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Abstract: *We give a geometric characterization for a plane set* $A \subset \mathbf{R}^2$ *to have the following linear bilipschitz extension property: For* $0 \leq \varepsilon \leq \delta$, *every* $(1+\varepsilon)$*-bilipschitz map* $f\colon A \to \mathbf{R}^2$ *has a* $(1+C\varepsilon)$*-bilipschitz extension to the whole plane* $\mathbf{R}^2$.

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1 Introduction

1.1. Let A be a subset of the euclidean n-space $\mathbf{R}^n$ and let $L \geq 1$. A map $f\colon A \to \mathbf{R}^n$ is *L-bilipschitz* if

$$|x-y|/L \leq |fx-fy| \leq L|x-y|$$

for all $x, y \in A$.

In general, an L-bilipschitz map $f\colon A \to \mathbf{R}^n$ cannot be extended to a bilipschitz map $F\colon \mathbf{R}^n \to \mathbf{R}^n$, not even to a homeomorphism, but this is often possible in the case the bilipschitz constant L is close to 1.

1.2. Let Φ be the set of increasing homeomorphisms $\varphi\colon \mathbf{R}^+ \to \mathbf{R}^+$. If $\varphi \in \Phi$ and $\delta > 0$, we say that a set $A \subset \mathbf{R}^n$ has the (φ,δ)-bilipschitz extension property, (φ,δ)-BLEP for short, if for $0 \leq \varepsilon \leq \delta$, every $(1+\varepsilon)$-bilipschitz map $f\colon A \to \mathbf{R}^n$ has an extension to a $(1+\varphi(\varepsilon))$-bilipschitz map $F\colon \mathbf{R}^n \to \mathbf{R}^n$. We say that a set $A \subset \mathbf{R}^n$ belongs to the class φ-BLEP if it has the (φ,δ)-BLEP for some $\delta > 0$. In the case $\varphi(\varepsilon) = C\varepsilon$ we say that A has the (C,δ)-linear BLEP.

It was shown in [ATV2] that a set $A \subset \mathbf{R}^n$ has (C,δ)-linear BLEP if it satisfies a geometric condition called sturdiness; see 2.2 for the definition. In this article we prove that the converse is true in the 2-dimensional case. More precisely, we obtain the following theorem.

1.3. **Theorem.** *Let $A \subset \mathbf{R}^2$ contain at least three points. Then the following assertions are quantitatively equivalent:*
(1) *A is c-sturdy.*
(2) *A has the (C,δ)-linear BLEP.*
Here quantitative equivalence means that C and δ depend only on c, and conversely, $c = c(C,\delta)$.

The proof is given in section 4.3. Note that a set $A \subset \mathbf{R}^n$ consisting of at most two points has the 1-linear BLEP but it is sturdy only in the cases $n = 1$ or $\#A = 1$.

For extension problems in higher dimensions and with more general bounds for the bilipschitz constant, see [Vä] and the references in [ATV2].

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2 Basic concepts

Notation follows closely our main reference [ATV2] and will not be repeated here except for the abbreviation $A(a,r) = A \cap \bar{B}(a,r)$.

However, we recall three geometric properties of sets that are needed in our main result.

2.1. *Thickness.* For each unit vector $e \in S^{n-1}$ we define the projection $\pi_e \colon \mathbf{R}^n \to \mathbf{R}$ by $\pi_e x = x \cdot e$. Let $A \neq \emptyset$ be a bounded set in $\mathbf{R}^n$. The *thickness* of A is the number

$$\theta(A) = \inf \{d(\pi_e A) : e \in S^{n-1}\}.$$

Alternatively, $\theta(A)$ is the infimum of all $t > 0$ such that A lies between two parallel hyperplanes F, F' with $d(F, F') = t$. We have always $0 \leq \theta(A) \leq d(A)$.

2.2. *Sturdiness.* Let $A \subset \mathbf{R}^n$. For $a \in A$ we set $s(a) = s_A(a) = d(a, A \setminus \{a\})$. Then $s(a) > 0$ if and only if a is isolated in A.

Let $c \geq 1$. We say that the set $A \subset \mathbf{R}^n$ is *c-sturdy* if

(1) $\theta(A(a,r)) \geq 2r/c$ whenever $a \in A,\ r \geq cs(a),\ A \not\subset B(a,r)$,

(2) $\theta(A) \geq d(A)/c$.

If A is unbounded, we omit (2), and the condition $A \not\subset B(a,r)$ of (1) is unnecessary.

