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ON A STOCHASTIC PARABOLIC INTEGRAL EQUATION

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To the memory of Günter Lumer

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Abstract: *In this article we analyze the stochastic parabolic integral equation*

$$u(t,x,\omega) = c_\alpha t^{-1+\alpha} * \Delta u + \sum_{k=1}^{\infty} \int_0^t g^k(s,x,\omega)\, dw_s^k,$$

where $t \geq 0$, $x \in \mathbb{R}^d$, $\alpha \in (\frac{1}{2}, 1)$ *and* $\omega \in \Omega$. *We take* $\{w_t^k \mid k = 1, 2, ...\}$ *to be a family of independent* $\mathcal{F}_t$*-adapted Wiener processes defined on a probability space* $(\Omega, \mathcal{F}, P)$. *Here* $\mathcal{F}_t \subset \mathcal{F}$ *and* $\mathcal{F}_t$ *an increasing filtration.*
By applying and modifying the method of Krylov we obtain existence and regularity results in L_p*-spaces,* $p \geq 2$.

AMS subject classifications: 60H20 45R05

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1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space, with $\{\mathcal{F}_t\}_{t\geq 0}$ an increasing filtration of σ-algebras satisfying $\mathcal{F}_t \subset \mathcal{F}$. Let $\mathcal{P}$ denote the predictable σ-algebra on $\mathbb{R}_+ \times \Omega$ generated by $\{\mathcal{F}_t\}_{t\geq 0}$, and assume $\{w_t^k \mid k = 1, 2, ...\}$ is a family of independent one-dimensional $\mathcal{F}_t$-adapted Wiener processes defined on $(\Omega, \mathcal{F}, P)$.

In this setting, we consider the stochastic parabolic integral equation

$$u(t, x, \omega) = \int_0^t k(t-s)\Delta u(s, x, \omega)\, ds + \sum_{k=1}^{\infty} \int_0^t g^k(s, x, \omega)\, dw_s^k, \qquad (1)$$

where the variables satisfy $t \geq 0$, $x \in \mathbb{R}^d$, $\omega \in \Omega$, and $k(t) = c_\alpha t^{-1+\alpha}$, with c_α, α given constants; $\alpha \in (\frac{1}{2}, 1)$; and g^k given functions. The infinite series of stochastic integrals on the right side of (1) converges in a weak sense made precise below. By modifying the analytic approach of Krylov [7], developed for stochastic parabolic partial differential equations, we obtain an existence and uniqueness result on (1). As in [7], the setting is L_p, with $p \geq 2$, thus a Hilbert space framework is not needed.

Before outlining the paper, we make some brief comments on the range of α-values.

With $\alpha = 1$, the equation (1) is a (much studied) parabolic stochastic partial differential equation. See, e.g., [7], for further references. Our proofs require $k \in L_2(0, 1)$, thus $\alpha > \frac{1}{2}$. For small α one may however formally argue as follows.

The equation (1) can be inverted to give

$$D_t^\alpha u = \Delta u + F, \qquad (2)$$

where $D_t^\alpha u \overset{\text{def}}{=} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt}(t^{-\alpha} * u)$, $t > 0$, is the fractional time derivative of order α of u (with $u(0) = 0$), and where $F = \frac{d}{dt}(t^{-\alpha} * G)$, with $G = \sum_k \int_0^t g^k\, dw_s^k$. Suppose that, in some sense, $G \in C^\delta$; then $F \in C^{\delta-\alpha}$. Assume that $\delta - \alpha > 0$. Equations of this type have been treated in Bessel potential spaces in [10], [11], and in Hölder spaces in [3] and [4].

The case $\alpha \in (1, 2)$ will be included in future work.

Equations of type (1) have been considered in Hilbert spaces in [1] and [2] by applying methods of [5]. In particular, certain regularity results on the stochastic convolution associated with (1) were obtained in [1].

Stochastic integral equations of type (1) or (2) occur in models of anomalous diffusion.

