Helsinki University of Technology, Institute of Mathematics, Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2008 A523

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Olavi Nevanlinna: *Circlets and diagonalization of real linear operators*; Helsinki University of Technology, Institute of Mathematics, Research Reports A523 (2008).

Abstract: *Real linear operators in* $\mathbb{C}^n$ *are considered as matrices of circlets* $\gamma \in \mathbb{S}$ *and* $\mathbb{C}^n$ *is considered as a subset of* $\mathbb{S}^n$. *Here "circlets" are scalar real linear operators. It is shown that considering* $\mathbb{S}^n$ *as a right* $\mathbb{S}$*-module yields a consistent structure where "ordinary" language from complex linear algebra can be used in a way which leaves most of the common facts true. However, since* $\mathbb{S}$ *is noncommutative and contains nontrivial nilpotents, some minor modifications are needed in standard algorithms. In this paper special attention is paid on linear independency, orthogonalization and diagonalization.*

AMS subject classifications: 15A33, 47S10

Keywords: real linear operators, circlets, noncommutative scalars

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ISBN 978-951-22-8761-1
ISSN 0784-3143

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Contents

1 Introduction

1.1 Motivation

Consider solving a system of equations of the form

$$Ax + B\bar{x} = f \tag{1}$$

where $x \in \mathbb{C}^n$ denotes the unknown complex vector and $\bar{x}$ its complex conjugate. In (1) the *real linear* mapping

$$x \mapsto Ax + B\bar{x} \tag{2}$$

has two parts, the matrix A representing the linear and B the conjugate linear part. Every real linear operator in $\mathbb{C}^n$ is characterized by such a pair (A, B). For situations where modelling in this form comes up naturally, see e.g. [13] and references in [2].

The customary approach is to take the real and imaginary parts of the equation and write in $\mathbb{R}^{2n}$

$$My = b \tag{3}$$

with $y = (\operatorname{Re} x, \operatorname{Im}\ x)^t$, $b = (\operatorname{Re} f, \operatorname{Im}\ f)^t$ and

$$M = \begin{pmatrix} \operatorname{Re} A + \operatorname{Re} B & -\operatorname{Im} A + \operatorname{Im} B \\ \operatorname{Im} A + \operatorname{Im} B & \operatorname{Re} A - \operatorname{Re} B \end{pmatrix}. \tag{4}$$

If one then uses standard linear algebra software to treat (3), much of the special structure of the problem gets lost. In [2] standard numerical algorithms were formulated directly for the real linear formulation (1).

If one, instead of solving (1), is interested in knowing for what complex λ the problem

$$\lambda x - Ax - B\bar{x} = f \tag{5}$$

is not uniquely solvable, then, as in the usual $\mathbb{C}-$linear case, these values are *eigenvalues*, as for every such λ there exists a nontrivial vector $a \in \mathbb{C}^n$ such that

$$Aa + B\bar{a} = \lambda a. \tag{6}$$

However, the set of eigenvalues is an algebraic curve on the plane of degree at most $2n$ which can even be empty, [2]. To take one step further would be to ask for what complex λ and μ the equation

$$\lambda x + \mu\bar{x} - Ax - B\bar{x} = f \tag{7}$$

becomes singular. In $\mathbb{R}^{2n}$ this corresponds to asking in place of $(\lambda, \mu) \in \mathbb{C}^2$ for a set of real $2 \times 2-$matrices. As we are accustomed to think in terms of complex eigenvalues for $\mathbb{C}-$linear problems, it is desirable to have a formalism in which the $\mathbb{R}-$linear situation would "bifurcate" naturally from the $\mathbb{C}-$linear one, rather than moving into a rather different set up. Another natural condition to be put for such a formalism is that most if not all of the usual facts in linear algebra could be stated in the usual language and would remain true. For example, in this paper we shall ask what is the real linear analogue of the fact that complex matrices can be diagonalized if the eigenvalues are all distinct.

In [6] we initiated a new notational approach. We introduced a special letter to denote the complex conjugation and then treated "scalar" operators

$$x \mapsto \alpha x + \beta \bar{x} \tag{8}$$

as scalars forming a real algebra in which the complex numbers is a subalgebra. This greatly helps in dealing with long algebraic expressions. We called this algebra "circlets" and denoted it by $\mathbb{S}$.

One can then go on in several directions. In [5] $\mathbb{C}^n$ is considered as a left module over $\mathbb{S}$. In this paper we shall not only deal $\mathbb{C}$ as a subalgebra of $\mathbb{S}$ but also consider $\mathbb{C}^n$ as a subset of $\mathbb{S}^n$. This becomes useful because we consider $\mathbb{S}^n$ as a *right* module over $\mathbb{S}$.

So, we shall consider real linear operators as matrices with circlet elements. Matrices with quaternion elements has been studied quite a lot, see e.g. [14]. The situation is only partly parallel. First, all nontrivial quaternions are invertible and secondly, since we treat circlets as operators, the norm structure is very different.

1.2 Definitions and notation

In order to be able to treat conveniently long products of real linear operators we denote with τ the *complex conjugation* operator

$$\tau: \ \mathbb{C} \to \mathbb{C}, \ \zeta \mapsto \bar{\zeta}. \tag{9}$$

Then clearly $\tau^2 = 1$ and $\tau^{-1} = \tau$. Now the real linear operator in (8) can be written as

$$\alpha + \beta\tau: \ x \mapsto \alpha x + \beta \bar{x}, \tag{10}$$

and that in (2) as

$$\mathcal{A} = A + B\tau, \tag{11}$$

where $A = \mathrm{Lin}(\mathcal{A})$ is the $\mathbb{C}$-*linear* part and $B = \mathrm{Con}(\mathcal{A})$ the *conjugate-linear* part of $\mathcal{A}$. Assuming the dimension of the matrices match, the sum and product of two operators are

$$\mathcal{A} + \mathcal{K} = (A + B\tau) + (K + L\tau) = A + K + (B + L)\tau$$

and

$$\mathcal{A}\mathcal{K} = AK + B\bar{L} + (AL + B\bar{K})\tau. \tag{12}$$

Convention 1. We shall write long products by "moving τ as far to the right as possible".

Definition 2. Real linear operators $\mathbb{C}^n \to \mathbb{C}^p$, represented in the form (11), are denoted by $\mathcal{M}_{p,n}$. When $n = p$ we write $\mathcal{M}_n$. Operators of the form $\gamma = \alpha + \beta\tau$ where $\alpha, \beta \in \mathbb{C}$ are called *circlets* and they can be considered as operators in $\mathbb{C}^n \to \mathbb{C}^n$ for any n. Circlets are denoted by $\mathbb{S}$. Real linear operators $\mathbb{C} \to \mathbb{C}^n$ are called *vectorands* and denoted by $\mathbb{S}^n$:

$$c = a + b\tau : \mathbb{C} \to \mathbb{C}^n, \quad \zeta \mapsto \zeta a + \bar{\zeta} b.$$

We shall discuss vectorands in Section 3 in more detail, but point out here already that we treat $\mathbb{S}^n$ as a *right* module over $\mathbb{S}$. This simply means that linear combinations of vectorands c, d are written as

$$c\gamma + d\delta$$

where $\gamma, \delta \in \mathbb{S}$. If we now denote a general real linear operator by $\mathcal{A}$ then this is not only $\mathbb{R}$-linear but even $\mathbb{S}$-linear since the mapping with $\mathcal{A}$ is realized as multiplication from *left*:

Proposition 1. *If $\mathcal{A} \in \mathcal{M}_{p,n}$, $c, d \in \mathbb{S}^n$ and $\gamma, \delta \in \mathbb{S}$ then*

$$\mathcal{A}(c\gamma + d\delta) = (\mathcal{A}c)\gamma + (\mathcal{A}d)\delta.$$

Remark 1. Observe, however, that in $\mathbb{S}^n$ we can give $\mathbb{R}-$linear operators which are not $\mathbb{S}-$linear but they do not arise from a pair (A, B). In fact, the complex conjugation

$$c \mapsto \tau c \tau = \bar{c}$$

is only $\mathbb{R}-$linear although multiplication by τ is $\mathbb{S}-$linear.

We identify complex numbers as a subset of circlets, $\mathbb{C} \subset \mathbb{S}$, vectors as a subset of vectorands, $\mathbb{C}^n \subset \mathbb{S}^n$, and $\mathbb{C}-$linear operators as a subset of real linear operators, $M_{p,n}(\mathbb{C}) \subset \mathcal{M}_{p,n} = M_{p,n}(\mathbb{S})$ in a natural way. But then we need to know what e.g. an expression like $c\zeta$ stands for. Following Convention 1 we obtain

$$c\zeta = \zeta a + \bar{\zeta} b\tau \in \mathbb{S}^n.$$

Convention 3. When we want to specify that an expression is a complex number, vector, or matrix we use the following notation:

$$[\mathcal{A}] = [A + B\tau] := A + B.$$

So, for example the mapping $x \mapsto Ax + B\bar{x}$ in (2) now reads

$$x \mapsto [\mathcal{A}x] = Ax + B\bar{x}$$

as

$$[\mathcal{A}x] = [(A + B\tau)x] = [Ax + B\bar{x}\tau] = Ax + B\bar{x}.$$

Lemma 2.

$$[\mathcal{A}[\mathcal{K}]] = [\mathcal{A}\mathcal{K}].$$

Proof. This follows immediately from (12). □

So, these brackets can be used "recursively from right". However, we can have $[[\mathcal{A}]\mathcal{K}] \neq [\mathcal{A}\mathcal{K}]$. In fact, $0 = [[1 - \tau]i] \neq [(1 - \tau)i] = 2i$.

1.3 Overview

We start by considering circlets in more details. As an associative real algebra $\mathbb{S}$ is isomorphic with the real Clifford algebra $\mathcal{C}l_{1,1}$ but as we view them as operators, their natural metric properties are different. In section 3 we discuss vectorands, linear independency and orthogonalization. For example, it turns out that a natural circlet-valued "inner" product is intimately linked with the norm of vectorands. Much of the usual properties of inner products hold, except the Pythagorean theorem. In Section 4 we discuss eigencirclets, eigenvectorands and their linear independency. The existence of linearly independent eigenvectorands depends on the solvability of a Sylvester's equation for the related eigencirclets and this is discussed in Section 5. In Section 6 we discuss different norms for real linear operators. In particular we show that $\mathcal{M}_n$ is a real C^*-algebra and this in turn implies that several seemingly different norms are identical. Finally, in Section 7 we discuss positivity, square roots of operators and polar decomposition.

The discussion is carried out on a rather detailed level in order to make the computations as explicit as possible. Notice, that sometimes a shorter proof could easily be available by treating the equivalent matrix problem obtained by representing $\mathbb{S}$ as $M_2(\mathbb{R})$, and using known properties of real (block) matrices.

There are a number of topics not discussed here at all as there are rather detailed discussions already available. For example, basic linear algebra decompositions can be found in [6], analytic properties of the cosolvent operators and the related functional calculus in [7], and the spectrum, in the sense of (6), as well as its computation in [9]. Finally, on low rank approximation of real linear operators see [8].

2 Circlets

2.1 Involutions, norms and orthogonality

In the following α, β, ζ are usually complex numbers and circlets are then written as follows:

$$\mathbb{S} = \{\gamma \mid \gamma = \alpha + \beta\tau, \text{ where } \alpha, \beta \in \mathbb{C}\}.$$

Further, ξ, η are usually real numbers and we sometimes represent a circlet also in the forms

$$\gamma = \xi_0 + \xi_1 i + \xi_2 \tau + \xi_3 i\tau = \xi_0 + \gamma_0,$$

with $\gamma_0 = \xi_1 i + \xi_2 \tau + \xi_3 i\tau$.

Proposition 3. *Circlets $\mathbb{S}$ is a real associative noncommutative algebra, with basis*

$$\{1, i, \tau, i\tau\},$$

which is isomorphic with the real Clifford algebra $\mathcal{C}l_{1,1}$.

Proof. Let e_1, e_2 be unit basis vectors in the *Minkowski plane* $\mathbb{R}^{1,1}$ such that $e_1^2 = 1$, $e_2^2 = -1$. Then a general element $x \in \mathcal{C}l_{1,1}$ can be represented as

$$x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_1 e_2$$

with $e_1 e_2 = -e_2 e_1$ a "bivector", [12]. If we identify e_1 with τ and e_2 with i we have

$$\tau^2 = 1,\ i^2 = -1, \quad \text{and } \tau i = -i\tau.$$

and the circlet corresponding to x is $x_0 + x_2 i + x_1 \tau - x_3 i\tau$. □

The norm of a circlet $\gamma = \alpha + \beta\tau$ is obtained by considering it as a linear operator $\mathbb{C} \to \mathbb{C}$.

Definition 4. Given $\mathcal{A} \in \mathcal{M}_{p,n}$ we define

$$\|\mathcal{A}\| = \sup_{\|x\|=1} \|[\mathcal{A}x]\|.$$

where $x \in \mathbb{C}^n$.

So, in particular,

$$\|\gamma\| = \sup_{\phi} |e^{i\phi}\alpha + e^{-i\phi}\beta| = |\alpha| + |\beta|. \tag{13}$$

Next we consider involutions. The common involutions in matrix algebra are complex conjugation $A \mapsto \bar{A}$, transpose $A \mapsto A^t$ and (hermitian) adjoint $A \mapsto A^*$. We shall choose and name the involutions for real linear operators and circlets in a consistent way. In Clifford algebra literature the language varies.

Definition 5. For $\gamma = \alpha + \beta\tau \in \mathbb{S}$ and more generally for $\mathcal{A} = A + B\tau \in \mathcal{M}_{p,n}$ we define

$$\begin{array}{rlll}
\text{(complex conjugate)} & \bar{\gamma} = \bar{\alpha} + \bar{\beta}\tau & \text{and} & \bar{\mathcal{A}} = \bar{A} + \bar{B}\tau \\
\text{(adjoint)} & \gamma^* = \bar{\alpha} + \beta\tau & \text{and} & \mathcal{A}^* = A^* + B^t\tau \\
\text{(circlet conjugate)} & \gamma^\sim = \bar{\alpha} - \beta\tau & \text{and} & \mathcal{A}^\sim = A^* - B^t\tau \\
\text{(transpose)} & \gamma^t = \alpha + \bar{\beta}\tau & \text{and} & \mathcal{A}^t = A^t + B^*\tau.
\end{array}$$

Notice in particular that the complex conjugate is obtained by conjugating all complex numbers appearing in $\mathcal{A} = (\alpha_{ij} + \beta_{ij}\tau)$ and that the adjoint is obtained by transposing the complex conjugate. The circlet conjugate appears in a natural way in the inversion of $\gamma \in \mathbb{S}$. In fact,

$$(\alpha + \beta\tau)^{-1} = \frac{1}{|\alpha|^2 - |\beta|^2}(\bar{\alpha} - \beta\tau),$$

so the circlet $\gamma = \alpha + \beta\tau$ is invertible if and only if $|\alpha| \neq |\beta|$. But $\gamma\gamma^\sim = |\alpha|^2 - |\beta|^2$ and we have

$$\gamma^{-1} = \frac{\gamma^\sim}{\gamma\gamma^\sim}.$$

The transpose *reverses* the product $i\tau$ to τi: $\gamma^t = \xi_0 + \xi_1 i + \xi_2\tau - \xi_3 i\tau$.

