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Abstract: *Assume that X is a continuous square integrable process with zero mean defined on some probability space (Ω, F, P). The classical characterization due to P. Lévy says that X is a Brownian motion if and only if X and $X_t^2 - t$, $t \geq 0$ are martingales with respect to the intrinsic filtration $\mathbb{F}^X$. We extend this result to fractional Brownian motion.*

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1 Introduction

In the classical stochastic analysis Lévy's characterization result for standard Brownian motion is a fundamental result. We extend Lévy's characterization result to fractional Brownian motion giving three properties necessary and sufficient for the process X to be a fractional Brownian motion. Fractional Brownian motion is a self-similar Gaussian process with stationary increments. However, these two properties are not explicitly present in the three conditions we shall give.

Fractional Brownian motion is a popular model in applied probability, in particular in teletraffic modelling and in finance. Fractional Brownian motion is not a semimartingale and there has been lot of research how to define stochastic integrals with respect to fractional Brownian motion. Big part of the developed theory depends on the fact that fractional Brownian motion is a Gaussian process. Since we want to prove that X is a special Gaussian process, we cannot use this machinery for our proof. Lévy's characterization result is based on Itô calculus. We cannot do computations using the process X. Instead, we use representation of the process X with respect to a certain martingale. In this way we can do computations using classical stochastic analysis.

Fractional Brownian motion

A continuous square integrable centered process X with $X_0 = 0$ is a *fractional Brownian motion* with self-similarity index $H \in (0,1)$ if it is a Gaussian process with covariance function

$$\mathbb{E}\left(X_s X_t\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1.1}$$

If X is a continuous Gaussian process with covariance (1.1), then obviously X has stationary increments and X is self-similar with index H. Mandelbrot named the Gaussian process X from (1.1) as *fractional Brownian motion*, and proved an important representation result for fractional Brownian motion in terms of standard Brownian motion in [2]. For a history on the research concerning fractional Brownian motion before Mandelbrot we refer to [3].

Characterization of fractional Brownian motion

Throughout this paper we work with special partitions. For $t > 0$ we put $t_k := t\frac{k}{n}$, $k = 0, \ldots, n$. $\mathbb{F}^X$ is the filtration generated by the process X.

Fix $H \in (0,1)$. Fractional Brownian motion has the following three properties:

(a) The sample paths of the process X are Hölder continuous with any $\beta \in (0, H)$.

(b) For $t > 0$ we have

$$n^{2H-1}\sum_{k=1}^{n}(X_{t_k}-X_{t_{k-1}})^2 \overset{L^1(P)}{\longrightarrow} t^{2H}, \tag{1.2}$$

as $n \to \infty$.

(c) The process

$$M_t = \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}dX_s \tag{1.3}$$

is a martingale with respect to the filtration $\mathbb{F}^X$.

If the process X satisfies **(a)**, we say that it is Hölder up to H. The property **(b)** is *weighted quadratic variation* of the process X, and the process M in **(c)** is the *fundamental martingale* of X. It follows from the property **(a)**, that the integral (1.3) can be understood as a Riemann-Stieltjes integral (see [4] and subsection 2.2 for more details).

Fractional Brownian motion satisfies the property **(a)**: From (1.1) we have that

$$\mathbb{E}(X_t - X_s)^2 = (t-s)^{2H}.$$

Since the process X is a Gaussian process we obtain from Kolmogorov's theorem [5, Theorem I.2.1, p.26] that the process X is Hölder continuous with $\beta < H$. Fractional Brownian motion satisfies also the property **(b)**. The proof of this fact is based on the self-similarity and on the ergodicity of the fractional Gaussian noise sequence $Z_k := X_k - X_{k-1},\ k \geq 1$. The fact that property **(c)** holds for fractional Brownian motion was known to Molchan [3], and recently rediscovered by several authors (see [4] and [3]). We summarize our main result.

Theorem 1.1. *Assume that X is a continuous square integrable centered process with $X_0 = 0$. Then the following are equivalent:*

- *The process X is a fractional Brownian motion with self-similarity index $H \in (0,1)$.*
- *The process X has properties* **(a)**, **(b)** *and* **(c)** *with some $H \in (0,1)$.*

Discussion

If $H = \frac{1}{2}$, then the assumption **(c)** means that the process X is a martingale. If X is a martingale, then the condition **(b)** means that $X_t^2 - t$ is a martingale. Hence we obtain the classical Lévy characterization theorem, when $H = \frac{1}{2}$. Note that in this case the property **(a)** follows from the fact that X is a standard Brownian motion.

Fractional Brownian motion X has the following property: for $T > 0$

$$\sum_{k=1}^{n}|X_{T\frac{k}{n}} - X_{T\frac{k-1}{n}}|^{\frac{1}{H}} \overset{L^1(P)}{\to} E|X_1|^{\frac{1}{H}}T \tag{1.4}$$

as $n \to \infty$. This gives another possibility to generalize the quadratic variation property of standard Brownian motion. However, it seems difficult to replace the condition **(b)** by the condition (1.4).

In the next section we explain the main steps in our proof. The rest of the paper is devoted to technical details of the proof, which are different for $H > \frac{1}{2}$ and $H < \frac{1}{2}$.

2 The proof of Theorem 1.1

2.1 A consequence of (b)

We use the following notation: $\stackrel{L^1(P)}{\to}$ means convergence in the space $L^1(P)$, $\stackrel{P}{\to}$ (resp. $\stackrel{\text{a.s.}}{\to}$) means convergence in probability (resp. almost sure convergence) and $B(a,b)$ is the beta integral $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$, defined for $a, b \geq 0$. The notation $X_n \leq Y + o_P(1)$ means that we can find random variables ϵ_n such that $\epsilon_n = o_P(1)$ and $X_n \leq Y + \epsilon_n$. If in addition $X = P - \lim X_n$, then we also have $X \leq Y$.

We fix now t and let $\mathcal{R}_t := \{s \in [0,t] : \frac{s}{t} \in Q\}$. Note that the set $\mathcal{R}_t$ is a dense set on the interval $[0,t]$. Fix now also $s \in \mathcal{R}_t$ and let $\tilde{n} = \tilde{n}(s)$ be a subsequence such that $\tilde{n}\frac{s}{t} \in \mathbb{N}$.

Lemma 2.1. *Fix $t > 0$ and $s \in \mathcal{R}_t$ and $\tilde{n}$ such that $\tilde{n}\frac{s}{t} \in \mathbb{N}$ and $\tilde{n} \to \infty$. Then*

$$\tilde{n}^{2H-1} \sum_{k=\tilde{n}\frac{s}{t}+1}^{\tilde{n}} \left(X_{t\frac{k}{\tilde{n}}} - X_{t\frac{k-1}{\tilde{n}}} \right)^2 \stackrel{P}{\longrightarrow} t^{2H-1}(t-s). \tag{2.1}$$

Proof. We have that

$$\begin{aligned}
\tilde{n}^{2H-1} \sum_{k=1}^{\tilde{n}\frac{s}{t}} (\Delta X_{t\frac{k}{\tilde{n}}})^2 &= \tilde{n}^{2H-1} \sum_{k=1}^{\tilde{n}\frac{s}{t}} (\Delta X_{\frac{sk}{\tilde{n}\frac{s}{t}}})^2 \\
&= \left(\tilde{n}\frac{s}{t}\right)^{2H-1} \cdot (\frac{t}{s})^{2H-1} \sum_{k=1}^{\tilde{n}\frac{s}{t}} (\Delta X_{\frac{sk}{\tilde{n}\frac{s}{t}}})^2 \\
&\stackrel{L^1(P)}{\to} s^{2H} \cdot (\frac{t}{s})^{2H-1} = st^{2H-1}.
\end{aligned}$$

Since $\tilde{n}^{2H-1} \sum_{k=1}^{\tilde{n}} (\Delta X_{t\frac{k}{\tilde{n}}})^2 \stackrel{L^1(P)}{\to} t^{2H}$, we obtain the proof. ☐

In what follows we shall write n pro $\tilde{n}$ and t_k pro $t\frac{k}{n}$.