2.3. *Relative connectivity* [TV, 4.6]. Let $A \subset \mathbf{R}^n$ and $M \geq 1$. A sequence $(x_0, x_1, \ldots, x_{N-1}, x_N)$ is proper if $x_{j-1} \neq x_j$ for all j. A sequence $(x_0, x_1, \ldots, x_{N-1}, x_N)$ in A is M-relative in A if it is proper and

$$|x_{j-1} - x_j|/M \leq |x_j - x_{j+1}| \leq M|x_{j-1} - x_j|$$

for all j. Such a sequence is said to join the pairs (x_0, x_1) and (x_{N-1}, x_N). The set A is M-relatively connected (abbr. RC) if every two proper pairs in A can be joined by an M-relative sequence in A.

The simplest examples of relatively connected sets are the connected ones, but also many totally disconnected sets like the Cantor middle-third set satisfy the RC-condition.

2.4. **Lemma.** *Let $A \subset \mathbf{R}^n$ be a closed c-sturdy set. Then A is c_1-RC for every $c_1 > c$.*

Proof. Let $a \in A$ and $r > 0$. Let $c_1 > c$ and assume that $A \cap \bar{B}(a,r) \neq \{a\}$ and $A \not\subset \bar{B}(a,r)$. If $R(a,r) = \{x \in A \mid r/c_1 \leq |x - a| \leq r\} = \emptyset$, then $\theta(A(a,r)) \leq \theta(\bar{B}(a, r/c_1)) \leq 2r/c_1 < 2r/c$, a contradiction with the c-sturdiness of A. It follows that, under the above assumptions, $R(a,r) \neq \emptyset$, and by [TV, 4.11], this implies the claim. $\square$

2.5. *Linear isometric approximation property.* Let $A \subset \mathbf{R}^n$. We say that A has the (C,δ)-linear isometric approximation property (IAP) if given $0 < \varepsilon \leq \delta$, a $(1+\varepsilon)$-bilipschitz map $f \colon A \to \mathbf{R}^n$, a point $a \in A$ and $r > 0$, there is an isometry $T = T_{a,r} \colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$|Tx - f(x)| \leq C\varepsilon r$$

for all $x \in A \cap \bar{B}(a,r)$.

2.6. **Theorem.** *Suppose that a set $A \subset \mathbf{R}^n$ has the (C, δ)-linear BLEP. Then it has the (C_1, δ)-linear IAP with $C_1 = C_1(C, n)$.*

Proof. Let $f\colon A \to \mathbf{R}^n$ be $(1+\varepsilon)$-bilipschitz with $0 < \varepsilon \le \delta$. Suppose that $a \in A$ and $r > 0$. Since A has the (C, δ)-linear BLEP, there is a $(1+C\varepsilon)$-bilipschitz extension $F\colon \mathbf{R}^n \to \mathbf{R}^n$ of f. Let $F_{a,r} = F \mid \bar{B}(a, r)$. Then $F_{a,r}$ is a $2C\varepsilon r$-nearisometry and since $\theta(\bar{B}(a, r)) = d(\bar{B}(a, r))$, [ATV1, 3.3] gives an isometry $T = T_{a,r}\colon \mathbf{R}^n \to \mathbf{R}^n$ such that

$$||T - F_{a,r}||_{\bar{B}(a,r)} \le 2c_n C\varepsilon r.$$

In particular, we have $|Tx - f(x)| \le 2Cc_n\varepsilon r$ for every $x \in A(a, r)$, and the proof is complete with $C_1 = 2c_n C$. $\square$

3 Triangle maps

Since we work with the planar case, we use complex numbers whenever it simplifies notation.

3.1. The basic triangle map $f\colon \{-1, 0, 1\} \to \mathbf{R}^2$ is defined by

$$f(\pm 1) = \pm 1 \quad \text{and} \quad f(0) = i\sqrt{\varepsilon}.$$

This map is $(1+\varepsilon)$-bilipschitz, but any approximation of f by an isometry T has an error at least $\sqrt{\varepsilon}/2$. This is seen by minimizing the distance from the image of f to the straight line $T\mathbf{R}$. The following elementary lemma generalizes this idea.

3.2. **Lemma.** *Let $0 \le \delta \le \delta' \le 1/4$, let $A = \{-1, a, 1\} \subset \mathbf{R}^2$ be such that $\theta(A) = |a_2| \le 2\delta$, and let $f\colon A \to \mathbf{R}^2$ satisfy $f(\pm 1) = \pm 1$ and $\theta(fA) = |f(a)_2| \ge 2\delta'$. If the disks $\bar{B}(\pm 1, \delta' - \delta)$ and $\bar{B}(f(a), \delta' + \delta)$ are disjoint, then every isometry $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ satisfies $||T - f||_A \ge \delta' - \delta$.*

Proof. We emphasize that the conditions $\theta(A) = |a_2|$ and $\theta(fA) = |f(a)_2|$ belong to the assumpions. In particular, they imply that $-1 < a_1 < 1$ and $-1 < f(a)_1 < 1$ so that the situation is not too far from the basic map above.

