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REGULARITY AND CONVERGENCE RESULTS IN THE CALCULUS OF VARIATIONS ON METRIC SPACES

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Niko Marola: *Regularity and convergence results in the calculus of variations on metric spaces;* Helsinki University of Technology, Institute of Mathematics, Research Reports A518 (2007); Article dissertation (summary + original articles).

Abstract: This dissertation studies regularity, convergence and stability properties for minimizers of variational integrals on metric measure spaces. The treatise consists of four articles in which the Moser iteration, Harnack's inequality and Harnack's convergence principle are considered in connection with quasiminimizers of the p-Dirichlet integral. In addition, we study a nonlinear eigenvalue problem in this setting. This is done in metric spaces equipped with a doubling measure and supporting a weak $(1, p)$-Poincaré inequality.

AMS subject classifications (2000): 35J20, 35J60, 35P30, 49J27, 49J52, 49N60

Keywords: Caccioppoli inequality, doubling measure, Harnack covergence theorem, Harnack inequality, metric space, minimizer, Moser iteration, Newtonian space, nonlinear eigenvalue problem, p-Dirichlet integral, p-Laplace equation, Poincaré inequality, quasiminimizer, quasisubminimizer, quasisuperminimizer, Rayleigh quotient, Sobolev space, subminimizer, superminimizer.

Niko Marola: *Variaatiolaskennan säännöllisyys- ja suppenemistuloksia metrisessä avaruudessa;* Teknillinen korkeakoulu, Matematiikan laitos, Tutkimusraportti A518 (2007); Yhdistelmäväitöskirja.

Tiivistelmä: Väitöskirjassa tutkitaan säännöllisyys-, suppenemis- ja stabiilisuusominaisuuksia variaatio-ongelmien ratkaisuille metrisessä avaruudessa. Työ koostuu neljästä artikkelista, jotka käsittelevät muun muassa Moserin menetelmää, Harnackin epäyhtälöä ja Harnackin suppenemisperiaatetta p-Dirichlet-integraalin kvasiminimoijille. Lisäksi tarkastelemme niin sanottua epälineaarista ominaisarvo-ongelmaa. Työssä tutkitaan metristä avaruutta, jonka mitta on tuplaava ja jossa on voimassa heikko $(1, p)$-Poincarén epäyhtälö.

Asiasanat: Caccioppolin epäyhtälö, epälineaarinen ominaisarvo-ongelma, Harnackin suppenemisperiaate, Harnackin epäyhtälö, kvasiminimoija, kvasisubminimoija, kvasisuperminimoija, metrinen avaruus, minimoija, Moserin iteraatio, Newtonin avaruus, p-Dirichlet-integraali, p-Laplacen yhtälö, Poincarén epäyhtälö, Rayleigh-osamäärä, Sobolevin avaruus, subminimoija, superminimoija, tuplaava mitta.

Preface

This dissertation has been mainly carried out at the Institute of Mathematics of Helsinki University of Technology during 2004–2006. It consists of this overview, a research report, and three articles in international journals with a referee practice. The aim of the overview is to introduce readers to the framework of the dissertation, that is to metric measure spaces equipped with a doubling measure and supporting a Poincaré inequality. A short introduction to Sobolev space theories in this setting is given in each of the included articles, therefore omitted in the overview.

Next, I would like to acknowledge those who have helped me during this project:

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Espoo, November, 2006 Niko Marola

List of included articles

This dissertation consists of this overview and the following publications:

[I] Marola, N., Moser's method for minimizers on metric measure spaces, *Report A478*, Helsinki University of Technology, Institute of Mathematics, 2004.

[II] Latvala, V., Marola, N. and Pere, M., Harnack's inequality for a nonlinear eigenvalue problem on metric spaces, *Journal of Mathematical Analysis and Applications* **321** (2006), 793–810.

[III] Björn, A. and Marola, N., Moser iteration for (quasi)minimizers on metric spaces, *Manuscripta Mathematica* **121** (2006), 339–366.

[IV] Kinnunen, J., Marola, N. and Martio, O., Harnack's principle for quasiminimizers, to appear in *Ricerche di Matematica*.

Throughout the overview these articles are referred to by their Roman numerals.

