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János Karátson Sergey Korotov

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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János Karátson Sergey Korotov

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

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Abstract: *The paper is devoted to the problem of verification of accuracy of approximate solutions obtained in computer simulations. This problem is strongly related to a posteriori error estimates, giving computable bounds for computational errors and detecting zones in the solution domain where such errors are too large and certain mesh refinements should be performed. A mathematical model embracing nonlinear elliptic variational problems is considered in this work. Based on functional type estimates developed in an abstract level, we present a general technology for constructing computable sharp upper bounds for the global error for various particular classes of elliptic problems. Here the global error is understood as a suitable energy type difference between the true and computed solutions. The estimates obtained are completely independent of the numerical technique used to obtain approximate solutions, and are sharp in the sense that they can be, in principle, made as close to the true error as resources of the used computer allow. The latter can be achieved by suitably tuning the auxiliary parameter functions, involved in the proposed upper error bounds, in course of the calculations.*

AMS subject classifications: 65J05, 65J15, 65M60, 65N15, 65N30, 65N50

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Correspondence

karatson@cs.elte.hu, sergey.korotov@hut.fi

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, FI-02015 TKK, Finland
email:math@tkk.fi http://math.tkk.fi/

1 Introduction

Many physical phenomena can be described by means of mathematical models presenting linear and nonlinear boundary value problems of elliptic type [10, 13, 28, 36, 45]. Various numerical techniques (such as the finite difference method, the finite element method, the finite volume method etc) are well developed for finding approximate solutions for such problems, see, e.g., [10, 28] and references therein.

However, in order to be practically meaningful, computer simulations always require an accuracy verification of computed approximations. Such a verification is the main purpose of a posteriori error estimation methods. Several approaches for deriving various a posteriori estimates for elliptic problems for errors measured in global (energy) norms ([1], [4], [5], [17], [20], [39], [46], [47], [50]), or in terms of various local quantities ([6], [12], [19], [25], [40], [45]) have been suggested (see also references in the above mentioned works). However, most of the estimates proposed there strongly use the fact that the computed solutions are true finite element (FE) approximations which, in fact, rarely happens in real computations, e.g., due to quadrature rules, forcibly stopped iterative processes, various round-off errors, or even bugs in computer codes.

A different approach, based on functional analysis background, has been first developed in [41], see the monograph [38] and also [42, 43], and the references therein. Hereby the estimation is developed independently of the numerical method used to obtain the approximation. One can thus obtain sharp estimates for linear problems and for certain nonlinear problems, however, for nonlinear problems in general, these estimates may fail to ensure the best upper bound [38, p. 236].

In this paper, based on a Banach space framework, we present another general functional type technology for obtaining *sharp computable guaranteed error bounds* needed for reliable control of the overall accuracy of computed approximations. Such bounds are again valid for *any conforming approximation* independently of the numerical method used to obtain them. The bounds obtained can be made *arbitrarily close* to the true error by tuning the auxiliary parameters involved. In real calculations this closeness only depends on resources of a concrete computer. Based on the general general theory, our error estimates are given for various classes of elliptic problems. We also discuss some issues of the practical realization of the proposed error estimation procedures.

We note that the estimates proposed in this work can be considered as a certain generalization of results presented for the first time in [41] and in a unified manner in [38], mentioned above. However, our way of constructing the error estimates is somewhat different and leads to sharp bounds for nonlinear problems. In particular, it does not require tools of the duality theory and is purely based on direct calculations. Estimates based on a similar direct approach are also obtained for linear convection-reaction-diffusion problems in [27, 29] and for Maxwell equations in [18].

Another advantage of the proposed approach is that it only uses a few (at most five) global constants, which do not depend on the computational process and must be computed only once in advance (or in parallel). These constants come from embedding theorems and the nonlinear coefficients of the equation. Many other existing estimation techniques (e.g., of the residual-type ones) normally involve much more unknown constants (usually related to patches of computational meshes used). Such constants are very hard to compute (or even sufficiently accurately estimate from above) and their evaluation normally leads to a very big overestimation of the error even in simple cases (cf. [9]). Moreover, those constants have to be always recomputed when we perform adaptive computations and change the computational mesh. On the contrary, our global constants do remain the same under any change of meshes during the whole computational process. Moreover, it suffices to estimate our constants only roughly from above (which can actually rely on a known explicit value or be cheaply calculated), since the terms which they multiply are normally decreasing towards zero during the tuning process of the error estimate.

Chapters 2-3 are devoted to the general estimation theory in Banach space. We consider operator equations of the form

$$F(u) + l = 0 \tag{1.1}$$

on a Banach space V with a given nonlinear operator $F : V \to V^*$ and a given bounded linear functional $l \in V^*$. We will assume certain monotonicity properties of F that both ensure well-posedness for (1.1) and allow a suitable measuring of the error. We give suitable background in Chapter 2, and present our sharp upper error bounds in Chapter 3. Applications to various classes of elliptic problems including second order problems with Dirichlet and mixed boundary conditions, systems, and fourth order equations are developed in Chapter 4.

2 Theoretical background

2.1 Some elementary definitions and properties

For the reader's convenience we define some basic notions and notations that we will use, together with some well-known properties. See, e.g., [13, 49] for related topics.

Definition 2.1 Let V be a given Banach space with norm $\|.\|_V$. Then its dual space V^* consists of the bounded linear functionals $l : V \to \mathbf{R}$ on V. If $l \in V^*$ and $u \in V$, then the value of l at u is denoted by $\langle l, u\rangle$, where $\langle ., .\rangle$ is the duality pairing.

Definition 2.2 The operator $F : V \to V^*$ is called a *monotone operator* if

$$\langle F(u) - F(v), u - v\rangle \geq 0 \qquad (u, v \in V),$$

a *strictly monotone operator* if

$$\langle F(u) - F(v), u - v\rangle > 0 \qquad (u, v \in V,\ u \neq v),$$

and a *uniformly monotone operator* if there exists a constant $m > 0$ such that

$$\langle F(u) - F(v), u - v\rangle \geq m\|u - v\|_V^2 \qquad (u, v \in V).$$

Definition 2.3 The nonlinear operator $F : V \to V^*$ has a *bihemicontinuous symmetric Gateaux derivative* if

(i) F is Gateaux differentiable;

(ii) F' is bihemicontinuous, i.e., for any $u, k, w, h \in V$ the mapping $(s,t) \mapsto F'(u + sk + tw)h$ is continuous from $\mathbf{R}^2$ to V^*;

(iii) for any $u \in V$ the operator $F'(u)$ is symmetric, i.e.,

$$\langle F'(u)h, v\rangle = \langle F'(u)v, h\rangle \qquad (u, h, v \in V).$$

Accordingly, a nonlinear functional $J : V \to \mathbf{R}$ has a *bihemicontinuous symmetric second Gateaux derivative* if J is Gateaux differentiable and J' satifies (i)-(iii) above.

Definition 2.4 The operator $F : V \to V^*$ is called a *potential operator* if there exists a Gateaux differentiable functional $\phi : V \to \mathbf{R}$ such that $\phi'(u) = F(u) \quad (u \in V)$. (Such a ϕ is called a potential of F).

The following characterization holds, see, e.g., [49]:

Proposition 2.1 *Let the operator $F : V \to V^*$ have a bihemicontinuous Gateaux derivative. Then F is a potential operator if and only if $F'(u)$ is symmetric for any $u \in V$.*

Then we have a related basic well-posedness result:

Theorem 2.1 *Let the operator $F : V \to V^*$ have a bihemicontinuous symmetric Gateaux derivative, and let there exist a constant $m > 0$ such that*

$$\langle F'(u)v, v\rangle \geq m\|v\|_V^2 \qquad (u, v \in V).$$

Then for any $l \in V^*$ the operator equation (1.1) *has a unique solution $u^* \in V$.*

For a proof see, e.g., [13, 49]. We note that the solution u^* is the unique minimizer of the functional $J(u) := \phi(u) + \langle l, u\rangle$, where ϕ comes from Definition 2.4.

2.2 Error functionals for monotone operators

Let us assume that the operator equation (1.1) has a unique solution $u^* \in V$. (Sufficent conditions will be given later when our main result is presented in section 3.)

In this paper we consider some approximate solution $u \in V$ of equation (1.1), i.e. $u \approx u^*$ where u^* is the exact solution. Our goal is to estimate the error arising from this approximation. For this purpose, we will use the following (energy type) error functional for equation (1.1):

$$E(u) := \langle F(u) + l,\, u - u^* \rangle \qquad (u \in V) \tag{2.1}$$

or in other form

$$E(u) = \langle F(u) - F(u^*),\, u - u^* \rangle \qquad (u \in V). \tag{2.2}$$

The following facts obviously hold. If F is monotone then $E(u) \geq 0 = E(u^*) \quad (u \in V)$. If F is also strictly monotone then $E(u) = 0$ if and only if $u = u^*$. If F is also uniformly monotone then

$$E(u) \geq m\|u - u^*\|_V^2 \qquad (u \in V). \tag{2.3}$$

We note that if F is a potential operator and ϕ is a potential of F, then another possible error functional has the form $\hat{E}(u) := J(u) - J(u^*)$, where $J(u) := \phi(u) + \langle l, u \rangle$. If the above monotonicity properties are assumed, then the corresponding statements on E also hold for $\hat{E}$, which can be verified via the corresponding convexity of ϕ. However, in this paper it will be more convenient to use the error functional E. We also note that E and $\hat{E}$ are just the same (up to a constant multiplier 2) for linear problems.

2.3 Error estimates in normed spaces via convex functionals

A thorough study of error estimation for nonlinear variational problems is given in the book [38], on which our paper also builds on. Here we briefly outline a setting from this book and some results.

In addition to the Banach space V, let us introduce another Banach space Y, assumed to be reflexive. We also introduce a linear operator $\Lambda : V \to Y$, for which there exist constants $c_2 \geq c_1 > 0$ such that

$$c_1\|u\|_V \leq \|\Lambda u\|_Y \leq c_2\|u\|_V \qquad (u \in V). \tag{2.4}$$

Then Λ has an adjoint operator $\Lambda^* : Y^* \to V^*$ satisfying

$$\langle y^*, \Lambda u \rangle = \langle \Lambda^* y^*, u \rangle \qquad (y^* \in Y^*,\ u \in V). \tag{2.5}$$

In [38, Chap. 7], functionals $J : V \to \mathbf{R}$ are considered in the form

$$J(u) := G(\Lambda u) + f(u) \qquad (u \in V),$$

where $G : Y \to \mathbf{R}$ and $f : V \to \mathbf{R}$ are given uniformly convex functionals, and error estimates are given with expressions based on duality theory.

More explicit estimates are found for the special case

$$J(u) := G(\Lambda u) + \langle l, u\rangle \qquad (u \in V), \tag{2.6}$$

in which $l \in V^*$, further, Y is assumed to be a Hilbert space and G has the form

$$G(y) = \tfrac{1}{2}\langle Ay, y\rangle + \Phi(y) \qquad (y \in Y) \tag{2.7}$$

where $A : Y \to Y$ is an invertible bounded self-adjoint linear operator and $\Phi : Y \to \mathbf{R}$ is a convex continuous functional. Denoting by u^* the minimizer of J, it is proved in [38, sec. 7.7] that

$$|||\tfrac{1}{2}(\Lambda(u^* - u))|||^2 \le (1+\beta)\, D_G(\Lambda u, y^*) + (1+\tfrac{1}{\beta})\, |\Lambda^* y^* + l|^2\,, \tag{2.8}$$

where

$$|||y||| := \langle Ay, y\rangle^{1/2}$$

is the A-norm, $\beta > 0$ is an arbitrary constant,

$$|\Lambda^* y^* + l| := \sup_{\substack{w\in V\\ w\neq 0}} \frac{\langle \Lambda^* y^* + l,\ w\rangle}{|||\Lambda w|||} \tag{2.9}$$

and

$$D_G(\Lambda u, y^*) = G(\Lambda u) + \sup_{y\in Y}\Big(\langle y^*, y\rangle - G(y)\Big) - \langle y^*, \Lambda u\rangle.$$

It is pointed out in [38, sec. 7.7] that estimate (2.8) is not sharp, and finding the best upper bound can only be expected from a further analysis of the particular problem considered.

An important special case of the above is formed by linear equations, i.e. when $\Phi \equiv 0$ and G is a quadratic functional to be minimized. Then, by [38, Chap. 6], an estimate similar to (2.8) holds:

$$\tfrac{1}{2}|||\Lambda(u^* - u)|||^2 \le (1+\beta)\, D(\Lambda u, y^*) + (1+\tfrac{1}{\beta})\, \tfrac{1}{2}\, |\Lambda^* y^* + l|^2\,, \tag{2.10}$$

where

$$D(\Lambda u, y^*) = \tfrac{1}{2}\ \langle A(\Lambda u) - y^*,\ \Lambda u - A^{-1}y^*\rangle.$$

Accordingly,

$$|||\Lambda(u^* - u)|||^2 \le (1+\beta)\ \langle A(\Lambda u) - y^*,\ \Lambda u - A^{-1}y^*\rangle + (1+\tfrac{1}{\beta})\, |\Lambda^* y^* + l|^2\,. \tag{2.11}$$

We note that in this special case, for $F(u) := J'(u)$, (2.2) coincides with $|||\Lambda(u^* - u)|||^2$.

Returning to the nonlinear case (2.6), the minimizer u^* of J is the solution of equation

$$\langle J'(u), v\rangle = \langle G'(\Lambda u), \Lambda v\rangle + \langle l, v\rangle = 0 \qquad (v \in V) \tag{2.12}$$

or $N(u) = 0$ where $N = J'$. It is important to note here that for twice Gateaux differentiable Φ, formula (2.7) implies

$$\langle G''(y)p, p\rangle = \langle Ap, p\rangle + \langle \Phi''(y)p, p\rangle \geq \langle Ap, p\rangle = |||p|||^2 \qquad (y, p \in Y). \tag{2.13}$$

The twice Gateaux differentiability of J (i.e. the Gateaux differentiability of N) is not a strong requirement and is satisfied in many practical cases, further, the corresponding condition (2.13) (up to a constant multiplier) is rather close to uniform convexity. Therefore, the assumption

$$\langle G''(y)p, p\rangle \geq m \,\|p\|_Y^2 \qquad (y, p \in Y)$$

(with $m > 0$) and the corresponding estimate (2.8) are a proper starting point for our search for sharp estimates. In fact, our goal is to obtain such an estimate as an extension of the more explicit formula (2.11).

2.4 Integral mean operators

Let Y be a Banach space and $A : Y \to Y^*$ an operator having a bihemicontinuous symmetric Gateaux derivative.

Definition 2.5 For any vectors $y, z \in Y$, we define $A'_{[y,z]} \in B(Y, Y^*)$, that is, a bounded linear operator $A'_{[y,z]} : Y \to Y^*$, by the formula

$$A'_{[y,z]} := \int_0^1 A'(y + t(z-y))\, dt \;. \tag{2.14}$$

This is an integral of a family of operators, understood via the corresponding bilinear forms:

$$\langle A'_{[y,z]}p, q\rangle = \int_0^1 \langle A'(y + t(z-y))p, q\rangle\, dt \qquad (p, q \in Y). \tag{2.15}$$

The unique existence of $A'_{[y,z]}$ (i.e., that this definition is correct) is ensured by the fact that

$$\int_0^1 \langle A'(y + t(z-y))p, q\rangle\, dt \leq \left(\max_{t\in[0,1]} \|A'(y + t(z-y))\|\right) \|p\|_Y \,\|q\|_Y \tag{2.16}$$

(where the maximum exists by the continuity of the mapping $t \mapsto A'(y + t(z-y))$ and of the operator norm), which means that the r.h.s. of (2.15) is a bounded bilinear form in p and q. Then we obtain by the definition of Y^* that this bilinear form can be represented as the bilinear form of a bounded linear operator from Y to Y^*.

The following properties are direct consequences of the above definition:

Proposition 2.2 *For any $y, z \in Y$*

(i) the operator $A'_{[y,z]}$ is symmetric, i.e.,

$$\langle A'_{[y,z]}p, q\rangle = \langle A'_{[y,z]}q, p\rangle \qquad (p, q \in Y). \tag{2.17}$$

(ii) $A'_{[y,z]} = A'_{[z,y]}$.

(iii) $A(z) - A(y) = A'_{[y,z]}(z - y)$.

3 A sharp global error estimate in normed spaces

3.1 Basic properties

We study the operator equation (1.1) in the Banach space V in the framework of subsection 2.3, in particular, for simplicity, property (2.4) can be replaced by the simplified version (3.2) (which is achieved just by redefining the norm of V by the equivalent norm $\|\Lambda u\|_Y$).

Following subsection 2.3, let $J : V \to \mathbf{R}$ be a functional of the form

$$J(u) := G(\Lambda u) + \langle l, u\rangle \qquad (u \in V) \tag{3.1}$$

under the following conditions:

Assumptions 3.1.

