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Abstract: *We study stagnation zones of $\mathcal{A}$-harmonic functions on canonical domains in the Euclidean n-dimensional space. Phragmen-Lindelöf type theorems are proved.*

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1 Introduction

In this article we investigate solutions of the $\mathcal{A}$-Laplace equation on canonical domains in the n-dimensional Euclidean space.

Suppose that D is a domain in $\mathbb{R}^n$, and let $f\colon D \to \mathbb{R}$ be a function. For $s > 0$, a subset $\Delta \subset D$ is called *s-zone* (*stagnation zone with the deviation s*) of f, if there exists a constant C such that the difference between C and the function f is smaller than s on Δ. We may, for example, consider difference in the sense of the Euclidean distance,

$$\|f(x) - C\|_{C(\Delta)} = \sup_{x\in\Delta} |f(x) - C| < s\,,$$

the L^p-norm

$$\|f(x) - C\|_{L^p(\Delta)} = \Bigg(\int\limits_{\Delta} |f(x) - C|^p \, d\mathcal{H}^n \Bigg)^{1/p} < s\,,$$

or the Sobolev norm

$$\|f(x)\|_{W^1_p(\Delta)} = \Bigg(\int\limits_{\Delta} |\nabla f(x)|^p d\mathcal{H}^n \Bigg)^{1/p} < s\,,$$

where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure in $\mathbb{R}^n$.

For discussion about the history of the question, recent results and applications, see [SS06, SS07].

Some estimates of stagnation zone sizes for solutions of the $\mathcal{A}$-Laplace equation on locally Lipschitz surfaces and behaviour of solutions in stagnation zones, were given in [Mik07]. In this paper we consider solutions of the $\mathcal{A}$-Laplace equation in subdomains of $\mathbb{R}^n$ of a special form. In two-dimensional case, such domains are sectors and strips. In higher dimensions, they are conical and cylindrical regions. The special form of domains allows us to obtain more precise results.

Below we study stagnation zones of generalized solutions of the $\mathcal{A}$-Laplace equation

$$\operatorname{div} \mathcal{A}(x, \nabla f) = 0$$

(see [HKM93]) with boundary conditions of types (see definitions 1.7 and 1.11 below):

$$\langle \mathcal{A}(x, \nabla f), \overline{\mathbf{n}} \rangle = 0\,, \quad x \in \partial D \setminus G$$

and

$$f\, \langle \mathcal{A}(x, \nabla f), \overline{\mathbf{n}} \rangle = 0\,, \quad x \in \partial D \setminus G$$

on canonical domains in the Euclidean n-dimensional space, where G is a closed subset of ∂D. We will prove Phragmén-Lindelöf type theorems for solutions of the $\mathcal{A}$-Laplace equation with such boundary conditions.

Canonical domains

Let $n \geq 2$. Fix an integer k, $1 \leq k \leq n$, and set

$$d_k(x) = \Big(\sum_{i=1}^{k} x_i^2 \Big)^{1/2}.$$

We call the set

$$B_k(t) = \{x \in \mathbb{R}^n : d_k(x) < t\}$$

a k-ball and

$$\Sigma_k(t) = \{x \in \mathbb{R}^n : d_k(x) = t\}$$

a k-sphere in $\mathbb{R}^n$. In particular, the symbol $\Sigma_k(0)$ denotes the k-sphere with the radius 0, i.e. the set

$$\Sigma_k(0) = \big\{x = (x_1, \ldots, x_k, \ldots, x_n) : x_{k+1} = \ldots = x_n = 0\big\}.$$

For every $0 < k < n$ we set

$$p_k(x) = \Big(\sum_{j=k+1}^{n} x_j^2 \Big)^{1/2}.$$

and

$$\Sigma_k^*(t) = \{x \in \mathbb{R}^n : p_k(x) = t\}, \qquad t \geq 0.$$

Let $0 < \alpha < \beta < \infty$ be fixed, and let

$$D_{\alpha,\beta}^k = \{x \in \mathbb{R}^n : \alpha < p_k(x) < \beta\}.$$

For $k = n-1$ we also assume that $x_n > 0$. Then for $k = n-1$ the $D_{\alpha,\beta}^{n-1}$ is the a layer between two parallel hyperplanes, and for $1 \leq k < n-1$ the boundary of the domain $D_{\alpha,\beta}^k$ consists of two coaxial cylindrical surfaces. The intersections $\Sigma_k(t) \cap D_{\alpha,\beta}^k$ are precompact for all $t > 0$. Thus, the functions $d_k(x)$ are *exhaustion functions* for $D_{\alpha,\beta}^k$.

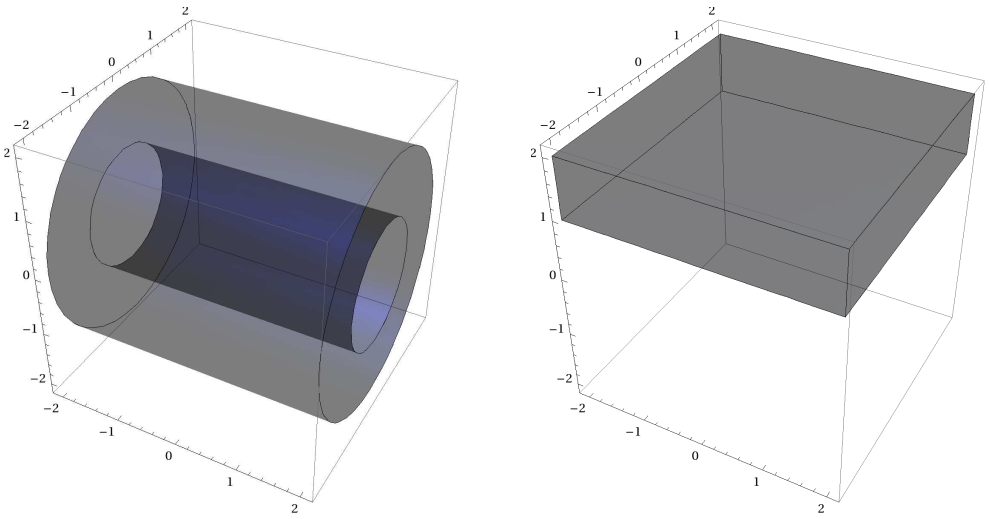

FIGURE: $D_{1,2}^1$ (left) and $D_{1,2}^2$ in $\mathbb{R}^3$.

Structure conditions

Let D be a subdomain of $\mathbb{R}^n$ and let

$$\mathcal{A}(x,\xi)\colon \overline{D}\times\mathbb{R}^n \to \mathbb{R}^n$$

be a vector function such that for a.e. $x\in\overline{D}$ the function

$$\mathcal{A}(x,\xi)\colon \mathbb{R}^n \to \mathbb{R}^n$$

is defined and continuous with respect to ξ. We assume that the function

$$x\mapsto \mathcal{A}(x,\xi)$$

is measurable in the Lebesgue sense for all $\xi\in\mathbb{R}^n$ and

$$\mathcal{A}(x,\lambda\xi)=\lambda\,|\lambda|^{p-2}\,\mathcal{A}(x,\xi)\,,\quad \lambda\in\mathbb{R}\setminus\{0\},\ p\geq 1. \tag{1.1}$$

Suppose that for a.e. $x\in D$ and for all $\xi\in\mathbb{R}^n$ the following properties hold:

$$\nu_1\,|\xi|^p\leq\langle\xi,\mathcal{A}(x,\xi)\rangle,\quad |\mathcal{A}(x,\xi)|\leq\nu_2\,|\xi|^{p-1} \tag{1.2}$$

with $p\geq 1$ and some constants $\nu_1,\ \nu_2>0$. We consider the equation

$$\operatorname{div}\mathcal{A}(x,\nabla f)=0\,. \tag{1.3}$$

An important special case of (1.3) is the Laplace equation

$$\Delta f=\sum_{i=1}^n\frac{\partial^2 f}{\partial x_i^2}=0\,.$$

As in [HKM93, Chapter 6], we call continuous weak solutions of the equation (1.3) $\mathcal{A}$-*harmonic* functions. However we should note that our definition of generalized solutions is slightly different from the definition given in [HKM93, p.56].

