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SELF-IMPROVING PHENOMENA IN THE CALCULUS OF VARIATIONS ON METRIC SPACES

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Outi Elina Maasalo: *Self–improving phenomena in the calculus of variations on metric spaces*; Helsinki University of Technology, Institute of Mathematics, Research Reports A543 (2008).

Abstract: *This dissertation studies the integrability properties of functions related to the calculus of variations on metric measure spaces that support a weak Poincaré inequality and a doubling measure. The work consists of three articles in which we study the higher integrability of functions satisfying a reverse Hölder inequality, quasiminimizers of the Dirichlet integral and superharmonic functions.*

AMS subject classifications: 31C05, 31C45, 42B25, 49N60, 49Q20.

Keywords: BMO function, Caccioppoli inequality, capacity, doubling measure, Gehring lemma, geodesic space, global integrability, higher integrability, Hölder domain, metric space, Muckenhoupt weight, Newtonian space, p–fatness, Poincaré inequality, quasiminimizer, reverse Hölder inequality, superharmonic function, superminimizer, stability.

Outi Elina Maasalo: *Variaatiolaskennan itseparantuvuusominaisuuksia metrisissä avaruuksissa*; Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja A543 (2008).

Tiivistelmä: *Väitöskirjassa tutkitaan variatiolaskentaan liittyvien funktioiden integroituvuusominaisuuksia metrisissä mitta–avaruuksissa, joilla on voimassa heikko Poincarén epäyhtälö ja joilla on määritelty tuplaava mitta. Työ koostuu kolmesta artikkelista, joissa tutkitaan korkeampaa integroituvuutta funktioille, jotka toteuttavat käänteisen Hölderin epäyhtälön, Dirichlet'n integraalin kvasiminimoijille sekä superharmonisille funktioille.*

Avainsanat: BMO–funktio, Caccioppolin epäyhtälö, Gehringin lemma, geodeettinen avaruus, globaali integroituvuus, Hölderin alue, kapasiteetti, korkeampi integroituvuus, kvasiminimoija, käänteinen Hölderin epäyhtälö, metrinen avaruus, Muckenhouptin paino, Newtonin avaruus, p–paksuus, Poincaré'n epäyhtälö, superharmoninen funktio, superminimoija, stabiilisuus, tuplaava mitta.

Preface

I have been fortunate in many ways when carrying out this thesis. I am honored to have had the opportunity to work with my instructor Juha Kinnunen. He has provided me with interesting topics, valuable discussions, and opportunities for international collaboration. Working with my co–author Anna Zatorska–Goldstein has shown me how rewarding teamwork can be. My supervisor Olavi Nevanlinna has supported me whenever I have needed it, which I truly appreciate. Ph.D. Heli Tuominen and Professor Zoltán Balogh gave up their valuable time to read the manuscript.

Additionally, during these last three years I have had the chance to fully concentrate on my research thanks to the funding I have received from the *Finnish National Graduate School in Mathematical Analysis and Its Applications* and from the *Finnish Academy of Science and Letters, the Vilho, Yrjö and Kalle Väisälä Foundation*.

In the future, I will also most certainly miss the research group, Daniel, Mikko, Niko, Teemu, Tuomo, and Riikka, for reasons one can understand best on Fridays (that is more a state of mind than a day of the week). I also have the privilege of being surrounded by a loving and inspiring group of friends and family. Considering the past few years in particular, the Champagne Chicks merit a mention. The Kärkikarvaajat have been of remarkable help in their own furry way. As for Valter – words fail me, if I try to express his value.

But finally, as the greatest fortune of all, I want to thank Marjatta and Niilo Kansanen for being the best parents on this globe (pardon me, you other people with offspring).

Otaniemi, February 25, 2008

Outi Elina Maasalo
(née Kansanen)

Included articles

The dissertation consists of the following publications:

[I] O.E. Maasalo. The Gehring lemma in metric spaces. *http://arxiv.org, arXiv:0704.3916v3*

[II] O.E. Maasalo, A. Zatorska–Goldstein. Stability of quasiminimizers of the p–Dirichlet integral with varying p on metric spaces. To appear in *J. Lond. Math. Soc. (2)*.

[III] O.E. Maasalo. Integrability of p–superharmonic functions on metric spaces. To appear in *J. Anal. Math.*

Author's contribution

The author has played a central role in all aspects of the work reported in this dissertation. Articles [I] and [III] are the results of the author's independent research and in [II] the author is responsible for a substantial part of the writing and analysis. The results in [II] are partly based on a result that is studied both in [I] and in an article by the second author, Anna Zatorska–Goldstein.

The author has presented the results of [I]-[III] in analysis seminars held at universities including those at Cincinnati, Naples (Frederico II) and Oulu and Helsinki University of Technology.

Self–improving phenomena in the calculus of variations on metric spaces

Outi Elina Maasalo

1 Introduction

In this dissertation our main interest is in extending some classical results of the calculus of variations to metric measure spaces. Our work is related to the calculus of variations, nonlinear partial differential equations, and harmonic analysis. In this section we introduce metric spaces equipped with a doubling measure and a weak Poincaré inequality. Furthermore, we give a short overview of analysis on metric spaces.

A typical nonlinear variational problem is to minimize the p–Dirichlet integral

$$\int_\Omega |Du|^p dx$$

in an open subset Ω of $\mathbb{R}^n$ among all functions $u\colon \Omega \to \mathbb{R}$ which belong to a suitable Sobolev space and have prescribed boundary values. Equivalently we can solve the p–Laplace equation

$$\operatorname{div}(|Du|^{p-2}Du) = 0,$$

which is the Euler–Lagrange equation of the p–Dirichlet integral. In a general metric measure space the latter approach may not be possible. The space has no a priori smooth structure and it may not be possible to consider directions or coordinates. Thus it is not clear how to define the partial derivatives of a function or what the counterpart of the p–Laplace equation should be. However, in the variational approach to the Dirichlet problem, the *modulus* of the gradient plays an essential role. Indeed, first–order Sobolev spaces on a metric measure space can be defined in terms of the modulus of the gradient without the notion of distributional derivatives. Hence, methods of the calculus of variations can be applied in this context.

An immediate consequence of this approach is that it covers a wide range of spaces at the same time. The results can be applied in manifolds, graphs, vector fields, and groups, to mention only a few areas. However, and perhaps more importantly, by giving up the linear structure of the space, we are able to study phenomena separately from geometry. This can offer us a better understanding of the phenomena and also lead to new results, even in the classical Euclidean case.