(i) Y is another Banach space and $\Lambda : V \to Y$ is a linear operator for which

$$\|\Lambda u\|_Y = \|u\|_V \qquad (u \in V), \tag{3.2}$$

(ii) $G : Y \to \mathbf{R}$ is a functional having a bihemicontinuous symmetric second Gateaux derivative (according to Definition 2.3),

(iii) there exists a constant $m > 0$ such that

$$\langle G''(y)p, p\rangle \geq m\,\|p\|_Y^2 \qquad (y, p \in Y), \tag{3.3}$$

(iv) the operator $F : V \to V^*$ has the form

$$\langle F(u), v\rangle = \langle G'(\Lambda u), \Lambda v\rangle \qquad (u, v \in V). \tag{3.4}$$

Proposition 3.1 *Under Assumptions 3.1, for any $l \in V^*$ the operator equation* (1.1) *has a unique solution $u^* \in V$.*

Proof. The assumptions yield that F has a bihemicontinuous symmetric Gateaux derivative that satisfies

$$\langle F'(u)v, v\rangle = \langle G''(\Lambda u)\Lambda v, \Lambda v\rangle \geq m\,\|\Lambda v\|_Y^2 = m\|v\|_V^2 \qquad (u, v \in V). \tag{3.5}$$

Then Theorem 2.1 implies well-posedness for (1.1). ■

We note that the solution u^* of (1.1) is the unique minimizer of J. However, from now on, our calculations will involve the operator G' in (3.4) rather than the functional G. Hence we study below the solution of equation (1.1) directly, instead of using the corresponding minimization problem.

3.2 Operator formulation and a preliminary estimate

3.2.1 Reformulation of the problem

We will replace the minimization problem for (3.1) by the corresponding operator equation, which is a more detailed form of (1.1) for this case. For this purpose, we introduce the operator

$$A := G' \,. \tag{3.6}$$

Then Assumptions 3.1 are equivalent to

Assumptions 3.2.

(i) Y is another Banach space and $\Lambda : V \to Y$ is a linear operator for which

$$\|\Lambda u\|_Y = \|u\|_V \qquad (u \in V); \tag{3.7}$$

(ii) the operator $A : Y \to Y^*$ has a bihemicontinuous symmetric Gateaux derivative (according to Definition 2.3);

(iii) there exists a constant $m > 0$ such that

$$\langle A'(y)p, p\rangle \geq m\,\|p\|_Y^2 \qquad (y, p \in Y); \tag{3.8}$$

(iv) the operator $F : V \to V^*$ has the form

$$\langle F(u), v\rangle = \langle A(\Lambda u), \Lambda v\rangle \qquad (u, v \in V). \tag{3.9}$$

Assumptions (ii)-(iii) imply in particular that A is bijective, i.e. $A^{-1} : Y^* \to Y$ exists. By (3.9), equation (1.1) can be written as

$$\langle A(\Lambda u), \Lambda v\rangle + \langle l, v\rangle = 0 \qquad (v \in V) \tag{3.10}$$

which has a unique solution $u^* \in V$ for any $l \in V^*$ by Proposition 3.1.

We will need some further related properties. First, Proposition 2.2 (i) and (3.8) imply

Proposition 3.2 *Under Assumptions 3.2, for any $y, z \in Y$ the mapping $p, q \mapsto \langle A'_{[y,z]}p, q\rangle$ is an inner product on Y.*

Proposition 3.3 *Under Assumptions 3.2, the following properties hold:*

(i) $E(u) = \langle A'_{[\Lambda u^*, \Lambda u]}\,\Lambda(u-u^*),\ \Lambda(u-u^*)\rangle \qquad (u \in V).$

(ii) $E(u) \geq m\,\|u-u^*\|_V^2 = m\,\|\Lambda(u-u^*)\|_Y^2 \qquad (u \in V).$

(iii) $\|A(z) - A(y)\|_{Y^*} \geq m\,\|z-y\|_Y \qquad (y, z \in Y).$

PROOF. (i) Using (3.9) and Proposition 2.2 (iii) for $z = \Lambda u$ and $y = \Lambda u^*$,

$$E(u) = \langle F(u) - F(u^*),\, u - u^* \rangle = \langle A(\Lambda u) - A(\Lambda u^*),\, \Lambda(u - u^*) \rangle \qquad (3.11)$$

$$= \langle A'_{[\Lambda u^*, \Lambda u]} \Lambda(u - u^*),\ \Lambda(u - u^*) \rangle \,.$$

(ii) Estimate (3.5) implies that F is uniformly monotone, hence (2.3) and (3.7) yield the required statement.

(iii) Estimate (3.8) implies

$$\langle A(z) - A(y), z - y \rangle \geq\ m \, \| z - y \|_Y^2 \qquad (y, z \in Y), \qquad (3.12)$$

whence we obtain the required statement using the Cauchy-Schwarz inequality. ■

For the V^*-norm of a linear functional $l \in V^*$, we introduce the notation of [38]:

$$|\, l \,| := \|\, l \,\|_{V^*} \qquad (l \in V^*). \qquad (3.13)$$

Here (3.7) yields

$$|l| = \sup_{w \in V} \frac{\langle l,\, w \rangle}{\|w\|_V} \ = \sup_{w \in V} \frac{\langle l,\, w \rangle}{\|\Lambda w\|_Y} \qquad (3.14)$$

i.e. we have an analogue of (2.9).

3.2.2 A preliminary estimate

In what follows, our goal is to find upper bounds for $E(u)$. We follow the setting of [26, 38]. Namely, we let $u \in V$ be arbitrary and look for a bound involving some other vector parameters.

First we let $y^* \in Y^*$ be an arbitrary vector. We give a preliminary estimate, which is a starting point for our study. In this subsection we show how a simple bound can be derived from this, which is a kind of analogue to those in subsection 2.3, however, is not sharp. We will point out that a sharp estimation requires a further assumption on the Lipschitz continuity of the derivative of the nonlinear operator. This will be our main result, developed in the next subsection.

Lemma 3.1 *Let Assumptions 3.2 hold and $u^* \in V$ be the solution of* (1.1). *Let $u \in V$ and $y^* \in Y^*$ be arbitrary, let $z^* := A^{-1}(y^*)$. Then*

$$E(u) \leq\ |\Lambda^* y^* + l| \, m^{-1/2} \, E(u)^{1/2} \ +\ \langle A'_{[z^*, \Lambda u]} (\Lambda u - z^*),\ \Lambda(u - u^*) \rangle \,. \quad (3.15)$$

PROOF. We have

$$E(u) = \langle F(u) + l,\, u - u^* \rangle = \langle \Lambda^* y^* + l, u - u^* \rangle + \langle F(u) - \Lambda^* y^*, u - u^* \rangle \,. \quad (3.16)$$

For the first term, we use (3.13) and Proposition 3.3 (ii) to obtain

$$|\langle \Lambda^* y^* + l, u - u^* \rangle| \leq |\Lambda^* y^* + l| \ \|u - u^*\|_V \leq |\Lambda^* y^* + l| \, m^{-1/2} \, E(u)^{1/2} \,. \quad (3.17)$$

The second term equals

$$\langle F(u)-\Lambda^* y^*,\, u-u^*\rangle = \langle A(\Lambda u)-y^*,\, \Lambda(u-u^*)\rangle = \langle A(\Lambda u)-A(z^*),\, \Lambda(u-u^*)\rangle$$

$$= \langle A'_{[z^*,\Lambda u]}\,(\Lambda u - z^*),\ \Lambda(u-u^*)\rangle \tag{3.18}$$

where (2.5), (3.9), and Proposition 2.2 (iii) have been used. ■

The r.h.s. of (3.15) becomes computable if the factor $\Lambda(u-u^*)$ is eliminated in above: this can be based on the estimate $\|u-u^*\|_V \le m^{-1/2}\,E(u)^{1/2}$ which was already used in the first part of Lemma 3.1. A straightforward way for this is to use a Cauchy-Schwarz inequality as follows.

Proposition 3.4 *Let Assumptions 3.2 hold and $u^* \in V$ be the solution of* (1.1)*. Let $u \in V$ and $y^* \in Y^*$ be arbitrary, let $z^* := A^{-1}(y^*)$. Then for any constant $\beta > 0$,*

$$E(u) \le\ (1+\beta)m^{-1}\,|\Lambda^* y^* + l|^2\ +\ (1+\tfrac{1}{\beta})\,\tfrac{M}{m}\ \langle A(\Lambda u)-y^*,\, \Lambda u - A^{-1}(y^*)\rangle\,. \tag{3.19}$$

Proof. Proposition 3.2 implies that the Cauchy-Schwarz inequality is valid for the inner product $p,q \mapsto \langle A'_{[z^*,\Lambda u]}\,p,q\rangle$. Hence (3.18) is estimated by

$$\langle A'_{[z^*,\Lambda u]}\,(\Lambda u - z^*),\ \Lambda(u-u^*)\rangle$$

$$\le \langle A'_{[z^*,\Lambda u]}\,(\Lambda u - z^*),\ \Lambda u - z^*\rangle^{1/2}\ \langle A'_{[z^*,\Lambda u]}\,\Lambda(u-u^*),\ \Lambda(u-u^*)\rangle^{1/2}.$$

Here Proposition 2.2 (iii) yields

$$\langle A'_{[z^*,\Lambda u]}\,(\Lambda u - z^*),\ \Lambda u - z^*\rangle = \langle A(\Lambda u)-A(z^*),\, \Lambda u - z^*\rangle\,,$$

and (2.15)–(2.16) and Proposition 3.3 (ii) yield

$$\langle A'_{[z^*,\Lambda u]}\,\Lambda(u-u^*),\ \Lambda(u-u^*)\rangle \le M\|\Lambda(u-u^*)\|_Y^2 \le \tfrac{M}{m}E(u)$$

where $M := \max_{t\in[0,1]}\|A'(\Lambda u + t(z^* - \Lambda u))\|$. Hence

$$\langle A'_{[z^*,\Lambda u]}\,(\Lambda u - z^*),\ \Lambda(u-u^*)\rangle \le (\tfrac{M}{m})^{1/2}\ \langle A(\Lambda u)-A(z^*),\, \Lambda u - z^*\rangle^{1/2}\,E(u)^{1/2}$$

and by (3.15),

$$E(u)^{1/2} \le\ m^{-1/2}\,|\Lambda^* y^* + l|\ +\ (\tfrac{M}{m})^{1/2}\ \langle A(\Lambda u)-A(z^*),\, \Lambda u - z^*\rangle^{1/2}\,.$$

Then the real inequality $(a+b)^2 \le (1+\beta)a^2 + (1+\frac{1}{\beta})b^2$ (where $\beta > 0$) and $z^* = A^{-1}(y^*)$ provide the desired statement. ■

Remark 3.1 Proposition 3.4 is an analogue of (2.11). The extra constant multiplier m in (3.19) comes from the fact that in (2.11) the norm $|||\cdot|||$ is used, for which (2.13) holds, whereas in (3.19) the norm $\|\cdot\|$ is used, for which (3.3) holds.

However, (3.19) is not sharp. To see this, let us consider the choice $y^* = A(\Lambda u^*)$ where u^* is the solution of (1.1) or (3.10). For linear A, the corresponding choice $y^* = A\Lambda u^*$ is known to be optimal, since it provides equality in (2.11) if $\beta \to \infty$, see [26, 38]. However, setting $y^* = A(\Lambda u^*)$ in (3.19) always overestimates the error. Namely, for all $w \in V$

$$\langle \Lambda^* y^* + l, w\rangle = \langle y^*, \Lambda w\rangle + \langle l, w\rangle = \langle A(\Lambda u^*), \Lambda w\rangle + \langle l, w\rangle = 0,$$

i.e. $\Lambda^* y^* + l = 0$ and the first term in (3.19) is zero. That is, in this case (3.19) becomes

$$E(u) \le \ (1 + \tfrac{1}{\beta})\, \tfrac{M}{m}\, \langle A(\Lambda u) - A(\Lambda u^*),\, \Lambda u - \Lambda u^*\rangle$$

$$= \ (1 + \tfrac{1}{\beta})\, \tfrac{M}{m}\, \langle F(u) - F(u^*),\, u - u^*\rangle \ = \ (1 + \tfrac{1}{\beta})\, \tfrac{M}{m}\, E(u)\,,$$

hence the infimum of the r.h.s. (as $\beta \to \infty$) is $\frac{M}{m}$ times larger than $E(u)$. The factor $\frac{M}{m}$ may be large for ill-conditioned problems where $m \approx 0$.

Remark 3.2 Our goal now is to eliminate the factor $\frac{M}{m}$, we will instead include another additional term that vanishes for $y^* = A(\Lambda u^*)$. This term will come from a Lipschitz condition on the derivatives.

3.3 The sharp error estimate

We complete Assumptions 3.2 by additional conditions:

Assumptions 3.3.

(i) There exists a subspace $W \subset Y$ with a new norm $\|.\|_W$ such that A' is Lipschitz continuous as an operator from Y to $B(W, Y^*)$.

(ii) There exists a constant $M > 0$ such that

$$\langle A'(y)p, p\rangle \le \ M\, \|p\|_Y^2 \qquad (y, p \in Y). \tag{3.20}$$

Assumption 3.3-(i) means that there exists a constant $L > 0$ such that

$$\|A'(z) - A'(y)\|_{B(W,Y^*)} \le \ L\, \|z - y\|_Y \qquad (y, z \in Y), \tag{3.21}$$

or in more detailed form,

$$|\langle (A'(z) - A'(y))\, w, p\rangle| \le \ L\, \|z - y\|_Y \|w\|_W \|p\|_Y \qquad (y, z, p \in Y,\ w \in W). \tag{3.22}$$

Lemma 3.2 *Let Assumption 3.3-(i) hold. Then the operators defined in* (2.14) *satisfy for all $y, v, z \in Y$*

$$\|A'_{[z,v]} - A'_{[y,v]}\|_{B(W,Y^*)} \leq \tfrac{L}{2} \|z-y\|_Y. \tag{3.23}$$

Proof. We have

$$\|A'_{[z,v]} - A'_{[y,v]}\|_{B(W,Y^*)} \leq \int_0^1 \|A'(z+t(v-z)) - A'(y+t(v-y))\|_{B(W,Y^*)}\, dt$$

$$\leq L \int_0^1 (1-t)\, \|z-y\|_Y\, dt = \tfrac{L}{2} \|z-y\|_Y. \qquad \blacksquare$$

In more detailed form (as in (3.22)), property (3.23) means that

$$|\langle (A'_{[z,v]} - A'_{[y,v]})w, p\rangle| \leq \tfrac{L}{2} \|z-y\|_Y \|w\|_W \|p\|_Y \qquad (y,v,z,p \in Y,\ w \in W). \tag{3.24}$$

Remark 3.3 For our error estimate in normed space, there is no restriction on the relation of the norms $\|.\|_W$ and $\|.\|_Y$. In practice the norm $\|.\|_W$ will be stronger, i.e. the range of the values $\|w\|_W / \|w\|_Y$ (where $w \in W$) will run from a positive constant to $+\infty$, see section 4.

Assumption 3.3-(ii) means that the constant M used in Proposition 3.4 is uniform for all u. Then the upper analogue of Proposition 3.3 (iii) holds:

$$\|A(z) - A(y)\|_{Y^*} \leq M \|z-y\|_Y \qquad (y, z \in Y). \tag{3.25}$$

Further, we will need the following inequality:

Lemma 3.3 *Let Assumptions 3.2-3.3 hold and $u^* \in V$ be the solution of* (1.1)*. Let $y^* \in Y^*$ be arbitrary and $z^* := A^{-1}(y^*)$. Then for any $h \in V$*

$$\|z^* - \Lambda u^*\|_Y \leq \tfrac{M}{m} \|z^* - \Lambda h\|_Y + \tfrac{1}{m} |\Lambda^* y^* + l|\,. \tag{3.26}$$

Proof. Let $w^* \in V$ satisfy $F(w^*) = \Lambda^* y^*$. By (3.9), w^* is the solution of equation

$$\langle A(\Lambda w^*), \Lambda v\rangle = \langle \Lambda^* y^*, v\rangle \qquad (v \in V). \tag{3.27}$$

We have

$$\|z^* - \Lambda u^*\|_Y \leq \|z^* - \Lambda w^*\|_Y + \|\Lambda(w^* - u^*)\|_Y\,. \tag{3.28}$$

Here (3.27) implies

$$\langle A(\Lambda w^*),\, \Lambda v\rangle = \langle y^*,\, \Lambda v\rangle = \langle A(z^*),\, \Lambda v\rangle \qquad (v \in V),$$

that is

$$\langle A(z^*) - A(\Lambda w^*),\, \Lambda v\rangle = 0 \qquad (v \in V). \tag{3.29}$$

Using (3.12), (3.29) and (3.25), respectively, we obtain for any $h \in V$ that

$$m \|z^* - \Lambda w^*\|_Y^2 \leq \langle A(z^*) - A(\Lambda w^*),\, z^* - \Lambda w^*\rangle = \langle A(z^*) - A(\Lambda w^*),\, z^* - \Lambda h\rangle$$

$$\leq\ M\,\|z^*-\Lambda w^*\|_Y\,\|z^*-\Lambda h\|_Y\,,$$

that is,

$$\|z^*-\Lambda w^*\|_Y \leq \tfrac{M}{m}\,\|z^*-\Lambda h\|_Y\,. \tag{3.30}$$

Further, using (3.7), (3.12), (3.27) and that u^* solves (3.10),

$$m\,\|w^*-u^*\|_V^2 = m\,\|\Lambda(w^*-u^*)\|_Y^2 \leq \langle A(\Lambda w^*)-A(\Lambda u^*),\,\Lambda w^*-\Lambda u^*\rangle =$$

$$= \langle \Lambda^* y^* + l,\, w^*-u^*\rangle \leq |\Lambda^* y^* + l|\,\|w^*-u^*\|_V\,,$$

hence

$$\|w^*-u^*\|_V \leq \tfrac{1}{m}\,|\Lambda^* y^* + l|\,. \tag{3.31}$$

Then (3.28), (3.30) and (3.31) give the desired estimate. ■

Now we can prove our main result.

