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Abstract: *The paper is concerned with deriving computable majorants and minorants of the difference between the exact solution of for the so–called three–field formulation of the generalized Stokes problem and any functions from the admissible (energy) spaces that contain velocity, pressure and stress fields. Physical motivation of this problem is related to models of viscous fluids with polymeric chains. For the the case of uniform Dirichlét boundary conditions this model and respective numerical approximation methods were analyzed in [14]. In the present paper, we consider the generalized Stokes problem with mixed Dirichlét/Neumann boundary conditions and variable viscosity in the context of a posteriori error analysis. For the velocity, pressure, and stress fields we derive two–sided functional a posteriori error estimates. The estimates are practically computable, sharp (i.e., have no gap between the left– and right–hand sides), and are valid for arbitrary functions from the respective functional classes. The estimates are derived by transformations of the integral identity that defines the solution (this method was suggested and used in [39, 40] for certain classes of elliptic type problems). Error majorants are given by weighted sums of the terms that present penalties for violations of all the relations of the problem considered with the weights defined by the constants in the* Friederichs–Poincáre *and* Ladyzhenskaja–Babuska–Brezzi *inequalities, respectively.*

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1 Introduction

A posteriori estimates present a necessary tool in the adaptive procedures used in computer simulation. A systematic investigation of a posteriori error estimation methods for FEM was started three decades ago (see [5, 6]) and was first of all focused on creation of adequate error indicators able to provide the information required for a successful improvement of a mesh (see, e.g., [1, 7, 8, 25, 49]).

A posteriori error estimates for finite element approximations of viscous flow problems were investigated in numerous publications. In this concise introduction it is impossible to give a complete overview of these results, so that confine ourselves to a short discussion of several papers that present main approaches. Readers will find more literature references in the papers cited. A systematic discussion of the numerical methods, mesh adaptive procedures, and a posteriori estimates used in computational fluid mechanics can be found in, e.g., [19, 20, 22, 23, 26, 31, 35, 45, 47]. Residual type a posteriori methods for finite element approximations are considered in, e.g., [3, 48, 49]. A posteriori analysis of approximations computed by a backward Euler scheme is presented in [11]. Error indicators for the Navier–Stokes equations in stream function and vorticity formulation are discussed in [2]. In [27], the authors investigate various a posteriori estimators for stabilized mixed approximations of the Stokes problem. A posteriori error estimators for some quasi–newtonian fluids are considered in [33] and for combined fluid–solid systems in [10]. Error indicators based on superconvergence of finite element approximations for Stokes and Navier–Stokes equations are studied in [50].

In this paper, we consider a generalized formulation of the Stokes problem. A motivation of the problem comes from the theory of viscous flow problems for fluids with polymeric chains. The problem was presented and investigated in [14] where the respective numerical methods were also suggested. The goal of the present paper is to analyze it in the context of a posteriori error analysis and drive two–sided a posteriori error estimates of a new type. These estimates are derived by purely functional analysis of the boundary–value problem considered and, therefore, are applicable to any conforming approximations that belong to the energy functional class. For this reason, they are called *functional* a posteriori estimates.

For elliptic type problems of the divergent type functional a posteriori estimates were derived in [36, 37, 38, 39, 40, 43, 44] and some other papers with the help of duality methods in the calculus of variations (see [30] for a consequent exposition of the approach). Computable upper bounds of approximation errors for the Stokes problem with Dirichlét boundary conditions were derived by this method in [41] and for some classes of generalized Newtonian fluids in [21, 40].

In [39, 40, 42], another method of the derivation of functional a posteriori estimates was suggested. The method is based on certain transformations of integral identities that define the respective generalized solution. It is easy to demonstrate its performance on the paradigm of the problem $\Delta u + f = 0$ in Ω with the condition $u = 0$ on the boundary $\partial\Omega$. Here, the generalized solution u is defined by the integral identity

$$\int\limits_{\Omega} \nabla u \cdot \nabla w \, dx = \int\limits_{\Omega} f w dx \quad \forall w \in \overset{\circ}{H}{}^1(\Omega),$$

which leads to the relation

$$\int_{\Omega} \nabla(u - v) \cdot \nabla w dx = \mathcal{F}_v(w),$$

where $\mathcal{F}_v(w) = \int_{\Omega} (\nabla v \cdot \nabla w - f w)$ is the error functional associated with the approximation $v \in \overset{\circ}{H}{}^1(\Omega)$. Let τ be a vector–valued function in the space $H(\Omega, \mathrm{div})$. Then,

$$|\mathcal{F}_v(w)| \, \leq \, \Big| \int\limits_{\Omega} (f w + \mathrm{div}\tau \, w) \, dx + \int\limits_{\Omega} (\tau - \nabla v) \cdot \nabla w \, dx \Big|.$$

We set $w = u - v$ and arrive at the estimate

$$\|\nabla(u - v)\| \leq \|\nabla v - \tau\| + c_F \|\mathrm{div}\tau + f\|, \tag{1}$$

where c_F is a constant in the Friederichs inequality. Estimate (1) is one of the simplest a posteriori estimates of the functional type (for the equation $\mathrm{div} A \nabla u + f = 0$ with positive definite symmetric matrix A such estimates are presented in [36, 37]). It is easy to observe that the right–hand side of (1) is nonnegative and vanishes if and only if $v = u$ and $\tau = \nabla u$. Moreover, it is exact in the sense that τ can be taken such that the right–hand side of (1) is equal to the left–hand one.

In the present paper, *two–sided* a posteriori error bounds for the generalized Stokes problem are derived from the respective integral identities. The estimates are obtained for the velocity, pressure, and stress fields. It is shown that the estimates are computable and sharp. Thus, the paper presents a complete analysis of the considered class of problems in the framework of the functional approach to a posteriori error estimation,

The paper is organized as follows. Section 2 presents a generalized formulation of the Stokes problem and its mathematically equivalent formulations. In Section 3, we prove some basic results necessary for the subsequent analysis. They follow from Lemma 1 that presents a fundamental fact in the theory of functions related to the operator div. It implies a simple proof of the existence of the generalized solution and stability estimates for the velocity and pressure fields (for the case of homogeneous Dirichlét boundary conditions these properties were earlier established in [14] but with the help of a somewhat different method). Moreover, we show that Lemma 1 implies estimates of the distance to the set of solenoidal fields (see also [40, 41]).

Two–sided a posteriori estimates for an approximation v of the velocity u are derived in Section 4. First, they are derived for the approximations that satisfy the condition $\mathrm{div} v = \phi$. In practice, such a condition may be difficult to exactly satisfy. Therefore, by Lemma 1 we derive two sided bounds for the approximations that may violate it. We outline that the constant c_Ω in Lemma 1 serves as a penalty for possible violation of this condition.

In Section 5, we derive functional a posteriori estimates for approximations of the pressure and stress fields. Again, an important role in the respective analysis plays Lemma 1 and the constant c_Ω appears in the estimates.

Final Section 6 is focused on the case of mixed Dirichlét-Neumann boundary conditions. Here we prove Lemma 3 that present a generalization of the estimate of the distance to the set of solenoidal functions to the case of functions vanishing on a part of the boundary. With help of Lemma 3 we derive a posteriori estimates for approximations of the velocity and pressure fields.

2 Generalized Stokes problem

Let Ω be a connected bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary $\partial\Omega$. In this paper, we analyze a generalized formulation of the classical Stokes problem that consists of finding (u, p, σ_p) such that

$$-\mathrm{Div}(\eta_s \varepsilon(u)) - \mathrm{Div}\sigma_p = f - \nabla p \qquad \text{in } \Omega, \tag{2}$$

$$\mathrm{div} u = \phi \qquad \text{in } \Omega, \tag{3}$$

$$\sigma_p = \eta_p(\text{æ} + \varepsilon(u)) \qquad \text{in } \Omega, \tag{4}$$

$$u = u_0 \qquad \text{on } \partial\Omega, \tag{5}$$

where div and Div denote the divergence of a vector– and tensor–valued function, respectively, $\eta_s \geq 0$, and $\eta_p > 0$. We assume that the given functions are such that

$$f \in L^2(\Omega, \mathbb{R}^d), \quad \phi \in L^2(\Omega), \quad \text{æ} \in L^2(\Omega, \mathbb{M}^{d\times d}) \tag{6}$$

and satisfy the compatibility relation

$$\int\limits_\Omega \phi\, dx = \int\limits_{\partial\Omega} u_0 \cdot n\, ds\,, \tag{7}$$

where n is the unit normal vector outward to $\partial\Omega$ Physical motivation of the system (2)–(5), its analysis, and numerical methods are presented in [14]. This Stokes type system is based on the usual splitting of the total stress for a polymeric liquid into three contributions: the pressure $-p\mathbb{I}$, the stress due to the Newtonian solvent $\eta_s\varepsilon(u)$, and the extra stress due to the polymeric chains σ_p. Here, p is the pressure function, σ_p is the extra stress arising due to polymer chains, v is the velocity field and u_0 is a given function that satisfies the relation $\mathrm{div} u_0 = \phi$ and defines the Dirichlét boundary conditions. In a more general case, η_s and η_p are positive functions. We also present the estimates applicable to such a situation.

It is not difficult to observe that (2)–(4) can be presented in the form

$$\mathrm{Div}\sigma + f = 0, \qquad \text{in } \Omega, \tag{8}$$

$$\mathrm{div} u = \phi \qquad \text{in } \Omega, \tag{9}$$

$$\sigma = -p\mathbb{I} + \mu æ + \nu\varepsilon(u) \quad \text{in } \Omega,, \tag{10}$$

where $\mu = \eta_p$ and $\nu = \eta_p + \eta_s$. Hereafter, we assume that μ and ν are positive functions such that $\mu \in [\mu_\ominus, \mu_\oplus]$ and $\nu \in [\nu_\ominus, \nu_\oplus]$.

Hereafter, we assume that æ satisfies the condition

$$\mathrm{tr}\,(\mu æ + \nu\varepsilon(u)) = 0, \tag{11}$$

where tr denotes the trace of a tensor. In essence, this assumption does not lead to a loss of generality because it is always possible to "shift" the functions and pass to an equivalent formulation that satisfies (11).

Let u_ϕ be a function such that $\mathrm{div} u_\phi = \phi$ and $u_\phi = u_0$ on $\partial\Omega$. Introduce the function $\bar{u} := u - u_\phi$. Then, the system can be represented in the form

$$\mathrm{Div}\sigma + f = 0, \qquad \text{in } \Omega, \tag{12}$$

$$\mathrm{div}\bar{u} = 0 \qquad \text{in } \Omega, \tag{13}$$

$$\sigma = -p\mathbb{I} + \mu\bar{æ} + \nu\varepsilon(\bar{u}), \quad \text{in } \Omega, \tag{14}$$

$$\bar{u} = 0 \qquad \text{on } \partial\Omega, \tag{15}$$

where $\bar{æ} := æ + \frac{\nu}{\mu}\,\varepsilon(u_\phi)$. Note that

$$\mathrm{tr}\mu\bar{æ} = -\nu\mathrm{div} u + \nu\mathrm{div} u_\phi = 0,$$

so that (14) decomposes $\bar{\sigma}$ into the spherical and deviatoric parts, respectively.

In what follows, we denote scalar product of vectors by $\cdot$ (i.e., $u\cdot v = u_i v_i$) and tensors by : (i.e., $\tau : \sigma = \tau_{ij}\sigma_{ij}$), where the agreement on the summation over the repeated indexes is adopted. All tensor–valued functions whose components are square summable in Ω form the space Σ with the norm $\|\tau\|^2 := \int\limits_\Omega |\tau|^2 dx$. Also, we use a special notation Q for the space $L^2(\Omega)$. Since no confusion may arise we denote the norm of Q and the norm of the space $L^2(\Omega, \mathbb{R}^d)$ (which contains all vector–valued functions with square summable components) by $\|\cdot\|$. $V_0(\Omega)$ is a subset of $H^1(\Omega)$ formed by the functions with zero traces on $\partial_1\Omega$ and

$$\overset{\circ}{L}{}^2(\Omega) := \left\{ q \in Q \,\middle|\, [\,q\,]_\Omega := \int_\Omega q\,dx = 0 \right\}.$$

By $V(\Omega)$ we denote the space $H^1(\Omega, \mathbb{R}^d)$. All the functions of $V(\Omega)$ that vanishes on $\partial\Omega$ form the space $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$. A subspace of $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$ that consists of solenoidal fields is denoted by $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$. If $\rho(x)$ is a positive bounded function then the relation $\|\tau\|^2_{(\rho)} := \int\limits_\Omega \rho|\tau|^2 dx$ defines another (weighted) norm in Σ.