The calculus of variations in metric spaces has mostly been developed during the past decade, and it is still a current and active research field. The monographs of Hajłasz and Koskela [23] and Heinonen [25, 26] are general reference works. A restricted list of papers contains, for example, the following works: related to Sobolev–type spaces on metric spaces, Cheeger [12], Hajłasz [22, 24], Heinonen and Koskela [28], Shanmugalingam [59, 60], and Koskela and MacManus [44]; for existence results for the Dirichlet problem, A. Björn, J. Björn, and Shanmugalingam [6] and Shanmugalingam [60]; for Sobolev– and Poincaré–type inequalities in metric spaces, Hajłasz and Koskela [23]; for regularity theory, Kinnunen and Shanmugalingam [42], and, finally, Kinnunen and Martio [39, 40] for the nonlinear potential theory. More references will be given in the following overview and in the research papers [I], [II], and, [III].

This dissertation is about various classes of functions related to the p–Dirichlet integral. We concentrate especially on quasiminimizers, superminimizers, and superharmonic functions. We study regularity, more precisely the integrability properties of the functions and their gradients. The integrability properties we are interested in are self–improving in the sense that they turn out to be better than it seems in the first place. We consider both local and global questions.

The Euclidean background of our research lies mainly in the works of Grandlund [21], Kilpeläinen and Koskela [36], Lindqvist [49, 50], Li and Martio [46, 47], and Reimann and Rychener [54]. We extend their results to the metric context. In the metric setting, an article by Buckley, [10], is important in our work. It provides a generalization of the Euclidean study by Smith and Stegenga [61]. We also prove a metric version of the celebrated Gehring lemma [18], which, besides being very interesting in itself, provides a powerful tool for solving regularity problems.

For the remainder of this chapter let (X, d, μ), or briefly X, denote a metric measure space.

1.1 Sobolev spaces on metric spaces

Several approaches to Sobolev spaces on metric spaces exist, but we will only consider two of them. We present the first only briefly before concentrating on the other, which we will adopt in this work. In general, the different definitions of Sobolev–type spaces do not lead to the same space, but there are a host of metric spaces where this is true, as we shall see.

An approach by Hajłasz, [22], is based on the observation that for $1 < p < \infty$ a p–integrable function is in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ if and only if there is a non–negative p–integrable function g such that

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y)) \tag{1.1}$$

for almost all x and y in $\mathbb{R}^n$. Any such function g is called a *Hajłasz gradient* of u. If u is a smooth function, we can choose g to be the Hardy–Littlewood maximal function of $|Du|$. For further properties we refer to [22], Heinonen and Koskela [28, 29], and Kinnunen, Kilpeläinen, and Martio [35]. A drawback of this characterization is that it applies only in the whole space $\mathbb{R}^n$ or in bounded open sets with the Sobolev extension property. An open bounded set with a Lipschitz boundary serves as a good example of such a set; see also Jones [31].

If the Euclidean distance is replaced by an arbitrary metric, Sobolev–type spaces on metric spaces can be defined as L^p–equivalence classes of functions that have p–integrable Hajłasz gradients. These are called *Hajłasz spaces*. It follows from the definition that a Hajłasz gradient is not unique, although if $1 < p < \infty$ there exists a unique g that minimizes the L^p–norm among all the p–integrable functions that satisty (1.1).

Sobolev spaces can also be defined in a metric setting also by introducing the notion of an upper gradient. These spaces are called *Newtonian spaces*. Let u be a function on X. A non–negative Borel measurable function g on X is said to be an *upper gradient* of u if, for all rectifiable paths γ joining points x and y in X, we have

$$|u(x) - u(y)| \leq \int_\gamma g ds \tag{1.2}$$

whenever $u(x)$ and $u(y)$ are both finite; otherwise, the path integral is defined asw being equal to infinity. Recall that a path is a continuous mapping from a compact interval of $\mathbb{R}$ to X and it is rectifiable if its length is finite. A path can thus be parametrized by arc–length. We also remind the reader, that a path is locally rectifiable if all of its closed subpaths are rectifiable. Upper gradients have been studied, for example, in Cheeger [12], Heinonen and Koskela [28], Koskela and MacManus [44], and Shanmugalingam [59, 60].

Inequality (1.2) implies immediately that, like a Hajłasz gradient, an upper gradient is not unique and that $g \equiv \infty$ is an upper gradient for every function. In $\mathbb{R}^n$ with the standard metric $g = |Du|$ is an upper gradient of a smooth function u.

Let Γ be a family of paths in X and $1 \leq p < \infty$. The p–modulus of Γ is defined as

$$\mathrm{Mod}_p(\Gamma) = \inf \int_X g^p d\mu,$$

where the infimum is taken over all Borel functions $g : X \to [0, \infty]$ satisfying

$$\int_\gamma g ds \geq 1$$

for all locally rectifiable $\gamma \in \Gamma$. For the definition of the path integral in metric spaces or further information on paths or the modulus, see Heinonen and Koskela [28], Shanmugalingam [59], or Väisälä [64].

If (1.2) fails only for a set of paths that is of zero p–modulus (i.e. holds for p–almost all paths), then g is said to be a *p–weak upper gradient*, or, in short, a *weak upper gradient*, of u. The set of all L^p–integrable weak upper gradients is exactly the L^p–closure of the set of L^p–integrable upper gradients of a function u; see Koskela and MacManus [44].

As well as for Hajłasz gradients, there exists a minimal weak upper gradient. Every function u that has a p–integrable weak upper gradient has a *minimal p–integrable weak upper gradient* denoted g_u. Here g_u is minimal in the sense that

$$\|g_u\|_{L^p(X)} = \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all weak upper gradients of u. Moreover, if g is a p–integrable upper gradient of u, then $g_u \leq g$ μ–almost everywhere in X; see Hajłasz [24].

Neither of the candidates for a gradient has all the good qualities of the Euclidean gradient, such as linearity. To illustrate this, let us consider the sum of two functions, u and v. Now the sum of their individual weak upper gradients is valid for a weak upper gradient of $u + v$. On the contrary, if g and h are weak upper gradients of u and v, respectively, the difference $g - h$ may not be valid for a weak upper gradient of $u - v$. A mild consolation is that the sum $g + h$ is fit for a weak upper gradient of $u - v$ as well. The same holds true for Hajłasz gradients.

