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Abstract: *This paper is devoted to the construction of a posteriori error estimators for problems in linear elasticity. The error control is performed in terms of linear (continuous) functionals, which are designed to verify the error between the exact solution and its finite element approximation in local subdomains of special interest with respect to certain quantities of interest (e.g., the J-integral in fracture mechanics). The approach employed has been analysed earlier in the author's works [12, 13] for a class of linear elliptic problems. It is based on the usage of an auxiliary (so-called adjoint) problem. In the framework of this approach, the original (primal) and adjoint problems are solved on noncoinciding meshes and averaging of gradients is used to evaluate the term in the estimator that cannot be computed directly. In the present paper, we consider a more difficult case of an elliptic system of partial differential equations arising in the theory of linear elasticity. Averaging procedures are applied to the field of strains (or stresses). Series of numerical tests show the asymptotic convergence of the proposed estimator if the number of nodes in the adjoint mesh grows and also demonstrate that, in many cases, a sufficiently accurate evaluation of the error in terms of a selected linear functional can be obtained even if the number of nodes in the adjoint mesh is essentially less than in the primal one.*

AMS subject classifications: 65N15, 65N30, 65N50

Keywords: a posteriori error estimation, local errors, superconvergence, finite element method, linear elasticity, fracture mechanics.

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1 Introduction

Linear elasticity problems, arising in various technical applications, are among the most interesting engineering problems. Numerical methods developed to find approximate solutions to these problems are well known and widely used in computational practice, see, e.g., [7, 9, 22].

However, calculations always require a reliable control of the accuracy of approximations obtained. The development of analytical and practical tools for such an error control is the main purpose of a posteriori error estimation analysis. Various approaches to derive estimates for elliptic-type boundary value problems for errors measured in the energy norm have been suggested by many authors (see, e.g., [1, 2, 3, 4, 11, 16, 20, 22] and references therein).

In recent years, a new line in a posteriori error estimation has been actively developed. It is based on the concept of error control in terms of special problem-oriented criteria (see, e.g., [1, 5, 11, 15, 19]) rather than (or in addition to) error control in the global energy norm. Error estimates of such type are strongly motivated by the needs of real-life problems, in which analysts are often interested not in the value of the overall error, but mainly in various local errors over certain "subdomains of special interest" or relative to some interesting characteristics (e.g. to the so-called J-integral in fracture mechanics, see [18] and references therein). A possible way of estimating such errors is to introduce a (linear) functional ℓ associated with the "problem-oriented criterion", also known as the quantity of interest, and to obtain an estimate for the value $\ell(\boldsymbol{u} - \bar{\boldsymbol{u}})$, where $\boldsymbol{u}$ is the exact solution and $\bar{\boldsymbol{u}}$ is its approximation. Known methods find estimates of $\ell(\boldsymbol{u} - \boldsymbol{u}^h)$ for a Galerkin approximation $\boldsymbol{u}^h$ by employing an auxiliary (adjoint) problem, whose right-hand side is formed by the functional ℓ.

For linear elasticity problems, the basic example of error control via linear functionals can be given in the most general case as follows

$$\int\limits_S \boldsymbol{\Phi}(s) \cdot (\boldsymbol{u}(s) - \bar{\boldsymbol{u}}(s))\, ds + \int\limits_S \nabla \boldsymbol{\Psi}(s) : \nabla(\boldsymbol{u}(s) - \bar{\boldsymbol{u}}(s))\, ds, \qquad (1)$$

where S is a certain subdomain in the problem domain Ω and $\bar{\boldsymbol{u}}$ is an approximation for the displacement field $\boldsymbol{u}$. Further examples of error control via functionals in linear elasticity can be found, e.g., in [18].

In the present paper, we use a new way of estimating the discretisation error via linear functionals, as proposed in our earlier work [13] (see also [12]) for linear scalar elliptic problems, and apply it to the case of elliptic systems in linear elasticity theory. The approach is essentially based on two principles: (a) original and adjoint problems are solved on non-coinciding meshes and (b) the term presenting the product of errors arising in the primal and adjoint problems is estimated by one of the gradient recovery techniques widely used in various applied problems (see [6, 10, 20, 21, 22]). This makes our approach different from others, where it is usually assumed that the Galerkin approximations of the primal and adjoint problems are computed in the same finite-dimensional subspaces. We analyse asymptotic convergence of the constructed estimator and verify its effectivity in a series of numerical tests.

