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Abstract: *We study differentiability in locally compact metric spaces, using point derivations of Lipschitz functions. The classical Riemannian manifold case can be generalized to those locally compact metric spaces* $(M,(x,y)\mapsto|xy|)$ *for which there are local Cauchy–Schwarz -like inequalities*

$$|wx\cdot yz|\leq C\ |wx|\ |yz|,$$

where $C<\infty$ *is a constant and* $wx\cdot yz\in\mathbb{R}$ *is defined by*

$$wx\cdot yz:=\frac{1}{2}\left(|wz|^2+|xy|^2-|wy|^2-|xz|^2\right).$$

These Cauchy–Schwarz spaces *behave well with respect to elementary operations, e.g. under Gromov–Hausdorff limits. The* dot product $wx\cdot yz$ *can be thought as a discrete analogue of the Riemannian metric tensor. Examples of good and bad spaces in this respect are given. Moreover, we get new interpretations of concepts like Gromov hyperbolicity, comparison angles, Aleksandrov spaces of non-positive curvature and Reshetnyak's quadruple comparison.*

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1 Introduction

In this paper we study differentiability in locally compact metric spaces from a metric geometry point of view. As it is well-known, Riemannian geometry was introduced by G. F. B. Riemann in his 1854 inaugural lecture, and metric spaces by M. Fréchet in his 1906 doctoral dissertation. As a starting point, we may consider how the Riemannian metric (tangent space inner product) is classically obtained from the distances on the manifold. Tangents on a manifold are bounded linear functionals satisfying the Leibniz product rule for C^1-functions at a point. In a locally compact metric space, Lipschitz point derivations serve as preliminary tangents; such derivations were studied by D. R. Sherbert [11] and L. Waelbroeck [13] in the 1960s, discovering the structure of the closed ideals in the normed algebra of Lipschitz functions. In generalizing the classical Riemannian observation to metric spaces, a natural restriction on the metric manifests itself. This appears to be related to "smoothness" and curvature bounds for metric spaces, as introduced by A. D. Aleksandrov [1], pioneering the theory in the 1950s. There were important further developments e.g. by Yu. G. Reshetnyak in the 1960s [9]. Later, the 1980s works of M. Gromov have been influential [6]. Monographs [2] and [3] are nice overviews on Cartan–Aleksandrov–Toponogov curvature bounds on metric spaces. Metric space curvature is also related to variance inequalities for probability measures, see e.g. K.T. Sturm's paper [12]. Measure theoretic aspects of differentiability have been studied extensively, see for instance works by G. David and S. W. Semmes [5], J. Cheeger [4], N. Weaver [15] and S. Keith [8]. However, the differentiability concept in this paper involves no measures in itself; Gromov–Hausdorff tangent cones are neither involved here.

Let us describe details of the situation: Let M be a Riemannian manifold and T_xM be the tangent space at $x \in M$. A tangent $a \in \mathrm{T}_xM$ is a bounded linear functional $a : C^1(M) \to \mathbb{R}$, satisfying the Leibniz rule at $x \in M$:

$$a(fg) = f(x)\, a(g) + a(f)\, g(x).$$

Let $\langle a, b\rangle_{\mathrm{T}_xM} \in \mathbb{R}$ denote the Riemannian inner product of tangents $a, b \in \mathrm{T}_xM$. The length of a smooth curve $\gamma : [0, 1] \to M$ is

$$\int_0^1 \|\gamma'(t)\|_{\mathrm{T}_{\gamma(t)}M}\, \mathrm{d}t,$$

where $\|a\|^2_{\mathrm{T}_xM} = \langle a, a\rangle_{\mathrm{T}_xM}$. The Riemannian distance $|xy|$ (metric of the metric space) between $x, y \in M$ is the infimum of the lengths of curves joining x, y. Conversely, the Riemannian inner product can be recovered from the distance by

$$\langle a, b\rangle_{\mathrm{T}_xM} = a\left(y \mapsto b(z \mapsto xy \cdot xz)\right),$$

where $xy \cdot xz = |xy|\, |xz| \cos(\tilde{\angle} yxz) = \frac{1}{2}(|xy|^2 + |xz|^2 - |yz|^2)$; here $\tilde{\angle} yxz$ is a (Euclidean) *comparison angle*, a basic notion in metric geometry. Actually, we can see that instead of $xy \cdot xz$ we could derivate $-|yz|^2/2$.

Now let M be a locally compact metric space; for convenience, assume it is even compact. We no longer have C^1-smoothness on M, but the space $\mathrm{Lip}(M)$

of Lipschitz functions $M \to \mathbb{R}$ will be our test function space. The *Lipschitz tangents* (or *Lipschitz point derivations*) at $x \in M$ are bounded linear functionals $D : \mathrm{Lip}(M) \to \mathbb{R}$ satisfying

$$D(fg) = f(x)\ D(g) + D(f)\ g(x).$$

On a manifold M, the restriction to $C^1(M)$ of a Lipschitz tangent is naturally a tangent; on the other hand, it turns out that a (non-unique) Hahn–Banach extension $\mathrm{Lip}(M) \to \mathbb{R}$ of a tangent $C^1(M) \to \mathbb{R}$ is always a Lipschitz tangent. On a (locally) compact metric space M, if D, E are Lipschitz tangents at $x \in M$, we might be tempted to define their "Riemannian product" by

$$D \cdot E := D(y \mapsto E(z \mapsto xy \cdot xz));$$

whereas on Riemannian manifolds this is sound, we may run into trouble on metric spaces where $y \mapsto E(z \mapsto xy \cdot xz)$ is not a Lipschitz function nearby $x \in M$. To exclude this misfortune, we denote $h(y, z) := xy \cdot xz$, define

$$\begin{aligned} y_1y_2 \cdot z_1z_2 &:= [h(y_2, z_2) - h(y_2, z_1)] - [h(y_1, z_2) - h(y_1, z_1)] \\ &= \frac{1}{2}\left(|y_1z_2|^2 + |y_2z_1|^2 - |y_1z_1|^2 - |y_2z_2|^2\right), \end{aligned}$$

and demand that a Cauchy–Schwarz -like inequality

$$|y_1y_2 \cdot z_1z_2| \le C\ |y_1y_2|\ |z_1z_2| \tag{1}$$

holds nearby $x \in M$; here $C < \infty$ is a constant on the neighborhood.

The structure and contents of the paper are briefly the following: Section 2 deals with the properties of the dot product $wx \cdot yz$ in a metric space in general, and in Section 3 we study *Cauchy–Schwarz spaces* satisfying local or global inequalities like (1): Examples of global Cauchy–Schwarz spaces are provided by metric spaces of global non-positive curvature (and perhaps surprisingly, several snow-flakings), and Riemannian manifolds are locally Cauchy–Schwarz spaces. Moreover, the Cauchy–Schwarz -like inequalities are inherited by subspaces, preserved in completion, also in reasonable Cartesian products and under Gromov–Hausdorff convergence. Meanwhile, singularities on manifolds may obstruct even the local Cauchy–Schwarz property, as well as non-unique geodesics may prevent the global Cauchy–Schwarz inequality; we shall also note that the doubling property of a measure does not guarantee the Cauchy–Schwarz properties at all. Finally, in Section 4 we examine infinitesimal metric geometry in Cauchy–Schwarz spaces: there tangent spaces and differentiability are studied.

2 Dot product in metric spaces

Let $(M, xy \mapsto |xy|)$ be a metric space; here $(x, y) \in M \times M$ is abbreviated by $xy \in M^2$. Let us define the *dot product* $wx \cdot yz \in \mathbb{R}$ of points $w, x, y, z \in M$ by

$$wx \cdot yz := \frac{1}{2}\left(|wz|^2 + |xy|^2 - |wy|^2 - |xz|^2\right). \tag{2}$$

To get a feeling of this, let $h(x,y) = -|xy|^2/2$, and write the iterated differences

$$\begin{aligned} wx \cdot yz &= [h(w,y) - h(w,z)] - [h(x,y) - h(x,z)] \\ &= [h(w,y) - h(x,y)] - [h(w,z) - h(x,z)]; \end{aligned}$$

the dot product seems a vague analogue of the Riemannian metric tensor. In this section, we study properties of the dot product in a general metric space. It will turn out that it resembles a vector space inner product, with its natural connection to comparison angles and a Cauchy–Schwarz -like inequality.

2.1 Dot product in general metric spaces

In this subsection the metric spaces satisfy no extra conditions. As the definition of the dot product itself is akin to the parallelogram law of inner product spaces, it is not surprising that there are inner-product-like features:

Theorem 2.1 *Let $(M, xy \mapsto |xy|)$ be a metric space. For every $v, w, x, y, z \in M$,*

$$\begin{aligned} xy \cdot xy &= |xy|^2, && (3) \\ wx \cdot yz &= yz \cdot wx && (4) \\ &= -xw \cdot yz && (5) \\ &= wv \cdot yz + vx \cdot yz, && (6) \\ 0 &= wx \cdot yz + wy \cdot zx + wz \cdot xy. && (7) \end{aligned}$$

A noteworthy case is $xx \cdot yz = 0$. □

A triple of points in a metric space can naturally be identified with a Euclidean plane triangle having the side-lengths corresponding to the distances between the points; the metric space is thereby compared to the flat Euclidean metric. Comparison triangles on surfaces of constant curvature are studied in the theory of Aleksandrov spaces of bounded curvature, see e.g. [3].

Euclidean comparison triangles. Let $(M, xy \mapsto |xy|)$ be a metric space. If $x, y, z \in M$ and $y \neq x \neq z$, the *comparison angle* $\tilde{\angle} yxz \in [0, \pi]$ is defined by

$$\cos\left(\tilde{\angle} yxz\right) := \frac{1}{2} \frac{|xy|^2 + |xz|^2 - |yz|^2}{|xy|\,|xz|}. \tag{8}$$

There is no harm in defining $\tilde{\angle} yxz := 0$ if $x \in \{y, z\}$. The geometric interpretation is that there exist a unique-up-to-isometry triangle in the Euclidean space $\mathbb{R}^2$ with side-lengths $|xy|, |xz|, |yz|$ such that $\tilde{\angle} yxz$ is the inner angle of the triangle at the vertex corresponding to the point $x \in M$. Consequently, for $yxz \in M^3$, let us define the *comparison points* $\bar{y}xz, y\bar{x}z, yx\bar{z} \in \mathbb{R}^2$ by

$$\bar{y}xz := (|xy|, 0), \quad y\bar{x}z := (0,0), \quad yx\bar{z} := |xz|\,(\cos(\phi), \sin(\phi)), \tag{9}$$

where $\phi = \tilde{\angle} yxz$. Comparison angles provide an interpretation for the dot product:

$$\begin{aligned} wx \cdot wz &= |wx|\,|wz| \cos(\tilde{\angle} xwz) && (10) \\ wx \cdot yz &= wx \cdot wz - wx \cdot wy && (11) \\ &= |wx| \left(|wz| \cos(\tilde{\angle} xwz) - |wy| \cos(\tilde{\angle} xwy) \right). && (12) \end{aligned}$$

Actually, one can define the dot product by formulas (10) and (11).

