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Espoo 2007 A521

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Sergey Korotov, Aleš Kropáč, Michal Křížek: *Strong regularity of a family of face-to-face partitions generated by the longest-edge bisection algorithm*; Helsinki University of Technology, Institute of Mathematics, Research Reports A521 (2007).

Abstract: *We examine the longest-edge bisection algorithm which chooses for bisection the longest edge in a given face-to-face simplicial partition of a bounded polytopic domain in* R^d*. Dividing this edge at its midpoint we define a locally refined partition of all simplices that surround this edge. Repeating this process, we obtain a family* $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ *of nested face-to-face partitions* $\mathcal{T}_h$*. For* $d = 2$ *we prove that this family is strongly regular, i.e., there exists a constant* $C > 0$ *such that* $\operatorname{meas} T \geq Ch^2$ *for all triangles* $T \in \mathcal{T}_h$ *and all triangulations* $\mathcal{T}_h \in \mathcal{F}$*. In particular, the well-known minimum angle condition is valid.*

AMS subject classifications: 65M50, 65N30

Keywords: Zlámal's minimum angle condition, simplicial elements, conforming finite element method, nested partitions

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ISBN 978-951-22-8707-9
ISSN 0784-3143

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1 Introduction

There are many methods developed and widely used in practice for refining a given simplicial partition of a closed bounded polytopic domain $\overline{\Omega} \subset R^d$, $d \in \{2, 3, \ldots\}$. One of such methods is the so-called bisection algorithm, which is very popular for its simplicity. Originally, this method was used (and analysed) for solving systems of nonlinear equations (see e.g. [5, 19] and relevant references therein). However, it was also found to be useful for refinement and adaptation of simplicial computational meshes for the finite element method. The longest-edge bisection algorithm chooses first the midpoint of the longest edge of each simplex in a given finite element partition. Then each simplex is bisected by a hyperplane passing through this midpoint and all its vertices that do not belong to the chosen longest edge. Thus, in each refining step the number of simplices is doubled. Repeating this process, we obtain a family of nested partitions that refine the initial partition of $\overline{\Omega}$ globally.

As an analysis of the above described algorithm reduces actually to monitoring the behaviour of the longest-edge bisection algorithm within each simplex of the initial partition, we briefly recall the classical results (convergence and nondegeneracy) obtained when the longest-edge algorithm is applied to a single simplex. First, Rosenberg and Stenger [18] for $d = 2$ showed that no angle of no triangle tends to zero for infinitely many steps of the above-described variant of the bisection algorithm. A somewhat stronger result has been achieved by Stynes [20, 21] (see also Adler [1]) who showed that the repeated bisection process yields only a finite number of similarity-distinct subtriangles. This number is bounded when the discretization parameter h tends to zero. In [7], Kearfott proved for the longest-edge bisection algorithm that the largest diameter of all simplices (i.e., the discretization parameter h) tends to zero for arbitrary dimension d.

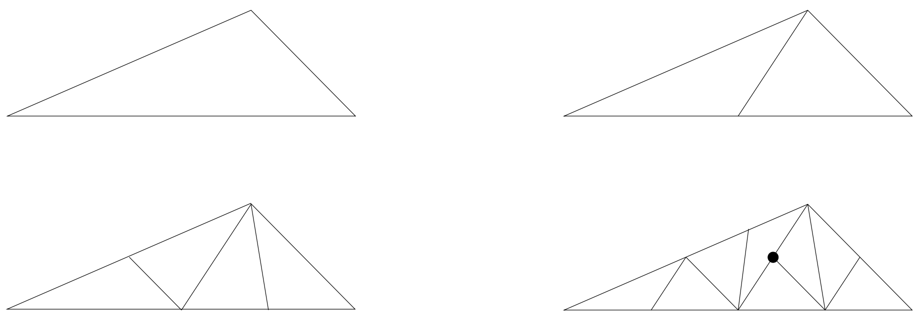

Figure 1

However, when all simplices are bisected simultaneously then so-called hanging nodes may appear (see the black dot in Figure 1), which is a certain disadvantage of this method, especially if we want to use conforming finite element discretizations over such partitions. Therefore, the above-described algorithm will be called a *nonconforming longest-edge bisection algorithm* in what follows. In order to avoid hanging nodes and at the same time to preserve the general idea and main properties of the bisection algorithm, there were developed a few techniques, see [2, 13, 14, 15, 16, 17]. For instance, Rivara in her series of papers started in 1984 has presented several global and local mesh refinement algorithms which always produce conforming triangular (and also tetrahedral – [17]) partitions. Each simplex that contains a hanging node is bisected. Numerical tests presented by Rivara show that all elements remain nondegenerating while refining partitions by any of her algorithms. However, proofs of the nondegeneracy property were only given for the global refinement algorithms and $d = 2$, i.e., for those which refine all triangles of the current partition (see [15]). Note that Horst in [6], and Liu and Joe in [11, 12] introduced and analyzed the generalized bisection algorithms which do not always halve the longest edge.

In this paper, we present a different type of the longest-edge bisection algorithm which does not produce hanging nodes. Consequently, we will call this algorithm a *conforming longest-edge bisection algorithm.* It yields face-to-face partitions and thus can be used in conforming finite element methods. The algorithm chooses the longest edge among all edges in a given simplicial partition (and not in each simplex separately). Using the midpoint of this edge, we can define a locally refined partition of all simplices that surround this edge as follows: each such simplex is bisected by a hyperplane passing through this midpoint and all its vertices that do not belong to the chosen longest edge. Repeating this process, we obtain a family $\mathcal{F}$ of nested face-to-face (i.e., conforming) partitions as indicated in Figure 2. In Figure 3 we observe subsequent partitions of a tetrahedron obtained by this algorithm.