2 Problem Formulation, Error Decomposition

In what follows, we denote the scalar product of vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ by the dot, $\boldsymbol{a} \cdot \boldsymbol{b} := \sum_{i=1}^{d} a_i b_i$. Similarly, the scalar product of symmetric tensors $\boldsymbol{\tau}, \boldsymbol{\varkappa} \in \boldsymbol{M}^{d\times d}_{sym}$ is denoted by $\boldsymbol{\tau} : \boldsymbol{\varkappa} := \sum_{i,j=1}^{d} \tau_{ij} \varkappa_{ij}$. The norm of a vector $\boldsymbol{a}$ is denoted by $\|\boldsymbol{a}\|$, $\|\boldsymbol{a}\| := (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$, the norm of a tensor $\boldsymbol{\tau}$ is denoted by $\|\boldsymbol{\tau}\|$ and is equal to $(\boldsymbol{\tau} : \boldsymbol{\tau})^{1/2}$.

To begin with, let us consider an elastic body occupying a bounded domain $\Omega \subset \mathbb{R}^d, d = 2, 3$, with a Lipschitz continuous boundary $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, where $\Gamma_1 \neq \emptyset$ and Γ_2 are disjoint and relatively open parts of $\partial\Omega$. The problem consists in finding the vector-valued function $\boldsymbol{u}$ (displacement) such that

$$-\mathrm{div}\, \boldsymbol{\sigma} = \boldsymbol{f} \qquad \text{in} \quad \Omega, \tag{2}$$

$$\boldsymbol{\sigma} = \boldsymbol{L}\, \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right), \tag{3}$$

$$\boldsymbol{u} = \boldsymbol{u}^0 \qquad \text{on} \quad \Gamma_1, \tag{4}$$

$$\boldsymbol{\sigma}\, \boldsymbol{n} = \boldsymbol{g} \qquad \text{on} \quad \Gamma_2, \tag{5}$$

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$, $\boldsymbol{f}$ and $\boldsymbol{g}$ denote the given volume loads and tractions, respectively, $\boldsymbol{u}^0$ is the prescribed displacement on Γ_1, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the stress and strain tensors, respectively, and $\boldsymbol{L} = \boldsymbol{L}(x) = (L_{ijkl}(x))^d_{i,j,k,l=1}$ is the fourth-order tensor of elastic moduli, which satisfies the following symmetry condition

$$L_{jikl} = L_{ijkl} = L_{klij}\ , \qquad i, j, k, l = 1, \ldots, d\ , \tag{6}$$

and the condition that there exists a positive constant C_1 such that

$$\boldsymbol{L}(x)\, \boldsymbol{\tau} : \boldsymbol{\tau} \geq C_1 \|\tau\|^2 \quad \forall \tau \in \boldsymbol{M}^{d\times d}_{sym} \tag{7}$$

holds almost everywhere in Ω. We also assume that

$$L_{ijkl} \in L^\infty(\Omega),\ \boldsymbol{f} \in (L^2(\Omega))^d,\ \boldsymbol{g} \in (L^2(\Gamma_2))^d,\ \boldsymbol{u}^0 \in (H^1(\Omega))^d. \tag{8}$$

By definition, we have

$$\boldsymbol{L}\, \boldsymbol{\tau} : \boldsymbol{\varkappa} := \sum_{i,j,k,l=1}^{d} L_{ijkl} \tau_{ij} \varkappa_{kl}. \tag{9}$$

The weak formulation of problem (2)–(5) is given below. It is called the *primal problem* and denoted as $\mathcal{PP}$ in what follows.

Primal Problem $(\mathcal{PP})$: Find $\boldsymbol{u} \in \boldsymbol{V}^0 + \boldsymbol{u}^0$ such that

$$a(\boldsymbol{u},\boldsymbol{w}) = F(\boldsymbol{w}) \qquad \forall \boldsymbol{w} \in \boldsymbol{V}^0, \tag{10}$$

where

$$\boldsymbol{V}^0 = \{\boldsymbol{v} \in (H^1(\Omega))^d \mid \boldsymbol{v} = 0 \quad \text{on} \quad \Gamma_1\}\ , \tag{11}$$

$$a(\boldsymbol{u},\boldsymbol{w}) := \int\limits_\Omega \boldsymbol{L}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{w})\,dx, \quad \boldsymbol{u},\boldsymbol{w} \in \boldsymbol{V}^0, \tag{12}$$

and

$$F(\boldsymbol{w}) := \int\limits_\Omega \boldsymbol{f}\cdot\boldsymbol{w}\,dx + \int\limits_{\Gamma_2} \boldsymbol{g}\cdot\boldsymbol{w}\,ds, \quad \boldsymbol{w} \in \boldsymbol{V}^0. \tag{13}$$

Since the bilinear form a defined above is continuous and coercive (due to Korn's inequality, see, e.g., [14]) and since the linear functional F is continuous, the primal problem (10)–(13) is uniquely solvable due to the Lax-Milgram theorem.