Partial Cauchy–Schwarz inequality. By (10) and (5), we get a *partial Cauchy–Schwarz inequality*

$$|xy \cdot yz| \le |xy|\,|yz|, \tag{13}$$

which is valid for every $x, y, z \in M$. However, often in a metric space there is no "general Cauchy–Schwarz inequality" like $|wx \cdot yz| \le |wx|\,|yz|$, nor even $|wx \cdot yz| \le C\,|wx|\,|yz|$ for a constant $C < \infty$; this is the starting point of the theory of Cauchy–Schwarz spaces, studied in Section 3.

Even though no algebraic structure for M is assumed, the product space M^2 has a trivial groupoid structure, which behaves well with respect to the dot product: let us denote $xy + yz := xz$ for $x, y, z \in M$. In this notation,

$$\begin{aligned} wx \cdot yz &= (wv + vx) \cdot yz \;=\; wv \cdot yz + vw \cdot yz, && (14) \\ |xz|^2 &= |xy + yz|^2 \;=\; |xy|^2 + |yz|^2 + 2\,xy \cdot yz; && (15) \end{aligned}$$

since $(|xy| - |yz|)^2 \le |xz|^2 \le (|xy| + |yz|)^2$ by the triangle inequality, we get a more direct proof for the partial Cauchy–Schwarz inequality (13).

Triangle equalities. In a metric space,

$$xy \cdot yz = |xy|\,|yz| \iff |xz| = |xy| + |yz|; \tag{16}$$

in this case, the intuition is that "y is located between x and z". Accordingly,

$$xy \cdot yz = -|xy|\,|yz| \iff |xz| = ||xy| - |yz||; \tag{17}$$

here either "z is located between x and y" or "x is located between y and z".

Lipschitz numbers. Let $(M, xy \mapsto |xy|)$ be a metric space. A function $f : M \to \mathbb{R}$ is called *Lipschitz*, denoted by $f \in \mathrm{Lip}(M)$, if its *Lipschitz seminorm*

$$\mathrm{Lip}(f) := \sup_{x,z \in M} \frac{|f(x) - f(z)|}{|xz|} \tag{18}$$

is finite (with convention $0/0 = 0$). If M is compact, the real algebra $\mathrm{Lip}(M)$ is given the Banach space norm $f \mapsto \|f\|_{\mathrm{Lip}(M)} = \max\{\|f\|_{C(M)}, \mathrm{Lip}(f)\}$, where the norm for continuous functions is $\|f\|_{C(M)} = \sup_{x \in M} |f(x)|$. The *pointwise* (or *local*) *Lipschitz number at* $x \in M$ is

$$\mathrm{Lip}_x(f) := \lim_{r \to 0^+} \sup_{y,z \in B(x,r)} \frac{|f(y) - f(z)|}{|yz|}, \tag{19}$$

where $B(x,r) := \{y \in M : |xy| < r\}$ is the *open ball of radius* $r > 0$ *centered at* $x \in M$. Notice that in some books and papers, various other (non-equivalent) pointwise Lipschitz numbers occur. Being defined in terms of squares of distances, the dot product has some natural Lipschitz properties:

Theorem 2.2 *Let* $\operatorname{diam}(M) := \sup_{x,y\in M} |xy|$ *be the* diameter *of a metric space* $(M, xy \mapsto |xy|)$. *The dot product satisfies*

$$|wx \cdot yz| \le (|wy| + |yx|)\,|yz| \tag{20}$$

for every $w, x, y, z \in M$, *and*

$$\begin{aligned}
\operatorname{Lip}(z \mapsto wx \cdot yz) &\le \sup_{v\in M}(|wv| + |vx|) \le 2\operatorname{diam}(M), && (21)\\
\operatorname{Lip}_v(z \mapsto wx \cdot yz) &\le |wv| + |vx|, && (22)\\
\operatorname{Lip}_w(z \mapsto wx \cdot yz) &\le |wx|. && (23)
\end{aligned}$$

Proof. By the triangle inequality and by (13),

$$|wx \cdot yz| = |wy \cdot yz - xy \cdot yz| \le |wy \cdot yz| + |xy \cdot yz| \le |wy|\,|yz| + |xy|\,|yz|,$$

yielding (20). Hence

$$|wx \cdot yz_1 - wx \cdot yz_2| = |wx \cdot z_2 z_1| \le (|wz_j| + |z_j x|)\ |z_1 z_2|$$

for each $j = 1, 2$. This gives the estimates for Lipschitz numbers. $\square$

2.2 Dot product in normed spaces and for paths

Next we deal with the dot product in presence of linear or length structures. Let M be a metric space and a vector space, with the origin $o \in M$. It is easy to see that $x \mapsto |ox|$ is a norm if and only if

$$|o(\lambda x)| = |\lambda|\,|ox| \quad \text{and} \quad |(x+z)(y+z)| = |xy|$$

for every $x, y, z \in M$ and for every scalar λ. A natural question is whether the dot product has something to do with inner product spaces:

Theorem 2.3 *Let* $(M, x \mapsto \|x\|)$ *be a normed space over* $\mathbb{R}$, *with the origin* $o \in M$, *and let* $|xy| := \|y - x\|$. *Then* $xy \mapsto ox \cdot oy$ *is an inner product if and only if* $o(-x) \cdot oy = -(ox \cdot oy)$ *for every* $x, y \in M$.

Proof. If $(x, y) \mapsto \langle x, y\rangle$ is an inner product for M then

$$\begin{aligned}
2wx \cdot yz &= |wz|^2 + |xy|^2 - |wy|^2 - |xz|^2\\
&= \|w - z\|^2 + \|x - y\|^2 - \|w - y\|^2 - \|x - z\|^2\\
&= -2\langle w, z\rangle - 2\langle x, y\rangle + 2\langle w, y\rangle + 2\langle x, z\rangle\\
&= 2\langle x - w, z - y\rangle;
\end{aligned}$$

the dot product $M^2 \times M^2 \to \mathbb{R}$ and the inner product $M \times M \to \mathbb{R}$ are naturally related. Conversely, suppose $o(-x) \cdot oy = -(ox \cdot oy)$ for every $x, y \in M$. Then

$$\begin{aligned} \|y+x\|^2 + \|y-x\|^2 &= |(-x)y|^2 + |xy|^2 \\ &= |(-x)o|^2 + |oy|^2 + 2(-x)o \cdot oy + |xo|^2 + |oy|^2 + 2xo \cdot oy \\ &= 2\|x\|^2 + 2\|y\|^2; \end{aligned}$$

the parallelogram law holds, so that M has an inner product given by

$$\langle x, y \rangle := \frac{\|x\|^2 + \|y\|^2 - \|x-y\|^2}{2} = \frac{|ox|^2 + |oy|^2 - |xy|^2}{2} = ox{\cdot}oy. \qquad \square$$

Remark. If M is a normed but not an inner product space then there are points for which $(w+v)(x+v) \cdot yz \neq wx \cdot yz$; the reasoning goes as in the proof above.

Intrinsic metrics [3] and dot product. Let $(M, xy \mapsto |xy|)$ be a metric space. The metric is called *intrinsic* (resp. *strictly intrinsic*) if for each $x, y \in M$ the distance $|xy|$ is the infimum (resp. minimum) of the lengths of paths between x and y. It well-known that the metric is intrinsic if and only if

$$\begin{aligned} \forall \varepsilon > 0\ \forall x_0, x_1 \in M\ \exists y \in M: \quad & |x_0 y|^2 + |x_1 y|^2 \leq 2^{-1}|x_0 x_1|^2 + 2\varepsilon, \\ \text{i.e.} \quad & (|x_0 x_1|/2)^2 \leq x_0 y \cdot y x_1 + \varepsilon; \end{aligned}$$

the metric is strictly intrinsic if and only if this inequality holds for $\varepsilon = 0$.

3 Cauchy–Schwarz spaces

The analogue $|wx \cdot yz| \leq |wx|\ |yz|$ of the classical Cauchy–Schwarz inequality does not hold in every metric space. In general, the best that can be said is

$$|wx \cdot yz| \leq (|wy| + |yx|)\ |yz| \geq |wx|\ |yz|.$$

The Cauchy–Schwarz inequality $|wx \cdot yz| \leq C\ |wx|\ |yz|$ is studied in the sequel.

3.1 Cauchy–Schwarz interpretations and first examples

Next we define the Cauchy–Schwarz constants and give them a meaning by comparison angles; later, dealing with the concept of differentiation, the emphasis is naturally on the local properties of the space. Some basic examples of good and bad spaces will be discussed: real inner product spaces, severely snow-flaked spaces and ℓ^p-metrics. Gromov δ-hyperbolicity will be dealt with the snow-flaked dot product. We shall see that locally Euclidean spaces, like a circle or torus, may behave globally badly in the Cauchy–Schwarz sense; in Section 4 it will turn out that Riemannian manifolds are still locally Cauchy–Schwarz.

Cauchy–Schwarz space. The *Cauchy–Schwarz constant* $C_M \in [0,\infty]$ of a metric space $(M, xy \mapsto |xy|)$ is

$$C_M := \sup_{w,x,y,z \in M} \frac{|wx \cdot yz|}{|wx|\,|yz|} \tag{24}$$

(with the convention $0/0 = 0$); a metric space $(M, xy \mapsto |xy|)$ is called a *Cauchy–Schwarz space* if $C_M < \infty$, i.e.

$$|wx \cdot yz| \leq C\,|wx|\,|yz| \tag{25}$$

for some $C < \infty$ for all $w,x,y,z \in M$. Notice that $C_M \geq 1$ if M contains at least two points. A metric space is *locally Cauchy–Schwarz* if it has an open cover of Cauchy–Schwarz subspaces; then the *Cauchy–Schwarz constant at* $z \in M$ is

$$C_z := \lim_{r \to 0^+} C_{B(z,r)}, \tag{26}$$

where $B(z,r) = \{y \in M : |xy| < r\}$ and $C_{B(z,r)}$ is the Cauchy–Schwarz constant of the metric subspace $(B(z,r), xy \mapsto |xy|)$.