2.2 Representation results

Throughout the paper we shall use the following notation. Put

$$Y_t = \int_0^t s^{\frac{1}{2}-H} dX_s; \tag{2.2}$$

then we have $X_t = \int_0^t s^{H-\frac{1}{2}} dY_s$ and we can write the fundamental martingale M as

$$M_t = \int_0^t (t-s)^{\frac{1}{2}-H} dY_s. \tag{2.3}$$

The equation (2.3) is a generalized Abel integral equation and the process Y can be expressed in terms of the process M:

$$Y_t = \frac{1}{\left(H-\frac{1}{2}\right) B_1} \int_0^t (t-s)^{H-\frac{1}{2}} dM_s \tag{2.4}$$

with $B_1 = B(H-\frac{1}{2}, \frac{3}{2}-H)$.

We work also with the martingale $W = \int_0^t s^{H-\frac{1}{2}} dM_s$. We have $[W]_t = \int_0^t s^{2H-1} d[M]_s$ and $[M]_t = \int_0^t s^{1-2H} d[W]_s$.

Note that all the integrals, even the Wiener integrals, can be understood as pathwise Riemann-Stieltjes integrals.

For $H > \frac{1}{2}$ we use the following representation result.

Lemma 2.2. *Assume that $H > \frac{1}{2}$ and* **(a)** *and* **(c)**. *Then the process X has the representation*

$$X_t = \frac{1}{B_1} \int_0^t \left(\int_u^t s^{H-\frac{1}{2}} (s-u)^{H-\frac{3}{2}} \, ds \right) dM_u, \tag{2.5}$$

Proof. Integration by parts in (2.4) gives:

$$Y_t = \frac{1}{B_1} \int_0^t (t-s)^{H-\frac{3}{2}} M_s ds.$$

Next, by using integration by parts and Fubini theorem we obtain

$$\begin{aligned}
X_t &= \int_0^t s^{H-\frac{1}{2}} dY_s = t^{H-\frac{1}{2}} Y_t - (H-\frac{1}{2}) \int_0^t s^{H-\frac{3}{2}} Y_s ds \\
&= \frac{t^{H-\frac{1}{2}}}{B_1} \int_0^t (t-s)^{H-\frac{3}{2}} M_s ds - \frac{H-\frac{1}{2}}{B_1} \int_0^t s^{H-\frac{3}{2}} \int_0^s (s-u)^{H-\frac{3}{2}} M_u du ds \\
&= \frac{t^{H-\frac{1}{2}}}{(H-\frac{1}{2})B_1} \int_0^t (t-s)^{H-\frac{1}{2}} dM_s - \frac{1}{B_1} \int_0^t s^{H-\frac{3}{2}} \int_0^s (s-u)^{H-\frac{1}{2}} dM_u ds \\
&= \frac{t^{H-\frac{1}{2}}}{(H-\frac{1}{2})B_1} \int_0^t (t-s)^{H-\frac{1}{2}} dM_s - \frac{1}{B_1} \int_0^t \left[\int_u^t s^{H-\frac{3}{2}} (s-u)^{H-\frac{1}{2}} ds \right] dM_u \\
&= \frac{1}{B_1} \int_0^t \left[\frac{t^{H-\frac{1}{2}}}{H-\frac{1}{2}} (t-u)^{H-\frac{1}{2}} - \int_u^t s^{H-\frac{3}{2}} (s-u)^{H-\frac{1}{2}} ds \right] dM_u \\
&= \frac{1}{B_1} \int_0^t \left[\int_u^t s^{H-\frac{1}{2}} (s-u)^{H-\frac{3}{2}} ds \right] dM_u.
\end{aligned}$$

This proves claim (2.5). ∎

For $H < \frac{1}{2}$ we use the following representation result, which can be proved as [4, Theorem 5.2].

Lemma 2.3. *Assume that $H < \frac{1}{2}$ and* **(a)** *and* **(c)**. *Then the process X has the representation*

$$X_t = \int_0^t z(t,s)dW_s, \tag{2.6}$$

with the kernel

$$z(t,s) = \left(\frac{s}{t}\right)^{1/2-H}(t-s)^{H-1/2} + (1/2-H)s^{1/2-H}\int_s^t u^{H-3/2}(u-s)^{H-1/2}du$$

2.3 The proof of the main result

We give the structure of the proof, and prove the main result. We know from Lemma 2.1

$$n^{2H-1}\sum_{k=n\frac{s}{t}+1}^{n}\left(X_{t\frac{k}{n}} - X_{t\frac{k-1}{n}}\right)^2 \xrightarrow{P} c_H t^{2H-1}(t-s).$$

We show, separately for $H < \frac{1}{2}$ and $H > \frac{1}{2}$, that the following asymptotic expansion holds

$$\begin{aligned} & n^{2H-1}\sum_{k=n\frac{s}{t}+1}^{n}\left(X_{t_k} - X_{t_{k-1}}\right)^2 \\ = \quad & n^{2H-1}\sum_{k=n\frac{s}{t}+1}^{n}\int_s^t \left(h_k^t(u)\right)^2 d[M]_u + o_P(1) \end{aligned} \tag{2.7}$$

with a sequence of deterministic functions h_k^t, depending on H. Here $o_P(1)$ means convergence to zero in probability.

Note that an H- fractional Brownian motion B^H also satisfies (**a**), (**b**) and (**c**), and hence it also satisfies the asymptotic expansion

$$\begin{aligned} & n^{2H-1}\sum_{k=n\frac{s}{t}+1}^{n}\left(B_{t_k}^H - B_{t_{k-1}}^H\right)^2 \\ = \quad & n^{2H-1}c_H(2-2H)\sum_{k=n\frac{s}{t}+1}^{n}\int_s^t \left(h_k^t(u)\right)^2 s^{1-2H}ds + o_P(1) \end{aligned} \tag{2.8}$$

with the same set of functions h_k^t.

Moreover, we show, that $[W] \sim \text{Leb}$ and the density $\rho^t(u) = \frac{d[W]_u}{du}$ satisfies $0 < c \le \rho^t(u) \le C < \infty$ with some constants c, C. With this information, we can finish the proof. Then

$$
\begin{aligned}
& P-\lim_n n^{2H-1}\sum_{k=n\frac{s}{t}+1}^{n}\int_s^t \left(h_k^t(s)\right)^2\rho^t(u)u^{1-2H}du \\
= & P-\lim_n n^{2H-1}\sum_{k=n\frac{s}{t}+1}^{n}\int_s^t \left(h_k^t(u)\right)^2 d[M]_u \\
= & t^{2H-1}(t-s) \\
= & P-\lim_n n^{2H-1}\sum_{k=n\frac{s}{t}+1}^{n}(B_{t_k}^H-B_{t_{k-1}}^H)^2 \\
= & P-\lim n^{2H-1}c_H(2-2H)\sum_{k=n\frac{s}{t}+1}^{n}\int_s^t \left(h_k^t(s)\right)^2 u^{1-2H}du.
\end{aligned}
$$

Since the set $\mathcal{R}_t$ is a dense set on the interval $[0,t]$ we can conclude from the above that $\rho^t(u)=c_H(2-2H)$. This means that the martingale M is a Gaussian martingale with the bracket $[M]_u=c_Hu^{2-2H}$ and by the pathwise representation results in the subsection 2.2 the process X is an H-fBm.

If M is a continuous square integrable martingale, then the bracket of M is denoted by $[M]$. Recall that in this case we have

$$
[M]_t=P-\lim_{|\pi^n|\to 0}\sum_{k=1}^{n}(M_{t_k}-M_{t_{k-1}})^2.
$$

2.4 Auxiliary lemmas

In the proof of (2.7) we use several times the following lemmas. Let M be a continuous martingale. Put $I_2(M)_t:=\int_0^t M_sdM_s$. Two continuous martingales M,N are *(strongly) orthogonal* if $[M,N]=0$; we write this as $M\perp N$. We use also notation $(N\cdot M)$ for the integral $(N\cdot M)_t=\int_0^t N_sdM_s$.

Lemma 2.4. *Assume that $M^{n,k}$ is a double array of continuous square integrable martingales with the properties*

(i) With n fixed and $k\neq l$ $M^{n,k}$ and $M^{n,l}$ are orthogonal martingales.

(ii) $\sum_{k=1}^{k_n}[M^{n,k}]_t\le C$, where C is a constant.

(iii) $\max_k[M^{n,k}]_t\xrightarrow{P}0$ as $n\to\infty$.

Then

$$
\sum_{k=1}^{k_n}I_2(M^{n,k})_t\overset{L^2(P)}{\to}0 \tag{2.9}
$$

as $k_n\to\infty$.