Let $\bar{\boldsymbol{u}} \in \boldsymbol{V}^0 + \boldsymbol{u}^0$ be an approximation of $\boldsymbol{u}$ (e.g., obtained by some numerical technique). Our main task is to derive an estimate for the following quantity of interest

$$\ell(\boldsymbol{u} - \bar{\boldsymbol{u}}), \tag{14}$$

where $\ell : \boldsymbol{V}^0 \to \boldsymbol{R}$ is a linear continuous functional selected to control the error. A common way to obtain an estimate for $\ell(\boldsymbol{u} - \bar{\boldsymbol{u}})$ is to introduce an auxiliary problem (often called the *adjoint problem* $\mathcal{AP}$ [5, 4, 15]) in the following way:

Adjoint Problem $(\mathcal{AP})$: Find $\boldsymbol{v} \in \boldsymbol{V}^0$ such that

$$a(\boldsymbol{v},\boldsymbol{w}) = \ell(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \boldsymbol{V}^0. \tag{15}$$

Under the above assumptions (6)–(8), the adjoint problem is uniquely solvable. However, in most cases, the exact (weak) solution of $(\mathcal{AP})$ is not known in analytical form and only its approximation $\bar{\boldsymbol{v}} \in \boldsymbol{V}^0$ is available.

Lemma 1. *The following error decomposition holds*

$$\ell(\boldsymbol{u} - \bar{\boldsymbol{u}}) = E_0(\bar{\boldsymbol{u}},\bar{\boldsymbol{v}}) + E_1(\bar{\boldsymbol{u}},\bar{\boldsymbol{v}}), \tag{16}$$

where

$$E_0(\bar{\boldsymbol{u}},\bar{\boldsymbol{v}}) = F(\bar{\boldsymbol{v}}) - \int\limits_\Omega \boldsymbol{L}\,\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\bar{\boldsymbol{v}})\,dx, \tag{17}$$

and

$$E_1(\bar{\boldsymbol{u}},\bar{\boldsymbol{v}}) = \int\limits_\Omega \boldsymbol{L}\,\boldsymbol{\varepsilon}(\boldsymbol{u} - \bar{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\boldsymbol{v} - \bar{\boldsymbol{v}})\,dx. \tag{18}$$

P r o o f : The decomposition (16) immediately follows from the following obvious integral identities

$$\ell(\boldsymbol{u}-\bar{\boldsymbol{u}}) = \int_\Omega \boldsymbol{L}\,\boldsymbol{\varepsilon}(\boldsymbol{v}) : \boldsymbol{\varepsilon}(\boldsymbol{u}-\bar{\boldsymbol{u}})\,dx =$$

$$= \int_\Omega \boldsymbol{L}\,\boldsymbol{\varepsilon}(\boldsymbol{v}-\bar{\boldsymbol{v}}) : \boldsymbol{\varepsilon}(\boldsymbol{u}-\bar{\boldsymbol{u}})\,dx + \int_\Omega \boldsymbol{L}\,\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}) : \boldsymbol{\varepsilon}(\boldsymbol{u}-\bar{\boldsymbol{u}})\,dx,$$

and

$$\int_\Omega \boldsymbol{L}\,\boldsymbol{\varepsilon}(\bar{\boldsymbol{v}}) : \boldsymbol{\varepsilon}(\boldsymbol{u}-\bar{\boldsymbol{u}})\,dx = \int_\Omega \boldsymbol{L}\,\boldsymbol{\varepsilon}(\boldsymbol{u}-\bar{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\bar{\boldsymbol{v}})\,dx =$$

$$= F(\bar{\boldsymbol{v}}) - \int_\Omega \boldsymbol{L}\,\boldsymbol{\varepsilon}(\bar{\boldsymbol{u}}) : \boldsymbol{\varepsilon}(\bar{\boldsymbol{v}})\,dx. \qquad \square$$

Remark 1. *We note that the term E_0 is directly computable once $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{v}}$ are known, whereas the term E_1 should be estimated. In Section 3, we suggest an idea that can be used to evaluate the term E_1 effectively in practical computations.*

Remark 2. *The term $E_1(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}})$ can also be represented in terms of stresses as follows:*

$$E_1(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}) = \int_\Omega \boldsymbol{L}^{-1}\,\boldsymbol{\sigma}(\boldsymbol{u}-\bar{\boldsymbol{u}}) : \boldsymbol{\sigma}(\boldsymbol{v}-\bar{\boldsymbol{v}})\,dx, \tag{19}$$

where $\boldsymbol{L}^{-1}$ is the tensor of elastic compliances.