Proposition 4. *In $\mathbb{S}$ complex conjugation is an automorphism while the other involutions are anti-automorphisms. More generally, if $\mathcal{A} \in \mathcal{M}_{q,p}, \mathcal{B} \in \mathcal{M}_{p,n}$ then $\bar{\mathcal{A}} \in \mathcal{M}_{q,p}$ while $\mathcal{A}^*, \mathcal{A}^\sim, \mathcal{A}^t \in \mathcal{M}_{p,q}$ and*

$$\begin{aligned} \overline{\mathcal{A}\mathcal{B}} &= \bar{\mathcal{A}}\bar{\mathcal{B}} \\ (\mathcal{A}\mathcal{B})^* &= \mathcal{B}^*\mathcal{A}^* \\ (\mathcal{A}\mathcal{B})^\sim &= \mathcal{B}^\sim\mathcal{A}^\sim \\ (\mathcal{A}\mathcal{B})^t &= \mathcal{B}^t\mathcal{A}^t. \end{aligned}$$

Proposition 5. *In $\mathbb{S}$*

$$\|\bar{\gamma}\| = \|\gamma^*\| = \|\gamma^\sim\| = \|\gamma^t\| = \|\gamma\|.$$

Proof. This is clear from (13): $\|\gamma\| = |\alpha| + |\beta|$. □

Proposition 6. *For $\gamma, \delta \in \mathbb{S}$*

$$\|\gamma^*\gamma\| = \|\gamma\|^2 \tag{14}$$

$$\|\delta^*\gamma\| \leq \|\gamma\|\|\delta\|. \tag{15}$$

Proof. We have $\gamma^*\gamma = |\alpha|^2 + |\beta|^2 + 2\bar{\alpha}\beta\tau$. Thus

$$\|\gamma^*\gamma\| = |\alpha|^2 + |\beta|^2 + |2\bar{\alpha}\beta| = (|\alpha| + |\beta|)^2 = \|\gamma\|^2.$$

By Lemma 2 we obtain

$$\|\delta^*\gamma\| = \sup_\theta |[\delta^*\gamma e^{i\theta}]| = \sup_\theta |[\delta^*[\gamma e^{i\theta}]]| \leq \|\delta^*\| \sup_\theta |[\gamma e^{i\theta}]| = \|\delta^*\| \ \|\gamma\|. \tag{16}$$

As $\|\delta^*\| = \|\delta\|$ we obtain (15) from (16). □

Remark 2. These properties may be compared with those of *quaternions* $\mathbb{H}$. A major difference with $\mathbb{S}$ is, that there are nontrivial nilpotent circlets while quaternions is a division ring. Further, if we denote $q \in \mathbb{H}$ in the form

$$q = x_0 + x_1 i + x_2 j + x_3 k$$

with $i^2 = j^2 = k^2 = -1$ with $x_l \in \mathbb{R}$ and the conjugation is given by $q^* = x_0 - x_1 i - x_2 j - x_3 k$ then the norm $\|q\| = \sqrt{q^* q} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$ makes $\mathbb{H}$ a 4-dimensional real euclidean space. Recall that a norm $\| \centerdot \|$ can be given by an inner product if the *parallelogram law*

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

holds. The norm in $\mathbb{S}$ does *not* satisfy this. In fact, with $x = 1$, $y = \tau$ we have

$$8 = \|1 + \tau\|^2 + \|1 - \tau\|^2 > 2(\|1\|^2 + \|\tau\|^2) = 4.$$

Although we do not have parallelogram law we still do have natural concept of orthogonality: $\delta^*\gamma = 0$. We shall see later that for vectorands the analogues of (14) and (15) hold, too. This is the main motivation to introduce in place of scalar or inner products a circlet valued product. We give the definition here already.

Definition 6. For $c, d \in \mathbb{S}^n$ we set

$$(c, d) := d^* c$$

and call it the *circlet product* of c and d. If $(c, d) = 0$ we say that the vectorands c and d are *orthogonal* .

The basic properties of the circlet product are given in the next section. However, for $n = 1$ they are obvious by the previous propositions.

At times one wants to consider $\mathbb{S}$ and $\mathbb{S}^n$ as inner product spaces in the usual sense. From (c, d) we can deduce two other products as follows.

Definition 7. For $c, d \in \mathbb{S}^n$ we set

$$(c, d)_{\mathbb{C}} := \mathrm{Lin}(d^* c)$$

and

$$(c, d)_{\mathbb{R}} := \mathrm{Re}(\mathrm{Lin}(d^* c))$$

The circlet valued, complex valued and real valued products induce different norms and orhogonalities. For example, for circlets $\gamma = \alpha + \beta\tau$ we get

$$(\gamma, \gamma)_{\mathbb{C}} = (\gamma, \gamma)_{\mathbb{R}} = |\alpha|^2 + |\beta|^2$$

and, if $\gamma = \alpha + \beta\tau = \xi_0 + \xi_1 i + \xi_2\tau + \xi_3 i\tau$, $\delta = \lambda + \mu\tau = \eta_0 + \eta_1 i + \eta_2\tau + \eta_3 i\tau$, then

$$(\gamma, \delta)_{\mathbb{C}} = \alpha\bar{\lambda} + \bar{\beta}\mu$$

and

$$(\gamma, \delta)_{\mathbb{R}} = \xi_0\eta_0 + \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3.$$

We call a circlet ω *unitary* if $\omega^*\omega = 1$ while $\mu = \rho + \sigma\tau$ is *positive* if $|\sigma| \leq \rho$.

Proposition 7. *Every circlet γ has a polar decomposition $\gamma = \omega\mu$ where ω is unitary and μ positive. The decomposition is unique if γ is invertible.*

Proof. If $\gamma = \alpha + \beta\tau$ with $|\alpha| \geq |\beta|$ then we can write $\gamma = \frac{\alpha}{|\alpha|}(|\alpha| + \frac{|\alpha|}{\alpha}\beta\tau)$. For $|\beta| > |\alpha|$ likewise

$$\gamma = \frac{\beta}{|\beta|}\tau(|\beta| + \frac{|\beta|}{\bar{\beta}}\bar{\alpha}\tau).$$

The decomposition is unique when $|\alpha| \neq |\beta|$ as all unitary circlets are of the form $e^{i\theta}$ or $e^{i\theta}\tau$. □

Proposition 8. *Every positive circlet has a unique positive square root.*

Proof. If $\alpha + \beta\tau$ is positive, then $\alpha \geq |\beta|$. We seek for $\rho + \sigma\tau$ satisfying

$$\rho^2 + |\sigma|^2 + 2\rho\sigma\tau = \alpha + \beta\tau$$

and $\rho \geq |\sigma|$. From $2\rho|\sigma| = |\beta|$ and $\rho^2 + |\sigma|^2 = \alpha$ we obtain

$$\rho^2 = \frac{\alpha}{2} + \frac{\sqrt{\alpha^2 - |\beta|^2}}{2}$$

with

$$|\sigma|^2 = \alpha - \rho^2$$

so that $\rho^2 - |\sigma|^2 = \sqrt{\alpha^2 - |\beta|^2} \geq 0$. To the end, choose the argument of σ to satisfy $2\rho\sigma = \beta$. $\square$

Since $\gamma^*\gamma = |\alpha|^2 + |\beta|^2 + 2\bar{\alpha}\beta\tau$ we conclude that $\gamma^*\gamma$ has a unique positive square root which was denoted by μ in Proposition 7.

Notions like "absolute value", "modulus" and quadratic form are used in Clifford algebras in many different ways. We shall reserve $|\gamma|$ for the square root of $\gamma^*\gamma$ while we use quadratic expression for $\gamma^{\sim}\gamma$.

Definition 8. Given $\gamma \in \mathbb{S}$ we denote by $|\gamma|$ the positive square root of $\gamma^*\gamma = |\alpha|^2 + |\beta|^2 + 2\bar{\alpha}\beta\tau$, that is

$$|\gamma| = |\alpha| + \frac{|\alpha|}{\alpha}\beta\tau, \text{ w}hen\ |\alpha| \geq |\beta|,$$

and

$$|\gamma| = |\beta| + \frac{|\beta|}{\bar{\beta}}\bar{\alpha}\tau \text{ w}hen\ |\alpha| \leq |\beta|.$$

We denote by $\nu(\gamma)$ the *norm form*:

$$\nu(\gamma) = \gamma\gamma^{\sim} = |\alpha|^2 - |\beta|^2.$$

Notice, in particular, that $\|\ |\gamma|\ \| = |\alpha| + |\beta| = \|\gamma\|$, and $\gamma^{-1} = \frac{\gamma^{\sim}}{\nu(\gamma)}$. Further, if $\gamma = \xi_0 + \gamma_0$ where $\gamma_0 = i\xi_1 + \beta\tau$, then $\nu(\gamma_0) = -\gamma_0^2$.

2.2 Subalgebras, similarity

There are several natural (real) subalgebras in $\mathbb{S}$. We set for convenience

$$\begin{array}{ll} \mathbb{S}_0 = \mathbb{R} = \text{span}\{1\}, & \mathbb{S}_3 = \text{span}\{1, i + \tau\}, \\ \mathbb{S}_1 = \mathbb{C} = \text{span}\{1, i\}, & \mathbb{S}_4 = \text{span}\{1, i\tau\}, \\ \mathbb{S}_2 = \text{span}\{1, \tau\}, & \mathbb{S}_5 = \text{span}\{1, i + \tau, i\tau\}. \end{array}$$

Proposition 9. *The sets $\mathbb{S}_i$, $i = 0, 1, \ldots, 5$ are all proper subalgebras of $\mathbb{S}$. $\mathbb{S}_5$ is noncommutative while all others are commutative. They all are invariant under the circlet conjugation "$\sim$".*

An operation of the form $\gamma \mapsto \delta^{-1}\gamma\delta$ is often called conjugation. We follow the language of linear algebra and call it *similarity.* For example, if $\mathbb{A} \subset \mathbb{S}$ is a subalgebra, then so is the set similar to it by a fixed invertible circlet δ

$$\delta^{-1}\mathbb{A}\delta = \{\delta^{-1}\gamma\delta \mid \gamma \in \mathbb{A}\}.$$

If γ is a given cirlet, then, since $\tau^2 = 1, i^2 = -1$ we have in particular

$$\tau^{-1}\gamma\tau = \bar{\gamma}$$

and

$$(\tau i)^{-1}\gamma(\tau i) = (\gamma^{\sim})^t$$

and we can say that complex conjugation of a circlet is obtained by similarity with the complex conjugation operator τ.

Theorem 10. *Every circlet is similar to a circlet in $\mathbb{S}_1$, in $\mathbb{S}_2$ or in $\mathbb{S}_3$.*

Proof. Given $\alpha + \beta\tau$ we first observe that we may assume β to be real and nonnegative. In fact, if $\beta = |\beta|e^{i\theta}$, then

$$e^{-i\theta/2}(\alpha + \beta\tau)e^{i\theta/2} = \alpha + |\beta|\tau \tag{17}$$

Denote $\alpha = \xi_0 + i\xi_1$. We show that $\alpha + |\beta|\tau$ is in turn similar to a complex number if

$$\frac{|\xi_1|}{|\beta|} > 1, \tag{18}$$

similar to $\xi_0 + |\beta|\tau$ if

$$\frac{|\xi_1|}{|\beta|} < 1, \tag{19}$$

and to $\xi_0 + |\beta|(i + \tau)$ when

$$|\xi_1| = |\beta|. \tag{20}$$

We can start from (20). If $\xi_1 < 0$ then complex conjugation brings it to the wanted form:

$$\tau(\xi_0 + |\beta|(-i + \tau))\tau = \xi_0 + |\beta|(i + \tau).$$

Suppose therefore that $|\xi_1| \neq |\beta|$. If t, s are real then

$$\begin{aligned} &(t - si\tau)(\alpha + |\beta|\tau)(t + si\tau) \\ =&(t^2 - s^2)\xi_0 + i(t^2 + s^2)\xi_1 - 2i|\beta|ts + [(t^2 + s^2)|\beta| - 2\xi_1 ts]\tau \end{aligned} \tag{21}$$

If (18) holds, then there are t, s such that $t^2 - s^2 = 1$ and

$$(t^2 + s^2)|\beta| - 2\xi_1 ts = 0$$

and we have conjugated the circlet to a complex number (such that the real part of α has stayed fixed. On the other hand, if (19) holds then the purely imaginary part in (21)

$$i(t^2 + s^2)\xi_1 - 2i|\beta|ts$$

can be made to vanish in the very same fashion. □

If we denote by $\sigma(\alpha + \beta\tau)$ the set of eigenvalues of the circlet $\alpha + \beta\tau$, (see Definition 14) then

$$\sigma(\alpha + \beta\tau) = \{\lambda \in \mathbb{C} \mid |\lambda - \alpha| = |\beta|\}.$$

Thus, the spectra of circlets are circles. The intersection of the spectra with the real line determines the subalgebra which the circlet can be taken by a similarity. In fact, the proof above gives:

Corollary 11. *Let $\gamma = \alpha + \beta\tau$ be given and put $\xi_0 = Re\,\alpha$. If*

$$\sigma(\gamma) \cap \mathbb{R} = \emptyset, \tag{22}$$

then γ is similar to $\xi_0 + i\eta \in \mathbb{C}$ with $\eta > 0$, if

$$\sigma(\gamma) \cap \mathbb{R} = \{\xi_0\}, \tag{23}$$

then γ is similar to $\xi_0 + |\beta|(i + \tau) \in \mathbb{S}_3$ and if

$$\sigma(\gamma) \cap \mathbb{R} = \{\xi_0 + \eta, \xi_0 - \eta\} \tag{24}$$

then γ is similar to $\xi_0 + \eta\tau \in \mathbb{S}_2$.

Write again $\gamma = \xi_0 + \gamma_0$.

Definition 9. We say that a nonreal circlet γ is of type 1 if $\nu(\gamma_0) > 0$, of type 2 if $\nu(\gamma_0) < 0$ and of type 3 if $\nu(\gamma_0) = 0$.

So, a circlet is of type 1 if (22) holds, of type 2 if (24) holds and of type 3 if (23) is the case.

Let $f(\zeta) = \sum_{j=0}^{\infty} \alpha_j \zeta^j$ be analytic in a disc $|\zeta| < R$ in the complex plane. Clearly, we can extend f as a convergent power series to circlets γ satisfying $\|\gamma\| < R$.

Proposition 12. *A power series f with real coefficients α_j preserves the type.*

Proof. Decompose $\gamma = \xi_0 + i\xi_1 + \beta\tau = \xi_0 + \gamma_0$. Thus, γ is of type 1, 2 or 3 depending whether $\gamma_0^2 = (i\xi_1 + \beta\tau)^2 = -\xi_1^2 + |\beta|^2$ is < 0, > 0 or $= 0$. Now, all even powers of γ_0 are real while all odd powers are of the form $\rho\gamma_0$ with $\rho \in \mathbb{R}$. Observe further that powers of ξ_0 commute with powers of γ_0. Thus, f is of the form

$$f(\gamma) = u(\xi_0, -\xi_1^2 + |\beta|^2) + v(\xi_0, -\xi_1^2 + |\beta|^2)\gamma_0$$

where u and v are real valued whenever f has real coefficients. As the type is determined by γ_0, the claim follows. $\square$

Example 3. Consider the exponential function

$$e^{\zeta} = \sum_{0}^{\infty} \frac{\zeta^n}{n!}.$$

We obtain

$$e^{\gamma} = e^{\xi_0} e^{\gamma_0} = e^{\xi_0}(\cos(\sqrt{\xi_1^2 - |\beta|^2}) + \frac{\sin(\sqrt{\xi_1^2 - |\beta|^2})}{\sqrt{\xi_1^2 - |\beta|^2}}(i\xi_1 + \beta\tau))$$

which as such covers all types but it is perhaps instructive to write for type 2 circlets

$$e^{\gamma} = e^{\xi_0} e^{\gamma_0} = e^{\xi_0}(\cosh(\sqrt{|\beta|^2 - \xi_1^2}) + \frac{\sinh(\sqrt{|\beta|^2 - \xi_1^2})}{\sqrt{|\beta|^2 - \xi_1^2}}(i\xi_1 + \beta\tau))$$

and for type 3

$$e^{\gamma} = e^{\xi_0} e^{\gamma_0} = e^{\xi_0}(1 + i\xi_1 + \beta\tau).$$

Example 4. Recall that an element e in a ring R is an *idempotent* if $e^2 = e$. One checks easily that in addition to the trivial idempotents $0, 1$ a circlet $\varepsilon \in \mathbb{S}$ is an idempotent if and only if

$$\varepsilon = \frac{1}{2}(1 + \eta i + e^{i\theta}\sqrt{1+\eta^2}\ \tau)$$

with $\eta, \theta \in \mathbb{R}$.