Theorem 3.1 *Let Assumptions 3.2-3.3 hold and $u^*\in V$ be the solution of* (1.1)*. Let $u\in V$ be an approximation of u^* such that $\Lambda u\in W$. Then for arbitrary $y^*\in Y^*$ such that $z^* := A^{-1}(y^*)\in W$ and for arbitrary $h\in V$,*

$$E(u)\leq\ EST(u;y^*,h) := \Big(m^{-1/2}\,|\Lambda^* y^*+l|\,+\,\tfrac{L}{2}\,m^{-3/2}\,D(u;y^*,h)$$
$$+\big(\langle A(\Lambda u)-y^*,\ \Lambda u - A^{-1}(y^*)\rangle\,+\,\tfrac{L}{2m}\,D(u;y^*,h)\,\|\Lambda u - A^{-1}(y^*)\|_Y\big)^{1/2}\Big)^2\,, \tag{3.32}$$

where

$$D(u;y^*,h) := \Big(M\,\|A^{-1}(y^*)-\Lambda h\|_Y + |\Lambda^* y^* + l|\Big)\,\|\Lambda u - A^{-1}(y^*)\|_W\,. \tag{3.33}$$

Proof. Lemma 3.1 provides

$$E(u)\leq\ |\Lambda^* y^*+l|\,m^{-1/2}\,E(u)^{1/2}\,+\,\langle A'_{[z^*,\Lambda u]}\,(\Lambda u - z^*),\ \Lambda(u-u^*)\rangle\,, \tag{3.34}$$

and our goal is to estimate the second term more accurately than it is done in Proposition 3.4. Firts, we observe that

$$\langle A'_{[z^*,\Lambda u]}\,(\Lambda u - z^*),\ \Lambda(u-u^*)\rangle$$

$$= \Big\langle \Big(A'_{[z^*,\Lambda u]} - A'_{[\Lambda u^*,\Lambda u]}\Big)\,(\Lambda u - z^*),\ \Lambda(u-u^*)\Big\rangle + \langle A'_{[\Lambda u^*,\Lambda u]}\,(\Lambda u - z^*),\ \Lambda(u-u^*)\rangle\,. \tag{3.35}$$

Using (3.24), the first term of (3.35) satisfies

$$\Big\langle \Big(A'_{[z^*,\Lambda u]} - A'_{[\Lambda u^*,\Lambda u]}\Big)\,(\Lambda u - z^*),\ \Lambda(u-u^*)\Big\rangle \leq$$

$$\leq \tfrac{L}{2}\,\|z^*-\Lambda u^*\|_Y\,\|\Lambda u - z^*\|_W\,\|\Lambda(u-u^*)\|_Y, \tag{3.36}$$

where $\|z^*-\Lambda u^*\|_Y$ fulfils (3.26) and $\|\Lambda(u-u^*)\|_Y \leq m^{-1/2}\,E(u)^{1/2}$ by Proposition 3.3 (ii), hence

$$\Big\langle \Big(A'_{[z^*,\Lambda u]} - A'_{[\Lambda u^*,\Lambda u]}\Big)\,(\Lambda u - z^*),\ \Lambda(u-u^*)\Big\rangle$$

$$\leq \tfrac{L}{2}\, m^{-3/2} \Big(M\, \|z^* - \Lambda h\|_Y + |\Lambda^* y^* + l| \Big)\, \|\Lambda u - z^*\|_W\, E(u)^{1/2}\,. \qquad (3.37)$$

The second term of (3.35) can be estimated with the Cauchy-Schwarz inequality:

$$\langle A'_{[\Lambda u^*, \Lambda u]}\, (\Lambda u - z^*),\ \Lambda(u - u^*)\rangle$$

$$\leq\ \langle A'_{[\Lambda u^*, \Lambda u]}\, (\Lambda u - z^*),\ \Lambda u - z^*\rangle^{1/2}\ \langle A'_{[\Lambda u^*, \Lambda u]}\, \Lambda(u - u^*),\ \Lambda(u - u^*)\rangle^{1/2}\,. \quad (3.38)$$

Proposition 3.3 (i) states that the second factor of (3.38) equals $E(u)^{1/2}$. For the first factor,

$$\langle A'_{[\Lambda u^*, \Lambda u]}\, (\Lambda u - z^*),\ \Lambda u - z^*\rangle$$

$$= \langle A'_{[z^*, \Lambda u]}\, (\Lambda u - z^*),\ \Lambda u - z^*\rangle\ +\ \langle (A'_{[\Lambda u^*, \Lambda u]} - A'_{[z^*, \Lambda u]})\, (\Lambda u - z^*),\ \Lambda u - z^*\rangle\,. \qquad (3.39)$$

Here Proposition 2.2 (iii) yields

$$\begin{aligned}\langle A'_{[z^*, \Lambda u]}\, (\Lambda u - z^*),\ \Lambda u - z^*\rangle\ &= \langle A(\Lambda u) - A(z^*),\ \Lambda u - z^*\rangle \\ &= \langle A(\Lambda u) - y^*,\ \Lambda u - A^{-1}(y^*)\rangle \qquad (3.40)\end{aligned}$$

and (3.24) and (3.26) imply

$$\langle (A'_{[\Lambda u^*, \Lambda u]} - A'_{[z^*, \Lambda u]})\, (\Lambda u - z^*),\ \Lambda u - z^*\rangle \leq$$

$$\leq \frac{L}{2}\, \|\Lambda u^* - z^*\|_Y \|\Lambda u - z^*\|_W \|\Lambda u - z^*\|_Y \leq$$

$$\leq \tfrac{L}{2m} \Big(M\, \|z^* - \Lambda h\|_Y + |\Lambda^* y^* + l| \Big)\, \|\Lambda u - z^*\|_W\, \|\Lambda u - z^*\|_Y\,. \qquad (3.41)$$

Summing up, (3.34), (3.35), (3.37), (3.39), (3.40) and (3.41) yield

$$E(u)^{1/2} \leq\ m^{-1/2} |\Lambda^* y^* + l| + \tfrac{L}{2}\, m^{-3/2} \Big(M \|z^* - \Lambda h\|_Y + |\Lambda^* y^* + l| \Big) \|\Lambda u - z^*\|_W\ +$$

$$\Big(\langle A(\Lambda u) - y^*, \Lambda u - A^{-1}(y^*)\rangle + \tfrac{L}{2m} \Big(M \|z^* - \Lambda h\|_Y + |\Lambda^* y^* + l| \Big) \|\Lambda u - z^*\|_W \|\Lambda u - z^*\|_Y \Big)^{1/2}$$

$$=\ \ m^{-1/2}\, |\Lambda^* y^* + l|\ +\ \tfrac{L}{2}\, m^{-3/2}\, D(u; y^*, h)$$

$$+ \big(\langle A(\Lambda u) - y^*,\ \Lambda u - A^{-1}(y^*)\rangle\ +\ \tfrac{L}{2m}\, D(u; y^*, h)\, \|\Lambda u - z^*\|_Y \big)^{1/2}\,. \qquad \blacksquare$$

The repeated application of the elementary inequality $(a+b)^2 \leq (1+\alpha)a^2 + (1+\frac{1}{\alpha})b^2$ (where $\alpha > 0$) yields

Corollary 3.1 *Under the assumptions and notations of Theorem 3.1, for any constants $\beta > 0$, $\gamma > 0$*

$$E(u) \leq\ \overline{EST}(u; y^*, h, \beta, \gamma) := \qquad (3.42)$$

$$(1+\beta)(1+\gamma) m^{-1}\, |\Lambda^* y^* + l|^2\ +\ (1+\beta)(1+\tfrac{1}{\gamma}) \tfrac{L^2}{4}\, m^{-3}\, D(u; y^*, h)^2$$

$$+ (1+\tfrac{1}{\beta}) \big(\langle A(\Lambda u) - y^*,\ \Lambda u - A^{-1}(y^*)\rangle\ +\ \tfrac{L}{2m}\, D(u; y^*, h)\, \|\Lambda u - z^*\|_Y \big)$$

where $D(u; y^*, h)$ is defined in (3.33).

Remark 3.4 It may be convenient to reformulate Theorem 3.1 for $z^* = A^{-1}(y^*)$ in order to avoid the computation of A^{-1}. Then for arbitrary $z^* \in W$ and for arbitrary $h \in V$,

$$E(u) \le \tilde{EST}(u; z^*, h) := \Big(m^{-1/2} \, |\Lambda^* A(z^*) + l| + \tfrac{L}{2} m^{-3/2} \tilde{D}(u; z^*, h) \tag{3.43}$$
$$+ \Big(\langle A(\Lambda u) - A(z^*), \ \Lambda u - z^* \rangle + \tfrac{L}{2m} \tilde{D}(u; z^*, h) \, \|\Lambda u - z^*\|_Y \Big)^{1/2} \Big)^2$$

where

$$\tilde{D}(u; z^*, h) := \Big(M \, \|z^* - \Lambda h\|_Y + |\Lambda^* A(z^*) + l| \Big) \, \|\Lambda u - z^*\|_W \, . \tag{3.44}$$

Remark 3.5 If A is a linear operator then A' is constant, hence its Lipschitz constant is $L = 0$. In this case all terms containing h vanish, and we have in (3.42)

$$\overline{EST}(u; y^*, h, \beta, \gamma) =$$
$$= (1+\beta)(1+\gamma) m^{-1} \, |\Lambda^* y^* + l|^2 \ + (1 + \tfrac{1}{\beta}) \left(\langle A(\Lambda u) - y^*, \ \Lambda u - A^{-1} y^* \rangle \ \right) ,$$

that is,

$$\hat{EST}(u; y^*, h, \beta) := \overline{EST}(u; y^*, h, \beta, 0)$$
$$= \ (1+\beta) m^{-1} \, |\Lambda^* y^* + l|^2 \ + (1 + \tfrac{1}{\beta}) \left(\langle A(\Lambda u) - y^*, \ \Lambda u - A^{-1} y^* \rangle \ \right) , \tag{3.45}$$

which is nothing but (2.11). (The factor m^{-1} is not present in (2.11) since it is built in the norm $|||.|||$.) This shows that our estimate is a direct extension of (2.11) for nonlinear problems.

Now we can turn to the problem of sharpness.

Proposition 3.5 *Estimates* (3.32) *and* (3.42) *are sharp in the following sense: denoting* $A(W) := \{A(v) : \ v \in W\}$, *we have*

$$\min_{\substack{y^* \in A(W), \\ h \in V}} EST(u; y^*, h) = E(u), \qquad \inf_{\substack{y^* \in A(W), \\ h \in V, \\ \beta, \gamma > 0}} \overline{EST}(u; y^*, h, \beta, \gamma) = E(u) \, ,$$

provided $\Lambda u^* \in W$.

Proof. Let us choose

$$y^* := A(\Lambda u^*) \quad \text{and} \quad h := u^*. \tag{3.46}$$

Then $z^* = A^{-1}(y^*) = \Lambda u^* \in W$, hence this y^* satisfies the assumption of Theorem 3.1. It was already shown in Remark 3.1 that $y^* = A(\Lambda u^*)$ satisfies $\Lambda^* y^* + l = 0$. Hence the first term in both $EST(u; y^*, h)$ and $\overline{EST}(u; y^*, h, \beta, \gamma)$ is zero in this case, further, $A^{-1}(y^*) - \Lambda h = \Lambda u^* - \Lambda u^* = 0$, therefore $D(u; A(\Lambda u^*), \Lambda u^*) = 0$ and thus the terms containing $D(u; y^*, h)$ are also zero in this case. That is,

$$EST(u; A(\Lambda u^*), \Lambda u^*) = \langle A(\Lambda u) - A(\Lambda u^*), \ \Lambda u - \Lambda u^* \rangle = E(u)$$

where (3.11) has been used. Similarly,

$$\overline{EST}(u; A(\Lambda u^*), \Lambda u^*, \beta, \gamma) =$$

$$= (1 + \tfrac{1}{\beta}) \langle A(\Lambda u) - A(\Lambda u^*),\ \Lambda u - \Lambda u^* \rangle = (1 + \tfrac{1}{\beta}) E(u),$$

hence

$$\inf_{\substack{y^* \in Y^*,\\ h \in V,\\ \beta, \gamma > 0}} \overline{EST}(u; y^*, h, \beta, \gamma) \le \inf_{\beta > 0} (1 + \tfrac{1}{\beta}) E(u) = E(u). \qquad \blacksquare$$

Remark 3.6 (Finding the optimal h in a Hilbert space.) In practice, y^* is percieved as an approximation of the optimal unknown value $A(\Lambda u^*)$ (cf. (3.46)). For given y^*, one can determine the optimal h via projection when Y is a Hilbert space. (In this case $\langle ., . \rangle$ means inner product.) This is achieved as follows. Let $z^* := A^{-1}(y^*)$ and let h_{opt} be the solution of the problem

$$\langle \Lambda h_{opt}, \Lambda v \rangle = \langle z^*, \Lambda v \rangle \qquad (v \in V), \tag{3.47}$$

i.e., h_{opt} is the orthogonal projection of z^* on the range of Λ. Then for all $h \in V$

$$z^* - \Lambda h = (z^* - \Lambda h_{opt}) + (\Lambda h_{opt} - \Lambda h),$$

where (3.47) for $v := h_{opt} - h$ shows that the terms on the right are orthogonal. Therefore

$$\|z^* - \Lambda h_{opt}\|_Y \le \|z^* - \Lambda h\|_Y \,.$$

That is, h_{opt} provides the smallest value of $\|z^* - \Lambda h\|_Y$ in (3.44).

Remark 3.7 (The Lipschitz condition for scalar nonlinearities.) The following class of operators A is an important example of the type discussed above, which occurs in many practical models (see section 4) and has the Lipschitz property from Assumption 3.3-(i).

Let E be a finite dimensional Euclidean space with scalar product $[., .]$, and let Y be the function space $L^2(\Omega, E)$, i.e.,

$$Y := \{p : \Omega \to E : \text{ the function } [p, p] \in L^2(\Omega)\} \,.$$

Then Y is a Hilbert space with inner product

$$\langle p, q \rangle = \int_\Omega [p, q] \,,$$

hence Y is a Banach space as well and $Y^* = Y$. Then we define the operator $A : Y \to Y$ as $A(p) := a([p, p])p$, or equivalently (in a test function form)

$$\langle A(p), q \rangle = \int_\Omega \Big(a([p, p])\, [p, q] \Big) \qquad (p, q \in Y), \tag{3.48}$$

where $a : \mathbf{R}^+ \to \mathbf{R}^+$ is a scalar C^2 function with the following properties: there exist constants $M \ge m > 0$ such that

$$0 < m \le a(t) \le M, \qquad 0 < m \le \tfrac{d}{dt} \Big(a(t^2) t \Big) \le M \qquad (t \ge 0), \tag{3.49}$$

further, there exists a constant $L > 0$ such that

$$\left|\tfrac{d^2}{dt^2}\Big(a(t^2)t\Big)\right| \le L \qquad (t \ge 0). \tag{3.50}$$

Then (3.49) implies that A has a bihemicontinuous symmetric Gateaux derivative satisfying

$$m\,\|p\|_Y^2 \le \langle A'(y)p, p\rangle \le M\,\|p\|_Y^2 \qquad (y, p \in Y) \tag{3.51}$$

(see, e.g., [13]), that is, Assumptions 3.2 (ii)-(iii) and Assumption 3.3 (ii) hold. Further, let

$$W := \{p \in Y : \; [p,p] \in L^\infty(\Omega)\}\,, \qquad \|p\|_W := \|\, |p|_E \,\|_{L^\infty(\Omega)}\,,$$

where $|x|_E := [x,x]^{1/2}$ $(x \in E)$. Then A' is Lipschitz continuous as an operator from Y to $B(W, Y^*)$. In fact, (3.50) and a simple calculation imply that the second Gateaux derivative A'' exists and satisfies

$$|A''(p)(y,y,y)|_E \le L \qquad (p, y \in Y,\; \|y\|_Y = 1).$$

Since $A''(p)$ is symmetric, by [48] we obtain $\|A''(p)\| \le L$ for all $p \in Y$, which implies the pointwise Lipschitz continuity

$$\|A'(p) - A'(q)\| \le L\,|p-q|_E \qquad (p, q \in Y). \tag{3.52}$$

Then for all $p, q, s \in Y$, $r \in W$

$$|\langle (A'(p) - A'(q))r, s\rangle| \le L \int_\Omega |p-q|_E\,|r|_E\,|s|_E$$

$$\le L\,\|\,|p-q|_E\,\|_{L^2(\Omega)}\,\|\,|r|_E\,\|_{L^\infty(\Omega)}\,\|\,|s|_E\,\|_{L^2(\Omega)} = L\,\|p-q\|_Y\|r\|_W\|s\|_Y\,,$$

which is (3.22), that is, Assumption 3.3-(i) holds as well.