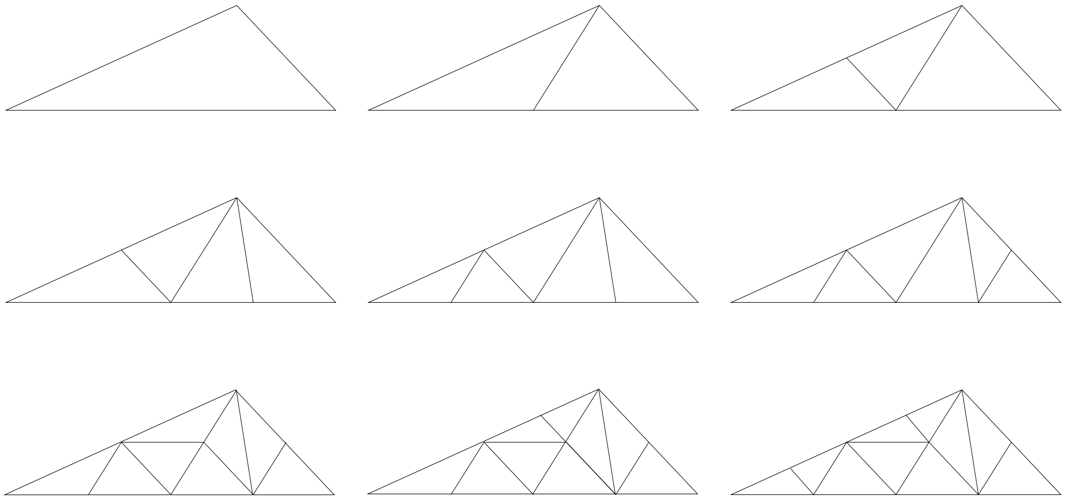

Figure 2

Clearly, the family $\mathcal{F}$ is never uniquely defined, since during the refinement process there may appear many new edges having the same length due to the bisections. For instance, the third bisection in Figure 2 is not uniquely determined (see the first illustration in the second line).

The use of the conforming algorithm produces partitions $\mathcal{T}_h$, in which all elements have approximately the same size for a sufficiently small parameter

h (cf. Figure 10). On the other hand, the size of elements when using the nonconforming-like algorithms essentially depend on the size of elements from the initial partition.

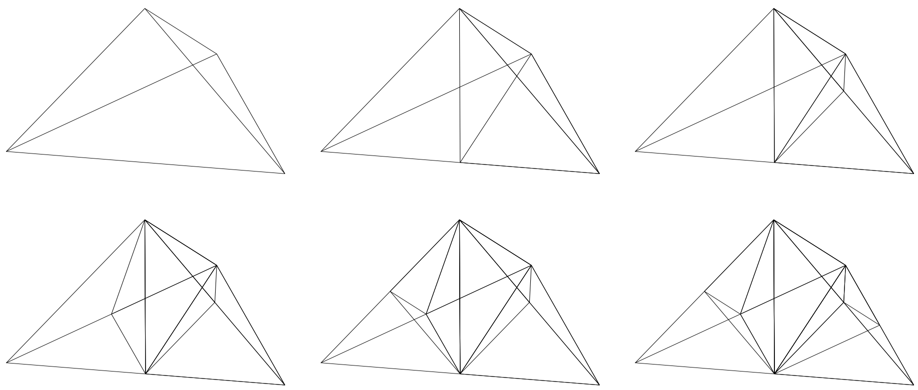

Figure 3

Note that the bisection algorithms are, in general, much simpler (especially for dimensions $d > 2$) than the popular finite element refinement of simplicial partitions that uses yellow, red, green, and red-green subdivisions (see e.g. [3, 9, 10]).

The structure of the paper is following. In Section 2, we analyze the behaviour of the proposed conforming longest-edge bisection algorithm within a single triangle from a given triangulation and present some auxiliary results. For simplicity the term "conforming" will be often omitted later on. In Section 3, we prove that the bisection algorithm produces a family $\mathcal{F}$ of nested triangulations which is strongly regular (see [4]), i.e., the famous inverse inequalities from finite element theory hold. In particular, the Zlámal minimum angle condition (see [4, 22]) is valid, i.e., all angles of all triangles from all triangulations are bounded from below by a fixed constant $\hat{\alpha} > 0$. Consequently, associated finite element functions have optimal interpolation properties in Sobolev spaces [8]. Moreover, the strong regularity of $\mathcal{F}$ enables us to construct convergent finite element approximations. Section 4 is devoted to numerical experiments.

We emphasize here that the proof of regularity of $\mathcal{F}$ while using the algorithm presented in this paper is more difficult that the regularity proofs given by Rivara in [15, Theorems 5 and 7] as it does not use the results of Rosenberg, Stenger and Stynes obtained for nonconforming algorithms.

2 Auxiliary lemmas

Throughout this paper assume that angles α, β, and γ of an arbitrary given triangle ABC are denoted so that

$$\alpha \leq \beta \leq \gamma, \tag{1}$$

and let

$$a \leq b \leq c \tag{2}$$

be lengths of the opposite sides. Now bisect the triangle by the median of length t to the longest side c. Denote the newly generated angles by $\alpha_1, \beta_1, \gamma_1$, and γ_2 as illustrated in Figure 4. If there are two or three sides having the maximum length, then the bisection is not uniquely determined. In this case, we will always bisect that side whose length is denoted by c.

