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Abstract: *In this note we consider a storage process with an inverse subordinator input. The basic characteristics of the process are computed. The paper is a generalisation of [5].*

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1 Introduction

A doubly infinite subordinator is defined as

$$X_s := \begin{cases} -X^{(1)}_{(-s)-}, & s \le 0, \\ X^{(2)}_s, & s \ge 0, \end{cases}$$

where $X^{(1)}$ and $X^{(2)}$ are two independent identically distributed subordinators started at zero. We call the right-continuous inverse T of X an inverse subordinator process. The paths of T are nondecreasing with flat pieces corresponding to the jumps of X. In the definition of X we can choose the initial distribution $(X_0^{(1)}, X_0^{(2)})$ such that the inverse subordinator T has stationary increments (see Proposition 3 below).

In this paper we consider T as an input in the storage model

$$S_t = \sup_{s \le t}\{T_t - T_s - \mu\,(t-s)\}. \tag{1}$$

The storage model given by (1), where T is the local time at zero of a reflecting Brownian motion with drift was introduced by [7]. In [5] T is the local time of an arbitrary diffusion. It is known that any inverse subordinator is the local time of some Markov process. Thus, this work generalises the results in [7] and [5]. The computations of the distributions of the ends of busy and idle periods of S are similar to those in [7] and [5].

In the present paper we consider also a second storage process with the same input T and a subordinator output. For this process we study its stationary distribution and give an example when output is a stable subordinator.

This paper is organised as follows. The inverse subordinator T as the input in (1) is introduced in Section 2. In Section 3 we compute the stationary distribution of S. The distributions of the ends of the busy and idle periods are given in Section 4. The storage process with the subordinator output is considered in Section 5. Finally, in Section 6 we summarise some needed results on exponential random variables.

2 Preliminaries

Before defining the storage process with an inverse subordinator input we construct the needed two-sided subordinator such that its inverse would have stationary increments. For details on subordinators we refer to [2], [3].

Let $\widetilde{X} = \{\widetilde{X}_t : t \ge 0\}$ be a subordinator (i.e., an increasing Lévy process) started at 0. The Laplace transform of $\widetilde{X}$ is given by

$$\mathbf{E}\big(e^{-\alpha \widetilde{X}_t}\big) = e^{-t\Phi(\alpha)} \quad (\alpha > 0).$$

The function $\Phi(\alpha)$ is expressed via the Lévy-Khintchine formula

$$\Phi(\alpha) = \int_0^\infty (1 - e^{-\alpha x})\,\nu(dx),$$

where ν is the Lévy measure. We assume that

$$m := \int_0^\infty \nu(x,\infty)dx = \int_0^\infty x\,\nu(dx) < \infty,$$

which is equivalent to $\mathbf{E}(\widetilde{X}_1) = m < \infty$. Assume also that $\nu(0,\infty) = \infty$, i.e., we exclude the case of compound Poisson processes.

Let the pair (ξ_1, ξ_2) be distributed as $(UV, (1-U)V)$, where U is a random variable uniformly distributed on $(0,1)$ and V is independent of U and has the distribution

$$\mathbf{P}(V \in dv) = \frac{1}{m}\, v\, \nu(dv), \quad v \geq 0.$$

The Laplace transform of (ξ_1, ξ_2) is then

$$\mathbf{E}\big(e^{-\alpha\,\xi_1 - \beta\,\xi_2}\big) = \frac{1}{m}\frac{\Phi(\alpha) - \Phi(\beta)}{\alpha - \beta}. \tag{2}$$

Following [6], we call a process $X = \{X_t : t \in \mathbb{R}\}$ a stationary two-sided subordinator if

- the pair $(X_0^{(1)}, X_0^{(2)}) := (-X_{0-}, X_0)$ has distribution given by the right-hand side of (2);
- the processes $\widetilde{X}^{(1)} = \{-X_{(-t)-} + X_{0-} : t \geq 0\}$ and $\widetilde{X}^{(2)} = \{X_t - X_0 : t \geq 0\}$ are independent (and independent of $(X_0^{(1)}, X_0^{(2)})$) copies of $\widetilde{X}$.

Denote by M the closed range of X, namely $\{X_t : t \in \mathbb{R}\}^{cl}$. The complement of M, say M^c, consists of countably many disjoint open intervals. For $t \in \mathbb{R}$, define

$$G_t := \sup\{s \leq t : s \in M\} \quad \text{and} \quad D_t := \inf\{s > t : s \in M\}.$$

In this case, G_t and D_t are the ends of the interval of M^c straddling t. Define the age and residual lifetimes by $A_t := t - G_t$, $R_t := D_t - t$. Before going on we need the following definitions.