The space $\mathrm{H}(\Omega, \mathrm{Div})$ is a subspace of Σ that contains tensor–valued functions with square–summable divergence, i.e.,

$$\mathrm{H}(\Omega, \mathrm{Div}) := \{\tau \in \Sigma \mid \mathrm{Div}\tau := \{\tau_{ij,j}\} \in L^2(\Omega, \mathbb{R}^d)\}.$$

Generalized solution $\bar{u}$ of the system (12)–(15) is a function in $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$ that satisfies the integral identity

$$\int\limits_\Omega \nu\varepsilon(\bar{u}) : \varepsilon(w) + \mu\,\bar{æ} : \varepsilon(w) dx = \int\limits_\Omega f \cdot w\, dx, \quad w \in \overset{\circ}{\mathrm{J}}{}^1(\Omega). \tag{16}$$

Existence and uniqueness of $\bar{u}$ is easy to prove if note that this function minimizes the functional

$$I(w) := \int\limits_\Omega \left(\frac{\nu}{2}|\varepsilon(w)|^2 + \mu\,\bar{æ} : \varepsilon(w)\right) dx - \int\limits_\Omega f \cdot w\, dx \tag{17}$$

over the space $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$ and (16) is the Euler equation for the minimizer $\bar{u}$. The functional I is evidently strictly convex and continuous on V_0. Moreover, I is coercive on $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$. The latter fact follows from the Korn's inequality and obvious estimate

$$\left|\int\limits_\Omega f \cdot w\, dx\right| \leq C_F \|\varepsilon(w)\|_{(\nu)}, \tag{18}$$

where C_F is a constant in the Friederichs type inequality

$$\|w\| \leq C_F \|\varepsilon(w)\|_{(\nu)}, \qquad \forall\, w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega). \tag{19}$$

Therefore, existence and uniqueness of $\bar{u}$ is easy to establish by known results in the calculus of variations (see, e.g., [18]).

Finally, we note that if a wider set of trial functions $w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ is considered, then $\bar{u}$ can be defined by the integral identity

$$\int\limits_\Omega (\nu\varepsilon(\bar{u}) : \varepsilon(w) + \mu\,\bar{æ} : \varepsilon(w))\, dx = \int\limits_\Omega p\, \mathrm{div} w\, dx + \int\limits_\Omega f \cdot w\, dx \tag{20}$$

that involves the pressure field $p \in \overset{\circ}{L}{}^2(\Omega)$.

Our goal is to derive upper and lower bounds for the energy norms of deviations $\bar{u} - \bar{v}$, $p - q$, where $\bar{v}$, and q are approximations of $\bar{u}$ and p, respectively. Also, we will obtain estimates for the difference $\sigma - \tau$ where $\tau \in \Sigma$ is an approximation of the true stress σ.

3 Stability Lemma and its corollaries

3.1 Stability Lemma

We begin with one important result in the theory of functions related to the operator div.

Lemma 1. *Let Ω be a bounded domain with Lipschitz continuous boundary. Then, a positive constant* c_Ω *exists (which depends only on Ω) such that for any function $f \in \overset{\circ}{L}{}^2(\Omega)$ one can find a function $w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ satisfying the relations* $\mathrm{div} w = f$ *and*

$$\|\nabla w\| \leq \mathrm{c}_\Omega \|f\|. \tag{21}$$

Readers will find the proof in [29]. Also, Lemma 1 can be considered as a special case of the *closed range lemma* (see, e.g., [16, 51]).

Lemma 1 means that the quantity $\inf\limits_{w \in \{\mathrm{div} w = f\}} \|\nabla w\|$ is uniformly bounded with respect to $\|f\|$. It implies several important results.

First, it leads to the key condition in the mathematical theory of incompressible fluids known in the literature as Inf–Sup (or *Ladyzhenskaya–Babuska–Brezzi* (LBB)) condition. The latter reads: there exists a positive constant $\mathbb{C}_\Omega$ such that

$$\inf_{\substack{q \in \overset{\circ}{L}{}^2(\Omega) \\ q \neq 0}} \ \sup_{\substack{w \in V_0 \\ w \neq 0}} \frac{\int_\Omega q \, \mathrm{div} w \, dx}{\|q\| \, \|\nabla w\|} \geq \mathbb{C}_\Omega. \tag{22}$$

Really, by Lemma 1 we know that for any $q \in \overset{\circ}{L}{}^2(\Omega)$ one can find a function $v_q \in V_0$ satisfying the conditions

$$\mathrm{div} v_q = q, \quad \|\nabla v_q\| \leq \mathrm{c}_\Omega \|q\|. \tag{23}$$

In this case,

$$\sup_{v \in V_0(\Omega), w \neq 0} \frac{\int_\Omega q \mathrm{div} v \, dx}{\|\nabla v\| \, \|q\|} \geq \frac{\int_\Omega q \mathrm{div} v_q \, dx}{\|\nabla v_q\| \, \|q\|} = \frac{\|q\|}{\|\nabla v_q\|} \geq \frac{1}{\mathrm{c}_\Omega}$$

and, consequently, (22) holds with $\mathbb{C}_\Omega = (\mathrm{c}_\Omega)^{-1}$. Inf-Sup condition (22) and its discrete analogs are used for proving stability and convergence of numerical methods in various problems related to the theory of viscous incompressible fluids. In [4] and [15], this condition was proved and used to justify the convergence of the so–called *mixed* methods, in which a boundary–value problem is reduced to a saddle–point problem for a certain Lagrangian. It is worth noting, that (22) can be also derived from the Nečas inequality, whose simple proof for domains with Lipschitz boundaries can be found in [13]. Estimates of the value of $\mathbb{C}_\Omega$ for various domains are discussed in, e.g., [17, 32, 40].

3.2 Existence of a solution and stability estimates

With help of Lemma 1 it is not difficult to prove existence of u, p, and σ that deliver a solution to the problem (12)–(16). For this purpose, we use general theorems in convex analysis concerning saddle points of Lagrangians. Consider the Lagrangian $L : \overset{\circ}{\mathrm{V}}{}^1(\Omega) \times \overset{\circ}{L}{}^2(\Omega) \to \mathbb{R}$ of the form

$$L(w,q) := \int\limits_{\Omega} \Big(\frac{\nu}{2}|\varepsilon(w)|^2 + \mu\bar{æ} : \varepsilon(w) + q\mathrm{div}w \Big)\, dx - \int\limits_{\Omega} f \cdot w\, dx.$$

and the saddle point problem

$$L(\bar{u},q) \leq L(\bar{u},p) \leq L(w,p) \qquad \forall w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega),\ q \in \overset{\circ}{L}{}^2(\Omega). \tag{24}$$

It is not difficult to verify that the saddle point $(\bar{u},p)$ is formed by the velocity field $\bar{u}$ and the pressure function p satisfying (12)–(16). Indeed, the left hand side of (24) means that $\mathrm{div}\bar{u} = 0$, while the right hand one leads to (20). Problem (24) is equivalent to two variational problems

$$(\mathcal{P}_u) \inf_{w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)} \sup_{q \in \overset{\circ}{L}{}^2(\Omega)} L(w,q) \qquad \text{and} \qquad (\mathcal{P}_p) \sup_{q \in \overset{\circ}{L}{}^2(\Omega)} \inf_{w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)} L(w,q).$$

Since

$$\inf_{w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)} \sup_{q \in \overset{\circ}{L}{}^2(\Omega)} L(w,q) = \inf_{w \in \overset{\circ}{\mathrm{J}}(\Omega)} I(w) = I(\bar{u}),$$

we observe that Problem $\mathcal{P}_u$ defines the velocity field $\bar{u}$. Problem $\mathcal{P}_p$ defines the pressure field, however the functional of this problem cannot be presented in an explicit form.

Existence of $\bar{u}$ and p follow from Lemma 1 and known theorems in the theory of saddle points. Evidently, L is convex and continuous with respect to the first variable and linear and continuous with respect to the second one. Therefore (see, e.g., [18] Chapter 4, §2) it suffices to show that

$$\exists \widetilde{q} \in \overset{\circ}{L}{}^2(\Omega) \qquad \text{such that} \qquad \lim_{\|w\|_{\overset{\circ}{\mathrm{V}}{}^1(\Omega)} \to +\infty} L(w,\widetilde{q}) = +\infty \tag{25}$$

and

$$\lim_{\|q\| \to +\infty} \inf_{w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)} L(w,q) = -\infty. \tag{26}$$

Set $\widetilde{q} = 0$, then (25) is satisfied. To prove (26) we select v_q such that $\mathrm{div}v_q = q$ and $\|\nabla v_q\| \leq \mathrm{c}_\Omega \|q\|$. Then

$$\begin{aligned}
&\inf_{w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)} L(w,q) \leq L(\lambda v_q, q) = \\
&\quad = \int\limits_{\Omega} \Big(\frac{\nu}{2}\lambda^2 |\varepsilon(v_q)|^2 + \mu\lambda\bar{æ} : \varepsilon(v_q) + \lambda|q|^2 \Big)\, dx - \lambda \int\limits_{\Omega} f \cdot v_q dx \leq \\
&\leq \left(\frac{\nu_\oplus \lambda^2}{2}\, c_\Omega^2 + \lambda \right) \|q\|^2 + \lambda \int\limits_{\Omega} \mu\bar{æ} : \varepsilon(v_q)\, dx - \lambda \int\limits_{\Omega} f \cdot v_q dx.
\end{aligned}$$

Since

$$\|\varepsilon(v_q)\| \leq \|\nabla v_q\| \leq \mathrm{c}_\Omega \|q\| \;\text{ and }\; \left|\int\limits_\Omega f\cdot v_q dx\right| \leq C_F\,\mathrm{c}_\Omega\|q\|\|f\|,$$

we set $\lambda = -\dfrac{1}{\nu_\oplus c_\Omega^2}$ and observe that

$$\inf_{w\in\overset{\circ}{\mathrm{V}}{}^1(\Omega)} L(w,q) \leq -\frac{\|q\|^2}{2\nu_\oplus c_\Omega^2} + \lambda(\mu_\oplus\|æ\|\mathrm{c}_\Omega + C_F\mathrm{c}_\Omega\|f\|)\|q\| \to -\infty \text{ as } \|q\|\to+\infty.$$

Thus, (26) holds and the saddle point $(\bar{u},p)$ exists.

From (16) we deduce the energy estimate for the velocity field

$$\|\varepsilon(\bar{u})\|_{(\nu)} \leq \|\frac{\mu}{\nu}\,\bar{æ}\|_{(\nu)} + C_F\|f\|. \tag{27}$$

Let $v_p \in\overset{\circ}{\mathrm{V}}{}^1(\Omega)$ be the function defined as a counterpart of p in Lemma 1. Then,

$$\int\limits_\Omega (\nu\varepsilon(\bar{u}):\varepsilon(v_p) + \mu\bar{æ}:\varepsilon(v_p))dx - \int\limits_\Omega f\cdot v_p\,dx = \int\limits_\Omega p\,\mathrm{div} v_p dx = \|p\,\|^2$$

and we obtain

$$\|p\,\| \leq \mathrm{c}_\Omega\left(\|\varepsilon(\bar{u})\|_{(\nu)} + \|\frac{\mu}{\nu}\bar{æ}\|_{(\nu)} + C_F\|f\|\right) \leq 2\mathrm{c}_\Omega\left(\|\frac{\mu}{\nu}\bar{æ}\|_{(\nu)} + C_F\|f\|\right), \tag{28}$$

which is the energy estimate for p. Estimate (27) and (28) show that the solution continuously depends on the external data and is stable.

3.3 Estimates of the distance to the set $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$

Approximations computed by numerical procedures may not belong to the space $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$. With help of Lemma 1 we can estimate the distance between such an approximation and the set of solenoidal fields. Subsequently, we will use such estimates and derive functional type a posteriori estimates valid for non-solenoidal approximations.

Lemma 2. *For any function $\widehat{v} \in\overset{\circ}{\mathrm{V}}{}^1(\Omega)$ there exists a function $v_0 \in\overset{\circ}{\mathrm{J}}{}^1(\Omega)$ such that*

$$\|\nabla(\widehat{v}-v_0)\| \leq \mathrm{c}_\Omega\|\mathrm{div}\widehat{v}\|. \tag{29}$$

Proof. Let $f = \mathrm{div}\widehat{v}$, where $\widehat{v}$ is a given function in $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$. Then, by Lemma 1 these exists a function $w_f \in\overset{\circ}{\mathrm{V}}{}^1(\Omega)$ such that

$$\mathrm{div}(\widehat{v}-w_f) = 0, \quad \|\nabla w_f\| \leq \mathrm{c}_\Omega\|\mathrm{div}\widehat{v}\|.$$

Hence, the function $v_0 := \widehat{v} - w_f \in\overset{\circ}{\mathrm{J}}{}^1(\Omega)$ satisfies the estimate (29). □

In other words, the distance between $\widehat{v} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ and the set of solenoidal fields $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$ is estimated from above by the quantity $\|\mathrm{div}\widehat{v}\|$ with the multiplier c_Ω that comes from Lemma 1.