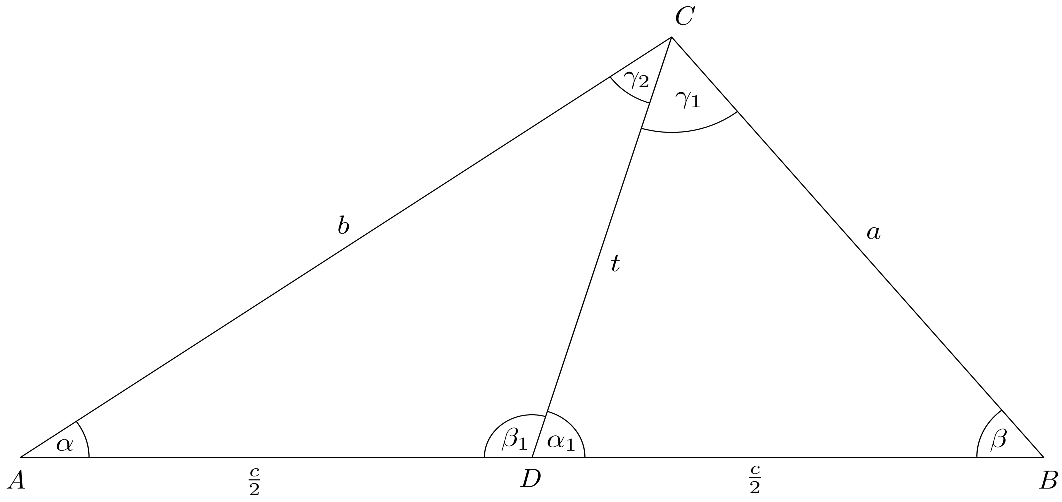

Figure 4

Lemma 1 *Under the above notation for any triangle we have*

$$\alpha \leq \frac{\pi}{3} \leq \gamma, \quad \beta < \frac{\pi}{2}, \tag{3}$$

$$\alpha_1 \leq \frac{\pi}{2} \leq \beta_1, \tag{4}$$

$$\alpha < \alpha_1, \tag{5}$$

$$\gamma_2 < \frac{\pi}{2}, \quad \gamma_2 \leq \gamma_1, \tag{6}$$

$$\frac{\pi}{6} \leq \gamma_1. \tag{7}$$

P r o o f : Absolute bounds (3) follow from (1) and the equality $\alpha+\beta+\gamma = \pi$.

By the Cosine theorem we see that

$$a^2 = t^2 + \left(\frac{c}{2}\right)^2 - tc\cos\alpha_1,$$

$$b^2 = t^2 + \left(\frac{c}{2}\right)^2 - tc\cos\beta_1.$$

From this and (2) we find that $\cos\alpha_1 \geq \cos\beta_1$. Since $\alpha_1 + \beta_1 = \pi$ and the function cosine is decreasing on the interval $[0, \pi]$, we get (4).

According to

$$\alpha < \alpha + \gamma_2 = \pi - \beta_1 = \alpha_1,$$

we get (5).

By (4), we immediately see that $\gamma_2 < \pi - \beta_1 \leq \frac{\pi}{2}$, i.e., the first inequality in (6) holds. Using (1), (3), and the Sine theorem, we find that

$$\frac{2\sin\gamma_2}{c} = \frac{\sin\alpha}{t} \leq \frac{\sin\beta}{t} = \frac{2\sin\gamma_1}{c},$$

which yields $\sin\gamma_2 \leq \sin\gamma_1$. From this and the first inequality of (6), we obviously get $\gamma_2 \leq \gamma_1$, because sinus is increasing in $[0, \frac{\pi}{2}]$.

Finally, the absolute bound in (7) follows from (3) and (6). ■

Remark 1 Denote vertices of a given triangle ABC as marked in Figure 4. Let D be the midpoint of the longest side AB and let E be such a point that D is the midpoint of the segment CE, i.e., $ACBE$ is a parallelogram. Using the triangle inequality for the triangle ACE and relation (2), we get $2t < a + b \leq 2b$, i.e.,

$$t < b. \tag{8}$$

From (2), (8), and the inequality

$$\frac{c}{2} < \frac{a+b}{2} \leq b,$$

we observe that the triangle ACD (which is never acute due to (4)) will always be bisected in the next step. Its side AC of length b will be halved.

Lemma 2 *For a nonacute triangle ABC we have*

$$\alpha \leq \gamma_2. \tag{9}$$

P r o o f : By the nonacuteness of the triangle and (1) we have $\gamma \geq \frac{\pi}{2}$, and therefore,

$$t \leq \frac{c}{2}. \tag{10}$$

Using the Sine theorem, we come to

$$\frac{\sin\alpha}{t} = \frac{2\sin\gamma_2}{c} \leq \frac{\sin\gamma_2}{t},$$

which implies (9) due to (3) and (6). ■

Corollary 1 *Let α be the smallest angle of a nonacute triangle ABC. Bisecting its longest side determines two triangles whose all angles are not less than α.*

P r o o f : The angles α_1, β, β_1, γ_1, and γ_2 (see Figure 4) can be estimated from below by α due to relations (5), (1), (4), (6), and (9). ■