Definition 1. A set $M \subset \mathbb{R}$ is called stationary if for all $t \in \mathbb{R}$, $M + t = \{s + t : s \in M\}$ has the same law as M.

Definition 2. A set $M \subset \mathbb{R}$ is called regenerative if

- for all $t \in \mathbb{R}$, $M \cap [G_t, \infty)$ and $M \cap (-\infty, G_t] - G_t$ are independent;
- for all $t \in \mathbb{R}$, $M \cap (-\infty, D_t]$ and $M \cap [D_t, \infty) - D_t$ are independent.

The set $M = \{X_t : t \in \mathbb{R}\}^{cl}$ defined above is a stationary regenerative set (see e.g., [10], [6]).

Define the inverse subordinator process $T = \{T_s : s \in \mathbb{R}\}$ as

$$T_s := \inf\{t \in \mathbb{R} : X_t > s\}, \quad s \in \mathbb{R}.$$

In terms of the two-sided subordinator X and its inverse T,

$$A_t = t - X_{T_t-}, \quad R_t = X_{T_t} - t.$$

The random set M is stationary if the process (A_t, R_t) is stationary.

The inverse processes of $\widetilde{X}^{(i)}$ and $X^{(i)} := \widetilde{X}^{(i)} + X_0^{(i)}$, $i = 1, 2$, are given by

$$\widetilde{T}_s^{(i)} = \inf\{t \geq 0 : \widetilde{X}_t^{(i)} > s\} \quad \text{and} \quad T_s^{(i)} = \inf\{t \geq 0 : X_t^{(i)} > s\}.$$

These processes are continuous and non-decreasing. As was mentioned before, the initial distributions $X_0^{(1)}$ and $X_0^{(2)}$ are chosen so that the process T has stationary increments.

Proposition 3. *The inverse subordinator process $T = \{T_s : s \in \mathbb{R}\}$ has stationary increments and $\mathbf{E}(T_s) = \frac{s}{m}$, $s \in \mathbb{R}$.*

Proof. 1) Fix $y \in \mathbb{R}$. By construction, $X^y = \{X_t^y := X_{T_y+t} : t \geq 0\}$ is a subordinator generating $M \cap [y, \infty)$, meaning $\{X_t^y : t \geq 0\}^{cl} = M \cap [y, \infty)$. Using the discussion above, the process $X^y - y$ is identical in law to $\{X_t : t \geq 0\}$. Indeed, $X^y - y$ can be written as $X_t^y - y = (X_{T_y} - y) + X_{T_y+t} - X_{T_y}$. By stationarity and regenerativity of M, the process $\{X_{T_y+t} - X_{T_y} : t \geq 0\}$ has the same law as $\widetilde{X}^{(2)}$ and the initial distribution $(X_{T_y} - y)$ is identical in law to $X_0^{(2)}$ and independent of $\{X_{T_y+t} - X_{T_y} : t \geq 0\}$. Since $\{T_{y+t} - T_y : t \geq 0\}$ is the right-continuous inverse of $\{X_t^y - y : t \geq 0\}$, it follows that $T_{y+t} - T_y$ and T_t have the same law, that is T has stationary increments.
2)Take e.g., $s > 0$. Let ξ be an independent of X exponential random variable with parameter α. Since $\mathbf{P}(T_s \leq t) = \mathbf{P}(X_t > s)$, we have

$$\begin{aligned}
\mathbf{E}(T_\xi) &= \int_0^\infty \mathbf{P}(T_\xi > t)\, dt \\
&= \int_0^\infty \mathbf{P}(X_t \leq \xi)\, dt \\
&= \int_0^\infty \mathbf{E}(e^{-\alpha X_t})\, dt \\
&= \int_0^\infty \frac{\Phi(\alpha)}{m\alpha} e^{-t\Phi(\alpha)}\, dt = \frac{1}{mu}.
\end{aligned}$$

Hence by the uniqueness of the Laplace transform, $\mathbf{E}(T_s) = \frac{s}{m}$. □

Remark 4. *In the case when T is the local time of a stationary diffusion Y as in [5], Proposition 3 can be proved by conditioning at Y_0 and integrating with respect to the speed measure of Y.*

Remark 5. *The process $T^{(1)} = \{T_s^{(1)} : s \geq 0\}$ has been studied in [4], [11].*