Nevertheless, some properties of a weak upper gradient make it more practical than a Hajłasz gradient. Indeed, it has better local properties. If a function is constant somewhere, say in an open set, we would like its gradient to be zero there. A weak upper gradient of a function can be chosen to be zero almost everywhere the function is constant; see A. Björn and J. Björn [5] or Shanmugalingam [59]. A Hajłasz gradient does not have this property. A weak upper gradient behaves somewhat like the norm of the gradient, while a Hajłasz gradient is more like a maximal function. Hence, the behavior of a Hajłasz gradient is more global.

Another useful property of the upper gradient approach is the following: every p–integrable function that has a p–integrable weak upper gradient is *absolutely continuous on almost all paths*, or briefly ACC_p; see [59]. This is the metric counterpart of the well–known ACL–property of Sobolev functions in $\mathbb{R}^n$, that is, they are absolutely continuous on almost all lines parallel to the coordinate axes.

More precisely, let $\ell(\gamma)$ denote the length of γ. A function u is said to be absolutely continuous on path γ if $u \circ \tilde{\gamma}$ is absolutely continuous on $[0, \ell(\gamma)]$, where $\tilde{\gamma}$ is the arc–length parametrization of γ. This property gives us a way, in some sense, to calculate weak upper gradients.

Indeed, if u is a p–integrable function and there is a Borel measurable function g such that for p–almost every path γ the function $h\colon s \mapsto u(\gamma(s))$ is absolutely continuous on $[0, \ell(\gamma)]$ and

$$|h'(s)| \leq g(\gamma(s)) \tag{1.3}$$

almost everywhere on $[0, \ell(\gamma)]$, then g is valid for an upper gradient of u. On the other hand, if g is a weak upper gradient of u, then (1.3) holds true almost everywhere on $[0, \ell(\gamma)]$ for p–almost every path γ. This is already a convenient property in $\mathbb{R}^n$. Instead of the norm of the gradient it is sufficient to find a suitable majorant.

Let us now take a closer look at the Newtonian spaces. We define for $1 \leq p < \infty$ the space $\widetilde{N}^{1,p}(X)$ to be the collection of all p–integrable functions u on X that have a p–integrable p–weak upper gradient g on X. The space is equipped with the seminorm

$$||u||_{\widetilde{N}^{1,p}(X)} = ||u||_{L^p(X)} + \inf ||g||_{L^p(X)},$$

where the infimum is taken over all p–weak upper gradients of u. Note that the norm in $\widetilde{N}^{1,p}(X)$ is precisely the sum of the L^p–norm of the function and of the L^p–norm of the minimal weak upper gradient.

We define an equivalence relation in $\widetilde{N}^{1,p}(X)$ by saying that $u \sim v$ if

$$||u - v||_{\tilde{N}^{1,p}(X)} = 0.$$

The *Newtonian space* $N^{1,p}(X)$ is then defined to be the quotient space $\tilde{N}^{1,p}(X)/\sim$ with the norm

$$||u||_{N^{1,p}(X)} = ||u||_{\widetilde{N}^{1,p}(X)}.$$

The normed space $(N^{1,p}(X), \|\cdot\|_{N^{1,p}(X)})$ is a Banach space, and, as is common, we call $u \in N^{1,p}(X)$ functions instead of speaking of equivalence classes. In $\mathbb{R}^n$ equipped with the n–dimensional Lebesgue measure and the Euclidean metric, this definition coincides with the classical definition of Sobolev spaces.

The concept of an upper gradient and thus of Newtonian spaces can be defined in any metric space. If the space has no rectifiable curves, or more generally the modulus of the family of rectifiable curves is zero, Newtonian spaces degenerate to $L^p(X)$. In contrast, in spaces with an abundance of rectifiable curves, an interesting analog to the theory of Sobolev spaces can be developed. Hence we need assumptions to guarantee that our approach is meaningful, and we have a sufficient number of tools of analysis available.

1.2 Doubling metric space with a Poincaré inequality

We make two rather standard, yet nontrivial, assumptions. We require that the metric space X supports a doubling measure μ and a weak $(1,p)$–Poincaré inequality. Let us discuss these notions.

1.2.1 Doubling measure

A metric space is said to be *doubling* if there exists a fixed number N such that every ball of radius $r > 0$ can be covered by at most N balls with radii $r/2$. This property is weaker than carrying a *doubling measure*; a positive Borel regular measure is said to be *doubling* if there exists a constant $c_d > 0$, called the *doubling constant*, such that

$$\mu(B(x, 2r)) \leq c_d \mu(B(x, r))$$

for every x in X and for all $r > 0$. Iterating the doubling condition we can prove the following growth condition: for all $x \in X$ and $R \geq r$ we have

$$\frac{\mu(B(x,R))}{\mu(B(x,r))} \leq c_d \left(\frac{R}{r}\right)^Q,$$

where $Q = \log_2 c_d$. The constant Q is called the *doubling dimension* of the space. Indeed, this implies that a metric space with a doubling measure is in some sense finite–dimensional.

A space supporting a doubling measure is always doubling as a metric space; the reader can see, for example, Semmes [58] for a proof. Conversely, a complete doubling metric space can be equipped with a doubling measure; see Luukkainen and Saksman [51], Vol´berg and Konyagin [65] and Wu [66]. There are, however, non-complete doubling metric spaces that do not support doubling measures; see Saksman [55]. From now on we will consider spaces with a doubling measure, even though in some cases the doubling property of the space itself would be sufficient.

A metric space equipped with a doubling measure has many useful properties. For instance, such a space is always locally compact. If the space is, in addition, complete, then it is proper, in other words its closed and bounded subsets are compact. This is a strictly stronger property than being locally compact. Furthermore, in a space with a doubling measure we have the Lebesgue theorem and Vitali–type covering theorems with a countabe number of balls; see Heinonen [25]. These important tools are needed, for example, in the proofs of various strong– and weak–type inequalities for maximal functions. The validity of the Vitali covering theorem, especially, is crucial in obtaining the main results of this work.

We will now give a couple of examples of doubling measures. The most typical ones are the n–dimensional Lebesgue measure or weighted Lebesgue measures on $\mathbb{R}^n$. If the Lebesgue measure is weighted, for example, with a Muckenhoupt weight or, more generally, with a function satisfying a reverse Hölder inequality, the resulting measure is doubling. In the case of Muckenhoupt weights this remains true if $\mathbb{R}^n$ is replaced by any metric space and the Lebesgue measure by any doubling measure. In [I] we prove that if the metric space satisfies an additional geometric assumption, a function satisfying a reverse Hölder inequality also induces a doubling measure.