Frequencies

Fix $t\geq 0$ and $p\geq 1$. Let O be an open subset of $\Sigma_k^*(t)$ (with respect to the relative topology of $\Sigma_k^*(t)$), and let $\mathcal{P}$ be a nonempty closed subset of ∂O. We let

$$\lambda_{p,\mathcal{P}}(O)=\inf_u\frac{\displaystyle\int_O|\nabla u|^p d\mathcal{H}^{n-1}}{\displaystyle\int_O u^p d\mathcal{H}^{n-1}}\,, \tag{1.4}$$

where $u\in\mathrm{Lip}_{\mathrm{loc}}(O)\cap C^0(\overline{O})$ with

$$u|_{\mathcal{P}}=0,\ \text{and}\ \langle\overline{\mathbf{n}},\mathcal{A}(x,\nabla u)\rangle=0,\ \ \mathcal{H}^{n-2}\ \text{-a.e. in}\ \ \partial O\setminus\mathcal{P}\,. \tag{1.5}$$

Here $\overline{\mathbf{n}}$ is the unit normal vector to ∂O. If $\mathcal{P} = \partial O$ we call $\lambda_p(O) \equiv \lambda_{p,\mathcal{P}}(O)$ the *first frequency* of the order $p \geq 1$ of the set O. If $\mathcal{P} \neq \partial O$ the quantity $\lambda_{p,\mathcal{P}}(O)$ is the *third frequency*.

The *second frequency* is the following quantity:

$$\mu_p(O) = \sup_C \inf_u \frac{\int\limits_O |\nabla u|^p d\mathcal{H}^{n-1}}{\int\limits_O (u - C)^p d\mathcal{H}^{n-1}}\,, \tag{1.6}$$

where the supremum is taken over all constants C and $u \in \mathrm{Lip}_{\mathrm{loc}}(O) \cap C^0(\overline{O})$. See also Pólya and Szegö [PS51], Lax [Lax57].

Generalized boundary conditions

Let $\varphi\colon D \subset \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. We denote by $D_b(\varphi)$ the set of all points $x \in D$ at which φ does not have the differential. Let $U \subset D$ be a subset and let $\partial' U = \partial U \setminus \partial D$ be its boundary with respect to D. If $\partial' U$ is $(\mathcal{H}^{n-1}, n-1)$-rectifiable, then it has locally finite perimeter in the sense of De Giorgi and $\mathcal{H}^{n-1}$-almost everywhere on ∂U, an unit normal vector $\overline{\mathbf{n}}$ exists [Fed69, Sections 3.2.14, 3.2.15].

Let $D \subset \mathbb{R}^n$ be a domain and let $G \subset \partial D$ be a subset of the boundary of D. Define the concept of a generalized solution of (1.3) with zero boundary conditions on $\partial D \setminus G$. A subset $U \subset D$ is called *admissible*, if $\overline{U} \cap \overline{G} = \emptyset$ and U has a $(\mathcal{H}^{n-1}, n-1)$-rectifiable boundary with respect to D.

Suppose that D is unbounded. Let $G \subset \partial D$ be a set closed in $\mathbb{R}^n \cup \{\infty\}$. We denote by (G, D) the set of all subdomains $U \subset D$ with $\partial U \subset (D \cup (\partial D \setminus G))$ and $(\mathcal{H}^{n-1}, n-1)$-rectifiable boundaries $\partial' U = \partial U \setminus \partial D$.

1.7 Definition. We say that a locally Lipschitz function $f\colon D \to \mathbb{R}$ is a generalized solution of (1.3) with the boundary condition

$$\langle \mathcal{A}(x, \nabla f), \overline{\mathbf{n}} \rangle = 0\,, \quad x \in \partial D \setminus G\,, \tag{1.8}$$

if for every subdomain $U \in (G, D)$,

$$\mathcal{H}^{n-1}\big[\partial' U \cap D_b(f)\big] = 0\,, \tag{1.9}$$

and for every locally Lipschitz function $\varphi\colon \overline{U} \setminus G \to \mathbb{R}$ the following property holds:

$$\int\limits_{\partial' U} \varphi\, \langle \mathcal{A}(x, \nabla f)\,, \overline{\mathbf{n}} \rangle\, d\mathcal{H}^{n-1} = \int\limits_U \langle \mathcal{A}(x, \nabla f)\,, \nabla \varphi \,\rangle\, d\mathcal{H}^n\,. \tag{1.10}$$

Here $\overline{\mathbf{n}}$ is the unit normal vector of $\partial' U$ and $d\mathcal{H}^n$ is the volume element on $\mathbb{R}^n$.

1.11 Definition. We say that a locally Lipschitz function $f\colon D\to\mathbb{R}$ is a generalized solution of (1.3) with the boundary condition

$$f\,\langle\mathcal{A}(x,\nabla f),\overline{\mathbf{n}}\rangle=0\,,\quad x\in\partial D\setminus G\,,\tag{1.12}$$

if for every subdomain $U\in(G,D)$ with (1.9), and for every locally Lipschitz function $\varphi\colon\overline{U}\setminus G\to\mathbb{R}$ the following property holds:

$$\int\limits_{\partial' U}\varphi\,f\,\langle\mathcal{A}(x,\nabla f)\,,\overline{\mathbf{n}}\rangle\,d\mathcal{H}^{n-1}=\int\limits_{U}\langle\mathcal{A}(x,\nabla f)\,,\nabla(\varphi\,f)\,\rangle\,d\mathcal{H}^{n}\,.\tag{1.13}$$

In the case of a smooth boundary ∂D, and $f\in C^2(D)$, the relation (1.10) implies (1.3) with (1.8) everywhere on $\partial D\setminus G$. This requirement (1.13) implies (1.3) with (1.12) on $\partial D\setminus G$. See [Mik06, Section 7.2.1].

The surface integrals exist by (1.9). Indeed, this assumption guarantees that $\nabla f(x)$ exists $\mathcal{H}^{n-1}$-a.e. on $\partial' U$. The assumption that $U\in(G,D)$ implies existence of a normal vector $\overline{\mathbf{n}}$ for $\mathcal{H}^{n-1}$-a.e. points on $\partial' U$ [Fed69, Chapter 2 Section 3.2]. Thus, the scalar product $\langle\mathcal{A}(x,\nabla f)\,,\overline{\mathbf{n}}\rangle$ is defined and finite a.e. on $\partial' U$.

2 Saint-Venant principle

In this section, we will prove the Saint-Venant principle for solutions of the $\mathcal{A}$-Laplace equation on canonical domains.

Let $0<k<n$. Fix a domain $D=D^k_{\alpha,\beta}\cap B_k(t_0)$ with $t_0>0$ and $0<\alpha<\beta<\infty$. We write $\mathcal{P}=\{x\in\partial D: p_k(x)=\alpha\}$ and $\mathcal{Q}=\{x\in\partial D: p_k(x)=\beta\}$ and $G=\mathcal{P}\cup\mathcal{Q}$. Let $t,\tau\in(\alpha,\beta)$, $t<\tau$, and

$$\Delta^k(t,\tau)=\{x\in D: t<p_k(x)<\tau\}\,.$$

For $s\geq 0$ we set

$$\sigma^k(s)=\big\{x\in\Delta^k(0,\infty): p_k(x)=s\big\}\,.$$

2.1 Theorem. *Let $\alpha<\tau'<\tau''<\beta$. If $f\colon D\to\mathbb{R}$ is a generalized solution of* (1.3) *with the generalized boundary condition* (1.8) *on $\partial D\setminus G$, then the inequality*

$$I_1(t,\tau')+C_1(t)/\nu_1\leq\big(I_1(t,\tau'')+C_1(t)/\nu_1\big)\,\exp\Bigg[-\frac{\nu_1}{\nu_2}\int\limits_{\tau'}^{\tau''}\mu_p(\sigma^k(\tau))\,d\tau\Bigg]\,.\tag{2.2}$$

holds for all $t\in(\alpha,\tau']$.