Comparison angles and Cauchy–Schwarz. Let $w,x,y,z \in M$. Recall the partial Cauchy–Schwarz inequality (13); now

$$\begin{aligned} wx \cdot yz &= wx \cdot wz - wx \cdot wy \\ &= |wx| \left(|wz| \cos(\tilde{\angle} xwz) - |wy| \cos(\tilde{\angle} xwy) \right), \end{aligned}$$

so that the Cauchy–Schwarz property becomes

$$\Big| |wz| \cos(\tilde{\angle} xwz) - |wy| \cos(\tilde{\angle} xwy) \Big| \leq C_M\,|yz|. \tag{27}$$

In terms of the comparison points $xw\bar{y} = (y_1, y_2), xw\bar{z} = (z_1, z_2) \in \mathbb{R}^2$,

$$|z_1 - y_1| \leq C_M\,|yz|. \tag{28}$$

Loosely speaking, if the "tetrahedron" determined by the points $w,x,y,z \in M$ is "flattened" into the Euclidean plane $\mathbb{R}^2$ so that five of the six "edges" preserve their lengths, then the length of the orthogonal shadow of the sixth "edge" onto the opposite edge is not "stretched" badly, only at most by factor $C_M < \infty$.

Proposition 3.1 *A real inner product space is Cauchy–Schwarz.* □

Proof. Since $\langle x - w, z - y \rangle = wx \cdot yz$, we get the result by the classical inner product Cauchy–Schwarz inequality $|\langle x - w, z - y \rangle| \leq \|x - w\|\,\|z - y\|$. □

Examples. Finite metric spaces are naturally Cauchy–Schwarz. A metric space $(M, xy \mapsto |xy|)$ is Cauchy–Schwarz if each subset $\{w,x,y,z\} \subset M$ of (at most) four points can be isometrically embedded into the Euclidean space $\mathbb{R}^3$; then $C_M \leq 1$; this happens e.g. for the discrete metric $xy \mapsto |xy| \in \{0,1\}$. This isometric embedding condition is not necessary for a space to be Cauchy–Schwarz: if $M = \{w,x,y,z\}$ has the metric $1 = |wx| = |xy| = |yz| = |zw|$ and $2 = |wy| = |xz|$, it cannot be embedded isometrically into any inner product space.

Circle. Torus $\mathbb{R}/\mathbb{Z}$ (or the circle $\mathbb{S}^1$) is clearly locally Cauchy–Schwarz, but it is not Cauchy–Schwarz, when endowed with the usual metric: For $v \in \mathbb{R}$, let $[v] := v + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}$. Let $0 < \varepsilon < 1/2$,

$$w := [0], \quad x_\varepsilon := [\varepsilon], \quad y_\varepsilon := [1/2 + \varepsilon], \quad z := [1/2].$$

Then $|x_\varepsilon - w|\,|z - y_\varepsilon| = \varepsilon^2$ and $wx_\varepsilon \cdot y_\varepsilon z = \varepsilon(1-\varepsilon)$, so that

$$\frac{|wx_\varepsilon \cdot y_\varepsilon z|}{|wx_\varepsilon|\,|y_\varepsilon z|} = \frac{1-\varepsilon}{\varepsilon} \to_{\varepsilon \to 0^+} \infty.$$

Distinct shortest paths. Let us generalize the circle example. Let M be a metric space in which $(s \mapsto x_s), (t \mapsto y_t) : [0,1] \to M$ are paths from $x_0 = y_0$ to $x_1 = y_1$ satisfying $|x_s x_t| = |s-t|\,|x_0 x_1| = |y_s y_t|$. Suppose $|x_s y_{1-s}| = (1 - 2f(s))|x_0 x_1|$, where $0 \leq f(s)/s \leq k < 1$ for a constant k, as $s \to 0^+$; this can happen on Riemannian manifolds. Then

$$\frac{x_0 x_s \cdot y_{1-s} y_1}{|x_0 x_s|\,|y_{1-s} y_1|} = \frac{2f(s)^2 - 2f(s) + 2s - s^2}{s^2} \geq -1 + \frac{2(1-k)}{s} \to_{s \to 0^+} \infty.$$

Thus sometimes non-unique geodesics may obstruct Cauchy–Schwarz property. On the circle, there are two shortest paths between a pair of most distant points: we may think that within the space, the antipode point is seen as a double image. Similarly, on the 2-sphere $\mathbb{S}^2$, the south pole is "seen in every direction" when viewed from the north pole: any longitude is a shortest path between the north and south poles. Broadly speaking, in the physical universe, the cosmic background radiation coming from every direction is the picture of a single point in the space-time, the big bang; in smaller scales, gravitational lenses may cause multiple images of distant stellar objects.

Proposition 3.2 *Let $0 < \alpha \leq 1/2$ and $(M, xy \mapsto |xy|)$ be a metric space. Then the "snow-flaked" metric space $M_\alpha = (M, xy \mapsto |xy|^\alpha)$ has the Cauchy–Schwarz constant $C_{M_\alpha} \leq 1$.*

Proof. Since $0 < \alpha \leq 1/2$, also $xy \mapsto |xy|^{2\alpha}$ is a metric; especially, there is the triangle inequality begetting the estimate

$$\begin{array}{rcl} wx \cdot^\alpha yz & := & 2^{-1}\left(|wz|^{2\alpha} + |xy|^{2\alpha} - |wy|^{2\alpha} - |xz|^{2\alpha}\right) \\ & \leq & 2^{-1}\left((|wy|^{2\alpha} + |yz|^{2\alpha}) + (|xz|^{2\alpha} + |zy|^{2\alpha}) - |wy|^{2\alpha} - |xz|^{2\alpha}\right) \\ & = & |yz|^{2\alpha}; \end{array}$$

thereby also $wx \cdot^\alpha yz \leq |wx|^{2\alpha}$, so that $|wx \cdot^\alpha yz| \leq |wx|^\alpha\,|yz|^\alpha$. $\square$

Remark. Snow-flaking $xy \mapsto |xy|^\alpha$ with $1/2 < \alpha < 1$ may not always produce a Cauchy–Schwarz space, as we will see later.

Gromov product [2, 3]. Let $wx \cdot^{\alpha} yz$ be as in the proof of Proposition 3.2. The *Gromov product of* $y, z \in M$ *with respect to* $x \in M$ is

$$(y \cdot z)_x := \frac{1}{2}\left(|yx| + |xz| - |yz|\right);$$

i.e. $(y \cdot z)_x = xy \cdot^{\frac{1}{2}} xz$. For $\delta \geq 0$, the space M is called δ-*hyperbolic* [2, 3] if

$$(x \cdot y)_w \geq \min\left\{(x \cdot z)_w, (y \cdot z)_w\right\} - \delta$$

for every $w, x, y, z \in M$, i.e. $\delta \geq \min\left\{wx \cdot^{\frac{1}{2}} yz, wy \cdot^{\frac{1}{2}} xz\right\}$.

Euclidean space distorted. Let $n \geq 2$ and $1 \leq p \leq \infty$. For $p < \infty$, let us endow $\mathbb{R}^n$ with the ℓ^p-metric $xy \mapsto |xy|_p := (\sum_{j=1}^n |x_j - y_j|^p)^{1/p}$. Let us also define $xy \mapsto |xy|_\infty := \max_{1 \leq j \leq n} |x_j - y_j|$. For $1 \leq p \leq \infty$, let $wx \cdot_p yz := 2^{-1}\left(|wz|_p^2 + |xy|_p^2 - |wy|_p^2 - |xz|_p^2\right)$. It is clear that $(\mathbb{R}^n, xy \mapsto |xy|_2)$ is Cauchy–Schwarz as an inner product space. If $1 \leq p < 2$, $\varepsilon > 0$ and $n = 2$, let

$$w = (0,0), \quad x = (\varepsilon^2, 0), \quad y = (0, \varepsilon), \quad z = (\varepsilon^2, \varepsilon),$$

so that

$$\frac{wx \cdot_p yz}{|wx|_p\, |yz|_p} = \frac{(\varepsilon^{2p} + \varepsilon^p)^{2/p} - \varepsilon^2}{\varepsilon^4} = \frac{(\varepsilon^p + 1)^{2/p} - 1}{\varepsilon^2} \approx \frac{2\, \varepsilon^p}{p\, \varepsilon^2} \to_{\varepsilon \to 0^+} \infty;$$

$(\mathbb{R}^n, xy \mapsto |xy|_p)$ is not even locally Cauchy–Schwarz. If $p = \infty$, let

$$w = (0,0), \quad x = (\varepsilon^2, -\varepsilon^2), \quad y = (\varepsilon, \varepsilon), \quad z = (\varepsilon + \varepsilon^2, \varepsilon - \varepsilon^2)$$

(a Euclidean rectangle, sides forming angles $\pi/4$ with coordinate axes), so that

$$\frac{wx \cdot_\infty yz}{|wx|_\infty\, |yz|_\infty} = \frac{\left(\varepsilon + \varepsilon^2\right)^2 - \varepsilon^2}{\varepsilon^4} = 1 + 2/\varepsilon \to_{\varepsilon \to 0} \infty;$$

thus $(\mathbb{R}^n, xy \mapsto |xy|_\infty)$ is not locally Cauchy–Schwarz. For $2 < p < \infty$, let $w, x, y, z \in \mathbb{R}^2$ be as in the $p = \infty$ case above, so that

$$\begin{aligned}
\frac{wx \cdot_p yz}{|wx|_p\, |yz|_p} &= \frac{((1+\varepsilon)^p + (1-\varepsilon)^p)^{2/p} - 2^{2/p}}{2^{2/p}\, \varepsilon^2} \\
&\to_{\varepsilon \to 0^+} \lim_{\varepsilon \to 0^+} \frac{((1+\varepsilon)^{p-1} - (1-\varepsilon)^{p-1})((1+\varepsilon)^p + (1-\varepsilon)^p)^{2/p-1}}{2^{2/p}\, \varepsilon} \\
&= p - 1,
\end{aligned}$$

where l'Hôpital was used; the local Cauchy–Schwarz constant is at least $p - 1 \geq 1$.

Conjecture: If $2 < p < \infty$ then $(\mathbb{R}^2, xy \mapsto |xy|_p)$ has the Cauchy–Schwarz constant $p - 1$.

Still snow-flaking. Let $xy \mapsto |xy|_\infty$ be the metric on $\mathbb{R}^2$ as in the preceding paragraph. Let $1/2 < \alpha < 1$, and let $wx \cdot_\infty^\alpha yz$ be the dot product with respect to the metric $xy \mapsto |xy|_\infty^\alpha$. If

$$w = (0,0), \quad x = (\varepsilon^2, -\varepsilon^2), \quad y = (\varepsilon, \varepsilon), \quad z = (\varepsilon + \varepsilon^2, \varepsilon - \varepsilon^2)$$

then

$$\frac{wx \cdot_\infty^\alpha yz}{|wx|_\infty^\alpha \, |yz|_\infty^\alpha} = \frac{(\varepsilon+\varepsilon^2)^{2\alpha} - \varepsilon^{2\alpha}}{\varepsilon^{4\alpha}} = \frac{(1+\varepsilon)^{2\alpha} - 1}{\varepsilon^{2\alpha}} \to_{\varepsilon \to 0^+} \lim_{\varepsilon \to 0^+} \frac{2\alpha\varepsilon}{\varepsilon^{2\alpha}} \overset{1/2<\alpha}{=} \infty.$$

Thus $xy \mapsto |xy|_\infty^\beta$ is a Cauchy–Schwarz metric if and only if $\beta \leq 1/2$. This same example shows that for every $\beta \in]1/2, 1[$ there is a metric space $(M, xy \mapsto |xy|)$ such that $xy \mapsto |xy|^\beta$ is a Cauchy–Schwarz metric, while $xy \mapsto |xy|^\alpha$ is not when $\alpha > \beta$. Similar studies on $(\mathbb{R}^2, xy \mapsto |xy|_p^\alpha)$ show that we do not have the Cauchy–Schwarz property if $1 \leq p < 2\alpha \leq 2$.

