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Abstract: *The one dimensional distribution of a Lévy process is not known in general even though its characteristic function is given by the famous Lévy-Khinchine theorem. This article gives an exact series representation for the one dimensional distribution of a Lévy process satisfying certain moment conditions. Moreover, this work clarifies an old result by Cramér on Edgeworth expansions for the distribution functions of Lévy processes.*

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1 Introduction

The Lévy-Khinchine theorem gives the characteristic function of a Lévy process. In spite of this, the distribution of a Lévy process is not analytically known, except in few special cases such as the Brownian motion, the Poisson process and the gamma process. For example, the distribution function of the compound Poisson process is not known in general despite its popularity as a risk process in insurance applications.

This article has two contributions. First of all, this article introduces some sufficient extra conditions to get an exact Edgeworth type series representation for the one dimensional distribution of a Lévy process in the presence of all moments. Secondly, this paper goes beyond an old result on Edgeworth approximation introduced without a proof by Cramér (1962) as an analogue to the i.i.d. case. This article clarifies the connection between the distribution functions of Lévy processes and classical approximation results of sums of independent random variables.

There are lots of approximation results in the literature. The normal approximation approximates well asymptotically the distribution function of a Lévy process when $t \to \infty$ if the third moment exists, see for instance Valkeila (1995). Several authors have considered asymptotic expansions in the central limit theorem (Edgeworth approximation) for the sums of independent random variables to improve the normal approximation, see e.g. Petrov (1995) or Cramér (1962). These approximation methods are also well known in statistics and insurance mathematics (Beard et al., 1977; Kolassa, 2006). Another approximation result is introduced for the distribution function of Lévy processes by Cramér (1962) as an analogue to the i.i.d. case but without a proof.

Beside the insurance applications, the results of this article could be applicable in the simulations of Lévy processes. In fact, the classical Edgeworth approximation has been used for getting error estimates for simulations of the small jumps of a Lévy process (Asmussen and Rosiński, 2001). Moreover, the exact series representations would maybe be useful tools also for proving theoretical results on Lévy processes.

2 Definitions

In this section, we define the concepts needed in the rest of the article.

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let X be a real valued random variable defined on this space. Let $v_X(s) = \mathbb{E}e^{isX}$ denote the characteristic function of X.

Definition 2.1 (Cramér's condition). *A random variable X is said to satisfy Cramér's condition if*

$$\limsup_{|s|\to\infty} |v_X(s)| < 1.$$

Remark 2.4 characterises Cramér's condition in the case of Lévy processes.

Definition 2.2 (Cumulants). *Let $k \in \mathbb{N} = \{1, 2, \dots\}$. The cumulant of order k of a random variable X is defined as*

$$\gamma_k^X = \frac{1}{i^k} \left[\frac{d^k}{ds^k} \log \upsilon_X(s) \right]_{s=0}.$$

Note that the cumulant of X of order k is finite if we have $\mathbb{E}|X|^k < \infty$.

We use the following definition for the (non-normalised) Hermite polynomial of order $n \in \mathbb{N}$

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}.$$

This choice of the definition makes the series representation much simpler than the normalised one. The same choice is done e.g. by Petrov (1995); Kolassa (2006). With this definition one gets the identities

$$\begin{aligned} H_{n+1}(x) &= xH_n(x) - nH_{n-1}(x), \\ H_n'(x) &= nH_{n-1}(x) \quad \text{and} \\ H_n(-x) &= (-1)^n H_n(x) \end{aligned}$$

analogous to those in Nualart (2006).

We set $V_X^2 = EX^2$. Let $\nu \in \mathbb{N}$ s.t. $\mathbb{E}|X|^{\nu+2} < \infty$. We are now ready to define the approximating function Q_ν^X to be used in the series approximations. We set

$$Q_\nu^X(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum H_{\nu+2l-1}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{\gamma_{m+2}^X}{(m+2)! V_X^{m+2}} \right)^{k_m}, \tag{1}$$

where the summation is extended over the non-negative integer solutions $(k_1, \dots, k_\nu)$ of the equation $k_1 + 2k_2 + \dots + \nu k_\nu = \nu$. Here we have $l = \sum_{j=1}^{\nu} k_j$. The first few of these functions are

$$\begin{aligned} Q_1^X(x) =& -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (x^2 - 1) \frac{\gamma_3^X}{6V_X^3}, \\ Q_2^X(x) =& -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} ((x^5 - 10x^3 + 15x) \frac{(\gamma_3^X)^2}{72V_X^6} + (x^3 - 3x) \frac{\gamma_4^X}{24V_X^4}), \\ Q_3^X(x) =& -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} ((x^8 - 28x^6 + 210x^4 - 420x^2 + 105) \frac{(\gamma_3^X)^3}{1296V_X^9} + \\ & (x^6 - 15x^4 + 45x^2 - 15) \frac{\gamma_3^X \gamma_4^X}{144V_X^7} + (x^4 - 6x^2 + 3) \frac{\gamma_5^X}{120V_X^5}). \end{aligned}$$

The approximating function of order zero is the cumulative distribution function of the standard normal distribution $\Phi(x)$.

In the remaining of this article the process $X = (X_t)_{t \geq 0}$ is assumed to be a Lévy process on $\mathbb{R}$. The standard definition for Lévy processes can be found for instance from Bertoin (1996).