We underline that (3.50) is a natural condition for functions satisfying (3.49). The latter almost implies (3.50), except for some pathological counterexamples. In particular, if $\frac{d^2}{dt^2}\Big(a(t^2)t\Big)$ is monotone for sufficiently large t, then it is elemetary to verify that (3.49) implies (3.50).

The above results (3.51) and (3.52) obviously remain valid under natural generalizations of the conditions (3.49)–(3.50). First, one can allow dependence on x: we let $a : \Omega \times \mathbf{R}^+ \to \mathbf{R}^+$ be a scalar-valued function that is measurable and bounded w.r. to the variable $x \in \Omega$ and C^2 in the variable $t \in \mathbf{R}$, and satisfies

$$0 < m \le a(x,t) \le M, \qquad 0 < m \le \tfrac{\partial}{\partial t}\Big(a(x,t^2)t\Big) \le M \qquad (x \in \Omega,\; t \ge 0), \tag{3.53}$$

$$\left|\tfrac{\partial^2}{\partial t^2}\Big(a(x,t^2)t\Big)\right| \le L \qquad (x \in \Omega,\; t \ge 0). \tag{3.54}$$

The operator A, where $a([p,p])$ in (3.48) is replaced by $a(x,[p,p])$, then satisfies (3.51) and (3.52). Further, the sum of such operators also inherits this property. For instance, the results hold for

$$\langle A(p), q\rangle = \int_\Omega \left(a(x,[p,p])\,[p,q] + b(x,\{p,p\})\,\{p,q\}\right) \qquad (p,q \in Y) \quad (3.55)$$

where $[.,.]$ and $\{.,.\}$ are two different semi-scalar products on E, such that the sum $[x,y] + \{x,y\}$ for $x,y \in E$ is already a scalar product on E, further, a and b are functions each satisfying (3.53)–(3.54). Finally, it is enough to require a to be C^2 except for finitely many points.

4 Sharp global error estimates for nonlinear elliptic problems

This chapter is the main part of the paper, where we apply the previous abstract results to obtain sharp global error estimates for various concrete nonlinear elliptic problems. The problems considered include second order problems with both Dirichlet and mixed boundary conditions, fourth order problems and second order systems. The restrictions are that they are in divergence form and consist of principal part only: however, as will be pointed out, we thus cover many important real-life models.

4.1 Second order Dirichlet problems

We consider the problem

$$\begin{cases} -\mathrm{div}\, f(\nabla u) = g \\ \quad u_{|\partial\Omega} = 0 \end{cases} \tag{4.1}$$

under the following assumptions:

Assumptions 4.1.

(i) $\Omega \subset \mathbf{R}^d$ is a bounded domain with piecewise C^2 boundary, locally convex at the corners.

(ii) $f \in C^1(\mathbf{R}^d, \mathbf{R}^d)$, the Jacobians $f'(\eta) := \frac{\partial f(\eta)}{\partial \eta}$ are symmetric and there exist constants $M \geq m > 0$ such that

$$m|\xi|^2 \leq f'(\eta)\,\xi\cdot\xi \leq M|\xi|^2 \qquad (\eta, \xi \in \mathbf{R}^d). \tag{4.2}$$

(iii) $f' : \mathbf{R}^d \to \mathbf{R}^{d\times d}$ is Lipschitz continuous with Lipschitz constant L.

(iv) $g \in L^2(\Omega)$.

Let $H_0^1(\Omega)$ denote, as usual, the Sobolev space $\{u \in H^1(\Omega) : \ u_{|\partial\Omega} = 0 \text{ in trace sense}\}$ with inner product

$$\langle u, v\rangle_{H_0^1} := \int_\Omega \nabla u \cdot \nabla v, \tag{4.3}$$

further, let

$$H(\mathrm{div}\,) := \{y \in L^2(\Omega)^d : \ \mathrm{div}\, y \in L^2(\Omega)\}.$$

We will also use the space $L^2(\Omega)^d$ with the usual inner product $\langle y, z\rangle_{L^2(\Omega)^d} := \int_\Omega y \cdot z$. Finally we recall the Friedrichs inequality: there exists a constant $C_\Omega > 0$ such that

$$\|v\|_{L^2(\Omega)} \le C_\Omega \, \|\nabla v\|_{L^2(\Omega)^d} \quad (v \in H_0^1(\Omega)). \tag{4.4}$$

Assumptions (ii) and (iv) imply that problem (4.1) has a unique weak solution $u^* \in H_0^1(\Omega)$, i.e., that satisfies

$$\int_\Omega f(\nabla u^*) \cdot \nabla v - \int_\Omega gv = 0 \qquad (v \in H_0^1(\Omega)). \tag{4.5}$$

We consider an approximate solution $u \in H_0^1(\Omega)$ and measure the error by the functional

$$E(u) := \int_\Omega (f(\nabla u) - f(\nabla u^*)) \cdot (\nabla u - \nabla u^*) = \int_\Omega f(\nabla u) \cdot (\nabla u - \nabla u^*) - \int_\Omega g(u - u^*). \tag{4.6}$$

We note that by (2.3),

$$\|u - u^*\|_{H_0^1}^2 \le m^{-1}\, E(u).$$

4.1.1 The error estimation

Now we formulate and prove our main result on the error estimation for (4.1) for the approximate solution u.

Theorem 4.1 *Let $u \in W^{1,\infty}(\Omega)$. Then for arbitrary $y^* \in H(\mathrm{div}\,) \cap L^\infty(\Omega)^d$ and arbitrary $h \in H_0^1(\Omega)$,*

$$E(u) \le \ EST(u; y^*, h) := \big(m^{-1/2}\, C_\Omega \, \|\mathrm{div}\, y^* + g\|_{L^2(\Omega)} \ + \ \tfrac{L}{2}\, m^{-3/2}\, D(u; y^*, h) \tag{4.7}$$

$$+ \big(\langle f(\nabla u) - y^*, \ \nabla u - f^{-1}(y^*)\rangle_{L^2(\Omega)^d} \ + \ \tfrac{L}{2m}\, D(u; y^*, h)\, \|\nabla u - f^{-1}(y^*)\|_{L^2(\Omega)^d}\big)^{1/2}\Big)^2 ,$$

where

$$D(u; y^*, h) := \Big(M\, \|f^{-1}(y^*) - \nabla h\|_{L^2(\Omega)^d} + C_\Omega \, \|\mathrm{div}\, y^* + g\|_{L^2(\Omega)}\Big)\, \|\nabla u - f^{-1}(y^*)\|_{L^\infty(\Omega)^d} . \tag{4.8}$$

Proof. Let $V := H_0^1(\Omega)$ and $Y := L^2(\Omega)^d$. We will use Theorem 3.1, to which end we must verify that Assumptions 3.2-3.3 hold for the corresponding spaces and operators.

First, Assumption 3.2 (i) is valid for the operator $\Lambda := \nabla$, since (4.3) just yields that (3.7) holds. Now let $A : L^2(\Omega)^d \to L^2(\Omega)^d$ be defined by

$$A(y) := f(y) \quad \text{(or, more precisely, } f \circ y), \tag{4.9}$$

that is, outer composition with f. Such an operator is often called a Nemyczki operator (see, e.g., [49]), and it follows in a standard way [13, 49] from our condition $f \in C^1(\mathbf{R}^d, \mathbf{R}^d)$ and from the assumed symmetry of the Jacobians that A has a bihemicontinuous symmetric Gateaux derivative according to Definition 2.3, i.e., Assumption 3.2 (ii) holds. The Gateaux derivative of A satisfies

$$\langle A'(y)p, q\rangle_{L^2(\Omega)^d} = \int_\Omega f'(y)\, p \cdot q \qquad (y, p, q \in L^2(\Omega)^d), \tag{4.10}$$

hence by (4.2) we have

$$m\, \|p\|^2_{L^2(\Omega)^d} \le \langle A'(y)p, p\rangle_{L^2(\Omega)^d} \le\ M\, \|p\|^2_{L^2(\Omega)^d} \qquad (y, p \in L^2(\Omega)^d). \tag{4.11}$$

The left-hand side of (4.11) coincides with Assumption 3.2 (iii). Finally, defining the operator $F : H_0^1(\Omega) \to H^{-1}(\Omega)$ via

$$\langle F(u), v\rangle \equiv \int_\Omega f(\nabla u) \cdot \nabla v \qquad (u, v \in H_0^1(\Omega)), \tag{4.12}$$

we obtain the equality (3.9), required for Assumption 3.2 (iv) to hold.

To verify Assumption 3.3 (i), let us define $W := L^\infty(\Omega)^d$ with the standard norm $\|y\|_{L^\infty(\Omega)^d} := \operatorname{ess\,sup}_\Omega |y|$. For the required Lipschitz continuity of A' from $L^2(\Omega)^d$ to $B(L^\infty(\Omega)^d, L^2(\Omega)^d)$, we must prove (3.22) for (4.9). In fact, we have imposed in Assumption 4.1 (iii) the Lipschitz continuity of f' with constant $L > 0$, i.e.,

$$\|f'(\xi) - f'(\eta)\| \le L|\xi - \eta| \qquad (\xi, \eta \in \mathbf{R}^d). \tag{4.13}$$

Therefore

$$|\langle (A'(z) - A'(y))\, w, p\rangle| = |\int_\Omega (f'(z) - f'(y))\, w \cdot p|$$

$$\le L \int_\Omega |z - y|\, |w|\, |p| \ \le\ L\, \|z - y\|_{L^2(\Omega)^d} \|w\|_{L^\infty(\Omega)^d} \|p\|_{L^2(\Omega)^d}, \tag{4.14}$$

$$y, z, p \in L^2(\Omega)^d, w \in L^\infty(\Omega)^d,$$

which is the desired estimate. Assumption 3.3 (ii) for (4.9) coincides with the right-hand side of (4.11).

It is left to check the remaining assumptions of Theorem 3.1. Defining the linear functional $l : H_0^1(\Omega) \to \mathbf{R}$ as

$$\langle l, v \rangle \equiv - \int_\Omega gv \qquad (v \in H_0^1(\Omega)) \tag{4.15}$$

and using (4.12), the weak formulation (4.5) of our problem becomes

$$\langle F(u^*), v \rangle + \langle l, v \rangle = 0,$$

i.e. u^* is the solution of (1.1) indeed. We have chosen u to satisfy $u \in W^{1,\infty}(\Omega)$, hence $u \in V = H_0^1(\Omega)$ and $\Lambda u = \nabla u \in W = L^\infty(\Omega)^d$. Further, we have assumed $y^* \in W = L^\infty(\Omega)^d$, and the left-hand side of (4.2) implies trivially that f^{-1} carries bounded sets into bounded sets (since it grows at most linearly with factor $1/m$), therefore $z^* := A^{-1}(y^*) = f^{-1}(y^*) \in L^\infty(\Omega)^d = W$. Finally, $h \in H_0^1(\Omega) = V$. That is, all the assumptions of Theorem 3.1 hold, therefore (3.32) is valid for our problem.

It remains to show that the general estimate (3.32) for our problem becomes estimate (4.7). Here, using $y^* \in H(\mathrm{div}\,)$,

$$\langle \Lambda^* y^*, v \rangle = \langle y^*, \Lambda v \rangle = \int_\Omega y^* \cdot \nabla v = - \int_\Omega (\mathrm{div}\, y^*) v \qquad (v \in H_0^1(\Omega)),$$

hence $\Lambda^* y^* = -\mathrm{div}\, y^*$. Then, by (3.14),

$$|\Lambda^* y^* + l| = \sup_{\|v\|_{H_0^1}=1} |\langle \Lambda^* y^* + l, v \rangle| = \sup_{\|v\|_{H_0^1}=1} \Big| - \int_\Omega (\mathrm{div}\, y^* + g) v \Big|$$

$$\leq \sup_{\|v\|_{H_0^1}=1} \|\mathrm{div}\, y^* + g\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \leq C_\Omega \, \|\mathrm{div}\, y^* + g\|_{L^2(\Omega)},$$

where C_Ω comes from the Friedrichs inequality (4.4), see also [38]. Plugging the latter into (3.32) and (3.33), and replacing V, Y, W, Λ and A by $H_0^1(\Omega)$, $L^2(\Omega)^d$, $L^\infty(\Omega)^d$, ∇ and f, respectively, we obtain (4.7). ∎

Remark 4.1 Following Remark 3.4, it is convenient to reformulate Theorem 4.1 for $z^* := f^{-1}(y^*)$ in order to avoid the computation of f^{-1}. Then, letting $u \in W^{1,\infty}(\Omega)$ be any approximate solution, for arbitrary $z^* \in L^\infty(\Omega)^d$ such that $f(z^*) \in H(\mathrm{div}\,)$, and for arbitrary $h \in H_0^1(\Omega)$,

$$E(u) \leq \tilde{E}ST(u; z^*, h) := \Big(m^{-1/2} C_\Omega \|\mathrm{div}\, f(z^*) + g\|_{L^2(\Omega)} + \tfrac{L}{2}\, m^{-3/2} \tilde{D}(u; z^*, h) \tag{4.16}$$

$$+ \Big(\langle f(\nabla u) - f(z^*),\ \nabla u - z^* \rangle_{L^2(\Omega)^d} \ + \ \tfrac{L}{2m}\, \tilde{D}(u; z^*, h)\, \|\nabla u - z^*\|_{L^2(\Omega)^d} \Big)^{1/2} \Big)^2 ,$$

where

$$\tilde{D}(u; z^*, h) := \Big(M \, \|z^* - \nabla h\|_{L^2(\Omega)^d} + C_\Omega \, \|\mathrm{div}\, f(z^*) + g\|_{L^2(\Omega)} \Big) \, \|\nabla u - z^*\|_{L^\infty(\Omega)^d} \, . \tag{4.17}$$

We note that one can further estimate (4.7) using quadratic terms as in Corollary 3.1, which we leave to the reader. Now we state the sharpness of the estimate:

Proposition 4.1 *Estimate* (4.7) *is sharp, that is,*

$$\min_{\substack{y^*\in H(\mathrm{div}\,)\cap L^\infty(\Omega)^d,\\ h\in H_0^1(\Omega)}} EST(u;y^*,h) = E(u).$$

PROOF. By [31], the weak solution of (4.1) satisfies $u^*\in C^{1,\alpha}(\overline{\Omega})$ with some $0<\alpha<1$, hence $\nabla u^*\in L^\infty(\Omega)^d=W$. Therefore we can apply Proposition 3.5 to obtain the desired statement. ∎

We note that by (3.46), the optimal values for "free" parameters in the estimate are

$$y^* := f(\nabla u^*) \quad \text{and} \quad h := u^*. \tag{4.18}$$

The practical approximations of these will be discussed in subsection 4.1.2.

Remark 4.2 Our result is a direct extension of earlier sharp error estimates obtained for linear problems first in [41] by means of the duality theory (see also [38]), and later in [42] (via the Helmholtz decomposition) and in [26] (via integral identities). Namely, if we have the linear equation

$$-\mathrm{div}\,(\mathcal{A}\nabla u) = g$$

for some symmetric and uniformly positive definite matrix $\mathcal{A}$ in (4.1), which corrersponds to the case $f(\eta):=\mathcal{A}\eta$, then we can use Remark 3.5. Now the Lipschitz constant is $L=0$, i.e. the terms containing L vanish, and (3.45) yields

$$E(u)\le(1+\beta)m^{-1}\|\mathrm{div}\,y^*+g\|^2_{L^2(\Omega)}+(1+\tfrac{1}{\beta})\langle\mathcal{A}\nabla u-y^*,\,\nabla u-\mathcal{A}^{-1}y^*\rangle_{L^2(\Omega)^d}.$$

4.1.2 Practical considerations

Finite element solution. A most important practical case is when finite element approximation is used to find an approximate solution. In general, let V_h be a given FEM subspace and $u_h\in V_h$ be the corresponding FEM approximation of the exact solution u^*. Then our error measure is

$$E(u_h) = \langle F(u_h)-F(u^*),\, u_h-u^*\rangle. \tag{4.19}$$

Here u_h is a continuous piecewise polynomial, hence condition $u_h\in W^{1,\infty}(\Omega)$ in Theorem 4.1 is satisfied. If we choose y^* to be any continuous piecewise polynomial function, e.g. a function from another FEM subspace, and arbitrary $w\in H_0^1(\Omega)$, then $y^*\in H(\mathrm{div}\,)\cap L^\infty(\Omega)^d$, hence Theorem 4.1 can be applied, which gives

$$E(u_h)\le\ EST(u_h;y^*,w):=\big(m^{-1/2}C_\Omega\|\mathrm{div}\,y^*+g\|_{L^2(\Omega)}+\tfrac{L}{2}m^{-3/2}D(u_h;y^*,w) \tag{4.20}$$

$$+\Big(\big(\langle f(\nabla u_h) - y^*, \nabla u_h - f^{-1}(y^*)\rangle_{L^2(\Omega)^d} + \tfrac{L}{2m} D(u_h; y^*, w)\|\nabla u_h - f^{-1}(y^*)\|_{L^2(\Omega)^d}\big)^{1/2}\Big)^2,$$

where

$$D(u_h; y^*, w) := \Big(M\|f^{-1}(y^*) - \nabla w\|_{L^2(\Omega)^d} + C_\Omega\|\operatorname{div} y^* + g\|_{L^2(\Omega)}\Big)\|\nabla u_h - f^{-1}(y^*)\|_{L^\infty(\Omega)^d}. \tag{4.21}$$

We note that it is useful to replace $f^{-1}(y^*)$ by z^* as in (4.16)–(4.17) to avoid the computation of f^{-1}. The obtained expressions are directly computable integrals.