Lemma 3 *For an acute triangle ABC we have*

$$\frac{\alpha}{2} \leq \gamma_2 < \alpha, \tag{11}$$

$$\frac{\pi}{4} < \beta. \tag{12}$$

P r o o f : For any triangle ABC (not necessarily acute), by (2) we have

$$t \leq \frac{\sqrt{3}}{2} c\,, \tag{13}$$

where the equality is attained for the equilateral triangle. From (13) and the Sine theorem for the triangle ACD, we get

$$\frac{2 \sin \alpha}{\sqrt{3} c} \leq \frac{\sin \alpha}{t} = \frac{2 \sin \gamma_2}{c}.$$

Therefore,

$$\sin \alpha \leq \sqrt{3} \sin \gamma_2. \tag{14}$$

Now, assume that ABC is acute. Using (6) and the fact that $\gamma < \frac{\pi}{2}$, we find that

$$\gamma_2 < \frac{\pi}{4}.$$

Consider now two cases:

1) Let $\gamma_2 \in (\frac{\pi}{6}, \frac{\pi}{4})$. Then by (3)

$$\frac{\alpha}{2} \leq \frac{\pi}{6} < \gamma_2,$$

and thus the first inequality of (11) holds.

2) Let $\gamma_2 \leq \frac{\pi}{6}$. By (14) and (6),

$$\sin \alpha \leq \sqrt{3} \sin \gamma_2 = 2 \cos \frac{\pi}{6} \sin \gamma_2 \leq 2 \cos \gamma_2 \sin \gamma_2 = \sin 2\gamma_2,$$

which implies that $\alpha \leq 2\gamma_2$, i.e., the first inequality of (11) holds again.

Further, we observe that

$$\frac{c}{2} < t, \tag{15}$$

since the triangle ABC is acute. From this and the Sine theorem for the triangle ACD we find that

$$\frac{2 \sin \gamma_2}{c} = \frac{\sin \alpha}{t} < \frac{2 \sin \alpha}{c}.$$

Hence, $\gamma_2 < \alpha$ and the second inequality of (11) holds for both cases 1) and 2).

Since $\gamma < \frac{\pi}{2}$, we observe that $\frac{\pi}{2} < \alpha + \beta \leq 2\beta$, which implies (12). ■

Corollary 2 *Let α be the smallest angle of an acute triangle ABC. Bisecting its longest side determines two triangles whose all angles are not less than $\frac{\alpha}{2}$. The lower bound $\frac{\alpha}{2}$ is attainable while bisecting the equilateral triangle.*

Before proving that the bisection algorithm guarantees the minimum angle condition, we present one more result.

Corollary 3 *Consider an acute triangle ABC such that $\alpha_1 > \beta$ after one bisection. Then the conforming longest-edge bisection algorithm yields only a finite number of similarity distinct subtriangles inside of this triangle.*

P r o o f : From (15), (4), and the Sine theorem, we see that

$$\frac{c}{2} < t < a. \tag{16}$$

Having in mind Remark 1, we find from (2) and (16) that the sides will be bisected in the following order: c, b, a, t, $\frac{c}{2}$, and $\frac{c}{2}$. After that we obtain a triangulation which consists of only two kinds of subtriangles (see Figure 5). They are similar to the two triangles produced after the first bisection of the original triangle ABC. ■

We prove one more lemma keeping the notation of Figure 4, i.e., γ_2 is the angle ACD, where D is the midpoint of the longest side AB.

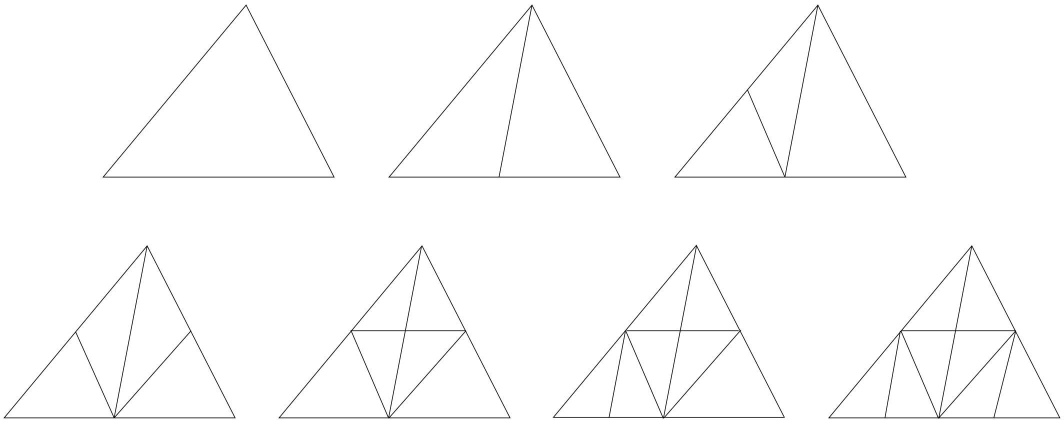

Figure 5

Lemma 4 *Let ABC be an acute triangle and let it be obtained by the longest-edge bisection of a mother triangle whose minimal angle is α'. Then*

$$\gamma_2 \geq \min(\alpha', \frac{\pi}{13}).$$

P r o o f : We can distinguish the following six cases sketched in Figures 6, 7, and 8:

1. Let ABC' be the mother triangle and $|AC'| = 2b$ (see Figure 6). We observe that the considered angle γ_2 is just equal to the angle at C' of the mother triangle ABC', i.e., $\alpha' = \gamma_2$ due to (11).