2.3 Matrix representations

Circlets $\mathbb{S}$ (and $\mathcal{C}l_{1,1}$) can be represented using 2×2-matrices. The *real matrix representation* of $\gamma = \xi_0 + i\xi_1 + \xi_2\tau + i\xi_3\tau = \alpha + \beta\tau$ is

$$\phi(\gamma) = \begin{pmatrix} \xi_0 + \xi_2 & -\xi_1 + \xi_3 \\ \xi_1 + \xi_3 & \xi_0 - \xi_2 \end{pmatrix},$$

and the *complex matrix representation* is likewise

$$\psi(\gamma) = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix},$$

[6]. Thus $\mathbb{S}$ is isomorphic to $M_2(\mathbb{R})$ and to a subalgebra of $M_2(\mathbb{C})$. For example,

$$\det\phi(\gamma) = \det\psi(\gamma) = \gamma^{\sim}\gamma = |\alpha|^2 - |\beta|^2.$$

It is instructive to compare the analogous *complex* representation of quaternions. If we write $q \in \mathbb{H}$ as $q = a + bj$ with $a = x_0 + ix_1$, $b = x_2 + ix_3 \in \mathbb{C}$, then the corresponding subalgebra of $M_2(\mathbb{C})$ consists of matrices of the form

$$\chi(q) = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

Now, for example,

$$\det\chi(q) = q^*q = |a|^2 + |b|^2.$$

2.4 Polynomials

We shall consider *monic* polynomials of the forms

$$p(\gamma) = \gamma^n + \sum_{j=0}^{n-1} \gamma^j \delta_j \tag{25}$$

and

$$q(\gamma) = \gamma^n + \sum_{j=0}^{n-1} \delta_j \gamma^j, \tag{26}$$

where the coefficients $\delta_j \in \mathbb{S}$ and $n \geq 1$.

Proposition 13. *We have for all $\gamma \in \mathbb{S}$ and integer $k > 0$*

$$\|\gamma^{2k} - \tau\| \geq 1$$

where equality holds e.g. with the singular idempotent $\varepsilon = \frac{1}{2}(1+\tau)$. Further, the polynomial $p(\gamma) = \gamma^{2k} - \tau$ is singular at ε:

$$p(\varepsilon) = \frac{1}{2}(1 - \tau).$$

Proof. Since

$$\inf_{\gamma \in \mathbb{S}} \|\gamma^{2k} - \tau\| \geq \inf_{\gamma \in \mathbb{S}} \|\gamma^2 - \tau\|$$

we can assume $k = 1$. Writing $\gamma = \xi + \eta i + \beta\tau$ we have

$$\|\gamma^2 - \tau\| = |\xi^2 - \eta^2 + |\beta|^2 + 2i\xi\eta| + |2\xi\beta - 1|$$

which is minimized over η at $\eta^2 = |\beta|^2 - \xi^2$. The claim then follows. The other claims are easy. $\square$

Thus, we cannot in general look for *roots* of a polynomial. However, we may ask whether the polynomial becomes singular somewhere.

Theorem 14. *Every monic polynomial of the forms* (25) *and* (26) *becomes singular at some circlet.*

Proof. Consider first polynomials of the form (25). Write

$$p(\gamma) = \gamma^n + \sum_{j=0}^{n-1} \gamma^j \delta_j = \gamma^n + P(\gamma)$$

with $\gamma = \alpha + \beta\tau$. If $\delta_j = \varepsilon_j + \nu_j\tau$ and if, for fixed γ, we decompose $P(\gamma)$ into linear and conjugate linear parts

$$P(\gamma) = P_L(\gamma) + P_C(\gamma)\tau,$$

then for complex α these are polynomials with complex coefficients:

$$P_L(\alpha) = \sum \alpha^j \varepsilon_j, \quad P_C(\alpha) = \sum \alpha^j \nu_j.$$

Let $n = 2k$ and fix $\alpha = \alpha_0$ such that $\alpha^{2k} + P_L(\alpha_0) = 0$. Then for $\gamma = \alpha_0 + \beta\tau$ with $|\beta| \to \infty$

$$\gamma^{2k} = |\beta|^{2k}(1 + o(1))$$

while

$$\|P(\gamma)\| = O(|\beta|^{2k-1}).$$

Thus $|\operatorname{Lin}(\gamma^{2k} + P(\gamma))|$ vanishes at $\beta = 0$, and since it grows faster than $|\operatorname{Con}(\gamma^{2k} + P(\gamma))|$, there, by continuity, exists β_0 such that with $\gamma_0 = \alpha_0 + \beta_0\tau$

$$|\operatorname{Lin}(\gamma_0^{2k} + P(\gamma_0))| = |\operatorname{Con}(\gamma_0^{2k} + P(\gamma_0))|$$

which means that the polynomial becomes singular at γ_0.

For $n = 2k + 1$ choose α_0 such that $P_C(\alpha_0) = 0$. Then

$$\mathrm{Con}(\gamma^{2k+1} + P(\gamma)) = |\beta|^{2k}\beta(1 + o(1))$$

while now

$$|\,\mathrm{Lin}(\gamma^{2k+1} + P(\gamma))| = O(|\beta|^{2k})$$

and the conclusion follows similarly as for n even.

Consider then polynomials of the form (26). Notice that a circlet is singular if and only if its transpose is singular. So, let $q(\gamma) = \gamma^n + \sum_{j=0}^{n-1} \delta_j \gamma^j$ be given and choose $\gamma_0 = \alpha_0 + \beta_0 \tau$ be such that

$$\gamma_0^n + \sum_{j=0}^{n-1} \gamma_0^j \delta_j^t$$

is singular. Then $q(\gamma_0^t)$ is singular. □

Remark 5. Notice that the set of points, at which a polynomial is singular, is in general large. For example, the polynomial $p(\gamma) = \gamma$ is singular at singular circlets which are determined by one quadratic equation

$$\xi_0^2 + \xi_1^2 = \xi_2^2 + \xi_3^2$$

in the 4-dimensional real space, while the polynomial $\delta\gamma$ is singular everywhere if δ is singular.

3 Vectorands

We treat $\mathbb{S}^n$ as a right $\mathbb{S}-$module with norm

$$\|c\| = \|a + b\tau\| = \sqrt{\|a\|^2 + \|b\|^2 + 2|(a, b)|},$$

Definition 4. The norm is not induced by any inner product but it is compatible with the circlet product $(c, d) = d^*c$, Definition 5, as follows:

$$\|(c, c)\| = \|c\|^2 \quad \text{and} \quad \|(c, d)\| \leq \|c\|\|d\|.$$

Much of the usual properties of inner product spaces still hold, with some exceptions. For example, the scalars (c, c) are in general circlets, not reals; vectorands are orthogonal in a natural way but the Pythagorean theorem ($\|x\|^2 + \|y\|^2 = \|x + y\|^2$ with orthogonal x, y) does not hold, and linear independency requires extra care since the scalars contain nontrivial nilpotent elements.

3.1 Vectorands as a right module

Let a, b be in $\mathbb{C}^n$ and denote by $c = a + b\tau \in \mathbb{S}^n$ a *vectorand*, that is, a real linear operator mapping $\mathbb{C} \to \mathbb{C}^n$

$$\zeta \mapsto [c\zeta] = \zeta a + \bar{\zeta} b.$$

Vectorands $\mathbb{S}^n$ could be viewed as usual vector spaces over $\mathbb{R}$ or $\mathbb{C}$. In fact, if $\{e_j\}_1^n$ is the standars basis for $\mathbb{R}^n$, then $\mathbb{S}^n$ is a $4n$-dimensional real space with basis $\{e_j, ie_j, e_j\tau, ie_j\tau\}_{j=1}^n$ and a $2n$-dimensional complex space with basis $\{e_j, e_j\tau\}_{j=1}^n$. However, as pointed out in Proposition 1, operators in $\mathcal{M}_{p,n}$ are automatically *right* $\mathbb{S}$*-linear*:

$$\mathcal{A}(c\gamma + d\delta) = (\mathcal{A}c)\gamma + (\mathcal{A}d)\delta$$

where $\gamma, \delta \in \mathbb{S}$ and $c, d \in \mathbb{S}^n$. It is therefore natural to consider $\mathbb{S}^n$ as a right module over $\mathbb{S}$.

A general *right module* over $\mathbb{S}$ consists of an additive group V and a map

$$V \times \mathbb{S} \to V, \quad (v, \gamma) \mapsto v\gamma$$

such that the usual distributivity and unity axioms hold and such that, for all $\gamma, \delta \in \mathbb{S}$ and $v \in V$,

$$(v\gamma)\delta = v(\gamma\delta).$$

We shall see that vectorands $\mathbb{S}^n$ as a right module are in a natural way "n-dimensional". Namely, the *right linear independency* respects faithfully the two aspects of a vectorand, that as an element in the $\mathbb{S}-$module and that as an operator $\mathbb{C} \to \mathbb{C}^n$.

Another natural set up for real linear operators would be to work in $\mathbb{C}^n$ and consider it as a left module over $\mathbb{S}$. Then, however, a complex nonzero vector can be linearly dependent. In the right linear set up a single vectorand can also be linearly dependent but this cannot happen for a vector, as a set of veectors is linearly independent as vectorands in $\mathbb{S}^n$ if and only if they are linearly independent as vectors in $\mathbb{C}^n$.

3.2 Linearly independent vectorands

We shall consider two kinds of right linear combinations with vectorands. First, if $\gamma_j = \alpha_j + \beta_j\tau \in \mathbb{S}$, then

$$\sum_{j=1}^{k} c_j\gamma_j = \sum_{j=1}^{k} (\alpha_j a_j + \bar{\beta}_j b_j + (\beta_j a + \bar{\alpha}_j b_j)\tau) \in \mathbb{S}^n.$$

In the other one the vectorands are considered as operators and evaluated at the complex "coefficients" ζ_j:

$$\sum_{j=1}^{k} [c_j\zeta_j] = \sum_{j=1}^{k} (\zeta_j a_j + \bar{\zeta}_j b_j) \in \mathbb{C}^n.$$

We use here the expression "in $\mathbb{C}$" to underline that we treat the vectorands as operating in $\mathbb{C}$.

Definition 10. Vectorands $\{c_1, \ldots, c_k\}$ are *linearly independent over* $\mathbb{S}$ if

$$\sum_{j=1}^{k} c_j\gamma_j = 0 \ \text{ with } \gamma_j \in \mathbb{S} \text{ implies } \ \gamma_j = 0 \ \text{ for } j = 1, \ldots, k.$$

They are *linearly independent in* $\mathbb{C}$ if

$$\sum_{j=1}^{k} [c_j\zeta_j] = 0 \ \text{ with } \zeta_j \in \mathbb{C} \text{ implies } \ \zeta_j = 0 \ \text{ for } j = 1, \ldots, k.$$

Example 6. A single nontrivial vectorand can be linearly dependent. For example, the vectorand $c = ia + a\tau$ is not linearly independent over $\mathbb{S}$ as $c\gamma = 0$ with $\gamma = i + \tau$. It is not linearly independent in $\mathbb{C}$, either, as $[c\zeta] = 0$ with $\zeta = e^{i\frac{\pi}{4}}$.

Example 7. We see that in the previous example, loosely speaking, multiplying from right with circlets or evaluated at complex numbers fit together. This is a general fact as Theorem 15, below, shows. On the other hand, the situation is *different* if we take linear combinations from *left*. In fact, if $c = ia + a\tau$ with $0 \neq a \in \mathbb{R}^n$, then

$$(i + \tau)c = (i + \tau)^2 a = 0$$

while for all $0 \neq \zeta \in \mathbb{C}$

$$\zeta c \neq 0 \ \text{ and } [\zeta c] = \zeta(i + 1)a \neq 0.$$

Theorem 15. *Vectorands* $\{c_1, \ldots, c_k\}$ *are linearly independent over* $\mathbb{S}$ *if and only if they are linearly independent in* $\mathbb{C}$*.*

Proof. Suppose there exist circlets γ_j, not all 0, such that

$$\sum_{j=1}^{k} c_j\gamma_j = 0. \tag{27}$$

If $\gamma_l \neq 0$ then $[\gamma_l e^{i\theta}]$ is a nonzero complex number for some θ. Denoting $\zeta_j := [\gamma_j e^{i\theta}]$ we obtain from (27)

$$\sum_{j=1}^{k} [c_j\zeta_j] = [\sum_{j=1}^{k} c_j\zeta_j] = [\sum_{j=1}^{k} c_j\gamma_j e^{i\theta}] = 0$$

with $\zeta_l \neq 0$. So, if the vectorands are not linearly independent over $\mathbb{S}$ then they are not in $\mathbb{C}$, either.

Reversely, assume

$$\sum_{j=1}^{k} [c_j\zeta_j] = 0$$

holds in $\mathbb{C}^n$. That is,

$$\sum_{j=1}^{k} \zeta_j a + \bar{\zeta}_j b_j = 0.$$

Set $\gamma_j = \zeta_j(1+\tau)$. Then

$$\sum_{j=1}^{k}(a_j + b_j\tau)\zeta_j(1+\tau) = \sum_{j=1}^{k} \zeta_j a + \bar{\zeta}_j b_j + (\sum_{j=1}^{k} \zeta_j a + \bar{\zeta}_j b_j)\tau = 0$$

and (27) holds in $\mathbb{S}^n$. $\square$

Theorem 15. allows us to talk shortly about the linear independency as the two concepts agree:

Definition 11. Vectorands $\{c_1, \ldots, c_k\}$ are *linearly independent* if they are linearly independent over $\mathbb{S}$ (and in $\mathbb{C}$).

Definition 12. For $\mathcal{A} \in \mathcal{M}_{p,n}$ we denote $\ker \mathcal{A} = \{x \in \mathbb{C}^n \mid [\mathcal{A}x] = 0\}$ and $\ker_{\mathbb{S}} \mathcal{A} = \{c \in \mathbb{S}^n \mid \mathcal{A}c = 0\}$.

Remark 8. Notice that $\ker \mathcal{A}$ is an $\mathbb{R}-$linear "subspace" of $\mathbb{C}^n$, while $\ker_{\mathbb{S}} \mathcal{A}$ is a right submodule of $\mathbb{S}^n$.

Lemma 16. *A vectorand $a + b\tau$ is linearly independent if and only if*

$$a + e^{i\theta} b \neq 0 \quad \textit{for all } \theta \in \mathbb{R}$$

which is equivalent with $\ker(a + b\tau) = \{0\}$.

Proof. If

$$[(a + b\tau)\rho e^{i\phi}] = 0$$

then either $\rho = 0$ or $a = -e^{-2i\phi}b$. The claim then follows from the definition. $\square$

So, it is important to notice that nonzero vectorands are not always linearly independent.

Proposition 17. *There exists, for a real linear operator $\mathcal{A} \in \mathcal{M}_n$ an inverse $\mathcal{A}^{-1} \in \mathcal{M}_n$ if and only if* $\ker \mathcal{A} = \{0\}$.

Given $\mathcal{A} = (\alpha_{ij} + \beta_{ij}\tau) \in \mathcal{M}_{p,n}$ form vectorands $c_j = a_j + b_j\tau$ from the columns of $\mathcal{A}$. Then the following are equivalent

(i) $\ker \mathcal{A} = \{0\}$

(ii) $\ker_{\mathbb{S}} \mathcal{A} = \{0\}$

(iii) the vectorands $\{c_1, \ldots, c_n\}$ are linearly independent.

Proof. The first claim on the existence of an inverse is standard, [2].

If $x = (x_1, \ldots, x_n)^t \in \mathbb{C}^n$ then $\mathcal{A}x = \sum_{j=1}^n (a_j + b_j\tau)x_j$. Now, $x \in \ker \mathcal{A}$ means $0 = [\mathcal{A}x] = \sum_{j=1}^n [c_j x_j] = \sum_{j=1}^n (a_j x_j + b_j \bar{x}_j)$ and the equivalence of (i) and (iii) follows.

If $x \in \ker \mathcal{A}$ then $x(1+\tau) \in \ker_{\mathbb{S}} \mathcal{A}$ and if $c \in \ker_{\mathbb{S}} \mathcal{A}$ then $[c] \in \ker \mathcal{A}$ as $[\mathcal{A}c] = [\mathcal{A}[c]]$, so (i) and (ii) are equivalent. $\square$

Proposition 18.

$$\textit{If } x \in \ker \mathcal{A}, \ \textit{then } x(1+\tau) \in \ker_{\mathbb{S}} \mathcal{A},$$

and

$$\textit{if } c \in \ker_{\mathbb{S}} \mathcal{A}, \ \textit{then } [c] \in \ker \mathcal{A}.$$

If $x, y \in \ker \mathcal{A}$ are linearly independent over $\mathbb{R}$, then $x(1+\tau) + y(1+\tau)i \in \ker_{\mathbb{S}} \mathcal{A}$ is a linearly independent vectorand.

