Helsinki University of Technology, Institute of Mathematics, Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2007 A515

Global a posteriori error estimates for convection-reaction-diffusion problems

Sergey Korotov

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

Helsinki University of Technology, Institute of Mathematics, Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2007 A515

Global a posteriori error estimates for convection-reaction-diffusion problems

Sergey Korotov

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

Sergey Korotov: *Global a posteriori error estimates for convection-reaction-diffusion problems* ; Helsinki University of Technology, Institute of Mathematics, Research Reports A515 (2007).

Abstract: *In this note we propose a nonstandard technique for constructing global a posteriori error estimates for the stationary convection-reaction-diffusion equation. In order to estimate the approximation error in appropriate weighted energy norms, which measures the overall quality of the approximations, the underlying bilinear form is decomposed into several terms which can be directly computed or easily estimated from above using elementary tools of functional analysis. Several auxiliary parameters are introduced to construct such a splitting and tune the resulting upper error bound. It is demonstrated how these parameters can be chosen in some natural and convenient for computations way so that the weighted energy norm of the error is almost recovered, which shows that the estimates proposed are, in fact, quasi-sharp. The presented methodology is completely independent of numerical techniques used to compute approximate solutions. In particular, it is applicable to approximations which fail to satisfy the Galerkin orthogonality, e.g., due to an inconsistent stabilization, flux limiting, low-order quadrature rules, round-off and iteration errors etc. Moreover, the only constant that appears in the proposed error estimates is of global nature and comes from the Friedrichs-Poincaré inequality.*

AMS subject classifications: 65N15, 65N50, 76M30

Keywords: convection-reaction-diffusion equation, a posteriori error estimation, Friedrichs-Poincaré inequality

Correspondence

sergey.korotov@hut.fi

ISBN-10 951-22-8567-3 // ISBN-13 978-951-22-8567-9

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1 Introduction

Mathematical models (and their numerical solution) consisting of convection-reaction-diffusion equations with various boundary conditions present a very important class of real-life problems [12, 16, 19]. The error incurred in the course of discretization and iterative solution of such problems is responsible for the defference between the computational results and the exact solutions of the models. A posteriori error estimates quantify this difference (also called the error) and constitute a modern reliable tool for constructing effective adaptive mesh refinement procedures aimed at reducing the error if necessary according to main computational goals. Currently, reliable error techniques are available, e.g., for finite element approximations of various elliptic problems (see [1, 3, 4, 5, 13, 21, 24, 25] and references therein). However, the derivation of reliable error estimates for convection-diffusion equations still represents a challenging open problem, although there exist a number of works on this topic written during the past two decades [9, 11, 26, 27].

A crucial limitation of many a posteriori error estimation techniques is the presence of a large set of so-called interpolation constants (see, e.g., [27]) which are extremely difficult to compute (or even to estimate, cf. [7] for a simpler elliptic problems), especially in the case of complex domains and unstructured computational meshes. The uncertainty involved in the computation of these constants may seriously reduce the actual reliability of the resulting estimates. Moreover, some popular methods rely on the existence of an equivalent minimization problem or assume the Galerkin orthogonality. For the residual to be orthogonal to the space of test functions, the discretization must be performed by a consistent (Petrov-)Galerkin method and the resulting algebraic equations must be solved exactly. These requirements are never satisfied in practice because of numerical quadrature, various round-off errors, slack tolerances for iterative solvers, programming bugs etc. The use of upwinding [2] or flux/slope limiters [17] in finite element codes may also violate the Galerkin orthogonality.

A very promising approach to error estimation was developed by S. Repin and his coauthors [21, 22, 23] during the last 10 years in the context of diffusion-type problems (and some other types of problems). Remarkably, it is applicable to any conforming approximation regardless of the numerical method used to compute it. The original derivation is based on rather sophisticated tools of functional analysis (duality theory, Helmholtz decomposition) but a simplified version was recently proposed and succesfully employed for reaction-diffusion problems in [10, 14].

The approach from [10, 14] with several modifications was further applied to convection-diffusion equations for the first time in the very recent work [15], where obtained estimates involve two free parameters for tuning the resulting bounds. However, the estimates from [15] may involve, in practice, a good resolution of the so-called adjoint problem, which can be sometimes unnecessary as this problem is of auxiliary nature in many situations.