We note that Lemma 2 can be equivalently derived from the LBB condition (22) (see [41]).

Remark 1. From Lemma 2 it follows that for any

$$\widehat{v} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_\phi := \{v \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) \mid v = v_0 + u_\phi,\ v_0 \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)\}$$

there exists

$$v_\phi \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi := \{v \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) \mid v = v_0 + u_\phi,\ v_0 \in \overset{\circ}{\mathrm{J}}{}^1(\Omega), \}$$

such that

$$\|\nabla(\widehat{v} - v_\phi)\| \leq \mathrm{c}_\Omega \|\mathrm{div}\widehat{v} - \phi\|. \tag{30}$$

Indeed, for $\widehat{v} - u_\phi \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ we can find a function $v_0 \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ such that

$$\|\nabla(\widehat{v} - u_\phi - v_0)\| \leq \mathrm{c}_\Omega \|\mathrm{div}(\widehat{v} - u_\phi)\| = \mathrm{c}_\Omega \|\mathrm{div}\widehat{v} - \phi\|.$$

Hence, $v_\phi = v_0 + u_\phi$ is the function required.

Remark 2. Sometimes, it is also required to estimate the distance between $\widehat{v}$ and the space of solenoidal H^1–functions in L^2–norm. Such an estimate follows from the solvability of the Dirichlét problem for the Lapalce operator. Indeed, the problem $\Delta w_g = g$, has a solution $w_g \in \overset{\circ}{H}{}^1(\Omega)$ for any $g \in L_2(\Omega)$ and meets the energy estimate $\|\nabla w_g\| \leq c_F \|g\|$. Therefore, there exists a vector–valued function $v_g = \nabla w_g$ such that $\mathrm{div} v_g = g$ and $\|v_g\| \leq c_F \|g\|$. Let $g = \mathrm{div}\widehat{v}$. Then, $v_0 = \widehat{v} - v_g$ is a solenoidal function and

$$\|\widehat{v} - v_0\| \leq c_F \|\mathrm{div}\widehat{v}\|, \tag{31}$$

where c_F is a constant in the Friederichs inequality.

4 A posteriori estimates

First, we derive functional a posteriori estimates for the approximations which are conforming in the sense that they exactly satisfy the relation (3). Let $v \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_\phi$. Then, the function $\bar{v} = v - u_\phi$ can be viewed as an approximation of $\bar{u}$ defined by the system (12)-(15). We will derive a computable upper bound for the quantity $\|\varepsilon(\bar{u} - \bar{v})\|_{(\nu)}$ from the integral identity (16). After that it is easy to obtain a similar estimate for $\|\varepsilon(u - v)\|_{(\nu)}$.

4.1 Upper bound of the error for $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$

Let τ be a tensor function in Σ. Introduce a linear continuous functional $\mathcal{L}_{\tau,f} : \overset{\circ}{\mathrm{V}}{}^1(\Omega) \to \mathbb{R}$ by the relation

$$\mathcal{L}_{\tau,f}(w) := \int\limits_\Omega f \cdot w \, dx - \int\limits_\Omega \tau : \varepsilon(w) \, dx.$$

Its norm is defined as follows:

$$[\![\mathcal{L}_{\tau,f}]\!] := \sup_{w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)} \frac{|\mathcal{L}_{\tau,f}(w)|}{\|\varepsilon(w)\|_{(\nu)}} \tag{32}$$

In view of (18), the functional is bounded and $[\![\mathcal{L}_{\tau,f}]\!] \leq C_F + \|\tau\|_{(\nu^{-1})}$. The kernel of $\mathcal{L}_{\tau,f}$ contains all the tensor–valued functions that satisfy (in a generalized sense) the equilibrium equation

$$\mathrm{Div}\tau + f = 0, \qquad \text{in } \Omega. \tag{33}$$

Theorem 1. *For any $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$, $q \in \overset{\circ}{L}{}^2(\Omega)$, and $\tau \in \Sigma$ the following estimate holds*

$$\|\varepsilon(u - v)\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\, æ - \nu\varepsilon(v)\|_{(\nu^{-1})} + [\![\mathcal{L}_{\tau,f}]\!]. \tag{34}$$

If τ belongs to a narrower set $\mathrm{H}(\Omega, \mathrm{Div})$ then the upper bound is expressed in terms of integrals, namely

$$\begin{aligned} &\|\varepsilon(u - v)\|_{(\nu)} \leq \\ &\quad \leq M^{(1)}_{\oplus}(v, \tau, q) := \|\tau + q\mathbb{I} - \mu æ - \nu\varepsilon(v)\|_{(\nu^{-1})} + C_F\|\mathrm{Div}\tau + f\|. \end{aligned} \tag{35}$$

Proof. First we derive estimates for the problem (16). Let $\bar{v}$ be a certain function in $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$. Insert it into both parts of (16). Then, for any $w \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$, we have

$$\int\limits_\Omega \nu\varepsilon(\bar{u} - \bar{v}) : \varepsilon(w) dx = -\int\limits_\Omega (\mu\, \bar{æ} : \varepsilon(w) + \nu\varepsilon(\bar{v}) : \varepsilon(w)) dx + \int\limits_\Omega f \cdot w \, dx.$$

Let $\tau \in \Sigma$. Then,

$$\begin{aligned} &\int\limits_\Omega \nu\varepsilon(u - \bar{v}) : \varepsilon(w) dx = \\ &= \int\limits_\Omega (\tau - \mu\, \bar{æ} - \nu\varepsilon(\bar{v})) : \varepsilon(w) dx + \int\limits_\Omega f \cdot w \, dx - \int\limits_\Omega \tau : \varepsilon(w) dx. \end{aligned} \tag{36}$$

It is easy to observe that

$$\begin{aligned} &\left| \int\limits_\Omega (\tau - \mu\bar{æ} - \nu\varepsilon(\bar{v})) : \varepsilon(w) dx \right| \leq \\ &\qquad \leq \|\tau + q\mathbb{I} - \mu\bar{æ} - \nu\varepsilon(\bar{v})\|_{(\nu^{-1})} \, \|\varepsilon(w)\|_{(\nu)}, \end{aligned} \tag{37}$$

where q is an arbitrary function in $\overset{\circ}{L}{}^2(\Omega)$. The second part of the right–hand side of (36) is formed by the functional $\mathcal{L}_{\tau,f}$ whose value is estimated from above by the quantity $|\!|\!|\mathcal{L}_{\tau,f}|\!|\!| \|\varepsilon(w)\|$.

From (36), (36), and (37) it follows that

$$\int\limits_\Omega \nu\varepsilon(\bar{u}-\bar{v}) : \varepsilon(w) dx \leq \tag{38}$$

$$\leq \Big(\|\tau + q\mathbb{I} - \mu\,\bar{æ} - \nu\varepsilon(\bar{v})\|_{(\nu^{-1})} + |\!|\!|\mathcal{L}_{\tau,f}|\!|\!| \Big) \|\varepsilon(w)\|_{(\nu)}.$$

Now, we set $w = \bar{u} - \bar{v}$ and arrive at the estimate

$$\|\varepsilon(\bar{u}-\bar{v})\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\,\bar{æ} - \nu\varepsilon(\bar{v})\|_{(\nu^{-1})} + |\!|\!|\mathcal{L}_{\tau,f}|\!|\!|. \tag{39}$$

Assume that $\tau \in \mathrm{H}(\Omega, \mathrm{Div})$. Then,

$$\mathcal{L}_{\tau,f}(w) = \int\limits_\Omega (\mathrm{Div}\tau + f)\cdot w\,dx \leq$$

$$\leq \|\mathrm{Div}\tau + f\|\|w\| \leq C_{F}\|\mathrm{Div}\tau + f\|\|\varepsilon(w)\|_{(\nu)}$$

and we find that

$$|\!|\!|\mathcal{L}_{\tau,f}(w)|\!|\!| \leq C_{F}\|\mathrm{Div}\tau + f\|. \tag{40}$$

By (39) we conclude that

$$\|\varepsilon(\bar{u}-\bar{v})\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\,\bar{æ} - \nu\varepsilon(\bar{v})\|_{(\nu^{-1})} + C_{F}\|\mathrm{Div}\tau + f\|. \tag{41}$$

To obtain estimates for the original problem we note that for $\bar{v} = v - u_\phi$

$$\|\varepsilon(u-v)\|_{(\nu)} = \|\varepsilon(\bar{u}-\bar{v})\|_{(\nu)}.$$

Since $\mu\bar{æ} = \mu æ - \nu\varepsilon(u_\phi)$ we use (39) and (41) and arrive at the estimates (34) and (35). $\square$

Estimates (34) and (35) have a clear meaning. Estimate (34) shows that the upper bound of the error consists of two parts. The first part vanishes if the functions $(\bar{v}, \tau, q)$ satisfy (14) in a strong (L^2) sense and the second one equals zero if τ satisfies (33) in a weak sense. In (35), the condition (33) is also considered in a strong sense. The majorant vanishes if and only if

$$\tau = -q\mathbb{I} + \mu\,æ + \nu\varepsilon(v)$$

and the relation $\mathrm{Div}\tau + f = 0$ holds almost everywhere in Ω. By the assumption v meets the Dirichlét boundary condition and satisfies the relation $\mathrm{div} v = \phi$, we conclude that in such a case $v = u$ and τ and q coincide with the exact stress and pressure fields, respectively.

$M^{(1)}_\oplus(v, \tau, q)$ is evidently continuous with respect to all the arguments. Therefore, it is not difficult to prove that

$$M^{(1)}_\oplus(v_k, \tau_k, q_k) \to 0$$

as $v_k \to u$ in $\overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_0$, $\tau_k \to \sigma$ in $\mathrm{H}(\Omega, \mathrm{Div})$, and $q_k \to p$ in $L^2(\Omega)$.

Remark 3. If $\text{æ} = 0$ then the estimate (41) comes in the form

$$\|\varepsilon(u - v)\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \nu\varepsilon(v)\|_{(\nu^{-1})} + C_F\|\mathrm{Div}\tau + f\,\|. \tag{42}$$

If $q \in H^1(\Omega)$, then it can be rewritten in another form

$$\|\varepsilon(u - v)\|_{(\nu)} \leq \|\tau - \nu\varepsilon(v)\|_{(\nu^{-1})} + C_F\|\mathrm{Div}\tau + f - \nabla q\,\|. \tag{43}$$

We note that (42) and (43) are the functional a posteriori estimates for the Stokes problem. They has been earlier derived in [40, 41].

Remark 4. By (11) we observe that

$$\begin{aligned}
&\|\tau + q\mathbb{I} - \mu\text{æ} - \nu\varepsilon(v)\|^2_{(\nu^{-1})} = \\
&= \int\limits_\Omega \frac{1}{\nu}\left(d\left(\frac{1}{d}\mathrm{tr}\tau + q\right)^2 + |\tau^D - \mu\text{æ}^D - \nu\varepsilon^D(v)|^2\right)dx.
\end{aligned}$$

If τ is selected such that $[\,\mathrm{tr}\tau\,]_\Omega = 0$ then we set $q = -\frac{1}{d}\mathrm{tr}\tau$ and obtain

$$\|\varepsilon(u - v)\|_{(\nu)} = \|\tau^D - \mu\text{æ}^D - \nu\varepsilon^D(v)\|_{(\nu^{-1})} + C_F\|\mathrm{Div}\tau + f\|. \tag{44}$$

Note that the right–hand side of (44) does not contain q. The right–hand side of (44) vanishes if

$$\begin{aligned}
\tau^D - \mu\,\text{æ}^D - \nu\varepsilon^D(v) = 0, &\quad \text{in} \quad \Omega,\\
\mathrm{Div}\tau + f = 0, &\quad \text{in} \quad \Omega.
\end{aligned}$$

Since $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$, we know that $\mathrm{div} v = \phi$ and satisfies the boundary condition. Besides, for the above tensor τ there exists a scalar function q with zero mean such that $\mathrm{tr}\tau = -dq$. This means that $v = u$, $\tau = \sigma$, and $q = p$.

