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Abstract: *We consider a stabilized finite element formulation for the Reissner-Mindlin plate bending model. The method, introduced in [18] uses standard bases functions for the deflection and rotation vector. Due to the stabilization the conditioning of the method is such that multigrid algorithms can readily been used. In the paper we first prove some error estimates needed for multigrid methods. Then we prove the a simple multigrid method has optimal complexity. Numerical results are also give.*

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1 Introduction

In this paper we will consider a family of finite element methods for the Reissner-Mindlin plate model, which was introduced in [18] and further analyzed in [12]. The origin of the method is in a "Galerkin-Least-Squares" method in introduced by Hughes and Franca [9]. In this paper the shear force was discretized independtly and locally condensed. In our paper [18] we showed that this step is unnecessary; it is possible to formulate the stabilized method directly in the displacement variables, the deflection and the rotation vector. For lowest order methods this was firs done by Pitkäranta [15]. This stabilized method has two advantages compared to more traditional methods. First, standard basis functions can used, i.e. no "bubble-function" are needed. Second, the condition number of the stiffness matrix is optimal which open the way for using direct multigrid and other iterative solvers.

So far, there has been relatively few works on multigrid methods for Reissner-Mindlin plate methods. The first are the work of Peisker, Rust and Stein [14], in which Pitkärantas method is analyzed. In a subsequent paper by Peisker [13] the Hughes-Franca method is analyzed. This work also contain an algorithm in which the shear force is kept as an independent unknown. This method has the disadvantage that the stiffness matrix of mixed form, not symmetric and positively definite as the engineering community is used to. The same holds for the multigrid methods analyzed in the paper by Arnold, Falk and Winter [1] and Brenner [7].

2 The plate model

In order to analyze the method in connection with multigrid algorithms we consider the plate model with a general loading. Let $\Omega \subset \mathbb{R}^2$ be the midsurface of the plate and suppose that the plate is clamped along the boundary Γ. The variational formulation of Reissner-Mindlin model is: find the deflection $w \in H_0^1(\Omega)$ and the rotation vector $\boldsymbol{\beta} = (\beta_x, \beta_y) \in [H_0^1(\Omega)]^2$ such that

$$a(\boldsymbol{\beta},\boldsymbol{\eta}) + t^{-2}(\nabla w - \boldsymbol{\beta}, \nabla v - \boldsymbol{\eta}) = (f, v) + (\boldsymbol{f}, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2. \tag{2.1}$$

Here t is the thickness of the plate and f is the transverse load acting on Ω. The bilinear form a represents bending energy and is defined as

$$a(\boldsymbol{\beta},\boldsymbol{\eta}) = \frac{1}{6}\Big\{(\boldsymbol{\varepsilon}(\boldsymbol{\beta}), \boldsymbol{\varepsilon}(\boldsymbol{\eta})) + \frac{\nu}{1-\nu}(\operatorname{div}\boldsymbol{\beta}, \operatorname{div}\boldsymbol{\eta})\Big\}, \tag{2.2}$$

where ν is the Poisson ratio, $\boldsymbol{\varepsilon}(\cdot)$ is the small strain tensor and "div" stands for the divergence, viz.

$$\boldsymbol{\varepsilon}(\boldsymbol{\beta}) = \frac{1}{2}\Big\{\nabla\boldsymbol{\beta} + (\nabla\boldsymbol{\beta})^T\Big\}, \tag{2.3}$$

$$\operatorname{div}\boldsymbol{\beta} = \frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y}. \tag{2.4}$$

The loading in the shear equilibrium equation (se below) will be needed for the multigrid analysis. Here and below we for $D \subset \mathbb{R}^2$ define the Sobolev spaces $H^s(D)$, with $s \geq 0$, in the usual way, i.e. first for integral values s and then for nonintegral values by interpolation, cf. [10]. The norms and seminorms will be denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. The L_2-inner products in $L_2(D)$, $[L_2(D)]^2$ or $[L_2(D)]^{2\times 2}$ are denoted by $(\cdot,\cdot)_D$. The subscript D will be dropped when $D = \Omega$.

By taking the scaled shear force

$$\boldsymbol{q} = t^{-2}(\nabla w - \boldsymbol{\beta}) \tag{2.5}$$

as an independent unknown in the space $[L_2(\Omega)]^2$ one gets the following mixed formulation: find $(w, \boldsymbol{\beta}, \boldsymbol{q}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2 \times [L^2(\Omega)]^2$ such that

$$\begin{aligned} a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) &= (f, v) + (\boldsymbol{f}, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2, \\ (\nabla w - \boldsymbol{\beta}, \boldsymbol{s}) - t^2(\boldsymbol{q}, \boldsymbol{s}) &= 0 \qquad \forall \boldsymbol{s} \in [L^2(\Omega)]^2. \end{aligned} \tag{2.6}$$

The distributional differential equations of this system are obtained by integrating by parts:

$$\begin{aligned} \boldsymbol{L\beta} + \boldsymbol{q} &= \boldsymbol{f} \quad \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{q} &= f \quad \text{in } \Omega, \\ -t^2\boldsymbol{q} + \nabla w - \boldsymbol{\beta} &= \boldsymbol{0} \quad \text{in } \Omega, \\ w = 0,\ \boldsymbol{\beta} &= \boldsymbol{0} \quad \text{on } \partial\Omega. \end{aligned} \tag{2.7}$$

Here the differential operator $\boldsymbol{L}$ is defined from

$$\boldsymbol{L\eta} = \frac{1}{6}\mathbf{div}\left\{\boldsymbol{\varepsilon}(\boldsymbol{\eta}) + \frac{\nu}{1-\nu}\operatorname{div}\boldsymbol{\eta}\boldsymbol{I}\right\} \tag{2.8}$$

and $\boldsymbol{m}$ is the moment tensor

$$\boldsymbol{m} = \frac{1}{6}\left\{\boldsymbol{\varepsilon}(\boldsymbol{\beta}) + \frac{\nu}{1-\nu}\operatorname{div}\boldsymbol{\beta}\boldsymbol{I}\right\}. \tag{2.9}$$

It holds

$$\boldsymbol{L\beta} = \mathbf{div}\,\boldsymbol{m}, \tag{2.10}$$

where we used the notation **div** for the divergence operator applied to a second order tensor:

$$\mathbf{div}\,\boldsymbol{m} = \left(\frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{xy}}{\partial y}, \frac{\partial m_{yx}}{\partial x} + \frac{\partial m_{yy}}{\partial y}\right). \tag{2.11}$$

The first two equations in (2.7) above are the local equilibrium equations between the moment, shear force and load. The third equation represents the constitutive relation between the shear strain and shear force.

In the limit $t \to 0$ the solution $(w, \boldsymbol{\beta}) = (w_t, \boldsymbol{\beta}_t)$ of the Reissner–Mindlin equations converges to the Kirchhoff solution with

$$\boldsymbol{\beta}_0 = \nabla w_0. \tag{2.12}$$

The limit solution w_0 satisfies the biharmonic equation in the domain Ω and only two boundary conditions on each part of the boundary, cf. [2]. This singularity gives rise to the boundary layers in the solution which complicates the convergence analysis of the methods.