Determining optimal y^* and w in $EST(u_h; y^*, w)$. Following (4.18), the optimal value of the parameter y^* should be a sufficiently accurate approximation of $f(\nabla u^*)$. For finite element solutions, a common and "computationally cheap" way to achieve this goal is to use an averaging procedure, i.e., to replace the unknown function ∇u^* (the gradient of the exact solution) by $G_h(\nabla u_h)$, where G_h is some averaging operator. For the case of linear finite elements, $G_h(\nabla u_h)$ is closer to ∇u^* than is ∇u_h by an order of magnitude, namely, the original approximation order $\|\nabla u^* - \nabla u_h\|_{L^2} = O(h)$ can be thus improved to $\|\nabla u^* - G_h(\nabla u_h)\|_{L^2} = O(h^2)$ if u^* is sufficiently smooth, see [8] or [21, Part I] for details. Accordingly, we can define

$$y^* := f\Big(G_h(\nabla u_h)\Big), \qquad z^* = f^{-1}(y^*) = G_h(\nabla u_h) \tag{4.22}$$

as a first candidate for the paramater y^* (or z^*).

Remark 4.3 The value (4.22) for the paramater y^* (or z^*) can still give a too rough bound in general, in which case we normally execute a minimization process for y^*, see [38, section 6.10] or [29, 18] for more details.

Next, using Remark 3.6, the optimal w for this z^* is given as the solution of the following linear auxiliary problem: find $w_{opt} \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla w_{opt} \cdot \nabla v = \int_\Omega z^* \cdot \nabla v \qquad (v \in H_0^1(\Omega)), \tag{4.23}$$

that is, the weak solution of the Poisson problem

$$\begin{cases} -\Delta w_{opt} = -\operatorname{div} z^* \\ \quad w_{opt\,|\partial\Omega} = 0. \end{cases} \tag{4.24}$$

This means that for given y^*, the optimal estimate for the second parameter w is found by solving a kind of adjoint or auxiliary equation; however, the latter is linear, hence its numerical solution costs much less than for the original one. For piecewise linear FEM, if (4.24) is solved numerically on the same mesh as used for u_h, then its right-hand side $-\operatorname{div} z^* = -\operatorname{div} G_h(\nabla u_h)$ is constant on each element, hence it requires minimal numerical integration and is therefore a cheap auxiliary problem. On the other hand, using a finer (or just different) mesh for (4.24) than the one used for u_h may considerably increase the accuracy of the estimate, similarly as for adjoint problems for linear equations [19, 25] (see also [45]), with low extra cost due to the linearity of (4.24).

Calculating the required constants. The constants used in estimate (4.7) are C_Ω, m, M and L. The only one depending on the domain is C_Ω, which can be easily estimated from above (it is sufficient for the estimation purposes) as in [33, p. 8]. Further, the three remaining constants m, M and L come from the given nonlinearity, see Assumptions 4.1 (ii)-(iii), where we note that a crucial point in our sharp estimates is the existence of L, i.e., the condition of Lipschitz continuity of the derivative of f. Based on Remark 3.7, one can see that this Lipschitz condition usually means no restriction in practice, since it is satisfied for most real problems. Namely, problems of the type (4.1) in real models are generally of the following special form, involving a scalar nonlinearity:

$$\begin{cases} -\mathrm{div}\left(a(|\nabla u|^2)\,\nabla u\right) = g \\ u_{|\partial\Omega} = 0 \end{cases} \tag{4.25}$$

(which corresponds to $f(\eta) = a(|\eta|^2)\,\eta$ in (4.1)), where $a : \mathbf{R}^+ \to \mathbf{R}^+$ is a scalar C^2 function with the following properties: there exist constants $M \geq m > 0$ such that

$$0 < m \leq a(t) \leq M, \qquad 0 < m \leq \tfrac{d}{dt}\left(a(t^2)t\right) \leq M \qquad (t \geq 0), \tag{4.26}$$

further, there exists a constant $L > 0$ such that

$$\left|\tfrac{d^2}{dt^2}\left(a(t^2)t\right)\right| \leq L \qquad (t \geq 0). \tag{4.27}$$

Such nonlinearities form the main examples for (4.1), arising, e.g., in elasto-plastic torsion [23], or in electromagneticity, see the presentation from nonlinear Maxwell equations in [28] and for nonlinear magnetostatic field in [11]. One may even have concrete formulas for the function a, such as

$$a(t) = \frac{1}{\mu_0}\left(\alpha + (1-\alpha)\frac{t^8}{t^8+\beta}\right) \qquad (t \geq 0), \tag{4.28}$$

which characterizes the reluctance of stator sheets in the cross-sections of an electrical motor [28], or

$$a(t) = \left(1 - (c-d)\frac{1}{t^2+c}\right) \qquad (t \geq 0), \tag{4.29}$$

which describes magnetostatic field [11]; the constants in these formulas are given positive characteristic physical values. Using Remark 3.7, condition (4.27) implies the Lipschitz continuity for f with the same constant L, that is, (4.13) holds. It has also been pointed out in Remark 3.7 that condition

(4.27) follows from the standard ellipticity property (4.26) except for some unrealistic special cases.

Summing up, it follows that the bounds m and M and the Lipschitz constant L, needed to calculate $E(u)$, can be determined as lower or upper bounds, respectively, for the scalar functions in (4.26)–(4.27). These only require an elementary numerical calculation. Moreover, if the parameters y^* and w are close to the optimal choice, then (using Proposition 3.5) all terms containing these constants (as well as C_Ω) in $EST(u_h; y^*, w)$ are close to zero, hence the global constants need not be estimated from above much accurately.

4.2 Second order mixed problems

Now we study second order problems with mixed boundary conditions. We also allow dependence of the nonlinearity f on x, which was not included in (4.1) for simpliciy. Let us therefore consider the problem

$$\begin{cases} -\operatorname{div} f(x, \nabla u) = g \\ u_{|\Gamma_D} = 0 \\ f(x, \nabla u) \cdot \nu_{\ |\Gamma_N} = \gamma \end{cases} \tag{4.30}$$

(where ν denotes the outer normal direction) under the following assumptions:

Assumptions 4.2.

(i) $\Omega \subset \mathbf{R}^d$ is a bounded piecewise C^1 domain; Γ_D, Γ_N are disjoint open measurable subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and $\Gamma_D \neq \emptyset$.

(ii) The function $f : \Omega \times \mathbf{R}^d \to \mathbf{R}^d$ is measurable and bounded w.r. to the variable $x \in \Omega$ and C^1 in the variable $\eta \in \mathbf{R}^d$. The Jacobians

$$f'(x,\eta) := \frac{\partial f(x,\eta)}{\partial \eta}$$

are symmetric and there exist constants $M \geq m > 0$ such that

$$m|\xi|^2 \leq f'(x,\eta)\ \xi \cdot \xi \leq M|\xi|^2 \qquad (x \in \Omega,\ \eta, \xi \in \mathbf{R}^d). \tag{4.31}$$

(iii) $f' : \Omega \times \mathbf{R}^d \to \mathbf{R}^{d\times d}$ is Lipschitz continuous in the variable η with Lipschitz constant L.

(iv) $g \in L^2(\Omega)$, $\gamma \in L^2(\Gamma_N)$.

Some more general assumptions will be considered in Remark 4.6.

The treatment of this problem is much similar to the Dirichlet problem in the previous section. As an appropriate modification, now we use the Sobolev space

$$H^1_D(\Omega) := \{u \in H^1(\Omega): \ u_{|\Gamma_D} = 0 \text{ in trace sense}\} \tag{4.32}$$

with inner product

$$\langle u, v\rangle_{H^1_D} := \int_\Omega \nabla u \cdot \nabla v, \tag{4.33}$$

further, let

$$H(\mathrm{div}\,, \Gamma_N) := \{y \in L^2(\Omega)^d: \ \mathrm{div}\, y \in L^2(\Omega), \ y \cdot \nu \in L^2(\Gamma_N)\}.$$

Instead of the Friedrichs inequality (4.4), we now use the estimates

$$\|v\|_{L^2(\Omega)} \le C'_\Omega \|\nabla v\|_{L^2(\Omega)^d}, \qquad \|v\|_{L^2(\Gamma_N)} \le C_{\Gamma_N} \|\nabla v\|_{L^2(\Omega)^d} \qquad (v \in H^1_D(\Omega)) \tag{4.34}$$

for some suitable constants $C'_\Omega, C_{\Gamma_N} > 0$, for the proofs see [35].

Similarly to section 4.1, the assumptions imply that problem (4.30) has a unique weak solution $u^* \in H^1_D(\Omega)$, i.e., that now satisfies

$$\int_\Omega f(x, \nabla u^*) \cdot \nabla v - \int_\Omega gv - \int_{\Gamma_N} \gamma v = 0 \qquad (v \in H^1_D(\Omega)). \tag{4.35}$$

We consider an approximate solution $u \in H^1_D(\Omega)$ and measure the error by the similar functional (4.6) as before (except that f also depends on x and the right-hand side expression in (4.6) is completed by the term $-\int_{\Gamma_N} \gamma(u-u^*)$).

Assumption 4.2 (ii) implies that for all fixed $x \in \Omega$, the function $f(x,.)$ is invertible on $\mathbf{R}^d$ w.r.t. η. We will denote by f^{-1} the inverse w.r.t. η, i.e.

$$f(x, \eta) = \xi \quad \Rightarrow \quad f^{-1}(x, \xi) := \eta. \tag{4.36}$$

Theorem 4.2 *Let $u \in W^{1,\infty}(\Omega)$. Then for arbitrary $y^* \in H(\mathrm{div}\,, \Gamma_N) \cap L^\infty(\Omega)^d$ and arbitrary $h \in H^1_D(\Omega)$,*

$$E(u) \le \ EST(u; y^*, h) := \tag{4.37}$$

$$\Big(m^{-1/2} C'_\Omega \|\mathrm{div}\, y^* + g\|_{L^2(\Omega)} + m^{-1/2} C_{\Gamma_N} \|y^* \cdot \nu - \gamma\|_{L^2(\Gamma_N)} + \tfrac{L}{2} m^{-3/2} D(u; y^*, h)$$

$$+ \Big(\langle f(x, \nabla u) - y^*, \nabla u - f^{-1}(x, y^*)\rangle_{L^2(\Omega)^d} + \tfrac{L}{2m} D(u; y^*, h) \|\nabla u - f^{-1}(x, y^*)\|_{L^2(\Omega)^d}\Big)^{1/2}\Big)^2$$

where

$$D(u; y^*, h) := \Big(M \|f^{-1}(x, y^*) - \nabla h\|_{L^2(\Omega)^d} + C'_\Omega \|\mathrm{div}\, y^* + g\|_{L^2(\Omega)} \tag{4.38}$$

$$+ C_{\Gamma_N} \|y^* \cdot \nu - \gamma\|_{L^2(\Gamma_N)}\Big) \|\nabla u - f^{-1}(x, y^*)\|_{L^\infty(\Omega)^d}\,.$$

PROOF. One can repeat the proof of Theorem 4.1 to verify that Theorem 3.1 holds in the spaces $V := H^1_D(\Omega)$ and $Y := L^2(\Omega)^d$ for $\Lambda := \nabla$. The definition of A in (4.9) is replaced by $A(y) := f \circ (id, y)$, i.e. $A(y)(x) := f(x, y(x))$ for $y \in L^2(\Omega)^d$, $x \in \Omega$; then $A^{-1}(y) = f^{-1} \circ (id, y)$ in the sense of (4.36). The main difference comes after this, when the term $|\Lambda^* y^* + l|$ is estimated. The last paragraph of the proof of Theorem 4.1 has to be modified as follows, in a similar vein as in [38, 26], to show that (3.32) now becomes (4.37). Here the linear functional l in (4.15) is replaced by $l : H^1_D(\Omega) \to \mathbf{R}$,

$$\langle l, v\rangle \equiv -\int_\Omega gv - \int_{\Gamma_N} \gamma v \qquad (v \in H^1_D(\Omega)). \tag{4.39}$$

Using $y^* \in H(\operatorname{div}, \Gamma_N)$,

$$\langle \Lambda^* y^*, v\rangle = \langle y^*, \Lambda v\rangle = \int_\Omega y^* \cdot \nabla v = -\int_\Omega (\operatorname{div} y^*) v + \int_{\Gamma_N} (y^* \cdot \nu)\, v \qquad (v \in H^1_D(\Omega)),$$

hence

$$|\Lambda^* y^* + l| = \sup_{\|v\|_{H^1_D}=1} |\langle \Lambda^* y^* + l, v\rangle| = \sup_{\|v\|_{H^1_D}=1} \left| -\int_\Omega (\operatorname{div} y^* + g) v + \int_{\Gamma_N} (y^* \cdot \nu - \gamma)\, v \right|$$

$$\leq \sup_{\|v\|_{H^1_D}=1} \Big(\|\operatorname{div} y^* + g\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|y^* \cdot \nu - \gamma\|_{L^2(\Gamma_N)} \|v\|_{L^2(\Gamma_N)} \Big)$$

$$\leq C'_\Omega \|\operatorname{div} y^* + g\|_{L^2(\Omega)} + C_{\Gamma_N} \|y^* \cdot \nu - \gamma\|_{L^2(\Gamma_N)} \tag{4.40}$$

where C'_Ω, C_{Γ_N} come from (4.34), see also [38, 26]. Plugging the latter into (3.32) and (3.33), and replacing V, Y, W, Λ and A by $H^1_D(\Omega)$, $L^2(\Omega)^d$, $L^\infty(\Omega)^d$, ∇ and f, respectively, we obtain (4.37). ■

Remark 4.4 Similarly to Remark 4.1, it is convenient to reformulate Theorem 4.2 for $z^* := f^{-1}(x, y^*)$ in order to avoid the computation of $f^{-1}(x, .)$. Here f^{-1} is understood w.r.t. η as in (4.36), hence $y^* = f(x, z^*)$. Then, letting $u \in W^{1,\infty}(\Omega)$ be any approximate solution, for arbitrary $z^* \in L^\infty(\Omega)^d$ such that $f(x, z^*) \in H(\operatorname{div}, \Gamma_N)$, and for arbitrary $h \in H^1_D(\Omega)$,

$$E(u) \leq \tilde{E}ST(u; z^*, h) := \tag{4.41}$$

$$\Big(m^{-1/2} C'_\Omega \|\operatorname{div} f(x, z^*) + g\|_{L^2(\Omega)} + m^{-1/2} C_{\Gamma_N} \|f(x, z^*) \cdot \nu - \gamma\|_{L^2(\Gamma_N)} + \tfrac{L}{2} m^{-3/2} D(u; z^*, h)$$

$$+ \Big(\langle f(x, \nabla u) - f(x, z^*),\ \nabla u - z^*\rangle_{L^2(\Omega)^d} + \tfrac{L}{2m} D(u; z^*, h)\, \|\nabla u - z^*\|_{L^2(\Omega)^d} \Big)^{1/2} \Big)^2$$

where

$$D(u; z^*, h) := \Big(M \|z^* - \nabla h\|_{L^2(\Omega)^d} + C'_\Omega \|\operatorname{div} f(x, z^*) + g\|_{L^2(\Omega)} \tag{4.42}$$

$$+ C_{\Gamma_N} \|f(x, z^*) \cdot \nu - \gamma\|_{L^2(\Gamma_N)} \Big) \|\nabla u - z^*\|_{L^\infty(\Omega)^d}\,.$$

Remark 4.5 Our result is a direct extension of earlier sharp error estimates obtained for linear mixed problems in [26] (see also [43]), which can be seen using Remark 3.5 in the same way as in Remark 4.2.

Remark 4.6 The analogues of Theorem 4.2 can be proved similarly if (4.30) is replaced by one of the following problems:

(a) Neumann problem. Allowing $\Gamma_D = \emptyset$ in Assumption 4.2. (i), we have

$$\begin{cases} -\mathrm{div}\, f(x, \nabla u) = g \\ f(x, \nabla u) \cdot \nu_{\;|\partial\Omega} = \gamma. \end{cases} \tag{4.43}$$

Then Theorem 4.2 remains true if we substitute the factorized space $V := \dot{H}^1(\Omega) := \{u \in H^1(\Omega) : \int_\Omega u = 0\}$ instead of $H^1_D(\Omega)$ and replace Γ_N by $\partial\Omega$ in the formulas. (With this factorization, (4.33) remains an inner product and the proof goes on in the same way.) In particular, the resulting constant $C_{\partial\Omega}$ to satisfy the second inequality in (4.34) for all $v \in \dot{H}^1(\Omega)$ is the smallest positive eigenvalue of $-\Delta$ with Neumann boundary conditions.