Author's contribution

The work presented in this dissertation has been mainly carried out at the Institute of Mathematics of Helsinki University of Technology during 2004–2006. The writing and analysis of [III] was in part done while the author was visiting Linköping University in February 2005.

The author has had a central role in all aspects of the work reported in this dissertation. In [I] the author's independent research is reported, while, in [II]–[IV], the author is responsible for the substantial part of the writing and analysis.

Some of the results both in [II] and [III] were reported previously in [I]. In [III] the first author, Anders Björn, is mainly responsible for the work concerning analysis of noncomplete metric spaces (Sections 4 and 6).

The thread of the proof of Lemma 4.1 and Theorem 4.3 in [IV] is based on the proof of Theorem 6.1 in Kinnunen–Martio [45].

In addition, the author has presented the results of [I]–[IV] in analysis seminars and colloquiums held at universities including those of Linköping, Helsinki, Joensuu and Oulu and Helsinki University of Technology.

Regularity and convergence results in the calculus of variations on metric spaces

Niko Marola

1. Introduction

This dissertation is about the calculus of variations on metric measure spaces. More precisely, we discuss regularity, stability and convergence of minimizers of variational integrals in the metric setting; Harnack's inequality and Harnack's convergence principle for quasiminimizers of the p-Dirichlet integral are studied. This section is devoted to giving a short overview of metric spaces equipped with a doubling measure and supporting a weak Poincaré inequality which is the framework of this treatise. It has been next to impossible, as well as unnecessary, to include all the relevant material here, hence, some additional results related to the topic are only appropriately cited.

1.1. Doubling metric spaces with a Poincaré inequality

Analysis in abstract metric spaces with no a priori smooth structure has been developed in recent years. In particular, abstract Sobolev space theories have been studied extensively. A short list, far from being exhaustive, includes the papers by Cheeger [14], Heinonen–Koskela [35], Hajłasz [28], Hajłasz–Koskela [30], Franchi et al. [20], Shanmugalingam [62, 63], Semmes [61], and the books by Ambrosio–Tilli [1], Hajłasz–Koskela [31] and Heinonen [33]. More references will be given in the course of the overview.

Motivation for such an abstract formulation comes from applications to Carnot–Carathéodory spaces and analysis on fractals, to mention only a few. One of the advantages of the metric space setting is that a wide variety of cases, such as manifolds, graphs, vector fields and groups, can be dealt with using the same universal method. Moreover, methods used in this general setup seem to open a new point of view in the Euclidean case also. By brushing aside everything that is not really needed in arguments, it is easier to see the essential phenomena behind the results and also obtain new results.

Tools required in these theories are a notion of first-order Sobolev space, a doubling measure and a suitably formulated Poincaré inequality for elements of such a space. The classical theory of Sobolev spaces is based on the notion of distributional derivatives. More precisely, distributional derivatives are defined in terms of an action on smooth functions via integration by parts. Hence, in general metric spaces, an alternative way of defining Sobolev spaces is needed. In cases where this mechanism can be defined, one can reasonably consider, for example, variational problems, partial differential equations and potential theory.

There are various approaches to the development of Sobolev-type spaces on metric spaces. We apply a geometric approach via the notion of upper gradients. (Cheeger [14] and Hajłasz [28] introduce alternative definitions of Sobolev spaces on metric spaces. These definitions, however, lead to the same space, see Shanmugalingam [62]. For a good survey of Sobolev-type spaces on metric spaces, see Hajłasz [29].)

Definition 1.1. A nonnegative Borel-measurable function g is an *upper gradient* of an extended real-valued function f on X if for all rectifiable paths $\gamma : [a, b] \to X$,

$$|f(\gamma(b)) - f(\gamma(a))| \leq \int_\gamma g\, ds$$

whenever both $f(\gamma(a))$ and $f(\gamma(b))$ are finite, and $\int_\gamma g\, ds = \infty$ otherwise.

A *path* (or a curve) in X is a continuous mapping from a compact interval, moreover, a path is *rectifiable* if its length is finite. A path can thus be parameterized by arc length.