Throughout the rest of the paper we will assume the domain Ω to be convex. The following regularity estimate is proved in [11].

Theorem 2.1. *Let Ω be a convex polygonal domain. Denote by $(w, \boldsymbol{\beta}, \boldsymbol{q})$ the Reissner–Mindlin solution for the clamped plate and let $w = w_0 + w_r$, where w_0 is the deflection obtained from the Kirchhoff model. With $f \in H^{-1}(\Omega)$, $tf \in L^2(\Omega)$ and $\boldsymbol{f} \in [L^2(\Omega)]^2$, it then holds*

$$\|w_0\|_3 + t^{-1}\|w_r\|_2 + \|\boldsymbol{\beta}\|_2 + \|\boldsymbol{q}\|_0 + t\|\boldsymbol{q}\|_1 \leq C(\|f\|_{-1} + t\|f\|_0 + \|\boldsymbol{f}\|_0). \tag{2.13}$$

In our analysis we will utilize the following t-dependent norms.

$$\|(v, \boldsymbol{\eta})\|_{1,t} = \|\boldsymbol{\eta}\|_0 + \inf_{v=v^0+v^r} \Big\{\|v^0\|_1 + t^{-1}\|v^r\|_0\Big\}, \tag{2.14}$$

$$\|(v, \boldsymbol{\eta})\|_{3,t} = \|\boldsymbol{\eta}\|_2 + \inf_{v=v^0+v^r} \Big\{\|v^0\|_3 + t^{-1}\|v^r\|_2\Big\} \tag{2.15}$$

and

$$\|(f, \boldsymbol{f})\|_{-1,t} = \|\boldsymbol{f}\|_0 + \|f\|_{-1} + t\|f\|_0. \tag{2.16}$$

Using these norms these the regularity estimate (2.13) gives

$$\|(w, \boldsymbol{\beta})\|_{3,t} \leq C\|(f, \boldsymbol{f})\|_{-1,t}. \tag{2.17}$$

Furthermore, the norms $\|(\cdot,\cdot)\|_{-1,t}$ and $\|(\cdot,\cdot)\|_{1,t}$ are dual. The following theorem (cf. the duality of the K- and J-functional in the theory of interpolation spaces [3]), where $\approx$ denotes equivalence of norms, is proved in Schöberl [16, 17].

Theorem 2.2.

$$\|(f, \boldsymbol{f})\|_{-1,t} \approx \sup_{(v,\boldsymbol{\eta})} \frac{(f, v) + (\boldsymbol{f}, \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}}. \tag{2.18}$$

2.1 Finite element subspaces

We will use standard notation from finite element analysis and we will assume that the domain Ω is polygonal and let $\mathcal{C}_h$ be the partitioning of $\bar{\Omega}$ into triangles or convex quadrilaterals satisfying the usual compatibility conditions. For generality we allow a mesh consisting of both triangles and quadrilaterals. As usual h_K denotes the diameter of $K \in \mathcal{C}_h$ and h stands for the global mesh parameter $h = \max_{K\in\mathcal{C}_h} h_K$. We define

$$R_m(K) = \begin{cases} P_m(K) \ \text{ when } K \text{ is a triangle,} \\ Q_m(K) \text{ or } Q'_m(K) \text{ when } K \text{ is a quadrilateral.} \end{cases} \tag{2.19}$$

The finite element subspaces for the deflection and the rotation is then defined as follows

$$
\begin{aligned}
W_h &= \{v \in H_0^1(\Omega) \mid v_{|K} \in R_{k+1}(K),\ \forall K \in \mathcal{C}_h\}, && (2.20)\\
\boldsymbol{V}_h &= \{\boldsymbol{\eta} \in [H_0^1(\Omega)]^2 \mid \boldsymbol{\eta}_{|K} \in R_k(K),\ \forall K \in \mathcal{C}_h\}, && (2.21)
\end{aligned}
$$

with the polynomial degree $k \geq 1$.

The finite element method is then defined as follows.

Method 2.1. ([*18*, *12*]) *Given the loading* $(f, \boldsymbol{f}) \in L^2(\Omega) \times [L^2(\Omega)]^2$, *find* $(w_h, \boldsymbol{\beta}_h) \in W_h \times \boldsymbol{V}_h$ *such that*

$$
\mathcal{A}_h(w_h, \boldsymbol{\beta}_h; v, \boldsymbol{\eta}) = \mathcal{F}_h(v, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{2.22}
$$

with the bilinear and linear forms defined as

$$
\begin{aligned}
&\mathcal{A}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta})\\
&\quad = a(\boldsymbol{\phi}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (\boldsymbol{L}\, \boldsymbol{\phi}, \boldsymbol{L}\, \boldsymbol{\eta})_K && (2.23)\\
&\qquad + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} (\nabla z - \boldsymbol{\phi} - \alpha h_K^2 \boldsymbol{L}\, \boldsymbol{\phi}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\, \boldsymbol{\eta})_K.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_h(v, \boldsymbol{\eta}) &= (f, v) + (\boldsymbol{f}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{f}, \boldsymbol{L}\, \boldsymbol{\eta})_K && (2.24)\\
&\quad - \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{f}, \nabla v - \boldsymbol{\eta})_K.
\end{aligned}
$$

Here and in throughout the paper α *is a positive parameter lying in the range* $0 < \alpha < C_I$, *where* C_I *is the constant in the following inverse inequality*

$$
C_I \sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{L}\, \boldsymbol{\phi}\|_{0,K}^2 \leq a(\boldsymbol{\phi}, \boldsymbol{\phi}) \qquad \forall \boldsymbol{\phi} \in \boldsymbol{V}_h.
$$

From the solution $(w_h, \boldsymbol{\beta}_h)$ *we then calculate the approximation for the shear by*

$$
\boldsymbol{q}_{h|K} = (t^2 + \alpha h_K^2)^{-1} \big(\nabla w_h - \boldsymbol{\beta}_h + \alpha h_K^2 (\boldsymbol{f} - \boldsymbol{L}\, \boldsymbol{\beta}_h)\big)_{|K} \qquad \forall K \in \mathcal{C}_h. \tag{2.25}
$$

Note that from (2.7) *we see that the exact shear satisfies*

$$
\boldsymbol{q}_{|K} = (t^2 + \alpha h_K^2)^{-1} \big(\nabla w - \boldsymbol{\beta} + \alpha h_K^2 (\boldsymbol{f} - \boldsymbol{L}\, \boldsymbol{\beta})\big)_{|K} \qquad \forall K \in \mathcal{C}_h. \tag{2.26}
$$

Remark 2.1. *For triangular elements with* $k = 1$ *it holds* $\boldsymbol{L}\, \boldsymbol{\phi} = \boldsymbol{0}$, $\forall \boldsymbol{\phi} \in \boldsymbol{V}_h$, *and the bilinear form is simply*

$$
\mathcal{A}_h(z, \boldsymbol{\phi}; v, \boldsymbol{\eta}) = a(\boldsymbol{\phi}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} (\nabla z - \boldsymbol{\phi}, \nabla v - \boldsymbol{\eta})_K \tag{2.27}
$$

and, furthermore, there is no upper limit for the parameter α. *This has been first prosed by Fried and Yang [8] and analyzed by Pitkäranta [15]. This formulation can be used for quadrilaterals as well, when* $k = 1$.