In Section 2, we introduce the necessary machinery and show how the stochastic Banach spaces developed in [7] can be modified in order to apply to the equations we consider.

In Section 3 we state and prove an existence result on (1). The fact that $\alpha < 1$ allows us to obtain additional time-regularity on the solution as compared to the case $\alpha = 1$. This we do in Section 4.

We will develop the present approach further in forthcoming work.

2 The Stochastic Machinery

Below, everywhere, $p \geq 2$.

Let $n \in \mathbb{R}$, and let $H_p^n(\mathbb{R}^d)$ be the Bessel potential space of distributions u such that $(1-\Delta)^{\frac{n}{2}} u \in L_p(\mathbb{R}^d)$, with norm

$$\|u\|_{n,p} \stackrel{\text{def}}{=} \|(1-\Delta)^{\frac{n}{2}} u\|_p.$$

Denote by l_2 the set of real-valued sequences $g = \{g^k \mid k = 1, 2, ...\}$ with norm $|g|_{l_2}^2 = \sum_k |g^k|^2$, and, for a function $g : \mathbb{R}^d \to l_2$, $\|g\|_p \stackrel{\text{def}}{=} \||g|_{l_2}\|_p$; $\|g\|_{n,p} \stackrel{\text{def}}{=} \||(1-\Delta)^{\frac{n}{2}} g|_{l_2}\|_p$.

For τ a bounded stopping time, write

$$(0,\tau]] \stackrel{\text{def}}{=} \{(\omega, t) \mid 0 < t \leq \tau(\omega)\},$$

$$\mathcal{H}_p^n(\tau) \stackrel{\text{def}}{=} L_p((0,\tau]], \mathcal{P}, H_p^n),$$

$$\mathcal{H}_p^n(\tau, l_2) \stackrel{\text{def}}{=} L_p((0,\tau]], \mathcal{P}, H_p^n(\mathbb{R}^d; l_2)).$$

The stochastic solution spaces $\hat{\mathcal{H}}_p^n(\tau)$ of (1) are then defined as follows.

Definition 1 *Let $u \in \cap_{T>0} \mathcal{H}_p^n(\tau \wedge T)$. Then $u \in \hat{\mathcal{H}}_p^n(\tau)$ if $u_{xx} \in \mathcal{H}_p^{n-2}(\tau)$, and there exist $f \in \mathcal{H}_p^{n-2}(\tau)$, $g \in \mathcal{H}_p^{n-1}(\tau, l_2)$ such that for any $\phi \in C_0^\infty(\mathbb{R}^d)$, the equality*

$$(u(t,\cdot), \phi(\cdot)) = \int_0^t k(t-s)(f(s,\cdot), \phi(\cdot))\, ds + \sum_{k=1}^\infty \int_0^t (g^k(s,\cdot), \phi(\cdot))\, dw_s^k, \quad (3)$$

holds for all $t \leq \tau$, a.s. The norm in the solution space is

$$\|u\|_{\hat{\mathcal{H}}_p^n(\tau)} \stackrel{def}{=} \|u_{xx}\|_{\mathcal{H}_p^{n-2}(\tau)} + \|f\|_{\mathcal{H}_p^{n-2}(\tau)} + \|g\|_{\mathcal{H}_p^{n-1}(\tau, l_2)}.$$

In (3), for $v \in H_p^n$, $\phi \in C_0^\infty$,

$$(v, \phi) \stackrel{\text{def}}{=} \big((1-\Delta)^{\frac{n}{2}} v, (1-\Delta)^{-\frac{n}{2}} \phi\big) = \int_{\mathbb{R}^d} \big((1-\Delta)^{\frac{n}{2}} v(x)\big)\big((1-\Delta)^{-\frac{n}{2}} \phi(x)\big)\, dx.$$

By the assumption on g, the series of stochastic integrals in (3) does converge (uniformly in t) in probability on $[0, \tau \wedge T]$, $T < \infty$.

Thus, if $u \in \hat{\mathcal{H}}_p^n(\tau)$, then u can be represented as the sum (in the weak sense (3)), of a Lebesgue convolution integral and a series of stochastic integrals. (For simplicity, we take $u(t=0) = 0$).