Proof. Since the martingales $M^{n,k}$ are pairwise orthogonal, when n is fixed, the same is true for the iterated integrals $I_2(M^{n,k})$. Hence

$$E\left(\sum_{k=1}^{k_n} I_2(M^{n,k})_t\right)^2 = \sum_{k=1}^{k_n} E\left(I_2(M^{n,k})_t\right)^2;$$

we can now use [1, Theorem 1, p. 354], which states that

$$E\left(I_2(M^{n,k})_t\right)^2 \leq B_{2,2}^2 E[M^{n,k}]_t^2.$$

But

$$\sum_{k=1}^{k_n}[M^{n,k}]_t^2 \leq \max_k [M^{n,k}]_t \sum_{k=1}^{k_n}[M^{n,k}]_t \xrightarrow{P} 0$$

as $n \to \infty$. The claim (2.9) now follows, since $\sum_{k=1}^{k_n}[M^{n,k}]_t^2 \leq C^2$. $\square$

Lemma 2.5. *Assume that $M^{n,k}$ and $N^{n,k}$ are double array of continuous square integrable martingales with the properties*

(i) With fixed n and $k \neq l$ $N^{n,l}$ and $N^{n,k}$ are orthogonal martingales, if $l < k$, then $M^{n,l} \perp N^{n,k}$, and for i, j, k, l we have $(N^{n,i} \cdot M^{n,j}) \perp (N^{n,k} \cdot M^{n,l})$.

(ii) $\sum_{k=1}^{k_n}[M_t^{n,k}] \leq C$ and $\sum_{k=1}^{k_n}[N_t^{n,k}] \leq C$.

(iii) The martingales $M^{n,k}$ are bounded by a constant K and $\max_k [N^{n,k}]_t \xrightarrow{P} 0$.

(iv) $[M^{n,k}N^{n,k}]_t = \left(\left(M^{n,k}\right)^2 \cdot [N^{n,k}]\right)_t$

Then

$$\sum_{k=1}^{k_n} M_t^{n,k} N_t^{n,k} \overset{L^2(P)}{\longrightarrow} 0 \tag{2.10}$$

as $k_n \to \infty$.

Proof. By the assumption (i) we obtain

$$E\left(\sum_{k=1}^{k_n} M_t^{n,k} N_t^{n,k}\right)^2 = \sum_{k=1}^{k_n} E\left(M_t^{n,k} N_t^{n,k}\right)^2 \tag{2.11}$$

By assumption (iv) we have

$$E\left(M_t^{n,k} N_t^{n,k}\right)^2 = E[M_t^{n,k} N_t^{n,k}] = E\left((M_t^{n,k})^2 [N^{n,k}]_t\right).$$

By assumption (ii) the sequence $\sum_k (M_t^{n,k})^2$ is tight, since it is dominated by $\sum_k [M^{n,k}]_t$, and since $\max_k [N^{n,k}]_t \xrightarrow{P} 0$, we have that $\sum_k (M_t^{n,k})^2 [N^{n,k}]_t \xrightarrow{P} 0$. By dominated convergence theorem we obtain the claim in (2.10). $\square$

3 The proof of Theorem 1.1: case of $H > \frac{1}{2}$

3.1 The basic estimation

For the proof we can assume that the martingales M and W, as well as their brackets $[M]$ and $[W]$ are bounded with a deterministic constant L. If this is not the case, we can always stop the processes.

We want to use expression

$$n^{2H-1} \sum_{k=n\frac{s}{t}+1}^{n} \left(X_{t_k} - X_{t_{k-1}}\right)^2$$

to obtain estimates for the increment of the bracket $[M]$, with the help of (2.5).

Use (2.5) to obtain

$$X_{t_k} - X_{t_{k-1}} = \frac{1}{B_1}\left(\int_0^{t_{k-1}} f_k^t(s)\, dM_s + \int_{t_{k-1}}^{t_k} g_k^t(s)\, dM_s\right), \tag{3.1}$$

where we used the notation

$$f_k^t(s) := \int_{t_{k-1}}^{t_k} u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}} du \tag{3.2}$$

and

$$g_k^t(s) := \int_s^{t_k} u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}} du.$$

Rewrite the increment of X as

$$\begin{aligned} & X_{t_k} - X_{t_{k-1}} \\ := \ & \frac{1}{B_1}\left(I_k^{n,1} + I_k^{n,2} + I_k^{n,3}\right) \\ := \ & \frac{1}{B_1}\left(\int_0^{t_{k-2}} f_k^t(s) dM_s + \int_{t_{k-2}}^{t_{k-1}} f_k^t(s) dM_s + \int_{t_{k-1}}^{t_k} g_k^t(s) dM_s\right). \end{aligned} \tag{3.3}$$

The random variables $I_k^{n,j}$ are the final values of the following martingales: put $m_v^1 := \int_0^{t_{k-2}\wedge v} f_k^t(u) dM_u$, $m_v^2 := \int_{t_{k-2}\wedge v}^{t_{k-1}\wedge v} f_k(u) dM_u$ and $m_v^3 := \int_{t_{k-1}\wedge v}^{t_k \wedge v} g_k^t(u) dM_u$, then $I_k^{n,i} = m_t^i$, $i = 1, 2, 3$. Hence we can use stochastic calculus and Itô formula to analyze these random variables.

Next, note the following upper estimate for the functions f_k^t:

$$\begin{aligned} f_k^t(s) &= \int_{t_{k-1}}^{t_k} u^{H-\frac{1}{2}}(u-s)^{H-\frac{3}{2}} du \\ &\leq (t_k)^{H-\frac{1}{2}} (t_{k-1} - s)^{H-\frac{3}{2}} \cdot (t_k - t_{k-1}) \\ &= \left(t\frac{k-1}{n} - s\right)^{H-\frac{3}{2}} \cdot \frac{t}{n} \cdot (t\frac{k}{n})^{H-\frac{1}{2}} \end{aligned} \tag{3.4}$$

note that this estimate is finite for $s \in (0, t_{k-1})$.

Lemma 3.1. *Fix $t>0$ and $s\in\mathcal{R}_t$ and $\tilde n$ such that $\tilde n\frac{s}{t}\in\mathbb{N}$ and $\tilde n\to\infty$. Then there exist two constants $C_1, C_2>0$ such that we have*

$$\begin{aligned}
C_1 t^{2H-1}\int_s^{t-2t/n} u^{2H-1} d[M]_u &\leq \tilde n^{2H-1}\sum_{k=\tilde n\frac{s}{t}+2}^{\tilde n}\int_0^{t\frac{k-2}{n}} (f_k^t(u))^2 d[M]_u \\
&\leq C_2 t^{4H-2}\left([M]_t-[M]_s\right)+R_n^t, \qquad (3.5)
\end{aligned}$$

where $R_n^t = o_P(1)$.

Proof. We continue to write n instead of $\tilde n$ and will not take care of the constants explicitly.

Upper estimate

At first we estimate

$$i^{n,1} := n^{2H-1}\sum_{k=n\frac{s}{t}+2}^{n}\int_0^{t_{k-2}} (f_k^t(u))^2 d[M]_u$$

from above.

From (3.4) we obtain the following estimate for $i^{n,1}$:

$$i^{n,1}\leq n^{2H-3}t^{2H+1}\sum_{k=n\frac{s}{t}+2}^{n}\int_0^{t_{k-2}} (t_{k-1}-u)^{2H-3} d[M]_u. \qquad (3.6)$$

We can assume that $0<s<t$ and $2\leq n\frac{s}{t}\leq n-3$, and rewrite

$$\begin{aligned}
\bar\imath^{n,1} &:= \sum_{k=n\frac{s}{t}+2}^{n}\sum_{i=1}^{k-2}\int_{t_{i-1}}^{t_i}(t_{k-1}-u)^{2H-3}d[M]_u \\
&= \Big(\sum_{i=1}^{n\frac{s}{t}}\sum_{k=n\frac{s}{t}+2}^{n} + \sum_{i=n\frac{s}{t}+1}^{n-2}\sum_{k=i+2}^{n}\Big)\int_{t_{i-1}}^{t_i}(t_{k-1}-u)^{2H-3}d[M]_u \qquad (3.7)\\
&= \sum_{i=1}^{n\frac{s}{t}}\int_{t_{i-1}}^{t_i}\Big(\sum_{k=n\frac{s}{t}+2}^{n}(t_{k-1}-u)^{2H-3}\Big)d[M]_u \\
&\quad + \sum_{i=n\frac{s}{t}+1}^{n-2}\int_{t_{i-1}}^{t_i}\Big(\sum_{k=i+2}^{n}(t_{k-1}-u)^{2H-3}\Big)d[M]_u.
\end{aligned}$$