Suppose that T is an isometry with $||T - f||_A < \delta' - \delta$ and let $L = T\mathbf{R}$. Writing $a' = (a_1, 0)$, we have

$$|Ta' - Ta| = |a' - a| = |a_2| \le 2\delta.$$

If L does not meet the disk $B(f(a), \delta' + \delta)$, then

$$|Ta - f(a)| \ge |Ta' - f(a)| - |Ta' - Ta| \ge (\delta' + \delta) - 2\delta = \delta' - \delta,$$

a contradiction.

It follows that the line L meets all three disks $\bar{B}(\pm 1, \delta'-\delta)$ and $B(f(a), \delta'+\delta)$. By assumption, these disks are disjoint, and by elementary geometry we get

$$(\delta' - \delta) + (\delta' + \delta) > |f(a)_2| = \theta(fA) \geq 2\delta',$$

which leads to a contradiction. The result follows from this. $\square$

Later on we will need maps that are defined on a narrow neighbourhood of a line but that still possess the essential features of the basic triangle map: they should be $(1 + c\varepsilon)$-bilipschitz but their approximation by isometries should produce an error of the order $\sqrt{\varepsilon}$. The following lemmas show how to construct these maps.

3.3. **Lemma.** *Let $0 \leq \varepsilon \leq 1/10$ and let $a, b \in [0,1]$ be such that $2\varepsilon \leq a \leq b/2$. Then there is a C^2 function $f\colon \mathbf{R} \to \mathbf{R}$ satisfying*

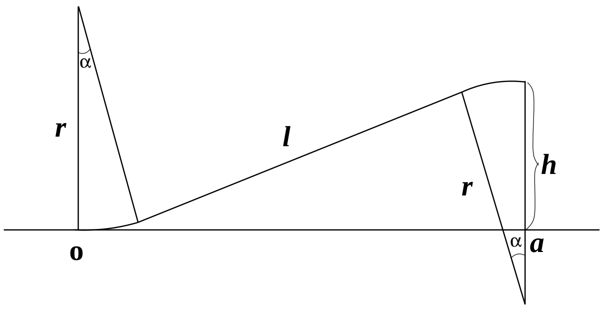
(i) *$f(x) = 0$ for $x \leq 0$ and $x \geq b$;*

(ii) *$f(a) = \varepsilon^{3/2}$;*

(iii) *f is $2\sqrt{\varepsilon}$-Lipschitz;*

(iv) *the curvature K of the graph $y = f(x)$ satisfies $K \leq 1/\sqrt{\varepsilon}$.*

Proof. Let $0 < o < a$ and consider first the interval $[o, a]$. One should think that $o \approx 0$, but we need $o > 0$ for technical reasons. Let $r = \sqrt{\varepsilon}$. The graph $y = f(x)$ consists of two circular arcs and a line segment. The construction is based on the diagram below, where also the notation is indicated.



Part of the graph $y = f(x)$ with $h = \varepsilon\sqrt{\varepsilon}$.

By elementary geometry the variables l and α must satisfy

$$\begin{cases} 2r\sin\alpha + l\cos\alpha = a - o \\ 2r(1 - \cos\alpha) + l\sin\alpha = \varepsilon^{3/2}, \end{cases}$$

and this system has the exact solution

$$l = \sqrt{(a-o)^2 - 4\varepsilon^2 + \varepsilon^3}, \ \alpha = \arcsin(\sqrt{\varepsilon}(2(a-o) + l\varepsilon - 2l)/(l^2 + 4\varepsilon)).$$

The Lipschitz condition requires that $\tan\alpha \leq 2\sqrt{\varepsilon}$. It is geometrically obvious that α is decreasing in a, and thus α attains its maximum at $a = 2\varepsilon$.

By substituting this value and choosing o small enough, we obtain $\alpha \le \arcsin\sqrt{\varepsilon} \le \arctan(2\sqrt{\varepsilon})$.