2. Let $A'BC$ be the mother triangle and $|A'C| = 2b$ (see Figure 6). Let s be the length of the altitude on the side AC from B and let

$$b_1 = \frac{s}{\tan\gamma}, \quad b_2 = \frac{s}{\tan\alpha}.$$

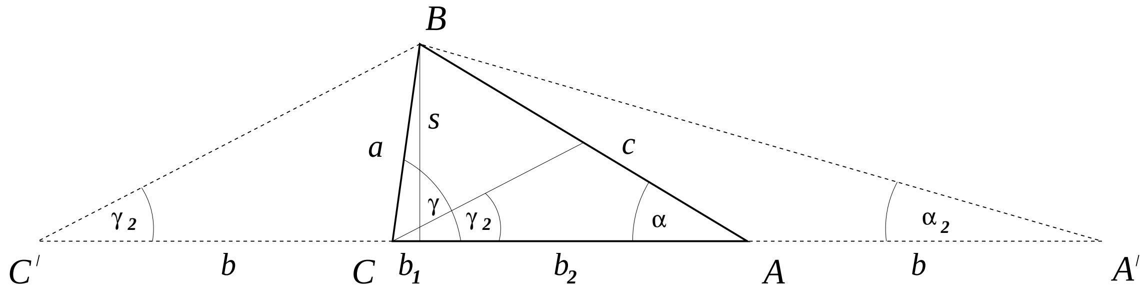

Figure 6

Then $b_1 + b_2 = b$, $b_1 \le b_2$, and

$$\tan\gamma_2 = \frac{s}{b+b_1} \ge \frac{s}{b+b_2} = \tan\alpha_2,$$

where α_2 is the angle at the vertex A'. Hence,

$$\gamma_2 \ge \alpha_2 = \alpha'. \tag{17}$$

3. Let $AB'C$ be the mother triangle and $|AB'| = 2c$ (see Figure 7). Let β_3 stand for the angle at B', which is acute. Denote by u the length of the median from B to the side AC. Then by the Cosine theorem and (2) we have

$$t^2 = \frac{1}{4}b^2 + \frac{3}{4}b^2 + \frac{1}{4}c^2 - bc\cos\alpha \le \frac{1}{4}b^2 + c^2 - bc\cos\alpha = u^2,$$

i.e.,

$$t \le u.$$

From this, the Sine theorem, and (2) we get

$$\frac{2\sin\beta_3}{b} = \frac{\sin\alpha}{u} \le \frac{\sin\alpha}{t} = \frac{2\sin\gamma_2}{c} \le \frac{2\sin\gamma_2}{b},$$

which implies that

$$\gamma_2 \ge \beta_3 \ge \alpha'.$$

(In fact, it is easy to find that $\beta_3 = \alpha'$.)

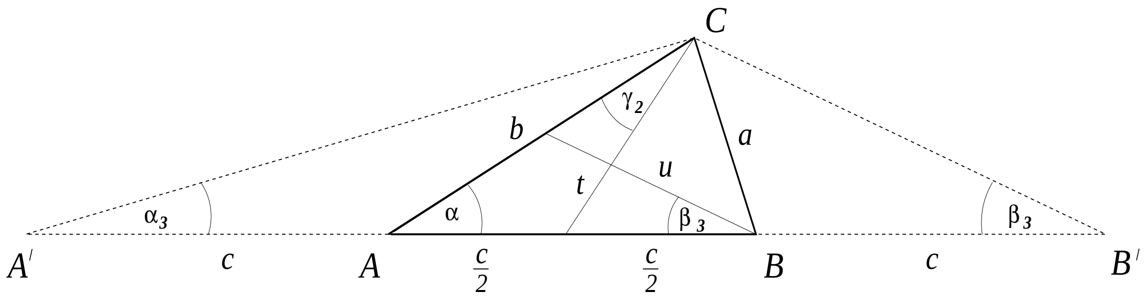

Figure 7

4. Let $A'BC$ be the mother traingle and $|A'B| = 2c$ (see Figure 7). Then similarly to (17), we find that $\beta_3 \ge \alpha_3$. Therefore,

$$\gamma_2 \ge \beta_3 \ge \alpha_3 = \alpha'.$$

5. Let $AB'C$ be the mother triangle and $|B'C| = 2a$ (see Figure 8). We observe that

$$2a \geq b, \tag{18}$$

since the longest side (which is of length $2a$) of the mother triangle is halved.

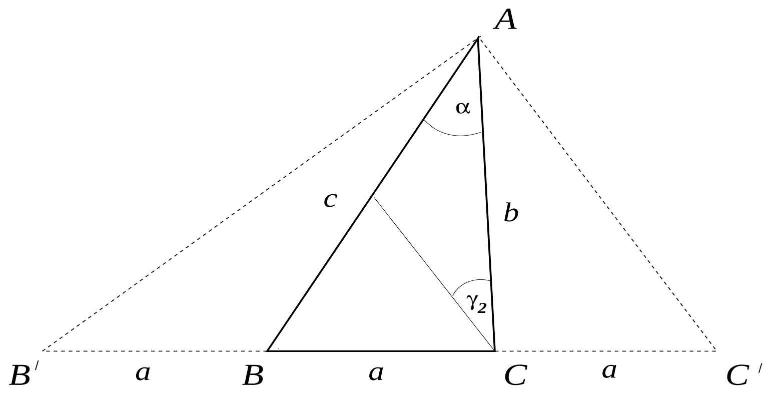

Figure 8

Without loss of generality, we may set $A = (0, 0)$ and $B = (1, 0)$. The shaded area in Figure 9 shows the admissible position of the vertex C. It is bounded by:

- the circle

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4} \tag{19}$$

with centre $M = (\frac{1}{2}, 0)$, since the triangle ABC is acute,

- the line $x = \frac{1}{2}$, since $a \leq b$,
- the circle $x^2 + y^2 = 1$, since $b \leq c$, and
- the circle

$$\left(x - \frac{4}{3}\right)^2 + y^2 = \frac{4}{9}, \tag{20}$$

since (18) holds. Its centre is $S = (\frac{4}{3}, 0)$ and any point P of this circle satisfies $|AP| = 2|BP|$.