In, e.g., Heinonen–Koskela [35] and Koskela–MacManus [48] upper gradients have been studied. However, this concept has recently been collected together independently by Cheeger [14] and Shanmugalingam [62, 63]. This approach gives a first-order theory that allows for applications of variational methods in potential theory and partial differential equations, see, e.g., Kinnunen–Shanmugalingam [47]. Primarily, this method provides metric spaces analogs of the classical Sobolev spaces $W^{1,p}$ for all values of p between 1 and ∞. However, there is a class of metric spaces not covered by this theory. In fact, the types of spaces for this approach that have nontrivial content are those which support sufficiently rich families of rectifiable paths. While this class includes a number of geometrically diverse examples, it nevertheless rules out possibilities such as classical self-similar fractals, the classical von Koch snowflake, for example, or other spaces lacking in rectifiable paths (except for the constant ones).

The framework is given by a metric space $X = (X, d, \mu)$ with a metric d and a positive complete Borel regular measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$, where $B = B(z_0, r) := \{z \in X : d(z, z_0) < r\}$. The main assumptions we make on the metric space X are:

1. the measure μ is doubling;
2. the space X supports a weak Poincaré inequality.

We want to emphasize that a notion of Sobolev space on metric spaces is also reasonable enough without the requirement of a doubling condition and a weak Poincaré inequality. With these very conditions, however, a host of properties true in the Euclidean case hold true in abstract metric spaces as well.

Let us comment these assumptions in brief.

Definition 1.2. The measure μ is said to be *doubling* if there exists a constant $c_\mu \geq 1$, called the *doubling constant* of μ, such that for all balls B in X,

$$\mu(2B) \leq c_\mu \mu(B),$$

where $2B = B(z_0, 2r)$.

By the doubling property there exists a lower bound for the density of the measure. Indeed, if $B(y,R)$ is a ball in X, $z \in B(y,R)$ and $0 < r \leq R < \infty$, then

$$\frac{\mu(B(z,r))}{\mu(B(y,R))} \geq c\Big(\frac{r}{R}\Big)^s$$

for $s = \log_2 c_\mu$ and some constant c only depending on c_μ. The exponent s serves as a counterpart of the dimension related to the measure. We point out that this is not the topological dimension of X, as it can be greater, and it depends on the measure μ and the metric d. The dimensions may change if we change the metric d.

Notice that the support of a nontrivial doubling measure is all the space X.

A metric space is *doubling* if there exists a constant $c < \infty$ such that every ball $B(z,r)$ can be covered by c balls with radii $\frac{1}{2}r$. It is now easy to see that every bounded set in a doubling metric space is totally bounded. Then, the notion of doubling metric space is intrinsically finite-dimensional. Moreover, a doubling metric space is *proper* (i.e., closed and bounded subsets are compact) if and only if it is complete. Observe that a complete metric space with a doubling measure is separable. Being proper is, furthermore, a stronger condition than being locally compact, as $\mathbf{R}^n \setminus \{0\}$ is locally compact but not proper.

A metric space equipped with a doubling measure is doubling and, conversely, any complete doubling metric space can be equipped with a doubling measure. There are, however, noncomplete doubling metric spaces that do not carry doubling measures. See [33], pp. 82–83 and Chapter 13, for more on doubling metric spaces.

Let us introduce the weak Poincaré inequality.

Definition 1.3. We say that X supports a *weak* $(1,p)$*-Poincaré inequality* if there exist constants $c > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all measurable functions f on X and for all upper gradients g of f,

$$⨍_B |f - f_B|\, d\mu \leq c(\operatorname{diam} B)\Big(⨍_{\lambda B} g^p\, d\mu\Big)^{1/p},$$

where $f_B := ⨍_B f\, d\mu := \int_B f\, d\mu/\mu(B)$. If $\lambda = 1$, then X supports a $(1,p)$-*Poincaré inequality.*

By the Hölder inequality, it is easy to see that, if X supports a weak $(1,p)$-Poincaré inequality, then it supports a weak $(1,q)$-Poincaré inequality for every $q > p$. If X is complete and μ doubling then it is shown in Keith–Zhong [42] that a weak $(1,p)$-Poincaré inequality implies a weak $(1,q)$-Poincaré inequality for some $q < p$. Observe that a weak $(1,1)$-Poincaré inequality is the strongest inequality in that it implies the weak $(1,p)$-Poincaré inequality for every $p > 1$.