3 Stationary distribution of S

The basic object of our interest in this paper is the storage process $S = \{S_t : t \in \mathbb{R}\}$ defined via

$$S_t := \sup_{-\infty < s \le t} \{T_t - T_s - \mu(t-s)\}.$$

Before stating the results we introduce the process

$$\widetilde{S}_0^{(1)} := \sup_{t \ge 0} \{\widetilde{T}_t^{(1)} - \mu t\}.$$

Consider next a spectrally positive (i.e., without negative jumps) Lévy process $\widetilde{X}_t^{(1)} - \frac{t}{\mu}$. Its Laplace transform is given by

$$\mathbf{E}\big(e^{-\alpha(\widetilde{X}_t - \frac{t}{\mu})}\big) = e^{-t\,(\Phi(\alpha) - \frac{\alpha}{\mu})} = e^{t\,\Psi(\alpha)}, \tag{3}$$

where $\Psi(\alpha) = \frac{\alpha}{\mu} - \Phi(\alpha)$. The following result is taken from [2] p. 190.

Lemma 6. *If* $\mathbf{E}\big(\widetilde{X}_1 - \frac{1}{\mu}\big) = \big(m - \frac{1}{\mu}\big) > 0$ *then the process* $\widetilde{X}_t - \frac{t}{\mu}$ *drifts to* $+\infty$*. Moreover, the equation*

$$\Psi(\alpha) = 0$$

has one positive solution α^* *and* $-\inf_{t\ge 0}\{\widetilde{X}_t - \frac{t}{\mu}\}$ *is exponentially distributed with parameter* α^**.*

Lemma 7. (see [8]) If $\mu > \frac{1}{m}$ then the random variable $\widetilde{S}_0^{(1)}$ is exponentially distributed with parameter $\frac{\alpha^*}{\mu}$.

Proof: From Lemma 6, we get

$$\begin{aligned}
\mathbf{P}\big(\widetilde{S}_0^{(1)} > s\big) &= \mathbf{P}\big(\exists t : \widetilde{T}_t^{(1)} - \mu t > s\big) \\
&= \mathbf{P}\Big(\exists t : \widetilde{X}_t^{(1)} - \frac{t}{\mu} < -\frac{s}{\mu}\Big) \\
&= \mathbf{P}\Big(-\inf_{t\ge 0}\big\{\widetilde{X}_t - \frac{t}{\mu}\big\} > \frac{s}{\mu}\Big) \\
&= e^{-\alpha^* \frac{s}{\mu}},
\end{aligned}$$

as claimed. $\square$

Due to Lemma 7 in what follows we shall only consider the case $\mu > \frac{1}{m}$. In the next proposition we show that S_t has the exponential distribution with an atom at 0 (cf. [8], where the diffusion case is considered).

Proposition 8. The stationary distribution of S is given by

$$\mathbf{P}(S_t > s) = \frac{1}{m\mu} e^{-\frac{\alpha^*}{\mu} s}, \quad s \ge 0. \tag{4}$$

In particular, $\mathbf{P}(S_t = 0) = 1 - \frac{1}{m\mu}$.

Proof: Since S_t is stationary, it is enough to consider the case $t = 0$. Fix $s > 0$. Since $\sup_{0 \le s < X_0^{(1)}} \{T_s^{(1)} - s\} = 0$ and $T_s^{(1)} = \widetilde{T}^{(1)}_{s - X_0^{(1)}}$ for $s \ge X_0^{(1)}$, we have

$$\begin{aligned}
S_0 &= \sup_{s \ge 0} \{T_s^{(1)} - \mu s\} \\
&= \sup_{s \ge X_0^{(1)}} \{\widetilde{T}^{(1)}_{s - X_0^{(1)}} - \mu s\} \vee 0 \\
&= \Big(\sup_{s \ge 0} \{\widetilde{T}_s^{(1)} - \mu s\} - \mu X_0^{(1)} \Big) \vee 0 \\
&= \Big(\widetilde{S}_0^{(1)} - \mu X_0^{(1)} \Big) \vee 0. \qquad (5)
\end{aligned}$$

Hence, using Lemma 7, we obtain for $s > 0$,

$$\begin{aligned}
\mathbf{P}(S_0 > s) &= \mathbf{P}\Big(\frac{\widetilde{S}_0^{(1)}}{\mu} > \frac{s}{\mu} + X_0^{(1)} \Big) \\
&= e^{-\frac{\alpha^* s}{\mu}} \, \mathbf{E}\big(e^{-\alpha^* X_0^{(1)}} \big) \\
&= \frac{1}{m\,\mu} \, e^{-\frac{\alpha^*}{\mu} s},
\end{aligned}$$

where in the last step we use that

$$\mathbf{E}\big(e^{-\alpha^* X_0^{(1)}} \big) = \frac{\Phi(\alpha^*)}{m\,\alpha^*} = \frac{1}{m\,\mu}.$$