In the present paper we describe another technique for constructing upper error estimtes for the same class of problems as in [15], which helps to get the upper bounds for the error which are valid for any admissible approximations without using any adjoint problems and which, in addition, hold in various global norms. The resulting upper bounds are shown be quasi-sharp in the sense that they almost reduce to the true error if the involved parameters are chosen in some quite natural and convenient for practical calculations way. Moreover, as in [15], there is just one global constant (due to the Friedrichs-Poincaré inequality) involved into the estimates, which depends on the geometry of the computational domain only and does not change during mesh adaptation.

2 Model problem

Consider the stationary convection-reaction-diffusion problem

$$\begin{cases} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. The constant diffusion coefficient ε, the velocity field $\mathbf{b}$, and the reaction rate c are supposed to satisfy the following conditions

$$\varepsilon > 0, \quad \mathbf{b} \in W^1_\infty(\Omega;\mathbb{R}^d), \quad c \in L_\infty(\Omega). \tag{2}$$

The weak formulation of the above problem reads: Find $u \in H^1_0(\Omega)$ such that

$$a(u,w) = F(w) \qquad \forall w \in H^1_0(\Omega), \tag{3}$$

where the bilinear form $a(\cdot,\cdot)$ and the linear functional $F(\cdot)$ are given by

$$a(v,w) = \int\limits_\Omega \varepsilon\nabla v\cdot\nabla w\,dx + \int\limits_\Omega \mathbf{b}\cdot\nabla v\, w\;dx + \int\limits_\Omega cvw\;dx, \tag{4}$$

$$F(w) = \int\limits_\Omega fw\;dx, \qquad u,w \in H^1_0(\Omega). \tag{5}$$

It is well known that the weak solution $u \in H^1_0(\Omega)$ of variational problem (3) exists and is unique provided that the following extra condition holds

$$\tilde{c}(x) := c(x) - \frac{1}{2}\nabla\cdot\mathbf{b}(x) \geq 0 \tag{6}$$

almost everywhere in Ω. Indeed, under this constraint, the bilinear form $a(\cdot,\cdot)$ is coercive

$$\begin{aligned} a(w,w) &= \int\limits_\Omega \varepsilon\nabla w\cdot\nabla w\,dx + \int\limits_\Omega (\mathbf{b}\cdot\nabla w)\,w\,dx + \int\limits_\Omega cw^2\,dx \\ &= \varepsilon\int\limits_\Omega |\nabla w|^2\,dx + \int\limits_\Omega \left(c - \frac{1}{2}\nabla\cdot\mathbf{b}\right)w^2\,dx \geq C\|w\|^2_{1,\Omega}, \end{aligned} \tag{7}$$

where C is a positive constant and $\|\cdot\|_{1,\Omega}$ denotes the standard norm in $H^1(\Omega)$. And the coercivity of the bilinear form $a(\cdot,\cdot)$ implies the unique solvability of problem (3) due to the Lax-Milgram lemma (see, e.g., [8]).

Remark 1: Here, we mention that we have no special restrictions on the coefficients of the problem as done, e.g., in [26, 27] (condition (A3)).

3 Error estimation technique

Let $\bar{u}$ be some function from $H_0^1(\Omega)$ considered as an approximation of u. In our work, we do not specify how $\bar{u}$ has been computed, it is just an arbitrary function from an admissible class.

The natural global measure of the error $e := u - \bar{u}$ is the value $\|e\|_{1,\Omega}$ since both, the exact solution and the approximation belong to the same space $H_0^1(\Omega)$. However, in what follows, it is more convenient to analyse the estimation of the error in terms of suitable global weighted energy norm defined as follows

$$|||e|||^2_{\lambda,\mu,\Omega} := \lambda \int\limits_\Omega |\nabla e|^2 \; dx + \mu \int\limits_\Omega \tilde{c}\, e^2 \; dx, \tag{8}$$

where the weights λ and μ are nonnegative real numbers (will be defined later), such that $\lambda + \mu > 0$. It is clear that $|||e|||_{\lambda,\mu,\Omega}$ is equivalent to $\|\cdot\|_{1,\Omega}$ under the above conditions on λ and μ.

In particular, we also have

$$a(e,e) = |||e|||^2_{\varepsilon,1,\Omega}, \tag{9}$$

which is the reason for calling error measures defined in (8) weighted energy norms.