We recall that a metric measure space is s–Ahlfors regular if there exists a constant $c \geq 1$ such that

$$c^{-1} r^s \leq \mu(B(x,r)) \leq c r^s$$

for all $x \in X$ and $r > 0$. It is a direct consequence of the definitions that Ahlfors–regular measures are doubling, but the converse is not necessarily true. The Ahlfors regularity of a measure means that all balls ”look alike” regardless of their size or their location in the space. In metric spaces with a doubling measure this is true only for balls located near each other, which is a weaker argument.

1.2.2 Weak Poincaré inequality

We say that the space supports a weak $(1,q)$–*Poincaré inequality* if there exist $c > 0$ and $\tau \geq 1$ such that

$$\fint_{B(x,r)} |u - u_{B(x,r)}| d\mu \leq cr \left(\fint_{B(x,\tau r)} g^q d\mu \right)^{1/q}$$

for all x in X, $r > 0$ and all pairs $\{u, g\}$ where u is a locally integrable function and g is a q–weak upper gradient of u. The above inequality is called weak since we allow a larger ball on the right–hand side. In $\mathbb{R}^n$ we can always take $\tau = 1$. For the sake of brevity, we sometimes call it a Poincaré inequality and omit "weak".

At first sight a Poincaré inequality may seem merely to be a way of integrating a function from its derivative. Indeed, the Poincaré inequality provides a connection between the infinitesimal and, on the other hand, larger–scale behavior of a function. This gives us a way to control a function by its weak upper gradient. Notice also that the measure μ does not appear explicitely in the definition of the weak upper gradient, and that they are linked together by the Poincaré inequality.

On the other hand, supporting a Poincaré inequality entails, perhaps surprisingly, many geometric properties for a metric space. Some of these implications are fundamental in our work. An immediate consequence is that a space supporting a Poincaré inequality has to be connected. Moreover, speaking loosely, we could say that supporting a Poincaré inequality guarantees for the space the existence of short rectifiable curves.

Next we will briefly present some of the geometric properties a doubling measure and a weak Poincaré inequality imply for the space. Furthermore, we will discuss the additional geometrical assumptions, such as geodecity and local linear connectivity, required in many problems of the variational calculus.

1.2.3 Poincaré inequality with a doubling measure

The following embedding theorem is from Hajłasz and Koskela [23], but see also [17] and the survey in [23] for related results.

In a doubling metric measure space a weak $(1,q)$–Poincaré inequality implies a weak (t,q)–Poincaré inequality for some $t > q$ and possibly a new τ. More precisely, there exist $c' > 0$ and $\tau' \geq 1$ such that

$$\left(\fint_B |u - u_B|^t d\mu \right)^{1/t} \leq c' r \left(\fint_{\tau' B} g^q d\mu \right)^{1/q}, \tag{1.4}$$

where

$$\begin{cases} 1 \leq t \leq Qq/(Q-q) & \text{if} \quad q < Q, \\ 1 \leq t & \text{if} \quad q \geq Q, \end{cases}$$

for all balls B in X, and Q is the doubling dimension.

Let $1 < q < \infty$. The smaller the exponent q, the stronger the $(1, q)$–Poincaré inequality. Indeed, if X supports a weak $(1, q)$–Poincaré, then it supports a weak $(1, q')$–Poincaré for all $q' > q$ by the Hölder inequality. The converse is not true in general. However, by a deep result of Keith and Zhong [34] a weak $(1, p)$–Poincaré implies a weak $(1, q)$–Poincaré for some $q < p$ in complete spaces that support a doubling measure. This plays an important role in the higher integrability result in [II].

A metric space equipped with a doubling measure and a Poincaré inequality offers fruitful ground for analysis. In this context the Hajłasz and Newtonian approaches lead to the same Sobolev space if $1 < p < \infty$. A number of properties of the Euclidean case hold good as well. For example, in the resulting Sobolev space Lipschitz functions form a dense set; see [59] or [60]. This is a counterpart of the classical result stating that smooth functions are dense in $W^{1,p}(\Omega)$ whenever Ω is an open set of $\mathbb{R}^n$.

The meaning of the two standard requirements is not yet thoroughly understood; for example, only a few sufficient conditions for a Poincaré inequality are known to this day. Nevertheless, the group of metric spaces satisfying these assumptions is large and interesting. We only give here a few examples here. Weighted Euclidean spaces, which we mentioned while discussing doubling measures, also support a $(1, p)$–Poincaré inequality; see the monograph of Heinonen, Kilpeläinen, and Martio [27]. Riemannian manifolds with non–negative Ricci curvature satisfy the (1, 2)-Poincaré inequality; see Saloff–Coste [56]. Additionally, many graphs support a weak Poincaré inequality and the counting measure is doubling on them; see, for example, Hajłasz and Koskela [23]. For every $s > 1$, Laakso [45] showed that there is an Ahlfors s-regular space satisfying the (1, 1)-Poincaré inequality. A longer list of examples with associated references can be found, for example, in Keith [33].

1.2.4 Length metrics and local linear connectivity

Sometimes we need more geometric structure than a doubling measure and a Poincaré inequality imply. In some cases we have to assume that a space is a length space or locally linearly connected. Let us discuss these and some related notions and their connection to the doubling property and the Poincaré inequality.

A metric space (X, d) is said to be *quasiconvex* if there exists a constant c such that every pair of points x and y in X can be joined by a path whose length is at most $cd(x, y)$. Furthermore, a metric d is called a *length metric* if for all x and y in X we have

$$d(x, y) = \inf \operatorname{length}(\gamma),$$

where the infimum is taken over all rectifiable paths joining x and y. If there exists a minimal curve whose length is equal to the distance, the space is called a *geodesic* one.

A length space is always quasiconvex and a geodesic space is always a length space, but the converse may not be true. A complete metric space supporting a doubling measure and a Poincaré inequality is always quasiconvex; see Cheeger [12] and Keith [32]. Additionally, a complete locally compact length space is always geodesic. This implies that in complete spaces that carry a doubling measure a length metric is actually geodesic. Finally, in a quasiconvex and proper metric space it is possible to define a new geodesic metric that is bi–Lipschitz equivalent to the original one; see Heinonen [25].