A similar construction is used on the interval $[a, b]$, and outside $[o, b-o]$ we define $f(x) = 0$. This function satisfies conditions (i)-(iv), but it is only piecewise C^2. However, at the six points where a circular arc is joined either to another arc or to a line segment, we use standard smoothing by clothoids (aka Cornu spirals), in an arbitrarily small neighbourhood of each joint, in such a way that the Lipschitz constant does not change, the curvature stays between the appropriate bounds, and the support of f does not expand outside $[0, b]$; see [Ad, p. 636] for the basic construction. □

Using the following lemma we can construct tubular neighbourhood extensions for mappings of the type $x \mapsto (x, f(x))$.

3.4. **Lemma.** *Let $0 < \varepsilon < 1/10$, let $I \subset \mathbf{R}$ be an interval and let $f\colon I \to \mathbf{R}$ be $\sqrt{\varepsilon}$-Lipschitz and C^2. Define $F\colon I \times [-\delta, \delta] \to \mathbf{R}^2$ by setting*

$$F(x, y) = x + if(x) + y\mathbf{n}(x),$$

where $\mathbf{n}(x)$ is the upper unit normal to the graph $y = f(x)$. Let K be the maximal curvature of $y = f(x)$. If $K\delta \le \varepsilon$, then F is $(1 + 4\varepsilon)$-bilipschitz. Moreover, if $f(x) = 0$ except for a subinterval of length l, then $|F(z) - z| \le \sqrt{\varepsilon}l + \delta$ for every $z \in I \times [-\delta, \delta]$.

Proof. Let $z_i = (x_i, y_i) \in I \times [-\delta, \delta]$, $i = 1, 2$. Note that

$$|y| \le \delta, \quad |f'(x)| \le \sqrt{\varepsilon} \;\text{ and }\; \frac{|f''(x)|}{(1 + f'(x)^2)^{3/2}} \le K$$

for all (x, y).

In complex form we have

$$\mathbf{n}(x) = \frac{1}{\sqrt{1 + f'(x)^2}}(-f'(x) + i).$$

Thus

$$\begin{aligned}
|F(z_1) - F(z_2)|^2 \;=\; & |x_1 - x_2|^2 + \left| \frac{y_1 f'(x_1)}{\sqrt{1 + f'(x_1)^2}} - \frac{y_2 f'(x_2)}{\sqrt{1 + f'(x_2)^2}} \right|^2 \\
& |f(x_1) - f(x_2)|^2 + \left| \frac{y_1}{\sqrt{1 + f'(x_1)^2}} - \frac{y_2}{\sqrt{1 + f'(x_2)^2}} \right|^2 \\
& -2(x_1 - x_2) \left(\frac{y_1 f'(x_1)}{\sqrt{1 + f'(x_1)^2}} - \frac{y_2 f'(x_2)}{\sqrt{1 + f'(x_2)^2}} \right) \\
& +2(f(x_1) - f(x_2)) \left(\frac{y_1}{\sqrt{1 + f'(x_1)^2}} - \frac{y_2}{\sqrt{1 + f'(x_2)^2}} \right).
\end{aligned}$$

Writing the right hand side above as $|x_1 - x_2|^2 + t_1 + t_2 + t_3 + t_4$, where t_4 contains the last two terms, we have to estimate each term. Since F is defined in a convex set, we can use the mean value theorem.

(i) To estimate t_1, let $g(x,y) = yf'(x)/\sqrt{1+f'(x)^2}$. Then

$$|\nabla g|^2 = \frac{y^2 f''(x)^2}{(1+f'(x)^2)^3} + \frac{f'(x)^2}{1+f'(x)^2} \le \delta^2 K^2 + \varepsilon \le 2\varepsilon,$$

which implies that $t_1 \le 2\varepsilon |z_1 - z_2|^2$.

(ii) The upper bound $t_2 \le \varepsilon |x_1 - x_2|^2$ follows from the Lipschitz condition.

(iii) We need both upper and lower bounds for t_3. Applying the mean value theorem for $h(x,y) = y/\sqrt{1+f'(x)^2}$, we get

$$t_3 = \left(-\frac{f'(u)f''(u)v}{(1+f'(u)^2)^{3/2}}(x_1 - x_2) + \frac{1}{\sqrt{1+f'(u)^2}}(y_1 - y_2) \right)^2$$

where (u,v) lies on the segment $[z_1, z_2]$. Using the estimate

$$2\varepsilon^{3/2}|x_1 - x_2||y_1 - y_2| \le 2\varepsilon |x_1 - x_2||y_1 - y_2| \le \varepsilon |x_1 - x_2|^2 + \varepsilon |y_1 - y_2|^2,$$

it follows that

$$\begin{aligned} t_3 &\le \varepsilon K^2 \delta^2 |x_1 - x_2|^2 + \frac{1}{1+f'(u)^2}|y_1 - y_2|^2 + 2\sqrt{\varepsilon} K\delta |x_1 - x_2||y_1 - y_2| \\ &\le \varepsilon^3 |x_1 - x_2|^2 + |y_1 - y_2|^2 + \varepsilon |x_1 - x_2|^2 + \varepsilon |y_1 - y_2|^2 \\ &\le 2\varepsilon |x_1 - x_2|^2 + (1+\varepsilon)|y_1 - y_2|^2. \end{aligned}$$