In our previous works [18, 12] we have analyzed the method for $\boldsymbol{f} = \mathbf{0}$. Hence, we will here prove the consistency for a general loading.

Theorem 2.3. *The solution* $(w, \boldsymbol{\beta})$ *to* (2.7) *satisfies the equation*

$$\mathcal{A}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) = \mathcal{F}_h(v, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h. \tag{2.28}$$

Proof. Recalling the first equation in (2.7), the expression (2.26), and the variational form (2.6), we get

$$\begin{aligned}
&\mathcal{A}_h(w, \boldsymbol{\beta}; v, \boldsymbol{\eta}) \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (\boldsymbol{L}\,\boldsymbol{\beta}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&\qquad + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} (\nabla w - \boldsymbol{\beta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\beta}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (\boldsymbol{q} - \boldsymbol{f}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&\qquad + \sum_{K \in \mathcal{C}_h} (\boldsymbol{q} - \alpha h_K^2 (t^2 + \alpha h_K^2)^{-1} \boldsymbol{f}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (\boldsymbol{q}, \boldsymbol{L}\,\boldsymbol{\eta})_K - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (\boldsymbol{f}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&\qquad + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (\boldsymbol{q}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&\qquad - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{f}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&= a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\boldsymbol{q}, \nabla v - \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (\boldsymbol{f}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&\qquad - \sum_{K \in \mathcal{C}_h} \alpha_K h^2 (t^2 + \alpha h_K^2)^{-1} (, \nabla \mathbf{v} - \boldsymbol{\eta} - \alpha \mathbf{h}_{\mathbf{K}}^{\mathbf{2}} \boldsymbol{L}\,\boldsymbol{\eta})_{\mathbf{K}} \\
&= (f, v) + (\boldsymbol{f}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (\boldsymbol{f}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&\qquad - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{f}, \nabla v - \boldsymbol{\eta} - \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&= (f, v) + (\boldsymbol{f}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{f}, \nabla v - \boldsymbol{\eta})_K \\
&\qquad - \sum_{K \in \mathcal{C}_h} (1 - \alpha h_K^2 (t^2 + \alpha h_K^2)^{-1}) (\boldsymbol{f}, \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K. \\
&= (f, v) + (\boldsymbol{f}, \boldsymbol{\eta}) - \sum_{K \in \mathcal{C}_h} \alpha h_K^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{f}, \nabla v - \boldsymbol{\eta})_K \\
&\qquad - \sum_{K \in \mathcal{C}_h} t^2 (t^2 + \alpha h_K^2)^{-1}) (\boldsymbol{f}, \alpha h_K^2 \boldsymbol{L}\,\boldsymbol{\eta})_K \\
&= \mathcal{F}_h(v, \boldsymbol{\eta}).
\end{aligned}$$

The following norms are the natural for the stability and error analysis.

Definition 2.1. *For $(v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2$ we define*

$$|||(v, \boldsymbol{\eta})|||_h^2 = \|v\|_1^2 + \|\boldsymbol{\eta}\|_1^2 + \sum_{K \in \mathcal{C}_h} (t^2 + h_K^2)^{-1} \|\nabla v - \boldsymbol{\eta}\|_{0,K}^2, \tag{2.29}$$

and for $\boldsymbol{r} \in [L^2(\Omega)]^2$

$$\|\boldsymbol{r}\|_h = (\sum_{K \in Ch} (t^2 + h_K^2) \|\boldsymbol{r}\|_{0,K}^2)^{1/2}. \tag{2.30}$$

The stability of the method is immediate (cf. [12]).

Theorem 2.4. *There is a positive constant C such that*

$$\mathcal{A}_h(v, \boldsymbol{\eta}; v, \boldsymbol{\eta}) \geq C |||(v, \boldsymbol{\eta})|||_h^2 \qquad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h.$$

From the stability, consistency and regularity result the following error estimate is is proved in [12] (which also contains some refined estimates).

Theorem 2.5. *For the solution $(w_h, \boldsymbol{\beta}_h)$ of (2.22) it holds*

$$|||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h + \|\boldsymbol{q} - \boldsymbol{q}_h\|_h \leq Ch \|(f, \boldsymbol{f})\|_{-1,t}. \tag{2.31}$$

For the multigrid analysis we additionally need estimates for the discrete solution with an inconsistent right hand side given by the following method

Method 2.2. *Given the loading $(f, \boldsymbol{f}) \in L^2(\Omega) \times [L^2(\Omega)]^2$, find $(w_h^*, \boldsymbol{\beta}_h^*) \in W_h \times \boldsymbol{V}_h$ such that*

$$\mathcal{A}_h(w_h^*, \boldsymbol{\beta}_h^*; v, \boldsymbol{\eta}) = (f, v) + (\boldsymbol{f}, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h. \tag{2.32}$$

For this one readily obtain the estimate.

Theorem 2.6. *It holds*

$$|||(w - w_h^*, \boldsymbol{\beta} - \boldsymbol{\beta}_h^*)|||_h \leq Ch \|(f, \boldsymbol{f})\|_{-1,t}. \tag{2.33}$$

We will also need the following estimates.

Theorem 2.7. *It holds*

$$\|(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_{1,t} \leq Ch^2 \|(f, \boldsymbol{f})\|_{-1,t}. \tag{2.34}$$

and

$$\|(w - w_h^*, \boldsymbol{\beta} - \boldsymbol{\beta}_h^*)\|_{1,t} \leq Ch^2 \|(f, \boldsymbol{f})\|_{-1,t} \tag{2.35}$$

Proof. Step 1. Let For $(l, \boldsymbol{l})$ given, let $(z, \boldsymbol{\theta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2$ be the solution to the problem

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) + t^{-2}(\nabla z - \boldsymbol{\theta}, \nabla v - \boldsymbol{\eta}) = (l, v) + (\boldsymbol{l}, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2. \tag{2.36}$$

Denoting $\boldsymbol{r} = t^{-2}(\nabla z - \boldsymbol{\theta})$, the regularity estimate (2.13) gives

$$\|(z, \boldsymbol{\theta})\|_{3,t} + \|\boldsymbol{r}\|_0 \leq C \|(l, \boldsymbol{l})\|_{-1,t}. \tag{2.37}$$

Note also that it holds

$$\boldsymbol{r}_{|K} = (t^2 + \alpha h_K^2)^{-1}\big(\nabla z - \boldsymbol{\theta} + \alpha h_K^2(\boldsymbol{l} - \boldsymbol{L}\,\boldsymbol{\theta})\big)_{|K} \qquad \forall K \in \mathcal{C}_h. \tag{2.38}$$

As in Theorem 2.3 we now have

$$\mathcal{A}_h(z, \boldsymbol{\theta}; v, \boldsymbol{\eta}) = \mathcal{L}_h(v, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{2.39}$$

with

$$\begin{aligned}\mathcal{L}_h(v, \boldsymbol{\eta}) \;\; &= \;\; (l, v) + (\boldsymbol{l}, \boldsymbol{\eta}) - \sum_{K\in\mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{l}, \boldsymbol{L}\,\boldsymbol{\eta})_K \\ &\quad - \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{l}, \nabla v - \boldsymbol{\eta})_K. \end{aligned}\tag{2.40}$$