In order to construct a posteriori estimates for the error in certain weighted energy norms of type (8), we first observe that

$$\begin{aligned} &a(e,e) = a(u-\bar{u}, u-\bar{u}) = \\ &= \varepsilon \int\limits_\Omega \nabla(u-\bar{u})\cdot\nabla(u-\bar{u}) \; dx + \int\limits_\Omega \mathbf{b}\cdot\nabla(u-\bar{u})(u-\bar{u}) \; dx + \int\limits_\Omega c(u-\bar{u})\,(u-\bar{u}) \; dx = \\ &= \int\limits_\Omega f(u-\bar{u}) \; dx - \varepsilon \int\limits_\Omega \nabla\bar{u}\cdot\nabla(u-\bar{u}) \; dx - \int\limits_\Omega \mathbf{b}\cdot\nabla\bar{u}\,(u-\bar{u}) \; dx - \int\limits_\Omega c\,\bar{u}\,(u-\bar{u}) \; dx, \end{aligned} \tag{10}$$

where the integral identity (3) with $w = u - \bar{u}$ has been used.

Further, we regroup terms in (10) and introduce a parametric vector-function $\mathbf{y}^\star \in H(\mathrm{div},\Omega)$ so that

$$a(u-\bar{u}, u-\bar{u}) =$$

$$= \int_\Omega (f - \mathbf{b}\cdot\nabla\bar u - c\,\bar u)(u-\bar u)\,dx - \int_\Omega (\varepsilon\nabla\bar u - \mathbf{y}^\star + \mathbf{y}^\star)\cdot\nabla(u-\bar u)\,dx = \quad (11)$$

$$= \int_\Omega (f - \mathbf{b}\cdot\nabla\bar u - c\,\bar u)(u-\bar u)\,dx - \int_\Omega (\varepsilon\nabla\bar u - \mathbf{y}^\star)\cdot\nabla(u-\bar u)\,dx - \int_\Omega \mathbf{y}^\star\cdot\nabla(u-\bar u)\,dx.$$

Using the Green formula for the last term in the right-hand side of (11), we finally get the following decomposition for the error in the standard energy norm

$$a(u-\bar u, u-\bar u) =$$

$$= \int_\Omega (f - \mathbf{b}\cdot\nabla\bar u - c\,\bar u + \nabla\cdot\mathbf{y}^\star)(u-\bar u)\,dx + \int_\Omega (\mathbf{y}^\star - \varepsilon\nabla\bar u)\cdot\nabla(u-\bar u)\,dx. \quad (12)$$

Now, we introduce another parameter, now being an arbitrary function v from the space $H_0^1(\Omega)$. Then we can represent (12) further as follows

$$a(u-\bar u, u-\bar u) =$$

$$= \int_\Omega (f - \mathbf{b}\cdot\nabla\bar u - c\,\bar u + \nabla\cdot\mathbf{y}^\star - cv - \mathbf{b}\cdot\nabla v)(u-\bar u)\,dx + \quad (13)$$

$$+ \int_\Omega (\mathbf{y}^\star - \varepsilon\nabla\bar u + \varepsilon\nabla v)\cdot\nabla(u-\bar u)\,dx +$$

$$+ \int_\Omega \Big((c\,v + \mathbf{b}\cdot\nabla v)(u-\bar u) - \varepsilon\nabla v\cdot\nabla(u-\bar u)\Big)\,dx = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3.$$

Thus, our further goal is to effectively estimate from above the foregoing three terms $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$.

First, we have

$$\mathbf{T}_1 \le \|f - \mathbf{b}\cdot\nabla\bar u - c\,\bar u + \nabla\cdot\mathbf{y}^\star - cv - \mathbf{b}\cdot\nabla v\|_{0,\Omega}\|u-\bar u\|_{0,\Omega} \le$$

$$\le C_\Omega\|f - \mathbf{b}\cdot\nabla\bar u - c\,\bar u + \nabla\cdot\mathbf{y}^\star - cv - \mathbf{b}\cdot\nabla v\|_{0,\Omega}\|\nabla(u-\bar u)\|_{0,\Omega},$$

where C_Ω is a constant from the Friedrichs-Poincaré inequality

$$\|w\|_{0,\Omega} \le C_\Omega\|\nabla w\|_{0,\Omega} \quad (14)$$

valid for all functions $w \in H_0^1(\Omega)$, and $\|\cdot\|_{0,\Omega}$ denotes the standard norm in $L_2(\Omega)$. Similarly, we get the estimate

$$\mathbf{T}_2 \le \|\mathbf{y}^\star - \varepsilon\nabla\bar u + \varepsilon\nabla v\|_{0,\Omega}\|\nabla(u-\bar u)\|_{0,\Omega}.$$