In the opposite direction, we have

$$\begin{aligned} t_3 &\ge \frac{1}{1+\varepsilon}|y_1 - y_2|^2 - 2\sqrt{\varepsilon} K\delta |x_1 - x_2||y_1 - y_2| \\ &\ge (1-2\varepsilon)|y_1 - y_2|^2 - \varepsilon |x_1 - x_2|^2. \end{aligned}$$

(iv) Rearranging and using the Taylor formula, we have

$$\begin{aligned} t_4 &= \frac{2y_1}{\sqrt{1+f'(x_1)^2}}(f(x_1) - f(x_2) - f'(x_1)(x_1 - x_2)) \\ &\quad + \frac{2y_2}{\sqrt{1+f'(x_2)^2}}(f'(x_2)(x_1 - x_2) - f(x_1) + f(x_2)) \\ &= \left(\frac{y_1 f''(\xi_1)}{\sqrt{1+f'(x_1)^2}} - \frac{y_2 f''(\xi_2)}{\sqrt{1+f'(x_2)^2}} \right) |x_1 - x_2|^2, \end{aligned}$$

where $\xi_1, \xi_2 \in [x_1, x_2]$. Since $|f''(\xi)| \le K(1+\varepsilon)^{3/2}$, this implies that

$$|t_4| \le 2K\delta(1+\varepsilon)^{3/2}|x_1 - x_2|^2 \le 3\varepsilon |x_1 - x_2|^2.$$

Using these estimates we obtain

$$\begin{aligned}|F(z_1)-F(z_2)|^2 &\le |x_1-x_2|^2+2\varepsilon|x_1-x_2|^2+2\varepsilon|y_1-y_2|^2+\varepsilon|x_1-x_2|^2\\ &\quad +2\varepsilon|x_1-x_2|^2+(1+\varepsilon)|y_1-y_2|^2+3\varepsilon|x_1-x_2|^2\\ &= (1+8\varepsilon)|x_1-x_2|^2+(1+3\varepsilon)|y_1-y_2|^2,\end{aligned}$$

so that $|F(z_1)-F(z_2)| \le \sqrt{1+8\varepsilon}\,|z_1-z_2| \le (1+4\varepsilon)|z_1-z_2|$.

For the lower bound, we discard irrelevant positive terms and get

$$\begin{aligned}|F(z_1)-F(z_2)|^2 &\ge |x_1-x_2|^2+t_3-|t_4|\\ &\ge (1-4\varepsilon)|x_1-x_2|^2+(1-2\varepsilon)|y_1-y_2|^2\\ &\ge (1-4\varepsilon)|z_1-z_2|^2.\end{aligned}$$

This implies that $|F(z_1)-F(z_2)| \ge \sqrt{1-4\varepsilon}\,|z_1-z_2| \ge |z_1-z_2|/(1+4\varepsilon)$.

The proof for the bilipschitz condition is now complete, and the last inequality is obvious. □

3.5. **Lemma.** *Let $A \subset \mathbf{R}^n$ and let $\varepsilon \le 1/10$. Suppose that $a \in A$, $r>0$ and let $f\colon A \to \mathbf{R}^n$ be $(1+\varepsilon)$-bilipschitz such that $|f(z)-z| \le \varepsilon r$ whenever $|z-a| \le r/2$ and $f(z)=z$ for $|z-a| \ge r/2$. Define $F\colon A \cup (\mathbf{R}^n \setminus B(a,r)) \to \mathbf{R}^n$ by setting*

$$F(z)=\begin{cases} f(z) & \text{for } z \in A,\\ z & \text{for } |z-a| \ge r.\end{cases}$$

Then F is $(1+3\varepsilon)$-bilipschitz.