(b) Interface problems. Let Γ_{int} be a piecewise smooth surface lying in the interior of Ω, and let us consider the problem

$$\begin{cases} -\mathrm{div}\, f(x, \nabla u) = g \\ u_{|\Gamma_D} = 0 \\ f(x, \nabla u) \cdot \nu_{\;|\Gamma_N} = \gamma_N \\ f(x, \nabla u) \cdot \nu_{\;|\Gamma_{int}} = \gamma_{int}, \end{cases} \tag{4.44}$$

where Assumptions 4.2 are modified such that the condition on γ in (iv) is replaced by $\gamma_N \in L^2(\Gamma_N)$ and $\gamma_{int} \in L^2(\Gamma_{int})$. The weak form of this problem is the same as (4.35) if Γ_N is replaced by $\Gamma := \Gamma_N \cup \Gamma_{int}$, see [24] for a related setting. Defining $\gamma \in L^2(\Gamma)$ such that its restrictions to Γ_N and Γ_{int} are γ_N and γ_{int}, respectively, Theorem 4.2 remains true if we replace Γ_N by Γ in the formulas.

Now we turn to the sharpness problem.

Proposition 4.2 *Estimate* (4.37) *is sharp, that is,*

$$\min_{\substack{y^* \in H(\mathrm{div}, \Gamma_N) \cap L^\infty(\Omega)^d, \\ h \in H^1_0(\Omega)}} EST(u; y^*, h) = E(u), \tag{4.45}$$

provided that the exact solution satisfies $u^* \in W^{1,\infty}(\Omega)$.

Proof. By assumption $\nabla u^* \in L^\infty(\Omega)^d = W$, therefore we can apply Proposition 3.5 to obtain the sharpness statement. ■

We note that by (3.46), analogously to (4.18), the optimal values are

$$y^* := f(x, \nabla u^*) \quad \text{and} \quad h := u^*. \tag{4.46}$$

As can be seen, in contrast to the Dirichlet problem before, the property $u^* \in W^{1,\infty}(\Omega)$ is not a priori known (to the authors' knowledge). Shortage of regularity may often arise in corners where the Dirichlet and Neumann boundaries Γ_D and Γ_N meet. For linear problems in 2D, it has been shown for problems with corners (see [16]) that the solution often has the representation

$$u^* = u_0 + \sum_{j=1}^{N} c_j r_j^{\beta_j} s_j(\theta_j), \tag{4.47}$$

where $B_1, \dots, B_N$ are the corners, u_0 is smooth on $\overline{\Omega}$ (in our case, it suffices that $u_0 \in W^{1,\infty}(\Omega)$), for all $j = 1, \dots, N$ the variables r_j and θ_j are the polar coordinates around the vertex B_j, further, $c_j \in \mathbf{R}$ and $\beta_j > 0$ are constants, $s_j \in C^1[0, 2\pi]$. We assume for simplicity that Ω is a polygon. In this case the assumption $u^* \in W^{1,\infty}(\Omega)$ in Proposition 4.2 can be weakened as follows:

Proposition 4.3 *Let the representation* (4.47) *hold, and let the exact solution and the right-hand sides satisfy* $u^* \in W^{1,4}(\Omega)$, $g \in L^4(\Omega)$ *and* $\gamma \in L^\infty(\Omega)$*, respectively, further, assume that* $f(x, 0) \equiv 0$. *Then estimate* (4.37) *is sharp in the sense of* (4.45).

Since the proof is technical, we only outline it in the Appendix.

Practical considerations. The application of (4.37) or (4.41) goes on similarly as discussed for Dirichlet problems in subsection 4.1.2. For finite element solutions, the error is measured by (4.19). Here the FEM approximation u_h is continuous piecewise polynomial, hence condition $u_h \in W^{1,\infty}(\Omega)$ in Theorem 4.2 is satisfied. If we choose y^* to be any continuous piecewise polynomial function, e.g. a function from another FEM subspace, and arbitrary $w \in H^1_D(\Omega)$, then $y^* \in H(\operatorname{div}, \Gamma_N) \cap L^\infty(\Omega)^d$, hence Theorem 4.2 can be applied. We obtain $E(u_h) \le \ EST(u_h; y^*, w)$, where the latter is a similar expression as in (4.20) with the modification that (as in (4.37)) the terms $C_\Omega \, \|\operatorname{div} y^* + g\|_{L^2(\Omega)}$ are replaced by the sum of two terms in (4.40).

To determine suitable y^* and w in $EST(u_h; y^*, w)$, following (4.46), first y^* should be some approximation of $f(x, \nabla u^*)$. For finite element solutions, using averaging as in (4.22), we can first let

$$y^* := f\Big(x, G_h(\nabla u_h)\Big), \quad z^* = f^{-1}(x, y^*) = G_h(\nabla u_h), \tag{4.48}$$

where G_h is some averaging operator and f^{-1} is understood w.r.t. η as in (4.36). Averaging for mixed boundary conditions is discussed, e.g., in [21, Part II]. More accurate error bounds can be obtained by suitable further bound minimization w.r.t. y^* (or z^*) as mentioned in Remark 4.3.

Then by Remark 3.6, the optimal w for this z^* to set in $EST(u_h; y^*, w)$ is given as the solution of a linear auxiliary problem, which is the modification of (4.23) for mixed boundary conditions: find $w_{opt} \in H^1_D(\Omega)$ such that

$$\int_\Omega \nabla w_{opt} \cdot \nabla v = \int_\Omega z^* \cdot \nabla v \qquad (v \in H^1_D(\Omega)), \tag{4.49}$$

that is, the weak solution of the Poisson problem

$$\begin{cases} -\Delta w_{opt} = -\operatorname{div} z^* \\ w_{opt\,|\Gamma_D} = 0 \\ \frac{\partial w_{opt}}{\partial \nu}_{\,|\Gamma_N} = z^* \cdot \nu \end{cases} \tag{4.50}$$

and one can therefore define w as the numerical solution of (4.50) on a suitable mesh (either the same as used for u_h or a finer/different mesh, as discussed in section 4.1).

The constants used in estimate (4.37) or (4.41) are C'_Ω, C_{Γ_N}, m, M and L. The ones depending on the domain are C'_Ω and C_{Γ_N}, we refer for discussion on their computation or estimation from above to [30, 33, 43] (and references therein). Further, the three constants m, M and L come from the given nonlinearity in Assumptions 4.2 (ii)-(iii). These constants can be obtained easily for most of the practical cases, as pointed out in subsection 4.1.2 for Dirichlet problems, using the scalar form of the nonlinearity (4.25). In view of possible x-dependence of f in (4.30), it is worth mentioning the similar analogue of (4.25). Namely, let our problem have the following special form, which covers most real examples for (4.30):

$$\begin{cases} -\operatorname{div}\Big(a(x,|\nabla u|^2)\,\nabla u\Big) = g \\ u_{|\Gamma_D} = 0 \\ a(x,|\nabla u|^2)\,\frac{\partial u}{\partial \nu}_{\,|\Gamma_N} = \gamma \end{cases} \tag{4.51}$$

(which corresponds to $f(x,\eta) = a(x,|\eta|^2)\,\eta$ in (4.30)), where the scalar function $a : \Omega \times \mathbf{R}^+ \to \mathbf{R}^+$ is as required in (3.53)–(3.54). Such x-dependent nonlinearities arise e.g. in electromagnetics (derived from nonlinear Maxwell equations, see [28]), where typically the function a is independent of x in some subdomain $\Omega_1 \subset \Omega$ and constant on the complement, where these subdomains correspond to ferromagnetic and other media, respectively. In the case of the nonlinearity (4.28) on Ω_1, we have

$$a(x,t) = \begin{cases} \frac{1}{\mu_0}\Big(\alpha + (1-\alpha)\frac{t^8}{t^8+\beta}\Big) & \text{if } x \in \Omega_1,\ t \ge 0 \\ \alpha \quad \text{if } x \in \Omega \setminus \Omega_1\,, \end{cases}$$

where $\alpha > 0$ is a constant magnetic reluctance [15, 28]. A further example of scalar nonlinearity, for mixed boundary conditions and in the x-independent form (4.25), describes air density in a subsonic potential flow, see, e.g., [3]:

$$a(t) \;=\; \varrho_\infty \,\Big(1 + \tfrac{1}{5}(M_\infty^2 - t)\Big)^{5/2} \qquad (t \ge 0),$$

where M_∞ is the Mach number at infinity. In the corresponding mixed problem, Γ_D is the wind inblow part and Γ_N consists of the other sides of the wind tunnel section.

Altogether, using Remark 3.7, conditions (3.53)–(3.54) provide the constants m, M and L for (4.37), and can be therefore determined by elementary numerical calculation. Moreover, as mentioned before, the global constants need not be estimated from above much accurately, since all terms multiplied by these constants in $EST(u_h; y^*, w)$ are close to zero for favourable parameters y^* and w.

.

4.3 Fourth order problems

In this section we study 4-th order Dirichlet problems. The concise presentation requires some basic notations: let D^2u denote the Hessian of a function $u : \Omega \to \mathbf{R}$ if $u \in H^2(\Omega)$, we define the elementwise matrix product and the corresponding Frobenius norm in the standard way

$$P : Q := \sum_{i,k=1}^{d} P_{ik}\, Q_{ik}, \qquad |P|_F := (P : P)^{1/2} \qquad (P, Q \in \mathbf{R}^{d\times d}), \qquad (4.52)$$

further, for a matrix-valued function $P : \Omega \to \mathbf{R}^{d\times d}$ we let

$$\operatorname{div}^2 P := \sum_{i,k=1}^{d} \frac{\partial^2 P_{ik}}{\partial x_i \partial x_k} \qquad (4.53)$$

(provided that these derivatives exist).

Now we can formulate the problems considered, defined via a matrix-valued nonlinearity B:

$$\left\{ \begin{array}{l} \operatorname{div}^2 B(x, D^2u) \;=\; g \\ u_{|\partial\Omega} = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0\,, \end{array} \right. \qquad (4.54)$$

such that the following assumptions are satisfied:

Assumptions 4.3.

(i) $\Omega \subset \mathbf{R}^d$ is a bounded piecewise C^1 domain;

(ii) The matrix-valued function $B : \Omega \times \mathbf{R}^{d\times d} \to \mathbf{R}^{d\times d}$ is measurable and bounded w.r. to the variable $x \in \Omega$ and C^2 in the matrix variable $\Theta \in \mathbf{R}^{d\times d}$. The Jacobian arrays

$$B'(x,\Theta) := \frac{\partial B(x,\Theta)}{\partial \Theta} = \left\{ \frac{\partial B_{rs}(x,\Theta)}{\partial \Theta_{ik}} \right\}_{i,k,r,s=1}^{d} \in \mathbf{R}^{(d\times d)^2}$$

are symmetric, i.e. $\partial B_{rs}/\partial\Theta_{ik} = \partial B_{ik}/\partial\Theta_{rs}$ for all i, k, r, s, and there exist constants $M \geq m > 0$ such that

$$m|\Phi|_F^2 \leq B'(x,\Theta)\Phi : \Phi \leq M|\Phi|_F^2 \qquad (x \in \Omega;\;\; \Theta, \Phi \in \mathbf{R}^{d\times d}). \quad (4.55)$$

(iii) $B' : \Omega \times \mathbf{R}^{d\times d} \to \mathbf{R}^{(d\times d)^2}$ is Lipschitz continuous in the matrix variable $\Theta \in \mathbf{R}^{d\times d}$, with Lipschitz constant L.

(iv) $g \in L^2(\Omega)$.

In the treatment of this problem we follow the previous sections. Now we use the Lebesgue space

$$L^2(\Omega)^{d\times d} := \{P : \Omega \to \mathbf{R}^{d\times d} : \; P_{ik} \in L^2(\Omega) \text{ for all } i,k = 1,\dots,d\} \tag{4.56}$$

with inner product

$$\langle P, Q\rangle_{L^2(\Omega)^{d\times d}} := \int_\Omega P : Q, \tag{4.57}$$

and the Sobolev space

$$H_0^2(\Omega) := \{u \in H^2(\Omega) : \; u_{|\partial\Omega} = \tfrac{\partial u}{\partial \nu}\big|_{\partial\Omega} = 0 \text{ in trace sense}\} \tag{4.58}$$

with inner product

$$\langle u, v\rangle_{H_0^2} := \langle D^2 u, D^2 v\rangle_{L^2(\Omega)^{d\times d}} = \int_\Omega D^2 u : D^2 v, \tag{4.59}$$

further, let

$$H(\operatorname{div}^2) := \{P \in L^2(\Omega)^{d\times d} : \; \operatorname{div}^2 P \in L^2(\Omega)\}.$$

The actual analogue of the Friedrichs inequality is as follows

$$\|v\|_{L^2(\Omega)} \le \tilde{C}_\Omega \, \|D^2 v\|_{L^2(\Omega)^{d\times d}} \qquad (v \in H_0^2(\Omega)) \tag{4.60}$$

for some suitable constant $\tilde{C}_\Omega > 0$. The assumptions imply that problem (4.54) has a unique weak solution $u^* \in H_0^2(\Omega)$, i.e., that now satisfies

$$\int_\Omega B(x, D^2 u^*) : D^2 v - \int_\Omega g v = 0 \qquad (v \in H_0^2(\Omega)). \tag{4.61}$$

We consider an approximate solution $u \in H_0^2(\Omega)$ and measure the error by the functional

$$\begin{aligned} E(u) &:= \int_\Omega (B(x, D^2 u) - B(x, D^2 u^*)) : (D^2 u - D^2 u^*) \qquad (4.62)\\ &= \int_\Omega B(x, D^2 u) : (D^2 u - D^2 u^*) - \int_\Omega g(u - u^*), \end{aligned}$$

where we note that (2.3) implies $\|u - u^*\|_{H_0^2}^2 \le m^{-1} E(u)$. Assumption 4.3 (ii) implies that for all fixed $x \in \Omega$, the function $B(x, .)$ is invertible on $\mathbf{R}^{d\times d}$ w.r.t. Θ. We will denote by B^{-1} the inverse w.r.t. Θ, i.e.

$$B(x, \Theta) = \Phi \quad \Rightarrow \quad B^{-1}(x, \Phi) := \Theta. \tag{4.63}$$

Theorem 4.3 *Let $u \in W^{2,\infty}(\Omega)$. Then for arbitrary $Y^* \in H(\operatorname{div}^2) \cap L^\infty(\Omega)^{d\times d}$ and arbitrary $h \in H_0^2(\Omega)$,*

$$\begin{aligned} E(u) \leq \; EST(u;Y^*,h) := \Big(& m^{-1/2}\,\tilde C_\Omega\,\|\operatorname{div}^2 Y^* - g\|_{L^2(\Omega)} \;+\; \tfrac{L}{2}\,m^{-3/2}\,D(u;Y^*,h) \\ &+ \big(\langle B(x,D^2u) - Y^*,\; D^2u - B^{-1}(x,Y^*)\rangle_{L^2(\Omega)^{d\times d}} \\ &+ \tfrac{L}{2m}\,D(u;Y^*,h)\,\|D^2u - B^{-1}(x,Y^*)\|_{L^2(\Omega)^{d\times d}}\big)^{1/2}\Big)^2 \end{aligned} \tag{4.64}$$

where

$$\begin{aligned} D(u;Y^*,h) := \Big(& M\,\|B^{-1}(x,Y^*) - D^2h\|_{L^2(\Omega)^{d\times d}} + \tilde C_\Omega\,\|\operatorname{div}^2 Y^* - g\|_{L^2(\Omega)}\Big)\times \\ &\times \|D^2u - B^{-1}(x,Y^*)\|_{L^\infty(\Omega)^{d\times d}}\,. \end{aligned} \tag{4.65}$$

Proof. One can repeat the proof of Theorem 4.1 to verify that Theorem 3.1 holds in the spaces $V := H_0^2(\Omega)$ and $Y := L^2(\Omega)^{d\times d}$ for $\Lambda := D^2$ and with the mapping $A(P) := B\circ(id,P)$, i.e. $A(P)(x) := B(x,P(x))$ for $P \in L^2(\Omega)^{d\times d}$, $x \in \Omega$; then $A^{-1}(P) = B^{-1}\circ(id,P)$ in the sense of (4.63). To estimate the term $|\Lambda^*Y^* + l|$ for the linear functional $\langle l,v\rangle \equiv -\int_\Omega gv$ on $H_0^2(\Omega)$, we use $Y^* \in H(\operatorname{div}^2)$ to obtain

$$\langle \Lambda^*Y^*, v\rangle = \langle Y^*, \Lambda v\rangle = \int_\Omega Y^* : D^2 v = \int_\Omega (\operatorname{div}^2 Y^*)v \qquad (v \in H_0^2(\Omega)),$$

whence

$$|\Lambda^*y^* + l| = \sup_{\|v\|_{H_0^2}=1} |\langle \Lambda^*y^* + l, v\rangle| = \sup_{\|v\|_{H_0^2}=1} \Big|\int_\Omega (\operatorname{div}^2 Y^* - g)v\Big|$$

$$\leq \sup_{\|v\|_{H_0^2}=1} \Big(\|\operatorname{div}^2 Y^* - g\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\Big) \leq \tilde C_\Omega\,\|\operatorname{div}^2 Y^* - g\|_{L^2(\Omega)} \tag{4.66}$$

where $\tilde C_\Omega$ comes from (4.60). Plugging the latter into (3.32) and (3.33), and replacing V, Y, W, Λ and A by $H_0^2(\Omega)$, $L^2(\Omega)^{d\times d}$, $L^\infty(\Omega)^{d\times d}$, D^2 and f, respectively, we obtain (4.64). ∎

Remark 4.7 Following [38, Chap. 6.6], the term $\tilde C_\Omega\,\|\operatorname{div}^2 Y^* - g\|_{L^2(\Omega)}$ in (4.64) can be replaced by

$$\hat C_\Omega\,\|\operatorname{div} Y^* - \eta^*\|_{L^2(\Omega)^{d\times d}} + \tilde C_\Omega\,\|\operatorname{div}\eta^* - g\|_{L^2(\Omega)}$$

for some new parameter function $\eta^* \in H(\operatorname{div})$. In this case the requirement $Y^* \in H(\operatorname{div}^2)$ can be weakened to $Y^* \in H(\operatorname{div})$ (understood row-wise).