An obvious question is whether this representation is unique. For $\alpha = 1$ the wellknown answer is yes. Below, in Lemma 2, we show that uniqueness holds also for $\alpha \in (\frac{1}{2}, 1)$.

Lemma 2 *Take $T > 0$, $\alpha \in (\frac{1}{2}, 1)$. Let f, $\{g^k\}$ satisfy*

$$f \in L_2((0,T) \times \Omega), \quad \{g^k\} \in L_2((0,T) \times \Omega, l_2),$$

and let both be adapted to $\{\mathcal{F}_t\}_{t \geq 0}$. Suppose that for $t \in [0,T]$,

$$\int_0^t (t-s)^{\alpha-1} f(s,\omega)\, ds = \sum_k \int_0^t g^k(s,\omega)\, dw_s^k,$$

a.s. Then $f = g^k = 0$ a.s.

Proof of Lemma 2. Both $\|f(t,\cdot)\|^2_{L_2(\Omega)}$ and $\|g(t,\cdot)\|^2_{L_2(\Omega;l_2)}$ are integrable over $(0,T)$. Let t_0 be a Lebesgue point of both functions. Consider the orthogonal projection P in $L_2(\Omega)$:

$$Pu = u - E(u \mid \mathcal{F}_{t_0}).$$

If $f_1(s,\cdot) \stackrel{\text{def}}{=} Pf(s,\cdot)$, then

$$P\Big(\int_0^t (t-s)^{\alpha-1} f(s)\, ds\Big) = \int_0^t (t-s)^{\alpha-1} f_1(s)\, ds = \int_{t_0}^t (t-s)^{\alpha-1} f_1(s)\, ds,$$

where we used the fact that since f is adapted to $\mathcal{F}_t$,

$$f(t) = E\big(f(t) \mid \mathcal{F}_{t_0}\big), \quad t \leq t_0.$$

The series $\sum_k \int_0^t g^k(s)\, dw_s^k$ has the martingale property:

$$E\Big(\sum_k \int_0^t g(s)\, dw_s^k \mid \mathcal{F}_{t_0}\Big) = \sum_k \int_0^{t_0} g^k(s)\, dw_s^k, \quad t \geq t_0.$$

We conclude that

$$P\Big(\sum_k \int_0^t g^k(s)\, dw_s^k\Big) = \sum_k \int_{t_0}^t g^k(s)\, dw_s^k, \quad t \geq t_0,$$

and therefore, a.s.,

$$\int_{t_0}^t (t-s)^{\alpha-1} f_1(s)\, ds = \sum_k \int_{t_0}^t g^k(s)\, dw_s^k, \quad t \in [t_0, T]. \tag{4}$$

Use Hölder and the fact that P is an orthogonal projection in $L_2(\Omega)$, to estimate the L_2-norms:

$$\begin{aligned} \| \int_{t_0}^t (t-s)^{\alpha-1} f_1(s)\, ds\|^2_{L_2(\Omega)} &\leq \Big(\int_{t_0}^t (t-s)^{2\alpha-2}\, ds\Big)\Big(\int_{t_0}^t \|f_1(s)\|^2_{L_2(\Omega)}\, ds\Big) \\ &\leq M(t-t_0)^{2\alpha-1} \int_{t_0}^t \|f(s)\|^2_{L_2(\Omega)}\, ds \leq M(t-t_0)^{2\alpha}, \end{aligned} \tag{5}$$

where the last inequality follows from t_0 being a Lebesgue point. By Itos identity,

$$\|\sum_k \int_{t_0}^t g^k(s)\, dw_s^k\|_{L_2(\Omega)} = \int_{t_0}^t \sum_k \|g^k(s)\|^2_{L_2(\Omega)}\, ds. \qquad (6)$$

Combine (4), (5) and (6), and use the fact that t_0 is a Lebesgue point of $\sum_k \|g^k(s)\|^2_{L_2(\Omega)}$, to get

$$\|g(t_0)\|^2_{L_2(\Omega;l_2)} = \lim_{t\to t_0} (t-t_0)^{-1} \int_{t_0}^t \|g(s)\|^2_{L_2(\Omega;l_2)}\, ds$$
$$\leq \lim_{t\to t_0} (t-t_0)^{-1} M(t-t_0)^{2\alpha} = 0,$$

where $2\alpha > 1$ was used. Lemma 2 follows.