If $f\colon D\to\mathbb{R}$ is a generalized solution of (1.3) *with the generalized boundary condition* (1.12) *then*

$$I_1(t,\tau')+C_2(t)/\nu_1\leq\big(I_1(t,\tau'')+C_2(t)/\nu_1\big)\,\exp\Bigg[-\frac{\nu_1}{\nu_2}\int\limits_{\tau'}^{\tau''}\lambda^{1/p}_{p,Z_f(\tau)}(\sigma^k(\tau))\,d\tau\Bigg]\tag{2.3}$$

holds for all $t \in (\alpha, \tau']$*. Here*

$$I_1(t,\tau) = \int\limits_{\Delta^k(t,\tau)} |\nabla f|^p \, d\mathcal{H}^n \,,$$

and

$$Z_f(\tau) = \{x \in \Sigma_k^*(\tau) \cap \Sigma_k(t_0) : \lim_{y \to x} f(y) = 0\}. \tag{2.4}$$

The inequalities of the form (2.3), (2.2) are called the Saint-Venant principle (see, for example, [OI77]).

Proof. The case A. At first we consider the case in which f is a generalized solution of (1.3) with the generalized boundary condition (1.12) on $\partial D \setminus G$. It is easy to see that a.e. on $D^k_{\alpha,\beta}$,

$$|\nabla p_k(x)| = 1 \,.$$

The domain $\Delta^k(t,\tau)$ belongs to (G,D). Let $\varphi \colon \overline{U} \setminus G \to \mathbb{R}$ be a locally Lipschitz function. By (1.13) we have

$$\int\limits_{\partial'\Delta^k(t,\tau)} \varphi \, f \, \langle \mathcal{A}(x, \nabla f) \,, \overline{\mathbf{n}} \rangle \, d\mathcal{H}^{n-1} = \int\limits_{\Delta^k(t,\tau)} \langle \mathcal{A}(x, \nabla f) \,, \nabla(\varphi \, f) \, \rangle \, d\mathcal{H}^n \,.$$

But

$$\partial' \Delta^k(t,\tau) = \sigma^k(t) \cup \sigma^k(\tau) \,.$$

For $\varphi \equiv 1$, we have by (1.2) and (1.13)

$$\begin{aligned} \nu_1 \, I_1(t,\tau) \;&\leq\; \int\limits_{\Delta^k(t,\tau)} \langle \mathcal{A}(x, \nabla f), \nabla f \rangle \, d\mathcal{H}^n \\ &=\; \int\limits_{\sigma^k(\tau)} f \, \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle \, d\mathcal{H}^{n-1} \\ &\quad - \int\limits_{\sigma^k(t)} f \, \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle \, d\mathcal{H}^{n-1} \,, \end{aligned}$$

since $\overline{\mathbf{n}} = \nabla p_k(x)$ for $x \in \sigma^k(\tau)$ and $\overline{\mathbf{n}} = -\nabla p_k(x)$ for $x \in \sigma^k(t)$. We obtain

$$\nu_1 \, I_1(t,\tau) + C_2(t) \leq \int\limits_{\sigma^k(\tau)} f \, \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle \, d\mathcal{H}^{n-1} \tag{2.5}$$

where

$$C_2(t) = \int\limits_{\sigma^k(t)} f \, \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle \; d\mathcal{H}^{n-1} \,.$$

Note that writing

$$\tilde{C}_2(\tau) = - \int\limits_{\sigma^k(\tau)} f\, \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle \; d\mathcal{H}^{n-1}\,,$$

we have

$$\nu_1\, I_1(t,\tau) + \tilde{C}_2(\tau) \le - \int\limits_{\sigma^k(t)} f\, \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle \, d\mathcal{H}^{n-1}. \tag{2.6}$$

Next we will estimate the right side of (2.5). By (1.2) and the Hölder inequality

$$\begin{aligned} &\Bigg| \int\limits_{\sigma^k(\tau)} f\, \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle \, d\mathcal{H}^{n-1} \Bigg| \\ &\qquad \le \int\limits_{\sigma^k(\tau)} |f|\, |\mathcal{A}(x, \nabla f)| \, d\mathcal{H}^{n-1} \le \nu_2 \int\limits_{\sigma^k(\tau)} |f|\, |\nabla f|^{p-1} \, d\mathcal{H}^{n-1} \\ &\qquad\qquad \le \nu_2 \Bigg(\int\limits_{\sigma^k(\tau)} |f|^p \, d\mathcal{H}^{n-1} \Bigg)^{1/p} \Bigg(\int\limits_{\sigma^k(\tau)} |\nabla f|^p \, d\mathcal{H}^{n-1} \Bigg)^{(p-1)/p}. \end{aligned}$$

We note that the surfaces $\sigma^k(\tau)$ are parallel to $\partial D^k_{\alpha,\beta}$. By using (1.4) we may write

$$\int\limits_{\sigma^k(\tau)} |f|^p \, d\mathcal{H}^{n-1} \le \lambda^{-1}_{p,Z_f(\tau)}(\sigma^k(\tau)) \int\limits_{\sigma^k(\tau)} |\nabla f|^p \, d\mathcal{H}^{n-1} \tag{2.7}$$

and

$$\Bigg| \int\limits_{\sigma^k(\tau)} f\, \langle \mathcal{A}(x, \nabla f), \nabla p_k(x) \rangle \, d\mathcal{H}^{n-1} \Bigg| \le \nu_2\, \lambda^{-1/p}_{p,Z_f(\tau)}(\sigma^k(\tau)) \int\limits_{\sigma^k(\tau)} |\nabla f|^p \, d\mathcal{H}^{n-1}\,.$$

By (2.5) and the Fubini theorem

$$\nu_1\, I_1(t,\tau) + C_2(t) \le \nu_2\, \lambda^{-1/p}_{p,Z_f(\tau)}(\sigma^k(\tau))\, \frac{dI_1}{d\tau}(t,\tau)$$

and

$$\frac{\nu_1}{\nu_2}\, \lambda^{1/p}_{p,Z_f(\tau)}(\sigma^k(\tau)) \, \le \frac{dI_1}{d\tau}(t,\tau) \Big/ \left(I_1(t,\tau) + \frac{C_2(t)}{\nu_1} \right).$$

By integrating this differential inequality we have

$$\exp\left[\frac{\nu_1}{\nu_2} \int\limits_{\tau'}^{\tau''} \lambda^{1/p}_{p,Z_f(\tau)}(\sigma^k(\tau))\, d\tau \right] \le \frac{I_1(t,\tau'') + C_2(t)/\nu_1}{I_1(t,\tau') + C_2(t)/\nu_1}$$

for arbitrary $\tau', \tau'' \in (\alpha, \beta)$ with $\tau' < \tau''$. We have shown that

$$I_1(t,\tau') + C_2(t)/\nu_1 \le \big(I_1(t,\tau'') + C_2(t)/\nu_1\big) \,\exp\left[-\frac{\nu_1}{\nu_2}\int\limits_{\tau'}^{\tau''} \lambda^{1/p}_{p,Z_f(\tau)}(\sigma^k_p(\tau))\,d\tau\right]. \tag{2.8}$$