The coordinates of the intersection Z of circles (19) and (20) are $Z = (\frac{4}{5}, \frac{2}{5})$. Therefore, the slope of the line AZ is $\overline{\alpha} = \arctan \frac{1}{2}$. According to (14), we find that

$$\sin \gamma_2 \geq \frac{\sin \alpha}{\sqrt{3}} > \frac{\sin \overline{\alpha}}{\sqrt{3}}.$$

Therefore,

$$\gamma_2 > \arcsin\left(\frac{\sin \overline{\alpha}}{\sqrt{3}}\right) > \frac{\pi}{13}, \tag{21}$$

where the absolute lower bound $\frac{\pi}{13}$ is obtained by a direct evaluation of the middle term in (21).

6. Let ABC' be the mother triangle and $|B'C| = 2a$ (see Figure 8). Since the longest side of the mother triangle is halved, we have $2a \geq c$, i.e., inequality (18) holds. Thus (21) is valid in this case, too. ■

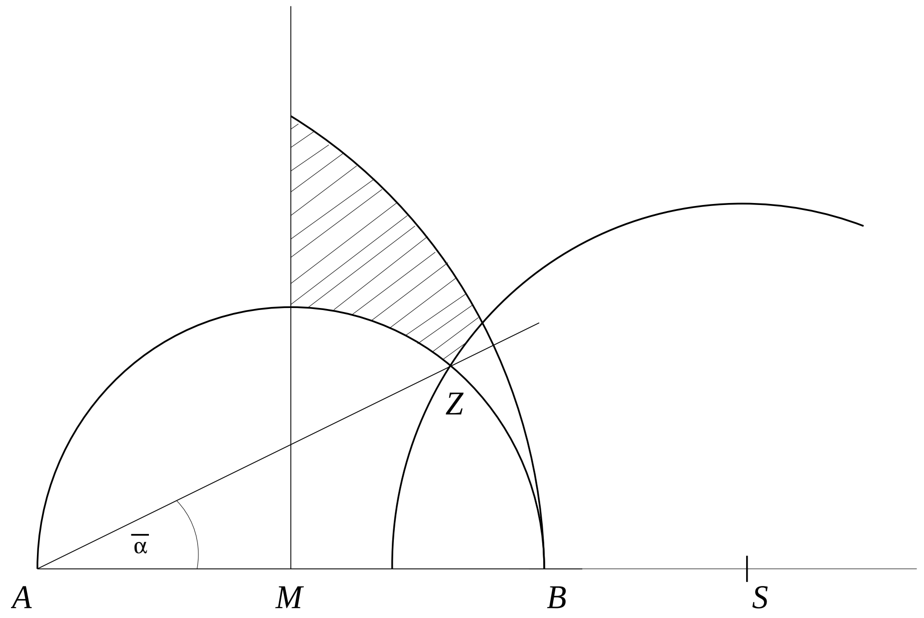

Figure 9

Remark 2 From (18) we can verify that the smallest angle α' of the triangle $AB'C$ sketched in Figure 8 is at vertex B'. We can also check that $\gamma_2 < \alpha'$. Therefore, we derived the lower bound (21).

3 Main results

Let $\overline{\Omega}$ be a given closed bounded polygonal domain. It is clear that the proposed conforming longest-edge bisection algorithm produces nested face-to-face triangulations $\mathcal{T}_h$, where h denotes the largest diameter of all triangles from $\mathcal{T}_h$. Now we prove that the discretization parameter h tends to zero as the algorithm proceeds.

Theorem 1 *The conforming longest-edge bisection algorithm yields a family of nested triangulations $\mathcal{F} = \{\mathcal{T}_h\}$, where h tends monotonically to zero.*

P r o o f : Let an initial triangulation of $\overline{\Omega}$ and an arbitrary $\varepsilon > 0$ be given. Then there exists only a finite number of sides whose lengths are greater than ε. Each of such sides of length $c > \varepsilon$ will be bisected and one or two new medians to this side will be constructed. The length of each of these new sides will be less than or equal to $\sqrt{3}c/2$, due to (13) (or (10)). Consequently, the discretization parameter h does not increase during refinements. Moreover, we observe that after a finite number of steps the lengths of all sides will be less than or equal to ε. ■

Definition 1 A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of triangulations is called *regular*, if there exists a constant $C > 0$ such that all triangulations $\mathcal{T}_h \in \mathcal{F}$ and for all triangles $T \in \mathcal{T}_h$ we have

$$\operatorname{meas} T \geq C(\operatorname{diam} T)^2.$$

It is well known (see e.g. [4]) that the regularity of $\mathcal{F}$ is equivalent to the Zlámal's minimum angle condition. Now we will provide a detailed analysis of the validity of this angle condition for $\{\mathcal{T}_h\}_{h\to 0}$.