We use the following version of the Lévy-Khinchine theorem to represent the characteristic function $\upsilon_{X_t}(s)$. The theorem can be found in one form or another for example in Bertoin (1996); Cont and Tankov (2004); Sato (1999).

Theorem 2.3 (Lévy-Khinchine). *There are unique $\sigma^2 \geq 0$, $\rho \in \mathbb{R}$ and a Radon measure μ on $\mathbb{R}\backslash\{0\}$ satisfying*

$$\int_{\mathbb{R}\backslash\{0\}} \min(u^2, 1) d\mu(u) < \infty$$

such that

$$\psi(s) = -\frac{1}{2}\sigma^2 s^2 + i\rho s + \int_{\mathbb{R}\backslash\{0\}} (e^{isu} - 1 - isu 1_{\{|u|\leq 1\}})\mu(du)$$

and

$$v_{X_t}(s) = e^{t\psi(s)}.$$

The measure μ is called the Lévy measure of X and (σ^2, ρ, μ) is the characteristic triplet of X.

Remark 2.4. *The random variable X_1 satisfies Cramér's condition iff we have $\sigma^2 \neq 0$ or the Lévy measure μ is not concentrated on a set of the form*

$$\{kh | k \in \mathbb{Z}\}, \quad \textit{for fixed } h > 0.$$

Moreover, if X_1 satisfies Cramér's condition, then X_t satisfies the same condition for all $t > 0$.

3 Approximation results

In the literature, there are lots of classical asymptotic expansion results for the i.i.d. sum case. I.i.d. sums are in some sense the discrete time analogues of the Lévy processes. The following theorem is presented in Petrov (1995).

Theorem 3.1. *Let $\{Y_j\}_{j=1}^n$ be a sequence of i.i.d. random variables satisfying Cramér's condition and $\mathbb{E}|Y_1|^k < \infty$ for some integer $k \geq 3$. Then*

$$\mathbb{P}\left(\sum_{j=1}^n Y_j < \sqrt{n} V_{Y_1} x\right) = \Phi(x) + \sum_{\nu=1}^{k-2} Q_\nu^{Y_1}(x) n^{-\frac{\nu}{2}} + o\left(n^{-\frac{k-2}{2}}\right)$$

uniformly in $x \in \mathbb{R}$.

This kind of results are presented also in Petrov (1975); Kolassa (2006); Cramér (1962). Generalisation of Theorem 3.1 is presented by Cramér (1962) as an analogue without a proof:

Theorem 3.2. *Let X_1 satisfy Cramér's condition and $k \geq 3$ be such an integer that $\mathbb{E}|X_1|^k < \infty$. Then*

$$\mathbb{P}(X_t < x V_{X_t}) = \sum_{\nu=1}^{k-3} Q_\nu^{X_1}(x) t^{-\frac{\nu}{2}} + o\left(t^{-\frac{k-2}{2}}\right).$$

In fact, Cramér (1962) introduces the form for the functions $Q_\nu^{X_1}(x)$ only implicitly. See Cramér (1962) pages 72, 98 and 99.

Note that we could include term $Q_{k-2}^{X_1}(x)t^{-\frac{k-2}{2}}$ to the result of Theorem 3.2 to get exactly analogous result to Theorem 3.1.

Next we are going to present some lemmata to scale the approximating functions $Q_\nu^{X_t}(x)$ with respect to t. The first of them is well-known but it is included here for convenience.

Lemma 3.3. *Let $k \in \mathbb{N}$ be s.t. $\mathbb{E}|X_1|^k < \infty$. Then*

$$\gamma_k^{X_t} = t\gamma_k^{X_1}.$$

Proof. Take $q \in \mathbb{Q}^+$. Now $q = \frac{m}{n}$ for some $m, n \in \mathbb{N}$. Now

$$\gamma_k^{X_{\frac{1}{n}}} = \frac{1}{i^k}\left[\frac{d^k}{ds^k}\log v_{X_1}(s)^{\frac{1}{n}}\right]_{s=0} = \frac{1}{n}\frac{1}{i^k}\left[\frac{d^k}{ds^k}\log v_{X_1}(s)\right]_{s=0} = \frac{1}{n}\gamma_k^{X_1}.$$

By repeating the previous argument we get

$$\gamma_k^{X_q} = m\gamma_k^{X_{\frac{1}{n}}} = \frac{m}{n}\gamma_k^{X_1} = q\gamma_k^{X_1}.$$

The general claim follows now by a simple density argument. □

Lemma 3.4. *Let $\nu \in \mathbb{N}$ be s.t. $\mathbb{E}|X_1|^{\nu+2} < \infty$, then*

$$Q_\nu^{X_t}(x) = t^{-\frac{\nu}{2}}Q_\nu^{X_1}(x), \quad \textit{for } x \in \mathbb{R}.$$

Proof. By definition,

$$Q_\nu^{X_t}(x) = f(x)\sum H_{\nu+2l-1}(x)\prod_{m=1}^{\nu}\frac{1}{k_m!}\left(\frac{\gamma_{m+2}^{X_t}}{(m+2)!V_{X_t}^{m+2}}\right)^{k_m},$$

where $f(x) = -\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ and the summation is extended over all non-negative integer solutions of the equation $\sum_{j=1}^{\nu} jk_j = \nu$, and we have $l = \sum_{j=1}^{\nu} k_j$.