In a view of a simple inequality for any $\alpha > 0$

$$pq \le \frac{\alpha}{2}p^2 + \frac{1}{2\alpha}q^2, \qquad (p,\ q \ge 0), \quad (15)$$

we immediately get that

$$\mathbf{T}_1+\mathbf{T}_2 \le \Big(\|\mathbf{y}^\star-\varepsilon\nabla\bar u+\varepsilon\nabla v\|_{0,\Omega}+C_\Omega\|f-\mathbf{b}\cdot\nabla\bar u-c\,\bar u+\nabla\cdot\mathbf{y}^\star-cv-\mathbf{b}\cdot\nabla v\|_{0,\Omega}\Big)\times$$

$$\times\|\nabla(u-\bar u)\|_{0,\Omega} \le \frac{\alpha}{2}\|\nabla(u-\bar u)\|^2_{0,\Omega}+$$

$$\frac{1}{2\alpha}\Big(\|\mathbf{y}^\star-\varepsilon\nabla\bar u+\varepsilon\nabla v\|_{0,\Omega}+C_\Omega\|f-\mathbf{b}\cdot\nabla\bar u-c\,\bar u+\nabla\cdot\mathbf{y}^\star-cv-\mathbf{b}\cdot\nabla v\|_{0,\Omega}\Big)^2,$$

where α is any positive number. For the second term in the right-hand side of the above inequality we can employ the following inequality

$$(p+q)^2 \le (1+\gamma)p^2+(1+\frac{1}{\gamma})q^2, \qquad (p,\ q\ge 0), \tag{16}$$

valid for any positive number γ. Finally, we get

$$\mathbf{T}_1+\mathbf{T}_2 \le \frac{\alpha}{2}\|\nabla(u-\bar u)\|^2_{0,\Omega}+\frac{1}{2\alpha}\Big((1+\gamma)\|\mathbf{y}^\star-\varepsilon\nabla\bar u+\varepsilon\nabla v\|^2_{0,\Omega}+ \tag{17}$$

$$+\Big(1+\frac{1}{\gamma}\Big)C^2_\Omega\|f-\mathbf{b}\cdot\nabla\bar u-c\,\bar u+\nabla\cdot\mathbf{y}^\star-cv-\mathbf{b}\cdot\nabla v\|^2_{0,\Omega}\Big).$$

Now, we shall estimate the third term $\mathbf{T}_3$. We observe that

$$\mathbf{T}_3 = \int\limits_\Omega \Big(c\,v(u-\bar u)+\mathbf{b}\cdot\nabla v(u-\bar u)+\varepsilon\nabla v\cdot\nabla\bar u-\varepsilon\nabla v\cdot\nabla u\Big)\,dx =$$

$$= \int\limits_\Omega \Big(c\,v(u-\bar u)+\mathbf{b}\cdot\nabla v(u-\bar u)+\varepsilon\nabla v\cdot\nabla\bar u-fv+\mathbf{b}\cdot\nabla uv+cuv\Big)\,dx =$$

$$= \int\limits_\Omega \Big(\mathbf{b}\cdot\nabla v\,(u-\bar u)+2\,c\,v\,(u-\bar u)+\mathbf{b}\cdot\nabla(u-\bar u)\,v\Big)\,dx+$$

$$+\int\limits_\Omega \Big(-f\,v+\mathbf{b}\cdot\nabla\bar u\,v-c\,v\,(u-\bar u)+\varepsilon\nabla v\cdot\nabla\bar u+cuv\Big)\,dx = \tag{18}$$

$$= \int\limits_\Omega \Big(\mathbf{b}\cdot\nabla\Big(v\,(u-\bar u)\Big)+2\,c\,v\,(u-\bar u)\Big)\,dx+\int\limits_\Omega \Big(\varepsilon\nabla v\cdot\nabla\bar u+\mathbf{b}\cdot\nabla\bar u\,v+c\,v\,\bar u-fv\Big)\,dx =$$

$$= \int\limits_\Omega \Big(\varepsilon\nabla v\cdot\nabla\bar u+\mathbf{b}\cdot\nabla\bar u\,v+c\,v\,\bar u-fv\Big)\,dx+2\int\limits_\Omega (c-\frac{1}{2}\nabla\cdot\mathbf{b})\,v\,(u-\bar u)\,dx =$$

$$= \mathbf{T}_{3,1}(v,\bar u)+\mathbf{T}_{3,2}(u,v,\bar u).$$

Obviously, the above term $\mathbf{T}_{3,1}$ is directly computable once we have the approximation $\bar u$ computed and fix the parameter v, but we should still estimate the term $\mathbf{T}_{3,2}$ containing the unknown exact solution u. For this purpose, employing the definition of $\tilde c$ in (6), we further apply the obvious estimate

$$\mathbf{T}_{3,2} \le \frac{1}{\beta}\|\sqrt{\tilde c}(u-\bar u)\|^2_{0,\Omega}+\beta\|\sqrt{\tilde c}\,v\|^2_{0,\Omega}, \tag{19}$$

where β is any positive number.