Theorem 2 *Let α_0 be the minimum angle of all triangles from an initial triangulation. Then the conforming longest-edge bisection algorithm yields the following lower bound upon any angle φ of any triangle from any triangulation $\mathcal{T}_h \in \mathcal{F}$*

$$\varphi \geq \hat{\alpha} := \min\Big(\frac{\alpha_0}{2}, \frac{\pi}{13}\Big). \qquad (22)$$

P r o o f : Without loss of generality we may investigate each triangle from the initial triangulation $\mathcal{T}_0$ separately. Denoting α_T the minimum angle of a particular triangle T from $\mathcal{T}_0$, we have

$$\alpha_0 = \min_{T\in\mathcal{T}_0} \alpha_T.$$

So let an arbitrary triangle $T \in \mathcal{T}_0$ be given. We keep the notation of Figure 4 for T. After the first step of the longest-edge bisection algorithm the minimum angle of the nonacute subtriangle ACD will be $\alpha = \alpha_T$ or γ_2 due to (4). Hence, by Lemmas 2 and 3, all angles of ACD are not less than $\alpha_T/2 \geq \hat{\alpha}$.

For the subtriangle BCD we have by (5), (1), and (7) that

$$\alpha_1 > \alpha, \quad \beta \geq \alpha, \quad \gamma_1 \geq \frac{\pi}{6},$$

i.e., its minimum angle is $\min(\alpha, \frac{\pi}{6})$ which is not less than $\alpha_T/2$ due to (3). Thus, we observe that all angles of the both subtriangles ACD and BCD are not less than $\alpha_T/2 \ \geq\ \hat{\alpha}$.

By Remark 1, the side b will be halved in the second bisection step. According to (4), the subtriangle ACD is nonacute, and therefore, by Corollary 1 all newly generated angles are again not less than $\alpha_T/2$.

When the second subtriangle BCD is bisected, then by Corollary 1 (if it is nonacute) and Lemma 4 (if it is acute) all newly generated angles are not less than $\hat{\alpha}$. Hence, the longest-edge bisection of the both subtriangles ACD and BCD guarantees the validity of (22).

Next, we will continue by induction. Consider now an arbitrary triangle ABC from a triangulation $\mathcal{T}_h$ obtained by the longest-edge algorithm. Assume that ABC will be bisected in the next step and that it does not belong to the initial triangulation $\mathcal{T}_0$. We will again keep the notation of Figure 4. Further, assume that all angles of all triangles (including the triangle ABC) from the considered triangulation $\mathcal{T}_h$ and from all previous triangulations of T are not less than $\hat{\alpha}$, i.e., (22) is valid.

First let ABC be nonacute. Then by Corollary 1, the bisection algorithm does not change the value of the minimum angle. This implies that all angles after bisection are not less than $\hat{\alpha}$.

Second assume that ABC is acute. Then by (5), (12), and (7) we come to

$$\alpha_1 > \alpha, \quad \beta > \frac{\pi}{4}, \quad \gamma_1 \geq \frac{\pi}{6}.$$

By the induction hypothesis, $\alpha \geq \hat{\alpha}$. The lower bounds for β and γ_1 are also greater than $\hat{\alpha}$. Hence, all angles of the subtriangle BCD are greater than $\hat{\alpha}$.

For the subtriangle ACD we have by the induction hypothesis and (4) that

$$\alpha \geq \hat{\alpha}, \quad \beta_1 \geq \frac{\pi}{2}.$$

So it remains to prove that

$$\gamma_2 \geq \hat{\alpha}.$$

Since ABC is not from the initial triangulation $\mathcal{T}_0$, there exists exactly one mother triangle whose longest-edge bisection produces ABC and which belongs to some previous triangulation of T. Therefore, the induction hypothesis holds also for the mother triangle. Thus all its angles are greater than or equal to $\hat{\alpha}$. Now the use of Lemma 4 completes the proof. ∎

Further, we will prove even a stronger result which, moreover, shows that all triangles have approximately the same size for a sufficiently small value of h (cf. Figure 10).

Definition 2 A family $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of triangulations is called *strongly regular*, if there exists a constant $C > 0$ such that all triangulations $\mathcal{T}_h \in \mathcal{F}$ and for all triangles $T \in \mathcal{T}_h$ we have

$$\operatorname{meas} T \geq Ch^2.$$

Theorem 3 *The conforming longest-edge bisection algorithm yields a strongly regular family of triangulations $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$.*

P r o o f : First we show that there exists a constant $C > 0$ such that for a sufficiently small discretization parameter h the lengths of all sides are bounded from below by Ch. Assume that all sides of all triangles from $\mathcal{T}_0$ were already halved at least one time. Denote such a triangulation by $\mathcal{T}_h$, where h is the length of the longest side. Let $T \in \mathcal{T}_h$ be that triangle with the shortest side (denoted by a) in the whole triangulation $\mathcal{T}_h$. Since all sides from the initial triangulation were already halved, there exists exactly one mother triangle T' from a previous triangulation $\mathcal{T}_{h'}$ such that the bisection of T' in the next step yields T and the diameter of T' is h'. Then we obtain either $h' = 2a$, or $h' = 2b$, or $h' = 2c$ (cf. Figures 6, 7, and 8). Therefore,

$$2c \geq h' \geq h.$$

Moreover, by the Sine theorem for the triangle T and Theorem 2 we see that

$$a = c\frac{\sin\alpha}{\sin\gamma} \geq c\sin\alpha \geq c\sin\hat{\alpha},$$

and thus,

$$\frac{\sin\hat{\alpha}}{2}h \leq a \leq b \leq c \leq h.$$

Form this we get that

$$\operatorname{meas} T = \frac{1}{2}bc\sin\alpha \geq \frac{\sin^3\hat{\alpha}}{8}h^2.$$

■

Remark 3 In a similar way as in the proof of Theorem 3 we could derive for $d \geq 2$ that the family of partitions obtained by the conforming longest-edge bisection algorithm is regular if and only if it is strongly regular. Theorem 1 can also be easily generalized for the case $d \geq 2$. However, a higher-dimensional analogue of Theorem 2, which guarantees a nondegeneracy of all simplices, is still an open problem.