$$\begin{aligned}
Q_\nu^{X_t}(x) &= f(x)\sum H_{\nu+2l-1}(x)\prod_{m=1}^{\nu}\left(\frac{t\gamma_{m+2}^{X_1}}{(m+2)!(\sqrt{t}V_{X_1})^{m+2}}\right)^{k_m} \\
&= f(x)\sum H_{\nu+2l-1}(x)\left(\prod_{m=1}^{\nu} t^{-\frac{1}{2}mk_m}\right)\cdot\left(\prod_{m=1}^{\nu}\frac{1}{k_m!}\left(\frac{\gamma_{m+2}^{X_1}}{(m+2)!V_{X_1}^{m+2}}\right)^{k_m}\right) \\
&= f(x)\sum t^{-\frac{1}{2}\sum_{m=1}^{\nu} mk_m} H_{\nu+2l-1}(x)\prod_{m=1}^{\nu}\left(\frac{\gamma_{m+2}^{X_1}}{(m+2)!V_{X_1}^{m+2}}\right)^{k_m} \\
&= t^{-\frac{\nu}{2}}Q_\nu^{X_1}(x).
\end{aligned}$$

In the last step, we used the fact that $\nu = k_1 + 2k_2 + \cdots + \nu k_\nu$. □

Corollary 3.5. *Let $k \geq 3$ be integer s.t. $\mathbb{E}|X_1|^k < \infty$ and let X_1 satisfy Cramér's condition. Then*

$$\begin{aligned}\mathbb{P}(X_t < xV_{X_t}) &= \Phi(x) + \sum_{\nu=1}^{k-2} Q_\nu^{X_1}(x) t^{-\frac{\nu}{2}} + o\left(t^{-\frac{k-2}{2}}\right) \\ &= \Phi(x) + \sum_{\nu=1}^{k-2} Q_\nu^{X_t}(x) + o\left(t^{-\frac{k-2}{2}}\right), \quad \textit{uniformly in } x \in \mathbb{R}.\end{aligned}$$

Proof. The result for rational t follows from Lemma 3.4 and Theorem 3.1. The result for general $t > 0$ follows by a continuity argument. □

From now on in this paper, we assume (if not otherwise stated) that X_1 satisfies Cramér's condition and has moments of all orders i.e.

$$\mathbb{E}|X_1|^\nu < \infty, \quad \text{for } \nu \in \mathbb{N}.$$

Now we have everything ready for introducing the main results of the article to get exact series representations. The proofs are in Section 4. In the following Theorems 3.6, 3.7 and 3.8, μ is assumed to be the Lévy measure of process X.

Theorem 3.6. *Let the Lévy measure of X have bounded support, then we get for $x_1 < x_2$ points of continuity of $\mathbb{P}(X_t < \cdot V_{X_t})$ that*

$$\begin{aligned}&\mathbb{P}\left(x_1 < \frac{X_t}{V_{X_t}} < x_2\right) = \mathbb{P}(X_t < x_2 V_{X_t}) - \mathbb{P}(X_t < x_1 V_{X_t}) \\ =&\Phi(x_2) - \Phi(x_1) + \sum_{\nu=1}^{\infty} \left(Q_\nu^{X_t}(x_2) - Q_\nu^{X_t}(x_1)\right) \\ =&\Phi(x_2) - \Phi(x_1) + \sum_{\nu=1}^{\infty} \left(Q_\nu^{X_1}(x_2) - Q_\nu^{X_1}(x_1)\right) t^{-\frac{\nu}{2}}.\end{aligned}$$

There is some discussion about the Lévy measures with bounded support for example in Sato (1999). In fact, this is a reasonable class to be considered in the simulations because of the practical limitations.

Nevertheless, the result of Theorem 3.6 is true with more general conditions:

Theorem 3.7. *Let μ be s.t. for some $a \geq 0$, $\mu(x)1_{\{|x|>a\}}$ is absolutely continuous with respect to the Lebesgue measure and for some $C, \epsilon > 0$*

$$\frac{d\mu(x)}{dx} \leq C \exp\{-|x|^{1+\epsilon}\}, \quad \textit{for } |x| \geq a.$$

Then the assertion of Theorem 3.6 holds.

And even more generally we get the following:

Theorem 3.8. *Assume that there are $a \geq 0$ and $C, \epsilon > 0$ s.t.*

$$\mu((-x-1,-x],[x,x+1)) \leq C\exp\{-x^{1+\epsilon}\}, \quad \text{for } x \geq a.$$

Then the representation of Theorem 3.6 holds.

Let us consider briefly Lévy processes with only positive (respectively negative) jumps and drift term. This is a reasonable class for risk processes, more precisely claim surplus processess in the sense of Asmussen (2000).

Remark 3.9 (Risk process case). *Consider a Lévy process satisfying conditions of Theorem 3.6, 3.7 or 3.8. Furthermore, assume that its Lévy measure is concentrated on positive reals and satisfies $\int_{\mathbb{R}\setminus\{0\}} |x|\mu(dx) < \infty$. Then there is some $x_1 \in \mathbb{R}$ s.t. $\mathbb{P}(X_t < x_1 V_{X_t}) = 0$ for all $t > 0$. Then we get easily a series representation for $\mathbb{P}(X_t < x_2 V_{X_t})$.*

Remark 3.10. *In the cases of Theorems 3.6, 3.7 and 3.8, we get some series representation also for other finite dimensional distributions since the series representation can be written for all increments separately.*

Moreover, we get a representation for the distribution function of the absolute value of a Lévy process as follows:

Corollary 3.11. *Assume that the assumptions of 3.6, 3.7 or 3.8 hold. Then we get for $x > 0$ and $-x$ points of continuity of $\mathbb{P}(X_t < \cdot V_{X_t})$ that*

$$\mathbb{P}(|X_t| < xV_{X_t}) = 2\Phi(x)-1+2\sum_{\nu=1}^{\infty} Q_{2\nu}^{X_t}(x) = 2\Phi(x)-1+2\sum_{\nu=1}^{\infty} Q_{2\nu}^{X_1}(x)t^{-\nu} \quad (2)$$

and

$$\mathbb{P}(|X_t| > xV_{X_t}) = 2-2\Phi(x)-2\sum_{\nu=1}^{\infty} Q_{2\nu}^{X_t}(x) = 2-2\Phi(x)-2\sum_{\nu=1}^{\infty} Q_{2\nu}^{X_1}(x)t^{-\nu}. \quad (3)$$

Proof.

$$\begin{aligned}
&\mathbb{P}(|X_t| < xV_{X_t}) = \mathbb{P}(X_t < xV_{X_t}) - \mathbb{P}(X_t < -xV_{X_t}) \\
=&\Phi(x) - \Phi(-x) + \sum_{\nu=1}^{\infty} \left(Q_\nu^{X_t}(x) - Q_\nu^{X_t}(-x)\right) \\
=&2\Phi(x) - 1 + \sum_{\nu=1}^{\infty} \left(Q_\nu^{X_t}(x) - Q_\nu^{X_t}(-x)\right).
\end{aligned}$$

We use the symmetry condition for Hermite polynomials and get

$$\begin{aligned}
&Q_\nu^{X_t}(x) - Q_\nu^{X_t}(-x) \\
=&-\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \sum \left(H_{\nu+2l-1}(x) - H_{\nu+2l-1}(-x)\right) \prod_{m=1}^{\nu} \left(\frac{\gamma_{m+2}^{X_t}}{(m+2)!V_{X_t}^{m+2}}\right)^{k_m} \\
=&2Q_\nu^{X_t}(x) 1_{\{\nu=2p|p\in\mathbb{N}\}}.
\end{aligned}$$

Equation (3) is a direct consequence of (2). ∎

If X_1 has density function, we get the following:

Corollary 3.12. *Assume besides the assumptions of 3.6, 3.7 or 3.8 that* $\frac{X_t}{V_{X_t}}$ *has density function* $g_{X_t}(s)$ *for* $t>0$. *Then*

$$g_{X_t}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} + \sum_{\nu=1}^{\infty} \frac{d}{dx} Q_\nu^{X_t}(x).$$

Corollary 3.12 gives us together with the following lemma an exact series representation for the density function.

Lemma 3.13. *For* $\nu \in \mathbb{N}$ *we have*

$$\frac{d}{dx} Q_\nu^{X_t}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum H_{\nu+2l}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{\gamma_{m+2}^{X_t}}{(m+2)! V_{X_t}^{m+2}} \right)^{k_m},$$

with the notation of (1).

Proof.

$$\begin{aligned}
\frac{d}{dx} Q_\nu^{X_t}(x) = & \left(\frac{d}{dx} \left(-\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right) \right) \sum H_{\nu+2l-1}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{\gamma_{m+2}^{X_t}}{(m+2)! V_{X_t}^{m+2}} \right)^{k_m} \\
& - \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum \frac{d}{dx} H_{\nu+2l-1}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{\gamma_{m+2}^{X_t}}{(m+2)! V_{X_t}^{m+2}} \right)^{k_m} \\
= & \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum (x H_{\nu+2l-1}(x) - (\nu+2l-1) H_{\nu+2l-2}(x)) \times \\
& \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{\gamma_{m+2}^{X_t}}{(m+2)! V_{X_t}^{m+2}} \right)^{k_m} \\
= & \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sum H_{\nu+2l}(x) \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{\gamma_{m+2}^{X_t}}{(m+2)! V_{X_t}^{m+2}} \right)^{k_m}.
\end{aligned}$$

In the last step, we used the recursion formula for the Hermite polynomials. □

4 Proofs

The following lemma gives us a representation formula for the cumulants of a Lévy process. The result may be well known but it is included in this paper for convenience. It is worth mentioning that Cramér's condition is not assumed in the following lemma. The condition (4) is used in the literature e.g. by Nualart and Schoutens (2000). This condition is enough to guarantee the existence of all moments.

Lemma 4.1. *Let (σ^2, ρ, μ) be the characteristic triplet of X. Furthermore, assume that for some $\lambda > 0$ and for all $\delta > 0$*

$$\int_{\mathbb{R}\setminus(-\delta,\delta)} e^{\lambda|x|}\mu(dx) < \infty. \tag{4}$$

Then

$$\gamma_\nu^{X_1} = \int_{\mathbb{R}\setminus\{0\}} x^\nu \mu(dx), \quad \nu \geq 3.$$

and

$$\gamma_2^{X_1} = \int_{\mathbb{R}\setminus\{0\}} x^2 \mu(dx) + \sigma^2.$$

Proof. Define μ^M and μ^Y s.t. $\frac{d\mu^M}{d\mu}(x) = 1_{\{|x|\leq 1\}}$ and $\frac{d\mu^Y}{d\mu}(x) = 1_{\{|x|>1\}}$. We make the following Lévy-Itô type decomposition

$$X_t = W_t + Y_t + M_t,$$

where W, Y and M are independent Lévy processes. The characteristic triplet of W is $(\sigma^2, \rho, 0)$. Y has the triplet $(0, 0, \mu^Y)$ and M has $(0, 0, \mu^M)$.