3 Construction of the Error Estimator

Let $\boldsymbol{V}^h$ and $\boldsymbol{V}^\tau$ be two finite-dimensional subspaces of $\boldsymbol{V}^0$, not necessarily coinciding, that are based on the finite element meshes $\mathcal{T}_h$ and $\mathcal{T}_\tau$, respectively. We shall use $\boldsymbol{V}^h$ and $\boldsymbol{V}^\tau$ for the construction of finite element approximations of the problems $(\mathcal{PP})$ and $(\mathcal{AP})$, respectively, i.e., we pose the following two problems:

Problem $(\mathcal{PP}^h)$: Find $\boldsymbol{u}^h \in \boldsymbol{V}^h + \boldsymbol{u}^0$ such that

$$a(\boldsymbol{u}^h, \boldsymbol{w}^h) = F(\boldsymbol{w}^h) \qquad \forall \boldsymbol{w}^h \in \boldsymbol{V}^h, \tag{20}$$

Problem $(\mathcal{AP}^\tau)$: Find $\boldsymbol{v}^\tau \in \boldsymbol{V}^\tau$ such that

$$a(\boldsymbol{v}^\tau, \boldsymbol{w}^\tau) = \ell(\boldsymbol{w}^\tau) \quad \forall \boldsymbol{w}^\tau \in \boldsymbol{V}^\tau. \tag{21}$$

Both above problems are, obviously, uniquely solvable due to the Lax-Milgram theorem.

Furthermore, upon setting $\bar{\boldsymbol{u}} = \boldsymbol{u}^h$ and $\bar{\boldsymbol{v}} = \boldsymbol{v}^\tau$ in the error decomposition (16)–(18), we obtain

$$\ell(\boldsymbol{u} - \boldsymbol{u}^h) = E_0(\boldsymbol{u}^h, \boldsymbol{v}^\tau) + E_1(\boldsymbol{u}^h, \boldsymbol{v}^\tau). \tag{22}$$

It is well known that the finite element solution of the linear elasticity problems possesses certain superconvergence properties (see, e.g., [10]), which makes it possible to prove that its averaged gradient often presents a good image of the true one. This fact made a posteriori error indicators based on gradient averaging techniques very popular. It is natural to exploit this property in the estimation of the term E_1, which contains unknown gradients of the solutions of the primal and adjoint problems.

Thus, let

$$\boldsymbol{G}^h, \boldsymbol{G}^\tau : (L_\infty(\Omega))^d \to (H^1(\Omega))^d$$

be some gradient averaging operators related to the meshes $\mathcal{T}_h$ and $\mathcal{T}_\tau$, respectively.

In the simplest case of linear finite elements, both gradient averaging operators can be defined as a mapping of a piecewise constant gradient ($\nabla \boldsymbol{u}^h$ or $\nabla \boldsymbol{v}^\tau$) into a tensor-valued piecewise linear function by setting each of its nodal values as the mean (or weighted mean) value of the gradient values on all elements than contain the corresponding nodal point (cf. [10]).

By means of $\boldsymbol{G}^h$ we may define an averaging operator for the strains (still denoted by the symbol $\boldsymbol{G}^h$) as follows

$$\boldsymbol{G}^h(\boldsymbol{\varepsilon}(\boldsymbol{u}^h)) := \frac{1}{2}\left(\boldsymbol{G}^h(\nabla \boldsymbol{u}^h) + (\boldsymbol{G}^h(\nabla \boldsymbol{u}^h))^T\right), \tag{23}$$

and, similarly,

$$\boldsymbol{G}^\tau(\boldsymbol{\varepsilon}(\boldsymbol{v}^\tau)) := \frac{1}{2}\left(\boldsymbol{G}^\tau(\nabla \boldsymbol{v}^\tau) + (\boldsymbol{G}^\tau(\nabla \boldsymbol{v}^\tau))^T\right). \tag{24}$$

Then, the term $E_1(\boldsymbol{u}^h, \boldsymbol{v}^\tau)$ can be replaced by the explicitly computable term

$$\widetilde{E}_1(\boldsymbol{u}^h, \boldsymbol{v}^\tau) = \int_\Omega \boldsymbol{L}\left(\boldsymbol{G}^h(\boldsymbol{\varepsilon}(\boldsymbol{u}^h)) - \boldsymbol{\varepsilon}(\boldsymbol{u}^h)\right) : \left(\boldsymbol{G}^\tau(\boldsymbol{\varepsilon}(\boldsymbol{v}^\tau)) - \boldsymbol{\varepsilon}(\boldsymbol{v}^\tau)\right) dx\,, \tag{25}$$

that leads to the error estimator in the form

$$\widetilde{E}(\boldsymbol{u}^h, \boldsymbol{v}^\tau) := E_0(\boldsymbol{u}^h, \boldsymbol{v}^\tau) + \widetilde{E}_1(\boldsymbol{u}^h, \boldsymbol{v}^\tau), \tag{26}$$

which is directly computable once the approximations $\boldsymbol{u}^h$ and $\boldsymbol{v}^\tau$ are defined.

Remark 3: We note that $E_0(\boldsymbol{u}^h, \boldsymbol{v}^\tau)$ asymptotically contains the major part of $\ell(\boldsymbol{u} - \boldsymbol{u}^h)$ as $\boldsymbol{v}^\tau \to \boldsymbol{v}$. For the special case $\boldsymbol{V}^h \equiv \boldsymbol{V}^\tau$, the term E_0 obviously vanishes.