Step 2. Next, we let $\tilde{z} \in W_h$ and $\tilde{\boldsymbol{\theta}} \in \boldsymbol{V}_h$ be the the solution of

$$\mathcal{A}_h(\tilde{z}, \tilde{\boldsymbol{\theta}}; v, \boldsymbol{\eta}) = \mathcal{L}_h(v, \boldsymbol{\eta}) \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{2.41}$$

and define $\tilde{\boldsymbol{r}}$ by

$$\tilde{\boldsymbol{r}}_{|K} = (t^2 + \alpha h_K^2)^{-1}\big(\nabla \tilde{z} - \tilde{\boldsymbol{\theta}} + \alpha h_K^2(\boldsymbol{l} - \boldsymbol{L}\,\tilde{\boldsymbol{\theta}})\big)_{|K} \qquad \forall K \in \mathcal{C}_h. \tag{2.42}$$

Hence, it holds

$$\mathcal{A}_h(z - \tilde{z}, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}; v, \boldsymbol{\eta}) = 0 \quad \forall (v, \boldsymbol{\eta}) \in W_h \times \boldsymbol{V}_h, \tag{2.43}$$

and

$$|||(z - \tilde{z}, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}})|||_h + \|\boldsymbol{r} - \tilde{\boldsymbol{r}}\|_h \le Ch\|(l, \boldsymbol{l})\|_{-1,t}. \tag{2.44}$$

From this it also follows that

$$\Big(\sum_{K\in\mathcal{C}_h} h_K^2 \|\boldsymbol{L}\,\tilde{\boldsymbol{\theta}}\|_{0.K}^2\Big)^{1/2} + \|\,\tilde{\boldsymbol{r}}\|_h \le Ch\|(l, \boldsymbol{l})\|_{-1,t}. \tag{2.45}$$

Step 3. We choose $v = w - w_h$ and $\boldsymbol{\eta} = \boldsymbol{\beta} - \boldsymbol{\beta}_h$ in (2.39) and obtain

$$\begin{aligned} &(l, w - w_h) + (\boldsymbol{l}, \boldsymbol{\beta} - \boldsymbol{\beta}_h) \\ &\quad = \mathcal{A}_h(z, \boldsymbol{\theta}; w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h) \\ &\qquad + \sum_{K\in\mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{l}, \boldsymbol{L}\,(\boldsymbol{\beta} - \boldsymbol{\beta}_h))_K \\ &\qquad + \sum_{K\in\mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{l}, \nabla(w - w_h) - (\boldsymbol{\beta} - \boldsymbol{\beta}_h))_K. \end{aligned}\tag{2.46}$$

From (2.22) and (2.28) we have

$$\mathcal{A}_h(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h; \tilde{z}, \tilde{\boldsymbol{\theta}}) = 0. \tag{2.47}$$

Using the symmetry of $\mathcal{A}_h$ we then get

$$\begin{aligned}
&(l, w - w_h) + (\boldsymbol{l}, \boldsymbol{\beta} - \boldsymbol{\beta}_h) \\
&\quad = \mathcal{A}_h(z - \tilde{z}, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}; w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h) \\
&\qquad + \sum_{K \in \mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{l}, \boldsymbol{L}\,(\boldsymbol{\beta} - \boldsymbol{\beta}_h))_K \\
&\qquad + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{l}, \nabla(w - w_h) - (\boldsymbol{\beta} - \boldsymbol{\beta}_h))_K.
\end{aligned} \tag{2.48}$$

The first term above we estimate using Theorem 2.5 and (2.44)

$$\begin{aligned}
&|\mathcal{A}_h(z - \tilde{z}, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}; w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)| \\
&\quad \leq \; C|||(z - \tilde{z}, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}})|||_h |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h \\
&\quad \leq \; Ch^2 \|(l, \boldsymbol{l})\|_{-1,t} \|(f, \boldsymbol{f})\|_{-1,t}.
\end{aligned} \tag{2.49}$$

The second term is treated as follows

$$\begin{aligned}
&|\sum_{K \in \mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{l}, \boldsymbol{L}\,(\boldsymbol{\beta} - \boldsymbol{\beta}_h))_K| \\
&\quad \leq \; Ch\|\boldsymbol{l}\|_0 \big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{L}\,(\boldsymbol{\beta} - \boldsymbol{\beta}_h)\|_{0,K}^2 \big)^{1/2} \\
&\quad \leq \; Ch^2 \|(l, \boldsymbol{l})\|_{-1,t} \|(f, \boldsymbol{f})\|_{-1,t},
\end{aligned} \tag{2.50}$$

where the last step follows from Theorem 2.5 and a scaling argument. The last term is readily estimated

$$\begin{aligned}
&|\sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{l}, \nabla(w - w_h) - (\boldsymbol{\beta} - \boldsymbol{\beta}_h))_K| \\
&\quad \leq \; Ch\|\boldsymbol{l}\|_0 |||(w - w_h, \boldsymbol{\beta} - \boldsymbol{\beta}_h)|||_h \\
&\quad \leq \; Ch^2 \|(l, \boldsymbol{l})\|_{-1,t} \|(f, \boldsymbol{f})\|_{-1,t}.
\end{aligned} \tag{2.51}$$

The estimate (2.34) now follows by combining (2.48) – (2.51).

Step 4. Finally, we turn to the estimate for $(w - w_h^*, \boldsymbol{\beta} - \boldsymbol{\beta}_h^*)$. From (2.39) we get

$$\begin{aligned}
&(l, w - w_h^*) + (\boldsymbol{l}, \boldsymbol{\beta} - \boldsymbol{\beta}_h^*) \\
&\quad = \mathcal{A}_h(z, \boldsymbol{\theta}; w - w_h^*, \boldsymbol{\beta} - \boldsymbol{\beta}_h^*) \\
&\qquad + \sum_{K \in \mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{l}, \boldsymbol{L}\,(\boldsymbol{\beta} - \boldsymbol{\beta}_h^*))_K \\
&\qquad + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{l}, \nabla(w - w_h) - (\boldsymbol{\beta} - \boldsymbol{\beta}_h^*))_K.
\end{aligned} \tag{2.52}$$

Next, adding and subtracting $\mathcal{A}_h(\tilde{z}, \tilde{\boldsymbol{\theta}}; w - w_h^*, \boldsymbol{\beta} - \boldsymbol{\beta}_h^*)$ gives

$$
\begin{aligned}
&(l, w - w_h^*) + (\boldsymbol{l}, \boldsymbol{\beta} - \boldsymbol{\beta}_h^*) \\
&= \mathcal{A}_h(z - \tilde{z}, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}; w - w_h^*, \boldsymbol{\beta} - \boldsymbol{\beta}_h^*) \\
&\quad + \sum_{K \in \mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{l}, \boldsymbol{L}(\boldsymbol{\beta} - \boldsymbol{\beta}_h^*))_K \\
&\quad + \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{l}, \nabla(w - w_h) - (\boldsymbol{\beta} - \boldsymbol{\beta}_h^*))_K \\
&\quad + \mathcal{A}_h(\tilde{z}, \tilde{\boldsymbol{\theta}}; w - w_h^*, \boldsymbol{\beta} - \boldsymbol{\beta}_h^*).
\end{aligned}
\tag{2.53}
$$