3.2 Basic properties of Cauchy–Schwarz spaces

In the sequel, we study the dot product in a Cauchy–Schwarz space as a Lipschitz mapping $z \mapsto wx \cdot yz$. We note the preservation of the Cauchy–Schwarz inequality in taking subspaces, in completion and in suitable countable Cartesian products. We study divergence of geodesics, and learn that the Cauchy–Schwarz inequality survives the Gromov–Hausdorff convergence of compact metric spaces.

Earlier we studied global and local Lipschitz behavior of the mapping $z \mapsto wx \cdot yz$ on a general metric space. By the following result, the Cauchy–Schwarz condition can be rewritten as a Lipschitz condition on the metric space:

Theorem 3.3 *Let $(M, xy \mapsto |xy|)$ be a Cauchy–Schwarz space. The mapping $(z \mapsto wx \cdot yz) : M \to \mathbb{R}$ has Lipschitz properties*

$$\begin{aligned} \mathrm{Lip}(z \mapsto wx \cdot yz) &\leq \min\left\{ C_M \, |wx|, \ \sup_{v \in M}(|wv| + |vx|) \right\}, && (29) \\ \mathrm{Lip}_v(z \mapsto wx \cdot yz) &\leq \min\left\{ C_M \, |wx|, \ |wv| + |vx| \right\}, && (30) \\ \mathrm{Lip}_w(z \mapsto wx \cdot yz) &\leq |wx|. && (31) \end{aligned}$$

Conversely, a metric space $(M, xy \mapsto |xy|)$ is a Cauchy–Schwarz space if

$$\mathrm{Lip}(z \mapsto wx \cdot wz) \leq C \, |wx| \qquad (32)$$

for every $w, x \in M$; the Cauchy–Schwarz constant is then the infimum of the possible constants C in the inequality.

Proof. This is actually a refinement of Theorem 2.2, when we notice that in a Cauchy–Schwarz space

$$|wx \cdot yz_1 - wx \cdot yz_2| = |wx \cdot z_2 z_1| \leq C_M \, |wx| \, |z_1 z_2|;$$

the inequalities for the Cauchy–Schwarz case follow. Conversely, suppose the metric space satisfies (32). Then

$$|wx \cdot yz| = |wx \cdot wz - wx \cdot wy| \leq \mathrm{Lip}(v \mapsto wx \cdot wv) \, |yz| \leq C \, |wx| \, |yz|. \quad \square$$

Violating Cauchy–Schwarz. Let M be bounded non-Cauchy–Schwarz. Let $w_n, x_n, y_n, z_n \in M$ such that $|w_n x_n \cdot y_n z_n|/(|w_n x_n|\ |y_n z_n|) \to \infty$. Then

$$\frac{|w_n x_n \cdot y_n z_n|}{|w_n x_n|\ |y_n z_n|} \leq \frac{|w_n x_n|\ (|y_n w_n| + |w_n z_n|)}{|w_n x_n|\ |y_n z_n|} \leq \frac{2\ \mathrm{diam(M)}}{|y_n z_n|},$$

i.e. $|y_n z_n| \to 0$; analogously, $|w_n x_n| \to 0$. If M is moreover compact then the violation of the Cauchy–Schwarz inequality can be isolated nearby at most two points: there exists $J \subset \mathbb{Z}^+$ such that

$$\lim_{J \ni n \to \infty} w_n = \lim_{J \ni n \to \infty} x_n \in M, \qquad \lim_{J \ni n \to \infty} y_n = \lim_{J \ni n \to \infty} z_n \in M.$$

Subspaces, completions and products. The Cauchy–Schwarz property is naturally preserved in taking subspaces, completions or Euclidean-like Cartesian products with at most countably many coordinate spaces. Comparing the Hilbert cube and an infinite-dimensional torus, we get an insight that a (locally) Cauchy–Schwarz space should not be "(locally) loopy".

Theorem 3.4 *Global (resp. local) Cauchy–Schwarzness is preserved in taking subspaces, in isometries, and in finite Cartesian products. Global Cauchy–Schwarz property is preserved in completion, and in countable Cartesian products if the Cauchy–Schwarz constants of the coordinate spaces are uniformly bounded.*

Proof. Trivially Cauchy–Schwarz properties are inherited by subspaces, and they are metric properties. It is evident that the global inequality (25) survives the completion with the same Cauchy–Schwarz constant; but if we remove the tip from a cone (see Section 3.4), we obtain a non-complete locally Cauchy–Schwarz space with a not-even-locally-Cauchy–Schwarz completion. Let $M_1, \ldots, M_n$ be Cauchy–Schwarz spaces. Let

$$|xy|^2 := \sum_{j=1}^{n} |x_j y_j|^2 \tag{33}$$

define a metric $xy \mapsto |xy|$ for $\prod_{j=1}^{n} M_j$. Then it is easy to see that

$$wx \cdot yz = \sum_{j=1}^{n} w_j x_j \cdot y_j z_j, \quad \text{and}$$

$$|wx \cdot yz| \leq \sum_{j=1}^{n} C_{M_j}\ |w_j x_j|\ |y_j z_j| \leq C \sum_{j=1}^{n} |w_j x_j|\ |y_j z_j| \leq C\ |wx|\ |yz|,$$

where $C = \max_{1 \leq j \leq n} C_{M_j}$; in the last step the Hölder inequality was applied. It is now clear that this way the local Cauchy–Schwarz property is also preserved in finite Cartesian products. If $\{M_j\}_{j=1}^{\infty}$ is a collection of bounded Cauchy–Schwarz spaces such that $\sup\{C_{M_j} : j \in \mathbb{Z}^+\} < \infty$ then we may (after rescaling) assume that $\mathrm{diam}(M_j) \leq 2^{-j}$, so that $|xy|^2 := \sum_{j=1}^{\infty} |x_j y_j|^2$ defines a Cauchy–Schwarz metric $xy \mapsto |xy|$ for $M = \prod_{j=1}^{\infty} M_j$. $\square$

Hilbert cube and torus. Let the Hilbert cube $M := \prod_{j=1}^{\infty}[0, 2^{-j}]$ be given the metric by $|xy| := (\sum_{j=1}^{\infty} |x_j y_j|^2)^{1/2}$ (here $|x_j y_j| = |x_j - y_j|$ is the absolute value). This is a compact path-connected non-doubling Cauchy–Schwarz space; recall that a metric space is called *doubling* if every closed ball of radius $2r$ has a covering family of at most $N \in \mathbb{Z}^+$ closed balls of radius r, where N is a constant. The countably infinite-dimensional torus $\prod_{j=1}^{\infty}(\mathbb{R}/(2^{-j}\mathbb{Z}))$ is a compact path-connected non-doubling not even locally Cauchy–Schwarz space, and it is not locally isometrically embeddable in an inner product space.

Geodetic properties. Suppose we have two paths of length $|xy|$ between points $x, y \in M$; we want estimate how far away from each other these two geodesics can be. It turns out that if the Cauchy–Schwarz constant is $C_M = 1$, such geodesics are unique (if they exist):

Theorem 3.5 *Let* $(M, ab \mapsto |ab|)$ *be a Cauchy–Schwarz space. Assume that* $(s \mapsto x_s), (t \mapsto y_t) : [0,1] \to M$ *are geodetic paths from* $x_0 = y_0$ *to* $x_1 = y_1$ *such that* $|y_0 y_t| = |x_0 x_t| = t\,|x_0 x_1| = |x_{1-t} x_1| = |y_{1-t} y_1|$ *for every* $t \in [0,1]$. *Then*

$$|x_t y_t| \leq \sqrt{2t(1-t)(C_M - 1)}\;|x_0 x_1|. \tag{34}$$

Especially, if M *is a complete length space and* $C_M = 1$ *then between any given two points there exists a unique shortest length path.*

Remark. Trivial inequalities

$$|x_t y_t| \leq 2t\,|x_0 x_1| \quad \text{and} \quad |x_t y_t| \leq 2(1-t)\,|x_0 x_1| \tag{35}$$

provide better estimates than the one given in Theorem above in the respective cases $0 < t < (C_M - 1)/(C_M + 1)$ and $1 - (C_M - 1)/(C_M + 1) < t < 1$. Consequently, if $C_M \geq 3$ then the estimate in Theorem is completely useless.

Proof of Theorem 3.5. Now

$$\begin{aligned} 2\,(x_t x_1 \cdot x_0 y_t) &= |x_t y_t|^2 + |x_1 x_0|^2 - |x_t x_0|^2 - |x_1 y_t|^2 \\ &= |x_t y_t|^2 + |x_0 x_1|^2\,\left(1 - t^2 - (1-t)^2\right) \\ &= |x_t y_t|^2 + 2(1-t)t\,|x_0 x_1|^2, \end{aligned}$$

and on the other hand

$$2\,(x_t x_1 \cdot x_0 y_t) \leq 2C_M\,|x_t x_1|\,|x_0 y_t| = 2(1-t)tC_M\,|x_0 x_1|^2;$$

therefore $|x_t y_t|^2 \leq 2t(1-t)(C_M - 1)\,|x_0 x_1|^2$. □

Remark. From the proof above we see that

$$x_t x_1 \cdot x_0 y_t \geq (1-t)t\,|x_0 x_1|^2,$$

so that $x_t x_1 \cdot x_0 y_t > 0$ whenever $x_0 \neq x_1$ and $0 < t < 1$.