To show that $\hat{\mathcal{H}}_p^n(\tau)$ is a Banach space, proceed as in [7], Theorem 3.7, and use $k \in L_2(0,1)$. We also recall the density result proved in [7], Theorem 3.10: If $g \in \mathcal{H}_p^n(l_2)$, then there exist $g_j \in \mathcal{H}_p^n(l_2)$; $j = 1, 2, ...$; such that $\|g - g_j\|_{\mathcal{H}_p^n(l_2)} \to 0$, as $j \to \infty$, and such that

$$g_j^k = \sum_{i=1}^j I_{(\tau_{i-1}^j, \tau_i^j]} g_j^{ik}(x), \quad k \leq j, \qquad (7)$$

and $g_j^k = 0$, for $k > j$. Here $g_j^{ik} \in C_0^\infty(\mathbb{R}^d)$.

3 Existence of Solutions

Our goal is now to prove the existence result Theorem 4, formulated at the end of this Section.

Take $n = 1$ in the definition of $\hat{\mathcal{H}}_p^n(\tau)$. Thus $g \in L_p = \mathcal{H}_p^0(\tau, l_2)$. Consider (1) with finitely many stochastic terms, each g^k being of the simple structure (7):

$$u(t,x,\omega) = \int_0^t k(t-s)\Delta u(s,x,\omega)\, ds + \sum_{k=1}^m \int_0^t g^k(s,x,\omega)\, dw_s^k. \qquad (8)$$

Define

$$u(t,x,\omega) \stackrel{\text{def}}{=} \sum_{k=1}^m \int_0^t S(t-s) g^k(s,x,\omega)\, dw_s^k. \qquad (9)$$

The resolvent $S(t) \subset B(X)$ (take, e.g., $X = L_p(\mathbb{R}^d)$) satisfies

$$S(t)y = y + \int_0^t k(t-s)\Delta S(s) y\, ds, \quad y \in D(\Delta), \quad t \geq 0. \qquad (10)$$

In fact, see [9], one has a kernel representation for S, such that $S(t-s)g^k(x)$ is bounded in $x \in \mathbb{R}^d$, $t \in [0,T]$. Hence u is welldefined. By the stochastic Fubini theorem, see, e.g., p. 159 of [8], and by (10), it follows that u as defined in (9) satisfies (8) a.s., $t \geq 0$.

Our next purpose is to obtain apriori bounds on u. In the case $\alpha = 1$, these are implied by the key result of [6]. This result is not immediately applicable in the case $\alpha < 1$, and so, to prove the needed estimates, we proceed differently.

Lemma 3 *Let $\alpha \in (\frac{1}{2}, 1)$, $g \in L_p([0,T] \times \mathbb{R}^d; l_2)$. Then*

$$\int_{\mathbb{R}^d} \int_0^T \Big(\int_0^t |\nabla S(t-s) g(s,x)|_{l_2}^2 \, ds \Big)^{\frac{p}{2}} \, dt \, dx \leq c \int_{\mathbb{R}^d} \int_0^T |g(t,x)|_{l_2}^p \, dt \, dx, \quad (11)$$

where $c = c(d, p, \alpha, T)$.

.

Proof of Lemma 3. Take the subadditive map

$$g \mapsto \Big(\int_0^t |\nabla S(t-s) g(s,x)|_{l_2}^2 \, ds \Big)^{\frac{1}{2}}.$$

If this is shown to map

$$L_\infty((0,T) \times \mathbb{R}^d; l_2) \quad \to \quad L_\infty((0,T) \times \mathbb{R}^d; \mathbb{R}), \quad (12)$$

and

$$L_2((0,T) \times \mathbb{R}^d; l_2) \quad \to \quad L_2((0,T) \times \mathbb{R}^d; \mathbb{R}); \quad (13)$$

then, by the Marcinkiewicz interpolation theorem, (11) follows.

