^{-1/q}.$$

The proof is complete. □

Remarks 5.11. (1) In the Euclidean case we have the symmetry between sub- and supersolutions of the p-Laplace equation. A function u is a supersolution, then $1/u$ is a subsolution of the equation. Theorem 5.9 follows directly from this and Theorem 5.4 in the Euclidean space. We do not know if this holds in a general metric measure space.

(2) As in the proof of Theorem 5.4, we fixed a parameter t to be greater or equal than one just for convenience. Any t strictly greater than zero would do nicely.

(3) As in the proof of Theorem 5.4 the minimizing property (2.10) was not needed in the proof. We used essentially estimate (4.4). Therefore the statement of the theorem can be restated to hold not for superminimizers but for functions which satisfy the Caccioppoli type estimate (4.4). This holds also for Lemma 5.12 below.

The following lemma will be crucial when we prove Theorem 5.19.

Lemma 5.12. *Suppose that $u > 0$ is a p-superminimizer in Ω and let $v = \log u$. Then $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$ and $g_v = g_u/u$ μ-almost everywhere in Ω. Furthermore, for every ball $B(z,r)$ with $B(z,2r) \subset \Omega$ we have*

$$\fint_{B(z,r)} g_v^p \, d\mu \leq c r^{-p}, \tag{5.13}$$

where $c = c_\mu(4p/(p-1))^p$.

Proof. We may assume that $u \geq \delta > 0$ for μ-almost all $x \in B(z,r)$. Hence v is bounded below in $B(z,r)$ and $v \in L^p(B(z,r))$. As in the proof of Lemma 4.1 we see that $g_v \leq g_u/u$ μ-almost everywhere in Ω. We obtain the reverse inequality, if we set $u = \exp(v)$, hence, $g_v = g_u/u$ μ-almost everywhere in Ω. It follows that $g_v \in L^p_{\mathrm{loc}}(\Omega)$ and consequently that $v \in N^{1,p}_{\mathrm{loc}}(\Omega)$.

Let η be a Lipschitz function such that $0 \le \eta \le 1$, $\eta = 1$ on $B(z,r)$, $\eta = 0$ in $X \setminus B(z,2r)$ and $g_\eta \le 4/r$. If we choose $\varepsilon = p-1$ in (4.4) we have

$$\int_\Omega g_v^p \eta^p \, d\mu = \int_\Omega g_u^p u^{-p} \eta^p \, d\mu \le \left(\frac{p}{p-1}\right)^p \int_\Omega g_\eta^p \, d\mu.$$

From this and the doubling property of μ we obtain

$$\begin{aligned} \int_{B(z,r)} g_v^p \, d\mu &\le \left(\frac{p}{p-1}\right)^p \int_{B(z,2r)} \frac{4^p}{r^p} \, d\mu \\ &\le c_\mu \left(\frac{4p}{p-1}\right)^p \frac{\mu(B(z,r))}{r^p}, \end{aligned}$$

which is the desired inequality. □

A locally integrable function v in Ω is said to belong to $\mathrm{BMO}(\Omega)$ if the inequality

$$⨍_B |v - v_B| \, d\mu \le c \tag{5.14}$$

holds for all balls B with $10B \subset \Omega$. The smallest bound c for which (5.14) is satisfied is said to be the "BMO-norm" of v in this space, and it is denoted by $\|v\|_*$.

In order to "jump" over the zero in exponents in (5.10), we use the John–Nirenberg lemma.

Theorem 5.15. *There exist two positive constants β and b such that for any $f \in \mathrm{BMO}(\Omega)$ and for every ball B with $10B \subset \Omega$, we have*

$$\mu\left(\{x \in B \ : \ |u - u_B| > \lambda\}\right) \le \beta \exp\left(\frac{-b\lambda}{\|u\|_*}\right) \mu(B) \tag{5.16}$$

for all $\lambda > 0$.