Using (2.44), (2.45) all except the last term are estimated as in Step 3. This last term we treat using (2.41), (2.32) and (2.24)

$$
\begin{aligned}
&\mathcal{A}_h(\tilde{z}, \tilde{\boldsymbol{\theta}}; w - w_h^*, \boldsymbol{\beta} - \boldsymbol{\beta}_h^*) = \mathcal{A}_h(\tilde{z}, \tilde{\boldsymbol{\theta}}; w, \boldsymbol{\beta}) - \mathcal{A}_h(\tilde{z}, \tilde{\boldsymbol{\theta}}; w_h^*, \boldsymbol{\beta}_h^*) \\
&\quad = \mathcal{F}_h(\tilde{z}, \tilde{\boldsymbol{\theta}}) - (f, \tilde{z}) + (\boldsymbol{f}, \tilde{\boldsymbol{\theta}}) \\
&\quad = -\sum_{K \in \mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{f}, \boldsymbol{L}\tilde{\boldsymbol{\theta}})_K \\
&\qquad - \sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{f}, \nabla\tilde{z} - \tilde{\boldsymbol{\theta}})_K.
\end{aligned}
\tag{2.54}
$$

Next, we use (2.45)

$$
\begin{aligned}
&\Big|\sum_{K \in \mathcal{C}_h} \alpha h_K^2 t^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{f}, \boldsymbol{L}\tilde{\boldsymbol{\theta}})_K\Big| \\
&\qquad \le Ch\|\boldsymbol{f}\|_0 \Big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{L}\tilde{\boldsymbol{\theta}}\|_{0.K}^2\Big)^{1/2} \le Ch^2 \|(l, \boldsymbol{l})\|_{-1,t} \|(f, \boldsymbol{f})\|_{-1,t}.
\end{aligned}
\tag{2.55}
$$

From (2.42) and (2.45) we get

$$
\begin{aligned}
&\Big|\sum_{K \in \mathcal{C}_h} (t^2 + \alpha h_K^2)^{-1} \alpha h_K^2 (\boldsymbol{f}, \nabla\tilde{z} - \tilde{\boldsymbol{\theta}})_K\Big| \\
&\quad = \Big|\sum_{K \in \mathcal{C}_h} \alpha h_K^2 \big(\boldsymbol{f}, \tilde{\boldsymbol{r}} + \alpha h_K^2 (t^2 + \alpha h_K^2)^{-1} (\boldsymbol{L}\tilde{\boldsymbol{\theta}} - \boldsymbol{l})\big)_K\Big| \\
&\quad \le Ch\|\boldsymbol{f}\|_0 \|\tilde{\boldsymbol{r}}\|_h + Ch\|\boldsymbol{f}\|_0 \Big(\sum_{K \in \mathcal{C}_h} h_K^2 \|\boldsymbol{L}\tilde{\boldsymbol{\theta}}\|_{0,K}^2\Big)^{1/2} + Ch^2 \|\boldsymbol{f}\|_0 \|\boldsymbol{l}\|_0 \\
&\quad \le Ch^2 \|(l, \boldsymbol{l})\|_{-1,t} \|(f, \boldsymbol{f})\|_{-1,t}.
\end{aligned}
\tag{2.56}
$$

The asserted estimate (2.35) now follows by combining the estimates in this step. □

3 The multigrid method

In this section we prove that a simple multigrid method leads to a solver with optimal complexity and which is robust with respect to the parameter t.

The stabilized bilinear-form $\mathcal{A}_h$ depends on the underlying mesh. Thus, the sequence of meshes lead to different operators on each level. Hence, apply the non-nested framework from [5] and adapt the notation from [4], Section 4.

Assume that we have a sequence of hierarchically refined meshes which we denote by $\mathcal{C}_1, \mathcal{C}_2, \ldots \mathcal{C}_J$. On each level k, $1 \leq k \leq J$, we the finite element spaces are denoted by $W_k \times \boldsymbol{V}_k$. We note that the spaces are nested, i.e.,

$$W_{k-1} \times \boldsymbol{V}_{k-1} \subset W_k \times \boldsymbol{V}_k$$

such that no special grid transfer operators have to be defined. On each level we denote the bilinear form by

$$\mathcal{A}_k : (W_k \times \boldsymbol{V}_k) \times (W_k \times \boldsymbol{V}_k) \to \mathbb{R}$$

in accordance to (2.23).

Since, we assume a hierarchy of meshes the are all uniform and we denote the corresponding mesh-size with h_k (or h when it is irrelevant which level is in question).

On each level k, an inner product $(\cdot;\cdot)_k : (W_k \times \boldsymbol{V}_k) \times (W_k \times \boldsymbol{V}_k) \to \mathbb{R}$ is defined as

$$(z, \boldsymbol{\delta}; v, \boldsymbol{\eta})_k := h_k^2 (h_k + t)^{-2} \, (z, v) + h_k^2 \, (\boldsymbol{\delta}, \boldsymbol{\eta}),$$

and $\|\cdot\|_k$ denotes the corresponding norm. We define the operator $A_k : W_k \times \boldsymbol{V}_k \to W_k \times \boldsymbol{V}_k$ by

$$(A_k(z, \boldsymbol{\delta}); v, \boldsymbol{\eta})_k = \mathcal{A}_k(z, \boldsymbol{\delta}; v, \boldsymbol{\eta}) \qquad \forall (v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k.$$

Furthermore, we define the projections $P_{k-1} : W_k \times \boldsymbol{V}_k \to W_{k-1} \times \boldsymbol{V}_{k-1}$ and $Q_{k-1} : W_k \times \boldsymbol{V}_k \to W_{k-1} \times \boldsymbol{V}_{k-1}$ by

$$\mathcal{A}_{k-1}(P_{k-1}(z, \boldsymbol{\delta}); v, \boldsymbol{\eta}) = \mathcal{A}_k(z, \boldsymbol{\delta}; v, \boldsymbol{\eta}) \qquad \forall (v, \boldsymbol{\eta}) \in W_{k-1} \times \boldsymbol{V}_{k-1},$$

and

$$(Q_{k-1}(z, \boldsymbol{\delta}); v, \boldsymbol{\eta})_{k-1} = (z, \boldsymbol{\delta}; v, \boldsymbol{\eta})_k \qquad \forall (v, \boldsymbol{\eta}) \in W_{k-1} \times \boldsymbol{V}_{k-1}.$$

Finally, let $R_k : W_k \times \boldsymbol{V}_k \to W_k \times \boldsymbol{V}_k$ be the smoothing operator defined by a scaled Jacobi iteration or by a Gauss-Seidel iteration. A symmetrized smoothing iteration is defined by setting $R_k^{(l)} = R_k$ if l is odd, and $R_k^{(l)} := R_k^t$ if l is even. Here, $(\cdot)^t$ denotes the adjoint operator with respect to $(\cdot;\cdot)_k$.