The statement follows. □

4 Busy and idle periods

Replacing the process X_t with $X_{\frac{t}{\mu}}$, it is seen that we can always consider the case $\mu = 1$ and $m = \mathbf{E}(X_1) < 1$. From now on we assume that $\mu = 1$.
If $S_0 = 0$ define the starting and the ending times of the on-going idle period at time zero as

$$g_i := \sup\{t < 0 : S_t > 0\}, \quad d_i := \inf\{t > 0 : S_t > 0\}.$$

If $S_0 > 0$ define the starting and the ending times of the on-going busy period at time zero as

$$g_b := \sup\{t < 0 : S_t = 0\}, \quad d_b := \inf\{t > 0 : S_t = 0\}.$$

As in [5], we consider the zero set of S. It is stationary since S is stationary. As it is shown in [5], given $d_b - g_b = v$, the random variable d_b is uniformly distributed on $(0, v)$ (and the same holds for $-g_b$). Hence it is enough to compute the marginal distribution of $-g_b$ (or d_b) and of $-g_i$ (or d_i).

Let $S_0 > 0$, that is there is a busy period at time zero. Consider the process S_t for $t \in [a_1, a_2]$, where

$$a_1 := \sup\{t < 0 : t - T_t = S_0\}, \quad a_2 := \inf\{t > 0 : t - T_t = S_0\}.$$

For $t \in (a_1, a_2)$, $t - T_t < S_0$ and hence

$$\begin{aligned} S_t &= T_t - t + \sup_{s \le t}\{s - T_s\} \\ &= T_t - t + S_0, \quad t \in (a_1, a_2), \end{aligned}$$

from which it is seen that $a_1 = g_b$, $a_2 = d_b$.
The distributions of $(-g_b, d_b)$ and $(-g_i, d_i)$ can be computed similarly to [5]. The computations for $-g_b$ and d_i can be simplified using the properties of exponential random variables given in Lemma 15 in Appendix.

Theorem 9. *Let η be the inverse of $\Psi(\alpha)$, $\alpha \ge \alpha^*$. Then the joint Laplace transform of $-g_b$ and d_b, given that $S_0 > 0$, is*

$$\mathbf{E}\big(e^{\alpha g_b - \beta d_b} \mid S_0 > 0\big) = \frac{\alpha^*}{\alpha - \beta}\left(\frac{\alpha}{\eta(\alpha)} - \frac{\beta}{\eta(\beta)}\right), \quad \alpha \ne \beta. \tag{6}$$

Proof: To compare, we give the computations for both $-g_b$ and d_b, which are taken with small changes from [7] and [5].
1) (Computation of the distribution of d_b)
Since d_b can be presented as

$$d_b = \tau_{(S_0 - X_0^{(2)})\vee 0} + S_0 = \tau_{(S_0 - X_0^{(2)})\vee 0} + (S_0 - X_0^{(2)}) + X_0^{(2)},$$

where

$$\tau := \inf\{s : \widetilde{X}_s^{(2)} - s > t\},$$

the crucial property in computation (cf. Corollary 3.3 in [5]) is that $S_0 - X_0^{(2)}$ is exponentially distributed and independent of $X_0^{(2)}$. In [5] this was shown using the Markov property of the diffusion Y at time zero and integrating with respect to the speed measure of Y. From (5) it follows that $\{S_0 > X_0^{(2)}\} = \{\widetilde{S}_0^{(1)} > X_0^{(1)} + X_0^{(2)}\}$. Since $\widetilde{S}_0^{(1)}$ and $(X_0^{(1)}, X_0^{(2)})$ are independent, it follows from Lemma 15(4) that given $S_0 > X_0^{(2)}$, the random variables $S_0 - X_0^{(2)}$ and $X_0^{(2)}$ are independent. Moreover, $S_0 - X_0^{(2)}$ given $S_0 > X_0^{(2)}$ is exponentially distributed with parameter α^* and

$$\mathbf{E}\big(e^{-\alpha X_0^{(2)}}; S_0 > X_0^{(2)}\big) = \mathbf{E}\big(e^{-(\alpha+\alpha^*)X_0^{(2)} - \alpha^* X_0^{(1)}}\big) = 1 - \frac{\Psi(\alpha + \alpha^*)}{\alpha}. \tag{7}$$

The rest of the computation of the distribution of d_b does not differ from one in the proof of Theorem 3.1 in [5] and we omit it.
2)(Computation of the distribution of g_b)