To prove (12), one argues as follows.

Suppose we can show that for any $h^k \in L^\infty(\mathbb{R}^d; l_2)$, and for $i = 1, ...d$;

$$\sup_{x \in \mathbb{R}^d} \Big| \frac{\partial}{\partial x_i} S(t) h^k(x) \Big|_{l_2}^2 \leq c t^{-\alpha} \sup_{x \in \mathbb{R}^d} |h^k(x)|_{l_2}^2, \quad (14)$$

with $c = c(\alpha, d)$. Replace t by $t - s$ in (14), and integrate in s over $[0,t]$. This gives

$$\sup_{x \in \mathbb{R}^d, 0 \leq t \leq T} \int_0^t |\nabla S(t-s) g(s,x)|_{l_2}^2 \, ds \leq c \sup_{x \in \mathbb{R}^d, 0 \leq t \leq T} |g(t,x)|_{l_2}^2, \quad (15)$$

which is (12).

To prove (14), take Laplace transforms in t in the resolvent equation, solve for the transform of $S(t)h^k(x)$, and invert. This results in

$$S(t) h^k(x) = (2\pi i)^{-1} \int_{\Gamma_{1,\psi}} e^{\lambda t} [I - \lambda^{-\alpha} \Delta]^{-1} \lambda^{-1} h^k(x) \, d\lambda, \quad (16)$$

where

$$\Gamma_{1,\psi} = \{e^{it} \mid |t| \leq \psi\} \cup \{\rho e^{i\psi} \mid 1 < \rho < \infty\} \cup \{\rho e^{-i\psi} \mid 1 < \rho < \infty\},$$

and $\psi \in (\frac{\pi}{2}, \pi)$. In (16), use analyticity, change variables and apply $\frac{\partial}{\partial x_i}$. This gives

$$\frac{\partial}{\partial x_i} S(t) h^k(x) = (2\pi i)^{-1} t^{-\alpha} \int_{\Gamma_{1,\psi}} e^s s^{\alpha-1} \frac{\partial}{\partial x_i} (\mu - \Delta)^{-1} h^k(x)\, ds, \qquad (17)$$

where $\mu = (\frac{s}{t})^\alpha$ is complex-valued. Consequently, $\frac{\partial}{\partial x_i}(\mu - \Delta)^{-1} h^k(x)$ needs to be evaluated. One obtains, after some calculations,

$$(\mu - \Delta)^{-1} h^k(x) = c(d) \Big(\frac{\mu^{\frac{\nu}{2}}}{r^\nu} K_\nu(\mu^{\frac{1}{2}} r) * h^k \Big)(x), \qquad (18)$$

where $\nu = \frac{d}{2} - 1$, $r^2 = \sum_{i=1}^d x_i^2$, and where $K_\nu(z)$ is the modified Bessel function of second kind of order ν.

For infinite rays Γ_τ originating at the origin one has

$$|\tau|^\nu K_\nu(\tau) \in L_1(\Gamma_\tau); \quad |\tau|^{\nu+1} K_\nu'(\tau) \in L_1(\Gamma_\tau), \qquad (19)$$

uniformly in $|\arg \Gamma_\tau| \le \theta < \frac{\pi}{2}$.

Now use (18) and (19) in (17), recall Hölders inequality, estimate, and sum in k. The relation (14) follows - hence also (12).

To obtain (13) one argues in much the same way. Lemma 3 is proved.