Now, combining (9) and (13) with estimates (17), (18), and (19), we prove the following theorem.

Theorem 1: Under conditions (2) and (6), for the solution u of the problem (3) and for an arbitrary function $\bar{u} \in H_0^1(\Omega)$, we have the following functional inequality

$$(\varepsilon - \frac{\alpha}{2})\|\nabla(u - \bar{u})\|_{0,\Omega}^2 + (1 - \frac{1}{\beta})\|\sqrt{\tilde{c}}(u - \bar{u})\|_{0,\Omega}^2 \leq$$

$$\leq \frac{1}{2\alpha}\Big((1+\gamma)\|\mathbf{y}^\star - \varepsilon\nabla\bar{u} + \varepsilon\nabla v\|_{0,\Omega}^2 + \tag{20}$$

$$+\Big(1 + \frac{1}{\gamma}\Big)C_\Omega^2\|f - \mathbf{b}\cdot\nabla\bar{u} - c\,\bar{u} + \nabla\cdot\mathbf{y}^\star - cv - \mathbf{b}\cdot\nabla v\|_{0,\Omega}^2\Big) +$$

$$+\int\limits_\Omega \Big(\varepsilon\nabla v\cdot\nabla\bar{u} + \mathbf{b}\cdot\nabla\bar{u}\,v + c\,v\,\bar{u} - fv\Big)\,dx + \beta\|\sqrt{\tilde{c}}\,v\|_{0,\Omega}^2,$$

which is valid for any positive numbers α, β, γ, and for any parameter-functions $\mathbf{y}^\star \in H(\mathrm{div}, \Omega)$ and $v \in H_0^1(\Omega)$.

Let us fix the numbers α and β and introduce a short denotation

$$\mathbf{EST}_{\alpha,\beta} := \mathbf{EST}_{\alpha,\beta}(\gamma, \mathbf{y}^\star, v, \bar{u})$$

for the functional on the right-hand side of the inequality (20). Now we formulate the main result of the paper.

Theorem 2: Let α and β be fixed positive numbers such that

$$2\varepsilon \geq \alpha > 0, \quad \beta \geq 1, \quad \varepsilon - \frac{\alpha}{2} + 1 - \frac{1}{\beta} > 0. \tag{21}$$

Then, in view of the inequality (20), we get the following a posteriori upper error estimates for the error in the weighted energy norms of type (8)

$$|||e|||_{\lambda,\mu,\Omega}^2 \leq \mathbf{EST}_{\alpha,\beta}(\gamma, \mathbf{y}^\star, v, \bar{u}), \tag{22}$$

where $\lambda := \varepsilon - \frac{\alpha}{2}$, $\mu := 1 - \frac{1}{\beta}$, and the parameters γ, $\mathbf{y}^\star$, and v are as defined in Theorem 1.

Remark 2: We see that under condition (21) in Theorem 2, we have $\lambda \geq 0$, $\mu \geq 0$, and $\lambda + \mu > 0$, i.e., the left-hand side of (22) really defines certain weighted norm of type (8). Taking different values of the numbers α and β within limits given in (21), we get the family of a posteriori error estimates in various weighted energy norms.

4 Final comments

4.1 On estimation of the constant C_Ω

We notice that the constant C_Ω that appears in the definition of the error functional $\mathbf{EST}_{\alpha,\beta}$ is global and depends solely on the geometry of the domain. Its usable upper estimate can be readily obtained by enclosing the domain Ω into a rectangular box as proposed by S. Mikhlin (see [18], p. 18)

$$C_\Omega \leq \frac{1}{\pi\sqrt{\frac{1}{a_1^2}+\cdots+\frac{1}{a_d^2}}}, \tag{23}$$

where $a_1, \ldots, a_d$ are the dimensions of the box. Note that C_Ω is independent of the mesh and needs to be evaluated only once for each particular domain.