To compute the distribution of g_b we write first

$$
\begin{aligned}
|g_b| &= \inf\{t > X_0^{(1)} : T_t^{(1)} - t = S_0\} \\
&= \inf\{t > X_0^{(1)} : \widetilde{T}^{(1)}_{t - X_0^{(1)}} - t = S_0\} \\
&= X_0^{(1)} + \inf\{t > 0 : \widetilde{T}_t^{(1)} - t = \widetilde{S}_0^{(1)}\} \\
&= X_0^{(1)} + \inf\{\widetilde{X}_{t-}^{(1)} : t - X_{t-}^{(1)} = \widetilde{S}_0^{(1)}\} \\
&= X_0^{(1)} - \widetilde{S}_0^{(1)} + \inf\{t : \widetilde{X}_{t-}^{(1)} - t = -\widetilde{S}_0^{(1)} = \inf_{t\geq 0}\{\widetilde{X}_t^{(1)} - t\}\} \qquad (8)
\end{aligned}
$$

Denote the last term in (8) by N. As in [7] we use the result from [1] which states that the process $\{\widetilde{X}_t^{(1)} - t : 0 \leq t < N\}$ is identical in law to $\{X_t^* : 0 \leq t < H_{-\theta}^*\}$, where $\{X_t^* : t \geq 0\}$, $X_0^* = 0$, is a spectrally positive Lévy process with

$$\mathbf{E}(e^{-\alpha X_t^*}) = e^{t\Psi(\alpha+\alpha^*)},$$

$\theta \sim Exp(\alpha^*)$ is independent of X^*, and $H_{-\theta}^*$ stands for the first hitting time of $-\theta$ for X^*. Let η^* be the inverse of $\Psi(\alpha + \alpha^*)$. Using Lemma 15(2), the definition of η and fact that $\alpha^* + \eta^*(\alpha) = \eta(\alpha)$, we have

$$
\begin{aligned}
\mathbf{E}\big(e^{-\alpha|g_b|}\,;\, S_0 > 0\big) &= \mathbf{E}\big(e^{-\alpha(X_0^{(1)} + H_{-\theta}^* - \theta)}\,;\, \theta > X_0^{(1)}\big) \\
&= \mathbf{E}\big(e^{-\alpha X_0^{(1)} - (\eta^*(\alpha) - \alpha)\,\theta}\,;\, \theta > X_0^{(1)}\big) \\
&= \frac{\alpha^*}{\eta^*(\alpha) - \alpha + \alpha^*}\mathbf{E}\big(e^{-(\alpha^* + \eta^*(\alpha))X_0^{(1)}}\big) \\
&= \frac{\alpha^*}{\eta(\alpha) - \alpha}\mathbf{E}\big(e^{-\eta(\alpha)X_0^{(1)}}\big) \\
&= \frac{\alpha^*}{\eta(\alpha) - \alpha}\frac{\Phi(\eta(\alpha))}{m\,\eta(\alpha)} \\
&= \frac{\alpha^*}{(\eta(\alpha) - \alpha)\,m}\Big(1 - \frac{\Psi(\eta(\alpha))}{\eta(\alpha)}\Big) \\
&= \frac{\alpha^*}{\eta(\alpha)\,m}.
\end{aligned}
$$

The fact that given $d_b - g_b = v$, the random variable $|g_b|$ has uniform distribution on $(0, v)$ completes the proof. $\square$

Remark 10. *Let the process X^* be as above. Let θ be an exponentially distributed random variable with parameter α^* independent of X^*. Then*

$$\mathbf{E}\big(e^{-\alpha H_{-\theta}^*}\big) = \frac{\alpha^*}{\alpha^* + \theta^*(\alpha)} = \frac{\alpha^*}{\eta(\alpha)}.$$

Theorem 11. *The joint Laplace transform of $-g_i$ and d_i, given that $S_0 = 0$, is*

$$\mathbf{E}(e^{\alpha g_i - \beta d_i} \mid S_0 = 0) = \frac{\alpha^* \mu}{(m-1)(\alpha - \beta)}\left(\frac{\Psi(\alpha)}{\alpha - \alpha^*} - \frac{\Psi(\beta)}{\beta - \alpha^*}\right), \quad \alpha \neq \beta. \quad (9)$$