To proceed, observe that Burkholder-Davis-Gundys inequality can be applied to the martingale

$$\nabla u = \sum_{k=1}^m \int_0^t \nabla S(t-s) g^k(s, x, \omega)\, dw_s^k.$$

This yields, when combined with (11),

$$E \int_{\mathbb{R}^d} \int_0^T \sup_{0 \le s \le t} |\nabla u|^p(t, x)\, dt\, dx \le c(p, \alpha, d, T) E \int_{\mathbb{R}^d} \int_0^T |g(s, x, \omega)|_{l_2}^p\, ds\, dx.$$

The solution u can be estimated in an analogous fashion, using modified Bessel functions, to obtain

$$E \int_{\mathbb{R}^d} \sup_{0 \le s \le t} |u|^p\, dx \le c(p, \alpha, d, t) E \int_{\mathbb{R}^d} \int_0^t |g(s, x, \omega)|_{l_2}^p\, ds\, dx. \qquad (20)$$

In addition, observe that $\|u_{xx}\|^p_{\mathcal{H}_p^{-1}} \le c \|u_x\|^p_{L_p}$, and so the right side (20) dominates $\|u_{xx}\|^p_{\mathcal{H}_p^{-1}}$.

Finally take an arbitrary $g \in \mathcal{H}_p^1(l_2)$, and approximate this g in the manner above by simpler functions g_j. Each g_j gives a solution u_j, and by the convergence of $\{g_j\}$ in $L_p(\Omega \times (0, T) \times \mathbb{R}^d, l_2)$, one has that $\{u_j\}$ is a Cauchy-sequence in $\hat{\mathcal{H}}_p^1$. By completeness, there exists u to which $\{u_j\}$ converges. Some additional analysis yields that u solves (1) in the sense (3). One has proved:

Theorem 4 *Let $\alpha \in (\frac{1}{2}, 1)$; $p \geq 2$. Assume that*

$$g \in L_p\big((0,T) \times \Omega, \mathcal{P}, L_p(\mathbb{R}^d; l_2)\big).$$

Then there exists a unique $u \in \hat{\mathcal{H}}_p^1$ such that

$$E \int_{\mathbb{R}^d} \sup_{0 \leq s \leq T} |u|^p \, dx + E \int_0^T \int_{\mathbb{R}^d} \sup_{0 \leq s \leq t} |\nabla u|^p \, dx \, dt$$
$$+ \|u_{xx}\|^p_{\mathcal{H}_p^{-1}} \leq cE \int_0^T \int_{\mathbb{R}^d} |g(t,x,\omega)|^p_{l_2} \, dx \, dt,$$

and such that, for $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$\big(u(t,\cdot), \phi(\cdot)\big) = \int_0^t k(t-s)\big(\Delta u(s,\cdot), \phi(\cdot)\big) \, ds + \sum_{k=1}^\infty \int_0^t \big(g^k(s,\cdot), \phi(\cdot)\big) \, dw_s^k,$$

a.s. for all $t \in [0,T]$.

4 Additional Time-regularity

It is not difficult to observe that some time-regularity is lacking in Theorem 4 above. To see this, argue as follows. In (1), a time-derivative of order α corresponds to a second order derivative in space. The stochastic series in (1) is, roughly, $C^{\frac{1}{2}}(L_p)$. But, by Theorem 4, $\Delta u \in \mathcal{H}_p^{-1}$ and the smoothing out (in time) by the kernel $t^{-1+\alpha}$ is not enough to give the deterministic integral the same degree of smoothness as the stochastic series. One therefore conjectures that Δu has some additional time-regularity. This is, in fact, the case:

Theorem 5 *Let p, α, g be as in the assumptions of Theorem 4. Let u be the solution given by Theorem 4. Take $\epsilon > 0$ arbitrary, but such that $\frac{1}{2} - \epsilon \neq \frac{1}{p}$, $\frac{1}{2} - \frac{\alpha}{2} - \epsilon \neq \frac{1}{p}$. Then*

$$(i) \quad u \in L_p\Big(\Omega; H_p^{\frac{1}{2}-\epsilon}\big([0,T]; L_p(\mathbb{R}^d)\big)\Big),$$

$$(ii) \quad u \in L_p\Big(\Omega; H_p^{\frac{1}{2}-\frac{\alpha}{2}-\epsilon}\big([0,T]; H_p^1(\mathbb{R}^d)\big)\Big),$$

$$(iii) \quad u \in L_p\Big(\Omega; H_p^{\frac{1}{2}-\alpha-\epsilon}\big([0,T]; H_p^2(\mathbb{R}^d)\big)\Big).$$

The norm of u in the respective space is bounded by (a constant times) the norm of g in $L_p\big((0,T) \times \Omega \times \mathbb{R}^d; l_2\big)$.