The above construction suggests the following numerical strategy, including suggestions for the relevant mesh adaptive procedure:

(a) Define $\boldsymbol{V}^\tau$ taking into account the nature of the functional ℓ (e.g., by putting extra trial functions in a subdomain associated with it), and calculate $\boldsymbol{v}^\tau$,

(b) Define $\boldsymbol{V}^h$ and calculate $\boldsymbol{u}^h$,

(c) Calculate $E_0(\boldsymbol{u}^h, \boldsymbol{v}^\tau)$ directly and use post-processed values of $\nabla \boldsymbol{u}^h$ and $\nabla \boldsymbol{v}^\tau$ to estimate $E_1(\boldsymbol{u}^h, \boldsymbol{v}^\tau)$, i.e. replace the unknown strains (stresses) by easily computable averaged strains (stresses),

(d) The estimator $\tilde{E}$ is, in fact, an integral over Ω, i.e.,

$$\tilde{E}(\boldsymbol{u}^h, \boldsymbol{v}^\tau) := \sum_{T \in \mathcal{T}_h^{(i)}} I_T, \tag{27}$$

where each contribution I_T is a value of the integral taken over a particular element T of the current mesh $\mathcal{T}_h^{(i)}$. To construct the next primal mesh $\mathcal{T}_h^{(i+1)}$ in order to decrease the error, we propose the following adaptive procedure. First, we find the maximum among all moduli $|I_T|$ and, secondly, mark up those elements T which have their contributions larger than the "user-given threshold" $\theta \in [0, 1]$ times that maximum value. Refining the marked elements (and making the mesh conforming), we obtain the next mesh, $\mathcal{T}_h^{(i+1)}$.

Note that computations made in the item (a) can be further used for the estimation of discretisation errors for approximations of the primal problem obtained on other meshes and with different $\boldsymbol{f}$, $\boldsymbol{u}^0$, $\boldsymbol{g}$.

4 Numerical Examples

In all numerical examples considered in this paper, we shall deal with plane stress problems without volume forces, i.e., $\boldsymbol{f} = \boldsymbol{0}$. The material parameters are chosen to correspond to glass, i.e. Young's modulus $E = 64000$ N/mm^2 and Poisson's ratio $\nu = 0.2$.

In the following, we present two numerical examples, in which we study how the technology presented above works for quantities of interest of the type (1). In the first test, the quantity of interest is chosen as the first part of (1), whereas in the second test, it is chosen as the second part of (1). Our aim here is to show how the error estimate works in these two very different situations.

To keep the focus on the estimator's behaviour for different quantities of interest, we have chosen the same domain and loading $F = 0.5$ N/mm^2 for both tests. The domain is presented in Figure 1 and it corresponds to a three point bending problem in fracture mechanics. Note that only half of the system needs to be modelled due to symmetry conditions.

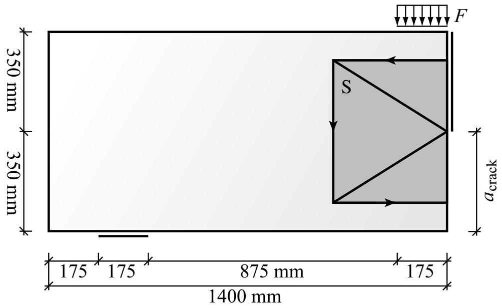

Figure 1: System and loading.

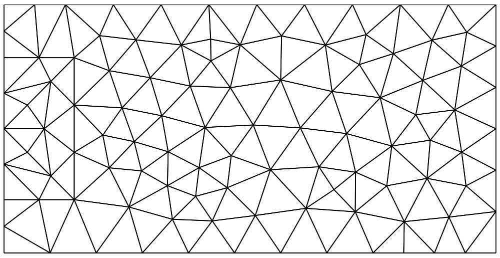

Figure 2: Initial primal mesh with 106 nodes and 173 elements.

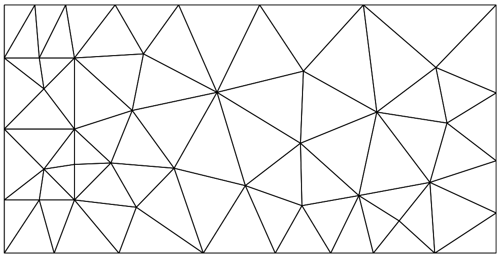

Figure 3: Adjoint mesh for run 1 with 47 nodes and 69 elements.

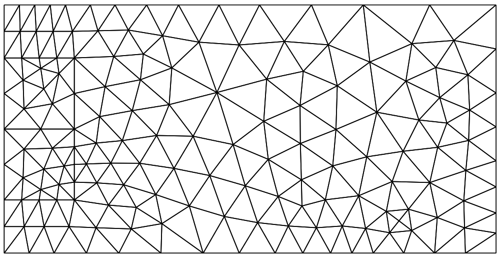

Figure 4: Adjoint mesh for run 2 with 162 nodes and 276 elements.