Proof: Suppose now that $S_0 = 0$, i.e., there is an idle period at time zero. From (5) it follows that $\{S_0 = 0\} = \{X_0^{(1)} > \widetilde{S}_0^{(1)}\}$. Notice that $-X_0^{(1)} < g_i < 0$ and write

$$\begin{aligned}
|g_i| &= \inf\Big\{0 < t < X_0^{(1)} : \sup_{s \geq t}\{T_s^{(1)} - (s-t)\} > 0\Big\} \\
&= \inf\Big\{0 < t < X_0^{(1)} : \sup_{s \geq X_0^{(1)}}\{T_s^{(1)} - s\} + t > 0\Big\} \\
&= \inf\Big\{0 < t < X_0^{(1)} : \sup_{s \geq X_0^{(1)}}\{\widetilde{T}^{(1)}_{s-X_0^{(1)}} - (s - X_0^{(1)})\} - X_0^{(1)} + t > 0\Big\} \\
&= \inf\Big\{0 < t < X_0^{(1)} : \widetilde{S}_0^{(1)} - X_0^{(1)} + t \ > 0\Big\},
\end{aligned}$$

from which we get $|g_i| = X_0^{(1)} - \widetilde{S}_0^{(1)}$. Thus, using Lemma 15(3), we obtain

$$\begin{aligned}
\mathbf{E}\big(e^{\alpha g_i}; S_0 = 0\big) &= \mathbf{E}\big(e^{-\alpha(X_0^{(1)} - \widetilde{S}_0^{(1)})}; X_0^{(1)} > \widetilde{S}_0^{(1)}\big) \\
&= \frac{\alpha^*}{\alpha - \alpha^*}\Big(\mathbf{E}\big(e^{-\alpha^* X_0^{(1)}}\big) - \mathbf{E}\big(e^{-\alpha X_0^{(1)}}\big)\Big) \\
&= \frac{\alpha^*}{\alpha - \alpha^*}\Big(\frac{\Phi(\alpha^*)}{m\,\alpha^*} - \frac{\Phi(\alpha)}{m\,\alpha}\Big) \\
&= \frac{\alpha^* \Psi(\alpha)}{(\alpha - \alpha^*)\,\alpha\, m}.
\end{aligned}$$

□

5 On a storage with subordinator output

Here we define a storage model with the input given in Section 3 and the output being a doubly infinite subordinator.

Let $\widetilde{Z} = \{\widetilde{Z}_t : t \geq 0\}$ be a subordinator started at 0 with the Laplace transform given by

$$\mathbf{E}\big(e^{-\alpha \widetilde{Z}_t}\big) = e^{-t\varphi(\alpha)} \quad (\alpha > 0).$$

Let $Z = \{Z_t : t \in \mathbb{R}\}$ be defined as

$$Z_t := \begin{cases} -Z^{(1)}_{(-t)-}, & t \leq 0, \\ Z^{(2)}_t, & t \geq 0, \end{cases}$$

where $Z^{(1)} = \{Z^{(1)}_t : t \geq 0\}$ and $Z^{(2)} = \{Z^{(2)}_t : t \geq 0\}$ are two independent copies of $\widetilde{Z}$. We assume next that Z is independent of X.

As in Section 3, we introduce

$$S_t := \sup_{-\infty < s \leq t}\{T_t - T_s - (Z_t - Z_s)\}$$

and

$$\widetilde{S}_0^{(1)} := \sup_{t \geq 0}\{\widetilde{T}_t^{(1)} - Z_t\}.$$

The following analog of Proposition 8 holds.

Proposition 12. *Assume that*

$$\mathbf{E}(\widetilde{X}_1)\,\mathbf{E}(Z_1) = \Phi^{'}(0)\,\varphi^{'}(0) > 1. \tag{10}$$

Then the process S is well-defined and its stationary distribution is given by

$$\mathbf{P}(S_t > s) = K\,e^{-\alpha^* s}, \quad s \geq 0,$$

where α^* is the unique positive solution of the equation

$$\alpha - \Phi(\varphi(\alpha)) = 0$$

and $K = \frac{\alpha^*}{m\,\varphi(\alpha^*)}$.

The following lemma will be useful for the proof of the proposition.

Lemma 13. *Assume that $\Phi^{'}(0)\,\varphi^{'}(0) > 1$. Then $\widetilde{S}_0^{(1)}$ is exponentially distributed with parameter α^*.*

Proof: Consider the process $Y = \{Y_t = \widetilde{Z}_{\widetilde{X}_t} : t \geq 0\}$. Clearly, Y is a subordinator with the Laplace transform given by

$$\mathbf{E}\big(e^{-\alpha Y_t}\big) = e^{-t\Phi(\varphi(\alpha))} \quad (\alpha > 0).$$