In Keith [41, Theorem 2], see also Heinonen–Koskela [36], it is shown that, if a weak Poincaré inequality holds for all compactly supported Lipschitz functions and their compactly supported Lipschitz upper gradients, then the complete metric space X with a doubling measure supports a weak Poincaré inequality.

Example 1.4. Let $X = A \cup B$ with $A, B \subset \mathbf{R}^n$ bounded open sets with $\operatorname{dist}(A,B) > 0$ and $\mu(A), \mu(B) > 0$, d Euclidean distance and μ the Lebesgue measure. Then $f = \chi_A$ is Lipschitz continuous on X, $|\nabla f| = 0$, but

$$0 < \fint_X |f - f_X| \, d\mu.$$

The example illustrates that a weak Poincaré inequality implies some kind of connectedness. Moreover, the Poincaré inequality implies the quasiconvexity of the complete metric space X, i.e., there exists a constant $c \geq 1$ such that every pair of points x and y in the space can be joined by a path whose length is at most $cd(x,y)$. Indeed, if the space X is doubling in measure and supports a weak Poincaré inequality, then it is *quasiconvex*. See, e.g., Keith [41]; the proof in [41] is based loosely on the argument of Semmes, see an exposition of Semmes' argument in Cheeger [14, Appendix].

To outline the geometric properties of spaces dealt with in this work, it is mentioned in passing that a quasiconvex and proper metric space can be turned into a geodesic one. That is to say, such a space is bi-Lipschitz to a geodesic metric space, see pp. 70–71 in Heinonen [33]. To digress slightly, recall that metric space X is said to be *geodesic* if every pair of points $x, y \in X$ can be joined by a path whose length is the very distance between the points.

There are a host of examples of metric spaces equipped with a doubling measure and satisfying a weak Poincaré inequality, see Ambrosio et al. [2], Björn–Björn [7], Coulhon et al. [16] and Theorem 4 in Keith [41]. We list here a few examples of such spaces.

Example 1.5.

1. Unweighted and weighted Euclidean spaces, i.e., spaces where the Lebesgue measure is replaced with a suitable absolutely continuous doubling measure, see Heinonen et al. [34].

2. This example shows that the (local) dimension of the metric space is not necessarily constant. Let $X_1 = [-1, 0]$, $X_2 = \{z \in \mathbf{C} : \ 0 \leq \operatorname{Re} z \leq 1 \text{ and } |\arg z| \leq \pi/4\}$ and $X = X_1 \cup X_2$. Further, let $\mu|_{X_1}$ be

the one-dimensional Lebesgue measure $\mathcal{L}^1$, and $d\mu|_{X_2} = |z|^{-1}\, d\mathcal{L}^2$, i.e., $d\mu|_{X_2} = dr\, d\theta$ in polar coordinates. It can be proved that μ is doubling and that X satisfies the $(1,1)$-Poincaré inequality, consult A. Björn [4].

3. Complete Riemannian manifolds with nonnegative Ricci curvature are doubling and satisfy the $(1,1)$-Poincaré inequality, see Buser [12] and Saloff-Coste [60].

4. Many graphs have the following two properties: the counting measure is doubling and a weak $(1,p)$-Poincaré inequality holds on the graph. For the potential theory on such graphs, see, e.g., Holopainen–Soardi [38], Shanmugalingam [64], Hajłasz–Koskela [31, Section 12] and the references cited therein.

5. One of the central applications of the theory of Sobolev spaces on metric spaces comes from Carnot–Carathéodory spaces and from the theory of Sobolev spaces associated with a family of vector fields. We refer the interested reader to the collection [3] of papers for a comprehensive introduction to the Carnot–Carathéodory spaces and geometry.
   In addition, Carnot groups are a special case of Carnot–Carathéodory spaces. An important example of Carnot groups is the first Heisenberg group $\mathbf{H}_1 = \mathbf{C} \times \mathbf{R}$ with the group operation
$$(z,t)\cdot(z',t') = (z+z', t+t' + 2\,\mathrm{Im}\,\bar{z}z').$$
   $\mathbf{H}_1$ is a doubling metric space and satisfies the $(1,1)$-Poincaré inequality. The proof can be found in Heinonen [33, Theorem 9.27]. For an extensive introduction to Carnot groups, see Folland–Stein [19]. See also, e.g., Garofalo–Nhieu [22], Capogna–Garofalo [13], Jerison [39], Heinonen [32], Manfredi [56] and the references therein.