All calculations are preformed using Matlab and the PDE Toolbox package. For details on the practical implementation of the estimator in the case of non-hierarchical primal and adjoint meshes we refer the reader to the recent paper [18].

4.1 Test 1

In the first numerical example, we define the quantity of interest as the first term in (1) with $\mathbf{\Phi} = \mathbf{1}$ and S as indicated in Figure 1. This type of functional has previously been applied to local error control in, e.g., [12, 13].

Our first aim is to study how the choice of adjoint meshes affects the performance of the estimator presented. This study is conducted by comparing the effectivity index I_{eff}, i.e., the ratio of the estimated error to the true error, when the adjoint problem is solved on a series of different adjoint meshes. These meshes are selected to contain no hierarchical structure. For comparison, we also include an adjoint mesh identical with the primal mesh.

The effectivity index plotted in Figure 6 clearly tends to one, when the adjoint mesh is made more dense. This behaviour can be explained by the development of the error components E_0 and $\tilde{E}_1$ as visualised in Figure 5.

Upon comparing the error components E_0 and $\tilde{E}_1$, we observe that the E_0 term dominates over $\tilde{E}_1$ for sufficiently dense adjoint meshes. This property is well known and has a positive impact on the effectivity of the estimator. Even if the component $\tilde{E}_1$ is estimated only roughly, it has a decreasing influence on the whole estimate as the ratio $E_0/\tilde{E}_1$ grows. It is important to note that

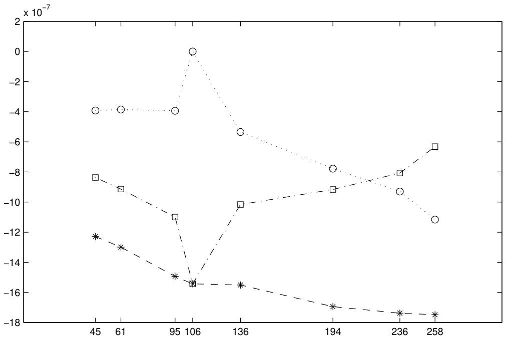

Figure 5: Terms E_0, $\tilde{E}_1$ and $E_0 + \tilde{E}_1$ for different adjoint meshes. Term E_0 is denoted by circles, $\tilde{E}_1$ by squares and $E_0 + \tilde{E}_1$ by stars.

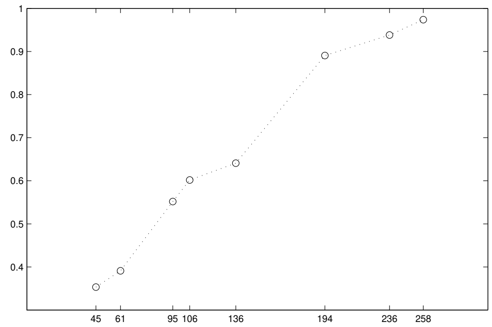

Figure 6: Effectivity index I_{eff} for error indicator computed using series of different adjoint meshes.

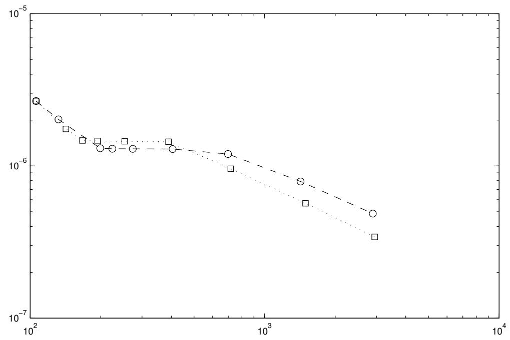

Figure 7: Error behaviour against number of nodes in Test 1, run 1 is denoted by circles and run 2 with squares.

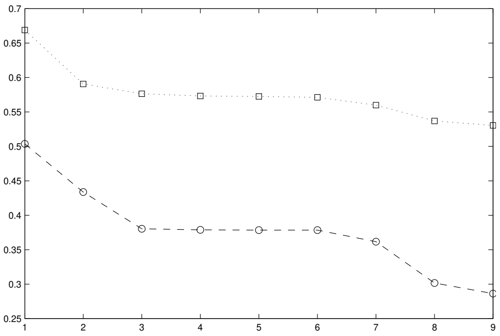

Figure 8: Effectivity index I_{eff} for each adaptive step in Test 1, run 1 is denoted by circles and run 2 with squares.

for hierarchical primal and adjoint meshes, the Galerkin orthogonality causes E_0 to vanish and we lose this desirable property.