Proof. Let $z_1 \in A \cap B(a,r/2)$ and $|z_2-a| \ge r$. Then $|z_1-z_2| \ge r/2$, which implies that

$$\begin{aligned}|F(z_1)-F(z_2)| &= |f(z_1)-z_2| \le |f(z_1)-z_1|+|z_1-z_2| \le \varepsilon r+|z_1-z_2|\\ &\le (1+2\varepsilon)|z_1-z_2|.\end{aligned}$$

In the opposite direction, we have

$$\begin{aligned}|F(z_1)-F(z_2)| &= |f(z_1)-z_2| \ge |z_1-z_2|-|f(z_1)-z_1| \ge |z_1-z_2|-\varepsilon r\\ &\ge (1-2\varepsilon)|z_1-z_2| \ge |z_1-z_2|/(1+3\varepsilon),\end{aligned}$$

since $\varepsilon \le 1/10$.

All other cases for z_1, z_2 are trivial, and the proof is complete. □

Finally, we need an estimate on the distortion of angles under bilipschitz maps.

3.6. **Lemma.** *Let $1<t\le 2$ and let $f\colon \{0,1,t\} \to \mathbf{R}^n$ be $(1+\varepsilon)$-bilipschitz with $\varepsilon \le 1/100$. Let $A=f(0), B=f(1), C=f(t)$ and $\alpha = \angle BAC$. Then $\alpha \le 2.1\sqrt{\varepsilon}$.*

Proof. Consider the triangle with vertices A, B, C. Elementary geometrical considerations show that α is maximal in the case $AB = 1 + \varepsilon$, $BC = (t-1)(1+\varepsilon)$, and $AC = t/(1+\varepsilon)$. Using trigonometry and Taylor approximation we obtain

$$\sin\alpha \le 2\sqrt{(t-1)\varepsilon} \le 2\sqrt{\varepsilon} \le 0.2.$$

Furthermore, for these values we have $\alpha \le 1.01 \sin\alpha \le 2.1\sqrt{\varepsilon}$, and the proof is complete. $\square$

4 Main proofs

We use triangle maps to prove the following theorem, which constitutes the first part of our main result.

4.1. **Theorem.** *Let $\lambda \ge 1$, $c > (30\lambda)^8$, and let $A \subset \mathbf{R}^2$ be λ-relatively connected but not c-sturdy. Then for $1/\sqrt{c} \le \varepsilon \le 1/(30\lambda)^4$ there is a $(1 + 48\varepsilon)$-BL map $f\colon A \to \mathbf{R}^2$ with the following property: there are $a \in A$ and $r > 0$ such that*

$$||T - f||_{A(a,r)} \ge (r/6000\lambda^3)\sqrt{\varepsilon}$$

for all isometries $T\colon \mathbf{R}^2 \to \mathbf{R}^2$.

Proof. Since A is not $(1/\varepsilon^2)$-sturdy, there are two possibilities.

Case 1: Condition 2.2(1) is not satisfied. In this case there are $a \in A$ and $r > 0$ such that $A \not\subset B(a,r)$, $s(a) \le \varepsilon^2 r$ and $\theta(A(a,r)) \le 2\varepsilon^2 r$. By scaling, we may assume that $a = 0$, $r = 1$, and then $A \not\subset B(1) = B(0,1)$, $s(0) \le \varepsilon^2$, $\theta(A(0,1)) \le 2\varepsilon^2$. Furthermore, we may assume that $A(0,1)$ is contained in the $2\varepsilon^2$-neighbourhood of $\mathbf{R} \subset \mathbf{R}^2$.

We apply [TV, 4.11(2)] with $c = 4\lambda$ to find points $u, v \in A$ as follows. Since $s(0) \le \varepsilon^2 < \varepsilon$, the set $A(0, 2\varepsilon)$ contains at least two points. Also $A \not\subset B(1)$, and thus there is a point $u \in A \cap B(8\lambda\varepsilon) \setminus B(2\varepsilon)$. Similarly, since $80\lambda^2\varepsilon \le 1$, there is $v \in A \cap B(80\lambda^2\varepsilon) \setminus B(20\lambda\varepsilon)$. There are six possibilities for the order of the points $0, u_1, v_1$ and of these only two are essentially different; we consider the case where $0 < u_1 < v_1 < 1$, the other cases being similar. However, the constants appearing below apply for all cases and may thus seem unnecessarily large for this special case.

We construct a bilipschitz map $f\colon A \to \mathbf{R}^2$ as follows:

- Apply Lemma 3.3 with substitutions $0 \mapsto 0$, $a \mapsto u_1$, $b \mapsto v_1$. This gives a $2\sqrt{\varepsilon}$-Lipschitz map $f_1\colon \mathbf{R} \to \mathbf{R}$ such that $f_1(x) = 0$ if $x \notin [0, v_1]$, $f_1(u_1) = \varepsilon^{3/2}$, and $K \le 1/\sqrt{\varepsilon}$.