First, we consider the compound Poisson process Y. It follows from the assumption that

$$\int_{\mathbb{R}\setminus(-1,1)} e^{\lambda|x|}\mu(dx) = \int_{\mathbb{R}\setminus(-1,1)} \sum_{\nu=0}^{\infty} \frac{(\lambda|x|)^\nu}{\nu!}\mu(dx) < \infty.$$

It follows by using the Lévy-Khinchine theorem and the dominated convergence theorem (Rudin, 1987) that

$$\log \mathbb{E}e^{iuY_1} = \int_{\mathbb{R}\setminus(-1,1)} (e^{iux} - 1)\mu(dx) = \int_{\mathbb{R}\setminus(-1,1)} \sum_{\nu=1}^{\infty} \frac{(iux)^\nu}{\nu!}\mu(dx)$$
$$= \sum_{\nu=1}^{\infty} \frac{(iu)^\nu}{\nu!} \int_{\mathbb{R}\setminus(-1,1)} x^\nu \mu(dx), \quad \text{for } |u| \leq \lambda.$$

Now we have

$$\gamma_\nu^{Y_1} = \int_{\mathbb{R}\setminus(-1,1)} x^\nu \mu(dx), \quad \text{for } \nu = 2, 3, \ldots$$

Next, we consider the jump martingale M. We begin by considering the process M^ϵ which is obtained by neglecting the jumps of M less than $\epsilon \in (0, 1)$ of absolute value. Rigorously, we define the measure μ^ϵ by

$$\frac{d\mu^\epsilon}{d\mu}(x) = 1_{\{\epsilon\leq|x|\leq 1\}}$$

and consider the Lévy process M^ϵ with the characteristic triplet $(0, 0, \mu^\epsilon)$. Now M^ϵ is a compensated compound Poisson process and we get like in the case of Y that

$$\gamma_\nu^{M_1^\epsilon} = \int_{\epsilon\leq|x|\leq 1} x^\nu \mu(dx).$$

By the condition (4) and the Lévy-Khinchine theorem, $\gamma_\nu^{M_1}$ is finite whenever $\nu \geq 2$. Furthermore,

$$\int_0^1 x^\nu \mu(dx), \quad \text{and} \quad \int_{-1}^0 x^\nu \mu(dx)$$

exist and are finite. Now we can use the monotone convergence theorem and obtain

$$\int_{\epsilon \leq |x| \leq 1} x^\nu \mu(dx) = \int_{-1}^{-\epsilon} x^\nu \mu(dx) + \int_{\epsilon}^{1} x^\nu \mu(dx) \to \gamma_\nu^{M_1},$$

when $\epsilon \downarrow 0$. Next, we use the facts that the cumulants of the normal distribution vanish when $\nu \geq 3$ and $\gamma_2^{W_1} = \sigma^2$ since W_1 is normally distributed with variance σ^2. The general result is now obtained by the additivity of cumulants. □

The next lemma gives us another characterisation of the condition on the Lévy measure in Theorem 3.6. From now on in this article, we will use the following notation of scaled cumulants $\lambda_\nu^{X_t} = \frac{\gamma_\nu^{X_t}}{V_{X_t}^\nu}$, for $\nu \in \mathbb{N}$.

Lemma 4.2. *The Lévy measure of process X is concentrated on some bounded interval is equivalent to the condition that there exists some $C > 0$ s.t.*

$$\lambda_\nu^{X_1} \leq C^\nu, \quad \textit{for all } \nu \in \mathbb{N}.$$

Proof. Let us first assume that such C exists. Now we can use Lemma 4.1 and we get for $\nu \geq 3$ that

$$\int_{\mathbb{R}\setminus\{0\}} x^\nu \mu(dx) \leq C^\nu V_{X_1}^\nu.$$

For even ν, $|\gamma_\nu| = \gamma_\nu$. We know also by Rudin (1987) page 71 that it holds for $L^p(\mu)$ norms that

$$||x||_{2n+1} \leq \max\left(||x||_{2n}, ||x||_{2n+2}\right), \quad \text{for } n \geq 1.$$

Hence there is some $D > 0$ s.t. $\int_{\mathbb{R}\setminus\{0\}} |x|^\nu \mu(dx) \leq D^\nu$ for all $\nu \geq 4$. Moreover, we get

$$D \geq ||x||_\nu \to ||x||_\infty \quad \text{as } \nu \to \infty.$$

Now $||\frac{x}{D}||_\infty \leq 1$ with respect to μ. In other words, μ is concentrated on some bounded interval.

The other way is even simpler. Because μ is concentrated on some bounded interval, it follows that $||x||_\infty < \infty$. We can choose $C = \frac{1}{V_{X_1}} \sup_\nu ||x||_\nu$. □

Now we have everything ready for the proofs of the main results.