Gromov–Hausdorff limits [7, 3]. Let $(M, wx \mapsto |wx|_M)$ and $(N, yz \mapsto |yz|_N)$ be compact metric spaces. The *Gromov–Hausdorff distance* $d_{GH}(M, N) \geq 0$ between them is defined to be less than $r > 0$ if and only if there exists a compact metric space $(K, ab \mapsto |ab|)$ such that there are isometric embeddings $g : M \to K$ and $h : N \to K$ such that the Hausdorff distance between $g(M), h(N) \subset K$ is less than r. The Gromov–Hausdorff distance provides a metric on the space of isometry classes of compact metric spaces. Let $\varepsilon > 0$, and let M, N be compact metric spaces. A (possibly even discontinuous) mapping $f : M \to N$ is called an *ε-isometry* if $f(M) \subset N$ is an ε-net and if

$$\sup_{w,x \in M} ||f(w)f(x)|_N - |wx|_M| \leq \varepsilon.$$

It can be proven that if $d_{GH}(M, N) < r$ then there exists a $2r$-isometry $M \to N$; and if there exists an r-isometry $M \to N$ then $d_{GH}(M, N) < 2r$. The Cauchy–Schwarz property behaves well with respect to the Gromov–Hausdorff limits:

Theorem 3.6 *Let $((M_k, x_k y_k \mapsto |x_k y_k|_k))_{k=1}^{\infty}$ be a sequence of compact metric spaces converging to a compact metric space $(M, xy \mapsto |xy|)$ in the Gromov–Hausdorff sense. Then the Cauchy–Schwarz constant C_M of M has an estimate*

$$C_M \leq \limsup_{k \to \infty} C_{M_k},$$

where $C_{M_k} \in [0, \infty]$ is the Cauchy–Schwarz constant of M_k.

Proof. Let $f_k : M \to M_k$ be an ε_k-isometry, where $\lim_{k\to\infty} \varepsilon_k = 0$. Suppose $C := \sup_{k \in \mathbb{Z}^+} C_{M_k} < \infty$. If $w, x, y, z \in M$ then

$$\begin{array}{rl} & f_k(w)f_k(x) \cdot f_k(y)f_k(z) \\ = & 2^{-1}\left(|f_k(w)f_k(z)|_k^2 + |f_k(x)f_k(y)|_k^2 - |f_k(w)f_k(y)|_k^2 - |f_k(x)f_k(z)|_k^2\right) \\ \to_{k\to\infty} & 2^{-1}\left(|wz|^2 + |xy|^2 - |wy|^2 - |xz|^2\right) \;=\; wx \cdot yz, \end{array}$$

and on the other hand

$$\begin{array}{rcl} |f_k(w)f_k(x) \cdot f_k(y)f_k(z)| & \leq & C_{M_k}\ |f_k(w)f_k(x)|_k\ |f_k(y)f_k(z)|_k \\ & \leq & C\ |f_k(w)f_k(x)|_k\ |f_k(y)f_k(z)|_k \\ & \to_{k\to\infty} & C\ |wx|\ |yz|. \end{array}$$

Hence $|wx \cdot yz| \leq C\ |wx|\ |yz|$. Generalizing this "sup-result", the claimed "$\limsup$ -result" is obvious. $\square$

Remarks. Above, it is possible that $\limsup_{k\to\infty} C_{M_k} = \infty$ but yet $C_M < \infty$; e.g. consider a sequence of circles shrinking to a point. There are many fundamental properties conserved in the Gromov–Hausdorff convergence: for instance, a complete limit of length (resp. boundedly compact, resp. boundedly compact complete length) spaces is a length (resp. boundedly compact, resp. boundedly compact complete length) space [2]. Notice also that a locally compact space is boundedly compact if and only if complete [3].

3.3 Spaces of non-positive curvature (NPC)

In an intrinsic metric space, one can try to define the angle between intersecting geodesics in a natural way using the limits of Euclidean comparison angles; the space is said to be of *non-positive curvature* (NPC spaces in sense of Aleksandrov) if these geodetic angles exist and are less than or equal to the involved comparison angles. Other curvature bounds require dealing with non-Euclidean surfaces of constant curvature [3].

Global NPC spaces [3, 12]. A *global NPC space* (or a *Hadamard space*) is a complete simply connected metric space of non-positive curvature. Alternatively, a complete metric space $(M, xy \mapsto |xy|)$ is a global NPC space if and only if

$$\forall x_0, x_1 \in M \ \exists x_{1/2} \in M \ \forall z \in M : \ |zx_{1/2}|^2 \leq \frac{1}{2}|zx_0|^2 + \frac{1}{2}|zx_1|^2 - \frac{1}{4}|x_0x_1|^2;$$

for an equivalent definition by a variance inequality, see [12].

Global NPC and dot product. According to Proposition 2.3 in [12], in a global NPC space M, from x_0 to x_1 there exists a unique geodesic $(t \mapsto x_t) : [0,1] \to M$ such that $|x_sx_t| = |s-t| \ |x_0x_1|$; moreover, for every $z \in M$,

$$|zx_t|^2 \leq (1-t) \ |zx_0|^2 + t \ |zx_1|^2 - t(1-t) \ |x_0x_1|^2.$$

In our notation, this NPC inequality is reduced to

$$zx_0 \cdot x_0x_t \leq t \ (zx_0 \cdot x_0x_1). \tag{36}$$

Proof. We get the result, since

$$\begin{aligned} |zx_t|^2 &= |zx_0 + x_0x_t|^2 \\ &= |zx_0|^2 + 2 \ (zx_0 \cdot x_0x_t) + |x_0x_t|^2 \\ &= |zx_0|^2 + 2 \ (zx_0 \cdot x_0x_t) + t^2 \ |x_0x_1|^2, \end{aligned}$$

and on the other hand

$$\begin{aligned} |zx_t|^2 &\leq (1-t) \ |zx_0|^2 + t \ |zx_1|^2 - t(1-t) \ |x_0x_1|^2 \\ &= |zx_0|^2 + t \ (|zx_1|^2 - |zx_0|^2 - |x_0x_1|^2) + t^2 \ |x_0x_1|^2 \\ &= |zx_0|^2 + 2t \ (zx_0 \cdot x_0x_1) + t^2 \ |x_0x_1|^2. \end{aligned} \qquad \square$$

Reshetnyak's Quadruple Comparison. In a global NPC space $(M, xy \mapsto |xy|)$,

$$|x_1x_3|^2 + |x_2x_4|^2 \leq |x_2x_3|^2 + |x_4x_1|^2 + 2|x_1x_2| \ |x_3x_4|$$

for every $x_1, x_2, x_3, x_4 \in M$ (see [12]), as shown originally by Reshetnyak. Using the dot product, this reads

$$-x_1x_2 \cdot x_3x_4 \leq |x_1x_2| \ |x_3x_4|,$$

meaning the Cauchy–Schwarz inequality with $C_M = 1$.

Riemannian manifolds. A Riemannian manifold M is a global NPC space if and only if it is complete, simply connected and has non-positive sectional curvature; then by Reshetnyak's Quadruple Comparison, the Cauchy–Schwarz constant $C_M = 1$. For the circle $\mathbb{S}^1$, we already saw that a manifold is not necessarily a Cauchy–Schwarz space. In Section 4, we show that a Riemannian manifold M is always locally Cauchy–Schwarz, even that $C_x = 1$ for any $x \in M$.

Geodesic comparison. Let $(M, ab \mapsto |ab|)$ be a global NPC space. In [12] it is shown that

$$|x_t y_t| \leq (1-t)\ |x_0 y_0| + t\ |x_1 y_1|, \tag{37}$$

where $t \mapsto x_t, t \mapsto y_t$ are geodesics satisfying $|x_s x_t| = |s-t|\ |x_0 x_1|$ and $|y_s y_t| = |s-t|\ |y_0 y_1|$; the Cauchy–Schwarz inequality and (36) clarify the proof:

$$\begin{aligned}
|x_t y_t|^2 &= |x_t x_0|^2 + |x_0 y_t|^2 + 2\ (x_t x_0 \cdot x_0 y_t) \\
&\leq |x_t x_0|^2 + |x_0 y_t|^2 + 2t\ (x_1 x_0 \cdot x_0 y_t) \\
&= (1-t)\ |x_0 y_t|^2 + t\ |x_1 y_t|^2 - (1-t)t\ |x_0 x_1|^2 \\
&= (1-t)\ \big(|x_0 y_0|^2 + |y_0 y_t|^2 + 2\ (x_0 y_0 \cdot y_0 y_t)\big) \\
&\quad + t\ \big(|x_1 y_1|^2 + |y_1 y_t|^2 + 2\ (x_1 y_1 \cdot y_1 y_t)\big) \\
&\quad - (1-t)t\ |x_0 x_1|^2 \\
&\leq (1-t)\ \big(|x_0 y_0|^2 + |y_0 y_t|^2 + 2t\ (x_0 y_0 \cdot y_0 y_1)\big) \\
&\quad + t\ \big(|x_1 y_1|^2 + |y_1 y_t|^2 + 2(1-t)\ (x_1 y_1 \cdot y_1 y_0)\big) \\
&\quad - (1-t)t\ |x_0 x_1|^2 \\
&= (1-t)^2\ |x_0 y_0|^2 + t^2\ |x_1 y_1|^2 + 2(1-t)t\ (x_0 y_0 \cdot x_1 y_1) \\
&\leq \left[(1-t)\ |x_0 y_0| + t\ |x_1 y_1|\right]^2 . \qquad \square
\end{aligned}$$

3.4 Singularities of spaces

We shall show that conical singularities (not to speak of sharper cusps) violate the Cauchy–Schwarz property; however, a rather standard doubling condition on measures may exclude only "infinitely sharp" cusps. The minimal doubling constant for the volume form of a compact smooth manifold reveals the precise dimension of the manifold, while the restriction of the Lebesgue measure to a compact submanifold of a Euclidean space can be non-doubling: doubling tells more about "sharpness" of singularities of the space than about the topological dimension. Cusps on two-dimensional surfaces exemplify these phenomena.