Reversely, if $c \in \ker_{\mathbb{S}} \mathcal{A}$ is linearly independent, then $[c], [ci] \in \ker \mathcal{A}$ are linearly independent over $\mathbb{R}$.

Proof. The first two statements are included in the proof of Proposition 17.

Let now $x, y \in \ker \mathcal{A}$ be linearly independent over $\mathbb{R}$. Then $z = x(1+\tau) + y(1+\tau)i \in \ker_{\mathbb{S}} \mathcal{A}$ is a linearly independent vectorand. To see this, suppose there is a θ such that $[ze^{i\theta}] = 0$ or

$$[(x(1+\tau) + y(1+\tau)i)e^{i\theta}] = 2x\cos\theta + 2y\cos(\theta + \pi/2) = 0.$$

Then, as x, y are linearly independent over $\mathbb{R}$, $\cos\theta = \cos(\theta + \pi/2) = 0$ which is a contradiction.

On the other hand, if $c = a + b\tau$ is a linearly independent vectorand in $\ker_{\mathbb{S}} \mathcal{A}$, then both $[c]$ and $[ci]$ are in $\ker \mathcal{A}$ and they are independent over $\mathbb{R}$. In fact,

$$\xi[c] + \eta[ci] = \xi(a+b) + i\eta(a-b) = (\xi + i\eta)a + (\xi - i\eta)b = [c(\xi + i\eta)] = 0$$

implies $\xi + i\eta = 0$. $\square$

3.3 Circlet product of vectorands and Gram-Schmidt orthogonalization

Circlet product and norm for vectorands were defined in Section 2.1. Their basic properties are summarized in the following.

Theorem 19. *Let $c, d, h \in \mathbb{S}^n$, $\gamma \in \mathbb{S}$, $\alpha \in \mathbb{C}$. Then*

$$\begin{aligned}
(c,d) &= (d,c)^* &(28)\\
(c\gamma, d\delta) &= \delta^*(c,d)\gamma &(29)\\
(c+d,h) &= (c,h)+(d,h) &(30)\\
\|(c,c)\| &= \|c\|^2 &(31)\\
\|(c,d)\| &\le \|c\|\|d\| &(32)\\
\|c\| &> 0, \qquad \textit{if } c \neq 0 &(33)\\
\|c+d\| &\le \|c\|+\|d\| &(34)\\
\|\alpha c\| = \|c\alpha\| &= |\alpha|\ \|c\| &(35)\\
\|\gamma c\| &\le \|\gamma\|\|c\|\ ,\ \|c\gamma\| \le \|c\|\|\gamma\|. &(36)
\end{aligned}$$

If additionally $(c,d)=0$, then

$$\frac{1}{2}(\|c\|^2+\|d\|^2) \le \|c+d\|^2 \le \|c\|^2+\|d\|^2. \tag{37}$$

Proof. We consider only (31), (32) and (37) as the others are simple consequences of the definitions. Let $c = a + b\tau$ and $d = e + f\tau$. Then

$$(c,d) = d^*c = (e+f\tau)^*(a+b\tau)$$

and

$$\begin{aligned}
\|(c,d)\| &= \sup_\phi |[(e^*a + f^t\bar{b} + (e^*b + f^t\bar{a})\tau)e^{i\phi}]|\\
&= |e^*a + f^t\bar{b}| + |e^*b + f^t\bar{a}|\\
&= |e^*a + b^*f| + |e^*b + a^*f|.
\end{aligned}$$

In particular,

$$\|(c,c)\| = |a^*a + b^*b| + |a^*b + a^*b|.$$

On the other hand,

$$\|c\|^2 = \sup_\phi \|\ [ce^{i\phi}]\ \|^2 = \sup_\phi \|a + e^{2i\phi}b\|^2 = \|a\|^2 + \|b\|^2 + 2|(a,b)| \tag{38}$$

and so $\|(c,c)\| = \|c\|^2$ which is (31).

Consider now (32). First, if $a \in \mathbb{C}^n$ then

$$[\mathcal{A}\mathcal{B}a] = [\mathcal{A}[\mathcal{B}a]] \tag{39}$$

which follows from Lemma 2 with $\mathcal{K} = \mathcal{B}a$. Denote $c(\phi) = [ce^{i\phi}] \in \mathbb{C}^n$. Then, with $x \in \mathbb{C}^n$ and using (39) with $\mathcal{A} = d^*$, $\mathcal{B} = c$, $a = e^{i\phi}$

$$\begin{aligned}
\|(c,d)\| &= \sup_\phi |[(d^*c)e^{i\phi}]|\\
&= \sup_\phi |[d^*c(\phi)]|\\
&\le \sup_\phi \sup_{\|x\|\le\|c(\phi)\|} |[d^*x]|\\
&= \sup_\phi \sup_{\|x\|\le\|c(\phi)\|} |e^*x + x^*f|
\end{aligned}$$

But $\|c\| = \sup_\phi \|c(\phi)\|$, so the claim follows if we can show for $x, e, f \in \mathbb{C}^n$ and $\|x\| = 1$ that

$$\|d\|^2 - |e^* x + x^* f|^2 \geq 0. \tag{40}$$

Denote $e_1 = x$ and let $\{e_j\}$ be orthonormal in $\mathbb{C}^n$ such that

$$\operatorname*{span}_{\mathbb{C}}\{x, e, f\} \subset \operatorname*{span}_{\mathbb{C}}\{e_1, e_2, e_3\}.$$

Then with $e = \sum_{j=1}^3 \alpha_j e_j$, $f = \sum_{j=1}^3 \beta_j e_j$

$$\begin{aligned}
&\|d\|^2 - |e^* x + x^* f|^2 \\
=&\sum_{j=1}^3 (|\alpha_j|^2 + |\beta_j|^2) + 2|\sum_{j=1}^3 \bar{\alpha}_j \beta_j| - |\bar{\alpha}_1 + \beta_1|^2 \\
=&\sum_{j=1}^3 (|\alpha_j|^2 + |\beta_j|^2) + 2|\sum_{j=1}^3 \bar{\alpha}_j \beta_j| - |\alpha_1|^2 - |\beta_1|^2 - 2\operatorname{Re}(\alpha_1 \beta_1) \\
=&\sum_{j=2}^3 (|\alpha_j|^2 + |\beta_j|^2) + 2|\sum_{j=1}^3 \bar{\alpha}_j \beta_j| - 2\operatorname{Re}(\alpha_1 \beta_1) \\
\geq&\sum_{j=2}^3 (|\alpha_j|^2 + |\beta_j|^2) + 2\ [\ |\bar{\alpha}_1 \beta_1| - |\sum_2^3 \bar{\alpha}_j \beta_j| - \operatorname{Re}(\alpha_1 \beta_1)] \\
\geq&\sum_2^3 (|\alpha_j|^2 + |\beta_j|^2) - 2\ |\sum_2^3 \bar{\alpha}_j \beta_j| \geq 0,
\end{aligned}$$

and (40) holds, completing the proof of (32).

Consider then (37) and assume that c, d are orthogonal. We have

$$(c + d, c + d) = (c, c) + (d, d) + (c, d) + (d, c) = (c, c) + (d, d).$$

Thus

$$\|c + d\|^2 = \|(c, c) + (d, d)\| \leq \|(c, c)\| + \|(d, d)\| = \|c\|^2 + \|d\|^2.$$

The lower bound is obtained from

$$\|c + d\|^2 \geq \|a\|^2 + \|b\|^2 + \|e\|^2 + \|f\|^2 \tag{41}$$

where $c = a + b\tau$ and $d = e + f\tau$ using (38). To derive (41) notice that orthogonality of c and d is equivalent with the two equations:

$$\begin{aligned}
(a, e) + (f, b) =&0 \\
(b, e) + (f, a) =&0
\end{aligned}$$

Utilizing these and (38) gives

$$\begin{aligned}
\|c + d\|^2 =&\|a + e\|^2 + \|b + f\|^2 + 2|(a + e, b + f)| \\
=&\|a\|^2 + \|b\|^2 + \|e\|^2 + \|f\|^2 + 2\operatorname{Re}((a, e) + (b, f)) + 2|(a, b) + (e, f)| \\
=&\|a\|^2 + \|b\|^2 + \|e\|^2 + \|f\|^2 + 2|(a, b) + (e, f)|.
\end{aligned}$$

∎

So, many of the "usual" properties hold but *not* the Pythagorean theorem, for which the proof gives the following exact replacement.

Corollary 20. *If $c = a + b\tau$ and $d = e + f\tau$ are orthogonal, then*

$$\|(a+e) + (b+f)\tau\|^2 = \|a\|^2 + \|b\|^2 + \|e\|^2 + \|f\|^2 + 2|(a,b) + (e,f)|$$

Example 9. Let $c = 1 + \tau$ and $d = 1 - \tau$, then $(c,d) = 0$ and

$$\|c + d\|^2 = 4$$

while

$$\|c\|^2 + \|d\|^2 = 4 + 4.$$

The proof above used identity (39). Let us draw another conclusion from it.

Proposition 21. *If $\mathcal{A} \in \mathcal{M}_{q,p}$ and $\mathcal{B} \in \mathcal{M}_{p,n}$, then*

$$\|\mathcal{A}\mathcal{B}\| \leq \|\mathcal{A}\|\|\mathcal{B}\|.$$

In particular, $\mathcal{M}_n$ is a real Banach algebra.

Proof. By (39)

$$\begin{aligned}\|\mathcal{A}\mathcal{B}\| &= \sup_{\|a\|=1} \|[\mathcal{A}\mathcal{B}a]\| \\ &= \sup_{\|a\|=1} \|[\mathcal{A}[\mathcal{B}a]]\| \\ &\leq \|\mathcal{A}\| \sup_{\|a\|=1} \|[\mathcal{B}a]\| = \|\mathcal{A}\|\|\mathcal{B}\|.\end{aligned}$$

∎

Definition 13. Vectorands $\{c_1, \ldots, c_m\}$ are *orthonormal* if

$$(c_j, c_k) = 0 \text{ for } j \neq k \text{ while } (c_j, c_j) = 1.$$

Proposition 22. *An orthonormal set of vectorands is linearly independent.*

Proof. If

$$\sum_{j=1}^{m} c_j \delta_j = 0$$

then for all l

$$0 = (\sum_1^m c_j \delta_j, c_l) = \sum_1^m (c_j, c_l)\delta_j = \delta_l.$$

∎

Lemma 23. *Let c, d be vectrorands such that*

$$(c, d) = \xi + \mu\tau$$

with $\xi > |\mu|$. Then there exists an invertible $\gamma \in \mathbb{S}$ such that

$$(c\gamma, d\gamma) = 1.$$

Proof. Without loss of generality we may assume that $\xi = 1$, so that $|\mu| < 1$. Let $\gamma = \alpha + \beta\tau$. Then

$$\begin{aligned}(c\gamma, d\gamma) =& \gamma^*(1 + \mu\tau)\gamma \\ =& (\bar{\alpha} + \beta\tau)(1 + \mu\tau)(\alpha + \beta\tau) \\ =& [|\alpha|^2 + |\beta|^2 + \bar{\alpha}\bar{\beta}\mu + \alpha\beta\bar{\mu}] + [2\bar{\alpha}\beta + \bar{\alpha}^2\mu + \beta^2\bar{\mu}]\ \tau.\end{aligned}$$

Thus, $(c\gamma, d\gamma) > 0$ if $2\bar{\alpha}\beta + \bar{\alpha}^2\mu + \beta^2\bar{\mu} = 0$. But this can be satisfied e.g. by setting $\alpha = 1$ and solving for β. Since $|\mu| < 1$ we have $|\beta| \neq 1 (= |\alpha|)$ so that γ is invertible. □

Lemma 24. *Let c be a linearly independent vectorand. Then there exists an invertible $\gamma \in \mathbb{S}$ such that $w = c\gamma$ is normalized: $(w, w) = 1$.*

Proof. If $c = a + b\tau$, then $(c, c) = \|a\|^2 + \|b\|^2 + 2a^*b\tau$. Since c is linearly independent, we have $\|a\|^2 + \|b\|^2 > |2a^*b|$ by Lemma 16 and the claim follows from Lemma 23. □

Theorem 25. *(Orthonormalization) Given a linearly independent set*

$$\{c_1, c_2, \ldots, c_m\} \subset \mathbb{S}^n$$

there exist orthonormal vectorands $\{w_1, \ldots, w_m\}$ such that for $k = 1, \ldots, m$

$$\{\sum_1^k c_j\gamma_j \mid \gamma_j \in \mathbb{S}\} = \{\sum_1^k w_j\gamma_j \mid \gamma_j \in \mathbb{S}\}. \tag{42}$$

Proof. We notice first that the claim holds for $m = 1$ by Lemma 23. What remains is just the familiar Gram-Schmidt process. Assuming that $\{w_1, \ldots, w_k\}$ is already orthonormalized, set

$$\hat{w} = c_{k+1} - \sum_1^k w_j(c_{k+1}, w_j) \tag{43}$$

so that for $l \leq k$

$$\begin{aligned}(\hat{w}, w_l) =& (c_{k+1}, w_l) - \sum_1^k (w_j, w_l)(c_{k+1}, w_j) \\ =& (c_{k+1}, w_l) - (w_l, w_l)(c_{k+1}, w_l) = 0.\end{aligned}$$

Finally, $\hat{w}$ is normalized to give w_{k+1} using Lemma 24. This to be possible, we need to check that $\hat{w}$ is linearly independent. To that end assume that there is a circlet γ such that $\hat{w}\gamma = 0$. From (43) we obtain

$$\hat{w}\gamma = c_{k+1}\gamma - \sum_1^k w_j(c_{k+1}, w_j)\gamma.$$

But by (42) there exist γ_j such that

$$\sum_1^k w_j(c_{k+1}, w_j) = \sum_1^k c_j\gamma_j$$

and thus

$$c_{k+1}\gamma = \sum_1^k c_j\gamma_j.$$

But $\{c_j\}$ are linearly independendent and thus $\gamma = 0$ and $\hat{w}$ can be normalized. □

3.4 Other orthogonalities

If $c = a + b\tau$, $d = e + f\tau$, then their circlet product is

$$(c, d) = e^*a + b^*f + \{e^*b + a^*f\}\tau.$$

With this product and the operator norm vectorands $\mathbb{S}^n$ can be viewed as an n- dimensional right module over $\mathbb{S}$. Clearly, vectorands can also be considered as a $2n$- dimensional space over $\mathbb{C}$ and a $4n$-dimensional space over $\mathbb{R}$.

In the complex (right linear) space we can use the following inner product:

$$(c, d)_{\mathbb{C}} = \mathrm{Lin}(c, d) = e^*a + b^*f.$$

We check easily that this is a complex inner product. Clearly, $(d, c) = \overline{(c, d)}$ and if $\alpha, \beta \in \mathbb{C}$, then by (29)

$$(c\alpha, d\beta) = \overline{\beta}(c, d)\alpha$$

and so

$$(c\alpha, d\beta)_{\mathbb{C}} = \alpha\overline{\beta}(c, d)_{\mathbb{C}}.$$

If we consider $\mathbb{S}^n$ as a $4n$-dimensional real space, then the right and left multiplications agree. There are two natural inner products available. The first one is a real valued analogue of $(c, d)_{\mathbb{C}}$:

$$(c, d)_{\mathbb{R}} = \mathrm{Re}(c, d)_{\mathbb{C}} = \mathrm{Re}\{e^*a + b^*f\}$$

while the second one is

$$\begin{aligned}((c,d)) =& \mathrm{Re}[(c,d)] = \mathrm{Re}\{e^*a + b^*f + e^*b + a^*f\}\\ =& \mathrm{Re}(a+b, e+f)\\ =& \mathrm{Re}([c],[d]).\end{aligned}$$

The second one is useful if one works with real linear problems which are formulated within $\mathbb{C}^n$ and results of operations taking vectors to vectorands c are always brought back to vectors $[c]$.