4.2 Lower bound of the error for $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$

Theorem 2. *For any* $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$

$$\|\varepsilon(u - v)\|^2_{(\nu)} \;\geq\; M^{(1)}_\ominus(v, w) \tag{45}$$

where

$$M^{(1)}_\ominus(v, w) := \left(2\int\limits_\Omega (f \cdot w - (\nu\varepsilon(v) + \mu\text{æ}) : \varepsilon(w))\,dx - \|\varepsilon(w)\|^2_{(\nu)}\right)^{1/2},$$

and w is an arbitrary function in $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$.

Proof. The proof is based upon the variational formulation of the problem (12)–(15). Let $\bar{v} \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$. Then

$$I(\bar{v}) - I(\bar{u}) = \int\limits_{\Omega} \frac{\nu}{2}\|\varepsilon(\bar{u}-\bar{v})\|^2 dx +$$
$$+ \int\limits_{\Omega} (\nu\varepsilon(\bar{u}) : \varepsilon(\bar{u}-\bar{v}) + \mu\bar{æ} : \varepsilon(\bar{u}-\bar{v}))\, dx - \int\limits_{\Omega} f\cdot(\bar{v}-\bar{u})\, dx.$$

Since $\bar{u}-\bar{v} \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ we arrive at the relation

$$I(\bar{v}) - I(\bar{u}) = \frac{1}{2}\|\varepsilon(\bar{u}-\bar{v})\|^2_{(\nu)}. \tag{46}$$

Therefore, for any $w \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$, we have

$$\|\varepsilon(\bar{u}-\bar{v})\|^2_{(\nu)} \ \geq 2(I(\bar{v}) - I(\bar{v}+w)) =$$
$$= \int\limits_{\Omega} \big(-\nu|\varepsilon(w)|^2 - 2(\nu\varepsilon(\bar{v}) + \mu\bar{æ}) : \varepsilon(w)\big)\, dx + 2\int\limits_{\Omega} f\cdot w\, dx.$$

We obtain (45) if set $\bar{v} = v - u_\phi$ and recall that $\nu\varepsilon(\bar{v}) + \mu\bar{æ} = \nu\varepsilon(v) + \mu æ$. □

4.3 Computability and efficiency of two–sided estimates

The majorant $M^{(1)}_{\oplus}(v,\tau,q)$ contains only known functions and the constant $C_F(\Omega)$. The latter can be estimated from above by the value $\nu_{\ominus}^{-1} c_F(\widehat{\Omega})$, where $\widehat{\Omega}$ is a square (cube) that contains Ω. Therefore, it is completely *computable.*

In the simplest case, we can set

$$\tau \ = \ \mathfrak{G}\,(\nu\varepsilon(v) - \mu æ - q\mathbb{I}),$$

where q is a computed pressure and $\mathfrak{G}$ a certain smoothing operator whose action is required to guarantee that $\tau \in \mathrm{H}(\Omega, \mathrm{Div})$. Then, the upper bound is directly computable but in general may be rather coarse. If it is desirable to obtain a better bound, then it is necessary to adjust the functions τ and q with the help of the procedure discussed below.

It is easily seen that if $\tau = \sigma$ and $q = p$, then the value of $M^{(1)}_{\oplus}(v,\sigma,p)$ coincides with $\|\varepsilon(u-v)\|_{(\nu)}$, i.e., the estimate (35) is *sharp* in the sense that there is no gap between its left and right hand sides. Therefore, in principle, for any v the respective upper bound can be computed with any desirable accuracy. The minorant $M^{(1)}_{\ominus}(v,w)$ possesses similar properties: it is directly computable and for $w = u - v$ coincides with the true error.

Let $\{V_h, \Sigma_h, Q_h\}$ be finite dimensional subspaces of $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$, $\mathrm{H}(\Omega,\mathrm{Div})$, and $\overset{\circ}{L}{}^2(\Omega)$ respectively. From the above analysis it follows that the numbers

$$m_{k\ominus} := \sup_{w_h\in V_h} M^{(1)}_{\ominus}(v,w_h) \text{ and } m_{k\oplus} := \inf_{\tau_h\in\Sigma_h, q_h\in Q_h} M^{(1)}_{\oplus}(v,\tau_h,q_h) \tag{47}$$

provide two–sided bounds for the quantity $\|\varepsilon(u-v)\|_{(\nu)}$. Note that the quantities $m_{k\ominus}$ and $m_{k\oplus}$ are defined with the help of finite dimensional problems and are indeed computable. The theorem below shows that two–sided estimates can be computed as close to the true error as it is required.

Theorem 3. *Let* $\{V_{hk}, \Sigma_{hk}, Q_{hk}\}_{k=1}^{+\infty}$ *be a sequence of finite dimensional spaces which be limit dense in the respective functional spaces. Then, for any* $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$

$$m_{k\ominus} \leq \|\varepsilon(u-v)\|_{(\nu)} \leq m_{k\oplus} \quad \text{and} \quad m_{k\ominus} \to m_{k\oplus} \text{ as } k \to +\infty.$$

Proof. The result immediately follows from the limit density property and above discussed sharpness of the estimates. □

For the classical Stokes problem, practical efficiency of the functional a posteriori estimates was studied and confirmed in [24].

4.4 Upper bound of the error for $v \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_\phi$

Let us now assume that approximate solution $\widehat{v}$ may not satisfy the relation $\mathrm{div}\widehat{v} = \phi$.

Theorem 4. *For any* $\widehat{v} \in V$ *such that* $\widehat{v} = u_0$ *on* $\partial\Omega$, $q \in \overset{\circ}{L}{}^2(\Omega)$, *and* $\tau \in \Sigma$ *the following estimate holds*

$$\|\varepsilon(u-\widehat{v})\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\,æ - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + \\ + [\![\mathcal{L}_{\tau,f}]\!] + 2\nu_\oplus^{1/2}\mathrm{c}_\Omega\|\mathrm{div}\widehat{v} - \phi\|. \tag{48}$$

If $\tau \in \mathrm{H}(\Omega, \mathrm{Div})$ *then*

$$\|\varepsilon(u-\widehat{v})\|_{(\nu)} \leq M_\oplus^{(2)}(\widehat{v}, \tau, q) := \\ = \|\tau + q\mathbb{I} - \mu æ - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + C_F\|\mathrm{Div}\tau + f\| + 2\nu_\oplus^{1/2}\mathrm{c}_\Omega\|\mathrm{div}\widehat{v} - \phi\|. \tag{49}$$

Proof. By Lemma 1, for the function $\bar{v} := (\widehat{v} - u_\phi) \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ one can find a function $w_0 \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ such that

$$\|\varepsilon(\bar{v} - w_0)\|_{(\nu)} \leq \nu_\oplus^{1/2}\|\varepsilon(\bar{v} - w_0)\| \leq \mathrm{c}_\Omega\nu_\oplus^{1/2}\|\mathrm{div}\bar{v}\| = \mathrm{c}_\Omega\nu_\oplus^{1/2}\|\mathrm{div}\widehat{v} - \phi\|. \tag{50}$$

Then,

$$\|\varepsilon(u-\widehat{v})\|_{(\nu)} = \|\varepsilon(u - \bar{v} - u_\phi)\|_{(\nu)} \leq \|\varepsilon(u - w_0 - u_\phi)\|_{(\nu)} + \|\varepsilon(\bar{v} - w_0)\|_{(\nu)}. \tag{51}$$

Note that $\mathrm{div}(w_0 + u_\phi) = \phi$, so that we can use (34) to estimate the first norm in the right–hand side of this inequality. Then, we arrive at the estimate

$$\|\varepsilon(u-\widehat{v})\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu æ - \nu\varepsilon(w_0 + u_\phi)\|_{(\nu^{-1})} + [\![\mathcal{L}_{\tau,f}]\!] + \|\varepsilon(\bar{v} - w_0)\|_{(\nu)}.$$

Since

$$\|\tau + q\mathbb{I} - \mu æ - \nu\varepsilon(w_0 + u_\phi)\|_{(\nu^{-1})} \leq$$
$$\leq \|\tau + q\mathbb{I} - \mu æ - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + \|\varepsilon(\bar{v} - w_0)\|_{(\nu)}$$

we apply (50) and arrive at (48).

Estimate (49) is derived from (35) by means of similar arguments. □

It is easy to see that the majorant $M^{(2)}_{\oplus}(\widehat{v}, \tau, q)$ has the same principal structure as $M^{(1)}_{\oplus}(v, \tau, q)$. The only difference is that it contains a new term. The latter can be thought of as a penalty for possible violation of the condition $\mathrm{div} u = \phi$.

Remark 5. In view of the relation

$$\|\tau + q\mathbb{I} - \mu æ - \nu\varepsilon(v)\|^2_{(\nu^{-1})} =$$
$$= \int\limits_\Omega \frac{1}{\nu}\left(\frac{1}{d}\left(\mathrm{tr}\tau + dq - \mu\,\mathrm{tr}æ - \nu\,\mathrm{div}\widehat{v}\right)^2 + |\tau^D - \mu æ^D - \nu\varepsilon^D(v)|^2\right) dx.$$

If we assume that $[\,\mathrm{tr}\tau\,]_\Omega = 0$ and select $q = -\frac{1}{d}\mathrm{tr}\tau$ then the estimate has the form

$$\|\varepsilon(u - \widehat{v})\|_{(\nu)} \leq \sqrt{\|\tau^D - \mu æ^D - \nu\varepsilon^D(\widehat{v})\|^2_{(\nu^{-1})} + \frac{1}{d}\|\mathrm{div}\widehat{v} - \phi\|^2_{(\nu^{-1})}} +$$
$$+ C_F\|\mathrm{Div}\tau + f\| + 2\nu_{\oplus}^{1/2}\mathrm{c}_\Omega\|\mathrm{div}\widehat{v} - \phi\|. \tag{52}$$

Remark 6. The majorants $M^{(1)}_{\oplus}(v, \tau, q)$ and $M^{(2)}_{\oplus}(\widehat{v}, \tau, q)$ generate new variational formulations of the generalized Stokes problem: minimize $M^{(1)}_{\oplus}$ or $M^{(2)}_{\oplus}$ on admissible velocity, pressure and stress fields. Both problems have the exact lower bound equal to zero. It is attained if and only if the above fields coincide with the exact ones.

4.5 Lower bound of the error for $v \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_\phi$

If $\widehat{v} - u_\phi \notin \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ then derivation of a computable lower bound of the energy norm of the error presents a more complicated task. However, it can be also derived.

Theorem 5. *For any* $\widehat{v} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_\phi$

$$\|\varepsilon(u - \widehat{v})\|^2_{(\nu)} \;\geq\; M^{(2)}_{\ominus}(\widehat{v}, \widehat{w}) := \frac{1}{1+\gamma}\left(\mathcal{M}(\widehat{v}, \widehat{w}) - \rho(\widehat{v}, \widehat{w}, \eta, \gamma)\right), \tag{53}$$

where

$$\mathcal{M}(\widehat{v}, \widehat{w}) := \int\limits_\Omega \left(2f \cdot \widehat{w} - \nu|\varepsilon(\widehat{w})|^2 - 2(\nu\varepsilon(\widehat{v}) + \mu æ) : \varepsilon(\widehat{w})\right) dx,$$

$\widehat{w}$ is an arbitrary function in $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$, $\eta \in \mathrm{H}(\Omega, \mathrm{Div})$, $\gamma > 0$,

$$\rho(\widehat{v}, \widehat{w}, \eta, \gamma) = m_2(\widehat{v}, \widehat{w}, \eta) + m_3(\widehat{v}, \widehat{w}) + \mathrm{c}_\Omega^2 \nu_\oplus \left(1 + \frac{1}{\gamma}\right) \|\mathrm{div}\widehat{v} - \phi\|^2$$

and the terms m_2 and m_3 are defined by the relations (55) and (56).