Our second aim is to study how the adaptive strategy presented performs in practice. Similar studies for quantities of interest can be found, e.g., in [12]. To take the influence into account, which the adjoint mesh has on the estimator, the adaptive procedure is repeated for two different adjoint meshes (Figures 3 and 4). In both runs, these adjoint meshes are kept unchanged during the refinement process.

The reference error behaviour in adaptive mesh refinements for two different adjoint meshes is plotted in Figure 7 and the effectivity index for corresponding adaptive meshes in Figure 8. From these figures, we notice two properties: both error developments are almost the same and both contain a plateau, where the convergence slows down.

The existence of the plateau can be traced down to the boundary term in the estimate. In the refinement, the boundary integral $\int_{\Gamma_N}$ is taken into account by transferring its value to the element T_i by adding the term $\int_{\Gamma_N \cap T_i}$ to the element error contribution. The boundary term does not converge to zero and elements near the Neumann boundary always have error contributions in the refinement process. However, after the plateau, elementwise contributions are sufficiently small and the convergence is restored to optimal speed.

4.2 Test 2

In this test, we choose our example functional from the field of linear elastic fracture mechanics. In his field, an engineer is interested in whether or not a pre-existing crack begins to propagate. The crack propagation can be deduced from the value of the J-integral as derived by Cherepanov [8] and Rice [17]. The J-integral is nonlinear, but it can be linearised, so that the estimator presented can be applied. The linearised J-integral can be described by the second of our terms in (1), where $\nabla\boldsymbol{\Psi}$ is defined as $\boldsymbol{L}_\Sigma \nabla(q\boldsymbol{e}_{||})$ with the tensor of elastic tangent moduli associated with the so-called Newton-Eshelby stress tensor $\boldsymbol{L}_\Sigma$, the direction of crack propagation $\boldsymbol{e}_{||}$ and a piecewise linear function q with support S and values 1 at the crack tip as well as 0 on the boundary of S. For further details on the derivation of the linearisation of the J-integral we refer the reader to [18].

For this quantity of interest, the behaviour of the estimator for different adjoint meshes is difficult to analyse due to the linearisation that depends on the (approximate) solution of the primal problem. Therefore, we only present results for adaptive refinement processes.

The reference error development in the adaptive refinement is presented in Figure 9 and the associated effectivity index in Figure 10. We observe similar phenomena as in Test 1. The plateau is again present and the same reasons are behind its existence. The effectivity index of the estimated error again depends on the adjoint meshes employed in the calculations.

Finally, in Figures 11 and 12 a comparison of the adaptively refined

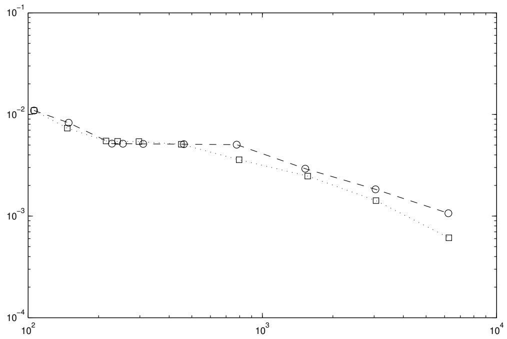

Figure 9: Error behaviour against the number of nodes in Test 2, run 1 is denoted by circles and run 2 with squares.

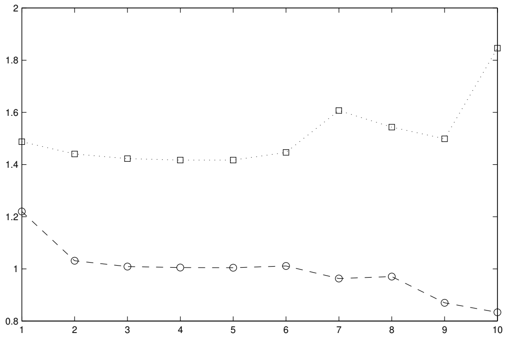

Figure 10: Effectivity index I_{eff} for each adaptive step in Test 2, run 1 is denoted by circles and run 2 with squares.

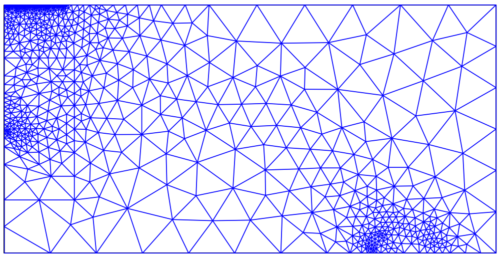

Figure 11: Primal mesh after 9 refinement steps from run 2 for Test 1.