The *local linear connectivity*, in brief the LLC property, of X means that there exist constants $c > 0$ and $r_0 > 0$ such that for all balls $B(x, r)$ in X with a radius at most r_0, every pair of points in the annulus $B(x, 2r) \setminus \overline{B}(x, r)$ can be connected by a curve lying in the annulus $B(x, 2cr) \setminus \overline{B}(x, c^{-1}r)$. The definition of LLC we assume here is the same as in J. Björn, MacManus, and Shanmugalingam [9] and it is stronger than the one in Heinonen and Koskela [28].

Although the definition is simple in a sense, it may be hard to see what restrictions it imposes on a space. One possibility is to compare it to the Poincaré inequality. What do they have in common, if anything? We stated above that the validity of a Poincaré inequality is in fact in a close relationship to the geometry of the space.

It is possible to construct examples of spaces that admit a Poincaré inequality but are not locally linearly connected. The Euclidean space $\mathbb{R}$ equipped with the one–dimensional Lebesgue measure offers a simple example. The space supports a $(1, 1)$–Poincaré inequality, but the annuli are disconnected. Thus Poincaré does not always imply LLC. Nonetheless, this is true in a complete metric space with a doubling measure that satisfies with some $s > 1$

$$\frac{\mu(B(x, r))}{\mu(B(x, R))} \leq c\Big(\frac{r}{R}\Big)^s$$

for all $x \in X$ and $0 < r \leq R$. With these assumptions a $(1, p)$–Poincaré implies LLC for all $p \leq s$; see Korte [43], as well as Hajłasz and Koskela [23].

As a conclusion we could say that a doubling measure and a Poincaré inequality do not always guarantee that a space is geodesic or LLC, but in a complete space these extra assumptions are not very restrictive.

2 Self–improving phenomena

This section is devoted to an overview of Papers [I], [II], and [III]. In particular, we focus on the covering arguments that we use in them. Throughout the chapter (X, d, μ), or briefly X, is a complete metric space, where μ is a doubling measure. If not otherwise mentioned, Ω is an open subset of X. We will impose additional requirements when they are needed. The main results of [I] and [II] are already known in the Euclidean case and we extend them to metric spaces. The main theorem of [III] is also new in the classical context.

We consider both local and global integrability questions. By a global property we mean that it holds true in an open proper subset of the space. When speaking of integrability, the term 'self–improving' has two meanings for us. On one hand, it stands for an integrability property that a class of functions already possesses, but that is actually better than it seems in the first place. This is the case in the first two papers. In [I] we prove the local higher integrability of a function that satisfies a reverse Hölder inequality, and in [II] the global higher integrability of quasiminimizers and their upper gradients. On the other hand, we consider functions that are not a priori integrable, but turn out to be so. We show in [III] that superharmonic functions are globally integrable to a small exponent. In both senses the better integrability property is built in in the class of functions, but it may be hard to see this from the definition.

Global problems naturally involve some constraints on Ω. For instance, it may be necessary to assume that the complement of the domain satisfies some type of a measure or a capacity thickness condition, or that the domain itself is, for example, a Hölder domain. These assumptions are already needed in the classical Euclidean case.

This is an essential part of what we could call a *from–local–to–global phenomenon*. For example, superharmonic functions are defined via the comparison principle, which implies that their definition is local. They are well known to be locally integrable, but why would this lead to global integrability? Indeed, in a general open subset Ω of X this may not be true. However, it turns out that in some cases the particular geometry of the set enables local properties to be transferred into global ones.

2.1 Self–improving of the reverse Hölder inequality

Every non–negative locally integrable function satisfies the Hölder inequality

$$\fint_B f\,d\mu \le \left(\fint_B f^q\,d\mu\right)^{1/q}$$

for all $1 < q < \infty$ and all balls B of X. Some of these functions also satisfy a reversed Hölder inequality for some exponent $1 \le p < \infty$. In other words there is a constant $c > 0$ such that the inequality

$$\left(\fint_B f^p\,d\mu\right)^{1/p} \le c\fint_B f\,d\mu \tag{2.5}$$

holds true for all balls B of X with the constant c independent of B. Such functions include Muckenhoupt weights and Jacobians of quasisymmetric mappings. If (2.5) holds true for p, it clearly holds true for all exponents smaller than p. It is thus natural to ask whether it holds true for any exponent $p' > p$, possibly with another constant c.

From a result obtained by Gehring, we know that this is true in $\mathbb{R}^n$ equipped with the n–dimensional Lebesgue measure; see [18]. Indeed, if a function satisfies (2.5), there exists $\varepsilon > 0$ such that

$$\left(\fint_B f^{p+\varepsilon} dx \right)^{1/p+\varepsilon} \leq c \fint_B f dx$$

for some other constant c. The proof is based on a covering argument and reverse weak–type inequalities.

It is mathematical folklore that the Gehring lemma also remains true in a metric space equipped with a doubling measure. Various versions of the lemma have been studied by, among others, D'Apuzzo and Sbordone [14], Fiorenza [16], Gianazza [19], Kinnunen [38, 37], Sbordone [57], Strömberg and Torchinsky [62], and Zatorska–Goldstein [67]. Our purpose is to prove the original version of the Gehring lemma.

The idea of the proof is the following. We fix a ball B_0 in X and consider a non–negative function f that satisfies (2.5) for $1 < p < \infty$. Since the inequality holds true in all balls with the same uniform constant, it is possible to transfer information to the distribution sets of the Hardy–Littlewood maximal function of f.

We take an arbitrary $q > p$ to begin with. One of the main steps in the proof is to estimate the integral of f^q over the intersection of B_0 and the level set $\{Mf > \lambda\}$, where λ is greater or equal to $\operatorname{ess\,inf}_{B_0} Mf$. The proof of this reverse weak–type inequality is rather standard after we have shown that

$$\int_{B_0 \cap \{Mf > \lambda\}} f^p d\mu \leq c\lambda^p \mu(100B_0 \cap \{Mf > \lambda\}). \tag{2.6}$$

This is always true if $p = 1$, but the case $p > 1$ requires a reverse Hölder inequality. The coefficient 100 could be replaced by any other sufficiently big constant, but the point is that working with balls we cannot avoid having a bigger ball on the right–hand side of the inequality. We will come back to this shortly.