It is clear from their definition that all these products lead to different orthogonalities. It is important that they also respect the *-involution in a natural way. In fact, a straightforward calculation shows that

$$(\mathcal{A}c, d) = (c, \mathcal{A}^*d)$$

from which the claim follows for the complex and real valued inner products. For $((c,d))$ we have

$$\begin{aligned}([\mathcal{A}c],[d]) =& (A[c] + B\overline{[c]}, [d])\\ =& ([c], A^*[d]) + \overline{([c], B^t\overline{[d]})}.\end{aligned}$$

Taking real parts we obtain

$$\begin{aligned}((\mathcal{A}c, d)) =& \mathrm{Re}([\mathcal{A}c],[d]) = \mathrm{Re}([c], A^*[d] + B^t\overline{[d]})\\ =& \mathrm{Re}([c],[\mathcal{A}^*d]) = ((c, \mathcal{A}^*d)).\end{aligned}$$

We summarize:

Proposition 26. *We have*

$$\begin{array}{rclcrcl}(\mathcal{A}c, d) & = & (c, \mathcal{A}^*d), & \quad & (\mathcal{A}c, d)_{\mathbb{C}} & = & (c, \mathcal{A}^*d)_{\mathbb{C}},\\ ((\mathcal{A}c, d)) & = & ((c, \mathcal{A}^*d)), & & (\mathcal{A}c, d)_{\mathbb{R}} & = & (c, \mathcal{A}^*d)_{\mathbb{R}}.\end{array}$$

Remark 10. Notice that the complex numbers $([\mathcal{A}c],[d])$ and $([c],[\mathcal{A}^*d])$ are generally *different* and therefore working with the usual inner product in $\mathbb{C}^n$ - when the operator is not $\mathbb{C}$-linear - requires extra care.

4 Eigencirclets and eigenvectorands

Our goal in this section is to consider the real linear analogue of the following fact in $\mathbb{C}$-linear algebra: eigenvectors related to different eigenvalues are linearly independent. From the noncommutativity of $\mathbb{S}$ it follows that there are several different ways to generalize the eigenvalue-eigenvector pair.

We shall only consider two concepts, namely eigencirclets (in $\mathbb{S}$) in the form

$$\mathcal{A}c = c\gamma \tag{44}$$

and eigenvalues (in $\mathbb{C}$) in the form

$$[\mathcal{A}x] = \lambda x$$

with $x \in \mathbb{C}^n$. The first is related to diagonalization while the second is the natural concept related to solvability of

$$\lambda x - Ax - B\bar{x} = f.$$

A third natural concept would be to ask for $\gamma \in \mathbb{S}$ and $c \in \mathbb{S}^n$ such that $\mathcal{A}c = \gamma c$.

In [14] a survey is given on analogous properties of matrices with quaternion elements. In that language the analogue of our eigencirclet would be called a right eigenvalue while γ satisfying $\mathcal{A}c = \gamma c$ would be a left eigenvalue. In [1] left eigenvalues are called singular eigenvalues while left eigenvalue is reserved for the situation $d\mathcal{A} = \delta d$ with d a "row vector".

4.1 Definitions and preliminaries

Definition 14. Given a real linear operator $\mathcal{A} \in \mathcal{M}_n$ a cirlect $\alpha + \beta\tau \in \mathbb{S}$ is an *eigencirclet* and a linearly independent vectorand $a + b\tau \in \mathbb{S}^n$ an *eigenvectorand* if in $\mathbb{S}^n$

$$\mathcal{A}(a + b\tau) = (a + b\tau)(\alpha + \beta\tau).$$

Further, a complex λ and a nonzero vector $x \in \mathbb{C}^n$ are called an *eigenvalue* and an *eigenvector*, respectively, if in $\mathbb{C}^n$

$$[\mathcal{A}x] = \lambda x.$$

Example 11. Consider the circlet τ as an operator in $\mathbb{C}$. Then ξ, τ with $\xi \in \mathbb{R}$ as vectorand and τ as eigencirclet satisfy

$$\tau\xi = \xi\tau$$

and there are no others. In contrast, there are a lot of eigenvalue-eigenvector pairs:

$$[\tau e^{i\theta}] = e^{-2i\theta} e^{i\theta}.$$

We conclude immediately that there cannot exist linear independency of eigenvectors related to different eigenvalues.

Example 12. Let

$$\mathcal{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tau,$$

then $\mathcal{B}$ has no eigenvalues at all, but

$$c = \begin{pmatrix} 1 + \tau \\ i - i\tau \end{pmatrix}$$

is an eigenvectorand with circlet i:

$$\mathcal{B}c = ci.$$

In $\mathbb{R}-$linear algebra similarities can be applied both to the operator and to the eigencirclet.

Proposition 27. *Eigencirclets are invariant in $\mathbb{R}-$linear similarities of $\mathcal{A}$, eigenvalues in $\mathbb{C}-$linear similarities. $\mathbb{R}$-linear similarity of an eigencirclet is an eigencirlet.*

Proof. Suppose (44). We have for any invertible $\mathcal{T}$ with $u+v\tau = \mathcal{T}(a+b\tau)$

$$\mathcal{T}\mathcal{A}\mathcal{T}^{-1}(u+v\tau) = (u+v\tau)(\alpha+\beta\tau)$$

and for invertible T with $y = Tx$

$$T\mathcal{A}T^{-1}y = \lambda y.$$

Notice that if $\lambda = \xi \in \mathbb{R}$ then, however,

$$\mathcal{T}\mathcal{A}\mathcal{T}^{-1}\mathcal{T}x = \mathcal{T}\xi x = \xi\mathcal{T}x,$$

but $\mathcal{T}\lambda \neq \lambda\mathcal{T}$ if $\mathcal{T}$ is not $\mathbb{C}-$linear and λ is not real.

Let $\mathcal{A}c = c\gamma$ and suppose $\gamma = \delta^{-1}\hat{\gamma}\delta$. Then

$$\mathcal{A}c = c\delta^{-1}\hat{\gamma}\delta$$

and γ_0 is an eigencirclet:

$$\mathcal{A}(c\delta^{-1}) = (c\delta^{-1})\hat{\gamma}.$$

□

Remark 13. Since every circlet is similar to a circlet in $\mathbb{S}_i$ with $i = 1, 2$ or 3, this can be assumed in a "normal form".

Definition 15. [6] Given $\mathcal{A} = A + B\tau$ the *complex matrix representation* of $\mathcal{A}$ is given by ψ:

$$\psi(\mathcal{A}) = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}.$$

Proposition 28. *We have*

$$\psi(\mathcal{A})\begin{pmatrix} a \\ \bar{b} \end{pmatrix} = \lambda\begin{pmatrix} a \\ \bar{b} \end{pmatrix} \tag{45}$$

if and only if

$$\mathcal{A}(a+b\tau) = (a+b\tau)\lambda. \tag{46}$$

Proof. Taking the complex conjugate of the lower row in (45) yields

$$Aa + B\bar{b} = \lambda a \qquad \text{and} \qquad Ab + B\bar{a} = \bar{\lambda}b.$$

But then $\mathcal{A}(a+b\tau) = Aa + B\bar{b} + [Ab + B\bar{a}]\tau = \lambda a + \bar{\lambda}b\tau = (a+b\tau)\lambda$. □

Remark 14. Since $\psi(\mathcal{A})$ always has at least one eigenvalue-eigenvector pair, there always exists at least one candidate for an eigencirclet-eigenvectorand pair for $\mathcal{A}$. Since $\psi(\mathcal{A})$ has generically 2n such pairs the whole set of these vectorands cannot be linearly independent and some of the candidates $a+b\tau$ may be linearly dependent.

Lemma 29. *Vectors* $\{\begin{pmatrix} a \\ \bar{b} \end{pmatrix}, \begin{pmatrix} b \\ \bar{a} \end{pmatrix}\}$ *are linearly independent in* $\mathbb{C}^{2n}$ *if and only if the vectorand* $a+b\tau$ *is linearly independent.*

Proof. If

$$\alpha_1 \begin{pmatrix} a \\ \bar{b} \end{pmatrix} + \alpha_2 \begin{pmatrix} b \\ \bar{a} \end{pmatrix} = 0$$

then equivalently

$$\begin{aligned} \alpha_1 a + \alpha_2 b &= 0 \\ \bar{\alpha}_1 b + \bar{\alpha}_2 a &= 0 \end{aligned}$$

which further requires $|\alpha_1| = |\alpha_2|$ (or $a = b = 0$).

$\square$

Corollary 30. *If* (45) *holds with* $\lambda \neq \bar{\lambda}$*, then* $a+b\tau$ *is an eigenvectorand.*

The structure of $\psi(\mathcal{A})$ implies that if $\{\lambda, \begin{pmatrix} a \\ \bar{b} \end{pmatrix}\}$ is an eigenpair, then so is $\{\bar{\lambda}, \begin{pmatrix} b \\ \bar{a} \end{pmatrix}\}$. Likewise, operating (46) from right with τ yields

$$\mathcal{A}(b + a\tau) = (b + a\tau)\bar{\lambda}$$

and we have two different circlet-vectorand pairs for $\mathcal{A}$. Clearly the two vectorands are linearly dependent:

$$(a + b\tau) + (b + a\tau)(-\tau) = 0$$

and hence only one from these vectorands can be included into our collection of linearly independent eigenvectorands. Notice that although λ and $\bar{\lambda}$ are (possibly) distinct they are always similar: $\bar{\lambda} = \tau\lambda\tau$.

Suppose now that λ is real. Clearly, if all eigenvalues of $\psi(\mathcal{A})$ would be real there would again be too many of them. Thus we need to pack two real eigenvalues into one eigencirclet.

Let $\xi, \eta \in \mathbb{R}$, $u, v \in \mathbb{C}^n$ be such that

$$[\mathcal{A}u] = (\xi + \eta)u \quad \text{and} \quad [\mathcal{A}v] = (\xi - \eta)v.$$

Put $x = u - iv$, $y = u + iv$.

Lemma 31. *If* $x, y \in \mathbb{C}^n$ *and* $\xi, \eta \in \mathbb{R}$ *are as above, then*

$$\mathcal{A}(x + y\tau) = (x + y\tau)(\xi + \eta\tau). \tag{47}$$

If $\eta \neq 0$*, then the vectorand* $x + y\tau$ *is linearly independent.*

Proof. Consider (47):

$$\begin{aligned}
&\mathcal{A}(x+y\tau)\\
=&Ax+B\bar{y}+(Ay+B\bar{x})\tau\\
=&(Au-iAv+B\bar{u}-iB\bar{v})+(Au+iAv+B\bar{u}+iB\bar{v})\tau\\
=&(\xi+\eta)u-i(\xi-\eta)v+((\xi+\eta)u+i(\xi-\eta)v)\tau\\
=&(u-iv)\xi+(u+iv)\xi\tau+(u+iv)\eta+(u-iv)\eta\tau\\
=&x\xi+y\xi\tau+y\eta+x\eta\tau\\
=&(x+y\tau)\xi+(x+y\tau)\eta\tau\\
=&(x+y\tau)(\xi+\eta\tau).
\end{aligned}$$

Suppose now that $\eta \neq 0$. We show that $[(x+y\tau)e^{i\theta}] \neq 0$ for all θ. In fact

$$[(u-iv+(u+iv)\tau)e^{i\theta}] = 2\cos\theta\ u + 2\sin\theta\ v = 0$$

would imply that either u or v would vanish or $u = tv$ with a real $t \neq 0$. This contradicts $\eta \neq 0$ as $[\mathcal{A}u] = (\xi+\eta)u$ and $[\mathcal{A}v] = (\xi-\eta)v$. $\square$

We have now considered eigencirclets in $\mathbb{C}$ and in $\mathbb{S}_2$. What remains is to have a look at eigencirclets in $\mathbb{S}_3$.

Example 15. Let

$$\mathcal{A} = \begin{pmatrix} 1+i+\tau & 1 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathcal{D} = \begin{pmatrix} 1+i+\tau & 0 \\ 0 & 1 \end{pmatrix}.$$

Then one checks easily that $\mathcal{A}^k$ grows quadratically while $\mathcal{D}^k = \mathcal{O}(k)$ and so $\mathcal{A}$ and $\mathcal{D}$ are not $\mathbb{R}-$linearly similar. Thus, there cannot be two linearly independent vectorands for $\mathcal{A}$, even so the diagonal elements are *not* similar. Let e_j denote the coordinate vectors in $\mathbb{C}^2$. Put

$$a+b\tau = \begin{pmatrix} 1+i+\tau \\ -i-\tau \end{pmatrix} \in \mathbb{S}^2$$

and

$$u = \begin{pmatrix} e^{i\pi/4} \\ 0 \end{pmatrix} \in \mathbb{C}^2.$$

Then e_1 and $a+b\tau$ are eigenvectorands for $\mathcal{A}$:

$$\mathcal{A}e_1 = e_1(1+i+\tau) \ \text{ and } \ \mathcal{A}(a+b\tau) = a+b\tau$$

while u is an eigenvetor:

$$[\mathcal{A}u] = u.$$

For $\mathcal{D}$ we have eigenvectorands:

$$\mathcal{D}e_1 = e_1(1+i+\tau) \ \text{ and } \ \mathcal{D}e_2 = e_2,$$

and eigenvectors

$$[\mathcal{D}u] = u \ \text{ and } \ [\mathcal{D}e_2] = e_2.$$

4.2 Linear independency of eigenvectorands

We have chosen to call a vectorand satisfying (44) an eigenvectorand only when the vectorand is linearly independent. Two eigenvectorands related to eigencirclets which are similar may well be linearly dependent. In Example 15 we found two eigenvectorands which are linearly dependent even though the corresponding eigencirclets δ_i are not similar. This is possible because there exists a nontrivial nilpotent $\gamma \in \mathbb{S}$ to the following equation:

$$\delta_1\gamma - \gamma\delta_2 = 0. \tag{48}$$

In fact, (48) holds with $\delta_1 = \xi + i + \tau$, $\delta_2 = \xi$ and $\gamma = i + \tau$.

The equation (48) is a special case of the *Sylvester's equation*

$$\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{B} = \mathcal{F} \tag{49}$$

where $\mathcal{A}, \mathcal{B}$ and $\mathcal{F}$ are given and $\mathcal{X}$ is unknown.

Definition 16. We call a pair $\{\mathcal{A}, \mathcal{B}\}$ *regular* if (49) has a unique solution for all $\mathcal{F}$.

Clearly, if δ_1, δ_2 are similar, then $\{\delta_1, \delta_2\}$ cannot be regular, as $\delta_2 = \sigma^{-1}\delta_1\sigma$ implies (48) with $\gamma = \sigma$. For quaternions, and division rings in general, a pair would be regular if and only if they are similar [10], but, as we have seen, a pair of circlet can fail to be regular even if they are not similar. We discuss the solvability of (49) in detail in Section 5. For sake of reference we give the result on circlets here.

Proposition 32. *A pair $\{\gamma, \delta\}$ of circlets is regular if and only if*

$$\{\xi_0 + \sqrt{\gamma_0^2},\ \xi_0 - \sqrt{\gamma_0^2}\} \cap \{\eta_0 + \sqrt{\delta_0^2},\ \eta_0 - \sqrt{\delta_0^2}\} = \emptyset$$

where ξ_0, η_0 denote the real parts of $\gamma = \xi_0 + \gamma_0$ and $\delta = \eta_0 + \delta_0$.

Proof. This is just a reformulation of Theorem 40. □

Now we can state the result on linear independency of eigenvectorands.