We estimate the first term in the last equation in (3.7):

$$\begin{aligned}
&\frac{1}{n}\Big(\sum_{k=n\frac{s}{t}+2}^{n}(t_{k-1}-u)^{2H-3}\Big) \\
= &\frac{t}{tn}\Big[(s+\frac{t}{n}-u)^{2H-3}+(s+\frac{2t}{n}-u)^{2H-3}+ \\
&+\cdots+(s+\frac{t(n-1)}{n}-u)^{2H-3}\Big] \\
\leq &\frac{1}{t}\int_{s-u}^{s+t-u}x^{2H-3}dx\leq\frac{1}{t(2-2H)}(s-u)^{2H-2};
\end{aligned}$$

next we estimate the second sum in the last equation of (3.7) similarly and obtain

$$\frac{1}{n}\sum_{k=i+2}^{n}(t_{k-1}-u)^{2H-3}\leq\frac{1}{n}(t_{i+1}-u)^{2H-3}+\frac{1}{(2-2H)t}(t_{i+1}-u)^{2H-2}.$$

We substitute these estimates into (3.7):

$$\begin{aligned}
\bar{i}^{n,1} &\leq \frac{1}{2-2H}n\sum_{i=1}^{\frac{ns}{t}}\int_{t\frac{i-1}{n}}^{t\frac{i}{n}}(s-u)^{2H-2}\frac{1}{t}d[M]_u \\
&+ n\sum_{i=\frac{ns}{t}+1}^{n}\int_{t\frac{i-1}{n}}^{t\frac{i}{n}}\left[\frac{1}{n}(t\frac{i+1}{n}-u)^{2H-3}\right. \\
&\qquad\qquad \left.+\frac{1}{(2-2H)t}(t\frac{i+1}{n}-u)^{2H-2}\right]d[M]_u \\
&\leq \frac{1}{2-2H}\frac{n}{t}\int_0^s(s-u)^{2H-2}d[M]_u+t^{2H-3}n^{-2H+3}([M]_t-[M]_s) \\
&+ \frac{n}{t}(\frac{t}{n})^{2H-2}\frac{1}{2-2H}([M]_t-[M]_s) \\
&\leq \frac{1}{2-2H}\frac{n}{t}\int_0^s(s-u)^{2H-2}d[M]_u+c_Ht^{2H-3}n^{3-2H}([M]_t-[M]_s)
\end{aligned}$$

with $c_H=\frac{1}{2-2H}+1$.

We continue from (3.6) and have

$$i^{n,1}\leq\frac{n^{2H-2}}{2-2H}t^{2H}\int_0^s(s-u)^{2H-2}d[M]_u+c_Ht^{4H-2}([M]_t-[M]_s). \tag{3.8}$$

From assumptions **(a)** and **(c)** we have that the martingale M is Hölder continuous up to $\frac{1}{2}$. This in turn implies that the bracket $[M]$ is Hölder continuous up to 1, and hence the random variable $\int_0^s(s-u)^{2H-2}d[M]_u$ is finite with probability one. This gives the upper bound for (3.5) with $R_n^t=n^{2H-2}t^{2H}\int_0^s(s-u)^{2H-2}d[M]_u$.

Lower bound in (3.5)

We finish the proof of Lemma 3.1 by giving the lower bound. Recall that $f_k^t(u)=\int_{t_{k-1}}^{t_k}v^{H-\frac{1}{2}}(v-u)^{H-\frac{3}{2}}dv$ and this gives the estimate

$$\left(f_k^t(u)\right)^2\geq(t_{k-1})^{2H-1}(t_k-u)^{2H-3}\cdot\frac{t^2}{n^2}. \tag{3.9}$$

We use (3.9) to estimate the sum $i^{n,1}$ from below:

$$\begin{aligned}
i^{n,1} &\geq n^{2H-3}t^2 \sum_{k=n\frac{s}{t}+2}^{n} \int_0^{t_{k-2}} (t_{k-1})^{2H-1} (t_k - u)^{2H-3}\, d[M]_u \\
&= n^{2H-3}t^2 \sum_{i=1}^{n\frac{s}{t}} \int_{t_{i-1}}^{t_i} \Big(\sum_{k=n\frac{s}{t}+2}^{n} (t_{k-1})^{2H-1} (t_k - u)^{2H-3} \Big) d[M]_u \\
&+ \; n^{2H-3}t^2 \sum_{i=n\frac{s}{t}+1}^{n-2} \int_{t_{i-1}}^{t_i} \Big(\sum_{k=i+2}^{n} (t_{k-1})^{2H-1} (t_k - u)^{2H-3} \Big) d[M]_u \\
&\geq n^{2H-3}t^2 \sum_{i=n\frac{s}{t}+1}^{n-2} \int_{t_{i-1}}^{t_i} \Big(\sum_{k=i+2}^{n} (t_{k-1})^{2H-1} (t_k - u)^{2H-3} \Big) d[M]_u
\end{aligned}$$

Next we estimate the last sum from below:

$$\begin{aligned}
&\frac{1}{n} \sum_{k=i+2}^{n} \left(t\frac{k-1}{n} \right)^{2H-1} \left(t\frac{k}{n} - u \right)^{2H-3} \\
\geq\; & \frac{1}{t} \int_{t\frac{i+2}{n}-u}^{t-u} x^{2H-3} \Big(x + u - \frac{1}{n}\Big)^{2H-1} dx \\
\geq\; & \frac{1}{t} \left(t\frac{i+1}{n} \right)^{2H-1} \int_{t\frac{i+2}{n}-u}^{t-u} x^{2H-3} dx \\
\geq\; & t^{2H-2} \left(\frac{i+1}{n} \right)^{2H-1} \frac{\left(t\frac{i+2}{n} - u\right)^{2H-2} - (t-u)^{2H-2}}{2-2H}.
\end{aligned}$$

With this estimate we continue and obtain

$$\begin{aligned}
i^{n,1} \geq \frac{t^{2H} n^{2H-2}}{2-2H} \quad \cdot \\
\cdot \sum_{i=n\frac{s}{t}+1}^{n-2} \left(\frac{i+1}{n} \right)^{2H-1} \int_{t\frac{i-1}{n}}^{t\frac{i}{n}} \left[\left(t\frac{i+2}{n} - u \right)^{2H-2} - (t-u)^{2H-2} \right] d[M]_u.
\end{aligned}$$

Consider the function $h(u) := \left(t\frac{i+2}{n} - u\right)^{2H-2} - (t-u)^{2H-2}$ and estimate it from below using the fact that $u \in (t\frac{i-1}{n}, t\frac{i}{n})$:

$$\begin{aligned}
h(u) &\geq \left(t\frac{i+2}{n} - t\frac{i-1}{n} \right)^{2H-2} - \left(t - t\frac{i}{n} \right)^{2H-2} \\
&\geq \left(\frac{3t}{n} \right)^{2H-2} - \left(\frac{4t}{n} \right)^{2H-2} = \frac{3^{2H-2} - 4^{2H-2}}{n^{2H-2}} t^{2H-2}.
\end{aligned}$$

So,

$$\begin{aligned}
i^{n,1} &\geq (3^{2H-2} - 4^{2H-2}) \frac{t^{2H} n^{2H-2}}{2-2H} \sum_{i=n\frac{s}{t}+1}^{n-2} \int_{t\frac{i-1}{n}}^{t\frac{i}{n}} \left(\frac{i+1}{n} \right)^{2H-1} t^{2H-2} d[M]_u \\
&\geq C_1 t^{2H-1} \sum_{i=n\frac{s}{t}+1}^{n-2} \int_{t\frac{i-1}{n}}^{t\frac{i}{n}} \Big(u + \frac{2t}{n}\Big)^{2H-1} d[M]_u,
\end{aligned}$$

and this gives the lower bound in (3.5). The proof of Lemma 3.1 is now finished. $\square$

Second upper bound

We estimate now the term

$$\int_{t_{k-2}}^{t_{k-1}} f_k^t(s)d[M]_s.$$

Lemma 3.2. *There exists a constant $C_3 > 0$ such that*

$$n^{2H-1} \sum_{k=n\frac{s}{t}+2}^{n} \int_{t_{k-2}}^{t_{k-1}} (f_k^t(u))^2 d[M]_u \le C_3 t^{4H-2}([M]_t - [M]_s). \tag{3.10}$$

Proof. We have the following upper estimate for the function f_k^t:

$$\begin{aligned}
f_k^t(u) &\le t_k^{H-\frac{1}{2}} \int_{t_{k-1}}^{t_k} (v-u)^{H-\frac{3}{2}} dv \\
&= \frac{1}{H-\frac{1}{2}} t_k^{H-\frac{1}{2}} \left((t_k - u)^{H-\frac{1}{2}} - (t_{k-1} - u)^{H-\frac{1}{2}} \right) \\
&\le \frac{1}{H-\frac{1}{2}} t^{H-\frac{1}{2}} \left(\frac{t}{n} \right)^{H-\frac{1}{2}}.
\end{aligned}$$

This gives the claim (3.10).