The case B. Now we assume that f is a generalized solution of (1.3) with the boundary condition (1.8) on $\partial D \setminus G$. Fix $t < \tau$. By choosing $\varphi \equiv 1$ in (1.10) we see that

$$\int\limits_{\sigma^k(t)\cup\sigma^k(\tau)} \langle \mathcal{A}(x,\nabla f)\,,\overline{\mathbf{n}}\rangle\, d\mathcal{H}^{n-1} = 0\,.$$

For an arbitrary constant C, we get from this and (1.10)

$$\int\limits_{\sigma^k(t)\cup\sigma^k(\tau)} (f-C)\,\langle \mathcal{A}(x,\nabla f)\,,\overline{\mathbf{n}}\rangle\, d\mathcal{H}^{n-1} = \int\limits_{\Delta^k(t,\tau)} \langle \mathcal{A}(x,\nabla f)\,,\nabla f\,\rangle\, d\mathcal{H}^{n}\,. \tag{2.9}$$

Thus

$$\int\limits_{\Delta^k(t,\tau)} \langle \mathcal{A}(x,\nabla f)\,,\nabla f\,\rangle\, d\mathcal{H}^{n} \le C_1(t) + \int\limits_{\sigma^k(\tau)} |f-C|\,|\mathcal{A}(x,\nabla p_k(x))|\, d\mathcal{H}^{n-1},$$

where

$$C_1(t) = \int_{\sigma^k(t)} |f-C|\,|\mathcal{A}(x,\nabla p_k(x))|\, d\mathcal{H}^{n-1},$$

or

$$\nu_1\, I_1(t,\tau) + C_1(t) \le \nu_2 \int\limits_{\sigma^k(\tau)} |f-C|\,|\nabla f|^{p-1}\, d\mathcal{H}^{n-1}\,. \tag{2.10}$$

As above, we obtain

$$\int\limits_{\sigma^k(\tau)} |f-C|\,|\nabla f|^{p-1}\, d\mathcal{H}^{n-1}$$

$$\le \Bigg(\int\limits_{\sigma^k(\tau)} |f-C|^p\, d\mathcal{H}^{n-1}\Bigg)^{1/p} \Bigg(\int\limits_{\sigma^k(\tau)} |\nabla f|^p\, d\mathcal{H}^{n-1}\Bigg)^{(p-1)/p}\,. \tag{2.11}$$

By using (1.6) we get

$$\Bigg(\int\limits_{\sigma^k(\tau)} |f-C_3|^p\, d\mathcal{H}^{n-1}\Bigg)^{1/p} \le \mu_p^{-1/p}(\sigma^k(\tau)) \Bigg(\int\limits_{\sigma^k(\tau)} |\nabla f|^p\, d\mathcal{H}^{n-1}\Bigg)^{1/p}, \tag{2.12}$$

where $C_3 = C_3(f)$ is the constant from (1.6). Then by (2.11) and (2.12),

$$\int\limits_{\sigma^k(\tau)} |f-C_3|\,|\nabla f|^{p-1}\, d\mathcal{H}^{n-1} \le \mu_p^{-1}(\sigma^k(\tau)) \int\limits_{\sigma^k(\tau)} |\nabla f|^p\, d\mathcal{H}^{n-1}\,,$$

and by (2.10) we have

$$\nu_1 \, I_1(t,\tau) + C_1(t) \le \nu_2 \, \mu_p^{-1}(\sigma^k(\tau)) \int\limits_{\sigma^k(\tau)} |\nabla f|^p \, d\mathcal{H}^{n-1}$$

or

$$\nu_1 \, I_1(t,\tau) + C_1(t) \le \nu_2 \, \mu_p^{-1}(\sigma^k(\tau)) \frac{dI_1}{dt}(t,\tau) \, .$$

By integrating this inequality we have shown that

$$I_1(t,\tau') + C_1(t)/\nu_1 \le (I_1(t,\tau'') + C_1(t)/\nu_1) \, \exp\left[-\frac{\nu_1}{\nu_2} \int\limits_{\tau'}^{\tau''} \mu_p(\sigma^k(\tau)) \, d\tau \right] . \tag{2.13}$$

□

3 Stagnation zones

Next we apply the Saint-Venant principle to obtain information about stagnation zones of generalized solutions of the equation (1.3). We first consider zones with respect to the Sobolev norm. Other results of this type follow immediately from well-known imbedding theorems.

Stagnation zones with respect to the W_p^1-norm

We rewrite (2.2) and (2.3) in another form. Fix $D^k_{\alpha,\beta}$. Let

$$p_k^*(x) = p_k(x) - \frac{\alpha+\beta}{2} \, .$$

For $x \in D^k_{\alpha,\beta}$ and

$$\beta^* = \frac{\beta - \alpha}{2} \tag{3.1}$$

we have

$$-\beta^* < p_k^*(x) < \beta^* \, ,$$

and we denote

$$D^{*,k}_{\beta^*} = \{x \in \mathbf{R}^n : -\beta^* < p_k^*(x) < \beta^*\} \, .$$

Let $t_0 > 0$ and $-\beta^* < \tau' \le \tau'' < \beta^*$. We write

$$\Delta^{*,k}(\tau',\tau'') = \{x \in B_k(t_0) : \tau' < p_k^*(x) < \tau''\} \tag{3.2}$$

and

$$I_2(\tau',\tau'') = \int\limits_{\Delta^{*,k}(\tau',\tau'')} |\nabla f|^p \, d\mathcal{H}^n \, .$$

Let $0 < \tau' < \tau'' < \beta^*$. By (2.3) we have for $t \in (-\tau, \tau)$

$$I_2(t,\tau')+C_4(t)/\nu_1 \le \big(I_2(t,\tau'')+C_4(t)/\nu_1\big)\ \exp\left[-\frac{\nu_1}{\nu_2}\int\limits_{\tau'}^{\tau''} \lambda^{1/p}_{p,Z^*_f(\tau)}(\sigma^{*,k}(\tau))\,d\tau\right],$$

where

$$Z^*_f(\tau) = \{x \in \Sigma_k(t_0) : p^*_k(x) = \tau \wedge \lim_{y\to x} f(y) = 0\}. \tag{3.3}$$

By choosing the estimate as in (2.6) we also have

$$\begin{aligned} &I_2(-\tau',t) + \tilde{C}_4(t)/\nu_1 \\ &\qquad \le \big(I_2(-\tau'',t) + \tilde{C}_4(t)/\nu_1\big)\ \exp\left[-\frac{\nu_1}{\nu_2}\int\limits_{-\tau''}^{-\tau'} \lambda^{1/p}_{p,Z^*_f(\tau)}(\sigma^{*,k}(\tau))\,d\tau\right], \end{aligned}$$

where

$$\sigma^{*,k}(s) = \{x \in \Delta^{*,k}(-\infty,\infty) : p^*_k(x) = s\}\,. \tag{3.4}$$

By adding these inequalities and noting that $C_4(t) + \tilde{C}_4(t) = 0$ we obtain

$$\begin{aligned} &I_2(-\tau',t) + I_2(t,\tau') \le \big(I_2(-\tau'',t) + I_2(t,\tau'')\big) \\ &\times\max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int\limits_{-\tau''}^{-\tau'} \lambda^{1/p}_{p,Z^*_f(\tau)}(\sigma^{*,k}(\tau))\,d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int\limits_{\tau'}^{\tau''} \lambda^{1/p}_{p,Z^*_f(\tau)}(\sigma^{*,k}(\tau))\,d\tau\right]\right\}. \end{aligned}$$