- Apply Lemma 3.4 with $\varepsilon \mapsto 4\varepsilon$, $\delta \mapsto 2\varepsilon^2$, $I \mapsto \mathbf{R}$ and $f \mapsto f_1$. Then $K\delta \le 2\varepsilon^{3/2} \le 4\varepsilon$, and the resulting map $F\colon \mathbf{R} \times [-\delta, \delta] \to \mathbf{R}^2$ is $(1 + 16\varepsilon)$-BL. Also, we have $l \le 160\lambda^2\varepsilon$ and therefore

  $$|F(z) - z| \le 160\lambda^2\varepsilon^{3/2} + 2\varepsilon^2 < \varepsilon$$

  for all z.

- We extend the definition of F outside $B(1)$ by $F(z) = z$. Substitute $\varepsilon \mapsto 16\varepsilon$ and $r = 1/2$ in Lemma 3.5. Since $160\lambda^2\varepsilon \le r/2$, we have $|F(z) - z| \le \varepsilon \le 16\varepsilon r$ for $|z| \le r/2$ and $F(z) = z$ for $|z| \ge r/2$. It follows that F is $(1 + 48\varepsilon)$-BL.

- The domain of definition for F contains the set A and by restriction we get the required $(1 + 48\varepsilon)$-BL map $f\colon A \to \mathbf{R}^2$.

It remains to show that f cannot be well approximated by isometries. For this it suffices to consider the restriction $f \mid \{0, u, v\}$ in the disk $B = \bar{B}(0, r_1)$, where $r_1 = 160\lambda^2\varepsilon$. Let $A' = \{0, u, v\}$ and let $h\colon \mathbf{R}^2 \to \mathbf{R}^2$ be a similarity such that $h(0) = -1, h(v) = 1$ and let $g = hfh^{-1}\colon hA' \to hfA'$. Since $f(0) = 0, f(v) = v$, Lemma 3.2 can be applied to g. The similarity ratio t of h satisfies $1/80\lambda^2\varepsilon \le t \le 1/10\lambda\varepsilon$, and thus $\theta(hA') \le 2\varepsilon^2/10\lambda\varepsilon = \varepsilon/5\lambda$ and $\theta(ghA')) \ge (\varepsilon^{3/2} - 2\varepsilon^2)/160\lambda^2\varepsilon > \sqrt{\varepsilon}/162\lambda^2$. Thus the error of approximation of g by an isometry is at least

$$\sqrt{\varepsilon}/324\lambda^2 - \varepsilon/10\lambda \ge \sqrt{\varepsilon}/340\lambda^2,$$

and therefore

$$||T - f||_{A(0,r_1)} \ge 10\lambda\varepsilon(\sqrt{\varepsilon}/340\lambda^2) = \varepsilon^{3/2}/34\lambda = \frac{r_1}{6000\lambda^3}\sqrt{\varepsilon}$$

for all isometries T. This completes the proof for Case 1.

Case 2: Condition 2.2(2) is not satisfied. This implies that A is bounded and $\theta(A) < \varepsilon^2 d(A)$. Using λ-relative connectedness, we can find points $a, b, c \in A$ such that $1 \le |a - b|/|b - c| \le \lambda$. Using Lemmas 3.3 and 3.4, we can construct a map $f\colon A \to \mathbf{R}^2$ that by 3.2 contradicts the requirements. The details are similar to Case 1 and are omitted.

This completes the proof. □

4.2. **Theorem.** *Let $\lambda \ge 1000$, let $A \subset \mathbf{R}^n$ be a closed set that is not λ-relatively connected. Then there is $\varepsilon \le 2/(\lambda - 2)$ and a $(1+\varepsilon)$-bilipschitz map $f\colon A \to \mathbf{R}^n$ with the following property: If $F\colon \mathbf{R}^n \to \mathbf{R}^n$ is a $(1 + \delta)$-bilipschitz extension of f, then*

$$\delta \ge 1/20 \ln^2 \varepsilon.$$

Proof. We use the concept of upper sets from [TV, 4.9]. Since A is not λ-relatively connected, the upper set $\tilde{A}$ consists of more than one $\ln \lambda$-component. Let γ be a $\ln \lambda$-component that is not the greatest element; see [TV, 3.2]. By [TV, 3.4(11) and 3.4(14)] the set $\pi\gamma$ is compact, and by [TV, 3.4(12)] we have $A \cap B(\pi\gamma, (\lambda - 1)d(\pi\gamma)) = \pi\gamma$. Choose $a, b \in \pi\gamma$ such that $|a - b| = d(\pi\gamma)$ and then $z \in A \setminus \pi\gamma$ such that $d(z, \pi\gamma)$ is minimal. We may assume that $|b - z| \le |a - z|$, and hence $\angle abz \ge \pi/3$. Using suitable similarities, we may assume that $b = 0$, $|a-b| = 1$ and $z = te_1$ with $t \ge \lambda - 1$.