The process $Y_t - t$ is a spectrally positive Lévy process with the Laplace transform

$$\mathbf{E}\big(e^{-\alpha(Y_t - t)}\big) = e^{t\Psi(\alpha)},$$

where $\Psi(\alpha) = \alpha - \Phi(\varphi(\alpha))$. If $\Psi^{'}(0) = 1 - \Phi^{'}(0)\,\varphi^{'}(0) < 0$ then (see [2] p. 190) the process $Y_t - t$ drifts to $+\infty$ and $-\inf_{t\geq 0}\{Y_t - t\}$ is exponentially distributed with parameter α^*, where α^* is the unique solution of $\Psi(\alpha) = 0$. Since $\widetilde{T}^{(1)}$ is the right-continuous inverse of $\widetilde{X}^{(1)}$ and $Z^{(1)}$ is increasing, we write

$$\begin{aligned}
\mathbf{P}(\widetilde{S}_0^{(1)} > s) &= \mathbf{P}(\exists t : \widetilde{T}_t^{(1)} - Z_t^{(1)} > s) \\
&= \mathbf{P}(\exists t : t - Z^{(1)}_{\widetilde{X}^{(1)}_{t-}} > s) \\
&= \mathbf{P}(\exists t : Y_{t-} - t < -s) \\
&= \mathbf{P}(-\inf_{t\geq 0}\{Y_t - t\} > s) \\
&= e^{-\alpha^* s},
\end{aligned}$$

as required. □

Proof of Proposition 12*:* First we notice that S_0 can be written as

$$\begin{aligned}
S_0 &= \sup_{s\geq X_0^{(1)}} \{\widetilde{T}^{(1)}_{s-X_0^{(1)}} - Z_s\} \vee 0 \\
&= \sup_{s\geq 0}\{\widetilde{T}_s^{(1)} - Z_{s+X_0^{(1)}}\} \vee 0 \\
&= \Big(\sup_{s\geq 0}\{\widetilde{T}_s^{(1)} - \widehat{Z}_s\} - Z_{X_0^{(1)}}\Big) \vee 0,
\end{aligned}$$

where $\{\widehat{Z}_s := Z_{s+X_0^{(1)}} - Z_{X_0^{(1)}} \,:\, s \geq 0\}$ is independent of $Z_{X_0^{(1)}}$ and has the same law as $\widetilde{Z}$. Denote

$$\widehat{S}_0 := \sup_{s\geq 0}\{\widetilde{T}_s^{(1)} - \widehat{Z}_s\}.$$

Using Lemma 13 , we obtain

$$\begin{aligned}
\mathbf{P}(S_0 > s) &= \mathbf{P}\big(\widehat{S}_0 - Z_{X_0^{(1)}} > s\big) \\
&= e^{-\alpha^* s}\,\mathbf{E}\big(e^{-\alpha^* Z_{X_0^{(1)}}}\big) \\
&= e^{-\alpha^* s}\,\mathbf{E}\big(e^{-\varphi(\alpha^*)X_0^{(1)}}\big) \\
&= e^{-\alpha^* s}\,\frac{\Phi(\varphi(\alpha^*))}{m\,\varphi(\alpha^*)} \\
&= e^{-\alpha^* s}\,\frac{\alpha^*}{m\,\varphi(\alpha^*)}.
\end{aligned}$$

The statement follows. □

Example 14. (Z is a stable subordinator)
Let $Z^{(1)}$ and $Z^{(2)}$ be stable subordinators with index γ, i.e.,

$$\mathbf{E}\big(e^{-\alpha \widetilde{Z}_t^{(i)}}\big) = e^{-t\,\alpha^\gamma}, \quad i = 1, 2.$$

For the stable subordinator $\widetilde{Z}$ we have the following identity in distribution:

$$\widetilde{Z}_t \sim t^{\frac{1}{\gamma}}\,\widetilde{Z}_1. \tag{11}$$

The stationary distribution of S is given by

$$\mathbf{P}(S_t > s) = \frac{(\alpha^*)^{1-\gamma}}{m}\,e^{-\alpha^* s}, s > 0,$$

where α^* is the unique positive solution of

$$\alpha - \Phi(\alpha^\gamma) = 0.$$

Next we compute the distribution of d_i given $S_0 = 0$. Clearly $d_i \geq X_0^{(2)}$ since for $t \in [0, X_0^{(2)})$,

$$S_t = \sup_{0\leq s\leq t}\{Z_s\} - Z_t = 0.$$

We write

$$\begin{aligned}
d_i &= \inf\Big\{t \geq X_0^{(2)} : \sup_{0\leq s\leq t}\{Z_s - T_s\} + T_t - Z_t > 0\Big\} \\
&= \inf\Big\{t \geq X_0^{(2)} : \sup_{X_0^{(2)}\leq s\leq t}\{Z_s - T_s\} + T_t - Z_t > 0\Big\} \\
&= \inf\Big\{t \geq X_0^{(2)} : Z_{X_0^{(2)}} + \sup_{X_0^{(2)}\leq s\leq t}\{Z'_{s-X_0^{(2)}} - \widetilde{T}^{(2)}_{s-X_0^{(2)}}\} \\
&\qquad + \widetilde{T}^{(2)}_{t-X_0^{(2)}} - Z'_{t-X_0^{(2)}} - Z_{X_0^{(2)}} > 0\Big\} \\
&= X_0^{(2)} + \inf\Big\{t \geq 0 : \sup_{0\leq s\leq t}\{Z'_s - \widetilde{T}_s^{(2)}\} + \widetilde{T}_t^{(2)} - Z'_t > 0\Big\}.
\end{aligned}$$