Doubling measure. A Borel measure μ on a metric space $(M, xy \mapsto |xy|)$ is called *doubling* if

$$0 < \mu(B(x, 2r)) \leq C_\mu\ \mu(B(x, r)) < \infty$$

for every $x \in M$ and $r > 0$, where $C_\mu < \infty$ is a constant. Then

$$\frac{\mu(B(x, R))}{\mu(B(x, r))} \leq C_\mu\ (R/r)^{\log_2 C_\mu},$$

when $0 < r < R$; in a sense, the "μ-dimension" of M at x cannot exceed $\log_2 C_\mu$. For instance, the Lebesgue measure μ of $\mathbb{R}^n$ with the Euclidean metric is doubling as $\mu(B(x,r)) = c(n)r^n$. Moreover, a complete metric vector space may carry a doubling measure if and only if the space is finite-dimensional [10]. The measure given by the volume form of a compact C^∞-smooth manifold is doubling; thus let us study some non-smooth manifolds:

Surfaces of rotation. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be non-decreasing and C^1-smooth such that $f(t) \to 0$ as $t \to 0^+$. Let M be the surface obtained by rotating the curve $x_2(x_1) = f(x_1)$ around the x_1-axis, with the origin $0 \in \mathbb{R}^3$ included. Then M is a two-dimensional submanifold of $\mathbb{R}^3$, C^1-smooth except possibly at $0 \in M$, and we endow M with the minimal arc length metric d. The surface area is our choice for the measure μ. Let $0 \neq x = (x_1, x_2, x_3) \in M$, i.e. $x_1 > 0$ and $f(x_1)^2 = x_2^2 + x_3^2$. Then

$$\begin{aligned} r_x &:= d(0,x) = \int_0^{x_1} \sqrt{1 + f'(t)^2}\,\mathrm{d}t, \\ A(r_x) &:= \mu(B(0,r_x)) = \int_0^{x_1} 2\pi f(t)\,\sqrt{1+f'(t)^2}\,\mathrm{d}t. \end{aligned}$$

Non-doubling cusp. Let $f(t) = \mathrm{e}^{-1/t}$; let M be defined by f as above. The cusp at $0 \in M$ is "infinitely sharp": $f^{(k)}(t) \to_{t\to 0^+} 0$ for every $k \in \mathbb{N}$. Now

$$r_x = \int_0^{x_1} \sqrt{1 + t^{-4}\mathrm{e}^{-2/t}}\,\mathrm{d}t \approx x_1$$

for $x \approx 0$, because $\sqrt{1 + t^{-4}\mathrm{e}^{-2/t}} \approx 1$ for $t \approx 0^+$. Hence for $r \approx 0^+$,

$$\begin{aligned} \frac{A(2r)}{A(r)} &\approx \frac{\int_0^{2r} 2\pi\mathrm{e}^{-1/t}\sqrt{1+t^{-4}\mathrm{e}^{-2/t}}\,\mathrm{d}t}{\int_0^{r} 2\pi\mathrm{e}^{-1/t}\sqrt{1+t^{-4}\mathrm{e}^{-2/t}}\,\mathrm{d}t} \approx \frac{\int_0^{2r} f(t)\,\mathrm{d}t}{\int_0^{r} f(t)\,\mathrm{d}t} \\ &\geq \frac{\frac{1}{2} f(2r)^2/f'(2r)}{r f(r)} = 2r\mathrm{e}^{1/(2r)} \to_{r\to 0^+} \infty. \end{aligned}$$

Thus μ is not doubling on M.

Doubling cusps. Let M be the surface of rotation corresponding to $f(t) = t^\alpha$, where $\alpha > 0$. Let $x \approx 0$. Now

$$r_x = \int_0^{x_1} \sqrt{1 + \alpha^2 t^{2\alpha-2}}\,\mathrm{d}t \approx \begin{cases} x_1, & \alpha > 1, \\ \sqrt{2}x_1, & \alpha = 1, \\ x_1^\alpha, & \alpha < 1, \end{cases}$$

$$A(r_x) = 2\pi \int_0^{x_1} \sqrt{t^{2\alpha} + \alpha^2 t^{4\alpha-2}}\,\mathrm{d}t \approx \begin{cases} \frac{2\pi}{\alpha+1} x_1^{\alpha+1}, & \alpha > 1, \\ \sqrt{2}\pi x_1^2, & \alpha = 1, \\ \pi x_1^{2\alpha}, & \alpha < 1, \end{cases}$$

so that for $r \approx 0^+$,

$$A(r) \approx \begin{cases} \frac{2\pi}{\alpha+1} r^{\alpha+1}, & \alpha > 1, \\ \pi r^2/\sqrt{2}, & \alpha = 1, \\ \pi r^2, & \alpha < 1. \end{cases}$$

The topological dimension of M is 2 for each α, but the μ-dimension at $0 \in M$ appears to be $\alpha + 1$ for $\alpha > 1$.

Cone is not Cauchy–Schwarz. Let $0 < 2\alpha < 2\pi$. Let us design a cone K by removing a slice of angle $2\pi - 2\alpha$ from the Euclidean plane $\mathbb{R}^2$, with the tip at the origin $o \in \mathbb{R}^2$; then identify the edges with each other in the natural way. The metric is the usual geodetic metric on the cone, i.e. the one obtained from the Euclidean plane (with taking in mind the removal of the slice). Let $\varepsilon > 0$. Choose points $w_\varepsilon, x_\varepsilon, y_\varepsilon, z_\varepsilon \in K$ so that

$$\begin{aligned} &|ow_\varepsilon| = |ox_\varepsilon| = |oy_\varepsilon| = |oz_\varepsilon| = \varepsilon, \quad |w_\varepsilon z_\varepsilon| = 2\varepsilon\ \sin(\alpha/2) = |x_\varepsilon y_\varepsilon|, \\ &|w_\varepsilon x_\varepsilon| = 2\varepsilon\ \sin(\varepsilon\alpha/2) = |y_\varepsilon z_\varepsilon|, \quad |x_\varepsilon z_\varepsilon| = 2\varepsilon\ \sin((1-\varepsilon)\alpha/2) = |w_\varepsilon y_\varepsilon|. \end{aligned}$$

Then using l'Hôpital's rule, we calculate

$$\frac{w_\varepsilon x_\varepsilon \cdot y_\varepsilon z_\varepsilon}{|w_\varepsilon x_\varepsilon|\ |y_\varepsilon z_\varepsilon|} = \frac{\sin^2(\alpha/2) - \sin^2((1-\varepsilon)\alpha/2)}{\sin^2(\varepsilon\alpha/2)} \to_{\varepsilon\to 0^+} \infty;$$

hence the cone K is not a Cauchy–Schwarz space, due to the singularity at the tip. Similarly, it can be proven that sharper cusps fail the Cauchy–Schwarz property.

4 Infinitesimal metric geometry

In the sequel, we deal with compact metric spaces, but many of the results hold in locally compact metric spaces. First, we review earlier theory of Lipschitz point derivations, and then combine it to the Cauchy–Schwarz property.

4.1 Lipschitz tangent space

Intuitively, given "an algebra $\mathcal{F}$ of test functions" $M \to \mathbb{R}$, a tangent at a point $x \in M$ is a linear functional $D : \mathcal{F} \to \mathbb{R}$ satisfying the Leibniz rule at x:

$$D(fg) = f(x)\ D(g) + D(f)g(x).$$

On a metric space M, the local Lipschitz property is the natural smoothness condition. So we choose the algebra of Lipschitz functions to be our "test function space". Then it is natural to require the tangents to be also bounded functionals. But even then, the resulting Lipschitz tangent space can be quite huge, as in the case of $M = ([0,1], xy \mapsto |x-y|)$; we will address to this problem later via the Cauchy–Schwarz property. In any case, Lipschitz tangents have been proven useful in characterizing closed ideals of the Lipschitz function algebra, see [13].

Lipschitz ideals

For a compact Hausdorff space X, there is the charming correspondence between the closed subsets of X and the closed ideals of $C(X)$: the ideal of the zero set of a closed ideal $\mathcal{J} \subset C(X)$ is $\mathcal{J}$, and the zero set of the ideal of a closed set $C \subset X$ is C. This is a well-known part of Gelfand theory. For the algebra $\mathrm{Lip}(M) \subset C(M)$ on a compact metric space M, the spectral synthesis is far more complicated.

Ideals. Let $K \subset M$ be a non-empty set. Obviously,

$$I(K) := \{f \in \mathrm{Lip}(M) \mid \forall x \in K : \ f(x) = 0\} \tag{38}$$

is a closed ideal of $\mathrm{Lip}(M)$. We denote $I(x) := I(\{x\})$. The *zero set* of an ideal $\mathcal{J} \subset \mathrm{Lip}(M)$ is

$$Z(\mathcal{J}) := \{x \in M \mid \forall f \in \mathcal{J} : \ f(x) = 0\}; \tag{39}$$

this is a closed subset of M, and $Z(I(K))$ is the closure of $K \subset M$. Let $K \subset M$ be a non-empty set. Then

$$J(K) := \{f \in I(K) \mid \forall x \in K : \ \mathrm{Lip}_x(f) = 0\} \tag{40}$$

is a closed ideal of $\mathrm{Lip}(M)$. We denote $J(x) := J(\{x\})$. We may think that $J(K)$ consists of "test functions" vanishing on K and having "zero gradient" on K. It is easy to prove that $fg \in J(x)$ if $f, g \in I(x)$. Moreover, $Z(J(K)) = Z(I(K))$, and if $\mathcal{J} \subset \mathrm{Lip}(M)$ is a closed ideal then $J(Z(\mathcal{J})) \subset \mathcal{J} \subset I(Z(\mathcal{J}))$.

Lipschitz tangents

There are at least three essentially different but in effect the same definitions for Lipschitz tangents. We are going to define them as bounded linear functionals satisfying the Leibniz rule at a point, but equivalently they are bounded linear functionals annihilating those functions that are locally constant at the reference point, or equivalently they can be approximated by "linear combinations of limits of difference quotients", as stated below precisely; these results have been collected from [14] (p. 128–130).

Lipschitz tangents [14]. A bounded linear functional $D : \mathrm{Lip}(M) \to \mathbb{R}$ is called a *Lipschitz tangent at* $x \in M$ (or a *point derivation at* $x \in M$), denoted by $D \in \mathrm{LipT}_x M$, if

$$D(fg) = f(x)\, D(g) + D(f)\, g(x) \tag{41}$$

for all $f, g \in \mathrm{Lip}(M)$. If $N \subset M$ is a compact subspace then $\mathrm{LipT}_x(N)$ has a natural embedding $D \mapsto \tilde{D}$ into $\mathrm{LipT}_x M$, given by $\tilde{D}(f) := D(f|_N)$, where $f|_N \in \mathrm{Lip}(N)$ is the restriction of $f \in \mathrm{Lip}(M)$.

Example [14]. Let $\tilde{M} := \{xy \mid x, y \in M,\ x \neq y\}$. Let

$$\beta\tilde{M} := \mathrm{Spec}(C_b(\tilde{M})) = \mathrm{Hom}(C_b(\tilde{M}), \mathbb{C}), \tag{42}$$

i.e. the maximal ideal space of the commutative C^*-algebra of bounded continuous complex-valued functions on $\tilde{M}$; to put this otherwise, $\beta\tilde{M}$ is the *Stone–Cech compactification* of $\tilde{M}$. Then there is a natural interpretation $\tilde{M} \subset \beta M$, and if $(x_j y_j)_{j\in J} \subset \tilde{M}$ is a net in $\beta\tilde{M}$ converging to $\xi \in \beta\tilde{M}$ then this net converges in $M \times M$ to a unique point xx; in this case ξ is said to *lie above x*. Moreover,

$$D_\xi f := \lim_j \frac{f(x_j) - f(y_j)}{|x_j y_j|} \tag{43}$$

defines a Lipschitz tangent D_ξ at $x \in M$.