An interpolation between (ii) and (iii) yields

$$u \in L_p\big(\Omega; L_p([0,T]; H_p^{\frac{1}{\alpha}-\epsilon})\big), \quad \frac{1}{2} < \alpha < 1.$$

For $\alpha = 1$ (the stochastic heat equation) the result obtained in [7] is $u \in L_p\big(\Omega; L_p([0,T]; H_p^1)\big)$. In forthcoming work we will analyze the apparent loss of regularity when moving from the stochastic heat equation to the stochastic integral equation.

Outline of proof of Theorem 5 (ii). Let $\epsilon > 0$ be such that $\frac{1}{2} - \frac{\alpha}{2} - \epsilon > 0$. We claim that, for fixed ω, $u_x \in H_p^{\frac{1}{2}-\frac{\alpha}{2}-\epsilon}\big([0,T]; L_p(\mathbb{R}^d)\big)$. By [10], p.29, this amounts to showing that

$$v \overset{\text{def}}{=} \big(\frac{d}{dt}\big)^{\frac{1}{2}-\frac{\alpha}{2}-\epsilon} u_x = \frac{d}{dt}\big(t^{-\frac{1}{2}+\frac{\alpha}{2}+\epsilon} * S * g_x\big) \in L_p\big((0,T)\times\mathbb{R}^d\big)$$

with $\|v\|_{L_p((0,T)\times\mathbb{R}^d)}$ being equivalent to $\|u_x\|_{H_p^{\frac{1}{2}-\frac{\alpha}{2}-\epsilon}([0,T];L_p(\mathbb{R}^d))}$.

Write $v = F * g_x$, where $F(t) \overset{\text{def}}{=} \frac{d}{dt}\big(t^{-\frac{1}{2}+\frac{\alpha}{2}+\epsilon} * S\big)$. The convolution $F * g_x$ is welldefined as an Ito integral, since $E\big\{\int_0^t |F(t-s)g_x(s)|_{l_2}^2\, ds\big\} < \infty$. Computing the Laplace transform of $F(t)$ gives

$$\tilde{F}(t) = \lambda^{-\frac{1}{2}-\frac{\alpha}{2}-\epsilon}\big(I - \lambda^{-\alpha}\Delta\big)^{-1},$$

and so

$$F(t)h^k(x) = (2\pi i)^{-1}\int_{\Gamma_{1,\psi}} \exp s\big(st^{-1}\big)^{-\frac{1}{2}+\frac{\alpha}{2}-\epsilon}(\mu-\Delta)^{-1}h^k(x)t^{-1}\, ds,$$

where, as in the proof of Theorem 4, $\mu = \big(st^{-1}\big)^{\alpha}$. Representing $(\mu - \Delta)^{-1}$ with Bessel functions, and estimating, results in

$$\sum_k \Big|\frac{\partial}{\partial x_i}F(t)h^k(x)\Big|^2 \le ct^{-1+2\epsilon}\sup_x \sum_k |h^k(x)|^2,$$

and so

$$\sup_{x\in\mathbb{R}^d}\Big(\int_0^t \Big|\frac{\partial}{\partial x_i}F(t-s)g(s,x)\Big|_{l_2}^2\, ds\Big)^{\frac{1}{2}} \le ct^{\epsilon}\|g(s,x)\|_{L_\infty((0,t)\times\mathbb{R}^d;l_2)}.$$

L_2-estimates are obtained in an analogous way.