It is clear that for $d = 3$ all triangles on surfaces of all tetrahedra (see Figure 3) in the partition can be bisected in the exactly same way as for $d = 2$, but we do not know yet, whether all dihedral angles of all resulting tetrahedra are bounded from below by a positive constant.

4 Numerical experiments

In Figure 10, we observe the initial triangulation and the result of the proposed conforming longest-edge bisection algorithm after 10 and 1000 refining steps. The list of sides in computer memory is ordered according to their lengths. Newly generated sides are included into this list by means of the standard quick-sort algorithm, whose one step requires to perform $\mathcal{O}(\log n)$ operations, where n is the number of sides. Thus, the longest-edge bisection algorithm requires asymptotically less computer time than solving the associated FE-system.

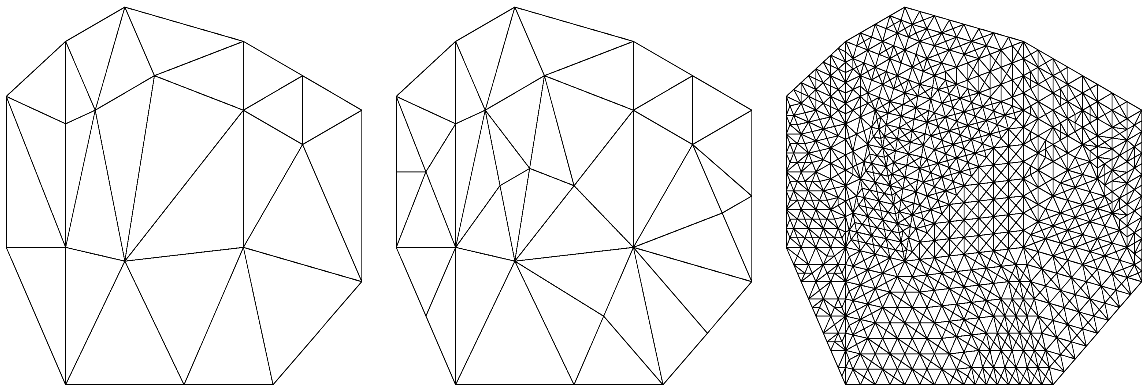

Figure 10

To illustrate the behaviour of the repeated bisection process we have chosen the obtuse triangle with vertices $A = (0,0)$, $B = (10,0)$, and $C =$

$(9,3)$ as the initial domain. In this case the assumptions of Corollary 3 are not valid. Nevertheless, numerical results in Figure 11 indicate that the number of similarity-distinct subtriangles seems to be bounded when $h \to 0$ (like for the nonconforming algorithm analysed in [21] which produces hanging nodes, in general). In this test we performed 1000 bisections. In Figure 12 we observe the behaviour of the maximal and minimal angles from the interval $(0°, 180°)$ during the 1000 bisections. The value of the minimal angle does not change. It is equal to $\gamma_2 \approx 18.5°$ obtained by the first bisection. The maximal angle does not exceed the angle $\beta_1 \approx 143°$ obtained by the first bisection.

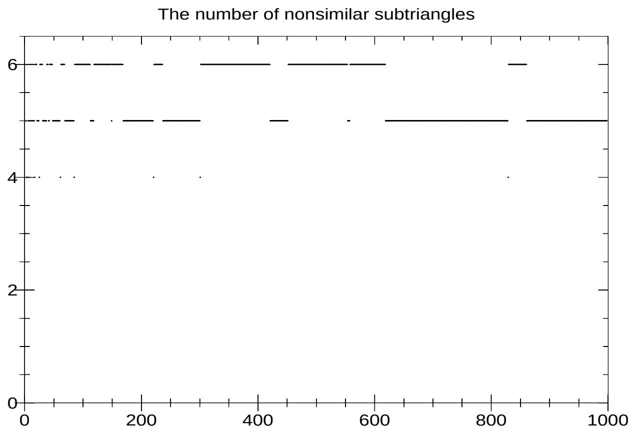

Figure 11

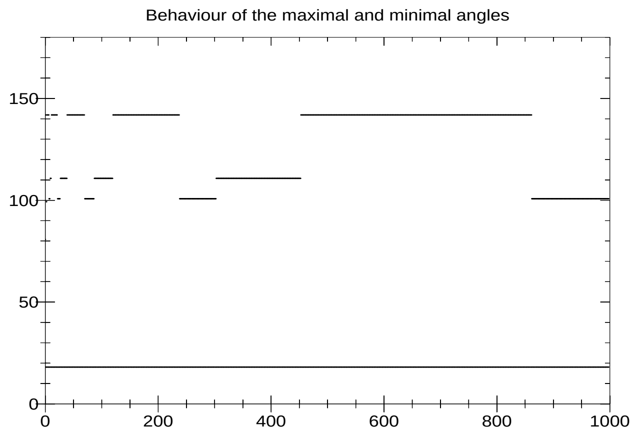

Figure 12

Acknowledgement: This paper was supported by the Academy Research Fellowship no. 208628 from the Academy of Finland, Institutional Research Plan nr. AV0Z 10190503 of the Academy of Sciences of the Czech Republic, and Grant nr. 1P05ME749 of the Ministry of Education of the Czech Republic. The authors are also indebted to Jan Brandts for valuable suggestions.

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ISBN 978-951-22-8707-9
ISSN 0784-3143