We define the multigrid operator B_J by induction. Set $B_1 = A_1^{-1}$. For $k = 2, \ldots, J$ we define $B_k : W_k \times \boldsymbol{V}_k \to W_k \times \boldsymbol{V}_k$ as follows.

Algorithm 3.1. *With $g_k \in W_k \times \boldsymbol{V}_k$ we define $B_k g_k$ by the following algorithm.*

> *Initialize* $x_k^0 \in W_k \times V_k$ *as* $x_k^0 = 0$
> *for* $l = 0 \ldots m_k - 1$ *do*

$$
\begin{aligned}
&\qquad x_k^{l+1} := x_k^l + R_k^{(l)}(g_k - A_k x_k^l)\\
&x_k^{m_k+1} = x_k^{m_k} + B_{k-1}Q_{k-1}(g_k - A_k x_k^{m_k})\\
&\textbf{for } l = m_{k+1}\dots 2m_k \textbf{ do}\\
&\qquad x_k^{l+1} := x_k^l + R_k^{(l-1)}(g_k - A_k x_k^l)\\
&B_k g_k := x_k^{2m_k+1}.
\end{aligned}
$$

We assume that the number of smoothing steps depends on the level as $m_k = 2^{J-k}$. This is the so called variable V-cycle multigrid algorithm.

Theorem 3.1. *Assume that the Reissner Mindlin plate problem satisfies the regularity estimate* (2.13)*. Then the multigrid algorithm provides an optimal preconditioner B_J, i.e.*

$$\mathit{cond}(B_J A_J) \le C.$$

The constant C does neither depend on the number of levels, nor on the parameter t.

Proof. We apply Theorem 4.6 from [4]. It is easily checked that there holds an inverse inequality

$$\mathcal{A}_k(v_k, \boldsymbol{\eta}_k; v_k, \boldsymbol{\eta}_k) \le \lambda_k \, \|(v_k, \boldsymbol{\eta}_k)\|_k^2 \qquad \forall (v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k$$

with $\lambda_k \simeq h^{-4}$. We have to check that the following conditions hold for all $(v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k$:

- (A.4) :
$$(\overline{R}_k(v_k, \boldsymbol{\eta}_k); v_k, \boldsymbol{\eta}_k)_k \ge c h_k^4 \, \|(v_k, \boldsymbol{\eta}_k)\|_k, \tag{3.1}$$
where $\overline{R}_k := (I - R_k A_k)(I - R_k^t A_k) A_k^{-1}$ is the symmetrized smoother.

- (A.10) with the choice $\alpha = 1/2$:
$$\mathcal{A}_k((I - P_{k-1})(v_k, \boldsymbol{\eta}_k); v_k, \boldsymbol{\eta}_k) \le c h_k^2 \, \|A_k v\|_k \, \mathcal{A}_k(v_k, \boldsymbol{\eta}_k; v_k, \boldsymbol{\eta}_k)^{1/2} \tag{3.2}$$

for all $(v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k$.
These conditions are proven in Lemma 3.1 and Lemma 3.4 below. $\square$

Lemma 3.1 (Smoothing Property). *Let the smoother be defined by a properly scaled Jacobi iteration, or by the symmetrized Gauss-Seidel iteration. Then condition* (3.1) *is satisfied.*

Proof. We apply Theorem 5.1 and Theorem 5.2 from [4], respectively. For this we have to show that the decomposition

$$(v_k, \boldsymbol{\eta}_k) = \sum_{i=1}^{\dim W_k} (v_i, 0) + \sum_{i=1}^{\dim \boldsymbol{V}_k} (0, \boldsymbol{\eta}_i)$$

into the one dimensional spaces generated by the finite element basis functions is stable with respect to the $\|\cdot\|_k$-norm, i.e.,

$$\sum_{i=1}^{\dim W_k} \|(v_i, 0)\|_k^2 + \sum_{i=1}^{\dim \boldsymbol{V}_k} \|(0, \boldsymbol{\eta}_i)\|_k^2 \le c \, \|(v, \boldsymbol{\eta})\|_k^2.$$

This holds true since both components of $\|\cdot\|_k$ are just scaled L^2-norms. Furthermore, we need that the number of overlapping finite element functions is uniformly bounded. $\square$

Lemma 3.2 (Approximation property). *Let $(z_k, \boldsymbol{\delta}_k) \in W_k \times \boldsymbol{V}_k$ be given. Define the coarse grid functions $(z_{k-1}, \boldsymbol{\delta}_{k-1}) \in W_{k-1} \times \boldsymbol{V}_{k-1}$ by the projection*

$$\mathcal{A}_{k-1}(z_{k-1}, \boldsymbol{\delta}_{k-1}; v_{k-1}, \boldsymbol{\eta}_{k-1}) = \mathcal{A}_k(z_k, \boldsymbol{\delta}_k; v_{k-1}, \boldsymbol{\eta}_{k-1}) \tag{3.3}$$

for all $(v_{k-1}, \boldsymbol{\eta}_{k-1}) \in W_{k-1} \times \boldsymbol{V}_{k-1}$. Then there holds the approximation estimate

$$\|(z_k - z_{k-1}, \boldsymbol{\delta}_k - \boldsymbol{\delta}_{k-1})\|_{1,t} \le Ch^2 \sup_{(v,\boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k} \frac{\mathcal{A}_k(z_k, \boldsymbol{\delta}_k; v, \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}}.$$

Proof. Let $(z_k, \boldsymbol{\delta}_k) \in W_k \times \boldsymbol{V}_k$ be given. Let Π and $\boldsymbol{\Pi}$ be Clément projection operators. Define $g \in L^2(\Omega)$ and $\boldsymbol{g} \in [L^2(\Omega)]^2$ by

$$(g, v) + (\boldsymbol{g}, \boldsymbol{\eta}) := \mathcal{A}_k(z_k, \boldsymbol{\delta}_k; \Pi v, \boldsymbol{\Pi}\boldsymbol{\eta}). \qquad \forall\, v \in L^2(\Omega),\ \forall\, \boldsymbol{\eta} \in [L^2(\Omega)]^2. \tag{3.4}$$

There holds

$$\begin{aligned}
\|(g, \boldsymbol{g})\|_{-1,t} &:= \sup_{(v,\boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k} \frac{(g, v) + (\boldsymbol{g}, \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}} \\
&= \sup_{(v,\boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k} \frac{\mathcal{A}_k(z_k, \boldsymbol{\delta}_k; \Pi v, \boldsymbol{\Pi}\boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}} \\
&= \sup_{(v,\boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k} \frac{\mathcal{A}_k(z_k, \boldsymbol{\delta}_k; \Pi v, \boldsymbol{\Pi}\boldsymbol{\eta})}{\|(\Pi v, \boldsymbol{\Pi}\boldsymbol{\eta})\|_{1,t}} \frac{\|(\Pi v, \boldsymbol{\Pi}\boldsymbol{\eta})\|_{1,t}}{\|(v, \boldsymbol{\eta})\|_{1,t}} \\
&= \sup_{(v,\boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k} \frac{\mathcal{A}_k(z_k, \boldsymbol{\delta}_k; v, \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}} \|(\Pi, \boldsymbol{\Pi})\|_{1,t}
\end{aligned}$$

Since Π is bounded in the L^2-norm as well as in the H^1-semi-norm, and $\boldsymbol{\Pi}$ is bounded in L^2-norm, the compound operator is bounded with respect to $\|\cdot\|_{1,t}$.