6. Let us recall that a measure μ in X is called s-regular if there exist two constants $c_i > 0, i = 1,2$, such that for every ball $B(z,r) \subset X$, $c_1 r^s \le \mu(B(z,r)) \le c_2 r^s$. If μ is s-regular then X is called an *Ahlfors s-regular* space. Ahlfors s-regular spaces are particular examples of doubling metric spaces in which there is also an upper bound for the density of the measure. Moreover, there is also a control from below on the dimension of the space; hence, there is a well-defined notion of dimension that is constant on the whole space, see pp. 61–62 in Heinonen [33].
   Laakso [49] showed that, for every real number $s > 1$ there is an Ahlfors s-regular space satisfying the $(1,1)$-Poincaré inequality.

There is a myriad of literature regarding Sobolev spaces, Sobolev functions, nonlinear potential theory and calculus of variations in metric spaces equipped with a doubling measure and supporting a weak Poincaré inequality. In addition to above references, see the papers by A. Björn [5, 6], A.

Björn et al. [8, 9], J. Björn [10], Buckley [11], Franchi et al. [21], Holopainen–Shanmugalingam [37], Kallunki–Shanmugalingam [40], Kilpeläinen et al. [43], Kinnunen–Martio [44, 45, 46], MacManus–Pérez [54], Shanmugalingam [64], to name but a few, and the numerous references in these papers.

2. Calculus of variations on metric spaces

This section is an overview of papers [I]–[IV]. This dissertation deals with the calculus of variations in doubling metric measure spaces supporting a weak Poincaré inequality and its applications to nonlinear partial differential equations. We discuss regularity, stability and convergence results for minimizers of variational integrals; Harnack's inequality and Harnack's convergence principle are considered in connection with quasiminimizers of the p-Dirichlet integral. In addition, we consider a nonlinear eigenvalue problem in this setting.

Some of our results seem to be new even in the Euclidean setting, but we study the question in metric spaces equipped with a doubling measure and supporting a weak $(1,p)$-Poincaré inequality. We have chosen this more general approach to emphasize the fact that the obtained properties hold in a very general context. Indeed, our approach covers weighted Euclidean spaces, Riemannian manifolds, Carnot–Carathéodory spaces, including Carnot groups such as Heisenberg groups, and graphs, see Example 1.5.

2.1. Moser iteration for (quasi)minimizers

In [I] and [III], the Moser iteration is considered in connection with both minimizers and quasiminimizers of the p-Dirichlet integral. We have chosen this more general approach to emphasize the fact that the method itself holds in a very general context. Paper [III] is a joint work with Anders Björn from Linköping University.

Let $\Omega \subset \mathbf{R}^n$ be a bounded open set and $1 < p < \infty$. A function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a Q-*quasiminimizer*, $Q \geq 1$, of the p-Dirichlet integral in Ω if for every open set $\Omega' \Subset \Omega$ and for all $\varphi \in W^{1,p}_0(\Omega')$ we have

$$\int_{\Omega'} |\nabla u|^p \, d\mathcal{L}^n \leq Q \int_{\Omega'} |\nabla (u+\varphi)|^p \, d\mathcal{L}^n.$$

In the Euclidean case, the problem of minimizing the p-Dirichlet integral

$$\int_{\Omega} |\nabla u|^p \, d\mathcal{L}^n$$

among all functions with given boundary values is equivalent to solving the p-Laplace equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

A minimizer, or 1-quasiminimizer, is a weak solution of the p-Laplace equation. Being a weak solution is clearly a local property; however, being a quasiminimizer is not a local property, see Kinnunen–Martio [45]. The theory for quasiminimizers, therefore, usually differs from the theory for minimizers.

Quasiminimizers were extensively studied by Giaquinta–Giusti, see [24] and [25]. See also DiBenedetto–Trudinger [18], Tolksdorf [66] and Ziemer [67]. The interest of this notion is mainly its unifying feature: it includes, among other things, minimizers of variational integrals, solutions of elliptic partial differential equations and systems, and quasiregular mappings.