Proof. Let $\widehat{w} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$. Then, for $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega) + u_\phi$ and $w \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ we can represent $M_\ominus^{(1)}(v, w)$ as follows:

$$\begin{aligned}
&M_\ominus^{(1)}(v, w) = \\
&\quad = 2 \int\limits_\Omega \Big(-\frac{\nu}{2} |\varepsilon(w - \widehat{w})|^2 - \frac{\nu}{2} |\varepsilon(\widehat{w})|^2 - \nu\varepsilon(v) : \varepsilon(w - \widehat{w}) - \nu\varepsilon(v) : \varepsilon(\widehat{w}) + \\
&+ \mu æ : \varepsilon(\widehat{w} - w) - \mu æ : \varepsilon(\widehat{w}) + f \cdot (w - \widehat{w}) + f \cdot \widehat{w} - \nu\varepsilon(w - \widehat{w}) : \varepsilon(\widehat{w}) \Big)\, dx = \\
&= \int\limits_\Omega \Big(-\nu |\varepsilon(\widehat{w})|^2 - 2(\nu\varepsilon(v) + \mu æ) : \varepsilon(\widehat{w}) + 2 f \cdot \widehat{w} \Big) dx - \nu_\oplus \|\varepsilon(\widehat{w} - w)\|^2 + \\
&+ 2 \int\limits_\Omega \Big((\nu\varepsilon(v) + \mu æ) : \varepsilon(\widehat{w} - w) - f \cdot (\widehat{w} - w) \Big) dx = J_1 + J_2 + J_3,
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \int\limits_\Omega \Big(2 f \cdot \widehat{w} - \nu |\varepsilon(\widehat{w})|^2 - 2(\nu\varepsilon(\widehat{v}) + \mu æ) : \varepsilon(\widehat{w}) \Big)\, dx, \\
J_2 &= 2 \int\limits_\Omega \Big((\nu\varepsilon(\widehat{v}) + \mu æ) : \varepsilon(\widehat{w} - w) - f \cdot (\widehat{w} - w) \Big) dx, \\
J_3 &= 2 \int\limits_\Omega (\nu\varepsilon(\widehat{v} - v) : \varepsilon(\widehat{w}) + \nu\varepsilon(\widehat{w} - w) : \varepsilon(\widehat{w})) dx + \\
&\qquad + 2 \int\limits_\Omega \nu\varepsilon(\widehat{v} - v) : \varepsilon(w - \widehat{w})\, dx - \nu_\oplus \|\varepsilon(\widehat{w} - w)\|^2.
\end{aligned}$$

Now, we apply the estimate

$$\begin{aligned}
&\|\varepsilon(u - \widehat{v})\|_{(\nu)}^2 \geq \frac{1}{1+\gamma} \|\varepsilon(u - v)\|_{(\nu)}^2 - \frac{1}{\gamma} \|\varepsilon(v - \widehat{v})\|_{(\nu)}^2 \geq \\
&\geq \frac{1}{1+\gamma} \left(J_1 + J_2 + J_3 - \left(1 + \frac{1}{\gamma}\right) \nu_\oplus \|\varepsilon(v - \widehat{v})\|^2 \right).
\end{aligned}$$

By Lemma 2 we can find the functions v_ϕ and w_0 such that

$$\|\varepsilon(\widehat{w} - w_0)\| \leq \mathrm{c}_\Omega \|\mathrm{div}\widehat{w}\|, \qquad \|\varepsilon(\widehat{v} - v_\phi)\| \leq \mathrm{c}_\Omega \|\mathrm{div}\widehat{v} - \phi\|. \quad (54)$$

Then,

$$\begin{aligned}
|J_3| \leq m_3(\widehat{v}, \widehat{w}) := 2\nu_\oplus \mathrm{c}_\Omega \Big(&(\|\mathrm{div}\widehat{v} - \phi\| + \|\mathrm{div}\widehat{w}\|) \|\varepsilon(\widehat{w})\| + \quad (55) \\
&+ \mathrm{c}_\Omega \|\mathrm{div}\widehat{v} - \phi\| \|\mathrm{div}\widehat{w}\| + \frac{\mathrm{c}_\Omega}{2} \|\mathrm{div}\widehat{w}\|^2 \Big)
\end{aligned}$$

To estimate J_2 we introduce a tensor–valued function $\eta \in \mathrm{H}(\Omega, \mathrm{Div})$. We have

$$
\begin{aligned}
&|J_2| = \qquad\qquad (56)\\
&= 2\Big|\int\limits_\Omega ((f + \mathrm{Div}\eta)\cdot(w - \widehat{w}) + ((\nu\varepsilon(\widehat{v}) + \mu æ) - \eta) : \varepsilon(\widehat{w} - w))\,dx\Big| \leq \\
&\qquad \leq m_2(\widehat{v}, \widehat{w}, \eta) := 2\mathrm{c}_\Omega\|\mathrm{div}\widehat{w}\|\Big(C_F\|f + \mathrm{Div}\eta\| + \|\eta - \nu\varepsilon(\widehat{v}) - \mu æ\|\Big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&(1+\gamma)\|\varepsilon(u - \widehat{v})\|^2_{(\nu)} \geq \\
&\qquad \geq J_1 - m_2(\widehat{v}, \widehat{w}, \eta) - m_3(\widehat{v}, \widehat{w}) - \mathrm{c}_\Omega^2\nu_\oplus\left(1 + \frac{1}{\gamma}\right)\|\mathrm{div}\widehat{v} - \phi\|^2
\end{aligned}
$$

and we arrive at (53). □

Remark 7. If $\mathrm{div}\widehat{v} = \phi$ and $\widehat{w} \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$, then $\rho(\widehat{v}, \widehat{w}, \eta, \gamma) = 0$ and

$$\mathcal{M}(\widehat{v}, \widehat{w}) = M^{(1)}_\ominus(\widehat{v}, \widehat{w}).$$

Thus, we set $\gamma = 0$ and observe that on this narrow class of functions (53) is equivalent to (45).

Remark 8. Let us evaluate the quality of the lower bound computed by the estimate (53) for an approximation $\widehat{v}$. Set $\widehat{w} = u - v_\phi$. Then $\mathrm{div}\widehat{w} = 0$, $m_2(\widehat{v}, \widehat{w}, \eta) = 0$ and

$$
\begin{aligned}
m_3(\widehat{v}, \widehat{w}) &= 2\nu_\oplus\mathrm{c}_\Omega\|\mathrm{div}\widehat{v} - \phi\|\|\varepsilon(u - v_\phi)\| \leq \\
&\leq 2\nu_\oplus\mathrm{c}_\Omega\|\mathrm{div}\widehat{v} - \phi\|(\|\varepsilon(u - \widehat{v})\| + \mathrm{c}_\Omega\|\mathrm{div}\widehat{v} - \phi\|).
\end{aligned}
$$

For the term $\mathcal{M}(\widehat{v}, \widehat{w})$ we have

$$\mathcal{M}(\widehat{v}, \widehat{w}) := \int\limits_\Omega \big(2f\cdot\widehat{w} - \nu|\varepsilon(\widehat{w})|^2 - 2(\nu\varepsilon(\widehat{v}) + \mu æ) : \varepsilon(\widehat{w})\big)\,dx.$$

Recall (16). We have

$$
\begin{aligned}
&\int\limits_\Omega \nu\varepsilon(u - u_\phi) : \varepsilon(u - v_\phi)dx = \qquad\qquad (57)\\
&\qquad = \int\limits_\Omega \Big(f\cdot(u - v_\phi) - (\mu\, æ + \mu\varepsilon(u_\phi)) : \varepsilon(u - v_\phi)\Big)\,dx.
\end{aligned}
$$

Since the choice of u_ϕ is restricted only by the boundary condition and the condition $\mathrm{div}u_\phi = \phi$, we can set $u_\phi = v_\phi$. Then,

$$
\begin{aligned}
\mathcal{M}(\widehat{v}, \widehat{w}) &:= \int\limits_\Omega \nu|\varepsilon(u - v_\phi)|^2dx \geq \frac{1}{1+\delta}\|\varepsilon(u - \widehat{v})\|^2_{(\nu)} - \frac{1}{\delta}\|\varepsilon(\widehat{v} - v_\phi)\|^2_{(\nu)} \geq \\
&\geq \frac{1}{1+\delta}\|\varepsilon(u - \widehat{v})\|^2_{(\nu)} - \frac{1}{\delta}\nu_\oplus\mathrm{c}_\Omega\|\mathrm{div}\widehat{v} - \phi\|^2
\end{aligned}
$$

and we obtain

$$M^{(2)}_{\ominus}(\widehat{v},\widehat{w}) \geq \frac{1}{1+\gamma}\Big(\frac{1}{1+\delta}\|\varepsilon(u-\widehat{v})\|^2_{(\nu)} - \left(1+\frac{1}{\delta}+\frac{1}{\gamma}\right)\nu_{\oplus}\mathrm{c}_{\Omega}\|\mathrm{div}\widehat{v}-\phi\|^2 - $$
$$-2\nu_{\oplus}\mathrm{c}_{\Omega}\|\mathrm{div}\widehat{v}-\phi\|(\|\varepsilon(u-\widehat{v})\| + \mathrm{c}_{\Omega}\|\mathrm{div}\widehat{v}-\phi\|)\Big). \qquad (58)$$

If $\mathrm{div}\widehat{v}=\phi$ then we set $\delta=\gamma=0$ and find that $M^{(2)}_{\ominus}(\widehat{v},\widehat{w})$ is equal to the error. Also, by (58) we conclude that the lower bound is good if $\mathrm{div}\widehat{v}$ is close to ϕ. If an approximate solution essentially violates this condition, then the quality of the lower bound deteriorates.

5 A posteriori estimates for approximations of pressure and stress fields

5.1 Estimates for the pressure

Estimates of $\|p-q\|$ can be also derived with the help of Lemma 1.

Theorem 6. *Let $q \in \overset{\circ}{L}{}^2(\Omega)$ be an approximation of the pressure field p. Then*

$$\frac{1}{2\mathrm{c}_{\Omega}\nu_{\oplus}^{1/2}}\|p-q\| \leq \|\nu\varepsilon(\widehat{v})+\mu æ-\tau-q\mathbb{I}\|_{(\nu^{-1})} + C_F\|\mathrm{Div}\tau+f\| +$$
$$+\nu_{\oplus}^{1/2}\mathrm{c}_{\Omega}\|\mathrm{div}\widehat{v}-\phi\|, \qquad (59)$$

where $\widehat{v}$ and τ are arbitrary functions in $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$ and $\mathrm{H}(\Omega,\mathrm{Div})$, respectively.

Proof. Since $(p-q)\in \overset{\circ}{L}{}^2(\Omega)$, by Lemma 1 we know that a function $\widetilde{w}\in\overset{\circ}{\mathrm{V}}{}^1(\Omega)$ exists such that

$$\mathrm{div}\widetilde{w}=p-q, \quad \text{and} \quad \|\varepsilon(\widetilde{w})\|\leq \mathrm{c}_{\Omega}\|p-q\|.$$

Hence,

$$\begin{aligned}
\|p-q\|^2 &= \int\limits_{\Omega}\mathrm{div}\widetilde{w}(p-q)\,dx = \\
&= \int\limits_{\Omega}(\nu\varepsilon(u):\varepsilon(\widetilde{w})+\mu\, æ:\varepsilon(\widetilde{w})-f\cdot\widetilde{w}-q\mathrm{div}\widetilde{w})\,dx = \\
&= \int\limits_{\Omega}\nu\varepsilon(u-\widehat{v}):\varepsilon(\widetilde{w})dx + \\
&+ \int\limits_{\Omega}(\nu\varepsilon(\widehat{v}):\varepsilon(\widetilde{w})+\mu\, æ:\varepsilon(\widetilde{w})-f\cdot\widetilde{w}-q\mathrm{div}\widetilde{w})\,dx.
\end{aligned}$$

Note that

$$\int\limits_{\Omega} \nu\varepsilon(u-\widehat{v}) : \varepsilon(\widetilde{w})\,dx \leq \mathrm{c}_\Omega \nu_\oplus^{1/2} \|\varepsilon(u-\widehat{v})\|_{(\nu)}\,\|p-q\|$$

and

$$\begin{aligned}\int\limits_{\Omega} &(\nu\varepsilon(\widehat{v}) : \varepsilon(\widetilde{w}) + \mu\,\text{æ} : \varepsilon(\widetilde{w}) - f\cdot\widetilde{w} - q\mathrm{div}\widetilde{w})\,dx = \\ &= \int\limits_{\Omega} (\nu\varepsilon(\widehat{v}) + \mu\text{æ} - \tau - q\mathbb{I}) : \varepsilon(\widetilde{w})dx - \int\limits_{\Omega} (\mathrm{Div}\tau + f)\cdot\widetilde{w}dx \leq \\ &\leq \Big(\|\nu\varepsilon(\widehat{v}) + \mu\text{æ} - \tau - q\mathbb{I}\| + C_F\nu_\oplus^{1/2}\|\mathrm{Div}\tau + f\|\Big)\mathrm{c}_\Omega\|p-q\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|p-q\| \leq\ & \mathrm{c}_\Omega\Big(\nu_\oplus^{1/2}\|\nu\varepsilon(\widehat{v}) + \mu\text{æ} - \tau - q\mathbb{I}\|_{(\nu^{-1})} + C_F\nu_\oplus^{1/2}\|\mathrm{Div}\tau + f\| + \\ &+\nu_\oplus^{1/2}(\|\tau + q\mathbb{I} - \mu\text{æ} - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + \\ &\qquad +C_F\nu_\oplus^{1/2}\|\mathrm{Div}\tau + f\| + 2\nu_\oplus\mathrm{c}_\Omega\|\mathrm{div}\widehat{v} - \phi\|)\Big)\end{aligned}$$

and we arrive at the estimate (59). □

It is easy to see that the right–hand side of (59) consists of the same terms as the right–hand side of (57) and vanishes if and only if,

$$\widehat{v} = u, \quad \tau = \sigma, \quad \text{and} \quad p = q.$$

However, in this case, the dependence of the penalty multipliers on the constant c_Ω is stronger.