Theorem 33. *Let c_j, $j = 1, 2, \ldots, m$ be eigenvectorands with eigencirclets δ_j for $\mathcal{A}$. Assume that all pairs $\{\delta_k, \delta_l\}$ with $k \neq l$ are regular. Then $\{c_1, \ldots, c_m\}$ is linearly independent.*

Proof. The proof is by induction. The statement holds by assumption for all subsets containing one vectorand. Suppose then that all subsets containing N vectorands are linearly independent and choose an arbitary subset with $N + 1$ vectorands. Assume that circlets γ_j are such that

$$\sum_{j=1}^{N+1} c_j\gamma_j = 0. \tag{50}$$

If one of the circlets, say $\gamma = \gamma_{N+1}$ is invertible, then

$$-c_{N+1} = \sum_1^N c_j \gamma_j \gamma^{-1}.$$

Applying this from left with $\mathcal{A}$ and with $\delta = \delta_{N+1}$ from right gives, as $\mathcal{A}c_j = c_j \delta_j$,

$$\sum_1^N c_j \delta_j \gamma_j \gamma^{-1} = \sum_1^N c_j \gamma_j \gamma^{-1} \delta$$

or

$$\sum_1^N c_j (\delta_j \gamma_j \gamma^{-1} - \gamma_j \gamma^{-1} \delta) = 0.$$

By induction hypothesis $\delta_j \gamma_j \gamma^{-1} - \gamma_j \gamma^{-1} \delta = 0$. But since $\{\delta_j, \delta\}$ is by assumption regular, we have $\gamma_j \gamma^{-1} = 0$ and further $\gamma_j = 0$ for $j = 1, \dots, N$. But then

$$c_{N+1} \gamma_{N+1} = 0$$

which contradicts the assumption on γ_{N+1} being invertible. Suppose therefore that none of the circlets γ_j are invertible but that (50) holds. If one of the circlets would vanish, then by induction hypothesis they all do. Thus we have $N+1$ nontrivial nilpotent circlets. Since nilpotent circlets have nontrivial kernels, let θ be such that $[\gamma_1 e^{i\theta}] = 0$. Again, by induction hypothesis, $[\gamma_j e^{i\theta}] = 0$ for all $j = 2, \dots, N+1$. But this simply means that all γ_j's are of the form

$$\gamma_j = \alpha_j (1 - e^{2i\theta} \tau)$$

with $0 \neq \alpha_j \in \mathbb{C}$ and (50) takes the form

$$(\sum_1^{N+1} c_j \alpha_j)(1 - e^{2i\theta} \tau) = 0.$$

But this implies that

$$0 = \sum_1^{N+1} c_j \alpha_j \in \mathbb{S}^n$$

where the complex nonvanishing constants α_j are treated as invertible circlets. The proof can now be completed by repeating the argument in the beginning of this proof. □

Theorem 34. *If $\mathcal{A} = \mathcal{A}^*$ and c_k, c_l are eigenvectorands related to a regular pair $\{\delta_k, \delta_l\}$ of eigencirclets, then the vectorands c_k, c_l are orthogonal.*

Proof. Suppose first that c is a normalized eigenvectorand so that $(c, c) = 1$. Then from $\mathcal{A}^* = \mathcal{A}$ and $\mathcal{A}c = c\delta$ we obtain

$$(c, c)\delta = (\mathcal{A}c, c) = (c, \mathcal{A}c) = \delta^*(c, c)$$

so that $\delta^* = \delta$. Thus

$$(c_k, c_l)\delta_k = (\mathcal{A}c_k, c_l) = (c_k, \mathcal{A}c_l) = \delta_l(c_k, c_l).$$

However, this is of the form $\gamma\delta_k = \delta_l\gamma$ and is, by assumption, satisfied with $\gamma = 0$ only.

If the eigenvectorands are not normalized, then by Lemma 35 below it can be done. So, suppose $c_j = \hat{c}\gamma_j$ where $(\hat{c}, \hat{c}) = 1$. Then $(\hat{c}_k, \hat{c}_l) = 0$ and

$$(c_k, c_l) = (\hat{c}_k\gamma_k, \hat{c}_l\gamma_l) = \gamma_l^*(\hat{c}_k, \hat{c}_l)\gamma_k = 0.$$

□

Lemma 35. *If*

$$\mathcal{A}c = c\delta$$

with a linearly independent c, then there exists a nonsingular $\gamma \in \mathbb{S}$ such that

$$\mathcal{A}\hat{c} = \hat{c}\hat{\delta}$$

where $\hat{\delta} = \gamma^{-1}\delta\gamma$ and $\hat{c} = c\gamma$ is normalized: $(\hat{c}, \hat{c}) = 1$.

If additionally $\mathcal{A}$ is self-adjoint, then γ can be so chosen that $\hat{\delta} \in \mathbb{S}_2$.

Proof. The first part follows immediately from Lemma 24.

Let then $\mathcal{A}$ be self-adjoint and assume c is already normalized. If $\delta \notin \mathbb{S}_2$ then it is of the form $\delta = \xi + \beta\tau$ with a real ξ and a complex β as by the proof above $\delta^* = \delta$. However, the similarity in (17) preserves the normalization of c

$$(ce^{i\theta/2}, ce^{i\theta/2}) = e^{-i\theta/2}(c, c)e^{i\theta/2} = 1$$

completing the proof. □

Finally, we formulate the analogue of the following result from linear algebra: left and right eigenvectors associated with different eigenvalues are orthogonal.

Theorem 36. *Assume that $\mathcal{A}c = c\gamma$, $d^*\mathcal{A} = \delta d^*$. If $\{\gamma, \delta\}$ is a regular pair, then c and d are orthogonal.*

Proof. From $\mathcal{A}c = c\gamma$ we obtain $d^*\mathcal{A}c = d^*c\gamma$. But since $d^*\mathcal{A} = \delta d^*$ we further have $d^*c\gamma = \delta d^*c$ or

$$(d^*c)\gamma - \delta(d^*c) = 0.$$

Since the pair is regular, $d^*c = 0$. □

5 Sylvester's equation

5.1 Preliminaries

Consider the equation

$$\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{B} = \mathcal{C} \tag{51}$$

where $\mathcal{A}, \mathcal{B} \in \mathcal{M}_n$ are given. We are interested in knowing when (51) has a unique solution $\mathcal{X} \in \mathcal{M}_n$ for all right hand sides $\mathcal{C} \in \mathcal{M}_n$. In other words, when the pair $\{\mathcal{A}, \mathcal{B}\}$ is regular, see Definition 16

The corresponding $\mathbb{C}-$linear Sylvester's equation

$$AX - XB = C$$

has a unique solution $X \in M_n$ for all $C \in M_n$ if and only if

$$\sigma(A) \cap \sigma(B) = \emptyset, \tag{52}$$

see [3].

The spectrum of a matrix generalizes in several ways to real linear operators. For an exposition, see [6]. The *eigenvalues* consist of singularities of the *resolvent* $\mathcal{R}(\lambda, \mathcal{A}) = (\lambda - \mathcal{A})^{-1}$:

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \ker(\lambda - \mathcal{A}) \neq \{0\}\}.$$

The set of eigenvalues is given by an algebraic curve of degree at most $2n$. The *proper values* $\sigma_c(\mathcal{A})$ are the singularities of the *cosolvent* which for large values of λ is given by

$$\mathcal{C}(\lambda, \mathcal{A}) = \sum_{j=0}^{\infty} \lambda^{-1-j} \mathcal{A}^j.$$

Since the cosolvent is analytic, we extend it and it turns out that (in the finite dimensional case) the cosolvent can be extended to the whole plane as a meromorphic function with a finite number of poles:

$$\sigma_c(\mathcal{A}) = \{\zeta \in \mathbb{C} \mid \zeta \text{ is a pole of } \mathcal{C}(\lambda, \mathcal{A})\}.$$

While $\sigma(\mathcal{A})$ can be empty, $\sigma_c(\mathcal{A})$ is never empty, containing at most $2n$ points.

The third generalization of eigenvalues would be *eigencirclets:*

$$\sigma_{\mathbb{S}}(\mathcal{A}) = \{\gamma \in \mathbb{S} \mid \gamma \text{ is an eigencirclet of } \mathcal{A}\}.$$

Finally, one can write a real linear operator $\mathcal{A} = A + B\tau \in \mathcal{M}_n$ as a matrix, e.g. in the form

$$\psi(\mathcal{A}) = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}$$

or as a real matrix $\phi(\mathcal{A}) = M$ with M as in (4). Now one can consider $\sigma(\psi(\mathcal{A}))$ which consists of $2n$ eigenvalues, counted with multiplicities. We have always $\sigma_c(\mathcal{A}) \subset \sigma(\psi(\mathcal{A}))$. Since $\psi(\mathcal{A})$ and $\phi(\mathcal{A})$ are $\mathbb{C}-$unitarily similar under

$$E = \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}$$

they have the same eigenvalues and can alternatively consider $\sigma(\phi(\mathcal{A}))$.

Proposition 37. *If $\{\mathcal{A}, \mathcal{B}\}$ is regular, so is $\{\mathcal{S}\mathcal{A}\mathcal{S}^{-1}, \mathcal{T}\mathcal{B}\mathcal{T}^{-1}\}$.*

Proof. Multiplying (51) from left with $\mathcal{S}$ and from right with $\mathcal{T}^{-1}$ yields

$$(\mathcal{S}\mathcal{A}\mathcal{S}^{-1})\mathcal{Y} - \mathcal{Y}(\mathcal{T}\mathcal{B}\mathcal{T}^{-1}) = \mathcal{E}$$

with $\mathcal{Y} = \mathcal{S}\mathcal{X}\mathcal{T}^{-1}$ and $\mathcal{E} = \mathcal{S}\mathcal{C}\mathcal{T}^{-1}$. □

We conclude that the generalization of (52) has the property that it is invariant under $\mathbb{R}-$linear similarities.

Proposition 38. *$\sigma(\mathcal{A})$, $\sigma_c(\mathcal{A})$ are invariant only under $\mathbb{C}-$linear similarities, while $\sigma_{\mathbb{S}}(\mathcal{A})$ and $\sigma(\psi(\mathcal{A}))$ are invariant under general $\mathbb{R}-$linear similarities.*

Proof. For eigencirclets this is in Proposition 27, while the other claims are in [6]. □

Example 16. Observe that since $\sigma_c(\mathcal{A}) \subset \sigma(\psi(\mathcal{A}))$, the proper values are "almost" invariant. However, if $\mathcal{A} = i$ we have $\tau^{-1} i \tau = -i$, and

$$\{i\} = \sigma_c(i) \neq \sigma_c(\tau^{-1} i \tau) = \{-i\}.$$

Since $\psi(i) = \begin{pmatrix} i & 0 \\ 0 & \bar{i} \end{pmatrix}$ we have $\sigma(\psi(i)) = \{i, -i\}$. Notice further, that $i1 = 1i$ and $i\tau = \tau(-1)$ and $\sigma_{\mathbb{S}}(i) = \{i, -i\}$.

5.2 Sylvester's equation for circlets

Let us denote

$$\delta_1 = \xi_0 + i\xi_1 + \xi_2\tau + i\xi_3\tau,$$

$$\delta_2 = \eta_0 + i\eta_1 + \eta_2\tau + i\eta_3\tau,$$

and

$$\gamma = \alpha_0 + i\alpha_1 + \alpha_2 + i\alpha_3\tau.$$

Substituting these into

$$\delta_1\gamma - \gamma\delta_2 = \nu$$

yields an equivalent system in $\mathbb{R}^4$

$$Ma = b$$

where $a = (\alpha_0, \dots, \alpha_3)^t$ is the unknown vector and

$$M = \begin{pmatrix} \xi_0 - \eta_0 & -\xi_1 + \eta_1 & \xi_2 - \eta_2 & \xi_3 - \eta_3 \\ \xi_1 - \eta_1 & \xi_0 - \eta_0 & \xi_3 + \eta_3 & -\xi_2 - \eta_2 \\ \xi_2 - \eta_2 & \xi_3 + \eta_3 & \xi_0 - \eta_0 & -\xi_1 - \eta_1 \\ \xi_3 - \eta_3 & -\xi_2 - \eta_2 & \xi_1 + \eta_1 & \xi_0 - \eta_0 \end{pmatrix}.$$

So, we need to calculate when $\det M \neq 0$. With some labor one finds

$$\det M = (\xi_0 - \eta_0)^2[(\xi_0 - \eta_0)^2 + 2(\xi_1^2 + \eta_1^2 - \xi_2^2 - \eta_2^2 - \xi_3^2 - \eta_3^2)] + (\xi_1^2 - \eta_1^2 - \xi_2^2 + \eta_2^2 - \xi_3^2 + \eta_3^2)^2.$$

However, by Theorem 10 and Proposition 37 we may assume that $\delta_j \in \mathbb{S}_k$ for j=1,2 and k=1,2 or 3. In particular, $\xi_3 = \eta_3 = 0$. Thus, we put

$$M_0 = \begin{pmatrix} \xi_0 - \eta_0 & -\xi_1 + \eta_1 & \xi_2 - \eta_2 & 0 \\ \xi_1 - \eta_1 & \xi_0 - \eta_0 & 0 & -\xi_2 - \eta_2 \\ \xi_2 - \eta_2 & 0 & \xi_0 - \eta_0 & -\xi_1 - \eta_1 \\ 0 & -\xi_2 - \eta_2 & \xi_1 + \eta_1 & \xi_0 - \eta_0 \end{pmatrix}.$$

Now, expanding we obtain

$$\det M_0 = (\xi_0 - \eta_0)^2[(\xi_0 - \eta_0)^2 + 2(\xi_1^2 + \eta_1^2 - \xi_2^2 - \eta_2^2)] + (\xi_1^2 - \eta_1^2 - \xi_2^2 + \eta_2^2)^2. \quad (53)$$

Lemma 39. *The pair* $\{\xi_0 + i\xi_1 + \tau\xi_2,\ \eta_0 + i\eta_1 + \tau\eta_2\}$ *is regular if and only if* $\det M_0 \neq 0$.

Observe that (53) stays invariant if we change the order of the pair, so regularity is independent of the order in which we list the pair. Thus, there are six different cases we need to check.

Case 1.1 If $\delta_1, \delta_2 \in \mathbb{C}$, then

$$\det M_0 = (\xi_0 - \eta_0)^2[(\xi_0 - \eta_0)^2 + 2(\xi_1^2 + \eta_1^2)] + (\xi_1^2 - \eta_1^2)^2$$

and the pair is regular if $\delta_1 \neq \delta_2$ and $\delta_1 \neq \bar{\delta}_2$, that is, $\sigma(\psi(\delta_1)) \cap \sigma(\psi(\delta_2)) = \emptyset$.

Example 17. Let

$$\mathcal{A} = \begin{pmatrix} i & \tau \\ 0 & -i \end{pmatrix}.$$

Now

$$\mathcal{A}^{4k} = \begin{pmatrix} 1 & 4ki\tau \\ 0 & 1 \end{pmatrix}$$

and we see that $\mathcal{A}$ cannot be similar to a diagonal operator.

Case 1.2 If $\delta_1 \in \mathbb{C}$, $\delta_2 \in \mathbb{S}_2$. Then

$$\begin{aligned}\det M_0 =&(\xi_0-\eta_0)^2[(\xi_0-\eta_0)^2+2(\xi_1^2-\eta_2^2)]+(\xi_1^2+\eta_2^2)^2\\ =&[(\xi_0-\eta_0)^2+(\xi_1^2-\eta_2^2)]^2+4\xi_1^2\eta_2^2.\end{aligned}$$

Thus, $\det M_0 = 0$ if and only if $\xi_1 = 0$ and $(\xi_0-\eta_0)^2 = \eta_2^2$. But this means that $\{\xi_0\} = \sigma(\psi(\delta_1)) \subset \sigma(\psi(\delta_2)) = \{\eta_0+\eta_2,\ \eta_0-\eta_2\}$.

Example 18. Let

$$\mathcal{A} = \begin{pmatrix} \tau & 1 \\ 0 & 1 \end{pmatrix}.$$

Now

$$\mathcal{A}^{2k} = \begin{pmatrix} 1 & k(1+\tau) \\ 0 & 1 \end{pmatrix}$$

and we see again, that $\mathcal{A}$ cannot be similar to a diagonal operator.

Case 1.3 If $\delta_1 \in \mathbb{C}$, $\delta_2 \in \mathbb{S}_3$. Then $\xi_2 = 0$ and $\eta_1 = \eta_2$, and

$$\det M_0 = (\xi_0-\eta_0)^2[(\xi_0-\eta_0)^2+2\xi_1^2]+\xi_1^4 = 0$$

if and only if $\xi_0 = \eta_0$ and $\delta_1 \in \mathbb{R}$. Since for general $\delta_1 \in \mathbb{C}$ we have $\sigma(\psi(\delta_1)) = \{\xi_0+i\xi_1,\ \xi_0+i\xi_1\}$ and for $\delta_2 \in \mathbb{S}_3$, $\sigma(\psi(\delta_2)) = \{\eta_0+\eta_2,\ \xi_0-\eta_2\}$ we conclude that the pair is regular if and only if $\sigma(\psi(\delta_1)) \cap \sigma(\psi(\delta_2)) = \emptyset$.