We prove (2.6) by a covering argument. First, we cover the intersection of B_0 and $\{Mf > \lambda\}$ by balls whose centers x are in the set and whose radii are

$$r_x = \operatorname{dist}(x, 100B_0 \setminus \{Mf > \lambda\}).$$

Using the Vitali 5–covering theorem, we get a countable covering by pairwise disjoint balls $\{B_i\}$ such that the intersection of B_0 and $\{Mf > \lambda\}$ is included in the union of $5B_i$. The advantage of this covering is that the intersection of $5B_i$ and $\{Mf \leq \lambda\}$ is not empty and that the union of $5B_i$ is included in $100B_0$. The first property enables us to bound above the integral averages of f over $5B_i$ by λ. The latter assures that we work in a fixed ball and thus all the balls we deal with are balls of the metric space (X, d).

Since the intersection of B_0 and $\{Mf > \lambda\}$ is open and bounded we may be tempted to try constructing a Whitney–type covering and thus avoid working in a larger ball. However, if we cover the intersection set with balls that stay inside the set, there may not exist $\sigma > 1$ such that the intersection of σB_i and $\{Mf \le \lambda\}$ is nonempty, which is relevant in order to control the integral averages.

With the covering that we have constructed we get

$$\int_{B_0 \cap \{Mf>\lambda\}} f^p d\mu \le \sum_{i=1}^{\infty} \mu(5B_i) \fint_{5B_i} f^p d\mu,$$

and (2.6) follows by the reverse Hölder inequality and estimating integral averages by λ^p.

We are now ready to make a rough sketch of the rest of the proof. Our method is to estimate

$$\int_{B_0} f^q d\mu = \int_{B_0 \cap \{Mf>\alpha\}} f^q d\mu + \int_{B_0 \cap \{Mf \le \alpha\}} f^q d\mu$$

in an arbitrary ball B_0 in X and with $\alpha = \operatorname{ess\,inf}_{B_0} Mf$. We apply the weak–type inequality to the first integral and obtain

$$\begin{aligned}\int_{B_0} f^q d\mu \le c\alpha^q \mu(100B_0 \cap \{Mf > \alpha\}) + c\frac{q-p}{q} \int_{100B_0} (Mf)^q d\mu \\ + \alpha^q \mu(100B_0) \cap \{Mf \le \alpha\}).\end{aligned}$$

Next we choose $0 < \varepsilon < 1$ and a possibly smaller q such that $c(q-p)/p < \varepsilon$. Then, by the choice of α and by using the reverse Hölder inequality together with basic estimations we get

$$\fint_{B_0} f^q d\mu \le \varepsilon \fint_{100B_0} f^q d\mu + c \left(\fint_{100B_0} f d\mu \right)^q.$$

Our second key lemma is an iteration lemma that gives us

$$\fint_{B_0} f^q d\mu \le c \left(\fint_{2B_0} f d\mu \right)^q, \tag{2.7}$$

and we are almost done.

The Calderón–Zygmund type argument we use in the proof produces a ball $2B$ on the right–hand side of (2.7). However, the measure induced by a function satisfying a reverse Hölder inequality turns out to be doubling in a metric space that satisfies the annular decay property. We say that a metric space satisfies the *annular decay property* for $0 < \alpha \le 1$ if there exists a constant $c \ge 1$ such that

$$\mu(B(x,r) \setminus B(x,(1-\delta)r)) \le c\delta^\alpha \mu(B(x,r)) \tag{2.8}$$

for all $x \in X$, $r > 0$ and $0 < \delta < 1$, see [11]. A typical example of such a space is a length space supporting a doubling measure.

Otherwise the proof is independent of the decay property. The author does not know if this assumption can be removed. It seems that even in the classical Euclidean case the annular decay property of $\mathbb{R}^n$ is needed if we consider balls instead of dyadic cubes.

The Gehring lemma is like the first domino that topples; it implies other self–improving properties. For instance, Muckenhoupt A_p–weights satisfy a reverse Hölder inequality. Because of the Gehring lemma, they satisfy a better reverse Hölder inequality, and thus belong to a better Muckenhoupt class with a smaller p. Since an A_p–weight always induces a doubling measure, the result holds true without the assumption of the annular decay property.

The better reverse Hölder inequality for Muckenhoupt weights in the metric setting can also be obtained by using a different covering argument. The proof by Aimar, Bernaedis, and Iaffei, [1], uses a construction of dyadic–type families introduced by Christ [13].

The Gehring lemma also has applications in the potential theory. In Article [II] we show that the minimal weak upper gradients of quasiminimizers satisfy a reverse Hölder inequality.

2.2 Global higher integrability of quasiminimizers

Article [II] is a joint work with Anna Zatorska–Goldstein. We extend the work of Granlund [21], Kilpeläinen and Koskela [36], and of Lindqvist [49] to the metric context. We suppose that the space has the LLC property and supports a weak $(1, p)$–Poincaré inequality for some $1 < p < \infty$.

We study the behavior of a sequence of p–Dirichlet integral quasiminimizers as p varies. A function $u \in N^{1,p}(\Omega)$ is called a *K–quasiminimizer* if it minimizes the Dirichlet functional up to a multiplicative constant K; that is, for all $\Omega' \subset\subset \Omega$ we have

$$\int_{\Omega'} g_u^p d\mu \leq K \int_{\Omega'} g_v^p d\mu$$

for all functions $v \in N^{1,p}(\Omega)$ which have the same boundary values as u. The notion of quasiminimizers in $\mathbb{R}^n$ was introduced by Giaquinta and Giusti in [20] as a tool for the unified treatment of variational integrals, elliptic equations and systems, obstacle problems, and quasiregular mappings. See also DiBenedetto and Trudinger [15].

Minimizers of the p–Dirichlet integral are 1–quasiminimizers, and in the Euclidean setting they are weak solutions of the p–Laplace equation. Naturally this is a local property. However, when $K > 1$, being a quasiminimizer is not a local property. Moreover, there is no uniqueness in the Dirichlet problem, nor any comparison principle for them. Quasiminimizers also lack a linear structure. The theory of quasiminimizers, therefore, differs from the theory of minimizers.