$\square$

The third estimation

Now we shall deal with terms of the form

$$\int_{t_{k-1}}^{t_k} (g_k^t(s))^2 d[M]_s.$$

Lemma 3.3. *There exists a constant C_4 such that*

$$n^{2H-1} \sum_{k=n\frac{s}{t}+1}^{n} \int_{t_{k-1}}^{t_k} (g_k^t(u))^2 d[M]_u \le C_4 t^{4H-2} \left([M]_t - [M]_s \right). \tag{3.11}$$

Proof. We have that

$$\begin{aligned}
g_k^t(z) &= \int_z^{t\frac{k}{n}} v^{H-\frac{1}{2}} (v-z)^{H-\frac{3}{2}} dv \le (t\frac{k}{n})^{H-\frac{1}{2}} \frac{(t\frac{k}{n} - z)^{H-\frac{1}{2}}}{H-\frac{1}{2}} \\
&\le C(t\frac{k}{n})^{H-\frac{1}{2}} (\frac{t}{n})^{H-\frac{1}{2}} \le C t^{2H-1} (\frac{1}{n})^{H-\frac{1}{2}}.
\end{aligned}$$

This gives the claim (3.11). $\square$

3.2 The proof for the asymptotic expansion

Recall that from (3.3) we have

$$\begin{aligned}
& X_{t_k} - X_{t_{k-1}} \\
= \; & \frac{1}{B_1}\left(\int_0^{t_{k-2}} f_k^t(s)dM_s + \int_{t_{k-2}}^{t_{k-1}} f_k^t(s)dM_s + \int_{t_{k-1}}^{t_k} g_k^t(s)dM_s\right) \\
=: \; & I_k^{n,1} + I_k^{n,2} + I_k^{n,3}.
\end{aligned}$$

Hence

$$\left(X_{t_k} - X_{t_{k-1}}\right)^2 = \left(I_k^{n,1} + I_k^{n,2} + I_k^{n,3}\right)^2.$$

Consider first the terms of the form $(I_k^{n,j})^2$, $j = 1,2,3$. From the Itô formula we have that (we will drop the constant B_1 in what follows)

$$(I_k^{n,1})^2 = \int_0^{t_{k-2}} (f_k^t(v))^2 d[M]_v + 2\int_0^{t_{k-2}} f_k^t(u)\left(\int_0^u f_k^t(v)dM_v\right)dM_u.$$

We shall show that

$$n^{2H-1}\sum_{k=n\frac{s}{t}+2}^{n}\int_0^{t_{k-2}} f_k^t(u)\left(\int_0^u f_k^t(v)dM_v\right)dM_u \xrightarrow{P} 0, \tag{3.12}$$

as $n\to\infty$. Note first that

$$\begin{aligned}
n^{2H-1}\sum_{k=n\frac{s}{t}+2}^{n} f_k^t(u)f_k^t(v) &\le Cn^{2H-1}\sum_{k=n\frac{s}{t}+2}^{n} f_k^t(u)\frac{1}{n}(s-v)^{H-\frac{3}{2}} \\
&\le Cn^{2H-2}(s-v)^{H-\frac{3}{2}}\int_s^t x^{H-\frac{1}{2}}(x-u)^{H-\frac{3}{2}}dx \to 0 \qquad (3.13)
\end{aligned}$$

for all $v < u < s$. Fix $u < s$ and write $w^n(v) := n^{2H-1}\sum_{k=n\frac{s}{t}+2}^{n} f_k^t(u)f_k^t(v)$. Then (3.13) gives that $\sup_{v\le u} w^n(v) \to 0$. We can now use [6, Theorem II.11, p.58], which says that if a predictable sequence of processes converges uniformly in probability to zero, then

$$\sup_{u<s}\left|\int_0^u w^n(v)dM_v\right| \xrightarrow{P} 0.$$

for all $s \le t$. Now we can apply the same theorem again and we get (3.12).

Consider next the sums

$$n^{2H-1}\sum_{k=n\frac{s}{t}+2}^{n}\int_{t_{k-2}}^{t_{k-1}} f_k^t(u)\int_{t_{k-2}}^{u} f_k^t(v)dM_v dM_u$$

and

$$n^{2H-1}\sum_{k=n\frac{s}{t}+1}^{n}\int_{t_{k-1}}^{t_k} g_k^t(u)\int_{t_{k-1}}^{u} g_k^t(v)dM_v dM_u.$$

It is quite straightforward to check that the assumptions of the Lemma 2.4 are satisfied with martingales

$$N_v^{n,k} := n^{H-\frac{1}{2}} \int_{t_{k-2}\wedge v}^{t_{k-1}\wedge v} f_k^t(u)dM_u$$

and

$$\tilde{N}_v^{n,k} := n^{H-\frac{1}{2}} \int_{t_{k-1}\wedge v}^{t_k\wedge v} g_k^t(u)dM_u.$$

Hence both sums are of the order $o_P(1)$.

Similarly, one can show that the cross product sums with $i \neq j$ satisfy $n^{2H-1}\sum_k I_t^{n,i} I_t^{n,j} = o_P(1)$. Indeed, define the martingales $M^{n,k}$ by

$$M_v^{n,k} := n^{H-\frac{1}{2}} \int_0^{t_{k-2}\wedge v} f_k^t(u)dM_u.$$

Note also that integration by parts gives

$$\sup_{s\leq t} |M_s^{n,k}| \leq 2Lt^{2H-2} n^{\frac{1}{2}-H} n^{H-\frac{1}{2}} \leq 2Lt^{2H-2}.$$

One can now use Lemma 2.5 to check that $n^{2H-1}\sum_k (I_k^{n,1} I_k^{n,2}) = \sum_k M_t^{n,k} N_t^{n,k}$ and $n^{2H-1}\sum_k (I_k^{n,1} I_k^{n,3}) = \sum_k M_t^{n,k} \tilde{N}_k^{n,k}$ are of the order $o_P(1)$ Finally, for the sum $n^{2H-1}\sum_k (I_k^{n,2} I_k^{n,3}) = \sum_k N_t^{n,k} \tilde{N}_t^{n,k}$ one can check again by integration by parts that $\sup_{s\leq t} |N_s^{n,k}| \leq 4Lt^{2H-1}$ and this sum is also of the order $o_P(1)$ by Lemma 2.5.

All this shows that we have the asymptotic expansion 2.7, and from the estimates (3.5), (3.10) and (3.11) we obtain the following inequality

$$C_1 t^{2H-1} \int_s^t u^{2H-1} d[M]_u \leq c_H t^{2H-1}(t-s) \leq C_2 t^{4H-2}([M]_t - [M]_s).$$

This in turn implies that $[W] \sim Leb$ on $[0,t]$, and the proof of Theorem 1.1 is finished with $H > \frac{1}{2}$.

4 The case of $H < \frac{1}{2}$

4.1 Starting point

The proof is similar to the case of $H > \frac{1}{2}$. It is more convenient to work with the martingale $W = \int_0^t s^{H-\frac{1}{2}} dM_s$. We shall indicate the main estimates in the proof. After this one can repeat the arguments of the proof of the case $H > \frac{1}{2}$ to finish the proof. Put

$$p_k^t(z) = \int_{t_{k-1}}^{t_k} (\frac{z}{u})^{\frac{1}{2}-H} (u-z)^{H-\frac{3}{2}} du$$

for $z < u$; and we have the estimate

$$p_k^t(z) \leq (t_{k-1} - z)^{H-\frac{3}{2}} \frac{t}{n}. \tag{4.1}$$

Note also that we have

$$p_k^t(z) = z^{\frac{1}{2}-H} f_k^t(z) \tag{4.2}$$

with f_k^t from (3.2).