Thus we have the estimate

$$\begin{aligned} &I_2(-\tau',\tau') \le I_2(-\tau'',\tau'') \\ &\times\max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int\limits_{-\tau''}^{-\tau'} \lambda^{1/p}_{p,Z^*_f(\tau)}(\sigma^{*,k}(\tau))\,d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int\limits_{\tau'}^{\tau''} \lambda^{1/p}_{p,Z^*_f(\tau)}(\sigma^{*,k}(\tau))\,d\tau\right]\right\}. \end{aligned} \tag{3.5}$$

Similarly, from (2.2) we obtain

$$\begin{aligned} &I_2(-\tau',\tau') \le I_2(-\tau'',\tau'') \\ &\times \max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int\limits_{-\tau''}^{-\tau'} \mu_p(\sigma^{*,k}(\tau))\,d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int\limits_{\tau'}^{\tau''} \mu_p(\sigma^{*,k}(\tau))\,d\tau\right]\right\}. \end{aligned} \tag{3.6}$$

From this we obtain the following theorem on stagnation W^1_p-zones:

3.7 Theorem. *Let $t_0 > 0$, $\beta > \alpha > 0$, and let $-\beta^* < \tau' \le \tau'' < \beta^*$ where β^* is as in* (3.1). *If f is a solution of* (1.3) *on $D = D^{*,k}_{\beta^*} \cap B_k(t_0)$ with* *the generalized boundary condition* (1.8) *on* $\partial D \setminus G$, *where* $G = \{x \in \partial D : p_k^*(x) = \pm\beta^*\}$ *and*

$$\max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'}\mu_p(\sigma^{*,k}(\tau))\,d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''}\mu_p(\sigma^{*,k}(\tau))\,d\tau\right]\right\} < s^{1/p}\,,$$

or a solution of (1.3) *on* D *with the generalized boundary condition* (1.12) *on* $\partial D \setminus G$ *and*

$$\max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'}\lambda^{1/p}_{p,Z_f^*(\tau)}(\sigma^{*,k}(\tau))\,d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''}\lambda^{1/p}_{p,Z_f^*(\tau)}(\sigma^{*,k}(\tau))\,d\tau\right]\right\} < s^{1/p}\,,$$

then the subdomain $\Delta^{*,k}(-\tau',\tau')$ *is a s-zone with respect to the* W_p^1*-norm i.e.,*

$$\int_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p\,d\mathcal{H}^n < s\,,$$

where $\Delta^{*,k}$ *is as in* (3.2).

Stagnation zones with respect to the L^p-norm

Let $t_0 > 0$, $\beta > \alpha > 0$, and let $-\beta^* < \tau' \le \tau'' < \beta^*$ where β^* is as in (3.1).

Denote by $C_5(k,p,\alpha,\beta,t_0)$ the best constant of the imbedding theorem from $W_p^1(D^{*,k}_{\beta^*})$ to $L^p(D^{*,k}_{\beta^*})$, i.e. in the inequality

$$\|g - C\|_{L^p(D^{*,k}_{\beta^*})} \le C_5(k,p,\alpha,\beta,t_0)\,\|g\|_{W_p^1(D^{*,k}_{\beta^*})}\,.$$

The domains $D^{*,k}_{\beta^*} \cap B_k(t_0)$ are convex and such a constant C_5 exists (see Maz'ya [Maz85] or [AF03, p.85]).

In this case we obtain from (3.5), (3.6)

$$\begin{aligned}&\|f - C\|^p_{L^p(\Delta^{*,k}(-\tau',\tau'))} \le C_5^p(k,p,\beta^*-\tau'',\beta^*+\tau'',t_0)\,I_2(-\tau'',\tau'')\\ &\times\max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'}\lambda^{1/p}_{p,Z_f^*(\tau)}(\sigma^{*,k}(\tau))\,d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''}\lambda^{1/p}_{p,Z_f^*(\tau)}(\sigma^{*,k}(\tau))\,d\tau\right]\right\},\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\|f - C\|^p_{L^p(\Delta^{*,k}_{\tau'})} \le C_5^p(k,p,\beta^*-\tau'',\beta^*+\tau'',t_0)I_2(-\tau'',\tau'')\\ &\times\max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int_{-\tau''}^{-\tau'}\mu_p(\sigma^{*,k}(\tau))\,d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int_{\tau'}^{\tau''}\mu_p(\sigma^{*,k}(\tau))\,d\tau\right]\right\}.\end{aligned} \tag{3.9}$$

These relations can be used to obtain information about stagnation zones with respect to the L^p-norm. Namely, we have:

3.10 Theorem. *If f is a solution of* (1.3) *on $D = D^{*,k}_{\beta^*} \cap B_k(t_0)$ with the generalized boundary condition* (1.8) *(or* (1.12)*) on $\partial D \setminus G$, where $G = \{x \in \partial D : p^*_k(x) = \pm\beta^*\}$, and the right side of* (3.8) *(or* (3.9)*) is smaller than $s > 0$, then the domain $\Delta^{*,k}(-\tau', \tau')$ is a stagnation zone with the deviation s^p in the sense of the L^p-norm on D.*

Stagnation zones for bounded, uniformly continuous functions

Let $t_0 > 0$, $\beta > \alpha > 0$, and let $-\beta^* < \tau' \le \tau'' < \beta^*$ where β^* is as in (3.1).

As above, denote by $C_6(k, p, \alpha, \beta, t_0)$ the best constant of the imbedding theorem from $W^1_p(D^{*,k}_{\beta^*})$ to $C(D^{*,k}_{\beta^*})$, i.e. in the inequality

$$\|g - C\|_{C(D^{*,k}_{\beta^*})} \le C_6(k, p, \alpha, \beta, t_0)\, \|g\|_{W^1_p(D^{*,k}_{\beta^*})} \,. \tag{3.11}$$

A domain $D^{*,k}_{\beta^*}$ is convex and hence (3.11) holds for $p > n$ (see Maz'ya [Maz85] or [AF03, p.85]).

In this case from (3.5), (3.6) we obtain

$$\begin{aligned} &\|f - C\|_{C(\Delta^{*,k}(-\tau',\tau'))} \le C_6^p(k, p, \beta^* - \tau', \beta^* + \tau', t_0)\, I_2(-\tau'', \tau'') \\ &\times\max\left\{ \exp\left[-\frac{\nu_1}{\nu_2} \int\limits_{-\tau''}^{-\tau'} \lambda^{1/p}_{p,Z^*_f(\tau)}(\sigma^{*,k}(\tau))\, d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2} \int\limits_{\tau'}^{\tau''} \lambda^{1/p}_{p,Z^*_f(\tau)}(\sigma^{*,k}(\tau))\, d\tau\right]\right\}, \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &\|f - C\|_{C(\Delta^{*,k}(-\tau',\tau'))} \le C_6^p(k, p, \beta^* - \tau', \beta^* + \tau', t_0) I_2(-\tau'', \tau'') \\ &\times \max\left\{ \exp\left[- \frac{\nu_1}{\nu_2} \int\limits_{-\tau''}^{-\tau'} \mu_p(\sigma^{*,k}(\tau))\, d\tau\right], \exp\left[- \frac{\nu_1}{\nu_2} \int\limits_{\tau'}^{\tau''} \mu_p(\sigma^{*,k}(\tau))\, d\tau\right]\right\}. \end{aligned} \tag{3.13}$$

These relations can be used to obtain theorems about stagnation zones for bounded uniformly continuous functions.

3.14 Theorem. *If f is a solution of* (1.3)*, $p > n$, on $D = D^{*,k}_{\beta^*} \cap B_k(t_0)$ with the generalized boundary condition* (1.8) *(or* (1.12)*) on $\partial D \setminus G$ where $G = \{x \in \partial D : p^*_k(x) = \pm\beta^*\}$, and the right side of* (3.12) *(or* (3.13)*) is smaller than $s > 0$, then the domain $\Delta^{*,k}(-\tau', \tau')$ is a stagnation zone with the deviation s in the sense of the norm $\|\cdot\|_{C^0(\Delta^{*,k}(-\tau',\tau'))}$.*

4 Other applications

Next we prove Phragmén-Lindelöf type theorems for the solutions of the $\mathcal{A}$-Laplace equation with boundary conditions (1.8) and (1.12).