Quasiminimizers have already been an active research subject for several years in the setting of a doubling metric measure space with a Poincaré inequality. We will mention only a few examples. A. Björn and Marola have studied the Moser iteration method for quasiminimizers [7]. The boundary continuity for quasiminimizers on a bounded set Ω with fixed boundary data has been examined by J. Björn [8]. It has been proved by Kinnunen and Shanmugalingam that quasiminimizers are (locally) Hölder continuous; see [42]. In [40], Kinnunen and Martio studied the nonlinear potential theory for quasiminimizers. They proved, for example, that the class of (local) quasisuperminimizers, for fixed p is closed under monotone convergence, provided that the limit function is bounded. Later, Kinnunen, Marola, and Martio proved that an increasing sequence of quasiminimizers converges locally uniformly to a quasiminimizer, provided that the limit function is finite at some point, even if the quasiminimizing constant and boundary values are allowed to vary.

Our main theorem is also a stability result. We consider a sequence (u_i) where $u_i\colon \Omega \to \mathbb{R}$ is a K–quasiminimizer of the p_i–Dirichlet integral in an open bounded subset Ω of X. We assume that all functions u_i have the same boundary data in $N^{1,p+\varepsilon}(\Omega)$ for a fixed $\varepsilon > 0$. Furthermore, we assume that the complement of Ω is uniformly p–fat, that is there exists a constant $c_0 > 0$ and $r_0 > 0$ such that for all x in $X \setminus \Omega$ and $0 < r < r_0$ we have

$$\operatorname{cap}_p\big((X \setminus \Omega) \cap B(x,r);\ B(x,2r)\big) \geq c_0 \operatorname{cap}_p(B(x,r); B(x,2r)).$$

Here $\operatorname{cap}_p(E,F)$ denotes the relative p–capacity of E with respect to F. We define this in more detail in Section 2.1.4. of [II].

We prove that if p_i converge to p, then there exists a K–quasiminimizer u of the p–energy integral in Ω with the same boundary data, such that u_i converges to u in $L^p(\Omega)$. There exists a similar Euclidean result for solutions of an obstacle problem and of a double obstacle problem; see Li and Martio [46] and [47], respectively.

Quasiminimizers and their minimal upper gradients are a priori integrable to the exponent p in Ω. To be able to prove the convergence theorem, we prove first that they are globally integrabile to a higher exponent in Ω. In the Euclidean case, Kilpeläinen and Koskela proved a similar result for solutions of the p–Laplace equation; see [36]. The idea of our proof is to show that the minimal upper gradients satisfy a weak reverse Hölder inequality, apply the Gehring lemma, and generalize the resulting local higher integrability to the whole Ω. To this end, we need a suitable covering argument.

Since we are considering quasiminimizers with boundary data, we are able to work near and on the boundary. In other words, if u is a quasiminimizer with a boundary function w, then $u - w \in N_0^{1,p}(\Omega)$ and we can set $u - w$ zero outside Ω. This gives us the opportunity to cover Ω by balls that are inside the set, together with those that intersect the complement.

Inside Ω, a Caccioppoli–type inequality by Kinnunen and Shanmugalingam [42] implies immediately that the minimal upper gradient satisfies a reverse Hölder inequality. Near the boundary, we have to work more. Here the p–fatness of the complement plays a role. Furthermore, we need two self–improving properties: that of the weak Poincaré inequality and that of the p–fatness condition. It is a result of J. Björn, MacManus and Shanmugalingam,

[9], that in a complete LLC metric space, that supports a doubling measure and a weak $(1, p)$–Poincaré inequality, the p–fatness condition implies a $(p - \delta)$–fatness for a $\delta > 0$. In addition to the Caccioppoli–type inequality, this allows us to use a capacity version of a Sobolev–Poincaré–type inequality by J. Björn [8], and obtain the reverse Hölder inequality. Finally, since Ω is bounded a finite number of the two types of balls suffices to cover it.

2.3 Global integrability of superharmonic functions

In Article [III] we generalize the result of Lindqvist [50] in $\mathbb{R}^n$ to the metric case. For preceding studies in $\mathbb{R}^n$ and on the complex plane see, for example, Armitage [2, 3], Masumoto [53], Maeda and Suzuki [52], and Suzuki [63].

We assume that the space supports a length metric and a weak $(1, p)$–Poincaré inequality for $1 < p < \infty$.

Imitating the Euclidean definition, we define p–superharmonic, or briefly superharmonic, functions in the metric context in the following way. A lower semicontinuous function $u \colon \Omega \to \mathbb{R}$ is called *p–superharmonic* in Ω if it obeys the comparison principle with respect to continuous minimizers of the p–Dirichlet integral. For other equivalent ways of defining p–superharmonic functions in the metric setting, we refer to A. Björn [4] and Kinnunen and Martio [40, 41].

There is a subtle difference between supersolutions and superharmonic functions. A superharmonic function is lower semicontinuous and defined at every point in its domain, but supersolutions are defined only up to a set of measure zero. Superharmonic functions do not a priori belong to a Sobolev space. Consequently, it is not evident how to relate them to the p–Laplace equation, whereas supersolutions have Sobolev derivatives. However, it turns out that all weak supersolutions have lower semicontinuous representatives and, in particular, lower semicontinuous supersolutions are superharmonic. By contrast, superharmonic functions are not supersolutions in general.

It has been shown, in the Euclidean setting by Lindvist [48] and in the metric setting by Kinnunen and Martio [41], that superharmonic functions are locally integrable to a small exponent. We are interested in their global integrability over open subsets of the space. We prove that if Ω is a Hölder domain in X and u a positive superharmonic function in Ω, then there exists $\beta_0 > 0$ such that u belongs to $L^\beta(\Omega)$ for all $0 < \beta \leq \beta_0$.

We remind the reader that a connected open subset Ω of X is a Hölder domain if there exists a constant c such that for all $x \in \Omega$ we can find a path γ_x joining x to a fixed point $x_0 \in \Omega$ such that

$$\int_{\gamma_x} \frac{ds(t)}{\operatorname{dist}(t, X \setminus \Omega)} \leq c \log\Big(\frac{c}{\operatorname{dist}(x, X \setminus \Omega)}\Big).$$

The idea of the Hölder condition is, loosely speaking, that all points of the domain can be connected to a fixed point by a chain of balls that is well inside the domain and is such that the consecutive balls of the chain intersect sufficiently with each other.