Using Lemma 2.3 we can now write the increment of X as

$$\begin{aligned}
& X_{t_k} - X_{t_{k-1}} \\
= \quad & \left(\frac{1}{2} - H\right) \int_0^{t_{k-2}} p_k^t(s) dW_s + \left(\frac{1}{2} - H\right) \int_{t_{k-2}}^{t_{k-1}} p_k^t(s) dW_s \\
& + \int_{t_{k-1}}^{t_k} \left(\frac{s}{t_k}\right)^{1/2-H} (t_k - s)^{H-1/2} \, dW_s \\
& + \left(\frac{1}{2} - H\right) \int_{t_{k-1}}^{t_k} s^{1/2-H} \int_s^{t_k} u^{H-3/2} (u-s)^{H-1/2} du dW_s \\
=: \quad & J_k^{n,1} + J_k^{n,2} + J_k^{n,3} + J_k^{n,4}.
\end{aligned}$$

We prove the asymptotic expansion using these four terms.

4.2 Upper estimate for the sum $n^{2H-1} \sum_{k=\frac{ns}{t}+2}^{n} \int_0^{t_{k-2}} \left(p_k^t(z)\right)^2 d[W]_z$

Put

$$j^{n,1} = n^{2H-1} \sum_{k=\frac{ns}{t}+2}^{n} \int_0^{t_{k-2}} \left(p_k^t(z)\right)^2 d[W]_z.$$

We decompose this sum as in the case of the proof $H > \frac{1}{2}$:

$$\begin{aligned}
j^{n,1} \quad := \quad & n^{2H-1} \left(\sum_{i=1}^{n\frac{s}{t}} \sum_{k=n\frac{s}{t}+2}^{n} + \sum_{i=n\frac{s}{t}+1}^{n-2} \sum_{k=i+2}^{n} \right) \int_{t_{i-1}}^{t_i} \left(p_k^t(u)\right)^2 d[W]_u \\
=: \quad & \tilde{j}^{n,1} + \bar{j}^{n,2}.
\end{aligned}$$

We continue first with using the estimate (4.1) for $\tilde{j}^{n,1}$, and then replacing the sum over the index k by integral:

$$\begin{aligned}
\tilde{j}^{n,1} &\le n^{2H-1}\sum_{i=1}^{n\frac{s}{t}}\int_{t_{i-1}}^{t_i}\sum_{k=n\frac{s}{t}+2}^{n}(t_{k-1}-u)^{2H-3}\,\frac{t^2}{n^2}d[W]_u\\
&\le n^{2H-1}\sum_{i=1}^{n\frac{s}{t}}\int_{t_{i-1}}^{t_i}\left[\frac{t}{(2-2H)n}\cdot\left(\left(s+\frac{t}{n}-u\right)^{2H-2}+\right.\right.\\
&\qquad\left.\left.+\left(s+\frac{t}{n}-u\right)^{2H-3}\frac{t^2}{n^2}\right)\right]d[W]_u\\
&\le \frac{tn^{2H-2}}{2-2H}\int_0^s\left(s+\frac{t}{n}-u\right)^{2H-2}d[W]_u\\
&\quad+t^2n^{2H-3}\int_0^s\left(s+\frac{t}{n}-u\right)^{2H-3}d[W]_u
\end{aligned}$$

According to the [4, Lemma 2.1] the martingale W is Hölder up to $\frac{1}{2}$, and so $[W]$ is Hölder up to 1. Integration by parts gives the estimate

$$\left|\int_0^s\left(s+\frac{t}{n}-u\right)^{2H-2}d[W]_u\right|\le C(\omega)\left(\frac{t}{n}\right)^{2H-2+\alpha},$$

for any $\alpha<1$. Hence the expression $tn^{2H-2}\int_0^s\left(s+\frac{t}{n}-u\right)^{2H-2}d[W]_u\to 0$ as $n\to\infty$ P-a.s. The same arguments apply to the integral

$$n^{2H-3}\int_0^s\left(s+\frac{t}{n}-u\right)^{2H-3}d[W]_u$$

and we obtain $\tilde{j}^{n,1}=o_P(1)$ and we can put $Q_n^t:=\tilde{j}^{n,1}$.

Using first the estimate (4.1) and then replacement of the sum by the integral we have

$$\begin{aligned}
\bar{j}^{n,2} &\le \frac{t^2}{n^2}n^{2H-1}\sum_{i=\frac{ns}{t}+1}^{n-2}\int_{t_{i-1}}^{t_i}\sum_{k=i+2}^{n}(t_{k-1}-u)^{2H-3}\,d[W]_u\\
&\le n^{2H-1}\sum_{i=\frac{ns}{t}+1}^{n-2}\int_{t_{i-1}}^{t_i}\left[\frac{t^2}{n}\left(\frac{1}{n}(t_{i+1}-u)^{2H-3}+\right.\right.\\
&\qquad\left.\left.+\frac{1}{2-2H}\frac{1}{t}(t_{i+1}-u)^{2H-2}\right)\right]d[W]_u\\
&\le t^2n^{2H-3}\left(\frac{t}{n}\right)^{2H-3}\sum_{i=\frac{ns}{t}+1}^{n-2}\int_{t_{i-1}}^{t_i}d[W]_u+\\
&\quad+\frac{t}{2-2H}\left(\frac{t}{n}\right)^{2H-2}n^{2H-2}\sum_{i=\frac{ns}{t}+1}^{n-2}\int_{t_{i-1}}^{t_i}d[W]_u\\
&\le t^{2H-1}C\left([W]_t-[W]_s\right),
\end{aligned}$$

where we also used the estimate $(t_{i+1}-u)^\alpha \le (\frac{t}{n})^\alpha$ for $\alpha<0$ and $u\le t_i$.

We have shown the following upper bound

$$j^{n,1} \le Ct^{2H-1}([W]_t-[W]_s)+Q_n^t. \tag{4.3}$$

4.3 Lower estimate for the sum $n^{2H-1}\sum_{k=\frac{ns}{t}+2}^{n}\int_0^{t_{k-2}}\left(p_k^t(z)\right)^2 d[W]_z$

Note first that by (4.2) and the definition of W we have

$$n^{2H-1}\sum_{k=\frac{ns}{t}+2}^{n}\int_0^{t_{k-2}}\left(p_k^t(z)\right)^2 d[W]_z = n^{2H-1}\sum_{k=\frac{ns}{t}+2}^{n}\int_0^{t_{k-2}}\left(f_k^t(z)\right)^2 d[M]_z.$$

Hence

$$\begin{aligned}
j^{n,1} &\ge t^2n^{2H-3}\sum_{k=\frac{ns}{t}+2}^{n}(t_k)^{2H-1}\int_0^{t_{k-2}}(t_k-z)^{2H-3}\,d[M]_z \\
&= t^2n^{2H-3}\left(\sum_{i=1}^{n\frac{s}{t}-1}\sum_{k=\frac{ns}{t}+2}^{n}+\sum_{i=\frac{ns}{t}+1}^{n-2}\sum_{k=i+2}^{n}\right)\left(\frac{tk}{n}\right)^{2H-1}\cdot \\
&\qquad\cdot\int_{t\frac{i-1}{n}}^{t\frac{i}{n}}\left(t\frac{k}{n}-z\right)^{2H-3}d[M]_z \\
&\ge t^{2H-1}t^2n^{2H-2}\sum_{i=\frac{ns}{t}+1}^{n-2}\int_{t\frac{i-1}{n}}^{t\frac{i}{n}}\frac{1}{n}\sum_{k=i+2}^{n}\left(t\frac{k}{n}-z\right)^{2H-3}d[M]_z \\
&\ge t^{2H+1}n^{2H-2}\sum_{i=\frac{ns}{t}+1}^{n-2}\int_{t\frac{i-1}{n}}^{t\frac{i}{n}}\int_{\frac{i+2}{n}}^{1}(tx-z)^{2H-3}\,dxd[M]_z \\
&= Ct^{2H+1}n^{2H-2}\sum_{i=\frac{ns}{t}+1}^{n-2}\int_{t\frac{i-1}{n}}^{t\frac{i}{n}}\frac{1}{t}\left(\left(t\frac{i+2}{n}-z\right)^{2H-2}-(t-z)^{2H-2}\right)d[M]_z.
\end{aligned}$$