Remark 9. If τ is subject to the condition $[\,\mathrm{tr}\tau\,]_\Omega = 0$ then the pressure can be excluded and instead of (59) we obtain

$$\begin{aligned}\|p-q\| \leq\ & 2\mathrm{c}_\Omega\nu_\oplus^{1/2}\Big(\sqrt{\|\tau^D - \mu\text{æ}^D - \nu\varepsilon^D(v)\|^2_{(\nu^{-1})} + \frac{1}{d}\|\mathrm{div}\widehat{v} - \phi\|^2_{(\nu^{-1})}} + \\ &+C_F\|\mathrm{Div}\tau + f\| + \nu_\oplus^{1/2}\mathrm{c}_\Omega\|\mathrm{div}v - \phi\|\Big).\end{aligned} \tag{60}$$

5.2 Estimates for stresses

Assume that $\widehat{v} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$, $\tau \in \Sigma$, and $q \in \overset{\circ}{L}{}^2(\Omega)$ approximate u, σ, and p, respectively. We have

$$\begin{aligned}&\|\tau - \sigma\| = \|\tau + \mathrm{p}\mathbb{I} - \mu\text{æ} - \nu\varepsilon(u)\| \leq \\ &\leq \|\tau + \mathrm{q}\mathbb{I} - \mu\text{æ} - \nu\varepsilon(\widehat{v})\| + \|\nu\varepsilon(\widehat{v} - u)\| + \sqrt{d}\|p-q\| \leq \\ &\leq \nu_\oplus^{1/2}\|\tau + \mathrm{q}\mathbb{I} - \mu\text{æ} - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + \nu_\oplus^{1/2}\|\varepsilon(\widehat{v} - u)\|_{(\nu)} + \sqrt{d}\|p-q\|.\end{aligned} \tag{61}$$

By (49) and (59) we obtain

$$\|\tau-\sigma\| \leq \nu_{\oplus}^{1/2}\Big(2(1+\sqrt{d}\mathrm{c}_\Omega)\|\tau+\mathrm{q}\mathbb{I}-\mu æ-\nu\varepsilon(\widehat{v})\|_{(\nu^{-1})}+ \tag{62}$$
$$+C_F(1+2\sqrt{d}\mathrm{c}_\Omega)\|\mathrm{Div}\tau+f\|+2\nu_{\oplus}^{1/2}\mathrm{c}_\Omega(1+\sqrt{d}\mathrm{c}_\Omega)\|\mathrm{div}\widehat{v}-\phi\|\Big).$$

Now it is not difficult to estimate the deviation $\tau-\sigma$ in the norm of $\mathrm{H}(\Omega,\mathrm{Div})$. However, it has a more symmetric form if the deviation is expressed in terms of the norm $\|\eta\|_{\mathrm{Div},C_F} := \|\eta\|+C_F\|\mathrm{Div}\eta\|$. In this case,

$$\|\tau-\sigma\|_{\mathrm{Div},C_F} \leq 2(1+\sqrt{d}\mathrm{c}_\Omega)\nu_{\oplus}^{1/2}\Big(\|\tau+\mathrm{q}\mathbb{I}-\mu æ-\nu\varepsilon(\widehat{v})\|_{(\nu^{-1})}+ \tag{63}$$
$$+\|\mathrm{Div}\tau+f\|+\nu_{\oplus}^{1/2}\|\mathrm{div}\widehat{v}-\phi\|\Big).$$

If τ is subject to the condition $\big[\,\mathrm{tr}\tau\,\big]_\Omega=0$ then the pressure field can be excluded from (63) and we arrive at the estimate

$$\|\tau-\sigma\|_{\mathrm{Div},C_F} \leq \tag{64}$$
$$\leq 2(1+\sqrt{d}\mathrm{c}_\Omega)\nu_{\oplus}^{1/2}\Big(\sqrt{\|\tau^D-\mu æ^D-\nu\varepsilon^D(\widehat{v})\|^2_{(\nu^{-1})}+\frac{1}{d}\|\mathrm{div}\widehat{v}-\phi\|^2_{(\nu^{-1})}}+$$
$$+\|\mathrm{Div}\tau+f\|+\nu_{\oplus}^{1/2}\|\mathrm{div}\widehat{v}-\phi\|\Big).$$

6 Mixed boundary conditions

6.1 Preliminaries

Consider the generalized Stokes equation with mixed Dirichlét–Neumann boundary conditions defined on two measurable nonintersecting parts $\partial_1\Omega$ and $\partial_2\Omega$ of $\partial\Omega$ such that $\overline{\partial\Omega}=\overline{\partial_1\Omega}\cup\overline{\partial_2\Omega}$ and $|\partial_1\Omega|>0$. We assume that

$$u=u_0 \quad \text{on } \partial_1\Omega, \qquad \sigma\, n=F \quad \text{on } \partial_2\Omega, \tag{65}$$

where $\mathrm{div}u_0=\phi$, (49) holds, and $F\in L^2(\partial_2\Omega,\mathbb{R}^d)$.

Now we define the space $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$ as follows

$$\overset{\circ}{\mathrm{V}}{}^1(\Omega):=\big\{v\in H^1(\Omega,\mathbb{R}^d)\ |\ v=0 \text{ on } \partial_1\Omega\big\}$$

and by $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$ mean the subspace of $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$ that consists of solenoidal fields.

Generalized solution of the system (8)–(10), (65) we define as $u=\bar{u}-u_\phi$, where $u_\phi=u_0$ on $\partial_1\Omega$, $\mathrm{div}u_\phi=\phi$, and $\bar{u}$ is a function in $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$ that satisfies the integral identity

$$\int\limits_\Omega(\nu\varepsilon(\bar{u}):\varepsilon(w)+\mu\,\bar{æ}:\varepsilon(w))\,dx=\ell(w)\quad\forall w\in\overset{\circ}{\mathrm{J}}{}^1(\Omega). \tag{66}$$

Here, $\ell :\overset{\circ}{\mathrm{V}}{}^1(\Omega)_0 \to \mathbb{R}$ is the linear continuous functional defined by the relation

$$\ell(w) := \int\limits_{\Omega} f \cdot w dx + \int\limits_{\partial_2\Omega} F \cdot w ds.$$

Existence and uniqueness of $\bar{u}$ is easy to prove by variational arguments if note that the problem is related to minimization of the functional

$$I(w) := \int\limits_{\Omega} \left(\frac{\nu}{2} |\varepsilon(w)|^2 + \mu\, \bar{\text{æ}} : \varepsilon(w) \right) dx - \ell(w) \tag{67}$$

over the space $\overset{\circ}{\mathrm{J}}{}^1(\Omega)$. Since $I(w)$ is strictly convex, continuous, and coercive on V_0 existence of a minimizer is proved by standard arguments.

It is easy to see that

$$|\ell(w)| \leq C_\ell \, \|\varepsilon(w)\|_{(\nu)} \qquad \forall w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega). \tag{68}$$

Note that C_ℓ depends on Ω and $\partial_2\Omega$ and $C_\ell \leq C_F \|f\| + C_T \|F\|_{\partial_2\Omega}$, where C_F and C_T comes from the Friederichs and trace inequalities for the functions vanishing at $\partial_1\Omega$:

$$\|w\| \leq C_F \|\varepsilon(w)\|_{(\nu)}, \qquad \|w\|_{\partial_2\Omega} \leq C_T \|\varepsilon(w)\|_{(\nu)} \qquad \forall\, w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega).$$

For any $\tau \in \Sigma$

$$\mathcal{L}_{\tau,\ell}\,(w) := \ell(w) - \int\limits_{\Omega} \tau : \varepsilon(w)\, dx$$

is a linear continuous functional on $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$, whose norm is

$$|\!|\!| \mathcal{L}_{\tau,\ell} |\!|\!| := \sup_{w \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)} \frac{|\mathcal{L}_{\tau,\ell}\,(w)|}{\|\varepsilon(w)\|_{(\nu)}} \leq C_\ell + \|\tau\|_{(\nu^{-1})}. \tag{69}$$

The set $\mathcal{K}_{\tau,\ell} = \mathrm{Ker}\, \mathcal{L}_{\tau,\ell}$ contains the tensor–valued functions that satisfy (in a generalized sense) the equilibrium equation

$$\mathrm{Div} \tau + f = 0 \qquad \text{in } \Omega \tag{70}$$

and the boundary condition

$$\tau\, n = F \qquad \text{on } \partial_2\Omega. \tag{71}$$

6.2 Estimates for approximations in $\overset{\circ}{\mathrm{J}}{}^1(\Omega)+u_\phi$

Theorem 7. *For any $v \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)+u_\phi$, $q \in Q$, and $\tau \in \Sigma$ the following estimate holds*

$$\|\varepsilon(u-v)\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\, æ - \nu\varepsilon(v)\|_{(\nu^{-1})} + [\![\mathcal{L}_{\tau,\ell}]\!]. \tag{72}$$

If $\tau \in \Sigma_{\mathrm{Div}} := \left\{\tau \in \Sigma, \mid\ \mathrm{Div}\tau \in L^2(\Omega,\mathbb{R}^d),\ \tau\, n \in L^2(\partial_2\Omega,\mathbb{R}^d)\right\}$ then

$$\begin{aligned}\|\varepsilon(u-v)\|_{(\nu)} \leq M_{\oplus}^{(1)}(v,\tau,q) := &\|\tau + q\mathbb{I} - \mu æ - \nu\varepsilon(v)\|_{(\nu^{-1})} + \\ &+C_F\|\mathrm{Div}\tau + f\| + C_T\|F-\tau\, n\|_{\partial_2\Omega}.\end{aligned} \tag{73}$$

Proof. From (66) we observe that

$$\int\limits_\Omega \nu\varepsilon(\bar{u}-\bar{v}) : \varepsilon(w)dx = -\int\limits_\Omega (\mu\,\bar{æ} : \varepsilon(w) + \nu\varepsilon(\bar{v}) : \varepsilon(w))dx + \ell(w).$$

Let $\tau \in \Sigma$. Then, for any $w \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$

$$\begin{aligned}&\int\limits_\Omega \nu\varepsilon(\bar{u}-\bar{v}) : \varepsilon(w)dx = \\ &= \int\limits_\Omega (\tau - \mu\,\bar{æ} - \nu\varepsilon(\bar{v})) : \varepsilon(w)dx + \ell(w) - \int\limits_\Omega \tau : \varepsilon(w)dx \leq \\ &\qquad\leq (\|\tau + q\mathbb{I} - \mu\bar{æ} - \nu\varepsilon(\bar{v})\|_{(\nu^{-1})} + [\![\mathcal{L}_{\tau,\ell}]\!])\|\varepsilon(w)\|_{(\nu)},\end{aligned}$$

where q is an arbitrary function in Q. Set $w = \bar{u}-\bar{v}$. Then we arrive at the estimate

$$\|\varepsilon(\bar{u}-\bar{v})\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\,\bar{æ} - \nu\varepsilon(\bar{v})\|_{(\nu^{-1})} + [\![\mathcal{L}_{\tau,\ell}]\!]. \tag{74}$$

Since

$$\|\varepsilon(u-v)\|_{(\nu)} = \|\varepsilon(\bar{u}-\bar{v})\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\,\bar{æ} - \nu\varepsilon(v) - \nu\varepsilon(u_\phi)\| + [\![\mathcal{L}_{\tau,\ell}]\!]$$

and $\mu\bar{æ} = \mu æ - \nu\varepsilon(u_\phi)$, we arrive at (72).