The general idea of the integrability proof is rather simple. We start by reminding the reader that a locally integrable function $u\colon \Omega \to \mathbb{R}$ is in $\mathrm{BMO}(\Omega)$ if there exists a constant c such that

$$\fint_B |u - u_B| \, d\mu \leq c$$

for all balls B in Ω. We say that u is in $\mathrm{BMO}_{\mathrm{loc}}(\Omega)$ if the inequality holds for all balls B in Ω such that $2B \subset \Omega$. From now on we call these balls *admissible*. It follows immediately from the definition that $\mathrm{BMO}(\Omega) \subset \mathrm{BMO}_{\mathrm{loc}}(\Omega)$.

By a result of Buckley [10] we know that BMO functions are exponentially integrable over Hölder domains. In the Euclidean case this was proved independently by Hurri–Syrjänen [30] and Smith and Stegenga [61]. Therefore, it is enough to show that the logarithm of a superharmonic function is a BMO function. First, using Caccioppoli–type inequalities we show that this holds true for superminimizers. Then, approaching a superharmonic function by an increasing sequence of superminimizers, the result can be proved for superharmonic functions. The argument is somewhat similar to the corresponding Euclidean proof by Lindqvist.

The proof is based on three essential steps. First of all, there is the connection between superminimizers and superharmonic functions and, second, the powerful exponential integrability theorem. However, we are able to deal with superminimizers and superharmonic functions only *locally*, that is in subsets which are compactly contained in Ω. Therefore we can only prove that the logarithms of superharmonic functions are local BMO functions. This is not sufficient in the exponential integrability theorem of Buckley.

In $\mathbb{R}^n$, the well–known theorem of Reimann and Rychener in [54] states that the definitions of local and global BMO spaces are actually equivalent for all open Ω. This imbedding theorem is also true in length metric spaces equipped with a doubling measure; see [10]. In [III], we present a transparent proof for this. To illustrate differences between the Euclidean and the metric settings, we will briefly sketch the covering argument that we use in the proof.

For expository purposes we first construct the Whitney–type covering in $\mathbb{R}^n$. Fix a cube Q_0 in $\mathbb{R}^n$. We want to cover Q_0 by dyadic cubes so that from any cube in the covering we are able to move to work in $\frac{1}{2}Q_0$, which is admissible. To this end, define $Q_i = (1 - 2^{-i})Q_0$, $i = 1, 2, \ldots$. Divide each Q_i dyadically into $(2^{i+1} - 2)^n$ pairwise disjoint cubes, which cover Q_i up to the measure zero. We call the obtained family C_i. Define a new family of disjoint cubes W_i such that W_1 is C_1 and when $i = 1, 2, \ldots$, W_{i+1} consists of those cubes in C_{i+1} that do not intersect with any cube in W_i.

Now the cubes in W_i form a "rectangular" annulus $Q_i \setminus Q_{i-1}$ and the cubes in the union of all W_i cover Q_0 up to a set of measure zero. From each Q in W_i we can form a chain of cubes to $Q_1 = \frac{1}{2}Q_0$ such that $\widetilde{Q}_1$ belongs to W_{i-1}, $\widetilde{Q}_2$ belongs to W_{i-2} and finally $\widetilde{Q}_{i-2}$ belongs to W_2. We choose the cubes in such a way that if we take a pair of consecutive cubes, there is at least one point, the corner, in the intersection of their closures. Now the length of the chain from Q in W_i is $i - 2$. This finishes the construction.

Let us now consider the metric setting. Given a ball $B_0 \subset \Omega$ with center x_0 and radius $R > 0$, we want to cover it by a countable family of admissible pairwise disjoint balls $\{B_i\}$ so that we can estimate

$$\fint_{B_0} |u - u_{B_0}| d\mu \leq 2 \frac{1}{\mu(B_0)} \int_{\cup_i B_i} |u - u_{\frac{1}{8}B_0}| d\mu,$$

where we have only admissible balls on the right–hand side. In the metric case, we allow ourselves a little space by choosing a smaller admissible ball $\frac{1}{8}B_0$ instead of $\frac{1}{2}B_0$. To estimate the integral on the right–hand side, we want to imitate the Euclidean case and construct a chain of admissible balls from each ball in the covering to $\frac{1}{8}B_0$. Then we need to control the length of these chains and, moreover, to estimate the difference between the integral averages of u over consecutive balls in the chain. Thus, every pair of consecutive balls has to have a nonempty intersection.

We start by covering B_0 with balls $B(x, r_x)$, where $x \in B_0$ and $r_x = (R - d(x_0, x))/40$. We choose r_x to be small enough that we can multiply it without losing the admissibility of the ball. Hence 40 can be replaced by any other sufficiently big constant. We use the Vitali 5–covering theorem to extract a countable subfamily of pairwise disjoint balls B_i so that the union of $5B_i$ covers B_0. It is important that $5B_i$ are still admissible.

Naturally, when constructing a chain of balls from $5B_i$ to $\frac{1}{8}B_0$ the length of the chain depends on the distance between the balls to be connected. In any metric space we can divide B_0 into annuli

$$B(x_0, (1 - 2^{-k})R) \setminus B(x_0, (1 - 2^{-(k-1)})R), \qquad k = 1, 2, \dots,$$

and then, in the spirit of the construction with cubes, cover them by subcovers of the original one for B_0. In this case the balls that cover the same annulus will possess almost equally long chains up to $\frac{1}{8}B_0$.

In a general situation, we are not able to estimate the measure of these annuli or their covers, even when the construction assures us that the balls near each other have about the same radii. However, in metric spaces that satisfy the annular decay property (2.8), defined in Section 2.1, this is possible.

Another issue is the actual construction of the chains. The convenience of the Euclidean setting is that the covering consists of pairwise disjoint cubes and the measure of each cube (with respect to Q_0) is known, as well as are the measures of "annuli" $Q_i \setminus Q_{i-1}$. Furthermore, the chains are constructed from cubes of the original covering, which is not possible in the metric case.

In order to connect two points by a chain of balls, there have to be, in some sense, enough points in the space between them. In our case, the points have to be connected by a path. This way we can choose balls with centers on the path and radii directly proportional to the distance of the centers to x_0. We need information on the length of the path to find out the number of balls that are needed.

This is why we choose to work in a length metric space. Equipped with a doubling measure, it satisfies the annular decay property and the length of a path between two points is equal to their distance. In this context we are able to calculate directly the length of the chain as a function of k from each ball in the covering with the center in the annulus

$$B(x_0, (1 - 2^{-k})R) \setminus B(x_0, (1 - 2^{-(k-1)})R).$$

This construction implies the equivalence of the two BMO–norms.

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