Proof. (Theorem 3.6)

Let us first work out the representation for the logarithm of the characteristic function i.e. the characteristic exponent of the Lévy process.

$$\sum_{\nu=2}^{\infty}\left|\frac{\lambda_\nu^{X_t}}{\nu!}(is)^\nu\right| = \sum_{\nu=2}^{\infty}\left|\frac{1}{\nu!}\frac{t\gamma_\nu^{X_1}}{t^{\frac{\nu}{2}}V_{X_1}^\nu}(is)^\nu\right|$$
$$=\sum_{\nu=2}^{\infty}\left|t^{-\frac{\nu-2}{2}}\frac{1}{\nu!}\frac{\gamma_\nu^{X_1}}{V_{X_1}^\nu}(is)^\nu\right|$$
$$=\sum_{\nu=2}^{\infty}\left|t\frac{\lambda_\nu^{X_1}}{\nu!}\left(\frac{is}{\sqrt{t}}\right)^\nu\right| \leq t\sum_{\nu=2}^{\infty}\frac{1}{\nu!}\left|\frac{Cs}{\sqrt{t}}\right|^\nu,$$

which is bounded when $t > \epsilon > 0$ and $|s| < K < \infty$ for arbitrary $\epsilon, K \in (0,\infty)$. In the last step, we used the characterisation of Lemma 4.2. Now this series is dominated by the series expansion of the exponential function and the series

$$\sum_{\nu=2}^{\infty}\frac{\lambda_\nu^{X_t}}{\nu!}(is)^\nu$$

converges to an analytical function when $t > 0$ is fixed. Now, define

$$f_{X_t}(s) = \upsilon_{X_t}\left(\frac{s}{V_{X_t}}\right).$$

By computing the cumulants, this notation gives for $n \in \mathbb{N}$

$$\left[\frac{d^n}{ds^n}\log f_{X_t}(s)\right]_{s=0} = \left[\frac{d^n}{ds^n}\log \upsilon_{X_1}\left(\frac{s}{\sqrt{t}V_{X_1}}\right)^t\right]_{s=0}$$
$$=t\left(\frac{1}{\sqrt{t}V_{X_1}}\right)^n\left[\frac{d^n}{ds^n}\log \upsilon_{X_1}(s)\right]_{s=0}$$
$$=t^{-\frac{n-2}{2}}i^n\frac{\gamma_n^{X_1}}{V_{X_1}^n} = t^{-\frac{n-2}{2}}i^n\lambda_n^{X_1} = i^n\lambda_n^{X_t}.$$

Now

$$\log f_{X_t}(s) = \sum_{\nu=2}^{\infty}\frac{\lambda_\nu^{X_1}}{\nu!}t^{-\frac{\nu-2}{2}}(is)^\nu.$$

We observe that $\lambda_2^{X_t} = 1$ for all $t > 0$. So we obtain

$$f_{X_t}(s) = e^{-\frac{s^2}{2}}\exp\left(\sum_{j=1}^{\infty}\frac{\lambda_{j+2}^{X_1}}{(j+2)!}t^{-\frac{j}{2}}(is)^{j+2}\right).$$

Next, consider a more general form

$$\exp\left(\sum_{j=1}^{\infty}\frac{\lambda_{j+2}^{X_1}}{(j+2)!}z^j u^{j+2}\right).$$

With fixed u, this series converges absolutely, uniformly in any compact set with respect to the parameter z. Thus in every compact set with respect to z, we rearrange the series of the exponential function and get a series representation with respect to z. Hence,

$$\exp\left(\sum_{j=1}^{\infty}\frac{\lambda_{j+2}^{X_1}}{(j+2)!}z^j u^{j+2}\right) = 1+\sum_{\nu=1}^{\infty}P_\nu(u)z^\nu$$

for some polynomials $(P_\nu)_{\nu=1}^{\infty}$ that can be computed formally by compounding these two series, which is possible due to the absolute convergence. Now

$$f_{X_t}(s) = e^{-\frac{s^2}{2}} + \sum_{\nu=1}^{\infty}P_\nu(is)e^{-\frac{s^2}{2}}t^{-\frac{\nu}{2}}.$$

By the inversion formula of the characteristic function (Petrov, 1995), we get for x_1, x_2 points of continuity of $\mathbb{P}(X_t < \cdot V_{X_t})$

$$\begin{aligned}
&\mathbb{P}(X_t < x_2V_{X_t}) - \mathbb{P}(X_t < x_1V_{X_t})\\
=&\frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T}\frac{e^{-isx_2}-e^{-isx_1}}{-is}\left(e^{-\frac{s^2}{2}}+\sum_{\nu=1}^{\infty}P_\nu(is)e^{-\frac{s^2}{2}}t^{-\frac{\nu}{2}}\right)ds.
\end{aligned}$$

With fixed $t > 0$, the series inside the integral is absolutely convergent uniformly in compact sets with respect to s. Thus the integral is always well-defined and can be computed term-wise. Moreover, the limit exists since

$$\begin{aligned}
&\lim_{S\to\infty}\int_{-S}^{S}\frac{e^{-isx_2}-e^{-isx_1}}{-is}f_{X_t}(s)ds\\
&-\int_{-T}^{T}\frac{e^{-isx_2}-e^{-isx_1}}{-is}\left(e^{-\frac{s^2}{2}}+\sum_{\nu=1}^{\infty}P_\nu(is)e^{-\frac{s^2}{2}}t^{-\frac{\nu}{2}}\right)ds\\
=&\lim_{S\to\infty}\int_{T<|s|<S}\frac{e^{-isx_2}-e^{-isx_1}}{-is}f_{X_t}(s)ds \to 0, \quad \text{when } T\to\infty.
\end{aligned}$$