Case 2.2 Assume both $\delta_j \in \mathbb{S}_2$. Then

$$\begin{aligned}\det M_0 =&(\xi_0-\eta_0)^2[(\xi_0-\eta_0)^2-2(\xi_2^2+\eta_2^2)]+(\xi_2^2-\eta_2^2)^2\\ =&[(\xi_0-\eta_0)^2-(\xi_2^2+\eta_2^2)]^2-4\xi_2^2\eta_2^2.\end{aligned}$$

Now $\det M_0 = 0$ if

$$\begin{aligned}(\xi_0-\eta_0)^2-(\xi_2^2+\eta_2^2) =& 2\xi_2\eta_2, \quad \text{or}\\ =&-2\xi_2\eta_2\end{aligned}$$

which in turn means that at least one of the following equations hold

$$\begin{aligned}\xi_0+\xi_2 &= \eta_0+\eta_2\\ \xi_0-\xi_2 &= \eta_0+\eta_2\\ \xi_0+\xi_2 &= \eta_0-\eta_2\\ \xi_0-\xi_2 &= \eta_0-\eta_2.\end{aligned}$$

But as $\sigma(\psi(\xi_0+\xi_2\tau)) = \{\xi_0+\xi_2,\ \xi_0-\xi_2\}$ we again have regularity if and only if $\sigma(\psi(\delta_1)) \cap \sigma(\psi(\delta_2)) = \emptyset$.

Case 2.3 Assume $\delta_1 \in \mathbb{S}_2$ and $\delta_2 \in \mathbb{S}_3$. Then

$$\det M_0 = (\xi_0-\eta_0)^2[(\xi_0-\eta_0)^2-2\xi_2^2]+\xi_2^4 = ((\xi_0-\eta_0)^2-\xi_2^2)^2.$$

This vanishes if and only if $\sigma(\psi(\delta_1)) = \{\xi_0+\xi_2,\ \xi_0-\xi_2\}$ and $\sigma(\psi(\delta_2)) = \{\eta_0\}$ share a point.

Case 3.3 Assume finally that both $\delta_j \in \mathbb{S}_3$. Then

$$\det M_0 = (\xi_0 - \eta_0)^4,$$

which is nonzero exactly when $\sigma(\psi(\delta_1)) \cap \sigma(\psi(\delta_2)) = \emptyset$. Collecting all the cases we have proved:

Theorem 40. *The pair $\{\delta_1, \delta_2\}$ of circlets is regular if and only if*

$$\sigma(\psi(\delta_1)) \cap \sigma(\psi(\delta_2)) = \emptyset.$$

Remark 19. Let us recall $\sigma(\psi(\gamma))$. For $\gamma = \alpha + \beta\tau = \xi_0 + i\xi_1 + \xi_2\tau + \xi_3 i\tau = \xi_0 + \gamma_0$ we have

$$\psi(\gamma) = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

Thus

$$\sigma(\psi(\gamma)) = \{\xi_0 + \sqrt{|\beta|^2 - \xi_1^2},\ \ \xi_0 - \sqrt{|\beta|^2 - \xi_1^2}\} = \{\xi_0 + \sqrt{\gamma_0^2},\ \ \xi_0 - \sqrt{\gamma_0^2}\}.$$

5.3 Diagonalization

We can start from the real linear Schur decomposition. By [2], given $\mathcal{A} \in \mathcal{M}_n$ there exist an $\mathbb{R}$-linear unitary $\mathcal{U}$ and an upper triangular $\mathcal{T}$ such that

$$\mathcal{A} = \mathcal{U}\mathcal{T}\mathcal{U}^*.$$

Let $\mathcal{T} = \mathcal{D} + \mathcal{N}$ where $\mathcal{D} = diag(\delta_k)$ and $\mathcal{N} = (\nu_{ij})$ is strictly upper triangular.

Theorem 41. *If for all $k, l = 1, \ldots, n$ with $k \neq l$ we have*

$$\sigma(\psi(\delta_k)) \cap \sigma(\psi(\delta_l)) = \emptyset,$$

then there exists an invertible $\mathcal{S} \in \mathcal{M}_n$ such that

$$\mathcal{A} = \mathcal{S}\mathcal{D}\mathcal{S}^{-1}.$$

In particular, if $\{e_j\}$ denotes the standard basis in $\mathbb{C}^n$, then

$$\mathcal{A}c_j = c_j\delta_j \ \ for \ \ j = 1, \ldots, n$$

with $c_j = \mathcal{S}e_j$ and $diag(\delta_j) = \mathcal{D}$.

Proof. Consider first diagonalizing a 2×2 triangular operator

$$\begin{pmatrix} \delta_k & \nu \\ 0 & \delta_l \end{pmatrix}.$$

Let γ be such that it solves the Sylvester's equation

$$\delta_k\gamma - \gamma\delta_l = \nu.$$

Such a unique γ exists by Theorem 40. But then

$$\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_k & \nu \\ 0 & \delta_l \end{pmatrix} \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \delta_k & 0 \\ 0 & \delta_l \end{pmatrix}.$$

A general $n \times n$ triangular real linear operator can likewise be diagonalized using repeatedly similarities by operators of the form

$$I + \Gamma_{kl}$$

where Γ_{kl} $(k < l)$ has just one nonzero element γ_{kl}, aimed to remove ν_{kl} from the strictly upper triangular part $\mathcal{N}$. □

5.4 Solvability of Sylverster's equation

We consider now the general Sylvester's equation

$$\mathcal{A}\mathcal{X} - \mathcal{X}\mathcal{B} = \mathcal{C} \tag{54}$$

where $\mathcal{A} \in \mathcal{M}_n$, $\mathcal{B} \in \mathcal{M}_p$ and $\mathcal{C} \in \mathcal{M}_{n,p}$ are given and $\mathcal{X} \in \mathcal{M}_{n,p}$ is the unknown.

Theorem 42. *The equation* (54) *has a unique solution* $\mathcal{X} \in \mathcal{M}_{n,p}$ *for all* $\mathcal{C} \in \mathcal{M}_{n,p}$ *if and only if*

$$\sigma(\psi(\mathcal{A})) \cap \sigma(\psi(\mathcal{B})) = \emptyset. \tag{55}$$

Proof. We may assume that $\mathcal{A}$ and $\mathcal{B}$ are upper triangular operators. In fact, Proposition 37 holds as such also in the case $n \neq p$ and so the real linear Schur decomposition permits the assumption. Suppose now that we denote $\mathcal{A} = (\gamma_{ij})$, $\mathcal{B} = (\delta_{ij})$, $\mathcal{C} = (\nu_{ij})$ and $\mathcal{X} = (\chi_{ij})$. Then (54) can be written elementwise as

$$\gamma_{ii}\chi_{ij} - \chi_{ij}\delta_{jj} = -\sum_{k=i+1}^{n} \gamma_{ik}\chi_{kj} + \sum_{l=1}^{j-1} \chi_{il}\delta_{lj} + \nu_{ij}. \tag{56}$$

But the equations

$$\gamma_{ii}\chi - \chi\delta_{jj} = \nu \tag{57}$$

are all solvable for all ν exactly when (55) holds. This follows from Theorem 40 and from the fact that for a triangular $\mathcal{A}$ its diagonal determines the eigenvalues of $\psi(\mathcal{A})$. So, if (55) holds, then (56) allows one to solve for χ_{ij}, setting first $i = n$, and solving for $j = 1, \ldots, p$. Then set $i = n - 1$ and again let $j = 1, \ldots, p$ and so on.

Assuming that (55) does not hold, then there are indeces i, j, assumed to be the first ones in the order applied above in the backsolve of (56) such that

$$\sigma(\psi(\gamma_{ii})) \cap \sigma(\psi(\delta_{jj})) \neq \emptyset.$$

Let now all other $\nu_{kl} = 0$ but $\nu_{ij} = \nu$ such that (57) has no solution. Then the right hand side of (56) reduces to ν and by arrangement the equation cannot be solved. Thus (55) is both sufficient and necessary. □

6 Operator norms

6.1 Notation and main result

We have defined the operator norm for $\mathcal{A} \in \mathcal{M}_{p,n}$ in Definition 4 as

$$\|\mathcal{A}\| = \sup_{\|x\|=1} \|[\mathcal{A}x]\|.$$

Here $x \in \mathbb{C}^n$ and the norms are the usual norms in $\mathbb{C}^n$ and in $\mathbb{C}^p$. Additionally, we can set, with $c \in \mathbb{S}^n$

$$\|\mathcal{A}\|_{\mathbb{S}} = \sup_{\|c\|=1} \|\mathcal{A}c\|.$$

Furthermore, in Definition 7 we introduced also $(,)_{\mathbb{C}}$ and $(,)_{\mathbb{R}}$. These induce the same norm into $\mathbb{S}^n$ as with $c = a + b\tau$

$$(c, c)_{\mathbb{C}} = (c, c)_{\mathbb{R}} = a^*a + b^*b.$$

Denoting $\|c\|_{\mathbb{C}} = \sqrt{(c, c)_{\mathbb{C}}}$ we can also define

$$\|\mathcal{A}\|_{\mathbb{C}} = \sup_{\|c\|_{\mathbb{C}}=1} \|\mathcal{A}c\|_{\mathbb{C}}.$$

Theorem 43. *For $\mathcal{A} \in \mathcal{M}_{p,n}$ we have*

$$\|\mathcal{A}\| = \|\mathcal{A}\|_{\mathbb{S}} = \|\mathcal{A}\|_{\mathbb{C}}.$$

Remark 20. Note however, that with $x \in \mathbb{C}^n$ we can have

$$\sup_{\|x\|_{\mathbb{C}}=1} \|\mathcal{A}x\|_{\mathbb{C}} < \|\mathcal{A}\|.$$

In fact, with $\mathcal{A} = 1 + \tau$ this inequality takes the form $\sqrt{2} < 2$.

Proof. To prove Theorem 43 we consider in the next section the case $n = p$. In particular, we show that $\mathcal{M}_n$ is a real C^*-algebra. Since the norm in a real C^*-algebra is unique, we conclude that the different orthogonalities in $\mathbb{S}^n$ all yield the same operator norm. So, suppose now that the claim holds for square operators. Then it immediately holds in the general case, as adding a suitable amount of zero rows or columns to the matrix does not affect the norms. $\square$

6.2 $\mathcal{M}_n$ as a real C^*-algebra

We show that $\mathcal{M}_n$ is a real C^*-algebra.

Here our general reference on real C^*-algebras is [11]. Let in a real Banach algebra "*" be an operation such that $a^{**} = (a^*)^* = a$ and $(ab)^* = b^*a^*$ hold for all $a, b \in A$.

Definition 17. A real Banach algebra A is called a real C^*-algebra if additionally it is hermitian and $\|a^*a\| = \|a\|^2$ for all $a \in A$.

Here A is called hermitian if the spectrum of any element satisfying $a = a^*$ is real - and by spectrum one means the spectrum of the element considered as an element in the standard complexification of the algebra.

Lemma 44. *If B is a real C^*-algebra, then there is a unique C^*-norm on $M_n(B)$ such that $M_n(B)$ is a real C^*-algebra.*

Proof. See Proposition 5.1.10 in [11]. □

Lemma 45. *$\mathcal{M}_n$ is hermitian.*

Proof. Obviously, $M_{2n}(\mathbb{C})$ is hermitian. Further, $\mathcal{M}_n$ as a real algebra can be viewed as the real algebra $M_{2n}(\mathbb{R})$ consisting of matrices of the form $M = \phi(\mathcal{A})$. Since $\phi(\mathcal{A})^* = \phi(\mathcal{A}^*)$ we conclude that $\mathcal{M}_n$ is hermitian. □

Lemma 46.

$$\|\mathcal{A}^*\mathcal{A}\| = \|\mathcal{A}\|^2.$$

Proof. We have

$$\|[Ax]\|^2 = (Ax, Ax) + (Ax, B\bar{x}) + (B\bar{x}, Ax) + (B\bar{x}, B\bar{x}).$$

Let x_0 be a unit vector in $\mathbb{C}^n$ maximizing $\|[Ax]\|$, and suppose u is orthogonal to x_0 in $\mathbb{C}^n$. Then with $x = x_0 + \varepsilon u$ we obtain

$$\begin{aligned}\|[Ax]\|^2 = \|[Ax_0]\|^2 &\\ &+ 2\operatorname{Re}\{\bar{\varepsilon}(A^*Ax_0, u) + \bar{\varepsilon}(A^*B\bar{x}_0, u) + \bar{\varepsilon}(B^t\bar{A}\bar{x}_0, u) + \bar{\varepsilon}(B^t\bar{B}x_0, u)\} \\ &+ o(\varepsilon).\end{aligned}$$

Thus, $A^*Ax_0 + A^*B\bar{x}_0 + B^t\bar{A}\bar{x}_0 + B^t\bar{B}x_0$ is orthogonal to u and since u was arbitrary in the oorthogonal complement of x_0 we conclude that there exists a $\lambda_0 \in \mathbb{C}$ such that

$$A^*Ax_0 + A^*B\bar{x}_0 + B^t\bar{A}\bar{x}_0 + B^t\bar{B}x_0 = \lambda_0 x_0. \tag{58}$$

On the other hand, for all x

$$\|[\mathcal{A}x]\|^2 = \operatorname{Re}\{([\mathcal{A}^*\mathcal{A}x], x)\} \tag{59}$$

where $\mathcal{A}^*\mathcal{A} = A^*A + B^t\bar{B} + (A^*B + B^t\bar{A})\tau$. With x_0 we obtain from (59)

$$\|\mathcal{A}\|^2 = \|[\mathcal{A}x_0]\|^2 = \operatorname{Re}\{([\mathcal{A}^*\mathcal{A}x_0], x_0)\} = \operatorname{Re}\lambda_0 \tag{60}$$

while (58) gives

$$\|\mathcal{A}^*\|\|\mathcal{A}\| \geq \|\mathcal{A}^*\mathcal{A}\| \geq \|[\mathcal{A}^*\mathcal{A}x_0]\| = |\lambda_0|. \tag{61}$$

From (60) and (61) we conclude that $\|\mathcal{A}\| \leq \|\mathcal{A}^*\|$ and since $\mathcal{A} = (\mathcal{A}^*)^*$ we also have $\|\mathcal{A}^*\| \leq \|\mathcal{A}\|$. It follows that $\|\mathcal{A}^*\mathcal{A}\| = \|\mathcal{A}\|^2 = \lambda_0$. □

Corollary 47. $\|\mathcal{A}\|^2$ *is an eigenvalue of* $\mathcal{A}^*\mathcal{A}$.

Lemma 48.

$$\|\mathcal{A}^*\mathcal{A}\|_{\mathbb{S}} = \|\mathcal{A}\|_{\mathbb{S}}^2.$$

Proof. In $\mathbb{S}^n$ we have $\|(d,d)\| = \|d\|^2$. Thus

$$\begin{aligned} \|\mathcal{A}\|_{\mathbb{S}}^2 &= \sup_{\|c\|=1} \|(\mathcal{A}c, \mathcal{A}c)\| \\ &= \sup_{\|c\|=1} \|(\mathcal{A}^*\mathcal{A}c, c)\| \\ &\leq \sup_{\|c\|=1} \|\mathcal{A}^*\mathcal{A}c\| = \|\mathcal{A}^*\mathcal{A}\|_{\mathbb{S}}. \end{aligned}$$

On the other hand, still by Theorem 19,

$$\|c\| = \sup_{\|d\|=1} \|(c,d)\|$$

and so

$$\begin{aligned} \|\mathcal{A}^*\mathcal{A}\|_{\mathbb{S}} &= \sup_{\|c\|=1, \|d\|=1} \|(\mathcal{A}^*\mathcal{A}c, d)\| \\ &= \sup_{\|c\|=1, \|d\|=1} \|(\mathcal{A}c, \mathcal{A}d)\| \\ &\leq \sup_{\|c\|=1} \|\mathcal{A}c\| \sup_{\|d\|=1} \|\mathcal{A}d\| = \|\mathcal{A}\|^2. \end{aligned}$$

□

Lemma 49.

$$\|\mathcal{A}^*\mathcal{A}\|_{\mathbb{C}} = \|\mathcal{A}\|_{\mathbb{C}}^2.$$

Proof. Essentially one can repeat the proof of the previous lemma, with (c,d) replaced by $(c,d)_{\mathbb{C}} = \mathrm{Lin}(c,d) = e^*a + f^t\bar{b}$ as this clearly satisfies the usual requirements of a (complex valued) inner product. □

7 Positive operators

7.1 Positivity

In (real) C^*-algebras positive elements are defined as those elements satisfying $a^* = a$ which have real nonnegative spectra. Since eigenvalues of $\phi(\delta)$ with $\delta^* = \delta$ are nonnegative if and only if δ is a positive circlet, we can define positive operators as follows.