Assume that $\tau \in \Sigma_{\mathrm{Div}}$ Then,

$$\begin{aligned}\mathcal{L}_{\tau,\ell}(w) = &\int\limits_\Omega (\mathrm{Div}\tau + f)\cdot w + \int\limits_{\partial_2\Omega} (F-\tau\, n)\cdot w ds \leq \\ &\leq (C_F\|\mathrm{Div}\tau + f\| + C_T\|F-\tau\, n\|_{\partial_2\Omega})\,\|\varepsilon(w)\|_{(\nu)}\end{aligned}$$

and, therefore,

$$[\![\mathcal{L}_{\tau,\ell}(w)]\!] \leq C_F\|\mathrm{Div}\tau + f\| + C_T\|F-\tau\, n\|_{\partial_2\Omega}. \tag{75}$$

Now (73) follows from (74) and (75). $\square$

The functional $M_{\oplus}^{(1)}(v,\tau,q)$ is *directly computable* provided that the constants C_F and C_T (or their upper bounds) are known. It vanishes if and only if

$$\tau = -q\mathbb{I} + \mu\,\text{æ} + \nu\varepsilon(v)$$

and the relations $\mathrm{Div}\tau + f = 0$ in Ω and $\tau\, n = F$ on $\partial_2\Omega$ hold almost everywhere. Since v meets the Dirichlét boundary condition on $\partial_1\Omega$ and satisfies the relation $\mathrm{div}v = \phi$, we conclude that in such a case $v = u$ and τ and q coincide with the exact stress and pressure fields, respectively.

Remark 10. For the stationary Stokes problem we have the following estimate

$$\begin{aligned}\|\varepsilon(u-v)\|_{(\nu)} \leq\ & \|\tau + q\mathbb{I} - \nu\varepsilon(v)\|_{(\nu^{-1})} + C_F\|\mathrm{Div}\tau + f\| + \\ & + C_T\|F - \tau\, n\|_{\partial_2\Omega}.\end{aligned} \tag{76}$$

Remark 11. A modification of the above a posteriori estimate is obtained if set $q = -\frac{1}{d}\mathrm{tr}\tau$. Then we obtain an estimate that does not contain q:

$$\begin{aligned}\|\varepsilon(u-v)\|_{(\nu)} \leq\ & \|\tau^D - \mu\text{æ}^D - \nu\varepsilon^D(v)\| + C_F\|\mathrm{Div}\tau + f\| + \\ & + C_T\|F - \tau n\|_{\partial_2\Omega}.\end{aligned} \tag{77}$$

Lower bound of the error can be derived by the arguments similar to those used in 4.2. It has the form

$$\|\varepsilon(u-v)\|^2_{(\nu)} \ \geq 2\ell(w) - \int\limits_{\Omega} \big(|\varepsilon(w)|^2 + 2(\varepsilon(v) + \bar{\text{æ}}) : \varepsilon(w)\big)\, dx,$$

where $w \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$.

6.3 Estimates for approximations in $\overset{\circ}{\mathrm{V}}{}^1(\Omega) + u_\phi$

First, we obtain an upper bound for $\|\varepsilon(\bar{v} - \bar{u})\|_{(\nu)}$ where $\bar{v} \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ and $\mathrm{div}\bar{v}$ may be not equal to zero. The assertion below is important for the subsequent analysis.

Lemma 3. *Assume that*

$$v \in \overset{\circ\circ}{\mathrm{V}}{}^1(\Omega) := \{v \in \overset{\circ}{\mathrm{V}}{}^1(\Omega) \ \mid\ [\,\mathrm{div}v\,]_\Omega = 0\}.$$

Then, there exists $v_0 \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ *such that*

$$\|\nabla(v - v_0)\| \leq\ \mathrm{c}_\Omega\|\mathrm{div}v\|. \tag{78}$$

Proof. For any $a \in H^{1/2}(\partial\Omega, \mathbb{R}^d)$ satisfying the condition $\int\limits_{\partial\Omega} a\cdot n ds = 0$ there exists a solution w_a of the Stokes problem

$$\begin{aligned} -\Delta w_a + \nabla p = 0 \qquad & \text{in } \Omega,\\ w_a + a = 0 \qquad & \text{on } \partial\Omega,\\ \mathrm{div}w_a = 0 \qquad & \text{in } \Omega.\end{aligned}$$

Let a be the trace of $v \in \overset{\circ\circ}{\mathrm{V}}{}^1(\Omega)$ on $\partial\Omega$. Then, $w_a + v = 0$ on $\partial\Omega$ and by Lemma 1 we know that there exists $w_0 \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ such that

$$\|\nabla(w_a + v) - \nabla w_0\| \leq \mathrm{c}_\Omega \|\mathrm{div}(w_a + v)\| = \mathrm{c}_\Omega \|\mathrm{div} v\|.$$

This estimate means that

$$\|\nabla v - \nabla(w_0 - w_a)\| \leq \mathrm{c}_\Omega \|\mathrm{div} v\|,$$

where the function $v_0 = w_0 - w_a$ is solenoidal and $v_0 = 0$ on $\partial_1\Omega$. □

Theorem 8. *For any $\widehat{v} \in V$ such that*

$$\widehat{v} = u_0 \quad \text{on } \partial_1\Omega \qquad \text{and} \qquad [\,\mathrm{div}\widehat{v} - \phi\,]_\Omega = 0, \tag{79}$$

$q \in Q$, and $\tau \in \Sigma$ the following estimate holds:

$$\begin{aligned} \|\varepsilon(u - \widehat{v})\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu\,æ - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + \\ + |\!|\!| \mathcal{L}_{\tau,\ell} |\!|\!| + 2\nu_\oplus^{1/2} \mathrm{c}_\Omega \|\mathrm{div}\widehat{v} - \phi\|. \end{aligned} \tag{80}$$

If $\tau \in \Sigma_{\mathrm{Div}}$ then

$$\begin{aligned} \|\varepsilon(u - \widehat{v})\|_{(\nu)} \leq M_\oplus^{(2)}(\widehat{v}, \tau, q) := \|\tau + q\mathbb{I} - \mu æ - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + \\ + C_F\|\mathrm{Div}\tau + f\| + C_T\|\tau n - F\|_{\partial_2\Omega} + 2\nu_\oplus^{1/2} \mathrm{c}_\Omega \|\mathrm{div}\widehat{v} - \phi\|. \end{aligned} \tag{81}$$

Proof. Let $\widehat{w} := (\widehat{v} - u_\phi)$. This function belongs to $\overset{\circ}{\mathrm{V}}{}^1(\Omega)$. In view of (79)

$$[\,\mathrm{div}\widehat{w}\,]_\Omega = [\,\mathrm{div}\widehat{v} - \phi\,]_\Omega = 0,$$

so that $\widehat{w} \in \overset{\circ\circ}{\mathrm{V}}{}^1(\Omega)$. By Lemma 3 there exists a function $v_0 \in \overset{\circ}{\mathrm{J}}{}^1(\Omega)$ such that

$$\|\varepsilon(\widehat{w} - v_0)\|_{(\nu)} \leq \mathrm{c}_\Omega \nu_\oplus^{1/2} \|\mathrm{div}\widehat{v} - \phi\|. \tag{82}$$

We have

$$\begin{aligned} \|\varepsilon(u - \widehat{v})\|_{(\nu)} = \quad & \|\varepsilon(u - \widehat{w} - u_\phi)\|_{(\nu)} \leq \\ & \leq \|\varepsilon(u - v_0 - u_\phi)\|_{(\nu)} + \|\varepsilon(\widehat{w} - v_0)\|_{(\nu)}. \end{aligned} \tag{83}$$

Since $\mathrm{div}(v_0 + u_\phi) = \phi$, we estimate the first norm by (72) and find that

$$\begin{aligned} \|\varepsilon(u - \widehat{v})\|_{(\nu)} \quad & \leq \|\tau + q\mathbb{I} - \mu æ - \nu\varepsilon(v_0 + u_\phi)\|_{(\nu^{-1})} + \\ & + |\!|\!| \mathcal{L}_{\tau,\ell} |\!|\!| + \|\varepsilon(\widehat{w} - v_0)\|_{(\nu)} \leq \|\tau + q\mathbb{I} - \mu æ - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} + \\ & + |\!|\!| \mathcal{L}_{\tau,\ell} |\!|\!| + 2\|\varepsilon(\widehat{w} - v_0)\|_{(\nu)}. \end{aligned}$$

By (82) we obtain (80).

Estimate (81) follows from (75) and (80). □

Estimates (80)and (81) have the same principal structure as (48) and (49). The only difference consists in the new term $C_T\|\tau n - F\|_{\partial_2\Omega}$ that serves as a penalty for possible violation of the Neumann boundary condition.

Remark 12. By the same arguments as was used in Remark 5 the pressure can be excluded from the upper bound and we obtain

$$\begin{aligned} \|\varepsilon(u - \widehat{v})\|_{(\nu)} \leq \quad & \sqrt{\|\tau^D - \mu æ^D - \nu\varepsilon^D(\widehat{v})\|^2_{(\nu^{-1})} + \frac{1}{d}\|\mathrm{div}\widehat{v} - \phi\|^2_{(\nu^{-1})}} + \\ & + C_F\|\mathrm{Div}\tau + f\| + C_T\|\tau n - F\|_{\partial_2\Omega} + 2\nu_\oplus^{1/2} \mathrm{c}_\Omega \|\mathrm{div}\widehat{v} - \phi\|. \end{aligned} \tag{84}$$

6.4 Estimates for the pressure

Theorem 9. *Let $q \in Q$ be an approximation of the pressure field p. Then*

$$\frac{1}{2\mathrm{c}^{\dagger}_{\Omega}}\|p-q\| \leq \|\nu\varepsilon(\widehat{v})+\mu æ-\tau-q\mathbb{I}\|_{(\nu^{-1})}+C_F\|\mathrm{Div}\tau+f\|+$$
$$+C_T\|\tau n-F\|_{\partial_2\Omega}+\nu_{\oplus}^{1/2}\mathrm{c}_{\Omega}\|\mathrm{div}\widehat{v}-\phi\|, \tag{85}$$

where $\widehat{v}$ and τ are arbitrary functions in $\overset{\circ\circ}{\mathrm{V}}{}^1(\Omega)$ and Σ_{Div}, respectively.

Proof. Since $(p-q-[\,p-q\,]_\Omega) \in \overset{\circ}{L}{}^2(\Omega)$, we have a function $w_0 \in V(\Omega)$ such that $w_0 = 0$ on $\partial\Omega$,

$$\mathrm{div} w_0 = p-q-[\,p-q\,]_\Omega, \quad \text{and} \quad \|\varepsilon(w_0)\| \leq \mathrm{c}_\Omega\|p-q-[\,p-q\,]_\Omega\|.$$

Let $v^\dagger \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$ be a vector–valued function such that $\mathrm{div} v^\dagger = 1$ in Ω. We note that many functions with such a property exist. Indeed, the nonhomogeneous Stokes problem

$$\begin{aligned}
-\Delta v+\nabla\widetilde{p} &= 0 && \text{in } \Omega,\\
\mathrm{div} v &= 1 && \text{in } \Omega,\\
v &= 0 && \text{on } \partial_1\Omega,\\
v &= a && \text{on } \partial_2\Omega, \qquad \int\limits_{\partial_2\Omega} a\cdot n ds = |\Omega|
\end{aligned}$$

has a solution (see, e.g., [46]). The latter can be taken as $v^\dagger$.