In the last step, we used the fact that f_{X_t} is a characteristic function. Hence, there are such functions $(R_\nu)_{\nu=1}^{\infty}$ that we can write

$$\mathbb{P}(X_t < x_2V_{X_t})-\mathbb{P}(X_t < x_1V_{X_t}) = \Phi(x_2)-\Phi(x_1)+\sum_{\nu=1}^{\infty}\left(R_\nu(x_2)-R_\nu(x_1)\right)t^{-\frac{\nu}{2}}.$$

We use the classical Theorem 3.1 and the scaling Lemma 3.4 and find out that for all $\nu = 1, 2, \ldots$

$$R_\nu(x) = Q_\nu^{X_1}(x) = t^{\frac{\nu}{2}}Q_\nu^{X_t}(x).$$

□

Proof. (Theorem 3.7)

The proof proceeds analogously to the proof of Theorem 3.6 but we have to argue why we can rearrange the series of

$$f_{X_t}(s) = e^{-\frac{s^2}{2}} \exp\left(\sum_{j=1}^{\infty} \frac{\lambda_{j+2}^{X_1}}{(j+2)!} t^{-\frac{j}{2}} (is)^{j+2}\right). \tag{5}$$

With these assumptions on the Lévy measure μ, we can use the representation Lemma 4.1 for the cumulants. Let $m \in \mathbb{N}$ be such that $\frac{1}{m} \leq \epsilon$. Observe now that

$$\int_0^\infty x^n e^{-x^{1+\frac{1}{m}}} dx = \int_0^\infty -\frac{m}{m+1} x^{n-\frac{1}{m}} \left(-\frac{m+1}{m} x^{\frac{1}{m}} e^{-x^{1+\frac{1}{m}}}\right) dx$$
$$= -\frac{m}{m+1}\left[x^{n-\frac{1}{m}} e^{-x^{1+\frac{1}{m}}}\right]_0^\infty + \int_0^\infty \frac{m}{m+1}\left(n-\frac{1}{m}\right) x^{n-1-\frac{1}{m}} e^{-x^{1+\frac{1}{m}}} dx$$
$$= \int_0^\infty \left(\frac{m}{m+1}\right)^2 \left(n-\frac{1}{m}\right)\left(n-1-\frac{2}{m}\right) x^{n-2-\frac{2}{m}} e^{-x^{1+\frac{1}{m}}} dx$$
$$= \left(\frac{m}{m+1}\right)^{\lfloor n\frac{m}{m+1}\rfloor} \prod_{j=1}^{\lfloor n\frac{m}{m+1}\rfloor} \left(n+1-j\left(1+\frac{1}{m}\right)\right) \times \int_0^\infty x^{n-\lfloor n\frac{m}{m+1}\rfloor\left(\frac{m+1}{m}\right)} e^{-x^{1+\frac{1}{m}}} dx$$
$$\leq \prod_{j=1}^{\lfloor n\frac{m}{m+1}\rfloor} \left(n+1-j\left(1+\frac{1}{m}\right)\right) \times D,$$

where

$$D = \max_{l=0,\dots,m} \int_0^\infty x^{l-\lfloor l\frac{m}{m+1}\rfloor\left(\frac{m+1}{m}\right)} e^{-x^{1+\frac{1}{m}}} dx.$$

Note that the constant D is finite and does not depend on n. Without loss of generality, we can assume $\tilde{X}$ to be compensated compound Poisson process with $a = 0$, since we can express general X as a sum of this kind of process and a process satisfying the conditions of Theorem 3.6. Then we get a bound for (5) by the additivity of cumulants.

Note that this decomposition can be made such a way that Cramér's condition does not fail here if the Lévy measure has unbounded support. This is due to the fact that the tail of the Lévy measure is absolutely continuous with respect to the Lebesgue measure. Now we have

$$\sum_{\nu=2}^{\infty} \left|\frac{\gamma_\nu^{\tilde{X}_t}}{V_{X_t}^\nu \nu!}(is)^\nu\right|$$
$$= \sum_{\nu=2}^{m} \left|\frac{\gamma_\nu^{\tilde{X}_t}}{V_{X_t}^\nu \nu!}(is)^\nu\right| + t \sum_{\nu=m+1}^{\infty} \frac{1}{\nu!}\left|\gamma_\nu^{\tilde{X}_1}\right| \left(\frac{|s|}{\sqrt{t}V_{X_1}}\right)^\nu$$
$$= \sum_{\nu=2}^{m} \left|\frac{\gamma_\nu^{\tilde{X}_t}}{V_{X_t}^\nu \nu!}(is)^\nu\right| + t \sum_{j=1}^{\infty}\sum_{k=0}^{m} \frac{1}{((m+1)j+k)!}\left|\gamma_{(m+1)j+k}^{\tilde{X}_1}\right| \left(\frac{|s|}{\sqrt{t}V_{X_1}}\right)^{(m+1)j+k}.$$

The first term is a finite sum of finite summands if $0 < t < \infty$. We get an estimate for the other sum as follows

$$
\begin{aligned}
&t\sum_{j=1}^{\infty}\sum_{k=0}^{m}\frac{1}{((m+1)j+k)!}\left|\gamma^{\tilde{X}_1}_{(m+1)j+k}\right|\left(\frac{|s|}{\sqrt{t}V_{X_1}}\right)^{(m+1)j+k}\\
\leq &t\sum_{j=1}^{\infty}\sum_{k=0}^{m}\frac{1}{((m+1)j+k)!}\times\\
&2CD\prod_{l=1}^{\lfloor((m+1)j+k)\frac{m}{m+1}\rfloor}\left((m+1)j+k+1-l\left(1+\frac{1}{m}\right)\right)\left(\frac{|s|}{\sqrt{t}V_{X_1}}\right)^{(m+1)j+k}.
\end{aligned}
$$