Definition 18. $\mathcal{A} \in \mathcal{M}_n$ is positive if it is self-adjoint and has positive eigencirclets.

Recall that $\delta = \xi + \eta\tau$ is positive if $\xi \geq |\eta|$.

We start with two Lemmas.

Lemma 50. *Let $\delta = \xi + \eta\tau$ be such that ξ is real. Then with $\gamma = \alpha + \beta\tau$ we have*

$$(Lin(\gamma^*\delta\gamma))^2 - |\,Con(\gamma^*\delta\gamma)|^2 = (\xi^2 - |\eta|^2)(|\alpha|^2 - |\beta|^2)^2.$$

Proof. Let $\alpha = |\alpha|e^{i\theta}, \beta = |\beta|e^{i\varphi}, \eta = |\eta|e^{i\psi}$ and put $\varkappa = \theta + \varphi - \psi$. Then

$$\begin{aligned}\gamma^*\delta\gamma =&(\bar\alpha + \beta)(\xi + \eta\tau)(\alpha + \beta\tau)\\ =&\xi(|\alpha|^2 + |\beta|^2) + 2\,\mathrm{Re}(\overline{\alpha\beta}\eta) + (2\xi\bar\alpha\beta + \bar\alpha^2\eta + \beta^2\bar\eta)\tau\\ =&\xi(|\alpha|^2 + |\beta|^2) + 2|\alpha\beta\eta|cos\varkappa\\ &+ (2\xi|\alpha\beta|e^{i(\varphi-\psi)} + |\eta|(|\alpha|^2e^{-2i\theta+i\psi} + |\beta|^2e^{2i\varphi-i\psi}))\tau\\ =&\,\mathrm{Lin}(\gamma^*\delta\gamma) + \mathrm{Con}(\gamma^*\delta\gamma)\tau.\end{aligned}$$

But

$$|\,\mathrm{Con}(\gamma^*\delta\gamma)| = |2\xi|\alpha\beta| + |\eta|(|\alpha|^2e^{-i\varkappa} + |\beta|^2e^{i\varkappa})|$$

and so

$$(\mathrm{Lin}(\gamma^*\delta\gamma))^2 - |\,\mathrm{Con}(\gamma^*\delta\gamma)|^2 = (\xi^2 - |\eta|^2)(|\alpha|^2 - |\beta|^2)^2.$$

□

Lemma 51. *$Lin(\mathcal{A}c, c) \in \mathbb{R}$ for all $c \in \mathbb{S}^n$ if and only if $\mathcal{A} = \mathcal{A}^*$.*

Proof.

$$\begin{aligned}\mathrm{Lin}(\mathcal{A}c, c) =&\,\mathrm{Lin}((A + B\tau)(a + b\tau), a + b\tau)\\ =&(Aa, a) + \overline{(Ab, b)} + (B\bar b, a) + \overline{(B^t\bar b, a)}.\end{aligned}$$

If $\mathcal{A}$ is self-adjoint, i.e. $A = A^*$, $B = B^t$, then clearly $\mathrm{Lin}(\mathcal{A}c, c)$ is real.

Reversely, to see that $\mathcal{A}$ has to be self-adjoint, choose first $b = 0$ to conclude from $\mathrm{Lin}(\mathcal{A}c, c) = (Aa, a) \in \mathbb{R}$ that $A = A^*$. Then $\mathrm{Lin}(\mathcal{A}c, c) \in \mathbb{R}$ if and only if

$$(B\bar b, a) + \overline{(B^t\bar b, a)} \in \mathbb{R}$$

which clearly happens exactly when $B = B^t$. □

Theorem 52. *Assume $\mathcal{A} \in \mathcal{M}_n$. Then the following are equivalent:*

(i) $\mathcal{A}$ is positive

(ii) $(\mathcal{A}c, c)$ is a positive circlet for all $c \in \mathbb{S}^n$

(iii) $Lin(\mathcal{A}c, c) \geq |\,Con(\mathcal{A}c, c)|$ for all $c \in \mathbb{S}^n$

(iv) $B = B^t$ and $(Ax, x) \geq |(B\bar x, x)|$ for all $x \in \mathbb{C}^n$.

Proof. Assume (i). Then there exist a unitary $\mathcal{U}$ and a positive diagonal $\mathcal{D}$ such that $\mathcal{A} = \mathcal{U}\mathcal{D}\mathcal{U}^*$. Suppose

$$c = \Sigma_{j=1}^n \mathcal{U}e_j\gamma_j.$$

and denote with δ_j the positive circlets on the diagonal of $\mathcal{D}$. Then

$$\begin{aligned}(\mathcal{A}c, c) =& (\mathcal{D}\mathcal{U}^* c, \mathcal{U}^* c)\\ =&\Sigma_j(\delta_j e_j\gamma_j, e_j\gamma_j)\\ =&\Sigma_j\gamma_j^*\delta_j\gamma_j.\end{aligned}$$

But by Lemma 50 each $\gamma_j^*\delta_j\gamma_j$ is positive and so is their sum. Thus, (ii) follows. Clearly, (ii) and (iii) are equivalent by the definition of a circlet being positive.

Suppose that (iii) holds. By Lemma 51 $\mathcal{A}$ is self-adjoint and (iv) follows from (iii) by choosing $c = x$.

Finally, if (iv) holds, we conclude first that $A = A^*$ and so $\mathcal{A}$ is self-adjoint. We need to show that $\sigma(\psi(\mathcal{A})) \subset \mathbb{R}$. But

$$\sigma(\psi(\mathcal{A})) = \sigma(\mathcal{A}) \cap \mathbb{R}.$$

Choose therefore $\xi \in \sigma(\mathcal{A}) \cap \mathbb{R}$ and and a nonzero $x \in \mathbb{C}^n$ such that

$$[\mathcal{A}x] = Ax + B\bar{x} = \xi x.$$

Then (iv) and

$$(Ax, x) + (B\bar{x}, x) = \xi\|x\|^2$$

imply $\xi \geq 0$, and (i) follows. □

7.2 Positive square root of a positive operator

By Proposition 8 every positive circlet has a unique positive square root. This generalizes immediately to operators. In fact, since a positive operator is self-adjoint by definition, we can assume that it is given in the form $\mathcal{A} = \mathcal{U}\mathcal{D}\mathcal{U}^*$ where the diagonal elements δ_j of $\mathcal{D}$ are positive. If ε_j denotes the unique positive square root of δ_j and $\mathcal{E}$ the corresponding diagonal operator, then clearly $\mathcal{B} = \mathcal{U}\mathcal{E}\mathcal{U}^*$ is positive and $\mathcal{B}^2 = \mathcal{A}$.

Proposition 53. *If $\mathcal{A}$ is positive, then there exists a unique positive $\mathcal{B}$ such that $\mathcal{B}^2 = \mathcal{A}$.*

Proof. What remains is to conclude that $\mathcal{B}$ is unique. So, let $\mathcal{C}$ be arbitrary self-adjoint such that $\mathcal{C}^2 = \mathcal{A}$. Since $\mathcal{C}$ is self-adjoint it has a full set of orthonormal eigenvectorands v_j

$$\mathcal{C}v_j = v_j\gamma_j.$$

Thus,

$$\mathcal{A}v_j = \mathcal{C}^2 v_j = v_j\gamma_j^2$$

and so, by assumption, γ_j^2 is positive. Since it has exactly one positive square root there is only one positive square root $\mathcal{B}$ of $\mathcal{A}$. □

Remark 21. Recall that in general, the number of square roots may vary. For example τ has no square roots while 1 has four, namely $1, -1, \tau, -\tau$ and 0 has infinitely many, namely $\xi(i + \tau)$, $\eta(-i + \tau)$ with $\xi, \eta \in \mathbb{R}$.

7.3 Polar decomposition and SVD

Assume given an arbitrary $\mathcal{A} \in \mathcal{M}_n$. Since (c, c) is positive for all vectorands c so is $(\mathcal{A}^*\mathcal{A}c, c) = (\mathcal{A}c, \mathcal{A}c)$ and we conclude, by Theorem 52 (i), that $\mathcal{A}^*\mathcal{A}$ is always positive.

Definition 19. We denote by $|\mathcal{A}|$ the unique positive square root of $\mathcal{A}^*\mathcal{A}$.

It follows immediately that $\||\mathcal{A}|c\| = \|\mathcal{A}c\|$ for all c and so

$$|\mathcal{A}|c \mapsto \mathcal{A}c$$

is an isometry on the range of $|\mathcal{A}|$. Denoting this isometry by $\mathcal{U}$ we obtain for all c

$$\mathcal{U}|\mathcal{A}|c = \mathcal{A}c.$$

Theorem 54. *Every $\mathcal{A} \in \mathcal{M}_n$ has a polar decomposition*

$$\mathcal{A} = \mathcal{U}|\mathcal{A}|$$

where $\mathcal{U}$ is an isometry. If $\mathcal{A}$ is invertible, then $\mathcal{U}$ is unitary and the polar decomposition is unique.

From *polar decomposition* we can now move over to a *singular value decomposition, SVD* in a standard manner.

Write $|\mathcal{A}|$ as $\mathcal{V}\mathcal{S}\mathcal{V}^*$ where $\mathcal{V}$ is unitary and $\mathcal{S}$ diagonal. Then

$$\mathcal{A} = \mathcal{U}|\mathcal{A}| = \mathcal{U}\mathcal{V}\mathcal{S}\mathcal{V}^* = \mathcal{W}\mathcal{S}\mathcal{V}^*.$$

Corollary 55. *Every $\mathcal{A} \in \mathcal{M}_n$ has a singular value decomposition*

$$\mathcal{A} = \mathcal{W}\mathcal{S}\mathcal{V}^*$$

where $\mathcal{W}$ is an isometry, $\mathcal{S}$ is positive and diagonal and $\mathcal{V}$ is unitary.

Remark 22. Although the polar decomposition is the natural generalization from the $\mathbb{C}$-linear concept, situation is more complicated with the SVD. The main use of SVD is in approximation of operators and ordering the singular values decreasingly one obtains best low rank approximations by just truncating. With real linear operators the eigencirclets can be combined in many ways from the eigenvalues of $\psi(|\mathcal{A}|)$ and consequently there are many ways to order them. Denoting the columns of $\mathcal{W}$ by w_j, elements of $\mathcal{S}$ by γ_j and the columns of $\mathcal{V}$ by d_j, then Corollary 55 points to approximations of $\mathcal{A}$ of the form

$$\Sigma_j c_j d_j^*,$$

where $c_j = w_j\gamma_j$. Another type of approximation which is truly particular for real linear operators, is obtained by restricting one of the vectorands c_j, d_j to be in $\mathbb{C}^n$. In fact, in [6] the expressions

$$(x + y\tau)z^*$$

with $x, y, z \in \mathbb{C}^n$ were called *operets* and approximations of $\mathcal{A}$ by sums of operets were considered, but for example, a good charaterization of the class of operators which have a decomposition

$$\mathcal{A} = \mathcal{U}\mathcal{D}V^*$$

with $\mathcal{U}$ unitary, $\mathcal{D}$ diagonal and V $\mathbb{C}$-linear unitary, is still lacking. Still another, again quite natural "SVD" is obtained from the SVD of $\psi(\mathcal{A})$ and it is unclear what would really "deserve" to be called the real linear SVD.

References

[1] P.M.Cohn: *Skew Field Constructions*, LMS Lect. Note Series 27, Cambridge Univ. Press 1977

[2] T.Eirola, M.Huhtanen, J.von Pfaler: *Solution methods for* $\mathbb{R}$*-linear problems in* $\mathbb{C}^n$. SIAM J. Matrix Anal. Appl., 25, pp. 804-828, 2004

[3] R.Horn, C. Johnson: *Topics in Matrix Analysis*, Cambridge Univ. Press 1986

[4] M.Huhtanen: Real linear Kronecker product operations. Linear Algebra Appl., 418, pp. 347-361, 2006

[5] M.Huhtanen: *Approximations with real linear modules*, submitted

[6] M.Huhtanen, O.Nevanlinna: *Real linear matrix analysis.* Banach Center Publ., 75, pp.171-189, 2007

[7] M.Huhtanen, O.Nevanlinna: *The real linear resolvent and cosolvent operators.* To appear in J. Operator Theory

[8] M. Huhtanen and O. Nevanlinna: *Approximating real linear operators.* Studia Math., 179, pp. 7-25, 2007

[9] M. Huhtanen, J.von Pfaler: *The real linear eigenvalue problem in* $\mathbb{C}^n$. Linear Algebra Appl., 394, pp. 169-199, 2005

[10] R.E.Johnson: *On the equation* $\chi\alpha = \gamma\chi + \beta$ *over an algebraic division ring.* Bull. Amer. Math. Soc. 50, pp. 202-207, 1944

[11] B.R.Lepmhi: *Real Operator Algebras*, World Scientific Publ., Singapore, 2003

[12] P. Lounesto: *Clifford Algebras and Spinors*, LMS Lect. Note Series 239, Cambridge Univ. Press 1997

[13] J.L.Mueller, S.Siltanen: *Direct reconstructions of conductivities from boundary measurements*, SIAM J. Sci. Comput., 24 (2003), pp. 1232-1266

[14] F. Zhang: *Quaternions and Matrices of Quaternions*, Linear Algebra Appl. 251 (1997), 21-57

Symbols and definitions

$\bar{\mathcal{A}}, \mathcal{A}^*, \mathcal{A}^\sim, \mathcal{A}^t$ Definition 5 (page 8)

$[\mathcal{A}] = A + B$ Convention 3 (page 6)

$|\mathcal{A}|$ Definition 19 (page 45) and Proposition 8 (page 10)

$\mathcal{C}(\lambda, \mathcal{A})$ Section 5.1 (page 34)

$\mathbf{Con}(\mathcal{A})$ Section 1.2 (page 5)

$(c, d)_{\mathbb{C}}$**,** $(c, d)_{\mathbb{R}}$ Definition 7 (page 10)

$\ker \mathcal{A}$**,** $\ker_{\mathbb{S}} \mathcal{A}$ Definition 12 (page 19)

$\mathbf{Lin}(\mathcal{A})$ Section 1.2 (page 5)

$\mathcal{R}(\lambda, \mathcal{A})$ Section 5.1 (page 34)

$\mathbb{S}$**,** $\mathbb{S}^n$ Definition 2 (page 6)

$\phi(\gamma)$**,** $\psi(\gamma)$ Section 2.3 (page 14)

$\sigma(\mathcal{A})$**,** $\sigma_c(\mathcal{A})$**,** $\sigma_{\mathbb{S}}(\mathcal{A})$ Section 5.1 (page 34)

τ Section 1.2 (page 5)

circlet Definition 2 (page 6), $\gamma = \alpha + \beta\tau \in \mathbb{S}$ where $\alpha, \beta \in \mathbb{C}$ and τ is the special real linear operator conjugating complex numbers

circlet product of vectorands, $(c, d) = d^*c$, Definition 6 (page 10)

cosolvent $\mathcal{C}(\lambda, \mathcal{A})$, Section 5.1 (page 34)

eigenvalue, eigenvector, eigencirclet, eigenvectorand Definition 14 (page 27)

involutions, complex conjugate, adjoint, $^\sim$-conjugate, transpose Definition 5 (page 8)

linearly independent over $\mathbb{S}$, in $\mathbb{C}$, Definition 10 (page 18), Definition 11 (page 19)

norm $\|\mathcal{A}\|$, Definition 4 (page 8)

norm form of circlet $\nu(\gamma)$, Definition 8 (page 11)

(continued from the back cover)

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ISBN 978-951-22-8761-1
ISSN 0784-3143