Remark 4.8 Similarly to Remark 4.1, it is convenient to reformulate Theorem 4.3 for $Z^* := B^{-1}(x, Y^*)$ in order to avoid the computation of $B^{-1}(x, .)$. Here B^{-1} is understood w.r.t. Θ as in (4.63), hence $Y^* = B(x, Z^*)$. Then, letting $u \in W^{2,\infty}(\Omega)$ be any approximate solution, for arbitrary $Z^* \in L^\infty(\Omega)^{d\times d}$ such that $B(x, Z^*) \in H(\operatorname{div}^2)$, and for arbitrary $h \in H_0^2(\Omega)$,

$$E(u) \leq \tilde{E}ST(u; Z^*, h) := \tag{4.67}$$

$$\Big(m^{-1/2}\tilde{C}_\Omega \|\operatorname{div}^2 B(x, Z^*) - g\|_{L^2(\Omega)} + \tfrac{L}{2} m^{-3/2} D(u; Z^*, h) +$$

$$\big(\langle B(x, D^2u) - B(x, Z^*), D^2u - Z^*\rangle_{L^2(\Omega)^{d\times d}} + \tfrac{L}{2m} D(u; Z^*, h) \|D^2u - Z^*\|_{L^2(\Omega)^{d\times d}}\big)^{1/2}\Big)^2$$

where

$$D(u; Z^*, h) := \Big(M \, \|Z^* - D^2h\|_{L^2(\Omega)^{d\times d}} + \tilde{C}_\Omega \, \|\operatorname{div}^2 B(x, Z^*) - g\|_{L^2(\Omega)}\Big) \times \\ \times \|D^2u - Z^*\|_{L^\infty(\Omega)^{d\times d}} \,. \tag{4.68}$$

Remark 4.9 Our result is a direct extension of earlier sharp error estimates obtained for linear fourth order problems [14, 37, 38]. (This is seen using Remark 3.5 in a similar way as in Remark 4.2.)

Proposition 4.4 *Estimate* (4.64) *is sharp, that is,*

$$\min_{\substack{Y^* \in H(\operatorname{div}^2) \cap L^\infty(\Omega)^{d\times d}, \\ h \in H_0^2(\Omega)}} EST(u; Y^*, h) = E(u), \tag{4.69}$$

provided that the exact solution satisfies $u^* \in W^{2,\infty}(\Omega)$.

Proof. By assumption $D^2u^* \in L^\infty(\Omega)^{d\times d} = W$, therefore we can apply Proposition 3.5 to obtain the sharpness statement. ■

In practice for finite element solutions, in order to have an approximate solution $u_h \in H_0^2(\Omega)$, one uses C^1-elements (i.e. $u_h \in C^1$ and is piecewise polynomial), see, e.g., [10]. In this case we automatically have $u \in W^{2,\infty}(\Omega)$, which was required for Theorem 4.3 to hold. (Another common FEM approach is to use mixed variables to have less smoothness for u_h. In this case one may expect to reformulate the terms containing D^2u in (4.64) via the mixed variables in a similar vein as in Remark 4.7; however, this is out of the scope of the present paper.) Next, following (3.46), Y^* should be some approximation of $B(x, D^2u^*)$. For finite element solutions, using averaging as before, we can first let

$$Y^* := B\Big(x, G_h(D^2u_h)\Big), \quad Z^* = B^{-1}(x, Y^*) = G_h(D^2u_h), \tag{4.70}$$

where G_h is some averaging operator that defines a C^1-approximation of D^2u_h, and B^{-1} is understood w.r.t. Θ as in (4.63). More accurate error bounds can be obtained by suitable further bound minimization w.r.t. Y^*

(or Z^*) as mentioned in Remark 4.3. Then by Remark 3.6, the optimal w for this Z^* to set in $EST(u_h; Y^*, w)$ is the solution of the following linear auxiliary problem: find $w_{opt} \in H^1_D(\Omega)$ such that

$$\int_\Omega D^2 w_{opt} : D^2 v = \int_\Omega Z^* : D^2 v \qquad (v \in H^2_0(\Omega)), \tag{4.71}$$

that is, the weak solution of the formal biharmonic problem

$$\begin{cases} \Delta^2 w_{opt} = \operatorname{div}^2 Z^* \\ w_{opt\,|\partial\Omega} = \frac{\partial w_{opt}}{\partial \nu}\Big|_{\partial\Omega} = 0\,. \end{cases} \tag{4.72}$$

Note that Z^* need not be in $H(\operatorname{div}^2)$ to pose (4.72): in general $\operatorname{div}^2 Z^*$ can be understood in a distributional sense in (4.72), which exactly means that we need to use the weak form (4.71). Thus the weaker condition $Y^* \in H(\operatorname{div})$ (or equivalently $Z^* \in H(\operatorname{div})$) can be used as mentioned in Remark 4.7. Altogether, one can define w as the numerical solution of (4.72) on a suitable mesh (either the same as used for u_h or a finer mesh, as discussed in section 4.1).

The most important real-life model that uses fourth order equations like (4.54) describes the elasto-plastic bending of a clamped thin plane plate $\Omega \subset \mathbf{R}^2$, see, e.g., [32]. This problem is as follows:

$$\begin{cases} \operatorname{div}^2 \left(\overline{g}(E(D^2u))\ \tilde{D}^2 u \right) \ = \ \alpha \\ u_{|\partial\Omega} = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \end{cases} \tag{4.73}$$

where

$$\tilde{D}^2 u := \tfrac{1}{2}\left(D^2 u + \Delta u \cdot I\right), \qquad E(D^2 u) := \tfrac{1}{2}\left(|D^2 u|_F^2 + (\Delta u)^2\right)$$

and $\overline{g}$ is a scalar material function satisfying (3.49)-(3.50) (with $\overline{g}$ substituted for a). This problem corresponds to the nonlinearity $B(x,\Theta) = \overline{g}(E(\Theta))\,\tilde{\Theta}$ in (4.54), where $E(\Theta) = \frac{1}{2}\left(|\Theta|_F^2 + (tr\,\Theta)^2\right)$, $\tilde{\Theta} = \frac{1}{2}\left(\Theta + tr\,\Theta \cdot I\right)$, and $tr\,\Theta$ denotes the trace of the matrix Θ. It is easy to see that the weak formulation of problem (4.73) leads to an operator like (3.48) with $[P,Q] = \frac{1}{2}\left(P : Q + (tr\,P)(tr\,Q)\right)$, i.e. for which $E(P) = [P,P]$. (See more details on this setting of the problem in [13]). Consequently, the bounds m and M and the Lipschitz constant L can be calculated numerically from (3.49)–(3.50) (with $\overline{g}$ substituted for a).

4.4 Second order elasticity systems

4.4.1 Formulation of the problem

Symmetric second order systems arise in the description of the elastic behaviour of a body. We follow the presentation of [7, 36] to describe the elasticity of a body $\Omega \subset \mathbf{R}^3$ with nonlinear behaviour of the material. The involved physical quantities are the displacement vector $u : \Omega \to \mathbf{R}^3$, the strain tensor $\varepsilon : \Omega \to \mathbf{R}^{3\times 3}$ and the stress tensor $\sigma : \Omega \to \mathbf{R}^{3\times 3}$. The basic system of equations is

$$\left. \begin{cases} -\operatorname{div} \sigma_i = \varphi_i & \text{in } \Omega \\ \sigma_i \cdot \nu = \tau_i & \text{on } \Gamma_N \\ u_i = 0 & \text{on } \Gamma_D \end{cases} \right\} \quad (i = 1, 2, 3) \tag{4.74}$$

where $\sigma_i = (\sigma_{i1}, \sigma_{i2}, \sigma_{i3})$ $(i = 1, 2, 3)$ is the ith row of the matrix σ, the functions $\varphi : \Omega \to \mathbf{R}^3$ and $\tau : \Gamma_N \to \mathbf{R}^3$ describe the body and boundary force vectors, respectively, further, $\partial\Omega = \Gamma_N \cup \Gamma_D$ is a disjoint measurable subdivision and $\Gamma_D \neq \emptyset$.

The problem (4.74) can be formulated as a second order system in terms of the displacement u. First, the strain tensor $\varepsilon = \varepsilon(u)$ is determined by the displacement via the relation

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla u^t\right)$$

where $\nabla u^t(x)$ denotes the transpose of the matrix $\nabla u(x) \in \mathbf{R}^{3\times 3}$ for $x \in \Omega$. The connection of strain and stress is given by a matrix-valued function T as follows. For any $\Theta \in \mathbf{R}^{3\times 3}$ let

$$\operatorname{vol}\Theta = \frac{1}{3} tr\Theta \cdot I, \qquad \operatorname{dev}\Theta = \Theta - \operatorname{vol}\Theta \tag{4.75}$$

where $tr\Theta = \sum_{i=1}^{3} \Theta_{ii}$ is the trace of Θ and I is the identity matrix. Using these notations, there holds

$$\sigma(x) = T(x, \varepsilon(u(x))) \tag{4.76}$$

with $T : \Omega \times \mathbf{R}^{3\times 3} \to \mathbf{R}^{3\times 3}$ given by

$$T(x,\Theta) = 3k(x, |\operatorname{vol}\Theta|^2)\operatorname{vol}\Theta + 2\mu(x, |\operatorname{dev}\Theta|^2)\operatorname{dev}\Theta \qquad (x \in \Omega,\ \Theta \in \mathbf{R}^{3\times 3}), \tag{4.77}$$

where $k(x, s)$ is the bulk modulus of the material and $\mu(x, s)$ is Lamé's coefficient. (Further properties of k and μ are give below in (4.82).) Then, substituting (4.76) into (4.74), we obtain the system

$$\left. \begin{cases} -\operatorname{div} T_i(x, \varepsilon(u)) = \varphi_i & \text{in } \Omega \\ T_i(x, \varepsilon(u)) \cdot \nu = \tau_i & \text{on } \Gamma_N \\ u_i = 0 & \text{on } \Gamma_D \end{cases} \right\} \quad (i = 1, 2, 3). \tag{4.78}$$

The weak formulation is done via the Sobolev space $H^1_D(\Omega)^3$, where $H^1_D(\Omega)$ has been defined in (4.32), with inner product

$$\langle u, v\rangle_{H^1_D(\Omega)^3} := \int_\Omega \nabla u : \nabla v, \tag{4.79}$$

using the matrix product (4.52). Using the symmetry of the matrices T, the weak form of system (4.78) reads as follows: find $u = (u_1, u_2, u_3) \in H^1_D(\Omega)^3$ such that

$$\int_\Omega T(x, \varepsilon(u)) \,:\, \varepsilon(v) - \int_\Omega \varphi \cdot v - \int_{\Gamma_N} \tau \cdot v \, d\sigma \;=\; 0 \qquad (v \in H^1_D(\Omega)^3). \quad (4.80)$$

Here, with the representation (4.77) for $T(x, \varepsilon(u))$, one can derive

$$T(x, \varepsilon(u)) \,:\, \varepsilon(v) =$$

$$= \; 3k(x, |\mathrm{vol}\, \varepsilon(u)|^2)\, \mathrm{vol}\, \varepsilon(u) \,:\, \mathrm{vol}\, \varepsilon(v) + 2\mu(x, |\mathrm{dev}\, \varepsilon(u)|^2)\, \mathrm{dev}\, \varepsilon(u) \,:\, \mathrm{dev}\, \varepsilon(v). \quad (4.81)$$

The functions $k, \mu : \Omega \times \mathbf{R}^+ \to \mathbf{R}$ are measurable and bounded w.r. to x and C^1 w.r. to the variable $t \in \mathbf{R}^+$, further, they satisfy

$$\begin{aligned} &0 < m \le \mu(x,t) < \tfrac{3}{2} k(x,t) \le M \,, \\ &0 < m \le \tfrac{\partial}{\partial t} \left(k(x,t^2)t\right) \le M \,, \qquad 0 < m \le \tfrac{\partial}{\partial t} \left(\mu(x,t^2)t\right) \le M \end{aligned} \quad (4.82)$$

with constants $M \ge m_0$ independent of (x, t), as described in [7]. We impose as an additional condition that k and μ are also piecewise C^2 (i.e. C^2 except for finitely many isolated points, which in practice typically separate the domain of linear and nonlinear behaviour), further, that there exists a constant $L > 0$ such that

$$\left| \tfrac{\partial^2}{\partial t^2} \Big(k(x,t^2)t \Big) \right| \le L, \qquad \left| \tfrac{\partial^2}{\partial t^2} \Big(\mu(x,t^2)t \Big) \right| \le L \qquad (x \in \Omega, \; t \ge 0). \quad (4.83)$$

We note that some concrete measurements or explicit expressions on k and μ are given, e.g., in [2, 36, 38], and k is often considered as constant. With the notations of (4.52) and (4.55), (4.77) and (4.82) imply the analogue of (4.55):

$$m|\Phi|_F^2 \le T'(x, \Theta)\Phi : \Phi \le M|\Phi|_F^2 \qquad (x \in \Omega; \;\; \Theta, \Phi \in \mathbf{R}^{3\times 3}). \qquad (4.84)$$

This property implies well-posedness for (4.80) in $H^1_D(\Omega)^3$ in view the famous Korn's inequality

$$\kappa \int_\Omega |\nabla u|^2 \le \int_\Omega |\varepsilon(u)|^2 \le \int_\Omega |\nabla u|^2 \qquad (u \in H^1_D(\Omega)^3) \qquad (4.85)$$

(where $\kappa > 0$), see more details, e.g., in [7, 13, 36].