Estimates for W_p^1-norms

Let $t_0 > 0$, $\beta > \alpha > 0$, and let be β^* is as in (3.1). First we will prove some estimates of the W_p^1-norm of a solution. Let f be a solution of (1.3) on $D_{\beta^*}^{*,k}$ with the generalized boundary condition (1.8) on $\partial D \setminus G$. Fix $0 < \tau' < \tau'' < \beta^*$ and estimate $\|f\|_{W_p^1(\Delta^{*,k}(-\tau',\tau'))}$.

Let $\psi : [\tau', \tau''] \to (0, \infty)$ be a Lipschitz function such that

$$\psi(\tau') = 1\,, \quad \psi(\tau'') = 0\,. \tag{4.1}$$

We choose

$$\phi(t) = \begin{cases} 1 & \text{for } |t| < \tau', \\ \psi(|t|) & \text{for } \tau' \leq |t| \leq \tau''. \end{cases} \tag{4.2}$$

The function $\varphi(x) = \phi(p_k^*(x))$ is admissible in Definition 1.7 for

$$U = \Delta^{*,k}(-\tau'', \tau'')\,.$$

As in (2.9) we may by (1.10) write

$$\int\limits_{\sigma^{*,k}(-\tau'') \cup \sigma^{*,k}(\tau'')} \phi^p(p_k^*(x))(f - C)\,\langle \mathcal{A}(x, \nabla f)\,, \overline{\mathbf{n}}\rangle \, d\mathcal{H}^{n-1}$$
$$= \int\limits_{\Delta^{*,k}(-\tau'',\tau'')} \langle \mathcal{A}(x, \nabla f)\,, \nabla\big(\phi^p(p_k^*(x))(f - C)\big)\,\rangle \, d\mathcal{H}^n\,.$$

By construction of ϕ, (4.1) and (4.2), the surface integral is equal to zero, and we have

$$\int\limits_{\Delta^{*,k}(-\tau'',\tau'')} \phi^p(p_k^*(x))\langle \mathcal{A}(x, \nabla f)\,, \nabla f\rangle \, d\mathcal{H}^n$$
$$= -p \int\limits_{\Delta^{*,k}(-\tau'',\tau'')} \phi^{p-1}(p_k^*(x))\,(f - C)\,\langle \mathcal{A}(x, \nabla f)\,, \nabla\phi(p_k^*(x))\,\rangle \, d\mathcal{H}^n\,.$$

Thus by (1.2)

$$\nu_1 \int\limits_{\Delta^{*,k}(-\tau'',\tau'')} \phi^p(p_k^*(x))|\nabla f|^p \, d\mathcal{H}^n$$
$$\leq p\nu_2 \int\limits_{\Delta^{*,k}(-\tau'',\tau'')} \phi^{p-1}(p_k^*(x))\,|f - C|\,|\nabla f|^{p-1}\,|\nabla\phi(p_k^*(x))|\, d\mathcal{H}^n\,. \tag{4.3}$$

Now we note that

$$|\nabla\phi(p_k^*(x))| = |\phi'(p_k^*(x))|$$

and by the Hölder inequality

$$\int\limits_{\Delta^{*,k}(-\tau'',\tau'')} \phi^{p-1}(p_k^*(x))\,|f-C|\,|\nabla f|^{p-1}\,|\nabla\phi(p_k^*(x))|\,d\mathcal{H}^n$$
$$= \int\limits_{\Delta^{*,k}(-\tau'',\tau'')} \phi^{p-1}(p_k^*(x))\,|\nabla f|^{p-1}\,|\phi'(p_k^*(x))|\,|f-C|\,d\mathcal{H}^n$$
$$\leq \Bigg(\int\limits_{\Delta^{*,k}(-\tau'',\tau'')} \phi^{p}(p_k^*(x))\,|\nabla f|^{p}\,d\mathcal{H}^n\Bigg)^{(p-1)/p}$$
$$\times \Bigg(\int\limits_{\Delta^{*,k}(-\tau'',\tau'')} |\phi'(p_k^*(x))|^p\,|f-C|^p\,d\mathcal{H}^n\Bigg)^{1/p}$$

From this inequality and (4.3) we obtain

$$\nu_1^p \int\limits_{\Delta^{*,k}(-\tau',\tau')} \phi^p(p_k^*(x))|\nabla f|^p\,d\mathcal{H}^n \leq p^p\nu_2^p \int\limits_{\Delta^{*,k}(-\tau'',\tau'')} |\phi'(p_k^*(x))|^p\,|f-C|^p\,d\mathcal{H}^n\,.$$

Because $\phi(p_k^*(x)) \equiv 1$ on $\Delta^{*,k}(-\tau',\tau')$ we have the following inequality:

$$\nu_1^p \int\limits_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p\,d\mathcal{H}^n \leq p^p\nu_2^p \int\limits_{\Delta^{*,k}(-\tau'',\tau'')\setminus\Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p\,|f-C|^p\,d\mathcal{H}^n\,. \tag{4.4}$$

Next we will find

$$\min_{\psi} \int\limits_{\Delta^{*,k}(-\tau'',\tau'')\setminus\Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p\,|f-C|^p\,d\mathcal{H}^n\,,$$

where the minimum is taken over all ψ in (4.2). We have

$$\int\limits_{\Delta^{*,k}(-\tau'',\tau'')\setminus\Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p\,|f-C|^p\,d\mathcal{H}^n$$
$$= \int\limits_{-\tau''}^{-\tau'} |\psi'(\tau)|^p\,d\tau \int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p\,d\mathcal{H}^{n-1}$$
$$+ \int\limits_{\tau'}^{\tau''} |\psi'(\tau)|^p\,d\tau \int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p\,d\mathcal{H}^{n-1}$$

and

$$\min_{\psi} \int\limits_{\Delta^{*,k}(-\tau'',\tau'')\setminus\Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p \, |f(x)-C|^p \, d\mathcal{H}^n$$

$$\leq \min_{\psi} \int\limits_{-\tau''}^{-\tau'} |\psi'(\tau)|^p \, d\tau \int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1}$$

$$+ \min_{\psi} \int\limits_{\tau'}^{\tau''} |\psi'(\tau)|^p \, d\tau \int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1} \equiv A_1 + A_2 \, . \quad (4.5)$$

Because by the Hölder inequality

$$1 \leq \Bigg(\int\limits_{\tau'}^{\tau''} |\psi'(\tau)| \, d\tau \Bigg)^p \leq \Bigg[\int\limits_{\tau'}^{\tau''} |\psi'(\tau)|^p d\tau \int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1} \Bigg]$$

$$\times \Bigg[\int\limits_{\tau'}^{\tau''} d\tau \Bigg(\int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1} \Bigg)^{1/(1-p)} \Bigg]^{p-1},$$

we have

$$\Bigg[\int\limits_{\tau'}^{\tau''} d\tau \Bigg(\int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1} \Bigg)^{1/(1-p)} \Bigg]^{1-p}$$

$$\leq \int\limits_{\tau'}^{\tau''} |\psi'(\tau)|^p d\tau \int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1},$$

and hence

$$A_2 \geq \Bigg[\int\limits_{\tau'}^{\tau''} d\tau \Bigg(\int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1} \Bigg)^{1/(1-p)} \Bigg]^{1-p}.$$

It is easy to see that here the equality holds for a special choice of ψ. Thus

$$A_2 = \Bigg[\int\limits_{\tau'}^{\tau''} d\tau \Bigg(\int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1} \Bigg)^{1/(1-p)} \Bigg]^{1-p}.$$