Quasiminimizers have been used as tools in studying the regularity of minimizers of variational integrals for quasiminimizers have a rigidity that minimizers lack: the quasiminimizing condition applies to the whole class of variational integrals at the same time. For example, if a variational kernel $F(x, \nabla u)$ satisfies the standard growth conditions

$$\alpha|h|^p \le F(x,h) \le \beta|h|^p$$

for some $0 < \alpha \le \beta < \infty$, then the minimizers of $\int F(x, \nabla u)$ are quasiminimizers of the p-Dirichlet integral. Apart from this, quasiminimizers have a fascinating theory in themselves, see, for example, Kinnunen–Martio [45].

Giaquinta–Giusti [24, 25] proved several fundamental properties for quasiminimizers, including the interior regularity result that a quasiminimizer can be modified on a set of measure zero so that it becomes Hölder continuous. Moreover, higher integrability of the gradient and boundary continuity has been studied. Some of these results have been extended to metric spaces, see A. Björn [5], Björn–Björn [7], J. Björn [10], [45], Kinnunen–Shanmugalingam [47].

In $\mathbf{R}^n$, minimizers of the p-Dirichlet integral are known to be locally Hölder continuous. This can be seen using either of the celebrated methods by De Giorgi, see [17], and Moser, see [57] and [58]. See also, for example, the books by Giaquinta [23], Giusti [27], Chen–Wu [15], Gilbarg–Trudinger [26], Heinonen et al. [34], Maly–Ziemer [55].

Moser's method gives Harnack's inequality first and then Hölder continuity follows from this in a standard way, whereas De Giorgi first proves Hölder continuity and then Harnack's inequality can be obtained as in DiBenedetto–Trudinger [18].

At first sight, it seems that Moser's technique is strongly based on the differential equation, whereas De Giorgi's method relies only on the minimization property. In Kinnunen–Shanmugalingam [47] De Giorgi's method was adapted to the metric setting. They proved that quasiminimizers are locally Hölder continuous, and satisfy the strong maximum principle and Harnack's inequality. The space was assumed to be complete, doubling in measure and to support a weak $(1, q)$-Poincaré inequality for some q with $1 < q < p$.

The purpose of the papers [I] and [III] is twofold. First, we shall adapt Moser's iteration technique to the metric setting, and, in particular, show that the differential equation is not needed in the background for the Moser iteration. On the other hand, we will study quasiminimizers and show that certain estimates, which are interesting in themselves, extend to quasiminimizers as well. We have not been able to run the Moser iteration for quasiminimizers completely, specifically because, there is one delicate step missing in the proof of Harnack's inequality using Moser's method. This is the so-called jumping over zero in the exponents related to the weak Harnack inequality. This is usually settled using the John–Nirenberg lemma for functions of bounded mean oscillation. More precisely, one has to show that a logarithm of a nonnegative quasisuperminimizer is a function of bounded mean oscillation. To prove this, the logarithmic Caccioppoli inequality, which has been obtained only for minimizers, is needed. However, for minimizers we prove Harnack's inequality using the Moser iteration.

We will impose slightly weaker requirements on the space than in Kinnunen–Shanmugalingam [47]. They assume that the space is equipped with a doubling measure and supports a weak $(1, q)$-Poincaré inequality for some $q < p$. We only assume that the space supports a weak $(1, p)$-Poincaré inequality (doubling is still assumed). It is noteworthy that according to the result of Keith and Zhong [42], a complete metric space equipped with a doubling measure that supports a weak $(1, p)$-Poincaré inequality admits a weak $(1, q)$-Poincaré inequality for some $q < p$. However, our approach is independent of the deep theorem of Keith and Zhong.

2.2. A nonlinear Rayleigh quotient on metric spaces

Article [II] is a joint work with Visa Latvala (University of Joensuu) and Mikko Pere (University of Helsinki).