We pose the plate problem: find $(z, \boldsymbol{\delta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2$ such that

$$\mathcal{A}(z, \boldsymbol{\delta}; v, \boldsymbol{\eta}) = (g, v) + (\boldsymbol{g}, \boldsymbol{\eta})$$

Using that $(\Pi, \boldsymbol{\Pi})$ is a projection on $W_k \times \boldsymbol{V}_k$, we recast (3.4) as

$$\mathcal{A}_k(z_k, \boldsymbol{\delta}_k; v, \boldsymbol{\eta}) = (g, v) + (\boldsymbol{g}, \boldsymbol{\eta}) \qquad \forall\, (v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k.$$

This means that $(z_k, \boldsymbol{\delta}_k)$ is the finite element solution obtained by of Method 2, where the consistency terms on the right hand side were skipped. Theorem 2.7 provides the estimate

$$\|(z - z_k, \boldsymbol{\delta} - \boldsymbol{\delta}_k)\|_{1,t} \le ch_k^2\ \|(g, \boldsymbol{g})\|_{-1,t}.$$

Using (3.3), we observe that

$$\mathcal{A}_{k-1}(z_{k-1}, \boldsymbol{\delta}_{k-1}; v, \boldsymbol{\eta}) = (g, v) + (\boldsymbol{g}, \boldsymbol{\eta}) \qquad \forall\, (v, \boldsymbol{\eta}) \in W_{k-1} \times \boldsymbol{V}_{k-1},$$

and again Theorem 2.7 proves

$$\|(z - z_{k-1}, \boldsymbol{\delta} - \boldsymbol{\delta}_{k-1})\|_{1,t} \leq c h_{k-1}^2 \, \|(g, \boldsymbol{g})\|_{-1,t}.$$

From the triangle inequality we obatin the result

$$\begin{aligned} \|(z_{k-1} - z_k, \boldsymbol{\delta}_{k-1} - \boldsymbol{\delta}_k)\|_{1,t} &\leq c h_k^2 \, \|(g, \boldsymbol{g})\|_{-1,t} \\ &\leq c h_k^2 \sup_{(v,\boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k} \frac{\mathcal{A}_k(z_k, \boldsymbol{\delta}_k; v, \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}}. \end{aligned}$$

□

The norms $\|\cdot\|_k$ and $\mathcal{A}_k(\cdot,\cdot)^{1/2}$ can be embedded into a scale of norms. For this we set

$$|||(v, \boldsymbol{\eta})|||_0 := \|(v, \boldsymbol{\eta})\|_k \qquad \text{and} \qquad |||(v, \boldsymbol{\eta})|||_2 := \mathcal{A}_k(v, \boldsymbol{\eta}; v, \boldsymbol{\eta})^{1/2}.$$

Norms in between are defined by interpolation [6, Chapter 12], i.e.

$$|||(v, \boldsymbol{\eta})|||_s := \|(v, \boldsymbol{\eta})\|_{[|||\cdot|||_0, |||\cdot|||_2]_{s/2}} \qquad s \in (0, 2).$$

Furthermore, the scale is extended by duality to the range $(2, 4]$

$$|||(v, \boldsymbol{\eta})|||_{2+s} := \sup_{(z,\boldsymbol{\delta})} \frac{\mathcal{A}_l(v, \boldsymbol{\eta}; z, \boldsymbol{\delta})}{|||(z, \boldsymbol{\delta})|||_{2-s}} \qquad s \in (0, 2].$$

In particular there holds

$$|||(v, \boldsymbol{\eta})|||_4 = \sup_{(z,\boldsymbol{\delta})} \frac{\mathcal{A}_k(v, \boldsymbol{\eta}; z, \boldsymbol{\delta})}{|||(z, \boldsymbol{\delta})|||_0} = \sup_{(z,\boldsymbol{\delta})} \frac{(A_k(v, \boldsymbol{\eta}); z, \boldsymbol{\delta})_k}{\|(z, \boldsymbol{\delta})\|_k} = \|A_k(v, \boldsymbol{\eta})\|_k.$$

Lemma 3.3. *The discrete* 1*-norm and the continuous* 1*-norm satisfy the following relation:*

$$|||(v, \boldsymbol{\eta})|||_1 \leq C \|(v, \boldsymbol{\eta})\|_{1,t} \qquad \forall\, (v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k. \tag{3.5}$$

Proof. Let $(v, \boldsymbol{\eta}) \in W_k \times \boldsymbol{V}_k$. By the definition of the $\|\cdot\|_{1,t}$ norm, there exists a decomposition $v = v_0 + v_r$ such that

$$\|v_0\|_1 + t^{-1}\|v_r\|_0 + \|\boldsymbol{\eta}\|_0 \leq \|(v, \boldsymbol{\eta})\|_{1,t}.$$

Although v is a finite element function, its decomposition will in general not remain in the finite element space. To return to the finite element space, we define Clément-interpolation operators $\Pi : L^2(\Omega) \to W_k$ and $\boldsymbol{\Pi} : [L^2(\Omega)]^2 \to \boldsymbol{V}_k$ with the following approximation properties:

$$\begin{aligned} \|v - \Pi v\|_s &\le h_k^{m-s}\|v\|_m, \qquad && 0 \le s \le 1,\ 0 \le m \le 2,\ s \le m, \\ \|\boldsymbol{\eta} - \boldsymbol{\Pi}\boldsymbol{\eta}\|_s &\le h_k^{m-k}\|\boldsymbol{\eta}\|_m, \qquad && 0 \le s \le m \le 1. \end{aligned}$$

The finite element functions are decomposed into finite element functions as

$$(v, \boldsymbol{\eta}) = (\Pi v_0, \boldsymbol{\Pi}\nabla\Pi v_0) + (\Pi v_r, \boldsymbol{\eta} - \boldsymbol{\Pi}\nabla\Pi v_0). \tag{3.6}$$

Applying the triangle inequality leads to

$$|||(v, \boldsymbol{\eta})|||_1 \le |||(\Pi v_0, \boldsymbol{\Pi}\nabla\Pi v_0)|||_1 + |||(\Pi v_r, \boldsymbol{\eta} - \boldsymbol{\Pi}\nabla\Pi v_0)|||_1. \tag{3.7}$$