We choose $\varepsilon = 2/(t - 1) \le 2/(\lambda - 2) < 0.01$ and construct a $(1 + \varepsilon)$-bilipschitz map $f\colon A \to \mathbf{R}^n$ as follows. Let $f \mid (A \setminus B(0, 1)) = \text{id}$, and let f

rotate $\bar{B}(0,1)$ so that $f(0)=0$ and $f(a)=e_1$. To calculate the bilipschitz constant L of f, we note that the worst case arises from $a=-e_1,\ f(a)=e_1$; this seems geometrically obvious and can be proved by solving an elementary extremal value problem. Thus

$$L \le \frac{t+1}{t-1} = 1 + \frac{2}{t-1} = 1+\varepsilon.$$

Suppose now that f can be extended to a $(1+\delta)$-bilipschitz map $F\colon \mathbf{R}^n \to \mathbf{R}^n$. We apply Lemma 3.6 to the map $F^{-1} \mid \{0, e_1, 2e_1, 4e_1, \ldots, 2^N e_1, z\}$, where $N = \lfloor \log_2 t \rfloor$. Let $a_i = F^{-1}(2^i e_1)$ for $i = 0,1,2,\ldots,N$ and $a_{N+1}=z$. The lemma implies that $\angle a_i 0 a_{i+1} \le 2.1\sqrt{\delta}$, and therefore

$$\begin{aligned} 1 \quad &\le \quad \frac{\pi}{3} \le \angle a0z \le \sum_{i=0}^{N} \angle a_i 0 a_{i+1} \le 2.1\sqrt{\delta}(N+1) \le 2.1\sqrt{\delta}(\log_2 t + 1) \\ &\le \quad 2.1\sqrt{\delta}(1.5\ln t + 1) \le 3.15\sqrt{\delta}\ln(2t). \end{aligned}$$

Since $t = 2/\varepsilon + 1 \le 2.1/\varepsilon$, we obtain

$$\delta \ge \frac{1}{10\ln^2(4.2/\varepsilon)} \ge \frac{1}{20\ln^2\varepsilon}.$$

This completes the proof. □

4.3. *Proof of Theorem* 1.3. The implication (1) ⇒ (2) was the main result of [ATV2].

For the converse part, suppose that A has the (C,δ)-linear BLEP. Let $\lambda \ge 2/\delta + 2$ so that $\varepsilon = 2/(\lambda-2) \le \delta$ and suppose that A is not λ-RC. We may assume that $\lambda \ge 1000$. Let $f\colon A \to \mathbf{R}^2$ be the $(1+\varepsilon)$-bilipschitz map given by Theorem 4.2. Since A has the (C,δ)-linear BLEP, we have

$$C\varepsilon \ge \frac{1}{20\ln^2\varepsilon}.$$

This leads to a contradiction unless $\varepsilon \ge \varepsilon(C) > 0$, which is equivalent to $\lambda \le \lambda(C,\delta) < \infty$.

It follows that A is λ-relatively connected with the above bound.

By Theorem 2.6 the set A has the (C_1,δ)-IAP with $C_1 = C_1(C)$. Supposing that A is not c-sturdy, we must find an upper bound for c, and may thus assume that $c > (30\lambda)^8 \vee 48^2/\delta^2$. Let $\varepsilon = 1/\sqrt{c}$ so that $48\varepsilon \le \delta$. Applying Theorem 4.1, we obtain a $(1+48\varepsilon)$-bilipschitz map $f\colon A \to \mathbf{R}^2$ such that

$$||T-f||_{A(a,r)} \ge (r/6000\lambda^3)\sqrt{\varepsilon}$$

for some $a \in A,\ r>0$ and for all isometries T. The IAP of A thus leads to the estimate

$$C_1 \cdot 48\varepsilon \ge \sqrt{\varepsilon}/6000\lambda^3,$$

which is a contradiction unless $\varepsilon \ge \varepsilon(C,\lambda)$, or equivalently, unless $c \le c(C,\delta)$.

It follows that A is c-sturdy with the above bound, and the proof of the main theorem is complete. □

4.4. *Remark.* The first part of the above proof can be easily modified to show that a planar set A having the φ-BLEP is relatively connected if

$$\lim_{\varepsilon \to 0} \varphi(\varepsilon) \ln^2 \varepsilon = 0.$$

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