Theorem 4.1 *Let $D : \mathrm{Lip}(M) \to \mathbb{R}$ be a bounded linear functional. Then the following conditions are equivalent:*

1. *D is a Lipschitz tangent at $x \in M$,*
2. *$Df = 0$ if f is locally constant at x,*
3. *D is a weak*-limit of linear combinations of some point derivations D_{ξ_j}, where the points ξ_j lie above x.*

Remarks. If D is a Lipschitz tangent at both x and y then either $x = y$ or $D = 0$. If $D \in \mathrm{LipT}_x M$ then actually

$$\|D\| = \sup_{f\in\mathrm{Lip}(M)} \frac{|Df|}{\|f\|_{\mathrm{Lip}(M)}} = \sup_{f\in\mathrm{Lip}(M)} \frac{|Df|}{\mathrm{Lip}_x(f)}. \tag{44}$$

Finally,

$$J(x) = \{f \in I(x) \mid \forall D \in \mathrm{LipT}_x M :\ Df = 0\}. \tag{45}$$

4.2 Differentiability in Cauchy–Schwarz spaces

In the following, we mainly study differentiability in compact Cauchy–Schwarz spaces, but due to the locality of the concepts, one can provide analogous theory for locally compact locally Cauchy–Schwarz spaces.

Tangent space in Cauchy–Schwarz space

As it has been pointed out, the Lipschitz tangent space $\mathrm{LipT}_x M$ can be awfully huge. This is due to that the test function space $\mathrm{Lip}(M)$ is quite large itself: e.g. $C^1([0,1]) \subset \mathrm{Lip}([0,1])$ is a proper closed subspace, and C^1-functions "do not see as many tangents" as Lipschitz functions do. The problem is that in a general metric space the best smoothness we can deal with is Lipschitz. However, using the metric space dot product, we are going to define a dot product between Lipschitz tangents in a Cauchy–Schwarz space. It turns out that this dot product may not distinguish between all the Lipschitz tangents, giving us the opportunity to collapse the Lipschitz tangent space $\mathrm{LipT}_x M$ modulo a dot equivalence to obtain a reasonable tangent space $\mathrm{T}_x M$. Later, we will be able to grasp the concepts of differentiability and gradients in a Cauchy–Schwarz space.

Dot product $xy \cdot D$**.** By Theorem (2.2), in **any** metric space, the dot product has the Lipschitz property $\mathrm{Lip}_x(z \mapsto xy \cdot xz) \leq |xy|$. Thereby, if $D \in \mathrm{LipT}_x M$ then

$$xy \cdot D := D(z \mapsto xy \cdot xz) \in \mathbb{R} \tag{46}$$

is well-defined and satisfies

$$|xy \cdot D| \leq |xy| \, \|D\|. \tag{47}$$

Notice that if $(x_j z_j)_{j \in J} \subset \tilde{M}$ converges to $\eta \in \beta\tilde{M}$ lying above $x \in M$ then

$$xy \cdot D_\eta = \lim_j \frac{xy \cdot z_j x_j}{|x_j z_j|}. \tag{48}$$

Lemma 4.2 *If $D \in \mathrm{LipT}_x M$ in a compact locally Cauchy–Schwarz space then*

$$\mathrm{Lip}_x(y \mapsto xy \cdot D) \leq C_x \, \|D\|. \tag{49}$$

Proof. Let $r > 0$ and $y_1, y_2 \in B(x, r)$. Now

$$\begin{aligned} |xy_1 \cdot D - xy_2 \cdot D| &= |D(z \mapsto y_2 y_1 \cdot xz)| \\ &\leq \|D\| \, \mathrm{Lip}_x(z \mapsto y_2 y_1 \cdot xz) \\ &\leq \|D\| \, C_{B(x,r)} \, |y_2 y_1|, \end{aligned}$$

where $C_{B(x,r)}$ is the Cauchy–Schwarz constant of the ball $B(x, r) \subset M$; in the last inequality, we applied Theorem 3.3. $\square$

Dot product $E \cdot D$**.** Let $(M, xy \mapsto |xy|)$ be a compact locally Cauchy–Schwarz space. If $D, E \in \mathrm{LipT}_x M$ then we can define

$$E \cdot D := E(y \mapsto xy \cdot D) = E(y \mapsto D(z \mapsto xy \cdot xz)) \tag{50}$$

by Lemma 4.2; this is called the *dot product of E and D*. Notice that

$$|E \cdot D| \leq C_x \, \|E\| \, \|D\|, \tag{51}$$

where C_x is the Cauchy–Schwarz constant at $x \in M$. If $(w_i x_i)_{i \in I}, (y_j z_j)_{j \in J} \subset \tilde{M}$ converge to respective points $\xi, \eta \in \beta\tilde{M}$ lying above $x \in M$ then

$$D_\xi \cdot D_\eta = \lim_i \lim_j \frac{w_i x_i \cdot y_j z_j}{|w_i x_i| \, |y_j z_j|}. \tag{52}$$

The Riemannian situation is well-behaved in this respect:

Theorem 4.3 *Let M be a Riemannian manifold of non-zero dimension. The Cauchy–Schwarz constant at $x \in M$ is $C_x = 1$.*

Proof. Since $x \in M$ is not isolated, $C_x \geq 1$. For each $n \in \mathbb{Z}^+$, choose $w_n, x_n, y_n, z_n \in B(x, 1/n)$ such that $w_n x_n \cdot y_n z_n / (|w_n x_n|\ |y_n z_n|) \to_{n\to\infty} C_x$. Choose a subsequence $(w_n x_n)_{n\in J} \subset (w_n x_n)_{n=1}^{\infty}$ converging to $\xi \in \beta\tilde{M}$. Choose a subsequence $(y_n z_n)_{n\in K} \subset (y_n z_n)_{n\in J}$ converging to $\eta \in \beta\tilde{M}$. We get point derivations $D_\xi, D_\eta \in \mathrm{LipT}_x M$, and we define corresponding tangents $a_\xi, a_\eta \in \mathrm{T}_x M$ by restriction $C^1(M) \subset \mathrm{Lip}(M)$. Let $h(y,z) := -|yz|^2/2$. Then

$$
\begin{aligned}
1 \ &\geq\ \|a_\xi\|_{\mathrm{T}_x M}\, \|a_\eta\|_{\mathrm{T}_x M}\ \geq\ \langle a_\xi, a_\eta\rangle_{\mathrm{T}_x M}\ =\ a_\xi\left(y \mapsto a_\eta\left(z \mapsto h(y,z)\right)\right) \\
&=\ \lim_{K\ni n\to\infty} \int_0^1 \int_0^1 \frac{\partial}{\partial s}\frac{\partial}{\partial t} f_n(s,t)\ \mathrm{d}s\ \mathrm{d}t,
\end{aligned}
$$

where function f_n is defined in the Riemann normal coordinates (at $x \in M$) by $f_n(s,t) = h(w_n + s(x_n - w_n), y_n + t(z_n - y_n))/(|w_n x_n|\ |y_n z_n|)$. Hence

$$
1 \geq \lim_{K\ni n} \left([f_n(1,1) - f_n(1,0)] - [f_n(0,1) - f_n(0,0)]\right) = \lim_{K\ni n\to\infty} \frac{w_n x_n \cdot y_n z_n}{|w_n x_n|\ |y_n z_n|},
$$

yielding $1 \geq C_x$. Thus $C_x = 1$. □

Remark. Let M be a compact Riemannian manifold, $D, E \in \mathrm{LipT}_x M$, and let $a = D|_{C^1(M)}, b = E|_{C^1(M)} \in \mathrm{T}_x M$. Then $D \cdot E = \langle a, b\rangle_{\mathrm{T}_x M}$. Hence let us generalize the notion of the tangent space to (locally) compact locally Cauchy–Schwarz spaces:

Tangent space. Let $(M, xy \mapsto |xy|)$ be a compact locally Cauchy–Schwarz space. Let us define an equivalence relation on $\mathrm{LipT}_x M$ by

$$
D \sim E \iff \forall F \in \mathrm{LipT}_x M:\ F \cdot D = F \cdot E. \tag{53}
$$

Let us call $\mathrm{T}_x M := (\mathrm{LipT}_x M)/\sim$ the *tangent space of M at x*. To simplify notation, let $D \in \mathrm{T}_x M$ still denote the equivalence class of $D \in \mathrm{LipT}_x M$.

Proposition 4.4 *$E \cdot D = D \cdot E$ for every $D, E \in \mathrm{LipT}_x M$.*

Proof. It is enough to show that $D_\xi \cdot D_\eta = D_\eta \cdot D_\xi$ where $\xi, \eta \in \beta\tilde{M}$ lie above $x \in M$, because every Lipschitz tangent can be weak*-approximated by linear combinations of the Lipschitz tangents of the form given by the limits of formal difference quotients. Let $\xi = \lim_i w_i x_i$ and $\eta = \lim_j y_j z_j$, where $(w_i x_i)_{i\in I}, (y_j z_j)_{j\in J} \subset \tilde{M}$. We have to show that

$$
\lim_i \lim_j \frac{w_i x_i \cdot y_j z_j}{|w_i x_i|\ |y_j z_j|} = \lim_j \lim_i \frac{w_i x_i \cdot y_j z_j}{|w_i x_i|\ |y_j z_j|}.
$$

The mapping $f: \tilde{M}\times\tilde{M} \to \mathbb{R}$, $f(ab, cd) := (ab\cdot cd)/(|ab|\ |cd|)$, is continuous and bounded, and it is symmetric: $f(ab, cd) = f(cd, ab)$. Then there is a unique $F \in C(X)$ on the Stone–Cech compactification $X := \beta(\tilde{M}\times\tilde{M}) \cong \beta(\tilde{M}) \times \beta(\tilde{M})$ extending f; moreover, F is symmetric, so that

$$
D_\xi \cdot D_\eta = F(\xi, \eta) = F(\eta, \xi) = D_\eta \cdot D_\xi. \qquad \square
$$

Corollary 4.5 *If $F \cdot F \geq 0$ for all $F \in \mathrm{LipT}_x M$ then for all $D, E \in \mathrm{LipT}_x M$*

$$
(D \cdot E)^2 \leq (D \cdot D)(E \cdot E).
$$

Proof. The case $D \cdot E = 0$ is trivial, so suppose $D \cdot E \neq 0$. If $\lambda \in \mathbb{R}$ then

$$0 \leq (D + \lambda E) \cdot (D + \lambda E) = D \cdot D + 2\lambda\, D \cdot E + \lambda^2\, E \cdot E;$$

hence $D \cdot E = 0$ if $E \cdot E = 0$. Otherwise, fixing $\lambda := -(D \cdot E)/(E \cdot E)$, we get

$$0 \leq D \cdot D - 2\frac{D \cdot E}{E \cdot E}\, D \cdot E + \left(\frac{D \cdot E}{E \cdot E}\right)^2\, E \cdot E. \qquad \square$$

Questions: When does $(E, D) \mapsto E \cdot D$ provide an inner product for $\mathrm{T}_x M$? Especially, if $E \not\sim 0$, is $E \cdot E > 0$?