4.4.2 Sharp error estimation

In the treatment of the elasticity problem we follow the previous sections. Now we use the Lebesgue space

$$L^2(\Omega)^{3\times 3}_{symm} := \{ P : \Omega \to \mathbf{R}^{3\times 3} : \;\; P_{ik} = P_{ki} \in L^2(\Omega) \;\text{ for all } i, k = 1, \ldots, 3 \} \quad (4.86)$$

with inner product

$$\langle P, Q\rangle_{L^2(\Omega)^{3\times 3}} := \int_\Omega P : Q, \tag{4.87}$$

using notation (4.52), Further, we endow the space $H^1_D(\Omega)^3$ with inner product

$$\langle u, v\rangle_\varepsilon := \langle \varepsilon(u), \varepsilon(v)\rangle_{L^2(\Omega)^{3\times 3}} = \int_\Omega \varepsilon(u) : \varepsilon(v), \tag{4.88}$$

which is equivalent to the standard inner product (4.79) owing to (4.85). Inequalities (4.34) and (4.85) then imply

$$\|v\|_{L^2(\Omega)^3} \le \kappa^{-1/2}\, C'_\Omega\, \|v\|_\varepsilon, \qquad \|v\|_{L^2(\Gamma_N)^3} \le \kappa^{-1/2}\, C_{\Gamma_N}\, \|v\|_\varepsilon \qquad (v \in H^1_D(\Omega)^3). \tag{4.89}$$

We define $L^\infty(\Omega)^{3\times 3}_{symm}$ analogously to (4.86), and finally let

$$H(\mathrm{div}\,, \mathbf{R}^3; \Gamma_N) := \{P \in L^2(\Omega)^{3\times 3}_{symm} :\ \mathrm{div}\, P \in L^2(\Omega)^3,\ P\cdot \nu \in L^2(\Gamma_N)^3\}.$$

We consider an approximate solution $u \in H^1_D(\Omega)^3$ and measure the error by the functional

$$E(u) := \int_\Omega (T(x,\varepsilon(u)) - T(x,\varepsilon(u^*))) : (\varepsilon(u) - \varepsilon(u^*)) \tag{4.90}$$

$$= \int_\Omega T(x,\varepsilon(u)) : (\varepsilon(u) - \varepsilon(u^*)) - \int_\Omega \varphi\cdot(u-u^*) - \int_{\Gamma_N} \tau\cdot(u-u^*),$$

where we note that (2.3) and (4.84)–(4.85) imply $\|u-u^*\|^2_{H^1_D(\Omega)^3} \le m^{-1}\kappa^{-1/2}\, E(u)$. Property (4.84) implies that for all fixed $x\in\Omega$, the function $T(x,.)$ is invertible on $\mathbf{R}^{3\times 3}$ w.r.t. Θ. We will denote by T^{-1} the inverse w.r.t. Θ, i.e.

$$T(x,\Theta) = \Phi \quad \Rightarrow \quad T^{-1}(x,\Phi) := \Theta. \tag{4.91}$$

Theorem 4.4 *Let $u \in W^{1,\infty}(\Omega)^3$. Then for arbitrary $Y^* \in H(\mathrm{div}\,, \mathbf{R}^3; \Gamma_N) \cap L^\infty(\Omega)^{3\times 3}_{symm}$ and arbitrary $h \in H^1_D(\Omega)^3$,*

$$\begin{aligned} E(u) \le\ & EST(u; Y^*, h) := \\ &\Big((\kappa m)^{-1/2}\, C'_\Omega\, \|\mathrm{div}\, Y^* + \varphi\|_{L^2(\Omega)^3} \ +\ (\kappa m)^{-1/2}\, C_{\Gamma_N} \|Y^*\cdot\nu - \tau\|_{L^2(\Gamma_N)^3} \\ &+ \tfrac{L}{2}\, m^{-3/2}\, D(u;Y^*,h) \ + \big(\langle T(x,\varepsilon(u)) - Y^*,\ \varepsilon(u) - T^{-1}(x,Y^*)\rangle_{L^2(\Omega)^{3\times 3}} \\ &+\ \tfrac{L}{2m}\, D(u;Y^*,h)\, \|\varepsilon(u) - T^{-1}(x,Y^*)\|_{L^2(\Omega)^{3\times 3}}\big)^{1/2}\Big)^2, \end{aligned} \tag{4.92}$$

where

$$\begin{aligned} D(u;Y^*,h) := \Big(& M\, \|T^{-1}(x,Y^*) - \varepsilon(h)\|_{L^2(\Omega)^{3\times 3}} + \kappa^{-1/2}\, C'_\Omega\, \|\mathrm{div}\, Y^* + \varphi\|_{L^2(\Omega)^3} \\ &+ \kappa^{-1/2}\, C_{\Gamma_N} \|Y^*\cdot\nu - \tau\|_{L^2(\Gamma_N)^3}\Big)\, \|\varepsilon(u) - T^{-1}(x,Y^*)\|_{L^\infty(\Omega)^{3\times 3}}\,. \end{aligned} \tag{4.93}$$

Proof. One can repeat the proof of Theorem 4.1 to verify that Theorem 3.1 holds in the spaces $V := H^1_D(\Omega)^3$ (endowed with inner product (4.88)) and $Y := L^2(\Omega)^{3\times 3}_{symm}$ for $\Lambda := \varepsilon(.)$ and with the mapping $A(P) := T\circ(id, P)$, i.e. $A(P)(x) := T(x, P(x))$ for $P \in L^2(\Omega)^{3\times 3}_{symm}$, $x \in \Omega$; then $A^{-1}(P) = T^{-1}\circ(id, P)$ in the sense of (4.91). To estimate the term $|\Lambda^*Y^* + l|$ for the linear functional $\langle l, v\rangle \equiv -\int_\Omega \varphi\cdot v - \int_{\Gamma_N}\tau\cdot v\, d\sigma$ on $H^1_D(\Omega)^3$, we use $Y^* \in H(\mathrm{div}\,, \mathbf{R}^3; \Gamma_N)$ and the symmetry of Y^* to obtain

$$\langle\Lambda^*Y^*, v\rangle = \langle Y^*, \Lambda v\rangle = \int_\Omega Y^* : \varepsilon(v) = \int_\Omega Y^* : \nabla v = -\int_\Omega \mathrm{div}\, Y^*\cdot v + \int_{\Gamma_N} Y^*\nu\cdot v\, d\sigma$$

($v \in H^1_D(\Omega)^3$), whence

$$|\Lambda^*y^* + l| = \sup_{\|v\|_{H^1_D(\Omega)^3}=1} |\langle\Lambda^*y^* + l, v\rangle| = \sup_{\|v\|_{H^1_D(\Omega)^3}=1} \left| -\int_\Omega (\mathrm{div}\, Y^* + \varphi)\cdot v + \int_{\Gamma_N} (Y^*\nu - \tau)\cdot v\, d\sigma \right|$$

$$\le \sup_{\|v\|_{H^1_D(\Omega)^3}=1} \Big(\|\mathrm{div}\, Y^* + \varphi\|_{L^2(\Omega)^3}\|v\|_{L^2(\Omega)^3} + \|Y^*\cdot\nu - \tau\|_{L^2(\Gamma_N)^3}\|v\|_{L^2(\Gamma_N)^3} \Big)$$

$$\le \kappa^{-1/2}\, C'_\Omega\, \|\mathrm{div}\, Y^* + \varphi\|_{L^2(\Omega)^3} + \kappa^{-1/2}\, C_{\Gamma_N} \|Y^*\cdot\nu - \tau\|_{L^2(\Gamma_N)^3}, \qquad (4.94)$$

where (4.89) has been used. Plugging the latter into (3.32) and (3.33), and replacing V, Y, W, Λ and A by $H^1_D(\Omega)^3$, $L^2(\Omega)^{3\times 3}_{symm}$, $L^\infty(\Omega)^{3\times 3}$, $\varepsilon(.)$ and T, respectively, we obtain (4.92). ∎

Remark 4.10 Similarly to Remark 4.1, it is convenient to reformulate Theorem 4.4 for $Z^* := T^{-1}(x, Y^*)$ in order to avoid the computation of $T^{-1}(x, .)$. Here T^{-1} is understood w.r.t. Θ as in (4.91), hence $Y^* = T(x, Z^*)$. Then, letting $u \in W^{1,\infty}(\Omega)^3$ be any approximate solution, for arbitrary $Z^* \in L^\infty(\Omega)^{3\times 3}_{symm}$ such that $T(x, Z^*) \in H(\mathrm{div}\,, \mathbf{R}^3; \Gamma_N)$, and for arbitrary $h \in H^1_D(\Omega)^3$,

$$\begin{aligned} E(u) \le\ & \tilde{E}ST(u; Z^*, h) := \\ & \Big((\kappa m)^{-1/2}\, C'_\Omega\, \|\mathrm{div}\, T(x, Z^*) + \varphi\|_{L^2(\Omega)} \qquad (4.95)\\ & + (\kappa m)^{-1/2} C_{\Gamma_N} \|T(x, Z^*)\cdot\nu - \tau\|_{L^2(\Gamma_N)^3} + \tfrac{L}{2} m^{-3/2} D(u; Z^*, h) + \\ & \Big(\langle T(x, \varepsilon(u)) - T(x, Z^*), \varepsilon(u) - Z^*\rangle_{L^2(\Omega)^{3\times 3}} + \tfrac{L}{2m} D(u; Z^*, h)\|\varepsilon(u) - Z^*\|_{L^2(\Omega)^{3\times 3}}\Big)^{1/2}\Big)^2 \end{aligned}$$

where

$$\begin{aligned} D(u; Z^*, h) := \Big(& M\,\|Z^* - \varepsilon(h)\|_{L^2(\Omega)^{3\times 3}} + \kappa^{-1/2}\, C'_\Omega\, \|\mathrm{div}\, T(x, Z^*) + \varphi\|_{L^2(\Omega)^3} \qquad (4.96)\\ & + \kappa^{-1/2}\, C_{\Gamma_N} \|T(x, Z^*)\cdot\nu - \tau\|_{L^2(\Gamma_N)^3}\Big)\, \|\varepsilon(u) - Z^*\|_{L^\infty(\Omega)^{3\times 3}}\,. \end{aligned}$$

Remark 4.11 Our result is a direct extension of earlier sharp error estimates obtained for linear elasticity problems [38, 34]. (This is seen using Remark 3.5 in a similar way as in Remark 4.2.) Further, quasi-sharp error estimates for nonlinear elasticity problems have been obtained earlier in [44].

Proposition 4.5 *Estimate* (4.92) *is sharp, that is,*

$$\min_{\substack{Y^* \in H(\mathrm{div}\,,\mathbf{R}^3;\Gamma_N)\cap L^\infty(\Omega)^{3\times 3}_{symm},\\ h\in H^1_D(\Omega)^3}} EST(u;Y^*,h) = E(u), \tag{4.97}$$

provided that the exact solution satisfies $u^* \in W^{1,\infty}(\Omega)^3$.

Proof. By assumption $\nabla u^* \in L^\infty(\Omega)^{3\times 3}$, hence $\varepsilon(u^*) \in L^\infty(\Omega)^{3\times 3}_{symm} = W$, therefore we can apply Proposition 3.5 to obtain the sharpness statement. ■

In practice, for finite element solutions, all three coordinate functions of the FEM approximation $u_h \in V_h \subset H^1_D(\Omega)^3$ are continuous piecewise polynomials, hence condition $u_h \in W^{1,\infty}(\Omega)^3$ in Theorem 4.4 is satisfied. If we choose Y^* to be a symmetric matrix function whose entries are also continuous piecewise polynomial functions, e.g. , functions from another FEM subspace, and arbitrary $w \in H^1_D(\Omega)^3$, then $Y^* \in H(\mathrm{div}\,,\mathbf{R}^3;\,\Gamma_N) \cap L^\infty(\Omega)^{3\times 3}_{symm}$, hence Theorem 4.4 can be applied. Next, following (3.46), Y^* should be some approximation of $T(x,\varepsilon(u^*))$. For finite element solutions, using averaging as before, we can first let

$$Y^* := T\Big(x, G_h(\varepsilon(u_h))\Big), \qquad Z^* = T^{-1}(x,Y^*) = G_h(\varepsilon(u_h)). \tag{4.98}$$

Here G_h is some averaging operator, based on [21] where averaging is discussed in the context of elasticity problems, and T^{-1} is understood w.r.t. Θ as in (4.91). More accurate error bounds can be obtained by suitable further bound minimization w.r.t. Y^* (or Z^*) as mentioned in Remark 4.3. Then by Remark 3.6, the optimal w for this Z^* to set in $EST(u_h;Y^*,w)$ is the solution of the following linear auxiliary problem: find $w_{opt} \in H^1_D(\Omega)^3$ such that

$$\int_\Omega \varepsilon(w_{opt}) : \varepsilon(v) = \int_\Omega Z^* : \varepsilon(v) \qquad (v \in H^1_D(\Omega)^3). \tag{4.99}$$

Hence one can define w as the numerical solution of (4.99) on a suitable mesh (either the same as used for u_h or a finer mesh, as discussed in section 4.1). Regarding the required constants, estimates for C'_Ω and C_{Γ_N} can be done similarly to [33, 43], several explicit values and estimates for Korn's constant κ are given in [22], finally, as pointed out at the end of Remark 3.7, the bounds m and M and the Lipschitz constant L can be calculated numerically from (4.82)–(4.83).

5 Appendix

The proof of Proposition 4.3 is outlined here. We cannot use Proposition 3.5 directly since that would need $u^* \in W^{1,\infty}(\Omega)$. Instead, we define $W^{1,\infty}$-approximations of u^* and prove that the terms in (4.37) that had vanished before will now converge to zero. Therefore, let $u_j := c_j r_j^{\beta_j} s_j(\theta_j)$. For all

$\varepsilon > 0$, let us define a truncated function u_{ε_j} to equal u_j if $u_j > \varepsilon$ and to equal ε if $u_j \le \varepsilon$. Let us denote the latter subset by H_{ε_j}, and denote by Ω_{R_j} the sector obtained by intersection of Ω with the disc around the vertex B_j with radius $R_j > 0$. By continuity, we have $H_{\varepsilon_j} \subset \Omega_{R_j}$ for some $R_j > 0$ such that $\varepsilon \to 0$ implies $R_j \to 0$. We finally define u_ε by replacing u_j in (4.47) by u_{ε_j}, and define $y^*_\varepsilon := f(x, \nabla u_\varepsilon)$ following (4.46).

Now we set y^*_ε in (4.37) and let $\varepsilon \to 0$. It is obvious from the definition of L^2 that the terms in (4.37) tend to zero except for $D(u; y^*_\varepsilon, h)$. Namely, these terms do not use any regularity of u^*, hence setting $y^* := f(x, \nabla u^*)$ as in (4.46) would make them vanish, and the fact $\nabla u_\varepsilon \to \nabla u^*$ in L^2-norm implies that this limit value zero is approached for these terms as $\varepsilon \to 0$. It is left to consider $D(u; y^*_\varepsilon, h)$, where the problem is that the last factor $\|\nabla u - f^{-1}(x, y^*_\varepsilon)\|_{L^\infty(\Omega)^d}$ may tend to infinity. We prove that the product still tends to zero. Here in fact we have $f^{-1}(x, y^*_\varepsilon) = \nabla u_\varepsilon$, and we can set $h := u^*$ as in (4.46), hence

$$D(u; y^*_\varepsilon, h) = \Big(M \, \|\nabla u_\varepsilon - \nabla u^*\|_{L^2(\Omega)^d} + C'_\Omega \, \|\mathrm{div}\, f(x, \nabla u_\varepsilon) + g\|_{L^2(\Omega)} \quad (5.1)$$

$$+ C_{\Gamma_N} \|f(x, \nabla u_\varepsilon) \cdot \nu - \gamma\|_{L^2(\Gamma_N)} \Big) \, \|\nabla u - \nabla u_\varepsilon\|_{L^\infty(\Omega)^d} .$$

It is enough to prove $D(u; y^*_{\varepsilon_j}, h) \to 0$ for fixed j since there are finitely many u_j. For technical simplicity, we may assume that $u^* = u_j$ for some fixed j. A direct calculation yields $|\nabla u_j| = r_j^{\beta_j - 1} \tilde{s}_j(\theta_j)$, where $\tilde{s}_j(\theta_j)^2 := \beta_j^2 s_j(\theta_j)^2 + s'_j(\theta_j)^2$. Then, for sufficiently small radius $R_j > 0$, we have $\int_{\Omega_{R_j}} |\nabla u_j|^4 = const. \cdot \int_0^{R_j} r_j^{4(\beta_j - 1)} \, r_j \, dr_j$, which is finite by assumption, hence $\beta_j > 1/2$. Here u_{ε_j} equals u_j outside H_{ε_j} and constant in H_{ε_j}, hence the first term of $D(u; y^*_{\varepsilon_j}, h)$ satisfies $\|\nabla u_{\varepsilon_j} - \nabla u_j\|_{L^2(\Omega)^d} = \|\nabla u_j\|_{L^2(H_{\varepsilon_j})^d} \le \|\nabla u_j\|_{L^2(\Omega_{R_j})^d} = const. \cdot R_j^{\beta_j}$. For the second term of $D(u; y^*_{\varepsilon_j}, h)$, using the assumption $u^* = u_j$, we have $\mathrm{div}\, f(x, \nabla u_{\varepsilon_j}) = \mathrm{div}\, f(x, \nabla u^*) = -g$ outside H_{ε_j} and $\mathrm{div}\, f(x, \nabla u_{\varepsilon_j}) = \mathrm{div}\, f(x, 0) = 0$ in H_{ε_j}, hence $\|\mathrm{div}\, f(x, \nabla u_\varepsilon) + g\|_{L^2(\Omega)} = \|g\|_{L^2(H_{\varepsilon_j})} \le \|g\|_{L^2(\Omega_{R_j})} \le \|g\|_{L^4(\Omega_{R_j})} \|1\|_{L^4(\Omega_{R_j})} \le \|g\|_{L^4(\Omega)} \|1\|_{L^4(\Omega_{R_j})} = const. \cdot R_j^{1/2}$, where Hölder's inequality was used. Similarly,

$$\|f(x, \nabla u_\varepsilon) \cdot \nu - \gamma\|_{L^2(\Gamma_N)} = \|\gamma\|_{L^2(H_{\varepsilon_j} \cap \Gamma_N)} \le$$

$$\le \|\gamma\|_{L^2(\Omega_{R_j} \cap \Gamma_N)} \le \|\gamma\|_{L^\infty(\Gamma_N)} \|1\|_{L^2(\Omega_{R_j} \cap \Gamma_N)} = const. \cdot R_j^{1/2}.$$

Finally, since u is fixed and using $|\nabla u_j| \le const. \cdot r_j^{\beta_j - 1}$, we have $\|\nabla u - \nabla u_{\varepsilon_j}\|_{L^\infty(\Omega)^d} \le \|\nabla u\|_{L^\infty(\Omega)^d} + \|\nabla u_{\varepsilon_j}\|_{L^\infty(\Omega)^d} \le const. \cdot (1 + R_j^{\beta_j - 1})$. Altogether, using the obtained estimates for each term of $D(u; y^*_{\varepsilon_j}, h)$, we obtain

$$D(u; y^*_{\varepsilon_j}, h) \le const. \cdot (R_j^{\beta_j} + R_j^{1/2})(1 + R_j^{\beta_j - 1}) \to 0 \quad \text{as } \varepsilon \to 0$$

owing to $\beta_j > 1/2$. We finally note that more exact approximations of u^* than the above u_ε can be obtained by C^1 extensions to H_{ε_j}.

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