Proof. In the proof we have to enlarge the ball $B(z_0, r), z_0 \in \Omega$, so that we take a constant σ such that if $z \in B(z_0, r)$ and $0 < \rho < r$, then $B(z,\rho) \subset B(z_0, \sigma r)$. (Take $\sigma = 1 + 12/5$.) Thus, we work with the balls B for which ten times larger balls are still in Ω. The proof can be found in [2, 24]. □

See for example [7] for the following corollary.

Corollary 5.17. *A function v is in $\mathrm{BMO}(\Omega)$ if and only if there are positive constants c_1 and c_2 such that*

$$⨍_B e^{c_1 |v - v_B|} \, d\mu \le c_2 \tag{5.18}$$

for every ball B with $10B \subset \Omega$.

Now we are ready to provide the proof for the following

Theorem 5.19. *If $u > 0$ is a p-superminimizer in $\Omega \subset X$, then there are $q > 0$ and $c > 0$ such that*

$$\left(\fint_{B(z,2r)} u^q \, d\mu \right)^{1/q} \le c \operatorname*{ess\,inf}_{B(z,r)} u \tag{5.20}$$

for every ball $B(z,r)$ such that $B(z, 10\tau r) \subset \Omega$.

Proof. By Theorem 5.9 we have

$$\begin{aligned} \frac{1}{c} \operatorname*{ess\,inf}_{B(z,r)} u &\ge \left(\fint_{B(z,2r)} u^{-q} \, d\mu \right)^{-1/q} \\ &= \left(\fint_{B(z,2r)} u^{-q} \, d\mu \fint_{B(z,2r)} u^q \, d\mu \right)^{-1/q} \left(\fint_{B(z,2r)} u^q \, d\mu \right)^{1/q}. \end{aligned}$$

To complete the proof, we have to show that

$$\fint_{B(z,2r)} u^{-q} \, d\mu \fint_{B(z,2r)} u^q \, d\mu \le c$$

for some $q > 0$. Write $v = \log u$. Then the weak $(1,p)$-Poincaré inequality, Lemma 5.12 and the doubling property of μ imply

$$\fint_{B(z,2r)} |v - v_{B(z,2r)}| \, d\mu \le cr \left(\fint_{B(z,2\tau r)} g_v^p \, d\mu \right)^{1/p} \le c.$$

We stress that instead of a weak $(1,q)$-Poincaré inequality – for some q with $1 < q < p$ – we applied only a weak $(1,p)$-Poincaré inequality. It follows from the John-Nirenberg lemma and (5.18) that there exist constants q and c such that

$$\begin{aligned} &\fint_{B(z,2r)} e^{-qv} \, d\mu \fint_{B(z,2r)} e^{qv} \, d\mu \\ = &\fint_{B(z,2r)} e^{q(v_{B(z,2r)} - v)} \, d\mu \fint_{B(z,2r)} e^{q(v - v_{B(z,2r)})} \, d\mu \\ \le &\left(\fint_{B(z,2r)} e^{q|v - v_{B(z,2r)}|} \, d\mu \right)^2 \le c, \end{aligned}$$

from which the claim follows. □

From this we easily obtain Harnack's inequality.

Theorem 5.21. *Suppose that $u > 0$ is a locally bounded p-minimizer in Ω. Then there exists a constant $c \ge 1$ so that*

$$\operatorname*{ess\,sup}_{B(z,r)} u \le c \operatorname*{ess\,inf}_{B(z,r)} u$$

for every ball $B(z,r)$ for which $B(z, 10\tau r) \subset \Omega$. Here the constant c is independent of the ball $B(z,r)$ and function u. (The constant $\tau \ge 1$ comes from the weak $(1,p)$-Poincaré inequality.)

Proof. By combining Theorem 5.4 and Theorem 5.19, the estimate follows. □

From Harnack's inequality it follows that p-minimizers are locally Hölder continuous and satisfy the strong maximum principle, see for example [12].

Acknowledgments

The work was funded by the Vilho, Yrjö and Kalle Väisälä Fund of Finnish Academy of Science and Letters. The author is grateful to professor Juha Kinnunen for valuable discussions and instructions. The author would also like to thank Anders and Jana Björn and Petteri Harjulehto for valuable comments.

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