It is easy to observe that $w_0^\dagger := w_0+[\,p-q\,]_\Omega v^\dagger \in \overset{\circ}{\mathrm{V}}{}^1(\Omega)$,

$$\int\limits_\Omega \mathrm{div} w_0^\dagger (p-q)dx = \|p-q\|^2,$$

and

$$\begin{aligned}
\|\varepsilon(w_0^\dagger)\|_{(\nu)} \leq\ & \|\varepsilon(w_0)\|_{(\nu)}+[\,p-q\,]_\Omega\|\varepsilon(v^\dagger)\|_{(\nu)} \leq \\
& \leq \mathrm{c}_\Omega\nu_\oplus^{1/2}\|p-q-[\,p-q\,]_\Omega\|+[\,p-q\,]_\Omega\|\varepsilon(v^\dagger)\|_{(\nu)} \leq \\
& \leq \mathrm{c}^\dagger_\Omega\|p-q\|,
\end{aligned} \tag{86}$$

where $\mathrm{c}^\dagger_\Omega = \mathrm{c}_\Omega\nu_\oplus^{1/2}+|\Omega|^{-1/2}\|\varepsilon(v^\dagger)\|_{(\nu)}$. By the integral identity

$$\int\limits_\Omega \left(\nu\varepsilon(u):\varepsilon(w_0^\dagger)+\mu\,æ:\varepsilon(w_0^\dagger)\right)dx = \ell(w_0^\dagger)+\int\limits_\Omega p\mathrm{div} w_0^\dagger dx \tag{87}$$

we find that

$$\begin{aligned}
\|p-q\|^2 &= \int\limits_\Omega \left(\nu\varepsilon(u):\varepsilon(w_0^\dagger)+\mu\,æ:\varepsilon(w_0^\dagger)-q\mathrm{div} w_0^\dagger\right)dx-\ell(w_0^\dagger) = \\
&= \int\limits_\Omega \nu\varepsilon(u-\widehat{v}):\varepsilon(w_0^\dagger)dx + \\
&+ \int\limits_\Omega \left(\nu\varepsilon(\widehat{v}):\varepsilon(w_0^\dagger)+\mu\,æ:\varepsilon(w_0^\dagger)-q\mathrm{div} w_0^\dagger\right)dx-\ell(w_0^\dagger).
\end{aligned}$$

Note that

$$\int\limits_{\Omega} \nu\varepsilon(u-\widehat{v}) : \varepsilon(w_0^\dagger)\,dx \leq \mathrm{c}_\Omega^\dagger \|\varepsilon(u-\widehat{v})\|_{(\nu)} \|p-q\|$$

and

$$\int\limits_{\Omega} \Big(\nu\varepsilon(\widehat{v}) : \varepsilon(w_0^\dagger) + \mu\,æ : \varepsilon(w_0^\dagger) - q\mathrm{div} w_0^\dagger\Big)\,dx - \ell(w_0^\dagger) =$$

$$= \int\limits_{\Omega} (\nu\varepsilon(\widehat{v}) + \mu æ - \tau - q\mathbb{I}) : \varepsilon(w_0^\dagger)dx - \int\limits_{\Omega} (\mathrm{Div}\tau + f)\cdot w_0^\dagger dx +$$

$$+ \int\limits_{\partial_2\Omega} (\tau n - F)\cdot w_0^\dagger ds \leq \Big(\|\nu\varepsilon(\widehat{v}) + \mu æ - \tau - q\mathbb{I}\|_{(\nu^{-1})} +$$

$$+ C_F\|\mathrm{Div}\tau + f\| + C_T\|F - \tau\, n\|_{\partial_2\Omega}\Big)\|\varepsilon(w_0^\dagger)\|_{(\nu)}.$$

Therefore,

$$\|p-q\| \leq \ \mathrm{c}_\Omega^\dagger\Big(\|\varepsilon(u-\widehat{v})\|_{(\nu^{-1})} + \|\nu\varepsilon(\widehat{v}) + \mu æ - \tau - q\mathbb{I}\|_{(\nu^{-1})} +$$

$$+ C_F\|\mathrm{Div}\tau + f\| + C_T\|F - \tau\, n\|_{\partial_2\Omega}\Big).$$

Now, we apply (81) and obtain the estimate

$$\|p-q\| \leq \ 2\mathrm{c}_\Omega^\dagger\Big(\|\tau + q\mathbb{I} - \mu æ - \nu\varepsilon(\widehat{v})\|_{(\nu^{-1})} +$$

$$+ C_F\|\mathrm{Div}\tau + f\| + C_T\|\tau n - F\|_{\partial_2\Omega} + \nu_\oplus^{1/2}\mathrm{c}_\Omega\|\mathrm{div}\widehat{v} - \phi\|\Big),$$

which is equivalent to (85). □

Remark 13. The constant $\mathrm{c}_\Omega^\dagger$ contains a subsidiary function $v^\dagger$ that must satisfy the condition $\mathrm{div} v^\dagger = 1$ and $v^\dagger = 0$ on $\partial_1\Omega$. Usually, such a function is not difficult to construct. For example, for polygonal domains $v^\dagger$ can be constructed with the help of Raviart–Thomas elements of the lowest order. It is desirable to have a function $v^\dagger$ such that $\|\varepsilon(v^\dagger)\|$ be as small as it is possible.

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References

[1] M. Ainsworth and J. T. Oden, *A posteriori error estimation in finite element analysis*, Wiley and Sons, New York, 2000.

[2] M. Amara, M. Ben Younes, and C. Bernardi, *Error indicators for the Navier-Stokes equations in stream function and vorticity formulation*, Numer. Math., 80(1998), pp. 181-206.

[3] R. E. Bank and B. D. Welfert, *A posteriori error estimates for the Stokes problem,* SIAM J. Numer. Anal., 28(1991), pp. 591-623.

[4] I. Babuška, *The finite element method with Lagrangian multipliers,* Numer. Math., 20, 179–192 (1973).

[5] I. Babuška and W. C. Rheinboldt, *A-posteriori error estimates for the finite element method,* Internat. J. Numer. Meth. Engrg. 12(1978), pp. 1597-1615.

[6] I. Babuška and W. C. Rheinboldt, *Error estimates for adaptive finite element computations,* SIAM J.Numer. Anal. 15(1978), pp. 736-754.

[7] I. Babuška and T. Strouboulis, *The finite element method and its reliability*, Oxford University Press, New York, 2001.

[8] W. Bangerth and R. Rannacher, *Adaptive finite element methods for differential equations,* Birkhäuser, Berlin, 2003.

[9] R. E. Bank and B. D. Welfert, *A posteriori error estimates for the Stokes problem,* SIAM J. Numer. Anal., N 3, 28, 591–623 (1991).

[10] A. Bermúdez, R. Durán and R. Rodríguez, *Finite element analysis of compressible and incompressible fluid-solid systems,* Math. Comput. 67(1998), pp. 221, 111-136.

[11] C. Bernardi and R. Verfurth, *A posteriori error analysis of the fully discretized time–dependent Stokes equations,* ECIAM:M2NL, 38(2004), 3, pp. 437–455.

[12] M. Bildhauer, M. Fuchs and S. Repin, *A posteriori error estimates for stationary slow flows of power–law fluids,* International Journal of Non-Newtonian Fluids (in press).

[13] J. Bramble, *A proof of the inf-sup condition for the Stokes equations on Lipschitz domains,* Math. Models Methods Appl. Sci,, 13 (2003), 3, pp. 361–371.

[14] J. Bonvin, M. Picasso, and R. Stenberg, *GLS and EVSS methods for a three–field Stokes problem arising from viscoelastic flows,* Comput. Methods Appl. Mech. Engrg. 190(2001), 3893–3914.

[15] F. Brezzi, *On the existence, uniqueness and approximation of saddle–point problems arising from Lagrange multipliers,* R.A.I.R.O., Annal. Numer. R2, 129–151 (1974).

[16] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer Series in Computational Mathematics, 15, New York, 1991.

[17] M. Dobrowolski, *On the LBB constant on stretched domains,* Math. Nachr., 254-255 (2003), pp. 64-67.

[18] I. Ekeland and R. Temam, *Convex analysis and variational problems*, North–Holland, New-York (1976).

[19] M. Feistauer. *Mathematical methods in fluid dynamics.* Longman, Harlow, 1993.

[20] L. P. Franca, J. Karam Filho, A. F. D. Loula, and R. Stenberg, *A convergence analysis of a stabilized method for the Stokes flow,* Mat. Apl. Comput. 10 (1991),1, pp. 19–26.

[21] M. Fuchs and S. Repin, *Estimates for the deviation from the exact solutions of variational problems modeling certain classes of generalized Newtonian fluids,* Mathematical Methods in Applied Sciences (M2AS) (in press).

[22] V. Girault and P. A. Raviart, *Finite element approximation of the Navier–Stokes equations*, Springer, Berlin (1986).

[23] R. Glowinski and O. Pironneau, *Numerical Methods for the First Biharmonic Equation and for the Two-Dimensional Stokes Problem,* SIAM Review, (21)1999, 2, pp. 167-212.

[24] E. Gorshkova, P. Neittaanmaki and S. Repin, *Comparative study of a posteriori error estimates for the Stokes problem,* Proc. of 6th European Conference on Numerical Mathematics and Advanced Applications, (Santiago de Compostela), 2005, Springer, Berlin, 2006, pp. 255-262.

[25] C. Johnson and P. Hansbo, *Adaptive finite elements in computational mechanics,* Comput. Methods Appl. Mech. Engrg. 101(1992), pp. 143-181.

[26] C. Johnson, R. Rannacher and M. Boman, *Numerics in hydrodynamic stability. Towards error control in CFD,* SIAM J. Numer. Anal., 32(1995), pp. 1058-1079.

[27] D. Kay and D. Silvester, *A-posteriori error estimation for stabilised mixed approximations of the Stokes equations,* SIAM J. Sci. Comput. 21(2000), 4, pp. 1321–1336.

[28] O. A. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, New York (1969).

[29] O. A. Ladyzenskaja and V. A. Solonnikov, *Some problems of vector analysis, and generalized formulations of boundary value problems for the Navier-Stokes equation,* Zap. Nauchn, Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), 59(1976), pp. 81–116 (in Russian).

[30] P. Neittaanmäki and S. Repin, *Reliable methods for computer simulation. Error control and a posteriori estimates*, Elsevier, New York, 2004.

[31] J. T. Oden, W. Wu, and M. Ainsworth, *An a posteriori error estimate for finite element approximations of the Navier–Stokes equations,* Comput. Methods. Appl. Mech. Engrg. 111(1994), pp. 185-202.

[32] M.A. Olshanskii and E.V. Chizhonkov, *On the best constant in the inf sup condition for prolonged rectangular domains,* Matematicheskie Zametki 67 (2000) 3, 387–396 (in Russian).

[33] C. Padra, *A posteriori error estimators for nonconforming approximation of some quasi-newtonian flows,* SIAM J. Numer. Anal., 34(1997), pp. 1600-1615.

[34] J. Pousin and J. Rappaz, *Consistance, stabilité, erreurs a priori et a posteriori pour des probl emes non linéaries,* C. R. Acad. Sci. Paris 312(1991), pp. 699-703.

[35] R. Rannacher, *Finite element methods for the incompressible Navier-Stokes equations,* Fundamental directions in mathematical fluid mechanics (G. P. Galdi ed.) Birkhauser, Basel, 2000, pp. 191-293.

[36] S. I. Repin, *A posteriori error estimation for nonlinear variational problems by duality theory,* Zapiski Nauchnych Semin. POMI, 243(1997), 201–214.

[37] S. I. Repin, *A posteriori error estimation for variational problems with uniformly convex functionals,* Math. Comput., 69(230), 2000, 481–500.

[38] S. I. Repin, *A unified approach to a posteriori error estimation based on duality error majorants,* Mathematics and Computers in Simulation 50(1999), 313–329.

[39] S. I. Repin, *Two-Sided estimates of deviation from exact solutions of uniformly elliptic equations,* Amer. Math. Soc. Transl. Series 2, 209(2003), 143–171.

[40] S. I. Repin, *Estimates of deviations from exact solutions for some boundary–value problems with incompressibility condition,* Algebra and Analiz, 16(2004), 5, 124–161.

[41] S. I. Repin, *Aposteriori estimates for the Stokes problem,* J. Math. Sci, (New York), 109 (2002), 5, 1950-1964.

[42] S. I. Repin, *Estimates of deviation from exact solutions of initial-boundary value problems for the heat equation,* Rend. Mat. Acc. Lincei, 13(2002), pp. 121-133.

[43] S. Repin, S. Sauter and A. Smolianski, *A posteriori error estimation for the Dirichlét problem with account of the error in the approximation of boundary conditions,* Computing, 70(2003), pp. 205-233.

[44] S. Repin, S. Sauter and A. Smolianski, *A posteriori error estimation for the Poisson equation with mixed Dirichlét/Neumann boundary conditions,* J. Comput. Appl. Math. 164/165 (2004), pp. 601–612.

[45] T. Strouboulis and J. T. Oden, *A posteriori error estimation of finite element approximations in fluid mechanics,* Comput. Meth. Appl. Mech. Engrg. 78(1990), pp. 201-242.

[46] R. Temam, *Navier–Stokes equations. Theory and numerical analysis,* Studies in Mathematics and its Applications, 2, North-Holland, Amsterdam, 1979.

[47] S. Turek, *Efficient solvers for incompressible flow problems: An algorythmic and computational approach*, Springer, Berlin, 1999.

[48] R. Verfürth, *A posteriori error estimators for the Stokes equations,* Numer. Math., 55(1989), pp. 309 - 326.

[49] R. Verfürth, *A review of a posteriori error estimation and adaptive mesh-refinement techniques,* Wiley and Sons, Teubner, New-York, 1996.

[50] J. Wang and X. Ye, *Superconvergence analysis for the Navier–Stokes equations,* Applied Numerical Mathematics, 41 (2002), pp. 515 - 527.

[51] K. Yosida, *Functional Analysis,* Springer, New Tork, 1996.

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