Similarly,

$$A_1 = \Bigg[\int\limits_{-\tau''}^{-\tau'} d\tau \Bigg(\int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1} \Bigg)^{1/(1-p)} \Bigg]^{1-p}.$$

From (4.5) we obtain

$$\min_{\psi} \int\limits_{\Delta^{*,k}(-\tau'',\tau'')\setminus\Delta^{*,k}(-\tau',\tau')} |\psi'(p_k^*(x))|^p \, |f-C|^p \, d\mathcal{H}^n$$
$$\le \left[\int\limits_{-\tau''}^{-\tau'} d\tau \Big(\int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1} \Big)^{1/(1-p)} \right]^{1-p}$$
$$+ \left[\int\limits_{\tau'}^{\tau''} d\tau \Big(\int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1} \Big)^{1/(1-p)} \right]^{1-p}.$$

By using (4.4) we obtain the inequality:

$$p^{-p}\Big(\frac{\nu_1}{\nu_2}\Big)^p \int\limits_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p \, d\mathcal{H}^n$$
$$\le \left[\int\limits_{-\tau''}^{-\tau'} d\tau \Big(\int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1} \Big)^{1/(1-p)} \right]^{1-p}$$
$$+ \left[\int\limits_{\tau'}^{\tau''} d\tau \Big(\int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1} \Big)^{1/(1-p)} \right]^{1-p},$$

where C is an arbitrary constant. From this we obtain

$$\int\limits_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p \, d\mathcal{H}^n \le C_7 \max\left\{ \left[\int\limits_{-\tau''}^{-\tau'} d\tau \Big(\int\limits_{\sigma^{*,k}(\tau)} |f(x){-}C|^p \, d\mathcal{H}^{n-1} \Big)^{1/(1-p)} \right]^{1-p}, \right.$$
$$\left. \left[\int\limits_{\tau'}^{\tau''} d\tau \Big(\int\limits_{\sigma^{*,k}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1} \Big)^{1/(1-p)} \right]^{1-p} \right\} \tag{4.6}$$

where $C_7 = 2p^p(\nu_2/\nu_1)^p$.

Similarly, for solutions of the $\mathcal{A}$-Laplace equation with the boundary condition (1.12) we may prove that

$$p^{-p}\Big(\frac{\nu_1}{\nu_2}\Big)^p \int\limits_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p \, d\mathcal{H}^n \le \left[\int\limits_{-\tau''}^{-\tau'} d\tau \Big(\int\limits_{\sigma^{*,k}(\tau)} |f(x)|^p \, d\mathcal{H}^{n-1} \Big)^{1/(1-p)} \right]^{1-p}$$
$$+ \left[\int\limits_{\tau'}^{\tau''} d\tau \Big(\int\limits_{\sigma^{*,k}(\tau)} |f(x)|^p \, d\mathcal{H}^{n-1} \Big)^{1/(1-p)} \right]^{1-p}.$$

It follows that

$$\int\limits_{\Delta^{*,k}(-\tau',\tau')} |\nabla f|^p \, d\mathcal{H}^n \le C_7 \max\left\{\left[\int\limits_{-\tau''}^{-\tau'} d\tau \left(\int\limits_{\sigma^{*,k}(\tau)} |f(x)|^p \, d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}, \left[\int\limits_{\tau'}^{\tau''} d\tau \left(\int\limits_{\sigma^{*,k}(\tau)} |f(x)|^p \, d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}\right\}. \quad (4.7)$$

Phragmén-Lindelöf type theorems I

We prove Phragmén-Lindelöf type theorems for cylindrical domains. Let $k = n-1$ and fix $t_0 > 0$. Consider the domain

$$D = B_k(t_0) = \left\{x = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n : d_{n-1}(x) < t_0\right\}.$$

Let $f_0 \colon D \to \mathbb{R}$ be a generalized solution of (1.3) with (1.1) and (1.2) satisfying the boundary condition (1.8) on $\Sigma_{n-1}(t_0)$.

Fix $\beta > \alpha > 0$, and let β^* be as in (3.1). Let $f(x) = f_0(x - \beta^* e_n)$, where e_n is the n:th unit coordinate vector, and let $0 < \tau' < \tau'' < \beta^* < \infty$. By (4.6)

$$\int\limits_{\Delta^{*,k}(-\tau'',\tau'')} |\nabla f|^p \, d\mathcal{H}^n \le C_7 \max\left\{\left[\int\limits_{-\tau''-1}^{-\tau''} d\tau \left(\int\limits_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p \, d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}, \left[\int\limits_{\tau''}^{\tau''+1} d\tau \left(\int\limits_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p \, d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}\right\}.$$

By using (3.6) we obtain from this the inequality:

$$I_2(-\tau',\tau') \le C_7 \max\left\{\left[\int\limits_{-\tau''-1}^{-\tau''} d\tau \left(\int\limits_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p \, d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}, \left[\int\limits_{\tau''}^{\tau''+1} d\tau \left(\int\limits_{\sigma^{*,n-1}(\tau)} |f(x) - C|^p \, d\mathcal{H}^{n-1}\right)^{1/(1-p)}\right]^{1-p}\right\}$$

$$\times \max\left\{\exp\left[-\frac{\nu_1}{\nu_2}\int\limits_{-\tau''}^{-\tau'} \mu_p(\sigma^{*,n-1}(\tau))\, d\tau\right], \exp\left[-\frac{\nu_1}{\nu_2}\int\limits_{\tau'}^{\tau''} \mu_p(\sigma^{*,n-1}(\tau))\, d\tau\right]\right\}.$$

We observe that in this case

$$\mu_p\big(\sigma^{*,n-1}(\tau)\big) \equiv \mu_p\big(\sigma^{n-1}(0)\big), \quad (4.8)$$

and hence

$$\int\limits_{\tau'}^{\tau''} \mu_p\big(\sigma^{*,n-1}(\tau)\big)\, d\tau = \mu_p\big(\sigma^{n-1}(0)\big)(\tau'' - \tau').$$

It follows that

$$
\begin{aligned}
I_2(-\tau',\tau') \le C_7 \max\Bigg\{ & \Bigg[\int\limits_{-\tau''-1}^{-\tau''} d\tau \Bigg(\int\limits_{\sigma^{*,n-1}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1}\Bigg)^{1/(1-p)}\Bigg]^{1-p}, \\
& \Bigg[\int\limits_{\tau''}^{\tau''+1} d\tau \Bigg(\int\limits_{\sigma^{*,n-1}(\tau)} |f(x)-C|^p \, d\mathcal{H}^{n-1}\Bigg)^{1/(1-p)}\Bigg]^{1-p}\Bigg\} \\
& \times \exp\Big[-\frac{\nu_1}{\nu_2}\,\mu_p(\sigma^{n-1}(0))(\tau''-\tau')\Big]. \qquad (4.9)
\end{aligned}
$$

By letting $\beta, \tau'' \to +\infty$ we obtain the following statement:

4.10 Theorem. *Let $t_0 > 0$, $f\colon B_{n-1}(t_0) \to \mathbb{R}$ be a generalized solution of* (1.3) *with* (1.1) *and* (1.2) *satisfying the boundary condition* (1.8) *on $\Sigma_{n-1}(t_0)$. If for a constant C the right side of* (4.9) *goes to* 0 *as $\tau'' \to \infty$, then $f \equiv \mathrm{const}$ on the cylinder $B_{n-1}(t_0)$.*