4.2 On the quality of the proposed error estimates

Let us assume that we have minimized the upper bound $\mathbf{EST}_{\alpha,\beta}$, i.e., that we have found the optimal parametres γ_{opt}, $\mathbf{y}^\star_{opt}$, v_{opt}, and define

$$\underline{\mathbf{EST}}_{\alpha,\beta} := \mathbf{EST}(\gamma_{opt}, \mathbf{y}^\star_{opt}, v_{opt}, \bar{u}).$$

In what follows, we shall demonstrate that the number $\underline{\mathbf{EST}}_{\alpha,\beta}$ does not give a too pessimistic overestimation of the error in (22) in principle.

First, let us prescribe

$$\mathbf{y}^\star := \varepsilon\nabla u \qquad \& \qquad v := u - \bar{u}, \tag{24}$$

which is really correct in a view of properties of u and $\bar{u}$. Then, using (3) with $w = u - \bar{u}$ and the Green formula, we get

$$\underline{\mathbf{EST}}_{\alpha,\beta} \leq \frac{1}{2\alpha}4\varepsilon^2(1+\gamma)\|\nabla(u-\bar{u})\|^2_{0,\Omega}+$$

$$+\int_\Omega \Big(\varepsilon\nabla(u-\bar{u})\cdot\nabla\bar{u}+\mathbf{b}\cdot\nabla\bar{u}\,(u-\bar{u})+c(u-\bar{u})\,\bar{u}-f(u-\bar{u})\Big)\,dx+\beta\|\sqrt{\tilde{c}}\,(u-\bar{u})\|^2_{0,\Omega} =$$

$$= \frac{2(1+\gamma)\varepsilon^2}{\alpha}\|\nabla(u-\bar{u})\|^2_{0,\Omega} + \beta\|\sqrt{\tilde{c}}\,(u-\bar{u})\|^2_{0,\Omega}$$

$$-\int_\Omega \Big(\varepsilon\nabla(u-\bar{u})\cdot\nabla(u-\bar{u})+\mathbf{b}\cdot\nabla(u-\bar{u})\,(u-\bar{u})+c\,(u-\bar{u})\,(u-\bar{u})+f\,(u-\bar{u})\Big)\,dx+$$

$$+\int_\Omega \Big(\varepsilon\nabla(u-\bar{u})\cdot\nabla u+\mathbf{b}\cdot\nabla u\,(u-\bar{u})+c\,(u-\bar{u})\,u\Big)\,dx =$$

$$= \Big(\frac{2(1+\gamma)\varepsilon^2}{\alpha}-\varepsilon\Big)\|\nabla(u-\bar{u})\|^2_{0,\Omega} + (\beta-1)\|\sqrt{\tilde{c}}\,(u-\bar{u})\|^2_{0,\Omega}.$$

Thus, for the choice of parameters (24), from above we get the following inequalities for the value of $\underline{\mathbf{EST}}_{\alpha,\beta}$

$$|||e|||^2_{(\varepsilon-\frac{\alpha}{2}),(1-\frac{1}{\beta}),\Omega} \le \underline{\mathbf{EST}}_{\alpha,\beta} \le |||e|||^2_{\left(\frac{2(1+\gamma)\varepsilon^2}{\alpha}-\varepsilon\right),(\beta-1),\Omega}. \tag{25}$$

Now we shall present examples on how sharp the proposed estimates are, in principle, for several particular choices of weights in the estimates (25).

Example 1: Let $\alpha := \varepsilon$ and $\beta := 1$. Then we observe that

$$|||e|||^2_{\frac{\varepsilon}{2},0,\Omega} \le \underline{\mathbf{EST}}_{\alpha,\beta} \le |||e|||^2_{\varepsilon(1+2\gamma),0,\Omega}. \tag{26}$$

Since γ can be taken arbitrarily small, we see that in this case $\underline{\mathbf{EST}}_{\alpha,\beta}$ can overestimate the error (which is, in fact, the L_2-norm of the difference $u-\bar{u}$)

$$|||e|||^2_{\frac{\varepsilon}{2},0,\Omega} = \frac{\varepsilon}{2}\|\nabla(u-\bar{u})\|^2_{0,\Omega}$$

at most twice, which is quite aceptable for the error control in real computation.