Put

$$\begin{aligned}
O_n^t &:= n^{2H-2}t^{2H}\sum_{i=\frac{ns}{t}+1}^{n-2}\int_{t\frac{i-1}{n}}^{t\frac{i}{n}}(t-z)^{2H-2}d[M]_z \\
&= n^{2H-2}t^{2H}\int_s^{t-2/n}(t-z)^{2H-2}d[M]_z.
\end{aligned}$$

One can show using integration by parts that $O_n^t\to 0$ P-a.s. as $n\to\infty$. Next, we estimate the sum

$$\hat{j}^{n,1} := n^{2H-2}t^{2H+1}\sum_{i=\frac{ns}{t}+1}^{n-2}\int_{t\frac{i-1}{n}}^{t\frac{i}{n}}\frac{1}{t}\left(t\frac{i+2}{n}-z\right)^{2H-2}d[M]_z$$

from below using the inequality $\left(t\frac{i+2}{n} - z\right)^{2H-2} \geq \left(\frac{3t}{n}\right)^{2H-2}$, valid for $z \in (t_{i-1}, t_i)$:

$$\hat{j}^{n,1} \geq t^{2H-3+1+2H} 3^{2H-2} n^{2H-2} n^{2-2H} ([M]_{t-t\frac{2}{n}} - [M]_s).$$

Combine this with the upper estimate from (4.3) and we have

$$Ct^{4H-2}([M]_{t-t\frac{2}{n}} - [M]_s) - O_n^t \leq j^{n,1} \leq Ct^{2H-1}([W]_t - [W]_s) + Q_n^t. \quad (4.4)$$

4.4 Auxiliary estimates

We know that $[W]$ is Hölder up to 1. In order to obtain further upper estimates we must prove that $[W]$ is Hölder continuous with index $= 1$.

Recall that we have

$$n^{2H-1} \sum_{k=n\frac{s}{t}+1}^{n} (X_{t_k} - X_{t_{k-1}})^2 \to c_H t^{2H-1}(t-s).$$

On the other hand we know that

$$\begin{aligned}
& n^{2H-1} \sum_{k=n\frac{s}{t}+1}^{n} (X_{t_k} - X_{t_{k-1}})^2 \\
= \ & n^{2H-1} \sum_k \left(J_k^{n,1}\right)^2 + n^{2H-1} \sum_k \left(J_k^{n,2} + J_k^{n,3} + J_k^{n,4}\right)^2 \\
& + n^{2H-1} \sum_k J_k^{n,1} \left(J_k^{n,2} + J_k^{n,3} + J_k^{n,4}\right) \qquad (4.5) \\
\geq \ & j^{n,1} + n^{2H-1} \sum_k \int_0^{t_{k-2}} p_k^t(u) \left(\int_0^u p_k^t(s) dW_s\right) dW_u - O_n^t \\
+ \ & n^{2H-1} \sum_k J_k^{n,1} \left(J_k^{n,2} + J_k^{n,3} + J_k^{n,4}\right).
\end{aligned}$$

We will show that as $n \to \infty$ we have

$$n^{2H-1} \sum_k \int_0^{t_{k-2}} p_k^t(u) \left(\int_0^u p_k^t(s) dW_s\right) dW_u \xrightarrow{P} 0 \quad (4.6)$$

and

$$n^{2H-1} \sum_k J_k^{n,1} \left(J_k^{n,2} + J_k^{n,3} + J_k^{n,4}\right) \xrightarrow{P} 0. \quad (4.7)$$

Let $n \to \infty$ and use (4.6), (4.7) in (4.5) to obtain

$$\int_s^t u^{1-2H} d[W]_u \leq C_H t^{1-2H}(t-s). \quad (4.8)$$

Integration by parts gives that

$$\begin{aligned}
& \int_s^t u^{1-2H} d[W]_u \\
= \ & t^{1-2H} \left([W]_t - [W]_s\right) + (1-2H) \int_s^t u^{-2H} \left([W]_s - [W]_u\right) du.
\end{aligned}$$

Use the Hölder continuity of $[W]$ to obtain the following estimate $[W]_u - [W]_s \leq K(u-s)^{2H}$ we can continue the estimation (4.8) and obtain

$$\begin{aligned} t^{1-2H}\left([W]_t - [W]_s\right) &\leq C_H t^{1-2H}(t-s) + K\int_s^t u^{-2H}(u-s)^{2H}du \\ &\leq (C_H t^{1-2H} + K)(t-s). \end{aligned}$$

Hence $[W]$ is Hölder continuous with index $= 1$.

First we prove (4.6). Note that we can show that for arbitrary $r \in \mathbb{N}$

$$n^{2H-1}\sum_{k=3}^{nr}\int_0^{t_{k-2}} p_k^t(z)\left(\int_0^z p_k^t(u)dW_u\right)dW_z \xrightarrow{P} 0$$

as $n \to \infty$, since $\sum_{k=n\frac{s}{t}+2}^{n} = \sum_{k=3}^{n} - \sum_{k=3}^{n\frac{s}{t}+1}$.

We have

$$\begin{aligned} & \sum_{k=3}^{nr}\int_0^{t_{k-2}} p_k^t(z)\left(\int_0^z p_k^t(u)dW_u\right)dW_z \\ = & \sum_{k=3}^{nr}\sum_{i=1}^{k-2}\int_{t_{i-1}}^{t_i} p_k^t(z)\left(\int_0^z p_k^t(u)dW_u\right)dW_z \\ = & \sum_{i=1}^{nr-2}\sum_{k=i+3}^{nr}\int_{t_{i-1}}^{t_i} p_k^t(z)\left(\int_0^z p_k^t(u)dW_u\right)dW_z. \end{aligned}$$

To prove (4.6) it is now sufficient to show that

$$n^{2H-1}\sum_{k=i+3}^{nr} p_k^t(u)p_k^t(z) \to 0$$

for all fixed $0 < u < z$, since then we can use again [6, Theorem II.1,p.58] and argue as in the proof for the case $H > \frac{1}{2}$. We have

$$\begin{aligned} & n^{2H-1}\sum_{k=i+3}^{nr} p_k^t(u)p_k^t(z) \\ \leq & n^{2H-1}\sum_{k=i+3}^{nr}\int_{t_{k-1}}^{t_k}(v-u)^{H-3/2}dv\int_{t_{k-1}}^{t_k}(v-z)^{H-3/2}dv \\ \leq & n^{2H-1}\sum_{k=i+3}^{nr}(t_{k-1}-u)^{H-3/2}\frac{1}{n}\int_{t_{k-1}}^{t_k}(v-z)^{H-3/2}dv \\ \leq & n^{2H-2}\left(t_{i+2}-u\right)^{H-3/2}\int_{t_{i+2}}^{tr}(v-z)^{H-3/2}dv \\ \leq & Cn^{2H-2}\cdot(z-u)^{H-3/2}\left(t_{i+2}-z\right)^{H-1/2} \\ \leq & Cn^{2H-2}(z-u)^{H-3/2}\left(\frac{t}{n}\right)^{H-1/2} \\ = & Cn^{H-3/2}(z-u)^{H-3/2} \to 0, \quad n \to \infty \end{aligned}$$

For the proof of (4.7) note that $n^{2H-1}\sum_{k=n\frac{s}{t}+1}^{n}(J_k^{n,1})^2$ is tight by (4.3) and (4.6). For $k \neq l$ we also have

$$E\left(J_k^{n,1}J_k^{n,2}J_l^{n,1}J_l^{n,2}\right)=0;$$

hence for

$$n^{2H-1}\sum_{k=n\frac{s}{t}+1}^{n}J_k^{n,1}J_k^{n,2}\xrightarrow{P}0 \tag{4.9}$$

it is sufficient to show that

$$n^{4H-2}\sum_{k=n\frac{s}{t}+1}^{n}\left(J^{n,1}J_k^{n,2}\right)^2\xrightarrow{P}0 \tag{4.10}$$

as $n\longrightarrow\infty$.