Now define

$$
g(l) = (m+1)j+k+1-l-\left\lfloor\frac{l}{m}\right\rfloor, \quad l = 1,\dots,\left\lfloor((m+1)j+k)\frac{m}{m+1}\right\rfloor.
$$

We observe that $g(l) > g(l+1)$ and the values of g are integers from 1 to $(m+1)j+k$. Nevertheless, g does not take every $(m+1)$th integer value. This fact is due to the jump of the floor function. So there is at least j terms missing in the product. By assuming them to be the j smallest ones, we get a rough estimate

$$
\prod_{l=1}^{\lfloor((m+1)j+k)\frac{m}{m+1}\rfloor}\left((m+1)j+k+1-l\left(1+\frac{1}{m}\right)\right) \leq \frac{((m+1)j+k)!}{j!}.
$$

And finally

$$
\begin{aligned}
&t\sum_{j=1}^{\infty}\sum_{k=0}^{m}\frac{1}{((m+1)j+k)!}\left|\gamma^{\tilde{X}_1}_{(m+1)j+k}\right|\left(\frac{|s|}{\sqrt{t}V_{X_1}}\right)^{(m+1)j+k}\\
\leq &t\sum_{j=1}^{\infty}\sum_{k=0}^{m}\frac{1}{j!}2CD\left(\frac{|s|}{\sqrt{t}V_{X_1}}\right)^{(m+1)j+k}\\
\leq &2CDt\left(\sum_{k=0}^{m}\left(\frac{|s|}{\sqrt{t}V_{X_1}}\right)^{k}\right)\sum_{j=1}^{\infty}\frac{1}{j!}\left(\left(\frac{|s|}{\sqrt{t}V_{X_1}}\right)^{m+1}\right)^{j} < \infty,
\end{aligned}
$$

as an exponential series when $0 < t < \infty$. The last part of the proof is analogous to the proof of Theorem 3.6. □

Proof. (Theorem 3.8)

We have to get a suitable estimate for the cumulants from above to be able to continue as in the proof of Theorem 3.7. Let us define function η on positive reals as follows

$$
\eta(x) = \mu((-x-1,-x],[x,x+1)).
$$

We can easily represent the growing condition for the Lévy measure using this function. Now we can estimate the cumulants in the spirit of Lemma 4.1. Without loss of generality, we can assume that $a \geq 1$. We get

$$\int_{-\infty}^{\infty} |x|^{\nu} \mu(dx) \leq \int_{-a}^{a} |x|^{\nu} \mu(dx) + \sum_{j=0}^{\infty} |x+a+1|^{\nu} \eta(a+j).$$

For $\nu \geq 2$, the first term is bounded by D^{ν} for some $D > 0$. For the second term we get

$$\sum_{j=0}^{\infty} |x+a+1|^{\nu} \eta(a+j) \leq \int_{a}^{\infty} (x+2)^{\nu} C e^{-x^{1+\epsilon}} dx \leq C \int_{a}^{\infty} (3x)^{\nu} e^{-x^{1+\epsilon}} dx.$$

The rest of the proof is analogous to the proof of Theorem 3.7. □

Proof. (Corollary 3.12)
Let $(P_{\nu})_{\nu=1}^{\infty}$ be the same polynomials as in the proof of Theorem 3.6. We can use the series representation for characteristic function of $\frac{X_t}{V_{X_t}}$ and get

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-isx} f_{X_t}(s) ds = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-isx} \left(e^{-\frac{s^2}{2}} + \sum_{\nu=1}^{\infty} P_{\nu}(is) e^{-\frac{s^2}{2}} t^{-\frac{\nu}{2}} \right) ds.$$

With fixed $t > 0$, the absolute convergence is uniform in compact sets with respect to s, as in the preceeding proofs. Thus, the integral is well-defined and can be computed term-wise. Moreover, with fixed $x \in \mathbb{R}$

$$\int_{|s|>T} e^{-isx} f_{X_t}(s) ds \to 0, \quad \text{as } T \to \infty,$$

since f_{X_t} is a characteristic function of some random variable with density function. Hence,

$$\begin{aligned} &\lim_{T\to\infty} \left| g_{X_t}(x) - \frac{1}{2\pi} \int_{-T}^{T} e^{-isx} \left(e^{-\frac{s^2}{2}} + \sum_{\nu=1}^{\infty} P_{\nu}(is) e^{-\frac{s^2}{2}} t^{-\frac{\nu}{2}} \right) ds \right| \\ &= \lim_{T\to\infty} \left| \frac{1}{2\pi} \int_{|s|>T} e^{-isx} f_{X_t}(s) ds \right| = 0. \end{aligned}$$

We have shown that there is some series representation but we still have to show that the limit equals to what is claimed. We have

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{-isx} P_{\nu}(is) e^{-\frac{s^2}{2}} ds = -\frac{1}{2\pi} \int_{\mathbb{R}} is \frac{e^{-isx}}{-is} P_{\nu}(is) e^{-\frac{s^2}{2}} ds = \frac{d}{dx} Q_{\nu}^{X_t}(x).$$

□

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