Example 2: Let now $\alpha := \varepsilon$ and $\beta := 2$. In this case, we get the same result as in Example 1, but now for another global error, which is nothing else, but the halved standard energy norm of the error

$$\frac{\varepsilon}{2}\|\nabla(u-\bar{u})\|^2_{0,\Omega} + \frac{1}{2}\|\sqrt{\tilde{c}}(u-\bar{u})\|^2_{0,\Omega}.$$

Example 3: Let us now take $\alpha := 2\varepsilon$ and $\beta := 2$. Then we have

$$|||e|||^2_{0,\frac{1}{2},\Omega} \le \underline{\mathbf{EST}}_{\alpha,\beta} \le |||e|||^2_{\varepsilon\gamma,1,\Omega}. \tag{27}$$

Since γ can be taken arbitrarily small, we see that in this case $\underline{\mathbf{EST}}_{\alpha,\beta} = \underline{\mathbf{EST}}_{2\varepsilon,2}$ can overestimate the value $\frac{1}{2}\|\sqrt{\tilde{c}}(u-\bar{u})\|^2_{0,\Omega}$ at most twice, too. It is worth noting that the number $\|\sqrt{\tilde{c}}(u-\bar{u})\|_{0,\Omega}$ can also serve as a global measure for the error if, for example, we have

$$0 < c_0 \le \tilde{c}(x) \le c_1$$

valid almost everywhere in Ω with positive constants c_0 and c_1.

Remark 3: All three above examples demonstrate that the global a priori error estimates proposed in the paper are, in principle, quasi-sharp. Moreover, theoretically there still exists a possibility that for the better choice of parametres $\mathbf{y}^\star$ and v, different from that one propose in (24), we can prove that the estimates perform, in fact, even better.

Remark 4: Situations with another choices of weights α and β can be also considered if required by real calculations following some more specific goal.

4.3 On choice of parameters in computations

A very comprehensive analysis on how to choose the optimal values for various parameters involved in the definition of the estimates of type (20) is the subject of our subsequent paper, see also [15, Sect. 5] for some relevant ideas.

However, for completeness we propose here one simple way of usage of our estimates in the situations when computations are performed on a series of meshes, which is a quite typical situation in the engineering practice. Let us have a sequence of computational meshes $\mathcal{T}^{h_1}$, $\mathcal{T}^{h_2}$, $\ldots$, $\mathcal{T}^{h_k}$ $(k \geq 2)$ with corresponding computed approximate solutions u^{h_1}, u^{h_2}, $\ldots$, u^{h_k}.

Assume that we want to control the error $e_i = u - u^{h_i},\ i = 1, \ldots, k$. Then as immediate candidates for the parameter v to be employed in our estimates we can take easily computable functions $v_{i,j} = u^{h_i} - u^{h_j},\ i, j = 1, \ldots, k$, and use functions $\varepsilon G_{h_i}(\nabla u_{h_i}),\ i = 1, \ldots, k$ as values for the parameter $\mathbf{y}^\star$. Here, G_{h_i} denotes some gradient averaging operator (see [6, 20] and references therein) for concrete definitions.

4.4 On mesh adaptive strategy

It is also natural to ask how we could use our estimates for the mesh adaptation purposes. Here, we describe a general strategy for such a goal. The upper estimate (20) is, in fact, an integral over the solution domain Ω. Thus, we can represent the value of this integral as the following sum

$$\sum_{T \in \mathcal{T}^{(i)}} I_T,$$

where each contribution I_T is a value of the total domain integral taken over a particular element T of the current mesh $\mathcal{T}^{(i)}$. To construct the next mesh $\mathcal{T}^{(i+1)}$ in order to obtain a more accurate approximation, we could use the following scheme. First, we find the maximum among all terms $|I_T|$ and, secondly, mark up those elements T which have their contributions larger than the "user-given threshold" θ $(\theta \in [0, 1])$ times that maximum value. Refining the marked elements (and making the mesh conforming), we obtain the next mesh $\mathcal{T}^{(i+1)}$.

Acknowledgements: The author was supported by the Academy Research Fellowship no. 208628 from the Academy of Finland. He would also like to thank Prof. Dmitri Kuzmin (Dortmund) and Prof. János Karátson (Budapest) for many valuable comments on the paper.

References

[1] M. Ainsworth and J. T. Oden. *A Posteriori Error Estimation in Finite Element Analysis.* John Wiley & Sons, 2000.

[2] L. Angermann, A posteriori error estimates for FEM with violated Galerkin orthogonality. *Numer. Methods Partial Differential Equations* **18** (2002) 241–259.

[3] I. Babuška and T. Strouboulis. *The Finite Element Method and Its Reliability.* Oxford University Press Inc., New York, 2001.

[4] W. Bangerth and R. Rannacher. *Adaptive Finite Element Methods for Differential Equations.* Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2003.

[5] R. Becker and R. Rannacher. A feed-back approach to error control in finite element methods: Basic approach and examples. *East-West J. Numer. Math.* **4** (1996) 237–264.