We have that $J_k^{n,2}=\int_{t_{k-2}}^{t_{k-1}}p_k^t(s)dW_s$. Note first that

$$p_k^t(s)\leq\left(\frac{1}{2}-H\right)(t_{k-1}-s)^{H-\frac{1}{2}} \tag{4.11}$$

and that W is Hölder up to $\frac{1}{2}$. Take $0<A<1$ and integration by parts gives

$$\begin{aligned}
&\int_{t_{k-2}}^{At_{k-1}}p_k^t(u)dW_u\\
=\;& p_k^t(At_{k-1})W_{At_{k-1}}-p_k^t(t_{k-2})W_{t_{k-2}}-\int_{t_{k-2}}^{At_{k-1}}W_u dp_k^t(u)\\
=\;& p_k^t(At_{k-1})\left(W_{At_{k-1}}-W_{t_{k-1}}\right)+p_k^t(t_{k-2})\left(W_{t_{k-1}}-W_{t_{k-2}}\right)\\
&\quad-\int_{t_{k-2}}^{At_{k-1}}\left(W_{t_{k-1}}-W_u\right)dp_k^t(u).
\end{aligned}$$

Now by (4.11) and Hölder continuity of W we have that for any $\alpha<1$

$$|p_k^t(At_{k-1})\left(W_{At_{k-1}}-W_{t_{k-1}}\right)|\leq K\left((1-A)\,t_{k-1}\right)^{H-\frac{1}{2}+\frac{\alpha}{2}},$$

and the same argument gives

$$|p_k^t(t_{k-2})\left(W_{t_{k-1}}-W_{t_{k-2}}\right)|\leq K\left(\frac{t}{n}\right)^{H-\frac{1}{2}+\frac{\alpha}{2}}.$$

Finally one can use integration by parts to check that

$$\begin{aligned}
&\left|\int_{t_{k-2}}^{At_{k-1}}\left(W_{t_{k-1}}-W_u\right)dp_k^t(u)\right|\\
\leq\;& K\left(\left((1-A)\,t_{k-1}\right)^{H-\frac{1}{2}+\frac{\alpha}{2}}+\left(\frac{t}{n}\right)^{H-\frac{1}{2}+\frac{\alpha}{2}}\right).
\end{aligned}$$

Finally, let $A \to 1$ and we have

$$n^{2H-1} \max_k (J_k^{n,2})^2 \le C n^{2H-1} \left(\frac{t}{n}\right)^{2H-1+\alpha};$$

this proves (4.10) and also (4.9). We can repeat the arguments to conclude that

$$n^{2H-1} \sum_{k=n\frac{s}{t}+1}^{n} J_k^{n,1} J_k^{n,3} \xrightarrow{P} 0 \tag{4.12}$$

as $n \to 0$. The last sum in (4.7) can be treated analogously, since $\int_s^{t_k} u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}} \le c_H s^{H-\frac{3}{2}}(t_k-s)^{H+\frac{1}{2}}$, and this gives the estimate

$$\begin{aligned} |J_k^{n,4}| &= \left| \int_{t_{k-1}}^{t_k} s^{1/2-H} \int_s^{t_k} u^{H-3/2}(u-s)^{H-1/2} du \, dW_s \right| \\ &\le C t^{H-\frac{1}{2}+\gamma} n^{-H+\frac{1}{2}-\gamma} \end{aligned}$$

with $0 < \gamma < \frac{1}{2}$.

We finally obtain, letting $n \to \infty$ in (4.5), that

$$C t^{4H-2}([M]_t - [M]_s) \le t^{2H-1}(t-s);$$

this means that the bracket $[M]$ is absolutely continuous with respect to Lebesgue measure, and since $[M]_t = \int_0^t s^{1-2H} d[W]_s$ the same holds true for the bracket $[W]$ and we have

$$[W]_t - [W]_s \le K(t-s) \tag{4.13}$$

with some constant $K = K(t, \omega, H)$.

4.5 Other estimates

Next we estimate the sums

$$j^{n,2} := n^{2H-1} \sum_{k=n\frac{s}{t}+1}^{n} \int_{t_{k-2}}^{t_{k-1}} \left(p_k^t(u)\right)^2 d[W]_u$$

and

$$j^{n,3} := n^{2H-1} \sum_{k=n\frac{s}{t}+1}^{n} \int_{t_{k-1}}^{t_k} \left(\left(\frac{s}{t_k}\right)^{\frac{1}{2}-H} (t_k-s)^{H-\frac{1}{2}}\right)^2 d[W]_s \tag{4.14}$$

from above.

The estimate (4.11) gives

$$j^{n,2} \le C_H n^{2H-1} \sum_{k=n\frac{s}{t}+1}^{n} \int_{t_{k-2}}^{t_{k-1}} (t_{k-1}-s)^{2H-1} d[W]_s. \tag{4.15}$$

Lemma 4.1. *For a fixed $A \in (0,1)$ we have the estimate*

$$\begin{aligned}
&\int_{t_{k-2}}^{t_{k-1}} (t_{k-1}-s)^{2H-1} d[W]_s \\
\leq \quad & \left(A\frac{t}{n}\right)^{2H-1} \left([W]_{t_{k-1}} - [W]_{t_{k-2}}\right) + C\left(A\frac{t}{n}\right)^{2H}.
\end{aligned} \tag{4.16}$$

Proof. Take $A \in (0,1)$ and use (4.16)

$$\begin{aligned}
&\int_{t_{k-2}}^{t_{k-1}} (t_{k-1}-s)^{2H-1} \, d[W]_s \\
\leq \quad & \int_{t_{k-2}}^{t_{k-1}-A\frac{t}{n}} + \int_{t_{k-1}-A\frac{t}{n}}^{t_{k-1}} (t_{k-1}-s)^{2H-1} \, d[W]_s \\
\leq \quad & \left(A\frac{t}{n}\right)^{2H-1} \left([W]_{t_{k-1}} - [W]_{t_{k-2}}\right) + C\left(A\frac{t}{n}\right)^{2H}.
\end{aligned}$$

□

We can now estimate the sums. For the estimate in (4.15) we obtain, using the inequality (4.16)

$$j^{n,2} \leq C(At)^{2H-1} \left([W]_t - [W]_s\right) + KA^{2H}t^{2H-1}(t-s). \tag{4.17}$$

Note that this estimate gives in the same way as in the case of $H > \frac{1}{2}$ that

$$Q_n^t := n^{2H-1} \sum_{k=n\frac{s}{t}}^{n} \int_{t_{k-2}}^{t_{k-1}} p_k^t(u) \int_{t_{k-2}}^{u} p_k^t(s) dW_s dW_u$$

satisfies $Q_n^t = o_P(1)$.

It is now quite obvious that for the sum (4.14) we have similar upper bound to (4.17):

$$j^{n,3} \leq C(At)^{2H-1} \left([W]_t - [W]_s\right) + KA^{2H}t^{2H-1}(t-s). \tag{4.18}$$

We can again repeat the arguments for iterated stochastic integrals and obtain

$$\begin{aligned}
& n^{2H-1} \sum_{k=n\frac{s}{t}+1}^{n} (J_k^{n,3})^2 \\
\leq \quad & C(At)^{2H-1} \left([W]_t - [W]_s\right) + KA^{2H}t^{2H-1}(t-s) + S_n^t
\end{aligned}$$

with $S_n^t = o_P(1)$.

We shall work with

$$j^{n,4} := n^{2H-1} \sum_{k=n\frac{s}{t}+1}^{n} \int_{t_{k-1}}^{t_k} v^{1-2H} \left(\int_v^{t_k} u^{H-\frac{3}{2}}(u-v)^{H-\frac{1}{2}} du\right)^2 d[W]_v.$$

With Schwartz inequality we obtain

$$\begin{aligned}
j^{n,4} &\leq n^{2H-1} \sum_{k=n\frac{s}{t}+1}^{n} \int_{t_{k-1}}^{t_k} v^{1-2H} \int_v^{t_k} u^{2H-3} du \int_v^{t_k} (u-v)^{2H-1} du \, d[W]_v \\
&\leq C \frac{t}{n} \sum_{k=n\frac{s}{t}+1}^{n} \int_{t_{k-1}}^{t_k} (v^{2H-2} - t_k^{2H-2}) d[W]_v \\
&\leq CK \frac{t}{n} \sum_{k=n\frac{s}{t}+1}^{n} \int_{t_{k-1}}^{t_k} (t_{k-1}^{2H-2} - t_k^{2H-2}) dv \\
&= CK \left(\frac{t}{n}\right)^2 \left(s^{2H} - t^{2H}\right).
\end{aligned}$$

This shows that $j_n^4 = o_p(1)$, and hence also $n^{2H-1} \sum_{k=n\frac{s}{t}}^{n} (J_k^{n,4})^2 = o_P(1)$; we see this by repeating the iterated integral arguments.

In order the asymptotic expansion to hold, we need to check that

$$n^{2H-1} \sum_{k=n\frac{s}{t}+1}^{n} J_k^{n,2} J_k^{n,3} = o_P(1).$$

We leave this to the reader.

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