Similarly for a solution f of (1.3) with (1.1) and (1.2), satisfying the boundary condition (1.12) we may write

$$
\begin{aligned}
I_2(-\tau',\tau') \le C_7 \max\Bigg\{ & \Bigg[\int\limits_{-\tau''-1}^{-\tau''} d\tau \Bigg(\int\limits_{\sigma^{*,n-1}(\tau)} |f(x)|^p \, d\mathcal{H}^{n-1}\Bigg)^{1/(1-p)}\Bigg]^{1-p}, \\
& \Bigg[\int\limits_{\tau''}^{\tau''+1} d\tau \Bigg(\int\limits_{\sigma^{*,n-a}(\tau)} |f(x)|^p \, d\mathcal{H}^{n-1}\Bigg)^{1/(1-p)}\Bigg]^{1-p}\Bigg\} \\
\times\max\Bigg\{\exp\Bigg[-\frac{\nu_1}{\nu_2}\int\limits_{-\tau''}^{-\tau'} \lambda^{1/p}_{p,Z^*_f(\tau)}(\sigma^{*,n-1}(\tau))\,d\tau\Bigg], & \exp\Bigg[-\frac{\nu_1}{\nu_2}\int\limits_{\tau'}^{\tau''} \lambda^{1/p}_{p,Z^*_f(\tau)}(\sigma^{*,n-1}(\tau))\,d\tau\Bigg]\Bigg\}.
\end{aligned}
\tag{4.11}
$$

However here we do not have any identity similar to (4.8). We have:

4.12 Theorem. *Let $t_0 > 0$, and let $f\colon B_{n-1}(t_0) \to \mathbb{R}$ be a generalized solution of* (1.3) *with* (1.1) *and* (1.2) *satisfying the boundary condition* (1.12) *on $\Sigma_{n-1}(t_0)$. If the right side of* (4.11) *tends to* 0 *as $\tau'' \to \infty$, then $f \equiv 0$ on the cylinder $B_{n-1}(t_0)$*

If $f(x) = 0$ everywhere on the boundary $\Sigma_{n-1}(t_0)$ of the cylinder, then an identity similar to (4.8) holds in the following form:

$$
\lambda_p^{1/p}(\sigma^{*,n-1}(\tau)) \equiv \lambda_p^{1/p}(\sigma^{n-1}(0))(\lambda)\,. \tag{4.13}
$$

As above, we find

$$I_2(-\tau',\tau') \le C_7 \max\Bigg\{ \Bigg[\int\limits_{-\tau''-1}^{-\tau''} d\tau \Bigg(\int\limits_{\sigma^{*,n-1}(\tau)} |f(x)|^p \, d\mathcal{H}^{n-1} \Bigg)^{1/(1-p)} \Bigg]^{1-p},$$

$$\Bigg[\int\limits_{\tau''}^{\tau''+1} d\tau \Bigg(\int\limits_{\sigma^{*,n-1}(\tau)} |f(x)|^p \, d\mathcal{H}^{n-1} \Bigg)^{1/(1-p)} \Bigg]^{1-p} \Bigg\}$$

$$\times \exp\Bigg[-\frac{\nu_1}{\nu_2} \lambda_p^{1/p}(\sigma^{n-1}(0))(\tau''-\tau') \Bigg]. \quad (4.14)$$

Thus we obtain:

4.15 Corollary. *Let $t_0 > 0$, and let $f\colon B_{n-1}(t_0) \to \mathbb{R}$ be a generalized solution of* (1.3) *with* (1.1) *and* (1.2) *satisfying the boundary condition $f = 0$ on $\Sigma_{n-1}(t_0)$. If the right side of* (4.14) *tends to* 0 *as $\tau'' \to \infty$, then $f \equiv \mathrm{const}$ on the cylinder $B_{n-1}(t_0)$*

See also Miklyukov [Mik81].

Phragmén-Lindelöf type theorems II

We prove Phragmén-Lindelöf type theorems for canonical domains of an arbitrary form. Let $1 \le k < n-1$ and let $t_0 > 0$ be fixed. We consider a domain

$$D = B_k(t_0) = \big\{ x = (x_1, \dots, x_k, x_{k+1}, \dots, x_n) : d_k(x) < t_0 \big\}.$$

Let f be a generalized solution of (1.3) with (1.1) and (1.2) satisfying the boundary condition (1.8) on

$$\Sigma_k(t_0) = \big\{ x \in \mathbb{R}^n : d_k(x) < t_0 \big\}\,.$$

Fix $\tau_0 > 0$. Let $\tau_0 < \tau' < \tau'' < \infty$. By (4.6) we may write

$$\int\limits_{D^k_{0,\tau'}} |\nabla f|^p \, d\mathcal{H}^n \le C_8 \Bigg[\int\limits_{\tau'}^{\tau''} d\tau \Bigg(\int\limits_{\sigma^k(\tau)} |f(x) - C|^p \, d\mathcal{H}^{n-1} \Bigg)^{1/(1-p)} \Bigg]^{1-p},$$

where $C_8 = C_7/2$. As in (3.6) we obtain from (2.2) the estimate

$$\int\limits_{D^k_{0,\tau_0}} |\nabla f|^p \, d\mathcal{H}^n \le \int\limits_{D^k_{0,\tau'}} |\nabla f|^p \, d\mathcal{H}^n \, \exp\Bigg[-\frac{\nu_1}{\nu_2} \int\limits_{\tau_0}^{\tau'} \mu_p(\sigma^k(\tau)) \, d\tau \Bigg].$$

By combining these inequalities we obtain

$$\int\limits_{D^k_{0,\tau_0}} |\nabla f|^p \, d\mathcal{H}^n \le C_8 \Bigg[\int\limits_{\tau'}^{\tau''} d\tau \Bigg(\int\limits_{\sigma^k(\tau)} |f(x) - C|^p \, d\mathcal{H}^{n-1} \Bigg)^{1/(1-p)} \Bigg]^{1-p} \times \exp \Bigg[-\frac{\nu_1}{\nu_2} \int\limits_{\tau_0}^{\tau'} \mu_p(\sigma^k(\tau)) \, d\tau \Bigg]. \quad (4.16)$$

The inequality (4.16) holds for arbitrary constant C and every $\tau'' > \tau'$. Thus the following statement holds:

4.17 Theorem. *Let $t_0 > 0$, and let $f \colon B_k(t_0) \to \mathbb{R}$ be a generalized solution of* (1.3) *with* (1.1) *and* (1.2) *satisfying the boundary condition* (1.8) *on $\Sigma_k(t_0)$, $1 \le k < n-1$. If for a constant C the right side of* (4.16) *tends to* 0 *as τ', $\tau'' \to +\infty$, then $f \equiv \mathrm{const}$ on $B_k(t_0)$.*

In the case if f satisfies (1.3) with (1.1), (1.2) and the boundary condition (1.12) on $\Sigma_k(t_0)$, then we have

$$\int\limits_{D^k_{0,\tau_0}} |\nabla f|^p \, d\mathcal{H}^n \le C_8 \Bigg[\int\limits_{\tau'}^{\tau''} d\tau \Bigg(\int\limits_{\sigma^k(\tau)} |f(x)|^p \, d\mathcal{H}^{n-1} \Bigg)^{1/(1-p)} \Bigg]^{1-p} \times \exp \Bigg[-\frac{\nu_1}{\nu_2} \int\limits_{\tau_0}^{\tau'} \lambda^{1/p}_{p,Z_f(\tau)}(\sigma^k(\tau)) \, d\tau \Bigg]. \quad (4.18)$$

We obtain:

4.19 Theorem. *Let $t_0 > 0$, and let $f \colon B_k(t_0) \to \mathbb{R}$ be a generalized solution of* (1.3) *with* (1.1) *and* (1.2) *satisfying the boundary conditions* (1.8) *on $\Sigma_k(t_0)$, $1 \le k < n-1$. If for a constant C the right side of* (4.18) *tends to* 0 *as τ', $\tau'' \to +\infty$, then $f \equiv 0$ on $B_k(t_0)$.*

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