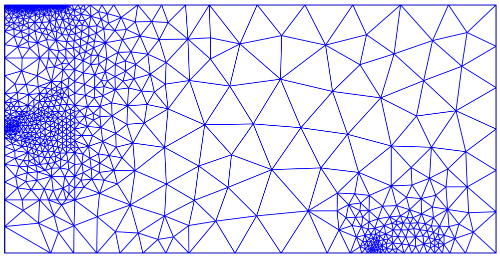

Figure 12: Primal mesh after 9 refinement steps from run 2 for Test 2.

meshes as obtained by Tests 1 and 2, respectively, is visualized for approximately the same number of nodes (2947 nodes for Test 1 and 3055 nodes for Test 2). As can be observed, the meshes look quite similar although the quantities of interest are different.

5 Conclusions

In this paper, we presented averaging-type goal-oriented a posteriori error estimators for the error of given goal quantities of interest within the framework of linear elasticity. Goal-oriented error estimators are based on the solution of an auxiliary adjoint problem which was, in this paper, solved on a different mesh than the primal problem. Furthermore, for the error estimator we obtain an additional term which, in turn, is always exactly computable. The strategy proposed in this paper has the obvious advantage that the adjoint solution does not have to be computed in every adaptive mesh refinement step and thus reduces the computational costs considerably compared to the common strategy to solve both the primal and the adjoint problem in each adaptive step on the same mesh. Moreover, in the numerical examples we obtained good numerical evidence for the effectivity of the error estimator presented in this paper.

References

[1] M. Ainsworth and J. T. Oden, *A Posteriori Error Estimation in Finite Element Analysis.* John Wiley & Sons, Inc., 2000.

[2] I. Babuška and W. C. Rheinbold, Error estimates for adaptive finite element computations. *SIAM J. Numer. Anal.* (1978) **15**, 736–754.

[3] I. Babuška and T. Strouboulis, *The Finite Element Method and its Reliability.* Oxford University Press Inc., New York, 2001.

[4] W. Bangerth and R. Rannacher, *Adaptive Finite Element Methods for Differential Equations.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2003.

[5] R. Becker and R. Rannacher, A feed-back approach to error control in finite element methods: Basic approach and examples. *East-West J. Numer. Math.* (1996) **4**, 237–264.

[6] J. Brandts and M. Křížek, Gradient superconvergence on uniform simplicial partitions of polytopes. *IMA J. Numer. Anal.* (2003) **23**, 489–505.

[7] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods. Second edition. Texts in Applied Mathematics, 15.* Springer-Verlag, New York, 2002.

[8] G. P. Cherepanov, Crack propagation in continuous media. *J. Appl. Math. Mech.* (1967) **31**, 503–512.

[9] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems.* North-Holland, Amsterdam, 1978.

[10] I. Hlaváček and M. Křížek, On a superconvergent finite element scheme for elliptic systems. Parts I, II, III. *Apl. Mat.* (1987) **32**, 131–154, 200–213, 276–289.

[11] C. Johnson and P. Hansbo, Adaptive finite elements in computational mechanics. *Comput. Methods Appl. Mech. Engrg.* (1992) **101**, 143–181.

[12] S. Korotov, A posteriori error estimation of goal-oriented quantities for elliptic type BVPs. *J. Comput. Appl. Math.* (2006) **191**, 216–227.

[13] S. Korotov, P. Neittaanmäki, and S. Repin, A posteriori error estimation of goal-oriented quantities by the superconvergence patch recovery. *J. Numer. Math.* (2003) **11**, 33–59.

[14] J. Nečas and I. Hlaváček, *Mathematical Theory of Elastic and Elasto-Plastic Bodies: An Introduction.* Elsevier Scientific Publishing Co., Amsterdam-New York, 1980.

[15] J. T. Oden and S. Prudhomme, Goal-oriented error estimation and adaptivity for the finite element method. *Comput. Math. Appl.* (2001) **41**, 735–756.

[16] S. Repin, A posteriori error estimation for variational problems with uniformly convex functionals. *Math. Comp.* (2000) **69**, 481–500.

[17] J. R. Rice, A path independent integral and the approximate analysis of strain concentration by notches and cracks. *J. Appl. Mech.* (1968) **35**, 379–386.

[18] M. Rüter, S. Korotov and C. Steenbock, Goal-oriented error estimates based on different FE-solution spaces for the primal and the dual problem with applications to linear elastic fracture mechanics. *Comput. Mech.* (2007) **39**, 787–797.

[19] M. Rüter and E. Stein, Goal-oriented a posteriori error estimates in linear elastic fracture mechanics. *Comput. Methods Appl. Mech. Engrg.* (2006) **195**, 251–278.

[20] R. Verfürth, *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques.* Wiley-Teubner, 1996.

[21] L. B. Wahlbin, *Superconvergence in Galerkin Finite Element Methods.* Vol. 1605 of Lecture Notes in Mathematics, Springer-Verlag, 1995.

[22] O. C. Zienkiewicz and J. Z. Zhu, A simple error estimator and adaptive procedure for practical engineering analysis. *Internat. J. Numer. Methods Engrg.* (1987) **24**, 337–357.

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