Next we use the result from [9] that

$$\mathbf{P}\big(Z'_{\widetilde{X}^{(2)}_t} < t \quad \text{for all sufficiently small } t > 0\big) = 1,$$

from which it follows that

$$\mathbf{P}\big(Z'_t < \widetilde{T}^{(2)}_t \quad \text{for all sufficiently small } t > 0\big) = 1$$

and hence $d_i = X_0^{(2)}$. Therefore,

$$\begin{aligned} \mathbf{E}\big(e^{-\alpha d_i}\,;\, S_0 = 0\big) &= \mathbf{E}\big(e^{-\alpha X_0^{(2)}}\,;\, \widehat{S}_0 < Z_{X_0^{(1)}}\big) \\ &= \mathbf{E}\Big(e^{-\alpha X_0^{(2)}}\,;\, X_0^{(1)} > \Big(\frac{\widehat{S}_0}{Z_1}\Big)^{\gamma}\Big). \end{aligned}$$

Next we show that $\Big(\frac{\widehat{S}_0}{Z_1}\Big)^{\gamma}$ is exponentially distributed. Using (11), we obtain

$$\begin{aligned} \mathbf{P}\Big(\Big(\frac{\widehat{S}_0}{Z_1}\Big)^{\gamma} > s\Big) &= \mathbf{P}\Big(\widehat{S}_0 > Z_1\, s^{\frac{1}{\gamma}}\Big) \\ &= \mathbf{E}\Big(e^{-\alpha^*\, s^{\frac{1}{\gamma}}\, Z_1}\Big) \\ &= e^{(\alpha^*)^{\gamma}\, s}. \end{aligned}$$

Using this and Lemma 15(3) yields

$$\begin{aligned} \mathbf{E}\big(e^{-\alpha d_i}\,;\, S_0 = 0\big) &= \mathbf{E}\big(e^{-\alpha X_0^{(2)}}\big) - \mathbf{E}\big(e^{-\alpha X_0^{(2)} - (\alpha^*)^{\gamma} X_0^{(1)}}\big) \\ &= \frac{\Phi(\alpha)}{m\,\alpha} - \frac{1}{m}\Big(\frac{\Phi(\alpha) - \Phi((\alpha^*)^{\gamma})}{\alpha - (\alpha^*)^{\gamma}}\Big) \\ &= \frac{\alpha\,\alpha^* - \Phi(\alpha)(\alpha^*)^{\gamma}}{m\,\alpha\,\big(\alpha - (\alpha^*)^{\gamma}\big)}. \end{aligned}$$

6 Appendix

Well-known results dealing with exponential random variables are given in the following lemma.

Lemma 15. Let $\tau \sim \mathit{Exp}(\lambda)$ and $X, Y \geq 0$ be two arbitrary random variables independent of τ. Then

1. $\mathbf{P}(X < \tau) = \mathbf{E}(e^{-\lambda X})$;
2. (the memoryless property) $\tau - X$ and X are conditionally independent, given $\tau > X$. Moreover, given $\tau > X$, $\tau - X$ is exponentially distributed with parameter λ and
   $\mathbf{E}\big(e^{-\beta X}; \tau > X\big) = \mathbf{E}\big(e^{-(\lambda+\beta)X}\big)$;
3. $\mathbf{E}\big(e^{-\alpha(X-\tau)}; \tau < X\big) = \dfrac{\lambda}{\alpha - \lambda}\Big(\mathbf{E}(e^{-\lambda X}) - \mathbf{E}(e^{-\alpha X})\Big)$;

4. Given $\tau > X + Y$, the random variables X and $\tau - X - Y$ are independent. Moreover, given $\tau > X + Y$, $\tau - X - Y$ is exponentially distributed with parameter λ and
$\mathbf{E}\big(e^{-\alpha X}; \tau > X + Y\big) = \mathbf{E}\big(e^{-(\alpha+\lambda)X-\lambda Y}\big)$.

The proof of Lemma 15 is straight-forward and is omitted.

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