Differentiability and Cauchy–Schwarz

Next we define differentiability and gradients in (locally) compact locally Cauchy–Schwarz spaces: the gradient of a function is a "vector field on M" or a "section of the tangent bundle $\mathrm{T}M$", differentiability being a pointwise directional derivative property. In Euclidean spaces, these concepts coincide with the traditional ones. Locally constant functions will be differentiable and have zero gradients. On a Cauchy–Schwarz space, differentiable functions form an algebra, but in general it is not clear whether there are non-trivial differentiable functions. Nevertheless, there are plenty of not-locally-constant functions that are differentiable at a given single point, and there are "first order Taylor expansions" for such functions, as well as gradient rule for a product of functions, and a chain rule for the composition of a Cauchy–Schwarz differentiable function and a smooth function $\mathbb{R} \to \mathbb{R}$.

Differentiability. A mapping $f : M \to \mathbb{R}$ is *differentiable at* $x \in M$ if $\mathrm{Lip}_x(f) < \infty$ and if there exists (necessarily unique) $\nabla f(x) \in \mathrm{T}_x M$ such that

$$Df = D \cdot \nabla f(x) \tag{54}$$

for every $D \in \mathrm{Lip}\mathrm{T}_x M$; here $\nabla f(x)$ is called the *gradient of f at x*. Notice that the requirement $\mathrm{Lip}_x(f) < \infty$ is for the operation $f \mapsto Df$ to be legal. A mapping $f : M \to \mathbb{R}$ is called *differentiable*, denoted by $f \in \mathrm{Diff}(M)$, if it is differentiable at every $x \in M$.

Remarks. If M is compact then $\mathrm{Diff}(M) \subset \mathrm{Lip}(M)$. In Euclidean space $(\mathbb{R}^n, xy \mapsto |xy|_2)$, the definition above is equivalent to the traditional differentiability. If $(\mathrm{T}_x M, (D, E) \mapsto D \cdot E)$ is a Hilbert space, $(E_j)_{j \in J}$ its orthonormal basis and $f : M \to \mathbb{R}$ is differentiable at x then $\nabla f(x) = \sum_{j \in J} (E_j f)\, E_j$:

$$D \cdot \nabla f(x) = Df = \left(\sum_{j \in J} (D \cdot E_j)\, E_j \right) f = \sum_{j \in J} (D \cdot E_j)\, E_j f = D \cdot \sum_{j \in J} (E_j f)\, E_j.$$

Theorem 4.6 *A mapping $f : M \to \mathbb{R}$ is differentiable at $x \in M$ if and only if*

$$f(y) = f(x) + xy \cdot E + R(x, y), \tag{55}$$

where $E \in \mathrm{LipT}_x M$ *and* $(y \mapsto R(x,y)) \in J(x)$*, i.e.* $R(x,x) = 0$ *and*

$$\inf_{r>0} \sup_{y,z \in B(x,r)} \frac{|R(x,y) - R(x,z)|}{|yz|} = 0. \tag{56}$$

Proof. If $f(y) = f(x) + xy \cdot E + R(x,y)$ as above and $D \in \mathrm{LipT}_x M$ then

$$Df = D(y \mapsto f(x)) + D(y \mapsto xy \cdot E) + D(y \mapsto R(x,y)) = D \cdot E,$$

so that $E \in \nabla f(x)$. Conversely, suppose f is differentiable at $x \in M$. Let $E \in \nabla f(x)$, and let us define $g : M \to \mathbb{R}$ by $g(y) := f(y) - (f(x) + xy \cdot E)$. Then $g(x) = f(x) - f(x) - xx \cdot E = 0$. Let $D \in \mathrm{LipT}_x M$. Then

$$Dg = Df - D(y \mapsto f(x)) - D(y \mapsto xy \cdot E) = D \cdot \nabla f(x) - 0 - D \cdot E = 0;$$

thereby $g \in J(x)$. By setting $R(x,y) := g(y)$, the proof is complete. □

Example. Let $c \in \mathbb{R}$, $E \in \mathrm{LipT}_x M$ and $R \in J(x)$. If $f : M \to \mathbb{R}$ is defined by

$$f(y) := c + xy \cdot E + R(y),$$

then it is differentiable at $x \in M$, $f(x) = c$, and $E \in \nabla f(x)$.

Examples. If f is locally constant at x then it is differentiable at x with zero gradient. Locally compact subsets of $(\mathbb{R}^n, xy \mapsto |xy|_2)$ carry plenty of differentiable functions. Moreover, the earlier example of the Hilbert cube also has separating sets of differentiable functions.

Snow-flaking. Let $(M, xy \mapsto |xy|)$ be a metric space and $0 < \alpha \leq 1/2$. Then $(M, xy \mapsto |xy|^\alpha)$ is Cauchy–Schwarz, without non-trivial rectifiable paths. If $f \in \mathrm{Lip}(M, xy \mapsto |xy|)$ then $f \in \mathrm{Diff}(M, xy \mapsto |xy|^\alpha)$ with zero gradient: original Lipschitz functions are differentiable in the snow-flaked metric.

Theorem 4.7 *Let* $\lambda \in \mathbb{R}$ *and* $f, g : M \to \mathbb{R}$ *be differentiable at* $x \in M$*. Then* $(y \mapsto \lambda), \lambda f, f+g, fg : M \to \mathbb{R}$ *are differentiable at* x*, and the gradients satisfy*

$$\begin{aligned} \nabla(y \mapsto \lambda)(x) &= 0, && (57)\\ \nabla(\lambda f)(x) &= \lambda \nabla f(x), && (58)\\ \nabla(f+g)(x) &= \nabla f(x) + \nabla g(x), && (59)\\ \nabla(fg)(x) &= f(x)\, \nabla g(x) + (\nabla f(x))\, g(x). && (60) \end{aligned}$$

Consequently, $\mathrm{Diff}(M)$ *is a subalgebra of* $\mathrm{Lip}(M)$.

Proof. Inclusion $\mathrm{Diff}(M) \subset \mathrm{Lip}(M)$ follows from Theorem 4.6. Let $D \in \mathrm{LipT}_x M$. Constant functions are differentiable, because $D(y \mapsto \lambda) = 0 = D \cdot 0$. Next, $D(\lambda f) = \lambda Df = \lambda(D \cdot \nabla f(x)) = D \cdot (\lambda \nabla f(x))$, and $D(f+g) = Df + Dg = D \cdot \nabla f(x) + D \cdot \nabla g(x) = D \cdot (\nabla f(x) + \nabla g(x))$; everything goes fine thanks to the linear structure. Recall that $I(x)I(x) \subset J(x)$ and that $J(x)$ is an ideal; now we can see that $fg \in \mathrm{Diff}(M)$:

$$\begin{aligned} D(fg) &= D(f)g(x) + f(x)D(g) \ = \ (D \cdot \nabla f(x))g(x) + f(x)(D \cdot \nabla g(x)) \\ &= D \cdot [f(x)\nabla g(x) + (\nabla f(x))g(x)]. \end{aligned}$$

□

Corollary 4.8 *If* $\mathrm{Diff}(M)$ *separates the points of* M *then it is dense in* $C(M)$*, the space of continuous functions* $M \to \mathbb{R}$*.*

Proof. This follows by the Stone–Weierstrass Theorem. □

Example. Coordinate projections of the Hilbert cube example are differentiable, so that there the differentiable functions approximate continuous functions. More generally, if $(M_j)_{j=1}^\infty$ is a sequence of compact Cauchy–Schwarz spaces with separating sets of differentiable functions then $\prod_{j=1}^\infty M_j$ can be endowed with a metric (as in the Hilbert cube case) such that coordinate projections are differentiable.

Lemma 4.9 (Pointwise Lipschitz chain inequality.) *Let* K, L *and* M *be metric spaces,* $x \in M$ *and* $f : M \to L$ *and* $g : L \to K$ *be Lipschitz mappings. Then*

$$\mathrm{Lip}_x(g \circ f) \leq \mathrm{Lip}_{f(x)}(g)\ \mathrm{Lip}_x(f). \tag{61}$$

Proof. If $A_r := B(x,r)$ and $B_r := B(f(x), \mathrm{Lip}(f)r)$ then

$$\lim_{r \to 0} \mathrm{Lip}(g \circ f|_{A_r}) \leq \lim_{r \to 0} [\mathrm{Lip}(g|_{B_r})\ \mathrm{Lip}(f|_{A_r})] = \mathrm{Lip}_{f(x)}(g)\ \mathrm{Lip}_x(f).$$

□

Theorem 4.10 (Chain rule.) *If* $f : M \to \mathbb{R}$ *is differentiable at* $x \in M$ *and* $g : \mathbb{R} \to \mathbb{R}$ *is differentiable at* $f(x) \in \mathbb{R}$ *then* $g \circ f : M \to \mathbb{R}$ *is differentiable at* $x \in M$ *and*

$$\nabla(g \circ f)(x) = g'(f(x))\ \nabla f(x). \tag{62}$$

Proof. Now $f(y) = f(x) + xy \cdot D + R(x,y)$, where $s = f(x)$ and $D \in \nabla f(x) \in \mathrm{T}_x M$, and $g(t) = g(s) + (t-s)g'(s) + R_g(s,t)$, so that

$$\begin{aligned} (g \circ f)(y) &= g(f(x)) + [f(y) - f(x)]g'(f(x)) + R_g(f(x), f(y)) \\ &= g(f(x)) + [xy \cdot D + R(x,y)]g'(f(x)) + R_g(f(x), f(y)) \\ &= g(f(x)) + xy \cdot g'(f(x))D \\ &\quad + g'(f(x))R(x,y) + R_g(f(x), f(y)). \end{aligned}$$

Let $R_{g \circ f}(x,y) := g'(f(x))R(x,y) + R_g(f(x), f(y))$. Clearly $R_{g \circ f}(x,x) = 0$ and $(y \mapsto g'(f(x))R(x,y)) \in J(x)$. By Lemma 4.9, we get

$$\mathrm{Lip}_x(y \mapsto R_g(f(x), f(y))) \leq \mathrm{Lip}_{f(x)}(t \mapsto R_g(f(x), t))\ \mathrm{Lip}_x(f) = 0.$$

Hence $x \mapsto R_{g \circ f}(x,y) \in J(x)$, and consequently $g'(f(x))D \in \nabla(g \circ f)(x)$. □

Corollary 4.11 *Let M be a compact Cauchy–Schwarz space. Suppose $f_z := (y \mapsto |yz|^2) : M \to \mathbb{R}$ is differentiable in a neighborhood of $w \in M$ for every $z \in M$. Then $\mathrm{Diff}(M) \subset C(M)$ is dense.*

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable such that $g(0) \neq 0$ and $g(t) = 0$ when $t > 1$. Suppose $f_z = (y \mapsto |yz|^2)$ is differentiable in $B(z, r_z) \subset M$. Then the family $\{y \mapsto g(f_z(y)/r^2) \mid 0 < r < r_z\} \subset \mathrm{Diff}(M)$ separates the point z from the set $M \setminus \{z\}$. Thus $\mathrm{Diff}(M)$ separates the points of M. □

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