We study a nonlinear eigenvalue problem, i.e., the eigenvalue problem of the p-Laplace equation on metric spaces. The problem is to find functions $u \in W_0^{1,p}(\Omega)$ that satisfy the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u, \quad 1 < p < \infty, \tag{2.1}$$

for some $\lambda \neq 0$ in a bounded domain (an open connected set) $\Omega \subset \mathbf{R}^n$. This problem was apparently first studied by Lieb in [50], see also de Thelin [65]. The *first eigenvalue* $\lambda_1 = \lambda_1(\Omega)$ is defined as the least real number λ for which the equation (2.1) has a non-trivial solution $u \in W_0^{1,p}(\Omega)$, i.e., there is $u \in W_0^{1,p}(\Omega), u \neq 0$, such that for all $\varphi \in C_0^\infty(\Omega)$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, d\mathcal{L}^n = \lambda \int_\Omega |u|^{p-2}u\varphi \, d\mathcal{L}^n.$$

The nontrivial solution u of (2.1) with $\lambda = \lambda_1$ is called the *first eigenfunction*. By approximation we may take any $\varphi \in W_0^{1,p}(\Omega)$ as an admissible test function above. In particular, the choice $\varphi = u$ implies that the first eigenvalue is obtained by minimizing the *Rayleigh quotient*

$$\lambda_1 = \inf_u \frac{\int_\Omega |\nabla u|^p \, d\mathcal{L}^n}{\int_\Omega |u|^p \, d\mathcal{L}^n} \tag{2.2}$$

with $u \in W_0^{1,p}(\Omega)$, $u \not\equiv 0$. It can be proved that the minimization problem (2.2) is equivalent to the corresponding Euler-Lagrange equation (2.1) with $\lambda = \lambda_1$.

In [II], we consider first eigenfunctions, i.e., solutions u of the eigenvalue problem (2.2), on a metric measure space X by replacing the standard Sobolev space $W_0^{1,p}(\Omega)$ with the Newtonian space $N_0^{1,p}(\Omega)$. Since differential equations are problematic in metric measure spaces, we use the variational approach. This has been previously studied in Pere [59], where it is proved that first eigenfunctions always exist in our setting and they have a locally Hölder continuous representative. The proof of the Hölder continuity in [59] is based on De Giorgi's method. We continue the study of [59] by proving that first eigenfunctions are bounded and nonnegative first eigenfunctions satisfy Harnack's inequality. The proof of the Harnack's inequality uses the Moser iteration, which was adapted to the metric setting in [I]. We also give a simple proof for the continuity of eigenfunctions by combining the weak Harnack estimates of the two different methods by De Giorgi and Moser.

The reader who wants to study this topic on bounded domains in $\mathbf{R}^n$ would do well by reading the articles by Lindqvist [51, 52, 53], for example, and checking the references cited therein.

2.3. Some convergence results for quasiminimizers

Paper [IV] is a joint work with Juha Kinnunen (University of Oulu) and Olli Martio (University of Helsinki) and is about stability properties of Q-quasiminimizers of the p-Dirichlet integral with varying Q in complete metric spaces equipped with a doubling measure and supporting a weak $(1,p)$-Poincaré inequality.

It is known that a sequence of locally bounded p-harmonic functions (continuous weak solutions of the p-Laplace equation) on a domain in $\mathbf{R}^n$ has a locally uniformly convergent subsequence that converges to a p-harmonic function on that domain, see Heinonen et al. [34]. The result has been extended to metric measure spaces by Shanmugalingam in [64].

In [IV], we prove the Harnack principle for Q-quasiminimizers with varying Q: an increasing sequence of Q_i-quasiminimizers in a domain converge locally uniformly, provided the limit function is finite at some point in that domain, to a Q-quasiminimizer with

$$Q = \liminf_{i\to\infty} Q_i.$$

Moreover, we show that a sequence (u_i) of Q_i-quasiminimizers in a domain, the sequence (u_i) is supposed to be locally uniformly bounded below, has a locally uniformly convergent subsequence that converges either to ∞ or a Q-quasiminimizer on that domain.

Let (f_i) be a uniformly bounded sequence of functions in an appropriate Newton–Sobolev space such that $f_i \to f$ as $i \to \infty$. Furthermore, we consider a sequence of Q_i-quasiminimizers in a bounded domain with boundary data

f_i and we study the stability of Q_i-quasiminimizers when Q_i tends to 1. We show that, in this case, the quasiminimizers converge locally uniformly to the unique minimizer of the p-Dirichlet integral with boundary values f. In the Euclidean case with the Lebesgue measure, we obtain convergence also in the Sobolev norm.

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