We estimate both terms by using that $|||.|||_1$ is the interpolation norm of $|||\cdot|||_0$ and $|||\cdot|||_2$ with parameter $1/2$. For $v_0 \in H_0^2(\Omega)$, the continuity and approximation properties of Π and an inverse inequality leads us to

$$\begin{aligned} &|||(\Pi v_0, \Pi\nabla\Pi v_0)|||_2^2 = \|\boldsymbol{\Pi}\nabla\Pi v_0\|_1^2 + (h+t)^{-2}\|(\boldsymbol{I} - \boldsymbol{\Pi})\nabla\Pi v_0\|_0^2 \\ &\quad \le \|\boldsymbol{\Pi}\nabla v_0\|_1^2 + \|\boldsymbol{\Pi}\nabla(v_0 - \Pi v_0)\|_1^2 \\ &\qquad + (h+t)^{-2}\|(\boldsymbol{I} - \boldsymbol{\Pi})\nabla v_0\|_0^2 + (h+t)^{-2}\|(\boldsymbol{I} - \boldsymbol{\Pi})\nabla(I - \Pi)v_0\|_0^2 \\ &\quad \le \|v_0\|_2^2. \end{aligned}$$

With an inverse inequalities and L^2-continuity we obtain

$$\begin{aligned} |||(\Pi v_0, \Pi\nabla\Pi v_0)|||_0^2 &= h^2\|\Pi\nabla\Pi v_0\|_0^2 + h^2(h+t)^{-2}\|\Pi v_0\|_0^2 \\ &\le \|v_0\|_0^2. \end{aligned}$$

The interpolation space $[L^2(\Omega), H_0^2(\Omega)]_{1/2}$ is $H_0^1(\Omega)$. Thus, we can apply operator interpolation to the linear operator $v \mapsto (\Pi v, \boldsymbol{\Pi}\nabla\Pi v)$ and obtain that

$$|||(\Pi v_0, \boldsymbol{\Pi}\nabla\Pi v_0)|||_1 \le \|v_0\|_1 \le \|(v, \boldsymbol{\eta})\|_{1,t}^2. \tag{3.8}$$

We continue with the second term of (3.7). From

$$\begin{aligned} &|||(\Pi v_r, \boldsymbol{\eta} - \boldsymbol{\Pi}\nabla\Pi v_0)|||_2^2 \\ &\quad = \|(\boldsymbol{\eta} - \boldsymbol{\Pi}\nabla\Pi v_0)\|_1^2 + (h+t)^{-2}\|\nabla\Pi v_r - \boldsymbol{\eta} + \boldsymbol{\Pi}\nabla\Pi v_0\|_0^2 \\ &\quad \le h^{-2}\{\|\boldsymbol{\eta}\|_0^2 + \|v_0\|_1^2\} + (h+t)^{-2}\{h^{-2}\|v_r\|_0^2 + \|\boldsymbol{\eta}\|_0^2 + \|v_0\|_1^2\} \\ &\quad \le h^{-2}\{\|v_0\|_1^2 + t^{-2}\|v_r\|_0^2 + \|\boldsymbol{\eta}\|_0^2\} \\ &\quad \le h^{-2}\|(v, \boldsymbol{\eta})\|_{1,t}^2 \end{aligned}$$

and

$$\begin{aligned} |||(\Pi v_r, \boldsymbol{\eta} - \boldsymbol{\Pi}\nabla\Pi v_0)|||_0^2 &= h^2\|\boldsymbol{\eta} - \boldsymbol{\Pi}\nabla\Pi v_0\|_0^2 + h^2(h+t)^{-2}\|\Pi v_r\|_0^2 \\ &\le h^2\left\{\|\boldsymbol{\eta}\|_0^2 + t^{-2}\|v_r\|_0^2 + \|v_0\|_1^2\right\} \\ &\le h^2\,\|(v, \boldsymbol{\eta})\|_{1,t}^2 \end{aligned}$$

we can conclude that

$$|||(\Pi v_r, \boldsymbol{\eta} - \boldsymbol{\Pi}\nabla\Pi v_0)|||_1^2 \le \|(v, \boldsymbol{\eta})\|_{1,t}^2.$$

□

Lemma 3.4. *The approximation property* (3.2) *holds.*

Proof. Applying Lemma 3.3 twice and Lemma 3.2 we obtain

$$\begin{aligned} |||(w_k - w_{k-1}, \boldsymbol{\beta}_k - \boldsymbol{\beta}_{k-1})|||_1 &\leq c\,\|(w_k - w_{k-1}, \boldsymbol{\beta}_k - \boldsymbol{\beta}_{k-1})\|_{1,t} \\ &\leq ch^2 \sup_{(v,\boldsymbol{\eta})} \frac{\mathcal{A}_k(w_k, \boldsymbol{\beta}_k; v; \boldsymbol{\eta})}{\|(v, \boldsymbol{\eta})\|_{1,t}} \\ &\leq ch^2 \sup_{(v,\boldsymbol{\eta})} \frac{\mathcal{A}_k(w_k, \boldsymbol{\beta}_k; v; \boldsymbol{\eta})}{|||(v, \boldsymbol{\eta})|||_1} \\ &= ch^2\,|||(w_k, \boldsymbol{\beta}_k)|||_3. \end{aligned}$$

This combined with

$$\begin{aligned} &\mathcal{A}_k(w_k - w_{k-1}, \boldsymbol{\beta}_k - \boldsymbol{\beta}_{k-1}; w_k, \boldsymbol{\beta}_k) \\ &\quad \leq |||(w_k - w_{k-1}, \boldsymbol{\beta}_k - \boldsymbol{\beta}_{k-1})|||_1\,|||(w_k, \boldsymbol{\beta}_k)|||_3 \\ &\quad \leq ch^2\,|||(w_k, \boldsymbol{\beta}_k)|||_3^2 \\ &\quad \leq ch^2\,|||(w_k, \boldsymbol{\beta}_k)|||_2\,|||(w_k, \boldsymbol{\beta}_k)|||_4 \\ &\quad = ch^2\,\mathcal{A}_k(w_k, \boldsymbol{\beta}_k; w_k, \boldsymbol{\beta}_k)^{1/2}\,\|A_k(w_k, \boldsymbol{\beta}_k)\|_k \end{aligned}$$

gives the asserted estimate (3.2). □

4 Computational results

We applied the proposed multigrid algorithm to a unit-square model problem. The plate is fully clamped on the boundary. The right hand side is the uniform load $f = 1$. The first $\mathcal{C}_1$ mesh consists of two triangles; the subsequent meshes $\mathcal{C}_2, \ldots, \mathcal{C}_J$ are obtained by regular refinement of one triangle into four.

We applied a conjugate gradient iteration with a multigrid preconditioner. We used the variable V-cycle with 2^{k-J} alternating Gauss-Seidel presmoothing and postsmoothing steps on the k^{th} level. Furthermore, we have computed the condition number of the preconditioned system matrix by the Lanczos algorithm.

Table 1 shows the condition number, and the required number of cg iterations for relative reduction of the error by a factor 10^{-8}. The error reduction was measured in the norm $(Br, r)^{1/2}$. We clearly see that the condition numbers and iteration numbers are bounded uniformly with respect to h and t. Note that the condition number of the matrix A behaves like $h^{-2}(h+t)^{-2}$ which was as high as 10^9.

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		$t = 0.1$		$t = 0.0001$	
Level	Elements	cond.numb.	cg	cond.numb.	cg
2	8	1.51	7	2.91	7
4	128	2.63	13	7.28	21
6	2048	3.32	14	8.88	27
8	32768	3.20	13	7.60	23

Table 1: Computational results

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