Hence, by the Burkholder-Davis-Gundy inequality and after applying the Marcinkiewicz interpolation theorem,

$$\begin{aligned}
E\|u_x\|^p_{H_p^{\frac{1}{2}-\frac{\alpha}{2}-\epsilon}([0,T];L_p(\mathbb{R}^d))} &\le cE\int_{\mathbb{R}^d}\int_0^T |v|^p\, dt\, dx \\
= c\int_{\mathbb{R}^d}\int_0^T E\Big(\sum_k \int_0^t \nabla F(t-s)g^k(s,x,\omega)\, dw_s^k\Big)^p\, dt\, dx \\
\le& c\int_{\mathbb{R}^d}\int_0^T E\Big(\int_0^t |\nabla F(t-s)g(s,x,\omega)|_{l_2}^2\, ds\Big)^{\frac{p}{2}}\, dt\, dx \\
=& cE\int_{\mathbb{R}^d}\int_0^T \Big(\int_0^t |\nabla F(t-s)g(s,x,\omega)|_{l_2}^2\, ds\Big)^{\frac{p}{2}}\, dt\, dx \\
\le& cE\int_{\mathbb{R}^d}\int_0^T |g(t,x,\omega)|_{l_2}^p\, dt\, dx,
\end{aligned}$$

which is (ii).

The relations (i), (iii) are proved in much the same fashion. Theorem 5 follows.

We finally remark that the statements (i)-(iii) of Theorem 5 can be slightly strengthened as follows. Take, e.g., (i), which states that

$$D_t^{\frac{1}{2}-\epsilon} S * g \in L_p\big((0,T) \times \Omega \times \mathbb{R}^d\big).$$

An examination of the proof reveals that one in fact has somewhat more, namely

$$\big(M(t)\big)^{-1} D_t^{\frac{1}{2}} S * g \in L_p((0,T) \times \Omega \times \mathbb{R}^d),$$

where $M(t) > 0$ is any function such that $\int_0^1 (tM^2(t))^{-1}\, dt < \infty$.

References

[1] Ph. Clément and G. Da Prato, *Some results on stochastic convolutions arising in Volterra equations perturbed by noise,* Rend.Mat.Acc.Lincei, ser. IX, vol.VII (1996), 147–153.

[2] Ph. Clément, G. Da Prato and J. Prüss, *White noise perturbation of the equations of linear parabolic viscoelasticity*, Rend.Istit.Mat.Univ.Trieste, XXIX, (1997), 207–220.

[3] Ph. Clément, G. Gripenberg and S-O. Londen, *Schauder estimates for equations with fractional derivatives*, Trans. A.M.S., 352 (2000), 2239-2260.

[4] Ph. Clément and S-O.Londen, *Regularity aspects of fractional evolution equations,* Rend.Istit.Mat.Univ.Trieste, XXXI, (2000), 19–30.

[5] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions.* Cambridge University Press, 1990.

[6] N. V. Krylov, *A parabolic Littlewood-Paley inequality with applications to parabolic equations*, Topological Methods in Nonlinear Analysis, Journal of the Juliusz Schauder Center 4 (1994), 355–364.

[7] N. V. Krylov, *An analytic approach to SPDEs.* In Stochastic Partial Differential Equations: Six Perspectives, R. A. Carmona and B. Rozovskii, eds. A.M.S. Mathematical Surveys and Monographs vol. 64 (1999), pp. 185–242.

[8] Ph. Protter, *Stochastic Integration and Differential Equations,* Springer-Verlag, 1990.

[9] J. Prüss, *Poisson estimates and maximal regularity for evolutionary integral equations in L_p-spaces*, Rend.Istit.Mat.Univ.Trieste, XXVIII, (1997), 287–321.

[10] R. Zacher, *Quasilinear parabolic problems with nonlinear boundary conditions*, Ph.D. thesis, Martin-Luther-Universität Halle-Wittenberg, 2003.

[11] R. Zacher, *Maximal regularity of type L_p for abstract parabolic Volterra equations*, J. Evol.Equ., 5 (2005), 79–103.

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