[6] J. Brandts and M. Křížek. Gradient superconvergence on uniform simplicial partitions of polytopes, *IMA J. Numer. Anal.* **23**, 489–505 (2003).

[7] C. Carstensen and S. A. Funken, Constants in Clément-interpolation error and residual based a posteriori error estimates in finite element methods. *East-West J. Numer. Math.* **8** (2000) 153–175.

[8] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam, 1978.

[9] K. Eriksson and C. Johnson, Adaptive streamline diffusion finite element methods for stationary convection-diffusion problems. *Math. Comp.* **60** (1993) 167–188.

[10] A. Hannukainen and S. Korotov, Computational technologies for reliable control of global and local errors for linear elliptic type boundary value problems. Technical Report **A494**, Helsinki University of Technology, 2006. Submitted to *J. Numer. Anal. Comput. Math.*

[11] V. John. A numerical study of a posteriori error estimators for convection-diffusion equations. *Comput. Methods Appl. Mech. Engrg.* **190** (2000) 757–781.

[12] C. Johnson, U. Nävert and J. Pitkäranta. Finite element methods for linear hyperbolic problems, *Comput. Methods Appl. Mech. Engrg.* **45**, 285–312 (1984).

[13] S. Korotov. A posteriori error estimation of goal-oriented quantities for elliptic type BVPs. *J. Comput. Appl. Math.* **191** 216–227 (2006).

[14] S. Korotov. A posteriori error estimation for linear elliptic problems with mixed boundary conditions. Technical Report **A495**, Helsinki University of Technology, 2006. To appear in *Appl. Math.*

[15] S. Korotov and D. Kuzmin. A new approach to a posteriori error estimation for convection-diffusion problems. I. Getting started. Technical Report **335**, University of Dortmund, 2006. Submitted to *SIAM J. Numer. Anal.*

[16] M. Křížek and P. Neittaanmäki. *Finite Element Approximation of Variational Problems and Applications*. Longman Scientific & Technical, Harlow, 1990.

[17] D. Kuzmin, R. Löhner and S. Turek (eds.) *Flux-Corrected Transport: Principles, Algorithms, and Applications*. Springer, 2005.

[18] S. G. Mikhlin. Constants in Some Inequalities of Analysis. A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1986.

[19] K. W. Morton. Numerical Solution of Convection-Diffusion Problems, Kluwer Academic Publishers, Dordrecht, 1995.

[20] M. Möller and D. Kuzmin, Adaptive mesh refinement for high-resolution finite element schemes. *Int. J. Numer. Meth. Fluids* **52** (2006) 545-569.

[21] P. Neittaanmäki and S. Repin, *Reliable Methods for Computer Simulation: Error Control and A Posteriori Estimates*. Elsevier, 2004.

[22] S. Repin, S. Sauter and A. Smolianski, A posteriori error estimation for the Dirichlet problem with account of the error in the approximation of boundary conditions. *Computing* **70** (2003) 205–233.

[23] S. Repin, S. Sauter and A. Smolianski, A posteriori error estimation for the Poisson equation with mixed Dirichlet/Neumann boundary conditions. *J. Comput. Appl. Math.* **164/165** (2004) 601–612.

[24] M. Rüter, S. Korotov and Ch. Steenbock, Goal-oriented error estimates based on different FE-solution spaces for the primal and the dual problem with applications to fracture mechanics, Comput. Mech. (in press).

[25] R. Verfürth. A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley-Teubner, 1996.

[26] R. Verfürth. A posteriori error estimators for convection-diffusion equations. *Numer. Math.* **80** (1998) 641–663.

[27] R. Verfürth. Robust a posteriori error estimates for stationary convection-diffusion equations. *SIAM J. Numer. Anal.* **43** (2005) 1766–1782.

(continued from the back cover)

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The list of reports is continued inside. Electronical versions of the reports are available at *http://www.math.hut.fi/reports/* .

A516 Sergey Repin , Rolf Stenberg
Two-sided a posteriori estimates for the generalized stokes problem
December 2006

A515 Sergey Korotov
Global a posteriori error estimates for convection-reaction-diffusion problems
December 2006

A514 Yulia Mishura , Esko Valkeila
An extension of the L'evy characterization to fractional Brownian motion
December 2006

A513 Wolfgang Desch , Stig-Olof Londen
On a Stochastic Parabolic Integral Equation
October 2006

A512 Joachim Schöberl , Rolf Stenberg
Multigrid methods for a stabilized Reissner-Mindlin plate formulation
October 2006

ISBN-10 951-22-8567-3 // ISBN-13 978-951-22-8567-9